REVIEWS.

I.

A Budget of Paradoxes ...... By Augustus De Morgan (Reprinted, with the Author's additions, from the Athenæum). London: Longmans. 1872 *.

It is an opinion current among librarians, that there is no such thing as trash: that the most foolish unconnected flysheet treating of nothing at all should in all cases be preserved and bound up with other such flysheets, not in view of any possible future investigator to whom it may be as gold among quartz, but because it is right that this thing should be done. No doubt a very great good comes of this absolute universalism in the conscience of all kinds of collectors. But still, for the purposes of the outer world, it remains true that there are books and books. It is obvious of some kinds of literary or scientific work that if *A. B.* had not done it, *C. D.* would have taken his place; and that at no loss to the world, even though *C. D.* were a person of mean capacities. The bricklayer who was to lay a certain course of bricks may fall off a ladder and yet the house be no worse in the end; while the skilled mason who carves a gargoyle may leave something which represents not merely his day's wages but himself (invaluable) so long as stone shall last, and therein something which no other man could exactly produce. The book before us is essentially a gargoyle. It is by very far the most individual book of the age—individual, not merely in its own singularity as a book, but as presenting with a marked degree of clearness and exactness the personality of one who was never quite a man among men, but always a man among other men.

The paradoxes herein treated of are those set forth by people ignorant of mathematics, who think themselves qualified to shew such as are not ignorant where they have gone astray. It might well have been conceived that a large book on such a subject would have been the dullest of the dull; that it would appeal only to mathematical readers, and even to them only for so long a time as the follies exposed in it were of recent interest. No anticipation could be more thoroughly wrong. The fault of the book is in the direction of a too incessant playfulness; it is excellent grotesque, which is only to be borne because it is so clearly an outwork of the beautiful. For while the pretenders here slaughtered are for the most part indeed nobodies, whose only use is for an example and a warning; while these jokes (with some notable exceptions) are small jokes, and such as we like chiefly in our idiotic moods; while even the character which so clearly shews through these pages, great and lovable

* [From *The Academy*, Vol. IV. No. 78. *August 15th*, 1873.]

as it is, is yet rather singular than pre-eminent, a study for comparative psychology rather than an ideal for the world to come; while all this is true, the book is an endeavour and a stretching forth towards right thinking and a protest against wrong thinking which is of infinite solemnity and weight to us of this present time. For we have no right to conclude that these paradoxers upon whom the *Budget* has fallen have been sinners above all that prate; we ought rather to learn that except we mend our ways we shall all likewise perish.

The word *paradox* is unfortunate; it includes under one name a rare thing and a common thing, and it brings upon the rare thing which is good some of the discredit that belongs to the common thing which is bad. "A paradox is something which is apart from general opinion, either in subject-matter, method, or conclusion." The "general opinion" must be that of people who have an opinion; not of all people indiscriminately, including those who have never considered the subject. The common form of paradox consists in ignorance of the subject-matter, powerlessness in the method, or incapacity to understand the conclusion. The rare form of paradox is an addition to the reasonable part of general opinion, which happens to contradict some of the unreasonable part. The older use of the word was strictly impartial, and it might be applied without any want of respect; De Morgan says the change came in the seventeenth century. It is certain that at present the epithet is a disparaging one; the overplus of wrong thoughts included under it has slowly sapped the moral constitution of the word, and it now sometimes stands in the way of a right appreciation of the nobler form of paradox.

For as it is hardly possible to lay too great a stress on the weight and worthiness of thought which diverges from the general opinion on account of its greater strength, which by its continual work in the world has in fact built up the present mind of man; so it is before all things necessary here to distinguish carefully from it that other divergence which comes of weakness and goes to destruction. It is true in all departments of human action that reform is the most precious and sacred prerogative of a citizen; but in order to have that prerogative one must be a citizen, not an alien; and one must act like a citizen in a legitimate and constitutional way. A man who should find an error in the value of π—even in the six hundredth place—would have all honour paid him as a true reformer by the brotherhood; but to this two things are necessary: he must not be ignorant of trigonometry, and he must work out the calculation. The belief of the weak paradoxer, on the contrary, is that things can be done by a flash; that a discovery is to start from his ignorant and untried mind like Pallas from the brain of Zeus. We know, of course, that the great discoveries—the true and noble paradoxes—have always come from men who by long prenticeship have so far mastered the tools forged by their fathers that they were not tied down to one particular way of using them; we know that Jove's head cannot crack with Minerva unless he have previously swallowed Metis. The time taken by distant discoveries—gravitation for instance, is foreshortened by perspective; but we have good cases immediately before our eyes. In Maxwell's theory of Electricity we have as instructive an example of the paradox of

right thinking as might well be; a conclusive victory over rival doctrines won by twenty years' patient proving (and improving) of the weapons wherewith previous battles had been gained; a testimony to all time that genius is a capacity for taking an infinite amount of the right sort of trouble. But your paradoxer of the *Budget* will master by a *coup-d'état* the republic of science, which allows no masters, but proved comrades only; he will climb by the back stairs into the house of knowledge, that has no back stairs. If there be any reward in the penal incurable blindness that follows such sacrilege, verily he has his reward.

And here is another important difference between the two kinds of heretics. The strong heretic is so because his ideas are living and plastic, and have an internal motion whereby they adapt themselves continually to new work; so that no man is so perfectly open to conviction as he is. But the weak heretic is so from the very narrowness of his range, which cannot grasp even established demonstration; he is hermetically sealed against all possible argumentative germs that might bring into his mind the lower forms of life.

In drawing this sharp distinction between two habits of mind, however, we must not forget what the *Budget* is specially calculated to impress upon us in a terrible and alarming manner; the exceedingly gradual transition from one to the other, and the possible coexistence of both in the same person in regard to different subjects. De Morgan has some very good remarks on the value of a study of logic in helping us to extend the habits of right thinking which we have got by practice in one subject over the whole range of our knowledge. A good specialist who is also a good logician can hardly be betrayed into gross paradoxy out of his proper range; for his special knowledge will make him cautious about facts, and his logic about conclusions. No man could have greater advantages in this respect than the author of the *Budget*, who had himself made important additions to logic, and was an excellent mathematician. And yet—this is the solemn warning of the book—he has in one case fallen into a sin to which we are all tempted, whether by the uncompromising precepts of theological systems, or by the insidious seductions of scientific text-books; the sin of making assumptions and then hiding from ourselves that they are assumptions and that we have no right to believe in them. Apropos of "From Matter to Spirit," he says that he refers certain phenomena "*either* to unseen intelligence *or something which man has never had any conception of.*" This apparently suspended judgment involves and hides the assumption that the said phenomena cannot possibly be referred to certain well known and commonly conceived things—the art of the conjuror, and the delusion of contagious excitement. This enormous assumption is, of course unconsciously, introduced and hidden under a brilliant display of candid impartiality and cautious scepticism. We point to this, not as throwing a stone thereat; but desiring that it should indicate the great and serious importance of the *Budget of Paradoxes.* To sum up, this is a book that should be read by those who care about circle-squarers and all manner of jokes, mathematical and other; by those who care to make the acquaintance of Augustus de Morgan, which it is well worth while to do; but above all by those who care to be led into right thinking and warned from wrong.

II.

A Treatise on some New Geometrical Methods, containing Essays on the Geometrical Properties of Elliptic Integrals, Rotatory Motion, the Higher Geometry, and Conics derived from the Cones; with an Appendix to the First Volume. In Two Volumes. Vol. II. By James Booth LL.D., F.R.S., F.R.A.S., &c., Vicar of Stone, Buckinghamshire. (*pp.* xxxii. + 440) *London: Longmans & Co., Paternoster Row.* 1877*.

If Rip van Winkle, instead of being an idle scapegrace, had been a most original and accomplished geometer; and if, instead of sleeping on the mountains for twenty years, he had from time to time applied himself in a sheltered cave to mathematical pursuits; he might, on rejoining his neighbours in the valley, have produced such a treatise as this. In the meantime, those neighbours have grown a great deal wiser; they know a great deal more than they did when he left them to live in solitude. For one thing, they have learned to appreciate what he did before he went away. And accordingly their first impulse, when they recognise the veteran explorer, is to think how very delighted he will be with what they can teach him. Those who read these pages on the representation and application of elliptic integrals, based on no later authority than the treatise of Verhulst, will think with sympathetic delight of the pleasure with which the author will read the *Fundamenta Nova*, and the setting forth and completion of that theory in Cayley's treatise, to say nothing of the works of Abel and the all-embracing method of Riemann. When he speaks of a hyperelliptic integral as a thing at present wholly beyond the powers of analysis, we at once think of a heap of volumes and memoirs which we must give him to revel in; and we wish we could see his face as he took in the discoveries of Göpel and Rosenhain, of Weierstrass, Hermite, and Königsberger. And so in reading these chapters about the "Higher Geometry" and "Conics," we feel that we should like to introduce him to Reye's *Geometrie der Lage*, and to the admirable volumes of Lindemann; wondering whether he would recognise his own children in their present developed and systematized condition.

But, on examining a little more closely, we find that this is a one-sided view of the matter; and that, instead of thinking of all the beautiful things that we can teach *him*, it would be more profitable to pay attention to many beautiful things which he can teach *us*. Those, indeed, whose knowledge of mathematics is derived from Tripos text-books, and who are accustomed to think of an elliptic differential as a thing whose "integral cannot be found†,"

* [From the *Educational Times*, June, 1877.]

† Cheyne, *The Earth's Motion of Rotation*, 1867, p. 3. Such language is very common; but when it is considered that the θ-series is as legitimate an algebraic form as the exponential of a continuous quantity, it will, we think, appear more proper to say that elliptic differentials have been integrated in finite terms; especially as they are now a part of the university curriculum.

will suppose that they are here reading the latest developments of an obscure subject, and may profit largely by every paragraph. But the freshness and originality of treatment which are here to be found will prove an effective stimulus to those who have endeavoured to keep up with the rapid progress of science. The geometrical applications of Elliptic and Abelian functions which have chiefly interested later geometers are those in which the function of a complex argument is used to represent the aggregate of the real and imaginary points of a curve. Such applications bear upon the projective theory of curves, and it is rarely that any metrical properties emerge from them. But Dr Booth considers the representation of functions of a real argument by the rectification and quadrature of curves; and it is well worth while to be recalled for a time to this aspect of the elliptic functions, which is, of course, the most important for physical purposes. The study of these chapters in the light of later ideas and with the help of later notation cannot fail to be highly profitable. With a small example of this we shall conclude a very meagre notice.

Of the two modes of representation of elliptic integrals which Dr Booth uses, that by the arcs of quadriquadric curves is undoubtedly the more important; but considerable interest attaches to the expression of elliptic integrals of the first kind by the arcs of the negative pedal of an ellipse in regard to the centre. If arc $AP=s$, $ZP=t$, $AC=a$, angle $ACZ=\phi$, then

$$\phi=\operatorname{am}\left(\frac{s-t}{a}, e\right),$$

the modulus e being the eccentricity of the ellipse. The quantity $s-t$ is called the *residual arc*, and a method is given for finding a curve whose residual arc shall represent any given integral. In fact, $s-t=\int p\,d\phi$, if $p=CZ$; so that the tangential polar equation of the curve which represents $\int f(\phi)\,d\phi$ by its residual arc is $p=f(\phi)$. There are two special cases of interest. When $e=0$, the ellipse becomes a circle, and the negative pedal coincides with the circle itself; in this case $t=0$, and the equation becomes $a\phi=s$. If, while CA is kept constant, e is made to increase from 0 to 1, the ellipse will lengthen out until it becomes the pair of tangents at the extremities of the minor axis. The negative pedal then becomes two parabolas, and it is clear that we need only attend to one of them. The elliptic amplitude becomes a function which is called by Cayley the Gudermannian (*Elliptic Functions*, p. 56), and which may be thus defined:—If $\cos\phi\cos(iu)=1$, then $\phi=\operatorname{gd} u$. It follows that $iu=\operatorname{gd}(i\phi)$; other formulæ are

$$\tan\phi=i\sin(iu),\quad\text{whence}\quad u=\int_0 \sec\phi\,d\phi=\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right).$$

We have

$$\tan\operatorname{gd}(u+v)=i\sin(iu+iv)=i\sin(iu)\cos(iv)+i\sin(iv)\cos(iu)$$

$$=\tan\operatorname{gd} u\sec\operatorname{gd} v+\tan\operatorname{gd} v\sec\operatorname{gd} u.$$

Dr Booth writes this formula thus:—

$$\tan(\phi \perp \chi) = \tan\phi \sec\chi + \tan\chi \sec\phi;$$

so that $$\phi \perp \chi = \mathrm{gd}(\mathrm{gd}^{-1}\phi + \mathrm{gd}^{-1}\chi);$$

similarly, $$\phi \top \chi = \mathrm{gd}(\mathrm{gd}^{-1}\phi - \mathrm{gd}^{-1}\chi).$$

Observe that the operations $\perp$, $\top$ satisfy the commutative and associative laws.

This notation appears convenient for some purposes, and the reader will find a number of very interesting developments obtained by means of it. It seems worth considering whether a similar notation might not be applied with advantage to the more general function $\mathrm{am}\, u$, which, like $\mathrm{gd}\, u$, has no proper addition-theorem.

PROBLEMS AND SOLUTIONS *.

1362. For every point A on a rectangular hyperbola, there exists a straight line BC, passing through the centre, such that if through any other point D on the curve, lines be drawn parallel to the asymptotes, cutting BC in B, C, the intercept BC subtends a right angle at A. [March, 1863, solved May, 1863. The proposer's solution, bearing date March 30, 1863, is given Reprint, Vol. XXXII. p. 32.]

Let O be the centre of the hyperbola; join OA, and draw BOC perpendicular to it. Let DB, DC, parallel to the asymptotes, cut them in F, E; and draw AM perpendicular to either of the asymptotes.

Then we have $CO : OB = CE : ED$,

$$\therefore\ CO^2 : CO \, . \, OB = OE \, . \, EC : OE \, . \, ED \text{ .... } (1).$$

But, by similar triangles OCE, OAM,

$$CO^2 : OE \, . \, EC = OA^2 : OM \, . \, MA \text{ .... } (2);$$

therefore, comparing (1) and (2), since

$$OE \, . \, ED = OM \, . \, MA, \quad OA^2 = CO \, . \, OB,$$

or BAC is a right angle.

The property holds for *any* hyperbola, but the line BC does not always pass through the centre; if it cut the asymptotes in P and Q, the angles APQ, AQP, are each equal to the angle between the asymptotes, and to BAC. I have not been able to find a geometrical construction for the line.

1373. Given a circle (C) and *any* point A, either within or without the circle: through A draw BAD cutting the circle in B, D. Then it is required to find another point E, such that, if LEM be drawn cutting the circle in L, M, we may always have $AE^2 = LE \, . \, EM \pm BA \, . \, AD$. [Proposed by T. T. Wilkinson, F.R.A.S. Solution sent 1863, printed in Reprint, Vol. XXXII. p. 22.]

* [From the *Educational Times* and from *Mathematical Questions with their Solutions from the Educational Times*. This latter work, which is cited under the title "Reprint," contains also many papers and solutions not published in the *Educational Times*. Professor Clifford contributed solutions to two questions which appeared in a weekly journal, *The Key*. They will be found in Vol. II. No. 34, August 22nd, p. 124, No. 35, August 29th, 1863, p. 140 (this last solution is given in full). The solutions are of no special interest.]

1. For the upper sign, when A is *without* the circle, and the lower, when A is *within*, E lies on the polar of A. [TET': AG perpendicular to CEG.]

For
$$AE^2 = AG^2 + GE^2 = AC^2 - CG^2 + GE^2$$
$$= AC^2 - CG^2 + (CG^2 + CE^2 - 2CE.CG)$$
$$= AC^2 + CE^2 - 2CT^2$$
$$= BA.AD - LE.EM,$$
when E is within the circle, as in the figure; when E is without the circle, $AE^2 = LE.EM + BA.AD$. By interchanging A and E, we get the second case.

2. For the same; E may obviously lie on a circle with CA as diameter.

3. For the upper sign, when A is *within* the circle, and the lower, when A is *without*; bisect CA in P, and with P as centre describe a circle whose radius is $\sqrt{(r^2 - 3CP^2)}$; E may lie on this circle. For, if EQ be perpendicular to AC,

then
$$r^2 - 3CP^2 = PE^2 = PQ^2 + QE^2,$$
or
$$2r^2 - \tfrac{3}{2}AC^2 = CE^2 + AE^2 - 2AP^2,$$
therefore
$$AE^2 = (r^2 - CE^2) \pm (r^2 \mp CA^2) = LE.EM + BA.AD.$$

4. In the same cases, E may evidently lie on a straight line through A perpendicular to CA.

1378. A *tangent* to an ellipse is a *chord* of a *concentric* circle, whose radius is equal to the distance between the ends of the axes of the ellipse; shew that the straight lines which join the ends of the chord to the centre are conjugate diameters. [April, 1863, solved June, 1863. Proposer's solution sent May 14th, 1863, printed in Reprint, Vol. XXXII. p. 31.]

Let the equations to the *ellipse*, the *circle*, and the *chord*, be respectively
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x^2 + y^2 = a^2 + b^2, \quad \frac{xh}{a^2} + \frac{yk}{b^2} = 1 \text{ .... } (1, 2, 3).$$

Then the equation
$$x^2 + y^2 = (a^2 + b^2)\left(\frac{xh}{a^2} + \frac{yk}{b^2}\right)^2 \text{ .... } (4)$$
represents the two straight lines passing through the origin and the intersections of (2) and (3). If these are conjugate diameters, we must have
$$-\frac{b^2}{a^2} = \frac{b^4}{a^4} \cdot \frac{a^4 - (a^2 + b^2)h^2}{b^4 - (a^2 + b^2)k^2},$$
which may easily be shewn to be the case, since (h, k) is on the ellipse, and therefore $a^2k^2 + b^2h^2 = a^2b^2$.

If we change b^2 into $-b^2$, we obtain a similar theorem for the hyperbola; but the conjugate diameters will be imaginary, if
$$\frac{h^2}{a^4} + \frac{k^2}{b^4} < \frac{1}{a^2 - b^2}.$$

1379. [If a curve of the third order have a double point A, and be cut by any straight line in B, C, D; and if, when ABC is taken as triangle of reference, the tangents at A are represented by the equation
$$P\beta^2 + Q\beta\gamma + R\gamma^2 = 0,$$
and the tangents at B, C, by the equations
$$P\alpha + N\gamma = 0, \text{ and } M\beta + R\alpha = 0;$$
shew that the equation to the straight line AD is
$$N\beta + M\gamma = 0,$$
and find the equation of the curve. Proposed by * * *, April, 1863, solved June, 1863.]

Since $P\beta^2 + Q\beta\gamma + R\gamma^2 = 0$ and $P\alpha + N\gamma = 0$ are the tangents at the points where $\gamma = 0$ meets the curve, its equation must be of the form
$$(P\beta^2 + Q\beta\gamma + R\gamma^2)(P\alpha + N\gamma) = \gamma^2\phi;$$
also of the form
$$(P\beta^2 + Q\beta\gamma + R\gamma^2)(M\beta + R\alpha) = \beta^2\chi.$$

It will be found that the equation
$$\alpha(P\beta^2 + Q\beta\gamma + R\gamma^2) + \beta\gamma(N\beta + M\gamma) = 0 \text{ .... } (1)$$
is of both these forms. For clearly the lines $P\beta^2 + Q\beta\gamma + R\gamma^2 = 0$ only meet the curve at the point $(\beta\gamma)$, and the lines $P\alpha + N\gamma = 0$, $M\beta + R\alpha = 0$, *touch* the curve at the points $(\alpha\gamma)$, $(\alpha\beta)$. It is now obvious that the other point where α meets the curve is in the line $N\beta + M\gamma = 0$, which is therefore the line AD.

We may notice that the tangents drawn from D to the curve are represented by
$$(Q \pm 2\sqrt{PR})\alpha + N\beta + M\gamma = 0 \text{ .... } (2);$$
there are only two because there is a double point. The lines drawn from A to their points of contact are represented by $\beta\sqrt{P} \pm \gamma\sqrt{R} = 0$; hence these form an *harmonic* pencil with AB, AC.

The equation to the tangent at D is
$$N^2M\beta + M^2N\gamma = (M^2P + N^2R - MNQ)\alpha \text{ .... } (3),$$
and that to the line joining A with the other point where (3) meets the curve is $MP\beta + NR\gamma = 0$; hence the condition that D may be a *point of inflexion* is
$$PM^2 = RN^2.$$

1387. Four common tangents are drawn to a circle and an ellipse which passes through the centre (O) of the circle; if A, B be *opposite* intersections of the tangents, shew that OA and OB are equally inclined to the tangent at O to the ellipse. [May, 1863, corrected to the above form in July, 1863, solved September, 1863: Reprint, Vol. I. pp. 19, 33.]

We use *rectangular tangential* coordinates (Ferrers, *Tril. Co.*, p. 130; Salmon, *Higher Plane Curves*, p. 2). It is easily shewn that the sum of the squares of the reciprocals of the intercepts made by any tangent to a circle on two diameters at right angles is constant. Hence the equation to a circle whose centre is the origin is

$$\xi^2+\eta^2=c^2 \quad\dots\dots\dots\dots (1).$$

The points $\xi=0$, $\eta=0$, are at an infinite distance, one on each of the axes; and $k=0$ (where k is a constant) represents the origin. From this it follows that the equation

$$\xi^2+bk\xi+ck\eta+dk^2=0 \quad\dots\dots\dots\dots (2)$$

(where the k may be left out at pleasure) represents a conic touching the axis of ξ at the origin. For if we seek the tangents drawn from $k=0$ to the curve, we find that they both coincide with the line $k\xi$, that is with the axis of ξ. Now if we put

$$\xi^2+\eta^2-c^2\equiv S,\ \ \xi^2+b\xi+c\eta+\delta\equiv T,$$

it is clear that the equation $S+\lambda T=0$ represents an envelope of the second class, touching all the common tangents of S and T. The discriminant of this equation is of the third degree in λ; hence there are three values of λ for which $S+\lambda T=0$ represents two points. But in every case the coefficient of $\xi\eta$ is zero; which is just the condition that the line joining the origin to the two points (which are evidently opposite intersections of the common tangents) should be equally inclined to the axis of ξ. For if $a\xi+b\eta=1$ be the equation of a point, (a, b) are its ordinary rectangular coordinates, and $(b:a)$ is the tangent of the angle which the line joining it to the origin makes with the axis of ξ. Hence if two points (a, b) and (c, d) are equally inclined to the point ξ, we must have

$$\frac{b}{a}=-\frac{d}{c},\ \text{or}\ ad+bc=0;$$

but $(ad+bc)$ is the coefficient of $\xi\eta$ in the product $(a\xi+b\eta-1)(c\xi+d\eta-1)$. The theorem is therefore proved.

It will be observed that the discriminant being of the third degree in λ, must always have one real root; but there will be four real common tangents only when the conic is an ellipse cutting the circle in four points.

It appears therefore that *any* two conics have two *real* intersections of real or imaginary common tangents, corresponding to the centres of similitude of two circles.

By projection we may shew that "If a straight line A join the poles of B with respect to two conics, then the lines joining AB to a pair of opposite intersections of common tangents, form, with A, B, an harmonic pencil."

And by reciprocation,—"If a point A be the intersection of the polars of B with respect to two conics, and AB be cut by a pair of common chords in C, D, then $ACBD$ is an harmonic range."

[A solution of this "elegant theorem," "included as a particular case in the known theorem—'given three conics inscribed in the same quadrilateral, the tangents from any points to these conics form a pencil in involution'"—is given by Prof. Cayley in the same volume of the Reprint on page 33.]

1389. [A curve of the third order, consisting of three symmetrical branches, is drawn so as to touch the sides of an equilateral triangle at their middle points.

These three points are joined so as to form a new equilateral triangle. Shew that if PA, PB, PC be the perpendiculars from any point P on the curve upon the sides of one equilateral triangle, and PD, PE, PF the perpendiculars from the same point on the sides of the other equilateral triangle, then the ratio vol. $PA\,.\,PB\,.\,PC$: vol. $PD\,.\,PE\,.\,PF$ is constant, wherever P be taken on the curve. Proposed by * * *. Reprint, Vol. I. p. 10, *E. T.* July, 1863.]

The general equation to a cubic touching $B+C=0$, $C+A=0$, $A+B=0$, where they meet $A=0$, $B=0$, $C=0$, is evidently

$$ABC=k(B+C)(C+A)(A+B) \quad\dots\dots\dots\dots (1).$$

In the case supposed, let $B+C\equiv\alpha$, $C+A\equiv\beta$, $A+B\equiv\gamma$, then (1) becomes

$$(\alpha+\beta-\gamma)(\alpha-\beta+\gamma)(-\alpha+\beta+\gamma)=k\alpha\beta\gamma,$$

which expresses the property in question.

The asymptotes are parallel to $\alpha\beta\gamma$, and there are three points of inflexion, all at an infinite distance.

The proof holds if the cubic touch the sides of any triangle in three points, such that the lines joining them to the opposite vertices meet in a point.

1399. From a point A two chords are drawn meeting a conic section in four points B, joined also by four straight lines a. These intersect two and two in two points P lying on the polar of A. At the points B are drawn four tangents b, which intersect in six points, two of which are on the polar of A, and the others lie two and two on the two straight lines AP. These tangents intersect the original chords in four points, which may be joined by four straight lines intersecting by pairs in the points P. The lines a and b intersect in eight points C, which may be joined by twenty lines c; four of these pass through A, and the others may be divided into groups of four. Each group has six intersections, two of which lie on the polar of A, and the others lie two and two on lines through A. Any two groups intersect in eight points, having properties like those of the points C. [June, 1863. Reprint, Vol. I. p. 14, *E. T.* August, 1863.]

Take the chords through A for the sides β, γ of the triangle of reference, one of the straight lines a as the side α; and let another of them be represented by

$$l\alpha+m\beta+n\gamma\,(=\delta)=0.$$

Let the equation to the conic be

$$\alpha\delta=k\beta\gamma \quad\dots\dots\dots\dots (1),$$

thus the four lines a are

$$\left.\begin{array}{l} a_1 \dots\dots\dots \alpha=0 \\ a_2 \dots\dots\dots \delta=0 \\ a_3 \dots\ l\alpha+m\beta=0 \\ a_4 \dots\ l\alpha+n\gamma=0 \end{array}\right\} \quad\dots\dots\dots\dots (2).$$

The polar of A is evidently

$$2l\alpha+m\beta+n\gamma=0 \quad\dots\dots\dots\dots (3).$$

The equations to the tangents b are

$$\left.\begin{aligned} b_1 \ldots m\alpha - k\gamma &= 0 \\ b_2 \ldots n\alpha - k\beta &= 0 \\ b_3 \ldots m\delta + lk\gamma &= 0 \\ b_4 \ldots n\delta + lk\beta &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots (4).$$

For instance, assume $\delta = p\gamma$ for the equation to b_3; then from (1) we find that this meets the curve where it meets $p\alpha = k\beta$; but as b_3 is a *tangent*, this must coincide with a_3, or $mp = -lk$; and thus b_3 becomes

$$m\delta + lk\gamma = 0.$$

The intersections of b_1b_2 and of b_3b_4 lie on $m\beta - n\gamma = 0$; of b_1b_4 and of b_2b_3 on $m\beta + n\gamma = 0$; and of b_1b_3 and b_2b_4 on $2l\alpha + m\beta + n\gamma = 0$.

The equations of the four lines joining intersections of the lines b with β and γ are

$$\left.\begin{aligned} mn\delta + lk(m\beta + n\gamma) &= 0 \\ mn\alpha - k(m\beta + n\gamma) &= 0 \\ mn(l\alpha + n\gamma) - lk(m\beta - n\gamma) &= 0 \\ mn(l\alpha + m\beta) + lk(m\beta - n\gamma) &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots (5).$$

The first pair meet $2l\alpha + m\beta + n\gamma = 0$ where it meets $m\beta + n\gamma = 0$; the second pair where it meets $m\beta - n\gamma = 0$.

The eight points C may be represented as follows:

$$\begin{matrix} a_4b_3, & a_3b_4, & a_2b_2, & a_1b_3, & a_3b_1, & a_4b_2, & a_1b_4, & a_2b_1 \\ E & F & G & H & K & L & M & N. \end{matrix}$$

With this notation, the five groups of lines C are

(EK, FL, GM, HN); (EF, GH, KL, MN); (EG, HL, KM, NF);
(EM, FH, NL, GK); (EM, FG, ML, KH).

All the lines of the first group pass through A, and have for equations

$$\left.\begin{aligned} EK, \ldots\ldots\ldots\; m^2\beta + kl\gamma &= 0 \\ FL, \ldots\ldots\ldots\; n^2\gamma + kl\beta &= 0 \\ GM, \ldots\; n^2\gamma + (mn + kl)\beta &= 0 \\ HN, \ldots\; m^2\beta + (mn + kl)\gamma &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots (6),$$

which may be easily verified.

As an example of the others, take the equations of the lines of the fourth groups:

$$\left.\begin{aligned} &EM, \ldots (l^2k^2 + 2klmn)(l\alpha + n\gamma) = (lk + mn)(m\delta + kl\gamma)\, n \\ &FH, \ldots (l^2k^2 + 2klmn)(l\alpha + m\beta) = (lk + mn)(n\delta + kl\beta)\, m \\ &NL, \ldots k(mn + kl)\,\delta = mn^2(m\alpha - k\gamma) \\ &GK, \ldots k(mn + kl)\,\delta = m^2n(n\alpha - k\beta) \end{aligned}\right\} \quad \ldots\ldots (7).$$

The intersections of EM and FH, and of NL and GK, lie on $m\beta - n\gamma = 0$; those of EM and NL, and of FH and GK, on $m\beta + n\gamma = 0$; those of EM and GK, and of NL and FH, on $2l\alpha + m\beta + n\gamma = 0$; and so with each of the other groups.

I was wrong in saying that *any* two groups intersect in eight points, &c.; this is true of the last four, for it will be found that any two of these form two quadrilaterals, the vertices of one resting on the sides of the other, two diagonals of each passing through A, and the others being identical $(2l\alpha + m\beta + n\gamma = 0)$; hence, by the converse of the first part of the equation, a conic may be inscribed in one so as to circumscribe the other, and the preceding reasoning applies. But the first group is an exception; there are only four new points formed by combining it with any of the others, and these may be joined by four lines meeting two and two in the polar of A.

All these theorems may be proved very readily by projecting the conic into a circle whose centre is the projection of A.

1393. [A shell formed of two equal paraboloids of revolution, having a common axis, is fixed with its vertex downwards, and axis vertical; and a heavy uniform rod of given length rests within it, in a vertical plane through the axis. Compare the pressures on the lower surface of the shell. Proposed by Mr J. R. Wilson, Jesus College, Cambridge, Reprint, Vol. I. p. 27, *E. T.* September, 1863.]

Let QPR be the rod, of length $4c$; draw tangents QT, RT, and normals QO, RO, to the outer parabola. We know by Geometry that $QP = PR$, and therefore PT is vertical. Now the rod is kept at rest by four forces, two of which, viz., gravity and the resistance at P, pass through P; therefore the resultant of the pressures at Q, R acts along OP. But OP, bisecting QR, is half the diagonal of the completed parallelogram (OQ, OR); hence the resistances at Q, R, are as the normals OQ, OR; that is, as $\sin ORQ : \sin OQR$, or as $\cos TRP : \cos TQP$. Now draw a tangent (ECF) to the outer parabola at the point (C) where TP meets it; then, putting $AB = k$, $4a$ = principal parameter of BRQ, and $\angle ECP = \theta$, the equation of BRQ, referred to CP, CQ, will be

$$y^2 = 4a \operatorname{cosec}^2\theta \,.\, x \quad \ldots\ldots\ldots (1).$$

At the point P, $x = k$, $y = 2c$,

$$\therefore\; c^2 \sin^2\theta = ak \quad \ldots\ldots\ldots (2).$$

The tangents QT, RT are represented by

$$k^2y^2 = c^2(x + a)^2 \quad \ldots\ldots\ldots (3).$$

Therefore by the usual formula for oblique axes

$$\frac{\cos TFC}{\cos TEC} = \left(\frac{c - k\cos\theta}{c + k\cos\theta}\right)\left(\frac{c^2 + k^2 + 2ck\cos\theta}{c^2 + k^2 - 2ck\cos\theta}\right)^{\frac{1}{2}}.$$

This, therefore, is the ratio required.

[The following general remarks accompany a solution of 1418, Reprint, Vol. I. p. 30, *E. T.* November, 1863.]

Consider any two conics, U, V; any point (ξ, η, ζ) has two polars, ΔU, ΔV, where Δ stands for

$$\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}.$$

These meet in a point which we shall call the *polar opposite* of (ξ, η, ζ). Similarly, the line joining the poles of a right line may be called its *polar opposite*.

Now consider the equations $S=0$, $S=LM$, where L, M are common chords.

The polars are $\Delta S=0$, $\Delta S=L\Delta M+M\Delta L$.

If $\Delta L=0$, these meet on L; that is,

(α) If any point lie on a common chord, its polar opposite lies on the same chord.

If also $\Delta M=0$, they coincide, or

(β) The intersection of a pair of common chords has only one polar. It is easily shewn that this is a line joining intersections of common tangents.

It follows that

(γ) The polar opposite of any point in a straight line with two opposite intersections of common tangents is an intersection of common chords.

Let $\Delta(L+KM)=0$, then the polars meet on $L-KM=0$. That is

(δ) Lines joining two polar opposites to an intersection of common chords, form, with the chords, an harmonic pencil.

Next, let the equations be $LM+N^2$, $LM+R^2$, so that L, M are common tangents. The polars are now

$$\left.\begin{array}{l}(L\Delta M+M\Delta L)+2N\Delta N=0\\(L\Delta M+M\Delta L)+2R\Delta R=0\end{array}\right\}.$$

If $\Delta N=0$, these intersect on R, or

(ϵ) If a point lie on one chord of contact of a pair of common tangents, its polar opposite lies on the other.

If $\Delta(N+KR)=0$, the polars meet on $R+KN=0$; or

(ζ) If the locus of a point is a line through the intersection of the chords of contact of a pair of common tangents, the locus of its polar opposite is another line through the same intersection.

Thirdly, consider the case of double contact, S, $S+L^2$. Here the polars are ΔS, $\Delta S+2L\Delta L$. These always meet on L, shewing that

(η) If two conics have double contact, the polar opposite of any point whatever lies on the chord of contact.

If $\Delta L=0$, they coincide, or

(θ) A point on the chord of contact has only one polar, which is also the locus of its polar opposites.

(ι) In general, if the locus of a point be a straight line,

$$l\xi+m\eta+n\zeta=0,$$

the locus of its opposite is the conic

$$\begin{vmatrix}\frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz}\\ \frac{dV}{dx}, & \frac{dV}{dy}, & \frac{dV}{dz}\\ l, & m, & n\end{vmatrix}=0,$$

which we may call the *polar conic* of the line (lmn). As the discriminant is of the third degree in (lmn), it appears that the envelop of lines whose polar conics break up into two right lines is a curve of the third class.

1409. For every point A on a conic section there exists a straight line BC, not meeting the curve, such that, if through any other point on the conic there be drawn any two straight lines meeting BC in B, C, and the curve in D, E, the angles BAC, DAE are either equal or supplementary. [July, 1863. Reprint, Vol. I. p. 33, *E. T.* December, 1863.]

Take the point A for origin, and the rectangular tangential equation used in Question 1387 [cf. supra], but in the more convenient form

$$(\xi-a)^2=4b(\eta-c) \quad \text{.............................. (1)},$$

which is evidently equivalent to the one there given.

The line BC is represented by

$$\xi-a=0, \ =\eta-b-c \quad \text{.............................. (2)};$$

it always passes through the pole of the normal, and is in fact the polar of the point of intersection of chords subtending a right angle at A.

If we assume for the general equation of a point on the curve

$$\xi-a=m(\eta-c)+\beta \quad \text{.............................. (3)},$$

then the equation

$$\{m(\eta-c)+\beta\}^2=4b(\eta-c)$$

must have equal roots for η, which gives $\beta=\frac{b}{m}$. We shall call this the point m.

Let the intersection of BD, CE, be the point m_1, and the points D, E, m_2, m_3. The equations of B, C, will therefore be

$$\left(m_2+\frac{1}{m_2}\right)\{(\xi-a)-m_1(\eta-b-c)\}=\left(m_1+\frac{1}{m_1}\right)\{(\xi-a)-m_2(\eta-b-c)\} \quad \text{...... (4)},$$

$$\left(m_3+\frac{1}{m_3}\right)\{(\xi-a)-m_1(\eta-b-c)\}=\left(m_1+\frac{1}{m_1}\right)\{(\xi-a)-m_3(\eta-b-c)\} \quad \text{...... (5)}.$$

These equations may be easily verified.

The angle between AB and the axis of ξ is therefore

$$\tan^{-1}\frac{m_2\left(m_1+\frac{1}{m_1}\right)-m_1\left(m_2+\frac{1}{m_2}\right)}{\left(m_2+\frac{1}{m_2}\right)-\left(m_1+\frac{1}{m_1}\right)}=\tan^{-1}\frac{m_1+m_2}{m_1m_2-1}.$$

Hence the angle BAC is

$$\tan^{-1}\frac{m_1+m_2}{1-m_1m_2}-\tan^{-1}\frac{m_1+m_3}{1-m_1m_3}.$$

That is, its tangent is equal to that of the angle between the lines joining the points m_2, m_3 to the origin. For the latter angle is clearly

$$\tan^{-1}\frac{m_2-m_3}{1+m_2m_3}.$$

Since, therefore, the angles BAC, DAE have their tangents equal, they are either equal or supplementary. If the conic be a circle, it is easily seen that the line BC is always at an infinite distance.

[A solution of this "very elegant theorem" is given by Prof. Cayley on p. 40 of the same volume.]

1442. [The same circle around the origin being employed in the operations of reciprocation and inversion, shew that the first positive and negative pedals of a given curve coincide, respectively, with the inverse of its reciprocal, and with the reciprocal of its inverse; further, that the reciprocal of the n^{th} pedal is the $(-n)^{\text{th}}$ pedal of the reciprocal, and the $(-n-1)^{\text{th}}$ pedal of the inverse of the primitive; and lastly, that the inverse of the n^{th} pedal is the $(-n)^{\text{th}}$ pedal of the inverse, and hence also the $(-n+1)^{\text{th}}$ pedal of the reciprocal of the primitive. Proposed by Dr Hirst, F.R.S. Reprint, Vol. I. pp. 41—3, *E. T.* January, 1864.]

1. Writing J for the operation of inversion, R for that of reciprocation, and P for that of taking the pedal, we have

$$J^2 = R^2 = 1,$$
$$P = JR;$$
$$JP = J^2R = R,$$
$$RJP = R^2 = 1,$$
$$RJ = P^{-1}.$$

2. These, then, are the laws of combination of the symbols R, J, P. We can now immediately prove the theorems in the question. For

$$R \,.\, P^n = R \,.\, (JR)^n = (RJ)^n \,.\, R = P^{-n}R = P^{-n} \,.\, RJ \,.\, J = P^{-n-1} \,.\, J \;\ldots\ldots\; (1),$$

$$J \,.\, P^n = J\,(JR)^n = J^2R \,.\, (JR)^{n-1} = R\,(JR)^{n-1}\,J \,.\, J = (RJ)^n \,.\, J = P^{-n} \,.\, J$$
$$= P^{-n} \,.\, JR \,.\, R = P^{-n+1} \,.\, R \;\ldots\ldots\ldots\ldots\ldots\; (2).$$

And we may write down any number of formulæ by this method. For instance, the identities

$$(JR)^n \,.\, J\,(RJ)^m = (JR)^{n+m} \,.\, J = J\,(RJ)^{n+m} \;\ldots\ldots\ldots\ldots\; (3),$$
$$(RJ)^n \,.\, R\,(JR)^m = (RJ)^{n+m} \,.\, R = R\,(JR)^{n+m} \;\ldots\ldots\ldots\ldots\; (4),$$

may be thus interpreted:

"The $(n)^{\text{th}}$ pedal of the inverse of the $(-m)^{\text{th}}$ pedal is the $(n+m)^{\text{th}}$ pedal of the inverse, and the inverse of the $(-n-m)^{\text{th}}$ pedal; and the $(-n)^{\text{th}}$ pedal of the reciprocal of the $(m)^{\text{th}}$ pedal is the $(-n-m)^{\text{th}}$ pedal of the reciprocal, and the reciprocal of the $(n+m)^{\text{th}}$ pedal."

3. Again, any formula may be transformed by interchanging R and J, and reversing the signs of all the indices of P. To derive in this way the second pair of theorems from the first, we shall have to make a further change from n to $-n$.

The formulæ (3) and (4) are immediately convertible.

4. The theory of Derived Surfaces and Curves is simply that of the interpretation of symbols. Let any straight line meet two rectangular axes Ox, Oy

in A, B, and draw OP perpendicular to AB, and PM, PN perpendicular to the axes. Then we have two systems of coordinates; (1) when $\frac{1}{OA}$, $\frac{1}{OB}$ are the coordinates of the point P, (2) when PM, PN are the coordinates of the line AB. The formulæ of transformation, between the first and Cartesian, and between the second and Tangential, coordinates, are

$$(\xi^2 + \eta^2)(x^2 + y^2) = 1, \;\; \xi y = \eta x \;\ldots\ldots\ldots\ldots\ldots\ldots\; (5).$$

These represent the operation of inversion in the two cases. But it is important to remember that, in *Tangential inversion*, the tangents, not the points, are inverted; that is, to every tangent of the primitive corresponds a line parallel to it, such that the rectangle under their distances from the origin is constant. Now let $U=0$ be an equation in x, y, and constants; and let CU denote the curve which is represented by $U=0$, when we interpret x, y as Cartesian coordinates, TU when we interpret x, y as *Tangential* coordinates, MU according to the *first* system of this article, and NU according to the *second*. Then, for instance, $TU = R \,.\, CU$, or, by separation of symbols, $T = RC$. In this way we have the equations

$$M = JC, \;\; N = RJC = RM = RJRT,$$

which serve to connect any two systems.

5. It appears from (4) that if the equation of any curve be written

$$u_n + u_{n-1} + \ldots + u_2 + u_1 + u_0 = 0,$$

then the equation of the inverse is

$$u_n + u_{n-1}(x^2+y^2) + u_{n-2}(x^2+y^2)^2 + \ldots + u_0(x^2+y^2)^n = 0.$$

This is of degree $2n$ in general, but reduces when the curve is circular, and when the origin is on the curve. If the curve be circular in the degree f, that is, if its equation be of the form

$$v_{n-2f}(x^2+y^2)^f + v_{n-2f+1}(x^2+y^2)^{f-1} + \ldots + u_1 + u_0 = 0,$$

the degree of the inverse is reduced $2f$, and if the origin be a multiple point of the order g, or if

$$u_0 = u_1 = \ldots = u_{g-1} = 0,$$

the degree is reduced by g. Hence generally the degree of the inverse is $2(n-f)-g$. It follows by reciprocation that if n is the class of any curve, and if the lines joining the origin to the circular points at infinity are multiple tangents of the order f, and if the line at infinity is a multiple tangent of the order g, then the degree of the first positive pedal is $2(n-f)-g$.

And again, if a curve has g points at infinity distinct from the two circular points at infinity, and has a multiple point of the order ϕ at the origin, being of degree v, then the *class* of the first negative pedal is $v - \phi + g$. This is easily obtained by inverting the result just proved; it being remarked that the inverse of a curve circular in the degree f has a multiple point of the order $n-2f$ at the origin. The *degree* of the negative pedal is the *class* of the inverse, and consequently is the same as the number of circles which can be drawn through an arbitrary point (ξ, η) and the origin to touch the primitive curve. To find this number, we must eliminate between

$$U=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$x^2+y^2+2Ax+2By=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$(x+A)\frac{dU}{dx}=(y+B)\frac{dU}{dy} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

A and B being converted by the linear relation

$$\xi^2+\eta^2+2A\xi+2B\eta=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

The degree of the eliminant in A and B, which is the degree of the first negative pedal, is *in general* $n(n+2)$, but will of course be reduced by peculiarities in the form of U.

1319. [It is announced at p. 205, Vol. II., 12th ed., Davies's Hutton, that "if a tetrahedron be drawn, formed of four tangent planes to a paraboloid, the sphere described about it will pass through the focus of the paraboloid." Prove or disprove this. Proposed by N'Importe. Reprint, Vol. I. p. 45, *E. T.* February, 1864.]

The statement is not true.

If perpendiculars be drawn from the foci of a conicoid of revolution on any tangent plane, the rectangle of these perpendiculars is equal to the square of the minor axis. If then a conicoid of revolution having foci $\alpha\beta\gamma\delta$, $\alpha_1\beta_1\gamma_1\delta_1$, touch the faces of the fundamental tetrahedron, we must have

$$\alpha\alpha_1=\beta\beta_1=\gamma\gamma_1=\delta\delta_1=b^2.$$

So that if one of the foci lies in the plane

$$l\alpha_1+m\beta_1+n\gamma_1+r\delta_1=0,$$

the locus of the other will be the surface of the third degree

$$\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}+\frac{r}{\delta}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which is otherwise interesting (Frost and Wolstenholme's *Solid Geometry*, p. 289). A particular case is when the surface of revolution is a paraboloid, one of whose foci lies on the plane at infinity,

$$A\alpha+B\beta+C\gamma+D\delta=0,$$

and the other on the surface

$$\frac{A}{\alpha}+\frac{B}{\beta}+\frac{C}{\gamma}+\frac{D}{\delta}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where $A\,B\,C\,D$ are the faces of the tetrahedron.

Now if the theorem of Davies's Hutton were true, we should have found for the locus the equation of the circumscribing sphere.

I write down one or two other instances of the application of this principle. (See Salmon's *Conics*, 4th ed., p. 261, Ex. 13, 15.)

Given five planes connected by the identical relation

$$a\alpha+b\beta+c\gamma+d\delta+e\epsilon=0,$$

the foci of any conicoid of revolution touching the "frustum" will lie in the surface of the fourth degree,

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}+\frac{d}{\delta}+\frac{e}{\epsilon}=0.$$

Given one focus and the intersection of the focal tangents of a parabola of the third class inscribed in the triangle of reference; the other focus moves on a conic (which is *never* a circle) circumscribing the triangle.

Given four tangents, a focus, and the intersection of the focal tangents, in a curve of the third class; the other two foci move on a curve of the third degree.

The general extension is sufficiently obvious.

1421. [If by the Harmonic centre, relative to a fixed plane, of A, C, points in a line meeting the fixed plane in D, be understood a point B between A and C, such that A, B, C, D form an harmonic system; prove that if through the harmonic centre of either diagonal of any of the three quadrilateral faces of the frustum of a triangular pyramid, and the harmonic centres of the two edges which meet but are not in the same face with that diagonal, a plane be drawn, the six planes thus obtained will all pass through one and the same point. Proposed by Professor Sylvester, F.R.S. Reprint, Vol. I. pp. 45, 46, *E. T.* February, 1864.]

The proposer has shewn, in the *Philosophical Magazine* for September, that the theorem is true when we put the arithmetical centre for the harmonic. His proof, by Cartesian coordinates, is exceedingly simple and elegant; but it will be found that the attempt to prove the same case by quadriplanar coordinates involves an enormous amount of algebraic work. However, it is clear that if the requisite operations were performed, the result must be the same as by the other method. Now when we find the arithmetic centre of a line by quadriplanar coordinates, the process is simply to find its harmonic centre with respect to the plane at infinity $A\alpha+B\beta+C\gamma+D\delta=0$. But the proof can make no mention of the meaning of $ABCD$, since the thing proved is general for any tetrahedron, and does not depend at all upon the areas of the faces. *Therefore the proof holds, whatever interpretation we give to the symbols* $ABCD$; that is to say, whatever plane is represented by

$$A\alpha+B\beta+C\gamma+D\delta=0.$$

This principle is evidently identical with the method of Projections in plane geometry. (See Salmon's *Higher Plane Curves*, Art. 246). Quadriplanar equations not connected with the absolute form of the fundamental tetrahedron, will hold good whatever tetrahedron we choose. Now it is analytically possible to choose a tetrahedron with reference to which a given conicoid shall be represented by a given equation. For a conicoid is determined by nine conditions, but each of the four planes involves three independent constants. We may, therefore, in addition, choose the tetrahedron so that the plane at infinity shall be represented by a given equation. Thus any property proved of any one conicoid and a plane, when expressed in quadriplanar coordinates, is true of any other conicoid and plane. The only limitation is that connecting properties which can be expressed in the coordinates will be retained in transformation. Again, we may analytically transform real lines and planes into imaginary, and *vice versâ*, without loss of continuity. Now, ruled conicoids only differ from others in containing real lines instead of imaginary; therefore, *the distinction between ruled and unruled conicoids is lost in transformation.*

In studying, then, the properties of any figure, the principal point will be the reduction of the figure to what (by an extension of the term) may be called the *canonical form*. For instance, the canonical form of a quadrilateral is a parallelogram; of a conic, a circle; of a conicoid, a sphere; and so on. In particular, we wish to find the canonical form of a tetrahedral frustum, which is the figure formed by the intersection of five planes $ABCDE$. For a quadrilateral $ABCD$ we proceed in this way; we join the vertex $\widehat{AB}$ to the vertex $\widehat{CD}$, and project the joining line to infinity. Hence by analogy in the frustum, we join the vertex $\widehat{ABC}$ to the edge $\widehat{DE}$, and project the joining plane to infinity. The figure is thus reduced to three parallel straight lines cut by two parallel planes. As an instance of the use of these canonical forms, we give the following properties, which may be easily proved: "From a point O, three chords o are drawn to a conicoid, meeting it in six points A. These may be joined again by four pairs of planes a, each pair intersecting in one of four lines β on the polar plane of O. At the points A six tangent planes b are drawn; if any three of these (whose points of contact are not in one plane through O) intersect in X and the other three in Y, then X, Y, O are in a straight line. Again, the planes b will intersect by pairs in three lines γ on the polar plane of O, and these will pass through the three intersections of one of the lines β with the other three. If the chords o cut the polar plane in three points, these will lie in three straight lines through the same intersections. There are thus three coaxal triangles on the polar plane, and their common pole is on the line joining O and two of the intersections XY. The tangent planes b cut the chords o in twelve new points C, four of which lie on each chord. Consider the eight C-points lying on two chords o; they may be divided into two groups, each group having two points on each of the chords. Lines joining points in *either* group intersect on the polar plane of O, but one group has for these two intersections, (1) a vertex of one of the three co-polar triangles, (2) the point where the opposite side cuts the common axis. Points in different groups may be joined by eight lines, intersecting in four points lying on the polar plane of O, and eight points lying on four lines through O, and so on.

"If a straight line be drawn through the vertex of either of the common tangent cones of two conicoids having double contact, to meet either of the planes of common section, and the two conicoids, it will be cut in involution, so that the equal anharmonic ratios of the involution are constant."

1443. [Shew that the locus of the centres of all the conics circumscribing a given quadrilateral is an ellipse if the quadrilateral is re-entrant, and an hyperbola if it is convex. Shew further that two real parabolæ may always be drawn through the angles of any convex quadrilateral. Proposed by Prof. Sylvester, F.R.S. Reprint, Vol. I. pp. 51—54, *E. T.* March, 1864.]

1. First, the locus of the centre is a conic.

Let U, V be the tangential equations of two conics through the four points; then the general equation of a conic through the points is

$$l^2pU + lmF + m^2qV = 0 \qquad\text{(1)},$$

where p, q are the discriminants of U, V and F is the conic touched by all the tangents to U and V at the four points of intersection; thus, if

$$U \equiv ax^2 + by^2 + cz^2, \qquad V \equiv a'x^2 + b'y^2 + c'z^2,$$

then

$$F \equiv aa'(bc' + b'c)\,x^2 + bb'(ca' + c'a)\,y^2 + cc'(ab' + a'b)\,z^2.$$

Now let (ξ, η, ζ) be a fixed straight line, and write Δ for

$$\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz};$$

then the pole of (ξ, η, ζ) with respect to (1) is

$$l^2p\Delta U + lm\Delta F + m^2q\Delta V = 0 \qquad\text{(2)},$$

whose locus is $4pq\Delta U \,.\, \Delta V = (\Delta F)^2$, a conic section.

We obtain the locus of *centres* by simply putting $\xi = \eta = \zeta$.

2. Solution by *trilinear* coordinates.

The equation to the conic is

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0 \qquad\text{(3)},$$

subject to the condition

$$\frac{l}{f} + \frac{m}{g} + \frac{n}{h} = 0. \qquad\text{(4)}.$$

The polar of a point (ξ, η, ζ) with respect to (3) is

$$\frac{\alpha}{\xi}\left(\frac{m}{\eta} + \frac{n}{\zeta}\right) + \frac{\beta}{\eta}\left(\frac{n}{\zeta} + \frac{l}{\xi}\right) + \frac{\gamma}{\zeta}\left(\frac{l}{\xi} + \frac{m}{\eta}\right) = 0.$$

If this coincide with a fixed line $x\alpha + y\beta + z\gamma = 0$, we must have

$$\frac{\frac{m}{\eta} + \frac{n}{\zeta}}{\xi x} = \frac{\frac{n}{\zeta} + \frac{l}{\xi}}{\eta y} = \frac{\frac{l}{\xi} + \frac{m}{\eta}}{\zeta z} = \frac{\frac{l}{\xi} + \frac{m}{\eta} + \frac{n}{\zeta}}{\frac{1}{2}(\xi x + \eta y + \zeta z)};$$

$$\therefore\ \frac{\frac{l}{\xi}}{-\xi x + \eta y + \zeta z} = \frac{\frac{m}{\eta}}{\xi x - \eta y + \zeta z} = \frac{\frac{n}{\zeta}}{\xi x + \eta y - \zeta z} \qquad\text{(5)}.$$

Substitute these values in (4), and we have for the equation to the locus

$$\frac{\xi}{f}(-\xi x + \eta y + \zeta z) + \frac{\eta}{g}(\xi x - \eta y + \zeta z) + \frac{\zeta}{h}(\xi x + \eta y - \zeta z) = 0 \qquad\text{(6)}.$$

To find the *locus* of *centres*, we may either consider the coordinates trilinear, and put a, b, c for x, y, z; or we may consider them triangular, and put $x = y = z = 1$.

It is clear that by varying the condition (4) we may easily find the locus in other cases. Thus, for instance, "the locus of the centres of all conics passing through three given points and touching a given straight line ($f\alpha + g\beta + h\gamma = 0$), is the curve of the fourth degree,

$$\sqrt{\{f\xi(-\xi+\eta+\zeta)\}} + \sqrt{\{g\eta(\xi-\eta+\zeta)\}} + \sqrt{\{h\xi(\xi+\eta-\zeta)\}} = 0,$$

the coordinates being triangular." (*Cambridge and Dublin Math. Journal*, Vol. v. p. 148.)

Another solution by trilinear coordinates has been proposed in the *Messenger of Mathematics*, Vol. II. p. 169; we give it here in order to notice one of the theorems which may be deduced from it. Consider the equations in Art. 1 as trilinear; then it may easily be proved that the locus of the pole of a line L, or $\xi x + \eta y + \zeta z = 0$, with respect to all the conics, $lU + mV = 0$, is the Jacobian of U, V, L, that is,

$$\begin{vmatrix} \frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz} \\ \frac{dV}{dx}, & \frac{dV}{dy}, & \frac{dV}{dz} \\ \xi, & \eta, & \zeta \end{vmatrix} = 0 \text{.........................(7).}$$

But this is precisely the equation which has been elsewhere obtained (see Art. ι, [p. 572, 1418]) as the locus of "*polar opposites*" of points in the line (ξ, η, ζ); that is, the polars of any point in this line with respect to all the conics $lU + mV$ pass through a fixed point in (7). The sides and diagonals of the quadrilateral are evidently cut harmonically by any line and its polar conic; and since the "locus of centres" is the polar conic of the line at infinity, it must coincide with the "nine-point conic," which *bisects* the sides and diagonals, besides passing through the points E, F, G (Fig. 120). That the conic (7) always *does* circumscribe the common self-conjugate triangle of U, V, may be shewn by putting it in the form

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \frac{\xi}{x}, & \frac{\eta}{y}, & \frac{\zeta}{z} \end{vmatrix} = 0 \text{...................................(8).}$$

It appears then that the nine-point conic possesses the following property: *the polars of any point of it, with respect to all the conics circumscribing the quadrilateral, are parallel.* And conversely, *all the diameters conjugate to a fixed straight line pass through a fixed point on the nine-point conic.*

The "curves of the third class," mentioned at the end of the solution of 1418 [p. 573] as the envelop of lines whose polar conics degenerate, is no other than the three vertices of the common self-conjugate triangle, as readily appears from geometrical considerations. By taking, then, the discriminant of the Jacobian (7) with respect to (x, y, z), we obtain a contravariant expression for these vertices, which enables us at once to reduce two quadrics to the canonical form.

3. Construction for the *directions of the asymptotes*. Let $ABCD$ (Fig. 121) be the four given points. Draw any line KL parallel to AB, meeting AD, BC in K, L respectively. Then DL, CK are parallel to the asymptotes of a certain conic through A, B, C, D. This follows immediately from Pascal's theorem. (See Gaskin's *Construction of a Conic Section*, &c., p. 39, Cor. 5.)

4. Construction for the *centre* of the last conic. Describe about $ABCD$ the parallelogram $\alpha\beta\gamma\delta$, having its sides parallel to DL, CK. Let $EFGH$ be the bisections of the sides of $ABCD$.

Then αE, βF, γG, δH will meet in a point X, which is the centre of the conic.

It may be observed that KL, MN, OP, QR are respectively parallel to BA, AD, DC, CB.

5. Construction for the directions of the axes of the two *parabolæ* through the four points. This would clearly be accomplished if we could draw KL so that CK, DL should be parallel. Let then the circle through DAB (Fig. 122) meet UBC in S, and the circle through ABC meet UAD in T. Take $UK^2 = US \,.\, UC$, and $UL^2 = UD \,.\, UT$; then KL is parallel to AB, and DL to CK. By treating V in the same way, we get another direction; but if the quadrilateral be re-entrant, it is easily seen that the construction fails.

This immediately determines the *species* of the conic found in Arts. 1 or 2. For if two parabolæ can be described through the four points, the locus of centres must have two points at infinity, that is, it must be an hyperbola. If no parabola can be so described, the locus has no point at infinity; that is, it is an ellipse.

6. We can now easily find a construction for the locus of centres when the quadrilateral is convex.

For the asymptotes are parallel to the axes of parabolæ found in Art. 5 and the locus must pass through the intersections of AB, CD, of AD, BC, and of AC, BD, each pair of lines being a conic through the four points. We have then the three finite points, and two points at infinity. Construct for the centre by Art. 4 (hence "if on the three sides of any triangle as diagonals, parallelograms be described, having their sides parallel to two given lines, the other diagonals of these parallelograms will meet in a point"); thus we can draw the axes and asymptotes; construct by Pascal's theorem the points where the curve meets the major axis, and the thing is done. Or, of course, the length of the axis may be found more simply by performing the geometric operations indicated by the equation

$$CA^2 = CN^2 - \left(PN \cdot \frac{CA}{CB}\right)^2,$$

P being one of the three given finite points.

7. It appears from Art. 2 that the locus will break up into two straight lines, if E or F (Fig. 120) be at infinity, that is, if two sides of the quadrilateral are parallel. This is also clear from the fact that, when a conic becomes two parallel straight lines, any point midway between them is a centre. The line at infinity is itself part of the locus when it contains two of the points A, B, C, D. If one of these four is at infinity, only one parabola (or rather two coincident parabolæ) can be drawn through them, and the locus of centres is a parabola.

When the quadrangle can be inscribed in two *different* equilateral hyperbolæ, the locus of centres is a circle; and when it can be inscribed in a circle, the locus of centres is an equilateral hyperbola, whose asymptotes are equally inclined to any pair of opposite sides, and to the two diagonals. If two equilateral hyperbolæ intersect in A, B, C, D, then AB is perpendicular to CD, AC to BD, and AD to BC, each of the points A, B, C, D, being, in fact, the intersection of perpendiculars of the triangles formed by the other three. It is easily seen that the locus of centres is, in this case, the nine-point circle of any of the four triangles ABC, BCD, CDA, DAB. See Note to solution of 1408 [Reprint, Vol. I. p. 28].

8. If we write the equation of the conic in the form $\alpha\beta=\mu\gamma\delta$, it is clear that we shall pass from a possible to an impossible region by changing the sign of *one* or *three* of the quantities $\alpha\beta\gamma\delta$. Attention to this and to Fig. [120] will shew that a given sign of μ the curve must lie wholly in the shaded regions, or wholly in the unshaded regions. We proceed to trace the cyclic succession of these curves, beginning with the pair of straight lines AB, CD, considered as the limit of a hyperbola. It is clear that these may separate into a finite hyperbola in two ways; so as to lie in the shaded regions, or in the unshaded regions. We begin with the former. One branch of the hyperbola lies entirely in (8), where also the centre is; the other branch lies in (6), (7), (10), (11), having its infinite parts in (7). The branch in (8), with the centre, moves off rapidly from E, and when the centre is at an infinite distance, we have a parabola in (6), (7), (10), (11), the infinite parts being in (7). When the parabola closes up into an ellipse, the centre reappears from the infinity of (7), and finally passes into (1). The ellipse again elongates itself, but in the direction of (6), into which the centre passes. In the limit we get another parabola, the centre going off to the infinity of (6). As the parabola merges into a hyperbola, the centre reappears from the infinity of (9), and the limit of the hyperbola is the pair of straight lines FA, FB, the centre being at F. We now pass into the unshaded regions, beginning with a small hyperbola, one branch lying in (2), (1), and (4), and the other in (3), (1), and (5). The centre is in (11), and moves down into (1). The parts of the hyperbola in (1) gradually approximate, giving as a limiting form, the lines AC, BD, when the centre is at G. The branches separate again in the other direction, one lying in (2), (1), and (5), and the other in (3), (1), and (4). The centre moves from (1) into (10), and gradually approaches the point E, where the hyperbola again becomes two straight lines. This is the point from which we started. The points E, F, G lie on the *same* branch of the hyperbola which is the locus of centres, and no part of the locus lies in the regions (2), (3), (4), (5).

The re-entrant quadrangle $ACEF$ may be treated in the same way; this case is simpler, all the conics of the series being hyperbolæ.

9. It may be worth while to notice a property of the nine-point conics of the quadrilateral faces of a tetrahedral frustum. With Prof. Sylvester's own notation, let $Oabc$ be a tetrahedron, the axes of coordinates being Oa, Ob, Oc; and let the plane $\alpha\beta\gamma$ cut off the frustum $\alpha\beta\gamma abc$. Put $4a$ for Oa, and similarly for the others; and consider the quadrilateral $ab\beta\alpha$ in the plane of xy. Its nine-point conic is easily found to be

$$\frac{x}{a\alpha}\{x-2(a+\alpha)\}=\frac{y}{b\beta}\{y-2(b+\beta)\},$$

and we draw through this a cylinder whose generating lines are parallel to the axis of z. There are three such cylinders, and they evidently have common to them the curve section

$$\frac{x}{a\alpha}\{x-2(a+\alpha)\}=\frac{y}{b\beta}\{y-2(b+\beta)\}=\frac{z}{c\gamma}\{z-2(c+\gamma)\}.$$

We are obviously entitled to conclude that, if we take instead the polar conics of the lines in which the faces are cut by any plane, and draw cones to the points where that plane cuts the opposite edges, these three cones will have a common section.

10. The theorem stated incidentally in Art. 6 is a particular case of Brianchon's theorem. It may be put a little more generally as follows:—

Take two points P, Q, and through each of them draw three straight lines. These triads will intersect in nine points, as in the following scheme,

$$\left.\begin{matrix} \overbrace{A\quad B\quad C}^{P} \\ D\quad E\quad F \\ G\quad H\quad K \end{matrix}\right\} Q.$$

Take now three points, one from each of the P-lines, and one from each of the Q-lines, as, for instance, B, F, G. To each pair of these take the opposite diagonal of the quadrilateral, *e.g.*, to BF corresponds CE; then these three lines CE, DK, AH will meet in a point. There are six such systems of lines.

The points P, Q may be considered as a conic inscribed in the hexagon $CKHEDA$, of which CE, DK, AH are the diagonals. The theorem is thus seen to be a particular case of Brianchon's. It will be found to involve also the following theorem of determinants: viz., the determinant whose constituents are the nine determinants,

$$\begin{Vmatrix} C & \frac{a}{b} & A \\ c & \frac{A}{B} & a \end{Vmatrix}$$

$$\begin{Vmatrix} A & \frac{b}{c} & B \\ a & \frac{B}{C} & b \end{Vmatrix}$$

$$\begin{Vmatrix} B & \frac{c}{a} & C \\ b & \frac{C}{A} & c \end{Vmatrix}$$

vanishes identically.

1416. [Shew that the area of the perspective representation, in a given picture, of a triangle of given area in a fixed plane, varies as the product of the distances of the angles of the perspective representation from the vanishing line. Proposed by Prof. Sylvester, F.R.S. Reprint, Vol. I. p. 77, *E. T.* June, 1864.]

Let ABC be the perspective representation of the triangle, DE the vanishing line. Let BC meet DE in D, and join AD. If abc is the triangle represented, AD is the picture of a line through a parallel to bc. If therefore B and C are fixed, the point A can only move along AD. But the area ABC varies as the perpendicular from A on BC, which is in a constant ratio to the perpendicular from A on DE, because A lies on a fixed line through the intersection of BC and DE. Since then when two of the perpendiculars on the vanishing line are fixed, the area varies directly as the remaining one; therefore when all vary, the area varies as the product of the three.

1479. Prove that the ordinary inverse of the *Tangential inverse* is the second positive pedal; and that the *Tangential inverse* of the ordinary inverse is the second negative pedal of the primitive.

[February, 1864. Solved, Reprint, Vol. I. p. 78.]

1497. (1) Given three points by equations of the form $lx+my+nz=0$, prove that the area of the triangle contained by them is

$$(l_1m_2n_3) \div (l_1+m_1+n_1)(l_2+m_2+n_2)(l_3+m_3+n_3),$$

that of the triangle of reference being unity.

(2) Also, if (123) denote the area of the triangle contained by the points 1, 2, 3, and so on, prove that

$$(123)(456) \equiv (156)(423) + (164)(523) + (145)(623).$$

[April, 1864. Solved, Reprint, Vol. I. p. 79; Proposer's solution, Reprint, Vol. IV. p. 53; September, 1865 (?).]

1. The area of the triangle formed by three points vanishes only when they are in a straight line, and becomes infinite only when one of them is at infinity. The condition that they may be in a straight line is

$$J \equiv \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} = 0, \text{ or } (l_1m_2n_3)=0;$$

and the condition that one of them may be at infinity is

$$P \equiv (l_1+m_1+n_1)(l_2+m_2+n_2)(l_3+m_3+n_3)=0.$$

Now the expression for the area must be of no dimensions in the coefficients, since we are only concerned with their ratios; and the equation obtained by equating this expression to a constant must be of the first order in each set of coefficients; since, two of the points being fixed, the locus of the other is then a straight line. The expression for the area is therefore some numerical multiple of $\frac{J}{P}$. By putting $x, y, z=0$ for the three points, we find that the area of the fundamental triangle, on the same scale, is unity.

2. To prove the second proposition, take (456) for the fundamental triangle. Then, by applying the above interpretation to the well-known theorem

$$(l_1m_2n_3) \equiv l_1(m_2n_3) + m_1(n_2l_3) + n_1(l_2m_3),$$

we find it equivalent to

$$(123)(456) \equiv (156)(423) + (164)(523) + (145)(623),$$

the factor P dividing out on both sides.

[Solutions were also sent to *E. T.* 1372, 1373, June, 1863: Reprint, Vol. I. 1380, p. 16: 1385, 1386, p. 9: 1402, p. 19: 1404, 1405, p. 21: 1406, p. 22: 1411, p. 43.]

1505. [If P, Q, 1, 2, 3, 4 be points on a conic, then the four points $P1$, $Q2$; $P2$, $Q1$; $P3$, $Q4$; $P4$, $Q3$ lie on a conic passing through the points P and Q. Proposed by Prof. Cayley. Reprint, Vol. II. pp. 9, 10, *E. T.* July, 1864.]

1. Let the four points $P1$, $Q2$; $P2$, $Q1$; $P3$, $Q4$; $P4$, $Q3$ be called S, T, U, V respectively; then

$$\{P.1234\} = \{Q.1234\};$$

but $\{P.1234\} = \{P.STUV\}$,

and $\{Q.1234\} = \{Q.TSVU\}$,

also $[TSVU] = [STUV]$*,

therefore $\{P.STUV\} = \{Q.STUV\}$,

which proves that the six points P, Q, S, T, U, V lie on a conic.

2. Let $A=0$, $B=0$, $C=0$, $D=0$, denote respectively the pairs of right lines ($P1$, $Q1$), ($P2$, $Q2$), ($P3$, $Q3$), ($P4$, $Q4$). Then we shall prove presently that there is an identical relation

$$A+B+C+D=0,$$

constant multipliers being supposed.

3. The Jacobian of any three of the four conics A, B, C, D is obviously the original conic $PQ1234$, together with the straight line PQ. Now the conic $A+B=0$ is identical with $C+D=0$ (by Art. 1); and it passes through all the intersections of A with B, and of C with D. It must therefore be the very conic $PQSTUV$. And there are clearly two more conics, namely, $A+C$ or $D+B$, and $A+D$ or $B+C$, obtained just in the same way. It may be as well to remark that the three are represented by the equation

$$\frac{1}{A}+\frac{1}{B}+\frac{1}{C}+\frac{1}{D}=0.$$

Moreover, the original conic may be reproduced by treating $STUV$ in the same way as we treated 1234. We have therefore four conics derived from the two points P, Q in a symmetrical manner. *Each of these conics is the Jacobian of the other three;* the line PQ being of course added. For the three conics $A+B$, $B+C$, $C+A$ have the same Jacobian as A, B, C, that is, the original conic and the line PQ.

* [ ] Auctoris.

The pole of PQ, with regard to the conic $STUV$, is the intersection of 12 and 34. For the chord ST is divided harmonically by PQ and 12, and the chord UV by PQ and 34. Hence *the poles of* PQ, *with regard to any three of the conics, form a self-conjugate triad with regard to the fourth.* For the poles with regard to $A+B$, $B+C$, $C+A$, are the intersections of (12, 34), (14, 23), (13, 42), which form a self-conjugate triad of any conic 1234.

4. By projecting the points P, Q into the circular points at infinity, we may prove M. Laguerre's theorems, proposed in the *Nouvelles Annales* for March, 1864 (p. 141, Question 698).

"Lorsqu'une courbe a quatre foyers sur un cercle, elle en a nécessairement douze autres situés par quatre sur trois autres cercles; tous ces cercles sont orthogonaux entre eux."

It follows that when four circles cut each other orthogonally, each centre is the intersection of perpendiculars of the triangle formed by the other three. Hence one of the circles must be imaginary.

5. We now proceed to prove the statement in (2).

LEMMA. When four conics have the same Jacobian, three and three, their equations are connected by an identical linear relation.

Any four conics can be reduced simultaneously to the form

$$a_1x^2+b_1y^2+c_1z^2+d_1w^2=0,$$
$$a_2x^2+b_2y^2+c_2z^2+d_2w^2=0,$$
$$\&c. \qquad \&c.$$

where $x+y+z+w=0$. This we can see by counting the constants. Let R be the determinant $(a_1\,b_2\,c_3\,d_4)$, and A_1, A_2, &c. its first minors. Then the Jacobians

$$\frac{A_1}{x}-\frac{B_1}{y}+\frac{C_1}{z}-\frac{D_1}{w}=0,$$
$$\&c. \qquad \&c.$$

and if these are all identical, we must have $(A_1B_2C_3D_4)=0$, which implies that $(a_1b_2c_3d_4)=0$, or the conics are connected by an identical linear relation.

For a metric interpretation, see Dr Salmon's *Conics*, 4th ed., Art. 94.

1514. [Let P be the points of intersection of the three perpendiculars, and G the centre of gravity of any triangle ABC; also let l, m, n be the middle points of the sides BC, CA, AB; S_l, S_m, S_n, the circles described upon Al, Bm, Cn as diameters, and S_1, S_2, S_3, the circles circumscribing the triangles PBC, PCA, PAB.

It is required to prove,

(α) That the circle which passes through l, m, n passes also through the points of intersection, real or imaginary, of the self-conjugate and circumscribing circles of each of the triangles PBC, PCA, PAB, ABC.

(β) That the six points common to the three pairs of circles S_l, S_1; S_m, S_2; S_n, S_3; lie on another circle Σ.

(γ) That the self-conjugate and circumscribing circles of the triangle, the circle which bisects its sides, the circle upon PG as diameter, the circle Σ, and the director of the maximum ellipse that can be inscribed in the triangle, all pass through the same two points, real or imaginary. Proposed by J. Griffiths, M.A. Reprint, Vol. II. p. 27, *E. T.* August, 1864.]

Let $\qquad U\equiv a^2yz+b^2zx+c^2xy,$

$$V\equiv(x+y+z)\,(bc\cos A\,.\,x+ca\cos B\,.\,y+ab\cos C\,.\,z);$$

then $U=kV$ represents respectively (the coordinates being triangular)

(1) when $k=\frac{1}{2}$, the *nine-point* circle of the triangle of reference;

(2) when $k=0$, the *circumscribing* circle;

(3) when $k=1$, the *self-conjugate* circle;

(4) when $k=\frac{1}{3}$, the *director of the maximum ellipse;*

(5) when $k=\frac{2}{3}$, the *circle on* PG as *diameter*.

The last two are the only ones which present any difficulty.

In (4) the tangential equations of the ellipse and the circular points at infinity are respectively

$$yz+zx+xy=0,$$
$$a^2x^2+b^2y^2+c^2z^2-2bc\cos A\,.\,yz-2ca\cos B\,.\,zx-2ab\cos C\,.\,xy=0.$$

Forming, by the rule, the *harmonic conic* of these two, we at once write down equation (4). To find equation (5), we make use of the following

THEOREM. If $A=0$, $B=0$, $C=0$, $D=0$ are any four lines in a plane, and if $\Psi AB=0$ denote the condition that A and B may be at right angles, then, the equation to the circle whose diameter is the line joining (AB), (CD) is

$$\begin{vmatrix} \Psi AC, & \Psi AD, & A \\ \Psi BC, & \Psi BD, & B \\ C\;, & D\;, & 0 \end{vmatrix}=0.$$

In the present case, take the four lines

$$A\equiv x-y;$$
$$B\equiv y-z;$$
$$C\equiv bc\cos A\,.\,x-ca\cos B\,.\,y;$$
$$D\equiv ca\cos B\,.\,y-ab\cos C\,.\,z;$$

then the condition of perpendicularity of $(l_1m_1n_1)\,(l_2m_2n_2)$ being

$$a^2l_1l_2+b^2m_1m_2+c^2n_1n_2-bc\cos A\,(m_1n_2+m_2n_1)-\&c.=0,$$

we have

$$\Psi AC=\Psi BD=abc\,(a\cos A+b\cos B+c\cos C)=2abc\,.\,a\sin B\sin C,$$
$$\Psi AD=\Psi BC=-abc\,.\,b\sin C\sin A=-\tfrac{1}{2}\,\Psi AC.$$

Expanding then the determinant, it becomes simply

$$2\,(AC+BD)+AD+BC=0,$$

and, substituting the values of A, B, C, D, we get equation (5).

1486. If two transversals ABC, DEF cut the sides of any triangle, then AE, BF, CD cut the sides in three points on a straight line X. If moreover the triangle touch a cubic at A, B, C, and cut it in D, E, F, the lines AE, BF, CD meet the curve in three points on a straight line Y, and the lines X, Y, ABC, meet in a point.

[*E. T.* March, 1864. Solved, Reprint, Vol. II. p. 40.]

1519. [ABC is a triangle having the three real points (P, Q, R) of inflexion of a cubic on the sides BC, CA, AB respectively, each of which also passes through two imaginary points of inflexion. The tangents at Q and R meet in D, those at R and P in E, and those at P and Q in F. Shew that AD, BE, CF meet in a point which is fixed for all the cubics having the same nine points of inflexion. Proposed by F. D. Thomson, M.A. Reprint, Vol. II. p. 48, *E. T.* September, 1864.]

Taking ABC for triangle of reference, the equation of the cubic is

$$a^3x^3+b^3y^3+c^3z^3-3dxyz=0,$$

which may also be written in the form

$$\left(\frac{d}{bc}x+by+cz\right)\left(\frac{d}{bc}x+\theta by+\theta^2cz\right)\left(\frac{d}{bc}x+\theta^2by+\theta cz\right)=\frac{d^3-a^3b^3c^3}{b^3c^3}x^3,$$

(where θ is an imaginary cube root of unity) shewing that the tangents at P, Q, R are

$$\frac{d}{bc}x+by+cz=0,$$

$$ax+\frac{d}{ca}y+cz=0,$$

$$ax+by+\frac{d}{ab}z=0.$$

The equation to AD is therefore $by=cz$, so that AD, BE, CF meet in the point $ax=by=cz$.

Now all the points of inflexion are on the axis $xyz=0$; consequently any other cubic having the same points of inflexion can only differ from the above in the coefficient of xyz, of which the point $ax=by=cz$ is independent.

This point is the pole of the line PQR with respect to the triangle ABC.

1517. [If each edge of a tetrahedron is perpendicular to the non-conterminous edge (it being observed that if two pairs of such edges be mutually perpendicular, the third pair will be so too); prove that the nine-point circles of the three triangular faces lie on a sphere; also that the nine-point circle of any triangular face, and the three points of intersection of the perpendiculars of the other three triangular faces, lie on a sphere; and find the equations of all these spheres. Proposed by H. R. Greer, M.A. Reprint, Vol. II. p. 79, November, 1864.]

The triangular equation of the nine-point circle is

$$a^2yz+b^2zx+c^2xy=2(x+y+z)(bc\cos A\,.\,x+ca\cos B\,.\,y+ab\cos C\,.\,z)\ldots(1).$$

Now when two opposite edges of a tetrahedron are perpendicular, a plane may be drawn through either perpendicular to the other, and will therefore contain the perpendiculars from the extremities of the first upon the second. Consequently, if a tetrahedron has *two* pairs of perpendicular edges, the perpendiculars from the vertices to opposite faces will meet in a point, and the foot of any one of them will be the intersection of perpendiculars of the face in which it lies. From this it is obvious that, if $ABCD$ be such a tetrahedron,

$$AB\,.\,AC\cos BAC=AC\,.\,AD\cos CAD=AD\,.\,AB\cos DAB=(A),\ \text{suppose.}$$

It follows at once that the nine-point circles of the four faces lie on a sphere; for the tetrahedral equation of this, the "twenty-four-point sphere," is

$$ab^2\,.\,xy+ac^2\,.\,xz+\ldots=2(x+y+z+w)\{(A)x+(B)y+(C)z+(D)w\}\ldots\ldots(2).$$

The equation to the sphere which contains the nine-point circle of the face a, and the polar centres of the other three faces, is got by changing the sign of (A) in equation (2). The equation of the self-conjugate sphere is

$$ab^2\,.\,xy+ac^2\,.\,xz+\ldots=(x+y+z+w)\{(A)x+(B)y+(C)z+(D)w\};$$

this, therefore, passes through the intersection of the circumscribed and the twenty-four-point spheres.

If through the middle point of each edge of a tetrahedron a line be drawn parallel to the opposite edge, the tetrahedron will be reproduced in an inverted position. In the present case, the two tetrahedra will have the same twenty-four-point sphere, and the sphere self-conjugate to one will circumscribe the other.

[The proposer in his solution refers to an article on this species of tetrahedron by Prof. Wolstenholme (*Quarterly Journal of Mathematics*, Vol. III.).]

1394. [Required a *direct* proof that an ellipse and its osculating circle have a contact of the third order at the ends of its axes; also prove that the deviations of the ellipse from the circle osculating it most closely at the ends of its axes are to each other inversely as the seventh powers of the axes. Proposed by Matthew Collins, B.A. Reprint, Vol. II. pp. 82—85, December, 1864.]

1. We know that if the osculating circle at a point P meet the ellipse again at Q, PQ and the tangent at P are equally inclined to the axes. (Salmon's *Conics*, 4th ed. Art. 244; Taylor's *Geometrical Conics*, p. 85). This shews that the equation of the osculating circle at (ξ, η) may be written in either of the forms

$$\left(\frac{\xi^2}{a^4}+\frac{\eta^2}{b^4}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left(\frac{x\xi}{a^2}+\frac{y\eta}{b^2}-1\right)\left\{\frac{\xi(x-\xi)}{a^2}-\frac{\eta(y-\eta)}{b^2}\right\}\ldots(1),$$

$$\left(\frac{\xi^2}{a^4}+\frac{\eta^2}{b^4}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left\{\frac{\xi^2}{a^4}(x-\xi)^2-\frac{\eta^2}{b^4}(y-\eta)^2\right\}\ldots\ldots\ldots(2).$$

When $\xi=0$, $\eta=\pm b$, (2) becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\left(\frac{1}{a^2}-\frac{1}{b^2}\right)(y-\eta)^2=0\ldots\ldots\ldots\ldots(3).$$

When $\eta=0$, $\xi=\pm a$, (2) becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1-\left(\frac{1}{a^2}-\frac{1}{b^2}\right)(x-\xi)^2=0\ldots\ldots\ldots\ldots(4).$$

This shews that in both these cases the curves meet only where they meet the common tangent, that is, they have *four* consecutive points common; or, what is the same thing, they have contact of the *third* order. This is also seen to follow at once from the property enunciated at the outset.

2. Consider now the geometrical meaning of equation (4). Take any point P on the circle, and draw P_1QQ_1T parallel to the major axis, meeting the ellipse in Q, Q_1, and the tangent at the vertex (A) in T. Then, if C is the centre of the ellipse and (x, y) the coordinates of P, the quantity $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ is equal to $\frac{PQ \,.\, PQ_1}{CA^2}$, and $\xi - x$ is PT. Hence (4) means that

$$\frac{PQ \,.\, PQ_1}{CA^2}=\left(\frac{1}{a^2}-\frac{1}{b^2}\right)PT^2.$$

But in the limit $PQ_1=2CA$, so that we may write

$$PQ=\frac{PT^2 \,.\, CA}{2}\cdot\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\text{.............................}(5).$$

If we make a similar construction with small letters near the extremity of the *minor* axis, we shall get from (3)

$$pq=\frac{pt^2 \,.\, CB}{2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right).$$

Hence $$\frac{PQ}{pq}=\frac{CA}{CB}\cdot\frac{PT^2}{pt^2}.$$

But PT, pt are as the reciprocals of the radii of curvature, or as

$$\frac{CA^2}{CB}:\frac{CB^2}{CA}.$$

Therefore $$\frac{PQ}{pq}=\frac{CA}{CB}\cdot\frac{CA^4}{CB^2}\cdot\frac{CA^2}{CB^4}=\frac{CA^7}{CB^7}.$$

3. We take this opportunity of setting down two other equations of the circle of curvature, which are easily deduced from (1) or (2).

The tangent at any point of an ellipse may be represented by the equation

$$lx+my=\sqrt{(l^2a^2+m^2b^2)}\text{.................................}(6).$$

Let $\sqrt{(l^2a^2+m^2b^2)}=p$, then the common chord of the ellipse and the circle of curvature at the points (l, m) is represented by

$$lx-my=\frac{l^2a^2-m^2b^2}{p}\text{ ....}(7).$$

Hence from (2) the equation to the osculating circle is

$$p^2(l^2+m^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\{l^2(px-la^2)^2-m^2(py-mb^2)^2\}\text{.....}(8).$$

Next, at a point whose eccentric angle is ϕ, the equations to the tangent and the common chord are, respectively,

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1\text{.................................}(9);$$

$$\frac{x}{a}\cos\phi-\frac{y}{b}\sin\phi=\cos 2\phi\text{........................}(10);$$

whence, putting $a^2-b^2=c^2$, we readily obtain the equation of the osculating circle in the form

$$x^2+y^2-2c^2\left(\frac{x}{a}\cos^3\phi-\frac{y}{b}\sin^3\phi\right)=a^2\sin^2\phi+b^2\cos^2\phi-c^2\cos 2\phi\text{.......}(11).$$

4. If we put $x=x_1+\alpha$, $y=y_1+\beta$, the equation of the tangent will still be of the form

$$l(x_1+\alpha)+m(y_1+\beta)=\sqrt{(l^2a^2+m^2b^2)}\text{.....................}(12),$$

and the line through (α, β) perpendicular to this is clearly $\frac{x_1}{l}=\frac{y_1}{m}$.

The locus of the foot of this perpendicular is therefore obtained by writing (x_1, y_1) for (l, m) in (12).

Hence, *if the equation of the tangent to any curve can be put in the form*

$$lx+my=F(l, m),$$

then the equation of the pedal with the point (α, β) *for origin is*

$$x(x-\alpha)+y(y-\beta)=F(x-\alpha, y-\beta)\text{ ....}(13).$$

For instance, the locus of the foot of the perpendicular from the centre of the ellipse on the common chord (7) is

$$(x^2+y^2)^2(a^2x^2+b^2y^2)=(a^2x^2-b^2y^2)^2\text{.....................}(14).$$

All that has been here said may be applied to the hyperbola by changing the sign of b^2.

5. The first pedal of the cycloid may be simply obtained by the method of Art. 4. For let the circle begin to roll on the axis of x at the origin, and consider the tangent at a point corresponding to a revolution ϕ of the circle. If its equation be written

$$lx+my=F(l,m),$$

we must have

$$\phi=2\cot^{-1}\left(-\frac{l}{m}\right);$$

now the tangent passes through the point $2x=a\phi$, $y=a$, where a is the diameter of the circle; whence the equation may be written

$$lx+my=a\left(m-l\cot^{-1}\frac{l}{m}\right),$$

and we immediately get the pedal with origin (α, β); viz.,

$$x(x-\alpha)+y(y-\beta)=a\left\{(y-\beta)-(x-\alpha)\cot^{-1}\frac{x-\alpha}{y-\beta}\right\}.$$

Putting $\alpha=0$ and $\beta=0$, we get the pedal from the origin, viz.,

$$x^2+y^2=a\left(y-x\cot^{-1}\frac{x}{y}\right),$$

or, in polar coordinates,

$$r=a(\sin\theta-\theta\cos\theta).$$

1468. Given the centre of a conic, and a conjugate triad; to construct for the directions of the asymptotes. [Proposed, *E. T.* January, 1864: solution, February, 1865.]

[If O be the given centre of the conic; A, B, C the three points of the self-conjugate triad; and OX, OY, OZ the three lines through O parallel to BC, CA, AB respectively; the two double rays (real or imaginary) OM and ON of the involution determined by the three angles AOX, BOY, COZ are the two asymptotes required. For the three pairs of conjugates, OA and OX, OB and OY, OC and OZ, determining the involution, being evidently pairs of conjugate diameters of the conic, divide, therefore, harmonically the angle (real or imaginary) MON determined by the two asymptotes OM and ON.

The same construction (with some slight and obvious modifications) applies also to the following more general problem, of which the above is evidently a particular case: viz., given a point and a line, pole and polar with respect to a conic and a conjugate triad; to construct the two tangents (real or imaginary) from the point to the curve, and the two intersections (real or imaginary) of the line with the curve. For if P and L be the point and line; A, B, C, as before, the three points of the triad; X, Y, Z and X', Y', Z' the six intersections of L with BC, CA, AB, and with PA, PB, PC, respectively; then, as X and X', Y and Y', Z and Z' are evidently pairs of conjugate points with respect to the conic, the two double points M and N of the involution determined by the three segments XX', YY', ZZ', as cutting them all harmonically, are the two intersections required; and as PX and PX', PY and PY', PZ and PZ', are evidently pairs of conjugate lines with respect to the conic, the two double rays PM and PN of the involution determined by the three angles XPX', YPY', ZPZ', as cutting them all harmonically, are the two tangents required.

The two corresponding problems in geometry of three dimensions, viz., Given, of a quadric, the centre and a self-conjugate tetrahedron, to construct the asymptotic cone of the surface; or, more generally: Given, of a quadric, a point and plane, pole and polar to each other, and a self-conjugate tetrahedron, to construct the tangent cone from the point to the surface, and the conic of intersection of the plane with the surface, may be readily solved by application of the above.]

Corollary i. Let any two straight lines parallel to two conjugate diameters be called *conjugate* with respect to a conic; then it is shewn above that the pairs of lines 12, 34; 13, 24; 14, 23, joining the points 1234, are conjugates with respect to the conic which has the point 1 for a centre, and 234 for a conjugate triad. But the symmetry of this statement shews that they are also conjugates with respect to the conic which has any other of the five points for centre, and the remaining three for a conjugate triad. We may draw four such conics; and since the asymptotes are determined in direction by *two* pairs of conjugates, it follows that these four conics are all similar and similarly situated. So, in the more general case, we shall have four conics intersecting in two points on the given straight line.

Corollary ii. Let a straight line and plane, drawn parallel to any diameter and its conjugate diametral plane, be called *conjugates* with respect to a conicoid. Then if we are given five points 12345, of which 1 is the centre, and 2345 a self-conjugate tetrahedron of a given conicoid, it is evident that since 2 is the pole of the plane 345, 12 is conjugate to 345, and so on. We thus get four pairs of conjugates. Again, since 23 is the polar line of 45, 123 is conjugate to 45, and 145 to 23, and so on. This gives us six more pairs of conjugates. But this amounts to saying that if we join any three of the five points by a plane, and the other two by a line, the line and plane are conjugates. This statement makes no mention of the particular point taken for centre; and we conclude as before, that if five conicoids are drawn, by taking each of five points in succession for centre, and the remaining four for a self-conjugate tetrahedron, these five conicoids will be similar and similarly situated. A line and its conjugate plane cut the plane at infinity in a point and line which are pole and polar with respect to the section which the plane at infinity makes of the conicoid. The problem is therefore equivalent to that of describing a conic, being given the poles of certain lines. Three points and their polars are sufficient to determine a conic; for let A, B, C be the points, and let AB meet the polar of A and B in P, Q respectively. Then the foci of the involution determined by AP, BQ, are evidently points on the conic. In this way we can determine six points on the sides of the triangle ABC, and six more on the sides of the reciprocal triangle; and it would be interesting to prove *a priori* that these twelve points must lie on the same conic, when the triangles are in perspective.

Corollary iii. In the plane case we are given three pairs of conjugates to determine two points at infinity; and we conclude that any transversal is cut in involution by the six lines joining four points. Similarly we conclude from the solid problem that "if five points in space are joined every way by ten lines and ten planes, the system will be cut by any plane in a system of points and lines which are poles and polars with respect to a certain conic." The analogy of this relation of points and lines with involution, may be illustrated analytically. Let $U \equiv ax^2 + by^2 = 0$ be a pair of points; then if we put Δ for

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right),$$

a point and its harmonic conjugate will be represented by the equations

$$\begin{vmatrix} x, & y \\ \xi, & \eta \end{vmatrix} = 0, \text{ and } \Delta U = 0.$$

And a system of such harmonic conjugates is of course a system in involution. Next let U represent a conic, and Δ stand for

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \zeta \frac{d}{dz}\right),$$

then a point and its polar will be represented by the equations

$$\left\| \begin{matrix} x, & y, & z \\ \xi, & \eta, & \zeta \end{matrix} \right\| = 0, \text{ and } \Delta U = 0,$$

and the analogy is obvious.

Corollary iv. Lastly, the four conics mentioned in Cor. i. are all similar to the *nine-point conic* of the quadrangle, or locus of centres of all conics through the four points. This proposition was set in a problem paper, at St John's College, Cambridge, in Dec. 1862; but I do not know to whom it is due.

It follows at once from the equation to the nine-point conic given in Art. 2 of the solution to Question 1443 [Reprint, Vol. I. p. 51, supra, p. 580]; for an equation of the second degree in x, y, 1, in which the coefficient of xy is zero, obviously represents a conic with respect to which the axes are conjugates. Thus the lines 12, 34; 13, 24; 14, 23, are conjugates with respect to the nine-point conic, and therefore its asymptotes are parallel to those of the other four. These, therefore, are ellipses when the quadrangle is re-entrant, and hyperbolas when it is convex.

[Reprint, Vol. III. pp. 35, 36. Prof. Townsend and the proposer are credited with the solution. I felt sure that the Corollaries were Clifford's work, and on asking Mr Miller I find that the part enclosed in brackets above was due to Prof. Townsend, and that Clifford instead of having his solution, a short one, printed, added the above Corollaries to the "proof."]

1652. Through the angles A, B, C of a plane triangle three straight lines Aa, Bb, Cc are drawn. A straight line AR meets Cc in R; RB meets Aa in P; PC meets Bb in Q; QA meets Cc in r; and so on. Prove that, after going twice round the triangle in this way, we always come back to the same point.

Shew that the theorem is its own reciprocal. Find the analogous properties of a skew quadrilateral in space, and of a polygon of n sides in a plane. [*E. T.* February, 1865. Reprint, Vol. III. p. 66, *E. T.* April, 1865, which contains a solution by Prof. Cayley.]

Let x, y, z be the sides of the triangle ABC, and let $ay=z$, $bz=x$, $cx=y$ be the three lines drawn through them. Start with the line AR or $y=z$, which meets Cc or $cx=y$ on $cx=z$, which meets $ay=z$ on $cx=ay$, which meets $bz=x$ on $cbz=ay$, which meets $cx=y$ on $bz=ax$, which meets $ay=z$ on $by=x$, which meets $bz=x$ on $y=z$; so that we have come round again. The extension of this is most easy; I write down two enunciations:—

Consider a plane polygon of an odd number of sides; let the two sides adjacent to any given side be produced to meet, and through their intersections let an arbitrary line be drawn; then treating these lines in the same way as Aa, Bb, Cc were treated in the case of the triangle, we may go twice round the polygon, and shall always come back to the same point.

Let $ABCD$ be a skew quadrilateral in space, and through the four sides AB, BC, CD, DA let arbitrary planes be drawn; let any line through A meet the plane through CD in a; aB meets the plane DA in b; and so on; after going *three* times round the quadrilateral we shall come back to the same point.

The theorem is not true for a plane polygon of an even number of sides; I have not been able to find an analogue in this case.

1679. [To find the envelope of the straight line joining the feet of the perpendiculars drawn on the sides of a triangle from a point in the circumference of the circumscribed circle. Proposed by H. R. Greer, B.A. Reprint, Vol. III. pp. 81, 2, *E. T.* May, 1865.]

The line in question is known to be the tangent at the vertex of a parabola which touches the sides of the triangle. Now this straight line, being always at right angles to the axis of the parabola, determines on the line at infinity a series of points in involution with the series determined by the parabola itself; we have then a series of conics touching four given lines, and a series of points on one of the lines, homographic with the series of conics; and we want to find the envelope of the remaining tangent, drawn from each point to its corresponding conic. Let then $U=kV$ be the tangential equation of the series of conics and $P=kQ$ of the series of points. We obtain the required envelope by eliminating k; it is $UQ=VP$, a curve of the third class touching the common tangents of U and V, and the line PQ. When, as in the case we are considering, the line PQ coincides with one of the common tangents of U, V, then it is a double tangent to the curve $UQ=VP$, and the points of contact are the double points of the involution; in this case, the circular points at infinity. Since the curve is of the third class, and has one double tangent (that is, all it can) it is of the fourth order; and because the double tangent has imaginary contacts, the curve has three *real* cusps. To determine the position of these cusps, and the general form of the curve, we have to study a most singular figure.

Consider four points, 1, 2, 3, 4, such that each is the intersection of perpendiculars of the triangle formed by the other three. About the triangles 234, 341, 412, 123 describe circles; it is known that these circles are all equal, and that their centres 1′, 2′, 3′, 4′ form another quadrangle, exactly similar and equal to 1 2 3 4, but in an inverted position, their centre of (inverse) similitude being the centre of the nine-point circle. Now suppose that the feet of the perpendiculars from any point in the circle 234 to the sides of the triangle 234 are joined by a line X. Then I say that *if at the points where the line X cuts the six connectors of the quadrangle* 1234, *perpendiculars be drawn to these six connectors respectively, the perpendiculars will concur three by three, in four points*, 1″, 2″, 3″, 4″, *situate one on each of the four circumscribing circles, and forming a quadrangle equal, similar, and similarly situated to* 1′ 2′ 3′ 4′. And *the centre of* (inverse) *similitude of* 1 2 3 4 *and* 1″ 2″ 3″ 4″ *is situated on the line* X, *and bisects the segment determined on it by any pair of connectors.* Hence we see (1), that the line X is connected with the whole quadrangle, and not with three particular points of it; (2), it is cut by the connectors in an involution, one double point of which is at infinity; and therefore is an *asymptote* of some conic passing through the points 1, 2, 3, 4.

Now, take any connector 1 2, and find a point on it, symmetrical in respect of 1, 2, with the point where it is cut by 34. Then the envelope of X touches all the connectors at the points thus determined.

Since writing the above, I have read a paper on the subject by Steiner, in the 53rd volume of Crelle's *Journal* *. He asserts that the curve is a hypocycloid of three branches, and gives a simple construction for the cusps.

The property of a quadrangle enunciated above, is in fact this:—If four parabolas be drawn, having their axes parallel, each inscribed in one of the four triangles determined by a quadrangle, these four will have a common tangent;

* [An abridgment of Steiner's paper is printed on pp. 97—100 of this Vol. III. of the *Reprint*.]

which is at once seen to be a particular case of the reciprocal of this:—The four circles, each circumscribing one of the triangles determined by a quadrilateral, have a common point. And this again is a particular case of that wonderful proposition, the involution of cubics:—All the cubics which pass through eight fixed points pass also through a ninth point.

Finally, reciprocate the whole figure in respect of the self-conjugate circle of any of the triangles 234, &c. We then get the locus of a point where the normal at (1) meets again a rectangular hyperbola circumscribing the quadrangle; it is a cubic having its asymptotes parallel to the sides of 234, and with a double point at (1), the tangents to which are the asymptotes of the polar circle. In fact, this problem is rather easier than its reciprocal.

1680. [(1) Prove that the envelope of the asymptotes of a rectangular hyperbola described about a given triangle is a curve of the third class, touching the sides of the triangle, the three perpendiculars, lines through the feet of the perpendiculars parallel to the opposite sides of the triangle formed by joining them, and also the line at infinity.

(2) Prove that the envelope of the asymptotes of a conic inscribed in a given quadrilateral, is a curve of the third class touching the sides and diagonals of the quadrilateral, the line at infinity, and the line joining the middle points of the diagonals. Proposed by F. D. Thomson, M.A. Reprint, Vol. III. pp. 82, 3, *E. T.* May, 1865.]

(1) It is shewn in the solution of 1679 that the line whose envelope is there considered is an asymptote of *some* rectangular hyperbola circumscribing the quadrangle; whence the two envelopes must be identical. This may also be proved thus: the proposition is that a rectangular hyperbola may circumscribe any triangle which circumscribes a parabola, and have for an asymptote the tangent at the vertex of the parabola. Let β be the axis of the parabola, α the tangent at its vertex, γ the line at infinity; then the respective equations to the hyperbola and parabola are

$$\gamma^2+2p\alpha\gamma=2\mu\alpha\beta, \qquad \beta^2=2\lambda\gamma\alpha\ ;$$

whence $\Theta=-p^2,\quad \Theta'=2p\lambda,\quad \Delta=-\mu^2,\quad \Delta'=-\lambda^2,$

and the condition $\Theta'^2=4\Theta\Delta'$ is satisfied. In fact, the triangle $(\alpha\beta\gamma)$ is inscribed in the hyperbola, and circumscribes the parabola.

Hence (i) the envelope of the asymptotes of all conics through four given points is a three-cusped quartic touching the six connectors of the given points, and the line at infinity at the points of contact of the parabola through them. (ii) If two tangents to a three-cusped quartic divide harmonically the double tangent, their intersection lies on a conic through the points of contact of the double tangent. This conic touches the quartic in three points. (iii) If a chord of a nodal cubic subtend harmonically the double point, its envelope is a conic touching the tangents at the double point, and the curve itself in three points.

M. Chasles gets the result (i) by his method of characteristics (Theor. XVI.). The envelope of the asymptotes is in general of class $\mu+\nu$, and has a ν-ple tangent at infinity; where μ is the number of conics of a system that can be

drawn through a given point, and ν the number that can be drawn to touch a given line.

(2) Here again M. Chasles's method shews that the envelope is of the third class, and touches *once* the line at infinity. Let U, V be two inscribed conics, and $(\xi,\ \eta,\ \zeta)$ the coordinates of the line at infinity; and write also Δ for $(\xi\delta_x+\eta\delta_y+\zeta\delta_z)$; then a conic of the system is $U=kV$, the centre $\Delta U=k\Delta V$, and the envelope required $U\Delta V=V\Delta U$, which is of the third class, touching the sides of the quadrilateral, and the line $\Delta U=0$, $\Delta V=0$, which joins the middle points of the diagonals. If for U, V we write AB, CD, the equation is

$$AB(C\,.\,\Delta D+D\,.\,\Delta C)=CD(A\,.\,\Delta B+B\,.\,\Delta A),$$

shewing that the curve touches the lines $(A=0,\ B=0)$ and $(C=0,\ D=0)$; that is the diagonals of the quadrilateral.

1724. The equations of three conics being given in the following forms, viz.,

$$a_1x^2+b_1y^2+c_1z^2+d_1w^2=0,$$
$$a_2x^2+\&c.=0,$$
$$a_3x^2+\&c.=0,$$

where $x+y+z+w\equiv 0,$

shew that a straight line $(\xi x+\eta y+\zeta z+\omega w=0)$ will be cut in involution by them, if

$$\Sigma\,.\begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}.\,(\xi-\eta)(\xi-\zeta)(\xi-\omega)\,.\,(\text{to four terms})=0.$$

[Proposed, *E. T.* May, 1865. Solved Reprint, Vol. IV. p. 52.]

1733. [To find the area of a triangle, the equations of whose sides in trilinear coordinates are

$$l_1\alpha+m_1\beta+n_1\gamma=0, \qquad l_2\alpha+\ldots=0, \qquad l_3\alpha+\ldots=0.$$

Proposed by W. A. Whitworth, M.A. Reprint, Vol. IV. pp. 55, 6, *E. T.* September, 1865.]

Call the three lines 1, 2, 3. Then we have to find the area of the triangle included by the points (23), (31), (12); that is, by the points

$$L_1x+M_1y+N_1z=0, \quad L_2x+\ldots=0, \quad L_3x+\ldots=0\ ;$$

where $L_1\ldots$ are the first minors of the determinant $(l_1m_2n_3)$. But by the Solution to 1497 [p. 584], this area is a numerical multiple of $\frac{J}{P}$, where

$$J\equiv(L_1M_2N_3)\equiv(l_1m_2n_3)^2,$$
$$P\equiv(aL_1+bM_1+cN_1)\,(aL_2+\ldots)\,(aL_3+\ldots)$$

in the Trilinear system.

In the case of the fundamental triangle we find that $\frac{J}{P}=\frac{1}{abc}$. Hence the *ratio* has the value given in the foregoing solutions* [and the area $=abc\,.\,\Delta\,.\,J\div P$].

* [Two other modes of solution were printed.]

The general formula for all systems (see my paper on "Analytical Metrics" [p. 90 supra]), is

$$\frac{\{J(ABC)\}^2}{J(BC\infty)\,.\,J(CA\infty)\,.\,J(AB\infty)},$$

where $J(ABC)=0$ is the condition that the lines $A, B, C, =0$ may meet in a point, and $J(BC\infty)=0$ is the condition that B and C may be parallel. The *ratio* to the area of the fundamental triangle may easily be found in any particular case by the method used above.

In the same way it may be shewn that the volume of a tetrahedron is

$$\frac{\{J(ABCD)\}^3}{J(BCD\infty)\,.\,J(CDA\infty)\,.\,J(DAB\infty)\,.\,J(ABC\infty)},$$

where $J(ABCD)=0$ is the condition that the planes $A, B, C, D, =0$ may meet in a point, and $J(BCD\infty)=0$ is the condition that B, C, D may be parallel to the same line.

In quadriplanar coordinates, for instance, if $\alpha, \beta, \gamma, \delta$, denote the areas of the faces of the fundamental tetrahedron, the equation to the plane at infinity is

$$\alpha x+\beta y+\gamma z+\delta w=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and the above expression for the volume, if calculated by means of (1), must be multiplied by the product $\alpha\beta\gamma\delta$ to give the ratio of the volume of the given tetrahedron to that of the fundamental one.

1732. [Prove that the characteristics of a system of conics, satisfying four conditions, remain unaltered when, in place of passing through a given point, each conic is required to divide a given finite segment harmonically. Proposed by T. A. Hirst, F.R.S. Reprint, Vol. IV. pp. 56, 7, *E. T.* September, 1865.]

In a system of conics satisfying four conditions (Z, Z', Z'', Z''') let μ be the number of conics that pass through an arbitrary point, and ν the number that touch an arbitrary line. Suppose that the polars of a point P, in respect of all the conics of the system, envelope a curve of class x. Then from the point P, x tangents can be drawn to the curve, that is to say, there are x polars of P which pass through P. But a point which lies on its polar in respect of a given conic is a point on the conic. Therefore x conics pass through P. But μ conics (by hypothesis) pass through P; so that $x=\mu$. Thus we get Chasles's Prop. xii., —*the polars of an arbitrary point, in respect of a system of conics* (μ, ν) *envelope a curve of class* μ. It follows that there are μ polars of P which pass through another arbitrary point Q; that is to say, there are μ conics of the system which divide harmonically a given segment PQ. This is Chasles's Prop. xxviii.

Suppose now that the condition Z''' is that the conics shall pass through a given point. Call S the condition that they shall divide harmonically the segment PQ. Then (by the above) the number of conics satisfying the conditions (Z, Z', Z'', S), and passing through the given point, is μ; that is to say, the *first characteristic* (μ') of the system (Z, Z', Z'', S) is equal to the *first characteristic* (μ) of the system (Z, Z', Z'', Z''') where Z''' is the condition of passing through a given point. In the next place let Z''' be the condition of touching a given line. Then the number of conics which satisfy the conditions

(Z, Z', Z'', Z''', S) is the same as the number which satisfy the conditions $(Z, Z', Z'', Z'''$, point); that is to say, the *second characteristic* (ν') of the system (Z, Z', Z'', S) is equal to the *second characteristic* (ν) of the system $(Z, Z', Z''$, point). Thus neither of the characteristics is altered when we substitute for the condition of passing through a given point, the condition S of dividing harmonically a given segment PQ.

By similar reasoning it may be shewn that neither characteristic is altered when we substitute for the condition of touching a given line, the condition of subtending harmonically a given angle.

1750. Given four straight lines whose equations are connected by the syzygy $x+y+z+w=0$; shew that the straight lines

$$lx+my+nz+sw=0, \quad \lambda x+\mu y+\nu z+\sigma w=0,$$

will be conjugates in respect of any conic touching (x, y, z, w), if

$$(l\mu+\lambda m)+(n\sigma+\nu s)=(l\nu+\lambda n)+(s\mu+\sigma m)=(l\sigma+\lambda s)+(m\nu+\mu n).$$

Shew also that if a quadrangle be formed from the quadrilateral by taking the pole of each line in respect of the triangle formed by the other three; then the relation between the two figures will be reciprocal; and if two straight lines be conjugates in respect of any conic inscribed in the quadrilateral, their poles in respect of the common connector-triangle will be conjugates in respect of any conic circumscribing the quadrangle.

[*E. T.* July, 1865. Solution Reprint, Vol. IV. p. 63.]

1775. If a straight line meet the faces of the tetrahedron $ABCD$ in the points a, b, c, d, respectively; the spheres whose diameters are Aa, Bb, Cc, Dd have a common radical axis.

[*E. T.* August, 1865. Solution Reprint, Vol. IV. pp. 66—8.]

1675. If a triangle abc be the reciprocal of ABC in respect of a parabola whose parameter is $4m$; and if n_1, n_2, n_3, be the normals at the vertices of diameters through ABC; then

$$\frac{(\text{area of } abc)^2}{bc\,.\,ca\,.\,ab}=\frac{2m^2}{n_1 n_2 n_3}(\text{area of } ABC).$$

[*E. T.* March, 1865. Solution Reprint, Vol. IV. pp. 90, 1.]

1823. The conicoids which pass through six fixed points in space, intersect any plane in a series of conics having a common self-conjugate quadrilateral. *Any four* conics have a common self-conjugate quadrilateral. (DEF. A quadrilateral is *self-conjugate* in respect of a conic which divides its diagonals harmonically.)

[*E. T.* October, 1865. Solution Reprint, Vol. IV. p. 110.]

1795. (1) Let P be the point in a homogeneous triangular lamina ABC at which the sides subtend equal angles. Shew that if the lamina be placed in a smooth prolate spheroid whose long axis is vertical, it will rest in equilibrium when the point P coincides with the lower focus of the spheroid.

(2) If the lamina be not homogeneous, and its centre of gravity be given, construct for the corresponding position of the point P.

[*E. T.* September, 1865. Solution Reprint, Vol. IV. p. 116.]

1638. Find the condition that the general equation of the third order may represent a cubic whose asymptotes form an equilateral triangle; and shew that this is always the case when the curve passes through three points and their three pairs of antifoci. [*E. T.* January, 1865. Solution Reprint, Vol. v. pp. 44, 5, February, 1866 ?]

Three lines forming an equilateral triangle meet the line at infinity in a point-cubic whose Hessian is the circular points. Now let

$$(a, b, c, d)(x, y)^3 \quad \text{.................} \quad \text{(i)}$$

be the terms of the highest order in the general equation of the third degree in Cartesian coordinates; then the three lines represented by (1) are parallel to the asymptotes. Now the Hessian of (1) is

$$(ac-b^2)x^2+(ad-bc)xy+(bd-c^2)y^2;$$

and in order that this may be identical with x^2+y^2, we must have

$$(ad=bc,\ ac-b^2=bd-c^2),$$

which are the conditions required.

To prove the proposition, let A, B, C be the three points, and P, Q the circular points at infinity. Let the equation to the three lines PA, PB, PC be $U=0$, and to the three lines QA, QB, QC, $V=0$. Then the nine intersections of the cubics U, V are the three points, and their three pairs of antifoci. Any other cubic through those intersections may be represented by $U=kV$. Let U', V' be the terms of highest order in U, V; then $U'-kV'$ will be the terms of highest order in $U-kV$. But U', V' must be perfect cubes, representing the circular points; say x^3, y^3. Then $U'-kV'$ is x^3-ky^3. But the Hessian of x^3-ky^3 is $-kxy$. That is, every cubic represented by $U=kV$ meets the line at infinity in a point-cubic whose Hessian is the circular points. Or, which is the same thing, the asymptotes of every such cubic form an equilateral triangle.

There is no difficulty in finding the conditions when the equation is given in a homogeneous form. We substitute for z, from the equation of the line at infinity, in the cubic and in any circle; let the former substitution give (1), and the latter, $Ax^2+2Bxy+Cy^2=0$; then the conditions are

$$\frac{ac-b^2}{A}=\frac{ad-bc}{2B}=\frac{bd-c^2}{C}.$$

1888. [(1) Amongst the conics which have three-pointic contact with a cubic at a given point, there are, in general, three which have a three-pointic contact elsewhere, and a fourth passes through the points of contact of these three with the cubic. The number of such conics is reduced to one, when the cubic has a cusp.

(2) Amongst the conics which have four-pointic contact with a cubic at a given point, there are three which touch the cubic elsewhere. There is but one

such conic when the cubic has a node, and none when it has a cusp. Proposed by E. de Jonquières. Reprint, Vol. v. p. 56, *E. T.* March, 1866.]

1. Let A be the given point on the cubic, and let F be any point of inflexion, or flex. Join AF, and let AF meet the curve again in B. Then a conic may be drawn having three-pointic contact with the cubic at the points A and B. For, consider these three cubic curves: (a) the cubic itself; (b) the line ABF taken three times over; (c) a conic having three-pointic contact at A and touching the cubic at B, together with the tangent at the flex F. The cubic (a) passes through eight out of the nine points of intersection of the cubics (a) and (b); consequently, by the theorem known as the involution of cubics, it passes through the ninth point. That is to say, a conic having three-pointic contact at A, and touching the cubic at B, will necessarily have three-pointic contact at B.

By joining the point A, therefore, to the *nine* flexes F, we shall obtain *nine* points B, and therefore nine conics fulfilling the required conditions; but only three of these points B will be real when the point A is real.

It remains to shew that a conic having three-pointic contact at A passes through the three real points B. Let F_1, F_2, F_3 be the three real flexes, which are known to be in one straight line; and let B_1, B_2, B_3 be the corresponding points B. Draw a conic U having three-pointic contact at A and passing through B_1, B_2. Then consider these three cubic curves: (a) the cubic itself; (b) the straight lines AB_1F_1, AB_2F_2, AB_3F_3; (c) the conic U and the line $F_1F_2F_3$. The cubic (c) passes through eight out of the nine intersections of the cubics (a) and (b); consequently it passes also through the ninth. That is to say, the conic U passes through the point B_3.

A cusped cubic has only one flex; in this case, therefore, the number of conics is reduced to one.

2. Let A be the given point. By COTTERILL's Theorem (which again is a particular case of the involution of cubics), if a conic have four-pointic contact with the cubic at A, its remaining chord of intersection with the cubic will pass through a fixed point M on the curve. Now the tangent at A, taken twice over, may be regarded as a conic having four-pointic contact at A; whence it appears that the point M is the second tangential of A. The number of conics of the system which touch the cubic at some other point is therefore the number of tangents that can be drawn from M to the curve; that is, four in general, two when the cubic has a node, and one when it has a cusp. But in this number there is always included that conic which is made up of the tangent at A taken twice over; and this is not a proper solution.

1996. If four circles $A=0$, $B=0$, $C=0$, $D=0$ are mutually orthotomic, the square of the radius of a circle $lA+mB+nC+sD=0$ is

$$(l^2r_1^2+m^2r_2^2+n^2r_3^2+s^2r_4^2)\div(l+m+n+s)^2,$$

where r_1, r_2, r_3, r_4 are the radii of A, B, C, D.

[*E. T.* July, 1866. Solutions Reprint, Vol. VI. pp. 74, 5.]

1878. A line of length a is broken up into n pieces at random; prove that (1) the chance that they cannot be made into a polygon of n sides is $n2^{1-n}$; and

(2) the chance, that the sum of the squares described on them does not exceed $\frac{a^2}{n-1}$, is

$$\left(\frac{\pi}{n^2-n}\right)^{\frac{1}{2}(n-1)} \cdot \frac{\Gamma(n)}{\Gamma\{\frac{1}{2}(n+1)\}} \cdot \frac{1}{n^{\frac{1}{2}}}.$$

[*E. T.* January, 1866. Reprint, Vol. VI. pp. 83—87, *E. T.* November, 1866. A solution by Prof. Wolstenholme is given Vol. XI. pp. 17, 18.]

1. Let us define as follows. A point is taken *at random* on a (finite or infinite) straight line, when the chance that the point lies on a finite portion of the line varies as the length of that portion. And, a line is broken up at random when the points of division are taken at random.

Now, the n pieces will always be capable of forming a polygon except when one of them is greater than the sum of all the rest; that is greater than half the line. The *first* part of the question may therefore be stated thus: $n-1$ points are taken at random on a finite line; to find the chance that some one of the intervals shall be greater than half the line.

2. *First* solution. Bisect the line AB at C. Then the chance that one of the points of division shall lie within BC is $\frac{1}{2}$; therefore the chance that all the $n-1$ points shall lie within BC is 2^{1-n}. But this is the chance that the *first piece* (reckoning from A) shall be greater than AC. Next, I say that the chance of the r^{th} piece being greater than half the line is equal to the chance of the $(r+1)^{th}$ piece being greater. For let PQ be the portion which is made up of the r^{th} and $(r+1)^{th}$ pieces. And take $PR=QS=AC$*. Then if the point of division between the r^{th} and $(r+1)^{th}$ pieces lies within RQ, the r^{th} piece is greater than AC; and if it lies within PS, the $(r+1)^{th}$ piece is greater than AC. But $RQ=PS$; therefore by definition the chances are equal. Consequently, the chance that any one of the n pieces shall be greater than AC is equal to the chance that any other of the n pieces shall be greater than AC. And all these n events are mutually exclusive; while we have proved that the chance of the first of them is 2^{1-n}. Therefore the chance that some one piece is greater than AC is $n2^{1-n}$.

3. *Second* solution. I am convinced that there is a fallacy in the above, and have therefore tried to get a rigorous proof in this way. Take P a point in AC, and let $AP=x$. Consider a small element dx at P. I want to find the chance that the r^{th} piece, reckoning from A, may begin at P (within the element dx) and be greater than AC. This requires, first, that one of the $n-1$ points of division shall be within dx; the chance of this is $(n-1)\frac{dx}{a}$; next, $r-2$ of the remaining points must be within AP, and the chance of this is

$$\frac{|\underline{n-2}}{|\underline{n-r}\;|\underline{r-2}}\left(\frac{x}{a}\right)^{r-2};$$

lastly, the $n-r+1$ points left must be within RB; whose chance is

$$\left(\frac{1}{2}-\frac{x}{a}\right)^{n-r+1}.$$

* [To draw the figure, take the points in the order A, P, S, C, R, Q, B.]

Therefore the chance required is

$$\frac{|\underline{n-1}}{|\underline{n-r}\;|\underline{r-2}}\left(\frac{x}{a}\right)^{r-2}\left(\frac{1}{2}-\frac{x}{a}\right)^{n-r+1}\frac{dx}{a}.$$

Now, if we integrate this with respect to x from 0 to $\frac{1}{2}a$, we shall get the entire chance that the r^{th} piece may be greater than AC. The integral is easily found to be 2^{1-n}. And as there are thus n equal chances, whose events are all mutually exclusive, the chance that some one of these events will happen is $n2^{1-n}$.

4. *Third* solution. To make this clear, I will state first the previously-known analogous solutions in the cases where $n=3$ and $n=4$. When the line is divided into three pieces, call them x, y, z, and take their lengths for the co-ordinates of a point P in geometry of three dimensions. Then, since

$$x+y+z=a \quad\text{.......................................}\quad (1),$$

and x, y, z are all positive, the point P must be somewhere on the surface of the equilateral triangle determined on the plane (1) by the coordinate planes. Now, consider those points on the triangle for which $x>\frac{1}{2}a$. These are cut off by the plane $x=\frac{1}{2}a$; and it is easy to see that this plane cuts off from one corner of the triangle a similar triangle of *half the linear dimensions*, and therefore of the fourth the area. Now, there are three corners cut off; their joint area is therefore three-fourths of the area of the triangle; and the chance required is accordingly $\frac{3}{4}$.

When the line is divided into *four* pieces, take the *first three* pieces as the co-ordinates of a point in space. Then we have $x+y+z<a$, and x, y, z all positive; so the point must lie within the content of the tetrahedron bounded by the plane $x+y+z=a$ and the coordinate planes. Now, if $x+y+z<\frac{1}{2}a$, the *fourth piece* must be *greater* than $\frac{1}{2}a$. The points for which this is the case are cut off by the plane $x+y+z=\frac{1}{2}a$; and it is easily seen as before that this plane cuts off from one corner of the tetrahedron a similar tetrahedron of half the linear dimensions, and therefore of one-eighth the volume. So also the plane $x=\frac{1}{2}a$ cuts off from another corner a similar tetrahedron of half the linear dimensions. Since therefore there are four corners cut off, their joint volume is ($\frac{4}{8}$ or) one half of the volume of the tetrahedron; and the chance required is accordingly $\frac{1}{2}$.

5. Now, consider the analogous cases in geometry of n dimensions. Corresponding to a closed area, and a closed volume, we have something which I shall call a *confine*. Corresponding to a triangle, and to a tetrahedron, there is a confine with $n+1$ corners or vertices, which I shall call a *prime* confine, as being the simplest form of confine. A prime confine has also $n+1$ faces, each of which is, not a plane, but a prime confine of $n-1$ dimensions. Any two vertices may be joined by a straight line, which is an *edge* of the confine. Through each vertex pass n edges. A prime confine may be *regular*, which it is when any three vertices form an equilateral triangle; or *rectangular*, which it is when the edges through some one vertex are all equal and at right angles to one another.

To solve the question for general values of n, we may adopt as a type either of the geometrical solutions given for the cases $n=3$ and $n=4$. First, take the lengths of the n pieces for the coordinates of a point in geometry of n dimensions. Then, since their sum is a, and they are all positive, the point must lie within a certain regular prime confine of $n-1$ dimensions. The supposition that a certain piece is greater than $\frac{1}{2}a$ cuts off from one corner of the confine a similar confine of half the linear dimensions, and therefore of 2^{1-n} times the content. And as there are n corners, their joint content is $n \cdot 2^{1-n}$ times the content of the confine; the chance required is consequently $n2^{1-n}$. Or, take the lengths of the *first* $n-1$ *pieces* as the coordinates of a point in geometry of $n-1$ dimensions; the point will then lie within a certain *rectangular* confine of $n-1$ dimensions; and the investigation proceeds as before, the n corners being cut off in the same manner.

6. It will be seen that this *third* solution involves in a geometrical form the assumption of which some sort of proof was given in the *first* solution. Let us make this extension of our fundamental definition:—A point is taken *at random* in a (finite or infinite) space of n dimensions, when the chance that the point lies in a finite portion of this space varies as the contents of that portion. The assumption is that when the lengths of the pieces into which a line is broken up are taken as coordinates of a point, then if the line is broken up at random the point is taken at random, and *vice versâ*. The proof of this assumption may be shewn to involve a geometrical proposition equivalent to the integration by parts of the differential in Art. (3).

Making this assumption, we may solve the second part of the question by the method of the *third* solution of the first part. I will first state the previously known analogous solution of the case where $n=3$. The question is in this case,—If a line of length a be broken into three pieces at random, find the chance that the sum of the squares of these pieces shall be less than $\frac{1}{2}a^2$. Take the lengths of the three pieces for coordinates x, y, z of a point P in geometry of three dimensions; then, as before, the point must lie somewhere in the area of the equilateral triangle determined on the plane $x+y+z=a$ by the coordinate planes. But if also the sum of the squares of the pieces is less than a certain quantity m^2, then the point P must lie within a certain circle determined on the plane $x+y+z=a$ by the sphere $x^2+y^2+z^2=m^2$. Now, in the case where $m^2=\frac{1}{2}a^2$ this circle is the circle inscribed in the equilateral triangle; so that the question reduces itself to this one:—

To find, in terms of the area of an equilateral triangle, the area of its inscribed circle.

Now let us go a little further, and consider the case in which $n=4$. Here we shall have to take a point P in geometry of four dimensions; the point must lie somewhere in the regular tetrahedron determined on the hyper-plane

$$x+y+z+w=a$$

by the coordinate hyper-planes. If also the sum of the squares of the pieces is less than a certain quantity m^2, then the point P must lie within a certain sphere determined on the hyper-plane $x+y+z+w=a$ by the quasi-sphere

$$x^2+y^2+z^2+w^2=m^2.$$

In the particular case where m is the perpendicular from the vertex on the base of a rectangular tetrahedron, each of whose equal edges is of length a, or

$$m^2=\tfrac{1}{3}a^2,$$

this sphere is the sphere inscribed in the regular tetrahedron. The question is therefore reduced to this one:—

To find, in terms of the volume of a regular tetrahedron, the volume of its inscribed sphere.

Now, a similar reduction holds in the general case; viz., the question can always be reduced to this one:—

To find, in terms of the contents of a regular prime confine of $n-1$ dimensions, the contents of its inscribed quasi-sphere.

This question I proceed to solve.

7. Let $n-1=p$. *The perpendicular from any vertex on the opposite face of a regular prime confine in* p *dimensions* $=\left(\frac{p+1}{2p}\right)^{\frac{1}{2}}$. (edge).

For, let O be the vertex in question, OA, OB, ... the p edges through O. Draw through each vertex A a space of $p-1$ dimensions parallel to the face opposite to A. The p spaces thus drawn will intersect in a point P, such that OP is the diagonal of a confine analogous to a parallelogram and to a parallelepiped. Then OP is p times the perpendicular from O on the opposite face of the regular confine; for the perpendicular is the projection of *one* edge at a certain angle, while OP is the projection at the same angle of a broken line consisting of p edges.

We have also

$$OP^2=OA^2+OB^2+OC^2+\ldots+2OA \cdot OB\cos AOB+\ldots$$
$$=\Sigma OA^2+\Sigma OA \cdot OB \text{ (since } \cos AOB=\tfrac{1}{2}, \text{ \&c.)},$$
$$=\{p+\tfrac{1}{2}p(p-1)\} \cdot OA^2=\tfrac{1}{2}p(p+1) \cdot OA^2,$$
$$\therefore \text{(perpendicular)}^2=\frac{OP^2}{p^2}=\frac{p+1}{2p} \cdot \text{(edge)}^2.$$

{If the confine were *rectangular*, or all the angles at O right angles, we should have $\cos AOB=0$, &c., and so

$$\text{(perpendicular)}^2=\frac{1}{p}\text{(edge)}^2=\frac{a^2}{n-1};$$

which proves that the question *does* always reduce itself to the one now under consideration}.

The content *of a regular prime confine in* p *dimensions whose edge is* a, *is*

$$=\frac{a^p}{|\underline{p}}\left(\frac{p+1}{2^p}\right)^{\frac{1}{2}}.$$

Suppose this formula true for $p-1$ dimensions; that is, let

$$V_{p-1}=\frac{a^{p-1}}{|\underline{p-1}}\left(\frac{p}{2^{p-1}}\right)^{\frac{1}{2}}.$$

Now, content of confine

$$=\frac{1}{p}\times\text{perpendicular}\times\text{content of face},$$

or

$$V_p=\frac{a}{p}\left(\frac{p+1}{2^p}\right)^{\frac{1}{2}}\cdot V_{p-1}=\frac{a^p}{\underline{|p}}\left(\frac{p+1}{2^p}\right)^{\frac{1}{2}}.$$

Hence the formula, if true for one value of p, is true for the next; now it can be immediately verified in the case of $p=1$; therefore it is generally true.

The radius of the inscribed quasi-sphere

$$=\frac{a}{\{2p\,(p+1)\}^{\frac{1}{2}}}.$$

We can divide the regular confine into $p+1$ equal confines, each having the centre of the inscribed quasi-sphere for vertex; and the content of one of these

$$=\frac{\rho}{p}\times\text{content of face};$$

but the sum of them all is equal to the content of the whole confine. Hence $(p+1)\,\rho=$ perpendicular of confine

$$=a\left(\frac{p+1}{2p}\right)^{\frac{1}{2}},\ \text{or},\ \rho=\frac{a}{\{2p\,(p+1)\}^{\frac{1}{2}}}.$$

The content of the quasi-sphere

$$=\rho^p\cdot\frac{\{\Gamma(\frac{1}{2})\}}{\Gamma(\frac{1}{2}p+1)}.$$

For it is the value of

$$\iiint\ldots dx\,dy\,dz\ldots$$

the integral being so taken as to give to the variables all values consistent with the condition that $x^2+y^2+z^2+\ldots$ is not greater than ρ^2. (See Todhunter's *Integral Calculus*, Art. 271.)

Let C_p denote this content; then

$$C_p=\rho^p\,\frac{\{\Gamma(\frac{1}{2})\}^p}{\Gamma(\frac{1}{2}p+1)}=\frac{a^p}{(2p^2+2p)^{\frac{1}{2}p}}\cdot\frac{\{\Gamma(\frac{1}{2})\}^p}{\Gamma(\frac{1}{2}p+1)},$$

therefore

$$\frac{C_p}{V_p}=\left(\frac{\pi}{p^2+p}\right)^{\frac{1}{2}p}\cdot\frac{\Gamma(p-1)}{\Gamma(\frac{1}{2}p+1)}\cdot\frac{1}{(p+1)^{\frac{1}{2}}}.$$

Restore $n-1$ for p, and we get the answer to the question, namely,

$$\left(\frac{\pi}{n^2-n}\right)^{\frac{1}{2}(n-1)}\cdot\frac{\Gamma(n)}{\Gamma\{\frac{1}{2}(n+1)\}}\cdot\frac{1}{n^{\frac{1}{2}}}.$$

8. The following are applications of the same method.

If a line be broken up at random into n pieces, the chance of an assigned two of them (the p^{th} and q^{th} from one end) being together greater than half the line, is $n2^{1-n}$.

If n pieces be cut off at random, one from each of n equal lines, the chance that the pieces cannot be made into a polygon is $\frac{1}{\underline{|n-1}}$.

2253. If four circles have a common radical centre, it is possible to find four planes which intersect, two and two, at angles equal to those at which the circles intersect, but not otherwise.

[*E. T.* October, 1866. Reprint, Vol. VII. p. 22.]

2220. A, B, C, D are four points on a circle, and through every pair, as AB, another circle (AB) is drawn; then the pair of circles (AB), (CD) intersects the pair (AC), (DB) in four new points on a circle U; the pair (AC), (DB) meets (AD), (BC) on a circle V; and the pair (AD), (BC) meets (AB), (CD) on a circle W; also the three circles U, V, W have a common radical axis. (This may be extended to spheres; and there are also analogous properties of rectangular hyperbolas).

[*E. T.* August, 1866. Reprint, Vol. VII. p. 37.]

[2135.* Reprint, Vol. VII. p. 45, is merely a repetition of 1378 (p. 566), with a different solution. It is noteworthy that there is no question so numbered in the *E. T.*, for the August (1866) No. gives 2110 in succession to 2009 and 2220 next to 2119.]

1962. Required the *characteristics* of the system of conics having five-pointic contact with a curve of order m and class n.

[*E. T.* May, 1866. Reprint, Vol. VII. p. 47.]

2343. A is any point within or without a conic, B any point on its polar, CD a fixed straight line. Tangents BC, BD are drawn cutting CD in C, D. AD, AC meet BC, BD in E, F; shew that EF is a fixed straight line and meets CD on the polar of A.

[*E. T.* February, 1867. Reprint, Vol. VIII. pp. 64, 5.]

2522. Prove (1) that the perpendiculars of a circular triangle have a common radical axis; and (2) that if the perpendiculars from the pairs of vertices of one circular triangle on the sides of another meet in a point, then *vice versâ*. (*Def.* A, B, C being circles, a circle coaxal with A, B, and orthogonal to C, is called the perpendicular from AB on C.)

[*E. T.* November, 1867. Reprint, Vol. IX. p. 42.]

2383. A and B are fixed points with regard to a conic, ACD a variable straight line passing through A and cutting the curve in C, D. The polar of A meets BC, BD in E, F; shew that DE and CF meet in a fixed point G, and that ABG is a straight line.

[*E. T.* April, 1867. Reprint, Vol. X. p. 81.]

2732. An epi- or hypo-cycloid is pushed through a very short fixed tube, so as to remain in one plane, shew that the locus of its centre is an ellipse.

[*E. T.* September, 1868. Reprint, Vol. X. p. 96.]

2748. If a circular cubic with a double point O be cut by a circle in four points A, B, C, D; and if OA, OB, OC, OD cut the circle again in E, F, G, H; shew that any pair of straight lines joining these four points will be equally inclined to the bisectors of the angles between the tangents at O.

[*E. T.* October, 1868. Reprint, Vol. x. pp. 105, 6.]

2301. A circle is drawn so that its radical axis with respect to the focus S of a parabola is a tangent to the parabola; if a tangent to the circle cut the parabola in A, B, and if SC, bisecting the angle ASB, cut AB in C, the locus of C is a straight line.

[*E. T.* December, 1866. Reprint, Vol. xi. p. 31.]

2776. Through A, the double point of a circular cubic, draw AB perpendicular to the asymptote; if chords be drawn to the curve subtending a right angle at the double point, shew that there is a fixed point in AB at which also they subtend a right angle.

[*E. T.* November, 1868. Reprint, Vol. xi. p. 64.]

2108. Required Analogues in Solid Geometry to the following propositions in Plane Geometry:—

(*a*) The perpendiculars of a triangle meet in a point.

(*b*) The middle points of the diagonals of the quadrilateral are in one straight line.

(*c*) The circles whose diameters are the diagonals of a quadrilateral have a common radical axis.

(*d*) Every rectangular hyperbola circumscribing a triangle passes through the intersection of perpendiculars.

(*e*) Every rectangular hyperbola to which a triangle is self-conjugate passes through the centres of the four touching circles.

(*f*) $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

(*g*) The sum of the angles of a triangle = two right angles.

(*h*) In any triangle

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

[*E. T.* July, 1864, where it is numbered 1526, cf. 2135, p. 607. Reprint, Vol. xi. pp. 102, 103, May, 1869.]

I have an analogue for each of the four (*d*), (*e*), (*f*), (*g*), and more than one of each of the others obtained by extensions of Mr Greer's methods*.

E.g.:—

(*c*) A straight line cuts the faces of a tetrahedron $ABCD$ in a, b, c, d; the spheres whose diameters are Aa, Bb, Cc, Dd, have a common radical axis. Hence the middle points of these four lines are in one plane.

* [? Reprint, Vol. ii. p. 80; cf. p. 583, supra.]

Let a conicoid whose asymptotic cone has three generating lines at right angles be called a rectangular conicoid.

(*d*), (*e*) Every rectangular conicoid circumscribing a tetrahedron whose perpendiculars meet in a point, passes through the point. And every rectangular conicoid to which a tetrahedron is self-conjugate, passes through the centres of the eight touching spheres.

(*f*) $$\sin(ABC) = \sin(ABD + BCD + CAD)$$

$$= \sin(BCD) \,.\, \cos\widehat{AD} + \sin(CAD) \,.\, \cos\widehat{BD} + \sin(ABD) \,.\, \cos\widehat{CD},$$

where A, B, C, D are four lines in space, and

$$\sin^2 ABC = \begin{vmatrix} 1, & \cos\widehat{AB}, & \cos\widehat{AC} \\ \cos\widehat{AB}, & 1, & \cos\widehat{BC} \\ \cos\widehat{AC}, & \cos\widehat{BC}, & 1 \end{vmatrix}.$$

(*g*) In the triangle case this should be written

$$(BC) + (CA) + (AB) = 0.$$

The analogue is then obviously

$$(BCD) - (CDA) + (DAB) - (ABC) = 0,$$

A, B, C, D being any four planes.

(*h*) In any tetrahedron,

$$\frac{AC \,.\, DB}{\sin\widehat{AC} \,.\, \sin\widehat{DB}} = \frac{abcd}{V^2} = \frac{a}{\cos A} = \frac{V^6}{(abcd)^2} \,.\, \cos A \cos B \cos C \cos D,$$

where a, b, c, d are the faces, and

$$\cos^2 A = \begin{vmatrix} 1, & \cos\widehat{BC}, & \cos\widehat{BD} \\ \cos\widehat{BC}, & 1, & \cos\widehat{CD} \\ \cos\widehat{BD}, & \cos\widehat{CD}, & 1 \end{vmatrix},$$

($\widehat{AB}$, &c. denoting angles between planes).

2793. C is the single focus of a semicubical parabola, and from any point O three tangents are drawn to the curve; if CD, CE, CF be perpendicular to them, shew that DE and CF are equally inclined to the direction of the infinite branches.

[*E. T.* December, 1868. Reprint, Vol. xii. p. 22.]

2932. [Given the inscribed and circumscribed circles of a triangle, the envelope of the polar circle is a bicircular quartic. Proposed by the Rev. J. Wolstenholme. Reprint, Vol. xii. p. 52.]

Let B, C, X be the inscribed, circumscribed, and polar circles respectively. The circle X has to be such that a triangle self-conjugate with regard to it can be circumscribed to B and inscribed to C; that is, it is subject to a condition of the first and a condition of the second degree in its coefficients. Two such circles can therefore be drawn through an arbitrary point.

Now, *any series of circles, such that two of them can be drawn through an arbitrary point, is one system of bitangent circles of a bicircular quartic.* For the equation of a circle of the series must contain a variable parameter in the second order; that is, it must be of the form

$$X+2\theta Y+\theta^2 Z=0^*,$$

where $X=0$, $Y=0$, $Z=0$ are circles. But the envelope of this is $XZ=Y^2$, a bicircular quartic.

This important remark is made by Cremona (*Teoria Geometrica*, ii. 21) in the case of a series of curves of any order and of index 2; that is, such that two of them can be drawn through an arbitrary point. The envelope is always what Prof. Cayley (*Edinb. Phil. Trans.*, 1868) calls a *trizomal* curve; viz. X, Y, Z being any three curves of the series, its equation may be written

$$\sqrt{\alpha X}+\sqrt{\beta Y}+\sqrt{\gamma Z}=0.$$

2923. [In a bicircular quartic, the points of contact of the four single tangents drawn from the centre of a circle on which four foci lie, are on the circle, and the corresponding points of contact of double tangents also lie on a circle. Proposed by S. Roberts, M.A. Reprint, Vol. XII. p. 57. *E. T.* September, 1869.]

A bicircular quartic is its own inverse with regard to any focal circle (Moutard, *Nouvelles Annales*, 1866). The bitangent circles divide themselves into four systems, all the circles of any one system being cut orthogonally by the corresponding focal circle. Through any point of the plane can be drawn two bitangent circles of each system. The two bitangent circles, then, that can be drawn through the centre of the focal circle of their system, are in fact straight lines touching the curve in two pairs of inverse points, which consequently lie on a circle.

The corresponding theory in anallagmatic surfaces is that the centre of each one of the five principal spheres is vertex of a quadric cone doubly tangent to the surface; the curve of contact being the intersection of this cone with a sphere. These five cones noticed by Moutard are independently arrived at by Kummer (*Berlin. Monatsber.*) in the case of the general quartic surface with a nodal conic.

2924. [On a focal chord PSQ of a parabola are taken p, q, on opposite sides of S, such that $Sp \,.\, Sq=SP \,.\, SQ$, and any parabola is described through p, q, and having its axis parallel to that of the former: prove that their chord of intersection will pass through S. Proposed by the Rev. J. Wolstenholme. Reprint, Vol. XII. pp. 62, 3. *E. T.* September, 1869.]

I consider the following more general question:—

Through a point a let a line B be drawn meeting a conic in l, m; then the quantity

$$al \,.\, am \sin BP \,.\, \sin BQ \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

* [With reference to this solution Mr Wolstenholme remarks (note p. 53), that "it is not necessary that X, Y, Z should be all circles; it is sufficient that one be a circle and the others straight lines. Thus the envelope might, so far as depends on this reasoning, be a circular cubic."]

(where P and Q are the asymptotes) is independent of the position of the line B, and may be called the *distance* of the point a from the conic. What now is the locus of a point equidistant from two given conics?

Let $C_2=0$, $D_2=0$ be the equations of the conics, and let a^2C_2 denote the result of substituting the coordinates of the point a for the variables in C_2; also let i, j be the circular points at infinity. Then I find that the distance (above defined) of the point a from C_2 is

$$\frac{a^2C_2}{(aij)^2 \,.\, \sqrt{(i^2C_2 \,.\, j^2C_2)}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where, of course, (aij) means the determinant formed with the coordinates of the points a, i, j.

This being so, the equation of the required locus is

$$\frac{C_2}{\sqrt{(i^2C_2 \,.\, j^2C_2)}}=\frac{D_2}{\sqrt{(i^2D_2 \,.\, j^2D_2)}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

shewing that the locus is a conic passing through the intersections of the two given ones.

Now if we are using Cartesian coordinates, and if the two conics are similar and similarly situated, it is easy to see that the terms of the second order have entirely disappeared from the equation (3); which indicates that the line at infinity is part of the locus. The remainder of it is then their finite chord of intersection; which is a true radical axis, in the sense that if any line whatever is met by the radical axis in a, by C_2 in l, m, and by D_2 in l', m', we must have always

$$al \,.\, am=al' \,.\, am' \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

To apply this to the question we have only to observe that parabolæ with parallel axes are homothetic, or similar and similarly situated curves; and that the equation

$$Sp \,.\, Sq=SP \,.\, SQ$$

indicates that the focus is situated on their radical axis.

The theorem of the radical axis of two homothetic conics may of course be proved for ellipses by orthogonal projection from the circle, and then extended by the doctrine of continuity to the rest.

2446. PQ is a chord of a conic, equally inclined to the axis with the tangent at P. Any circle through PQ cuts the conic in RS. Shew that the harmonic conjugate of RS relative to P lies on the straight line joining Q to the other extremity of the diameter through P. Hence shew by inversion that, if chords be drawn to a circular cubic through the point where the asymptote cuts the curve, the locus of their middle points is a circle through the double point.

[*E. T.* July, 1867. Reprint, Vol. XII. p. 91.]

2960. The envelope of a series of surfaces of order n, such that two of them can be drawn through an arbitrary point, is a surface of order $2n$, whose equation may be written in the form

$$\sqrt{(\alpha X)}+\sqrt{(\beta Y)}+\sqrt{(\gamma Z)}=0,$$

where $X=0$, $Y=0$, $Z=0$ are equations of any three surfaces of the series.

The envelope of a net of surfaces of order n, such that two of them can be drawn through *two* arbitrary points, is a surface of order $2n$, whose equation referred to any four surfaces of the net is of the same form as the equation of a quadric referred to four tangent planes.

[*E. T.* September, 1869. Reprint, Vol. XII. p. 96.]

2942. Let p, q be the foci, and P, Q the asymptotes of a conic; θ the angle it subtends at a point a, and $\{A\}$ the chord it cuts off from a line A. Then

1. If a line B is drawn through the point a meeting the conic in l, m,

$$al\,.\,am\,.\,\sin BP \sin BQ = \frac{ap^2\,.\,aq^2\,.\,\sin^2\theta}{pq^2}.$$

2. If from a point b on the line A tangents L, M are drawn to the conic,

$$\sin AL \sin AM\,.\,bp\,.\,bq = \frac{\sin^2 AP \sin^2 AQ\,.\,\{A\}^2}{\sin^2 PQ}.$$

(Here al means the distance between the points a, l, and BP means the angle between the lines B, P.)

3. Find analogous propositions for a curve of any order on a plane or on a sphere.

[*E. T.* August, 1869; 4724, July, 1875. Reprint, Vol. XII. pp. 99—101. *E. T.* November, 1869. Solutions given by the Rev. J. Wolstenholme and Mr J. J. Walker, and Prof. Clifford remarks]

The extensions for a plane are

(curve of class n)...$al\,.\,am\,.\,an...\sin BP\,.\,\sin BQ...$

$$= \frac{(ap\,.\,aq\,.\,ar...)^{2(n-1)} \sin^2 LM\,.\,\sin^2 LN...}{pq^2\,.\,pr^2\,.\,qr^2},$$

(curve of order m) ..$\sin AL\,.\,\sin AM...bp\,.\,bq...$

$$= \frac{(\sin AP\,.\,\sin AQ...)^{2(m-1)}\, lm^2\,.\,ln^2\,.\,mn^2...}{\sin^2 PQ\,.\,\sin^2 PR\,.\,\sin^2 QR},$$

where P, Q, R, ... are the asymptotes, and p, q, r, ... the real foci. These give me ideas of the "distance" of a point from a line or surface, and they may be extended so as to give the distance of two curves from one another.

3021. The three pairs of foci of a sphero-conic are a, a'; b, b'; c, c'; and p is any point on the sphere.

Prove the formulæ

$$\sin aa'\,.\,\sin bb'\,.\,\sin cc' = 8 \quad\text{.........(1)},$$

$$(\sin aa')^{-\frac{2}{3}} + (\sin bb')^{-\frac{2}{3}} + (\sin cc')^{-\frac{2}{3}} = 0 \quad\text{.........(2)},$$

$$\frac{(\sin pa\,.\,\sin pa')^3}{\sin^2 aa'} = \frac{(\sin pb\,.\,\sin pb')^3}{\sin^2 bb'} = \frac{(\sin pc\,.\,\sin pc')^3}{\sin^2 cc'} \quad\text{.........(3)}.$$

[*E. T.* December, 1869. Reprint, Vol. XIII. p. 50.]

2979. Two triads of points abc, $\alpha\beta\gamma$ being taken on a line, let the two triads be called *harmonic* of one another when

$$a\alpha\,.\,b\beta\,.\,c\gamma + a\beta\,.\,b\gamma\,.\,c\alpha + a\gamma\,.\,b\alpha\,.\,c\beta + a\gamma\,.\,b\beta\,.\,c\alpha + a\beta\,.\,b\alpha\,.\,c\gamma\,. + a\alpha\,.\,b\gamma\,.\,c\beta = 0;$$

then (1) the envelope of a line cut harmonically by two cubics is of the third class. (The contravariant $\overline{\mathrm{a}11'}^3$)—(2) this line is also cut harmonically by every pair of cubics through the intersections of the first two. (3) The envelope of a line cut harmonically by a given cubic and the cubic made up by the polar line and conic of a given point is the mixed concomitant $\overline{\mathrm{a}12}\,.\,\overline{\mathrm{a}13}^2$. (4) Two cubics having the same inflexions cut harmonically any line whatever.

[*E. T.* October, 1869. Reprint, Vol. XIII. p. 52.]

3197. If the epicycloid described by a point on the circumference of a circle rolling on an equal fixed circle be loaded with matter proportional to its curvature at every point, the centre of gravity of the whole will be at the centre of the fixed circle.

[*E. T.* August, 1870. Reprint, Vol. XIV. p. 98.]

3282. It is known that the circles circumscribing the triangles formed by four lines meet in a point, and that the points so belonging to the five tetragrams formed by five lines lie in a circle. Prove that the circles so belonging to the six pentagrams formed by six lines meet in a point, and so on; the series of theorems being interminable.

To every $2n+1$ lines there belongs in this way a circle. If from any point p on this circle perpendiculars be let fall on the straight lines, their feet will all lie on a curve of order n, having a $(n-1)$-ple point at p.

[*E. T.* December, 1870. Reprint, Vol. XV. p. 47.]

3885. If A be the single focus of a semi-cubical parabola, there exists a straight line BC, such that if two tangents at right angles cut it in B, C, the angle BAC is also a right angle.

[*E. T.* October, 1872; 2674, June, 1863. Reprint, Vol. XVIII. p. 82.]

3876. [Shew that there are 5184 positions in a cubic curve such that at each of them curves of the 50th order may be drawn having 90-point contact with the cubic. Proposed by J. J. Sylvester, F.R.S. Reprint, Vol. XIX. p. 46. April, 1873?]

The gross number of points where a curve of order n can have $3n$-point contact with a cubic is $9n^2$. The problem is in fact the same as that of the divisions of the periods of an elliptic function by $3n$, and as there are two periods, there are $9n^2$ solutions. (Clebsch, *Anwendung der Abelschen Functionen in der Geometrie*, Crelle, LXII.) But in the case when n is a composite number, all the curves whose order is a division of n, and which have complete contact with the cubic, are included in the result. Thus each inflexional tangent, taken n times over, constitutes a curve of order n having $3n$-point contact with the cubic; and the nine inflexional tangents are thus always included in the $9n^2$ solutions. To obtain the number of *proper* solutions, then, we must subtract all these improper ones. When $n=30$, the result is

$$9\{30^2-15^2-10^2-6^2+5^2+3^2+2^2-1^2\}=9\times 576=5184.$$

Here it is to be observed that the curves of order 5 are *twice* subtracted, with the curves of order 15 and 10; so that they have to be added in again. The same remark applies to the orders 3 and 2. The curves of order 1 (inflexional tangents) having been thrice subtracted and thrice added, must finally be subtracted again.

4010. Prove that the lines of curvature of a quadric surface are projected from an umbilic on a plane parallel to its tangent plane into a series of confocal Cartesian ovals.

[*E. T.* March, 1873. Reprint, Vol. XIX. pp. 73, 4.]

4034. Prove that the forty umbilics of a cubic surface which passes once, or of a quartic surface which passes twice, through the imaginary circle at infinity, lie by fives upon sixteen straight lines.

[*E. T.* April, 1873; 3308, January, 1871. Reprint, Vol. XIX. p. 77.]

2020. [Reprint, Vol. XIX. p. 84. The question and solution are identical with those of 1379, p. 567 supra: here the Authorship of the Question is ascribed to N'Importe.]

4097. If about a prolate conicoid of revolution there be described an octahedron so that its three diagonals pass through a focus, shew that they must be at right angles to each other.

[*E. T.* June, 1873; 1415, August, 1863. Reprint, Vol. XIX. p. 100.]

2022. [Reprint, Vol. XIX. p. 108. This is 1378, of which see Solution p. 566 supra.]

4069. 1. Curves of order $2n+1$ pass n times through each circular point and through n^2+4n+1 other fixed single points (or their equivalent in multiple points); shew that the envelope of their asymptotes is a tricusped hypocycloid.

2. Curves of order $2n+2$ pass n times through each circular point and through n^2+6n+4 other fixed points, and their real asymptotes are at right angles; shew that the envelope of their asymptotes is a tricusped hypocycloid.

[*E. T.* May, 1873. Reprint, Vol. XX. pp. (31) 50—53.]

Prof. Wolstenholme's remark in his solution of this question, given on p. 31 of this volume of the *Reprint*, that the first part "is not quite true as it stands," has led me to examine the whole with the help of his method; and it turns out, singularly enough, that it is the *second* part that requires correction, not the first. The way in which this comes about is instructive, and the corrected theorem leads us to consider a somewhat interesting series of curves.

1. I will first state the grounds on which I originally concluded that these theorems were true. It is required to find the envelope of the asymptotes of a pencil of curves which if of odd order have *one* real point at infinity besides

the circular points, if of even order *two* which are at right angles or harmonic of the circular points. The intersections with the line infinity at the circular points are due to multiplicity of these points, not to contact with the line infinity.

Now, first, *the line infinity is a tangent to this envelope at each of the circular points and no otherwhere.* For the line infinity can only become an asymptote by the variable one point or one of the variable two points at infinity coming to coincide with one of the circular points. In the second case the variable two points being harmonic of the circular points, if one of them coincide with a circular point, the other must coincide with it. In both cases, then, there are two curves of the pencil which have the line infinity for asymptote; and it is clear that the intersection of the line infinity with the next consecutive asymptote (i.e. its point of contact with the envelope) is the circular point at which it is an asymptote.

Next, from any point at infinity not a circular point, one tangent distinct from the line infinity can be drawn to the envelope. For there is one curve of the pencil that passes through this point.

If, then, the line infinity is an ordinary tangent at each of the circular points, we see that from any point at infinity three tangents may be drawn to the envelope; viz., the line infinity counting twice, and one other. The envelope therefore is of the third class, having the line infinity for double tangent whose points of contact are the circular points; that is to say, a hypocycloid of three branches.

In fact, the tangential equation of the curve may be at once written down. Let $i=0$, $j=0$ be the equations to the circular points, $k=0$ that to some other point; then the equation is $ijk+(i,j)^3=0$. It is, in fact, of the same form as the equation of a cubic curve having a node at the origin to which the axes are tangents. If for k we write $k+\lambda i+\mu j$, it is clear that by proper choice of λ, μ we can get rid of the two middle terms of $(i,j)^3$; the equation then becomes $ijk+\alpha i^3+\beta j^3=0$, which is the same as $p=a\cos 3\theta$, where p is the distance of a tangent from the origin $k=0$, and θ the angle it makes with a fixed line. (Salmon, *Higher Plane Curves*, p. 271, Ex. 5.)

This result is true *if the line infinity is an ordinary tangent at each of the circular points.* Now this holds good in the *first case* of the question; for in this the one variable point at infinity is made to move up to a multiple point, and so only one branch acquires an ordinary contact; in virtue of this, then, the line infinity counts only once as an asymptote for each circular point. It also holds good in the already well-known case of curves of the second order, i.e. in the second case of the question when $n=0$. For in this only the two variable points at infinity coincide at a circular point, making again an ordinary contact.

But in the second case of the question, when n is not zero, something different happens. Here the two variable points at infinity simultaneously approach a circular point which is already multiple on the curve; they approach it on the same branch, and *produce a point of inflexion on that branch.* In respect of each circular point, therefore, the line infinity counts for *two* asymptotes; the envelope is raised to the *fifth* class, and *has the line infinity for inflexional tangent at each circular point.*

2. This synthetic discussion shall now be confirmed by analysis. In the first case, the equation of a curve of the pencil is

$$(x+\lambda y)(x^2+y^2)^n+k(\lambda,\ 1\between x,\ y)^2\,.\,(x^2+y^2)^{n-1}+\ldots=0,$$

and its real asymptote is

$$(x+\lambda y)(1+\lambda^2)+k(\lambda,\ 1\between \lambda,\ -1)^2=0,$$

whose envelope is of the third class, touched by $k=0$ (the line infinity) for the two values $\lambda=\pm(-1)^{\frac{1}{2}}$; whence as before.

In the second case, it is convenient to write the equation of the variable curve in the form

$$\left\{x^2+\left(\lambda-\frac{1}{\lambda}\right)xy-y^2\right\}.\,(x^2+y^2)^n+k\left(\lambda-\frac{1}{\lambda},\ 1\between x,\ y\right)^3.\,(x^2+y^2)^{n-1}+\ldots=0.$$

The two real asymptotes are

$$\left(x-\frac{y}{\lambda}\right)(1+\lambda^2)^2+k\left(\lambda-\frac{1}{\lambda},\ 1\between 1,\ \lambda\right)^3=0,$$

$$(x+\lambda y)(1+\lambda^2)^2+\lambda k\left(\lambda-\frac{1}{\lambda},\ 1\between \lambda,\ -1\right)^3=0.$$

These have the same envelope, as one equation is got from the other by writing $-\lambda^{-1}$ for λ. The envelope is of the fifth class, touched by $k=0$ twice for each of the values $\lambda=\pm(-1)^{\frac{1}{2}}$. The line infinity is therefore a double tangent with united contacts (*i.e.* an inflexional tangent, just as a cusp is a double point with united branches) at each of the circular points.

3. It remains to investigate the nature of a curve of the fifth class having the line infinity for inflexional tangent at each of the circular points. This singularity being equivalent to two inflexions and four double tangents, Plücker's equations at once tell us that the curve is of the sixth order and has five cusps and five nodes. Its tangential equation may be at once written down, being of the same form as that of a quintic curve having a quadruple point at the origin, two of whose branches coincide with each of the axes; namely, it is $i^2j^2k+(i,j)^5=0$, where $i=0$, $j=0$ are the circular points, and $k=0$ is some other point. As before, we may suppose k to have been so selected as to get rid of the two middle terms of $(i,j)^5$. Now a particular case of the equation is

$$i^2j^2k+\alpha i^5+\beta j^5=0,\ \text{or}\ p=a\cos 5\theta,$$

which represents the hypocycloid [Fig. 123] described by a point on a rolling circle whose radius is two-fifths of the radius of the fixed circle.

The general equation may be transformed into

$$p=a\cos 5\theta+b\cos 3\theta+c\sin 3\theta,$$

or, if we write

$$p_1=2a\cos 5\theta,\ p_2=2b\cos 3\theta+2c\sin 3\theta,$$

the equation is $$2p=p_1+p_2.$$

Now p_2 and p_1 are the distances from the origin of parallel tangents to a three-cusped and a sextic five-cusped hypocycloid respectively; whence we learn that *the curve in question is the envelope of a line midway between parallel tangents to two such hypocycloids.*

These hypocycloids have only to be concentric; and their relative size and orientation are the two variable elements in the equation of our curve. The method of description by tangents, however, gives us immediately a description by points, since it is clear that the point of contact of the variable tangent bisects the line joining the points of contact of the two tangents to which it is parallel and intermediate. In this way it is easy to draw roughly a few typical forms.

4. A hypocycloid in which the radii of the rolling and fixed circles are to one another as n to $2n+1$ is a curve of order $2n+2$, class $2n+1$, with $2n+1$ cusps, $(n-1)(2n+1)$ nodes, and has $(n+1)$-pointic contact with the line infinity at each circular point. Its tangential equation is

$$i^nj^nk+\alpha i^{2n+1}+\beta j^{2n+1}=0;$$

or, which is the same thing, $p=a\cos(2n+1)\theta$.

To this simplest class of roulettes, all whose tangential singularities are at infinity, it may be permissible to give the name "stars." Thus an ordinary tricusp is a three-rayed star, the curve in Fig. [123] is a five-rayed star, and so on. We may now state the following proposition:

Every curve of class $2n+1$, which has $(n+1)$-pointic contact with the line infinity at each circular point, is the envelope of a line which is the mean of the parallel tangents of n concentric stars of all odd classes up to $2n+1$.

Namely, its equation is $\quad i^nj^nk+(i,j)^{2n+1},$

or $$p=a_3\cos 3\theta+b_3\sin 3\theta+\ldots+a_{2n+1}\cos(2n+1)\theta,$$

from which the proposition is obvious. The curve has the same number of nodes and cusps as a star of class $2n+1$; only they need not, as in the case of the star, be all real.

I remark, in conclusion, first, that the point-equation of one curve of the fifth class is

$$\text{Discrt.}\ (a,b,\ x+iy,\ x-iy,\ e,\ f\between \lambda,\ \mu)^5=0,$$

which is worked out in Dr Salmon's *Higher Algebra**; and secondly, that the Hessian of the tangential equation is easily calculated and shews the cuspidal tangents to be common tangents of a three- and a five-rayed star.

4236. C is the double point of a circular cubic, and a straight line cuts the curve in DEF; join CD, CE, CF, and on the two latter lines take A, B, so that $CA\,.\,CE=CB\,.\,CF$; then prove that AB and CD are equally inclined to the tangents at C.

[*E. T.* November, 1873; 2817, January, 1869. Reprint, Vol. xx. p. 88.]

* [Pp. 208, &c., Third Edition.]

4199. Three ternary quadrics U, V, W, break up into linear factors 1, 1′; 2, 2′; 3, 3′ respectively. Prove that

$$\square(U, V, W) \equiv 123 \,.\, 1'2'3' + 1'23 \,.\, 12'3' + 12'3 \,.\, 1'23' + 123' \,.\, 1'2'3,$$

where $\square(U, V, W)$ is the coefficient of $\lambda\mu\nu$ in the discriminant of $\lambda U + \mu V + \nu W$, and 123 means the determinant formed with the coefficients of the linear factors 1, 2, 3. Required developments and interpretations.

[*E. T.* October, 1873; 1907, February, 1866. Reprint, Vol. XXI. p. 38.]

4641. If a circular cubic with a double point O be cut by a circle in four points A, B, C, D; and if OA, OB, OC, OD cut the circle again in E, F, G, H; shew that any pair of straight lines joining these four points will be equally inclined to the bisectors of the angles between the tangents at O.

[*E. T.* April, 1875. Reprint, Vol. XXIII. p. 59.]

4696. Six circles pass through twelve points on a conic in the following order,

(*a*) $A_1A_2A_3A_4$, (*b*) $B_1B_2B_3B_4$, (*c*) $C_1C_2C_3C_4$,
(*d*) $A_1A_2B_3C_4$ (*e*) $B_1B_2C_3A_4$, (*f*) $C_1C_2A_3B_4$;

prove that two circles and another point may be taken arbitrarily, and that the circles *abc* meet the circles *def* in six new points which lie on the circumference of another circle.

[*E. T.* June, 1875; 2281, November, 1866. Solution of the *first* part given in Reprint, Vol. XXIV. pp. 42, 43. The solver called in question the truth of the second part, but, on seeing the following solution on pp. 76, 77, admitted the correctness of this 'beautiful' theorem.]

Three circles taken together constitute a sextic curve passing three times through each of the circular points at infinity. Now the sextic *def* passes through all the twelve points of intersection of the sextic *abc* with the conic, which we may call k; hence, by a well-known theorem, there must be an identical equation of the form

$$\mu \,.\, def = \lambda \,.\, abc + q \,.\, k.$$

Here λ and μ are numerical ratios, and q is a quartic function of the coordinates. The equation may also be written

$$-q \,.\, k = \lambda \,.\, abc - \mu \,.\, def,$$

and in this form it shews that the equation $qk = 0$ represents a curve of the sixth order passing three times through each circular point. But, by hypothesis, the conic k does not pass through either circular point. The quartic curve q has therefore two triple points on the line infinity, it must therefore contain that line. The rest of it is a cubic having two double points on the line infinity; it also must therefore contain that line. The final remainder is a conic passing once through each circular point, that is to say, a circle. Calling this circle s, we reduce our equation to the form

$$s \,.\, k \,.\, \infty^2 = \lambda \,.\, abc - \mu \,.\, def,$$

which shews that the remaining six intersections of *abc* with *def* lie on a circle s.

It will be observed that the proof holds good if we substitute for the conic in the enunciation a circular cubic or a bicircular quartic. From the latter extension we may obtain a transformed theorem of some interest. Invert the whole figure in regard to a point not in its plane; the bicircular quartic becomes a section of a sphere by an arbitrary quadric surface, and every circle becomes a section of the sphere by a plane. In this form we may substitute for the sphere any quadric surface, and the transformed theorem may then be stated and proved as follows:

If six planes pass through twelve points on a quadriquadric curve in the order above stated, the six lines of intersection ae, af, bf, bd, cd, ce *will meet every quadric surface passing through the curve in six points which lie in one plane.*

It is to be observed that these six lines already meet the quadriquadric curve in the six points A_3, A_4, B_3, B_4, C_3, C_4. Let h_2, k_2 be two quadric surfaces passing through the curve; then the cubic surface *abc* passes through all the twelve intersections of the cubic *def* and the quadrics h_2, k_2. We must therefore have an identical equation of the form

$$\lambda \,.\, abc = \mu \,.\, def + uh_2 + vk_2,$$

where λ and μ are numerical ratios, while u and v are expressions of the first order in the coordinates. Writing this identity in the form

$$vk_2 = \lambda \,.\, abc - \mu \,.\, def - uh_2,$$

we see that the nine lines of intersection of *abc* and *def* must meet the quadric h_2 either on the quadric k_2 (i. e. on the quadriquadric curve) or on the plane v. Now three of these, *ad*, *be*, *cf* meet the curve in two points each, and the rest, *ae*, *af*, *bf*, *bd*, *cd*, *ce* in one point each; consequently these latter must meet the quadric h_2 on the plane v.

The construction of the figure depends first on that of the hexagon $A_4A_3B_4B_3C_4C_3$. In the case of the plane conic the opposite sides of this hexagon are parallel, and the possibility of the construction is assured by Pascal's theorem. When the hexagon has been drawn, it is easy to make a pair of circles pass through the ends of two opposite sides and intersect on the conic. In this way I have drawn the figure as carefully as I can, and it seems to come right. In the case of the quadriquadric curve, each pair of opposite sides is such that a quadric surface can be drawn through them to contain the curve. In the first instance they are given as two chords which are both met by the same third chord; thus the lines A_4A_3, B_3C_4 are both met by A_1A_2. Now the problem, to draw a straight line meeting two given straight lines and a quadriquadric curve twice, admits in general of eight solutions; but in the case where the two given lines are chords of the curve, the four lines joining their points of intersection count for two solutions each, and if there is one other solution, there must be an infinite number; i.e. the two lines and the curve must lie on the same quadric surface. The possibility of the inscription of a hexagon whose opposite sides possess this property may be shewn by a method analogous to that used for the plane conic [pp. 42, 43]. The quadriquadric is a curve of deficiency *one*, and therefore the coordinates of any point on it may be expressed as elliptic functions of a parameter; this may be so taken that the sum of the parameters of four points in one plane shall be congruent to zero (Clebsch "On the application of Abel's functions to Geometry," *Crelle's Journal*). Using the letters A_1, A_2, &c., to represent these parameters, we shall have

$$A_1+A_2+A_3+A_4\equiv 0 \text{ (mod. } \omega, \omega', \text{ where } \omega, \omega' \text{ are the periods)},$$

$$A_1+A_2+B_3+C_4\equiv 0;$$

and therefore, $A_3+A_4\equiv B_3+C_4$ (mod. ω, ω'), as the condition to be satisfied by two opposite sides of the hexagon. Now, if this condition is satisfied by two pairs of opposite sides, it will be satisfied by the third pair; for the congruence $B_4+B_3\equiv C_3+A_4$ follows from the congruences

$$A_4+A_3\equiv B_3+C_4, \qquad C_4+C_3\equiv A_3+B_4.$$

The theorem states that, when such a hexagon has been constructed, lines may be drawn through its vertices which shall meet every quadric surface passing through the curve in six points on one plane. *As the surface varies, this plane passes through a fixed line;* for

$$uh_2+vk_2=u(h_2+\rho k_2)+(v-\rho u)k_2.$$

Lastly, I observe that not every skew hexagon can have a quadriquadric curve drawn through it so that each pair of opposite sides shall be generators of the same quadric passing through the curve. Let the hexagon $A_4A_3B_4B_3C_4C_3$ be given; through the lines A_4A_3, B_3C_4 and the points B_4C_3, a singly infinite number of quadrics can be drawn, which will intersect in a quadriquadric curve; and one condition is necessary in order that the chords A_3B_4, C_4C_3 may possess the required property, or, which is the same thing, that the three curves which we may then get from the three pairs of opposite sides may be identical. The hexagon therefore possesses a geometrical property which can doubtless be expressed in terms of its diagonal lines or planes; this expression, however, I have not as yet been able to find.

4972. Let $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ be any eight tangents of a conic, and let a cubic pass through all the intersections of A's with B's excepting A_1B_1, A_2B_2, A_3B_3, A_4B_4. Then (1) there is a singly infinite number of such octograms inscribed in the cubic and circumscribed to the conic; (2) the groups of eight tangents form an involution of the eighth order; (3) the quadrilaterals $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are totally inscribed in a second fixed cubic; (4) the points A_1B_1, A_2B_2, A_3B_3, A_4B_4 are on a fixed straight line.

[*E. T.* May, 1876. Reprint, Vol. xxv. p. 76.]

4996. If the series

$$1+a\frac{1-x}{1-r}+a^2.\frac{1-x}{1-r}.\frac{1-rx}{1-r^2}+a^3.\frac{1-x}{1-r}.\frac{1-rx}{1-r^2}.\frac{1-r^2x}{1-r^3}+\ldots\ldots$$

be called $\phi(a, x)$; then prove that

$$\phi(1, a).\phi(a, x)=\phi(1, ax).$$

[*E. T.* June, 1876. Reprint, Vol. xxvi. p. 18.]

5304. Prove that the negative pedal of an ellipse, in regard to the centre, has six cusps and four nodes; find their positions, and the length of the arc external to the ellipse between two real cusps; and account fully for the apparent reduction of the curve to a circle and two parabolas respectively, in special cases.

[*E. T.* June, 1877. Reprint, Vol. xxix. p. 47.]

4871. Let U, V be any two cubic functions of x; shew that a quantic function $f(x)$ may always be found, such that, by the substitution $y=U:V$, the elliptic differential $dx:\{f(x)\}^{\frac{1}{2}}$ will be transformed into $Mdy:\{\phi(y)\}^{\frac{1}{2}}$, where $\phi(y)$ is a quartic function of y, and M a constant.

[*E. T.* January, 1876. Reproposed as 6475, October, 1880. Solved, Reprint, Vol. xxxv. (in progress). *E. T.* January, 1881.]

3980. It is known that if four lines be given, the circles circumscribing the four triangles so formed meet in a point; and that if five lines be given, the five points so belonging to their five tetragrams lie on a circle, (Miquel's Theorem; see *Diary* for 1861, p. 55 [viii. supra, pp. 38—54]).

Shew that this series of propositions is interminable; so that if $2n$ lines be given, they determine $2n$ circles which meet in a point; and if $2n+1$ lines be given, they determine in this manner $2n+1$ points which lie on a circle.

[*E. T.* February, 1873; 5423, October, 1877; 6441, September, 1880. Solved, Reprint, Vol. xxxiv. p. 80, *E. T.* November, 1880; cf. 3282, p. 613.]

5626. The circles doubly normal to a bicircular quartic arrange themselves in four systems, each system cutting orthogonally a principal circle; find the envelope of all the binormal circles of one system.

[*E. T.* May, 1878. Proposed November, 1869, as 3000; July, 1873, as 4123. Reprint, Vol. xxxii. p. 17.]

4143. Three elastic strings without weight, whose natural lengths are OA, OB, OC, are joined together at O, the centre of the circumscribing circle of the horizontal triangle ABC; and a smooth sphere of given radius and weight is placed with its centre vertically above O, and allowed to descend until the centre rests at O. Find the moduli of elasticity in the three strings.

[*E. T.* August, 1873. Reprint, Vol. xxxiii. p. 18. Originally proposed as 1459, December, 1863.]

1433. [Prove the following reciprocal cases of involution:—

a. The three sides of every triangle, and every three concurrent lines through its three vertices, intersect every axis in six points in involution.

a'. The three vertices of every triangle, and every three collinear points on its three sides, subtend every vertex in six rays in involution.

b. The six perpendiculars on the six lines from any point in the former case determine at the point a pencil of six rays in involution.

b'. The six perpendiculars from the six points upon any line in the latter case determine on the line a system of six points in involution. Proposed by W. J. C. Miller, B.A. Reprint, Vol. xxxiii. pp. 50, 51.]

Six points are in involution when the anharmonic ratio of any four is equal to that of their four conjugates.

a. Let then ABC be the triangle [Fig. 124], D the point of concourse; and let a straight line meet the sides and corresponding lines in $a, b, c, \alpha, \beta, \gamma$ respectively; then

$$[abc\gamma]=\{A\,.\,DCB\gamma\}=\{A\,.\,DCF\gamma\}=\{B\,.\,DCF\gamma\}=[\beta ac\gamma]=[a\beta\gamma c],^{*}$$

which proves the proposition.

a'. Let now ADF be the triangle, C, B, E the collinear points, and take any point O; then we have

$$\{O\,.\,ABDE\}=\{B\,.\,AODE\}=\{B\,.\,FODC\}=\{O\,.\,FBDC\}=\{O\,.\,CDBF\},$$

which proves the second case.

b. The line at infinity is cut in involution, by prop. a; hence, lines parallel to the given six through any point will form a pencil in involution; turn this pencil through a right angle, and it coincides with the perpendiculars.

b'. Any point at infinity is subtended in involution, by a'; whence the theorem immediately follows.

* [ ] Auctoris.

UNSOLVED QUESTIONS.

1423. Shew that

$$\int_0^{\frac{1}{2}\pi} \cos(a \tan x)\, \epsilon^{\beta \tan x}\, dx = \tfrac{1}{2}\pi\epsilon^{-a}(\cos\beta+\sin\beta).$$

[*E. T.* September, 1863. Reproposed as 2316, January, 1867; 3941, December, 1872; 4794, October, 1875; 5330, July, 1877.]

1448. The sides of a triangle repel with a force varying inversely as the cube of the distance; find the position in which a particle will rest.

Also, supposing the faces of a tetrahedron to repel according to the same law, find where a particle will rest.

[*E. T.* November, 1863. Reproposed as 3336, February, 1871; 4171, September, 1873, and 6120, November, 1879.]

1507. Consider six planes $ABCDEF$, and join the point ABC to the point DEF, and so on; we have thus ten finite straight lines, and their middle points lie in a plane.

[*E. T.* May, 1864.]

1585. If three circles are mutually orthotomic, prove that the circles on their common chords as diameters have a common radical axis.
[*E. T.* October, 1864.]

1605. Required the area of the triangle included by three points in space, given by equations of the form

$$lx+my+nz+sw=0.$$

[*E. T.* November, 1864.]

1691. If the radii of two spheres be ρ_1, ρ_2, and D the distance between their centres; and if a tetrahedron be inscribed in each; prove that the product of the volumes of the tetrahedra into $(D^2-\rho_1^2-\rho_2^2)$ may be expressed as an integral function of the squares of the distances between the vertices of the tetrahedra. Hence deduce the condition $(\Theta=0)$ that four points in a plane may lie in a circle. If they do *not* lie in a circle, what is the meaning of Θ?

[*E. T.* April, 1865.]

1724. The equations of three conics being given in the forms:

$$a_1x^2+b_1y^2+c_1z^2+d_1w^2=0,$$
$$a_2x^2+b_2y^2+c_2z^2+d_2w^2=0,$$
$$a_3x^2+b_3y^2+c_3z^2+d_3w^2=0,$$

where $x+y+z+w\equiv 0$, shew that a straight line

$$(\xi x+\eta y+\zeta z+\omega w=0)$$

will be cut in involution by them, if

$$\Sigma.\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}.(\xi-\eta)(\xi-\zeta)(\xi-\omega)\text{ (to four terms)}=0.$$

[*E. T.* May, 1865.]

1748. Let $X, Y, Z, U, V=0$ be the Cartesian equations, and r_1, r_2, r_3, r_4, r_5, the radii, of five spheres, cutting each other orthogonally; then identically

$$\frac{X^2}{r_1^2}+\frac{Y^2}{r_2^2}+\frac{Z^2}{r_3^2}+\frac{U^2}{r_4^2}+\frac{V^2}{r_5^2}=0,\quad \frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}+\frac{1}{r_5^2}=0.$$

[*E. T.* June, 1865.]

1918. It is known that the conic of five pointic contact at any point A of a cubic meets the curve again in a point B, constructed by joining the point A to its second tangential; let this point be called the *conic tangential* of A. Then the conic tangential of B will be the second conic tangential of A, and so on. Shew how, having given the conic tangential of any order, and also the line tangential of any order, we can construct for the original point A by the ruler alone.

[*E. T.* March, 1866. Reproposed as 4299, January, 1874.]

1929. 1. If 1 2 3 4 be four concyclic foci of an anallagmatic (bi-circular) quartic curve, on a plane or on a sphere, and P any point of the curve; the arc at P is equally inclined to the circles $P12$, $P34$.

2. The bitangent circles of an anallagmatic are arranged in four systems, orthotomic respectively of the four focal circles. Two bitangents of the same system can be drawn to cut orthogonally a given circle A. The four points of contact of these lie on a circle B_1 (*polar circle of* A *in that system*). There are polar circles B_2, B_3, B_4 in the other three systems, and the circles A, B_1, B_2, B_3, B_4 have a common radical axis.

3. The two bitangents of the same system through any given point are equally inclined to each of the two confocal anallagmatics which pass through that point.

4. The bitangent spheres of an anallagmatic quartic surface are arranged in five systems, orthotomic respectively of the five focal spheres. An infinite number of bitangents of the same system can be drawn to cut orthogonally a given sphere A; these envelope a cyclide, whose curve of contact with the anallagmatic surface lies on a sphere B, polar sphere of A in that system. The five polar spheres of A have with A a common radical plane.

5. The five tangent cyclides to an anallagmatic from any point have their focal spheres touched by the three confocal anallagmatics through that point.

6. A tangent cyclide from any focus is a *Tore*.

Def. A *cyclide* is the envelope of a sphere touching three fixed spheres. When the centres of the three fixed spheres are in one straight line, the cyclide becomes a *tore*, or anchor-ring.

[*E. T.* April, 1866. Reproposed as 4340, March, 1874.]

2229. 1. The distances (r, s, t) of a variable point on *one* focal curve of an anallagmatic quartic surface from any three fixed points on another focal curve of the same surface, are connected by a relation of the form $lr+ms+nt=0$ (*i.e.* a relation of the same form as that which connects the distances of a variable point on a circle from three fixed points on the circle).

2. The distances of a variable point on an anallagmatic quartic surface from four fixed foci of the surface are connected by a relation of the same form as that which connects the distances of a variable point on a sphere from four fixed points on the sphere.

3. The four points in which an anallagmatic quartic curve is cut by any circle may be taken as the foci of an anallagmatic quartic curve which passes through any four concyclic foci of the original curve. (This is true on a plane or on a sphere.)

4. The curve in which any anallagmatic quartic surface is cut by any sphere may be taken as the focal curve of an anallagmatic quartic surface which passes through any one focal curve of the original surface.

5. It is required to find the property of anallagmatic quartic curves which corresponds to the property of conics, $SP.HP=CD^2$.

6. If two anallagmatic quartic curves or surfaces, A and B, are such that a confocal to A can be inscribed or subinscribed to B: then also a confocal to B can be inscribed or subinscribed to A.

Def. One anallagmatic curve is *inscribed* to another when it touches it in four points on a circle. One surface is inscribed to another when it touches it all along its curve of intersection with a sphere. One surface is *subinscribed* to another when it touches it in four points on a circle.

[*E. T.* September, 1866. Reproposed as 4754, August, 1875.]

2510. If a conic be inverted into a circular cubic with a double point, the foci and directrices of the conic will invert into foci and directing circles of the cubic.

[*E. T.* October, 1867. Reproposed as 4667, May, 1875.]

2858. If the intersections of two circles $A=0$, $B=0$ are concentric with the antifoci of the intersections of $C=0$, $D=0$, then *vice versâ*; and if this property hold for the pairs AB, CD, and also for the pairs AC, DB, it will hold for the pairs AD, CB.

[*E. T.* March, 1869. Reproposed as 4513, October, 1874; 5691, July, 1878.]

3255. Two planes A, B, are said to have an (x, y) correspondence, when to every point on the plane A correspond y points on the plane B, and to every point on B correspond x points on A.

On each plane there is in general a locus of points, two of whose correspondents coincide: this is called the *cross-curve* (Uebergangscurbe, Clebsch in *Math. Annalen*).

On each plane there is also a locus of these united correspondents; this is called the *node-curve*.

1. If a curve touch the cross-curve in either plane, the corresponding curve in the other plane will have a node lying on the node-curve in that plane.

2. The correspondence may be represented as a (1, 1) correspondence of two multiple planes A', B'; A' consisting of y sheets, and B' of x sheets, which are connected together along the cross-curves.

3. In a $(1, y)$ correspondence, in which to two straight lines in the plane A correspond curves of deficiency p in the plane B, the order of the cross-curves in A is $=2(y+p-1)$.

[*E. T.* November, 1870.]

3308. Prove that the 40 umbilici of an anallagmatic surface lie by fives on 16 straight lines.

[*E. T.* January, 1871. Reproposed as 5274, May, 1877: cf. however 4034, p. 614.]

3961. In a polyhedron having n summits and only triangular faces (Δ-faced n-acron, Cayley), let every plane which contains three summits, but is not a face, be called a diagonal plane; and let every plane which contains two summits, but is not an edge, be called a diagonal line: then (a) there is a surface of class $n-4$ touching all the diagonal planes, (b) this surface contains all the diagonal lines; (c) the conditions of passing through the diagonal lines and touching the diagonal planes are just sufficient to determine the surface and no more, and (d) when the surface touches the plane at infinity, the volume of the polyhedron is zero.

[*E. T.* January, 1873. Reproposed as 5210, March, 1877. Cf. xviii. supra, pp. 168—176.]

4819. A spherical curve of class n has in general n^2 foci. Let n foci such that no two are on a line touching the absolute be called a *set*, and denoted by $p, q, r, \ldots$ If x be any point of the sphere, the quantity $\dfrac{(\Pi \sin xp)^{n+1}}{\Pi \sin^2 pq}$ is the same for all sets.

[*E. T.* November, 1875. Reproposed as 6890, November, 1881.]

4843. If, in regard to a system of n quadric surfaces, the two systems of n polar planes in regard to any two points of space are projective to one another, either the quadrics have a common Jacobian or each of them is a doubled plane.

[*E. T.* December, 1875. Reproposed as 5825, December, 1878.]

4897. Let $U, V=0$ be the point-equations, and $u, v=0$ the line-equations of the same two conics. If a tangent to U and a tangent to V are conjugate in respect of $u \pm \lambda v=0$, they will intersect on $U-\lambda^2 V=0$. This last conic passes through the points of contact of the conics $u \pm \lambda v=0$ with the common tangents of u and v.

[*E. T.* February, 1876.]

4925. Let $U, V, W=0$ be the point-equations, and $u, v, w=0$ the plane-equations of three quadrics inscribed in the same developable, and let $u+v+w$ be identically zero. Then, if a tangent plane to U, a tangent plane to V, and a tangent plane to W, are mutually conjugate in respect of

$$au+bv+cw=0,$$

they will intersect on

$$\frac{U}{(b-c)^2}+\frac{V}{(c-a)^2}+\frac{W}{(a-b)^2}=0,$$

which passes through the curves of contact of the developable with $au+bv+cw$ and one other quadric.

[*E. T.* March, 1876.]

4950. Prove that every matrix of the second order may be expressed in the form $aI+bJ$, where I is the matrix unity, and J a matrix such that $J^2=-1$. Hence find an expression for any power of such a matrix. (See Cayley on Matrices, *Phil. Trans.* 1858.) Required a geometrical representation for a non-self-conjugate linear and vector function.

[*E. T.* April, 1876.]

5457. A triangle ABC has its vertices A, B, jointed to two rods AD, BE, which can turn about the fixed points D, E; express the coordinates of the point C in terms of elliptic functions of a single parameter.

[*E. T.* November, 1877.]

[N.B. 1724, p. 624, has been solved, see p. 597.]

LECTURE I. ON BOUNDARIES IN GENERAL*.

Syllabus.

Every body distinguishes two adjacent regions of space, one inside and one outside.

(*a*) The surface of the body is surface to both of these regions.

(*b*) It takes up no solid room, or has no thickness.

(*c*) When the body is moved continuously, the surface is moved continuously with it.

(*d*) And yet a surface remains in the same place when the body is taken away.

Congruent regions are those which can be filled at different times by a body which does not alter in size or shape.

The remarks (*a*) (*b*) (*c*) (*d*) are true also of a line, boundary between two adjacent surface-regions, and of a point, boundary between two adjacent line-regions.

A line is also the intersection of two surfaces.

A point is the intersection of two lines, a line and a surface, or three surfaces.

A line is the path or locus of a moving point, a surface of a moving line, and a solid of a moving surface.

The number of points in a piece of line is singly infinite; the number in a piece of surface doubly infinite; and the number in a piece of solid space triply infinite.

A point on a line has one variation; on a surface, two; in solid space, three.

Questions.

1. Can two regions be partly adjacent and partly not? (Distinguish between solid, surface, and line-regions.)

2. Explain how a point is the intersection of three surfaces, and give an example.

* [From information furnished to me by Mr F. Pollock, Mr W. J. Ritchie and others, I find that these Syllabuses belong to a series of lectures given to a class of ladies at South Kensington in the spring and summer of 1869. The "proofs," for a complete set of which I am indebted to Mr C. J. Clay, bear dates ranging from April 8 to June 11, 1869. Lecture 1 is evidently that printed in "Seeing and Thinking," pp. 127—156, NATURE Series: *Macmillan's Magazine*, Vol. XL. No. 238, pp. 359—368, Aug. 1879.]

3. State clearly what is meant by the assertion: A point can be moved continuously from one position to another, with the bodies of whose surfaces it is the intersection; and yet remains when they are taken away.

4. Is the motion of a shadow always continuous?

5. An infinite number of circles can be drawn upon a piece of paper. Is this number singly, doubly, or triply infinite?

LECTURE II. ON PLANE SURFACES AND STRAIGHT LINES.

Syllabus.

Space exists independently of the things in it, but allows them to be moved about without altering their size or shape.

A plane surface may be slid about upon itself or another plane surface, and will always fit.

It may also be turned over and applied to itself so as to fit.

A plane is of infinite extent.

The intersection of two planes (a straight line) is also the intersection of an infinite number of planes: or a plane may turn round it and slide along it.

A straight line divides a plane into two congruent regions.

A straight line is fixed by two points, a plane by three.

Two straight lines, or three planes, can meet only in one point.

Two straight lines, meeting, divide a plane into four regions congruent two and two. If all four are congruent, each is called a right angle, and the lines are perpendicular.

Only one straight line can be drawn perpendicular to a given straight line through a given point.

Only one straight line can be drawn parallel to a given straight line through a given point. It is then parallel to the given line at all points on it.

Regions at an infinite distance in opposite directions on the same straight line are adjacent and separated by one point.

All points at an infinite distance on one plane are in a straight line.

All points at an infinite distance in space are in a plane.

A straight line and a plane are respectively a line and surface *of the first order*.

Questions.

1. Two bodies which have plane surfaces may have those surfaces applied to each other in an infinite number of ways. Is this number singly, doubly, or triply infinite?

2. How do we know that the spaces on the two sides of a plane surface are of the same shape?

3. One plane divides another into two congruent surface regions. Explain what this means, and how you see it to be true.

4. How many straight lines can be drawn through a given point of space to meet each of two given straight lines?

5. In what sense is it true that there is only one point at an infinite distance on a straight line?

Rouché et De Comberousse, p. 12, Ex. 1—4.
Wright, p. 12, Ex. 1—4.
Wilson, p. 10, Ex. 1—7.

LECTURE III. ON THE ROTATION OF PLANE FIGURES.

Syllabus.

The properties of plane figures are divided into *projective* properties which are retained in the shadows of the figures, and *non-projective* properties, which are not so retained.

Properties connected with Rotation are non-projective.

A straight line turning about a point in it, by equal amounts of rotation generates congruent angles.

If a plane figure turn about any point in its plane, the directions of all lines in the figure are altered by the same amount.

The direction of a figure is equally altered by the same amount of turning about two different points.

Every change of position of a figure in its plane may be produced by a single rotation.

(The external angles of a polygon are together equal to four right angles. A triangle is determined by three independent elements. A parallelogram is congruent to itself in one way; a rectangle in three.)

The path of a rotating point is a *circle*. The direction of the point's motion is always perpendicular to the radius.

Equal angles at the centre of a circle cut off equal arcs of its circumference.

Questions.

1. Parallel lines (1) do not meet, (2) make congruent intersections with a third line. Shew that any two lines which possess either of these properties possess also the other.

2. Prove that the exterior angle of a regular hexagon is equal to the interior angle of an equilateral triangle.

3. Shew how to construct a triangle of which you know the height and the two base angles.

4. Can *any* two congruent triangles in the same plane be made to coincide by rotating one of them about some point?

5. How does it appear that the tangent at any point of a circle is perpendicular to the radius?

Rouché et De Comberousse, p. 22, Ex. 4, 7; p. 28, Ex. 1—3; p. 36, Ex. 1—6.
Wright, p. 22, Ex. 4, 7; p. 30, Ex. 1—3; p. 39, Ex. 1—7.
Wilson, p. 17, Ex. 1—7; p. 34, Ex. 1—7.

LECTURE IV. ON SIMILAR FIGURES.

Syllabus.

When a figure is enlarged so as to remain still of the same shape, every straight line in it remains a straight line, and every angle remains congruent to itself.

All the parts of the figure are equally enlarged.

When one figure is an enlarged copy of another, the two are said to be *similar*.

The degree of enlargement necessary to make one figure equal to the other is called their *ratio of similitude*.

The ratio of two lines in the one figure is equal to the ratio of the two corresponding lines in the other.

* If four quantities are proportional, and the first is greater or less than any fraction of the second, the third is greater or less than the same fraction of the fourth.

If two quantities are so connected that each, being given, determines the other: and if to the sum of two values of one corresponds the sum of the two corresponding values of the other: then the ratio of any two values of the first quantity is equal to the ratio of the two corresponding values of the second.

Triangles are similar which have (1) their sides proportional, (2) an angle in one equal to an angle in the other, and the sides about them proportionals.

Similar rectilinear figures are made up of similar triangles.

Questions.

1. Prove that all circles are of the same shape.

2. If two parallelograms have the angles of one equal respectively to the angles of the other, and if the ratio of the two diagonals of one is equal to the ratio of the two diagonals of the other, the parallelograms are similar.

3. Shew that two regular polygons of the same number of sides are similar figures.

4. What is Euclid's definition of proportion? Prove that it follows from the fact (*) stated in the Syllabus.

5. There are two similar triangles such that the first and second sides of one of them are four and five feet long respectively; and the first and third sides of the other are twelve and twenty-one feet long respectively. Find the lengths of the remaining sides.

LECTURE V. THE FIRST PRINCIPLES OF CALCULATION.

Syllabus.

Numbers may be changed into other numbers by the operations of addition, subtraction, multiplication, and division.

Operations of addition or subtraction may be changed into others by the operations of multiplication, division, and reversion.

Multiplication is *commutative* $[ab = ba]$, and *distributive* $[a(b+c) = ab + ac]$.

A *quantity* is measured by the ratio which it bears to some fixed quantity of the same kind, called the unit.

Quantities may be changed into others by the four fundamental operations, which are subject to the same laws as in the case of numbers.

Ratios are approximately represented by the ratios of numbers; viz. (1) by decimal fractions, (2) by continued fractions.

The *position* of a point on a straight line may be represented by its distance from a fixed point on the line; or by the quantity of motion necessary to carry it from that fixed point to its position. Positive and negative motion are in opposite directions.

Questions.

1. What theorem is implied in the idea of number? and what in that of the numerical ratio of two quantities?

2. Distinguish between the multiplication of numbers and the multiplication of operations. Prove that multiplication is distributive in the former case.

3. In what cases can subtraction and division be performed with (*a*) numbers, (*b*) quantities, (*c*) motions?

4. Translate into English:

$$5(3+2) = 25; \qquad -7+4 = -3; \qquad (-4)(-5) = +20.$$
$$a+b-c = a-c+b; \qquad AB+BC+CA = 0.$$

5. Write in symbolic language:

If the operation of adding one number and subtracting another be multiplied by the sum of the numbers, the result will be equivalent to the operation of adding the square of the first number and subtracting that of the second.

6. Shew how to express the ratio of the diagonal of a square to its side in the form of a continued fraction.

LECTURE VI. THE THEOREM OF PYTHAGORAS.

Syllabus.

If squares are described on the sides of a right-angled triangle, the two squares on the sides containing the right angle are together equal to the square on the side opposite the right angle.

I. Clairaut's Proof.

When several figures can be fitted together in different ways, they always cover the same area. The square on the hypothenuse can be cut up into pieces out of which the other two squares may be formed.

II. The Proof in the First Book of Euclid.

Parallelograms of equal base and height have equal areas. The area of a triangle is half that of a rectangle of the same base and height.

The squares on the sides of the right-angled triangle are severally equal to the two parts into which the square on the hypothenuse is cut by the perpendicular.

III. The Proof in the Sixth Book of Euclid.

If one side of a rectangle is altered in any ratio, the rectangle is altered in the same ratio.

The perpendicular from the vertex of the right angle on the hypothenuse of a right-angled triangle divides it into two triangles similar to each other and to the original triangle.

The areas of similar figures are as the squares on their corresponding sides.

The theorem of Pythagoras is true of any three similar figures described on the sides of a right-angled triangle, *e.g.*, three semicircles; the Lune of Hippocrates.

IV. The Figure of the Bride's Chair*.

The area of a rectangle is measured by the product of the measure of its sides.

Squares on the sum and difference of two lines:

$$(a+b)^2 = a^2+b^2+2ab; \quad (a-b)^2 = a^2+b^2-2ab.$$

The theorem follows from the latter formula.

Numbers proportional to the sides of a right-angled triangle may be obtained from the formula

$$(a^2-b^2)^2 + (2ab)^2 = (a^2+b^2)^2.$$

Questions.

1. Can a length be equal to an area? Can two areas be proportional to two lengths? Can an area have any ratio to a length?

* [I have not been able to trace the origin of this title: it appears to be the case (3) of De Morgan's Article, "Hypothenuse," in the *English Cyclopædia*, where the demonstration is said to be derived from the Hindu treatises on Algebra. See Note, p. 637.]

2. Give examples of the method of proving two areas equal by cutting them up and rearranging the pieces.

3. Shew how with an inch measure to construct a square containing five square inches; and cut it up into square inches.

4. The square whose height is equal to that of an equilateral triangle is three-fourths of the square on the side of the triangle.

5. One side of a right-angled triangle is 24 inches long. Find the triangle so that each of the other sides may be an exact number of inches.

6. Prove geometrically that

$$(x+a)(x+b)=x^2+(a+b)x+ab.$$

LECTURE VII. THE PROPERTIES OF ONE CIRCLE.

Syllabus.

I. The construction of a circle which passes through three given points.

Lines bisecting at right angles the sides of a triangle meet in a point.

The perpendiculars of a triangle meet in a point.

No point but the centre is equidistant from three points in the circumference of a circle.

II. *All the angles in the same segment of a circle are equal to one another.*

A rectangle is the only parallelogram that can be inscribed in a circle. The diagonals of the rectangle are then diameters of the circle. Every angle in a semicircle is a right angle.

If the perpendiculars AF, BG, CH of a triangle ABC meet in the point O, then there are nine points, viz.: the points F, G, H, the middle points of the sides BC, CA, AB, and the middle points of the lines OA, OB, OC, which are all upon the circumference of the same circle.

The equality of angles in the same segment may be proved either, as in Euclid, from the fact that the angles of a triangle are together equal to two right angles, or by the properties of rotation.

Four straight lines make four triangles, and their circumscribing circles meet in a point.

III. If through a point P two lines PAB, PCD are drawn to cut a circle in A, B, C, D, the rectangle contained by PA, PB is equal to the rectangle contained by PC, PD.

Questions.

1. The perpendiculars of a triangle ABC meet in a point D. Shew, by examining the figure, that each of the four points A, B, C, D is the intersection of perpendiculars of the triangle formed by the other three.

2. What is the locus of the centres of all circles that pass through two fixed points?

3. And what is the locus of the intersection of the two tangents to each of the circles at those points?

4. Tangents are drawn to a circle at the angular points of an inscribed rectangle. Prove that the quadrilateral thus formed has all its sides equal.

5. AF, BG, CH are the perpendiculars of a triangle ABC, meeting in O. Shew that circles can be described about the quadrilaterals $AGOH$, $BHOF$, $CFOG$; and that the angles BFH, CFG are equal to one another.

LECTURE VIII. THE PROPERTIES OF ONE CIRCLE—*continued.*

Syllabus.

The following are different statements of the same projective property of a circle:—

1. If a series of lines are drawn through a fixed point to meet a circle, and if at the two points of intersection of each of these lines with the circle tangents are drawn; the intersections of all these pairs of tangents will lie on a fixed straight line.

2. If a series of points are taken on a fixed straight line, and from each of them two tangents are drawn to the circle; the several chords of contact of these pairs of tangents will all pass through a fixed point.

Two points A and B are called *inverse* points in respect of a circle whose centre is O, when OAB is a straight line, and the rectangle OA, OB is equal to the square of the radius.

A line through B perpendicular to OAB is then called the *polar* of the point A; and a line through A perpendicular to OAB is called the *polar* of the point B.

When a line is the polar of a point, the point is called the *pole* of the line.

If a point L lies on the polar of M, then M lies on the polar of L.

The polar of a point outside the circle is the chord of contact of tangents from it to the circle.

Questions.

1. Two inverse points in respect of a circle are distant from the centre two inches and eight inches respectively; what is the length of the radius?

2. A circle has its radius one foot long, and a certain straight line is sixteen inches distant from the centre; how far is this straight line from its own pole?

3. Explain in what way every diameter is the polar of some point at infinity.

4. Prove that if one point lies on the polar of a second, the second will lie on the polar of the first. How does it follow from this that when three points are in a straight line, their three polars meet in a point?

5. Shew that a circle can be drawn through any two pairs of inverse points.

LECTURE IX. ON THE SHADOWS OF A CIRCLE.

Syllabus.

The *shadow* of a solid body may mean either the solid region which it deprives of light, or the portion of any surface which is within that region. And the *shadow* of a curve may mean either the conical surface composed of straight lines passing through the luminous point and meeting the curve, or the new curve in which this cone cuts any surface.

Besides the *real* shadow, or darkened part of space, it is necessary to consider also the *ideal* or geometrical shadow, obtained by producing the rays of light backwards behind the luminous point.

The surface-shadow of a circle is a cone of the second order.

The shadow of a circle cast on a plane is of one of three shapes, called respectively the Ellipse, Parabola, and Hyperbola.

The Ellipse and Hyperbola have connected with them two points called the *foci*, such that in the Ellipse the sum of their distances from every point of the curve is the same, and in the Hyperbola the difference of their distances from every point of the curve is the same. The Parabola has one focus, which is so related to a certain straight line called the *directrix* that the distance of every point on the curve from the focus is equal to its distance from the directrix.

All these curves are of the *second order*, and all the polar properties of the circle belong to them.

Questions.

1. Explain what is meant by the statement that a right cone is a surface of the second order. Describe the surface, and shew that the statement is true.

2. In how many directions can a circle be cut from an oblique cone?

3. Describe the shape of an ellipse, and shew how it can be practically drawn.

4. If the two foci of an ellipse coincide, what does it become?

5. Why are the two branches of a hyperbola regarded as one curve?

LECTURE X. ON THE ORDER OF GEOMETRICAL PROBLEMS.

Syllabus.

When a point has to be found on a straight line, and the problem of finding it is of the second order, so that it may have two solutions; then in certain particular cases these two solutions coincide and become one, and in other cases disappear altogether.

Geometers however are accustomed to say in the first case that there are two coincident solutions, rather than one solution; and in the second case that the two solutions have become *imaginary* or *invisible*, rather than that they have altogether ceased to exist.

This language contains a reference to a class of problems different from the one directly considered, in which lengths are to be measured, not on a line in one of two directions only, but in a plane and in any direction. In these problems the solutions never disappear, and the number of them is always exactly equal to the order of the problem.

By this use of language belonging to a different branch of science, not only is great generality introduced into the enunciations and proofs of theorems, but an explanation is afforded of the very different forms under which the same theorem presents itself.

[Bride's chair, Lect. VI, see below.*]

[The following, Prof. Henrici tells me, are notes of a course of lectures on Synthetic Geometry and Graphical Statics. I have thought them worthy of a place here as the treatment is somewhat novel, at least, to English readers.]

I. Shadows of a circle, of parallel lines.
Perspective range and pencils of lines and planes.
Range perspective to itself with two points interchanged.
1. is called *harmonic*, construction by quadrilateral.

II. Projective ranges and pencils (assuming only Euclid).
2. If two point-ranges are projective in one position they are in all positions (two cases).

Projective correspondence.

Case 1. Projective correspondence is determined by three pairs of corresponding points.

Case 2. Two on same line may have two united points but not more.

3. Two ranges projective to same range are projective to each other.

* [I am indebted to Mr H. C. Levander for a reference to Dyer's English *Folk-Lore*, p. 204. Mr Dyer quotes *Antiquarian Repertory*, 1807, Vol. I. p. 107, and Harland and Wilkinson's *Lancashire Folk-Lore*, 1867, p. 265, which contain accounts of two Bride's Chairs. The former work describes the chair of the Venerable Bede, at Jarrow Church, Northumberland. "It is preserved in the vestry of the Church, whither all Brides repair immediately the marriage service is over, to seat themselves upon it. .. The Chair, which is very rude and substantial, is made of oak; is 4 feet 10 inches high; having an upright back, and sides that shape off for the arms." Mr Levander also points out, in a figure, the resemblance of the diagram to the Constellation of Cassiopeia, or 'the Lady in her Chair.']

4. Construction for corresponding point to given one when three pairs of correspondents are given.

5. Line-pencils standing on same range are projective.

6. Line-pencils on projective ranges are projective.
Plane pencils on same range are called projective.

7. Plane pencils on projective ranges are projective.

III. Same propositions proved without assuming Euclid.

4*. Two co-basic prime-forms having three coincident pairs of corresponding elements are identical.

IV. Position-vectors: ratios of the same.

8. Cross-ratio of pencil = that of range cut by it.
(By areas of triangles.)

9. $Ix \,.\, Jy = k$ (similar triangles).

10. Two projective ranges on same line have two united points, visible, coincident or invisible.

11. One pair of symmetrical correspondents makes all pairs symmetrical.
Involution. Specialities of harmonic.

V. Unique Correspondence.

Postulate:—Two uniquely corresponding quantities are connected by an equation of the form

$$axy + bx + cy + d = 0.$$

12. Two uniquely corresponding quantities which vanish and become infinite together are proportional.

13. Two uniquely corresponding quantities which vanish and become infinite together alternately are reciprocal.

14. Uniquely corresponding prime-forms are projective.

9*. $Ix \,.\, Jy = k$.

VI. *Secondary Forms.*

15. Locus of intersection of corresponding rays of two projective pencils is of second order.
Envelop of connectors of corresponding points of two projective ranges is of second class.

16. Locus of intersections of corresponding planes of two projective pencils is of second order.

A. Figure described by join of corresponding elements of two projective prime-forms of same kind is form of second degree.
Construction from five points or five tangents.

17. Form of second order may be described by pencils having vertices at any two of its points.

18. From any point on tangent one and one only other tangent can be drawn.
Construction of simultaneous points and tangents: definition of conic.

19. Pascal, Brianchon.

VII. Poles and Polars.

VIII. Metrical properties (centre and diameters).

IX. Systems of conics, foci, etc.
Preceded by re-treatment of conics by (1, 1) correspondence.

X. Surfaces, corresponding to VII. and VIII.

Appendix A. On the representation of Solid Figures, commonly called Descriptive Geometry.

Appendix B. On Graphical Calculations.

Part 2. Graphical Statics,

I. Polygon of forces, plane and space, stresses on frame-work by method of *sections* (?).

II. Tie-Polygon (plane only).

III. Moments and Parallel Forces. Couples.

IV. Centre of Parallel Forces.

For Appendix B. On Graphical Calculation. (Integration of x^n.)

1. Sum of Geometric Series by anti-parallels *

$$\frac{12 + 23 + 34 + 45}{12} = \frac{56 - 12}{23 - 12} = \frac{Oe - Oa}{Ob - Oa},$$

which is obvious.

2. Differentiation of $y = ax^n$

$$\frac{y_1 - y_2}{x_1 - x_2} = a\frac{x_1^n - x_2^n}{x_1 - x_2} = \text{(in limit) } a \,.\, nx^{n-1}.$$

3. Ditto for n fractional.

4. If ordinate of one curve represents area of another, $\frac{\delta y}{\delta x}$ of this curve will represent ordinate of other.

5. Area for $y = ax^n$.

* [Draw *O*123456 horizontal and *Oabcde* inclined to it, so that 1*a*, 2*b*, 3*c*, 4*d*, 5*e* form one set of parallels and *a*2, *b*3, *c*4, *d*5, *e*6 the other set of parallels.]

NOTES.

The paper *XVI. was printed before I had examined the question 2942 (p. 612) and the solutions of it. I have since submitted it to Mr J. J. Walker, who has placed at my service the following remarks:

Greater clearness is attained, and some errors are avoided, in establishing these fundamental formulæ by starting at once with homogeneous co-ordinates $a_1a_2a_3$ of the point a. The constant $(\Delta \div R)$

$$a_1 \sin\alpha + a_2 \sin\beta + a_3 \sin\gamma = \tfrac{\iota}{2} \begin{vmatrix} a_1 & a_2 & a_3 \\ -1 & e^{-\iota\gamma} & e^{\iota\beta} \\ -1 & e^{\iota\gamma} & e^{-\iota\beta} \end{vmatrix}$$

so that calling i the point whose co-ordinates are proportional to

$$-1, \quad e^{-\iota\gamma}, \quad e^{\iota\beta},$$

and j the point whose co-ordinates are proportional to

$$-1, \quad e^{\iota\gamma}, \quad e^{-\iota\beta},$$

the constant may be written symbolically

$$\Delta \div R = \tfrac{\iota}{2}\,(aij).$$

The numerator in Faure's expression for $\overline{ab}^2$: viz.

$$\overline{ab}^2 = \frac{\Sigma\,(a_2b_3 - a_3b_2)^2 - 2\Sigma\,(a_3b_1 - a_1b_3)\,(a_1b_2 - a_2b_1)\cos\alpha}{(a_1\sin\alpha + a_2\sin\beta + a_3\sin\gamma)^2\,(b_1\sin\alpha + \ldots)^2},$$

may also be expressed as the product of the determinants

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ -1 & e^{\iota\gamma} & e^{-\iota\beta} \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ -1 & e^{-\iota\gamma} & e^{\iota\beta} \end{vmatrix}$$

or (abi), (abj); so that

$$\overline{ab}^2 = 16\,\frac{(abi)\,(abj)}{(aij)^2\,(bij)^2} \text{ or } \overline{ab} = 4\,\frac{\sqrt{(abi)\,(abj)}}{(aij)\,(bij)}.$$

Again the distance of a from B is

$$\frac{B_1a_1 + B_2a_2 + B_3a_3}{\sqrt{B_1^2 + B_2^2 + B_3^2 - 2B_2B_3\cos\alpha - \ldots}},$$

and the denominator may be written as the square root of the product

$$(-B_1 + B_2e^{\iota\gamma} + B_3e^{-\iota\beta})\,(-B_1 + B_2e^{-\iota\gamma} + B_3e^{\iota\beta});$$

thus

$$\text{distance } aB = \frac{aB}{\sqrt{iB\,.\,jB}}.$$

The well-known formula for $\sin AB$ is

$$\frac{\Sigma\,(A_2B_3 - A_3B_2)\sin\alpha}{\sqrt{iA\,.\,jA\,.\,iB\,.\,jB}} \text{ or } \frac{AB\overline{ij}}{\ldots},$$

wherein $\overline{ij}$ stands for the line containing i, j, viz., as has been shewn, the line

$$x_1\sin\alpha + x_2\sin\beta + x_3\sin\gamma = 0.$$

Also

$$\cos AB = \frac{A_1B_1 + A_2B_2 + A_3B_3 - (A_2B_3 + A_3B_2)\cos\alpha - \ldots}{\ldots \quad \ldots \quad \ldots} =$$

$$\tfrac{1}{2}\,\frac{(-A_1 + A_2e^{\iota\gamma} + A_3e^{-\iota\beta})(-B_1 + B_2e^{-\iota\gamma} + B_3e^{\iota\beta}) + (-A_1 + A_2e^{-\iota\gamma} + A_3e^{\iota\beta})(-B_1 + \ldots)}{\ldots \quad \ldots \quad \ldots},$$

or

$$2\cos AB = \frac{Ai\,.\,Bj + Aj\,.\,Bi}{\sqrt{Ai\;Aj\;Bi\;Bj}}.$$

I may also state that I have recently received lecture-notes on the subject from a pupil of Prof. Clifford's at University College*, who has also furnished me with the accompanying proof, given also in lecture, of Ivory's Theorem†.

Def. Corresponding points on two confocal ellipsoids are such as coincide when either ellipsoid is deformed by a homogeneous strain, so as to coincide with the other.

Statement. Let corresponding points Pp be taken on two homogeneous confocal ellipsoids Ee. The x component of the attraction of E on p is to that of e on P as the area of the section of E by the plane yz is to the area of the coplanar section of e, that is in the ratio $\beta\gamma : bc$.

This theorem is true for any law of force.

First we shew that for any law of force the attraction of a straight line AB of uniform thickness and density on an external point P depends only on the distances PA, PB. (Fig. 125.)

$$\frac{dr}{dx} = \cos PMB.$$

Thus, when $f(r)$ determines the law of force, attraction of element dx resolved parallel to $AB = f(r)\,\rho dx \cos PMB$.

Thus whole attraction parallel to AB is

$$\int_u^v f(r)\,\rho\,dr = \rho\,\{F(v) - F(u)\}.$$

Divide up both ellipsoids (Fig. 126) into strips parallel to the axis of x.

Let Qq be corresponding points.

Then clearly $Q'q'$ are also corresponding points.

* Mr G. W. von Tunzelmann, who says that Prof. Clifford informed him that he had found it easy to extend the method to higher plane curves, but that surfaces were much more difficult and he had not made much progress in its application to them. This pupil has also put at my service other Notes which I hope to make use of hereafter.

† See *Lectures and Essays*, Vol. I. p. 4.

The attraction of the strip qq' on any point P on the outer ellipsoid E depends on the thickness of the strip, and on Pq, Pq'; and attraction of QQ' on p, corresponding point on inner ellipsoid e, depends on the thickness of the strip, and on Qp, $Q'p$. But $Qp = Pq$ and $Q'p = Pq'$.

Therefore the whole attractions are in the ratios of the sections by the plane yz. Therefore the Theorem is proved.

XXV. is referred to in Maxwell's *Electricity*, Vol. I. p. 171. Nothing further on the subject has been met with in the papers that have been submitted to me.

*XXXVII. On a loose sheet I find some work slightly differing from that in the text, with the result

$$\cos(2x\cos\phi) = f(-x^2) - 2x^2 f_2(-x^2)\cos 2\phi + 2x^4 f_4(-x^2)\cos 4\phi - \&c.,$$

$$\therefore \int_0^\pi \cos(2x\cos\phi)\cos 2n\phi\, d\phi = (-)^n \pi x^{2n} f_{2n}(-x^2).$$

The following notes are given here as additional illustrations to *XLII., p. 369.

[i] If a line meeting two polar lines be displaced equally along both of them through an angle α, the two positions of the line will be called *parallel* and said to have the same direction. The normal distance from *any* point on one of them to the other is α. Two lines parallel to the same line are parallel to one another, and if a line meet two parallel lines it meets them at equal angles. According as the twist converting a line into a parallel line is right-handed or left-handed, the common direction of the two lines will be called a right or left direction; thus every line has two directions, and through an arbitrary point two lines may be drawn having with it right and left parallelism respectively. All lines having the same right direction meet the same two generators of one system of the absolute; all having the same left direction meet two generators of the other system. If a series of parallel lines be drawn through all the points of any line, they will trace out a surface of zero curvature and finite extent, which is in fact a quadric having quadruple contact with the absolute or meeting it in four straight lines. Starting with three lines meeting at right angles, we may determine a triple series of such surfaces, intersecting everywhere at right angles in lines parallel to the original three; thus every point in space will be characterized by co-ordinates measured parallel to three given axes. If any solid body receive simultaneous equal rotations about two polar lines, all the points of the body will move in parallel lines, and any one of these lines may be regarded as an axis of the motion, viz. the body has equal rotations about this line and its polar. Such a motion may be called a *Vector*; it may be represented by a finite straight line having given magnitude and direction, and will be a *right* or *left* vector according as the parallelism is right or left. The ratio of two vectors of the same side (i.e. both right or both left) is a *quaternion* of the same side; viz., one can be converted into the other by a certain rotation about an axis perpendicular to both but of indeterminate position. Every motor is the sum of a right and a left vector; for we have identically

$$A = \frac{1+\omega}{2}A + \frac{1-\omega}{2}A,$$

XIII.

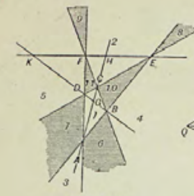


Fig. 120.

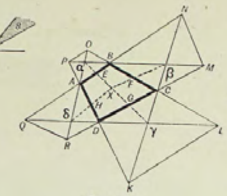


Fig. 121.

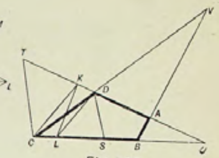
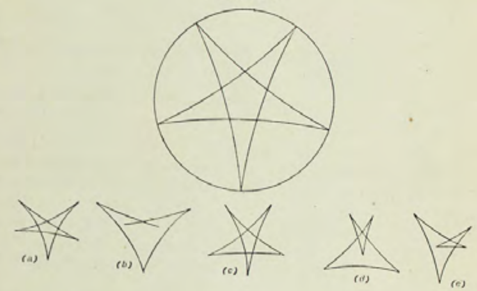


Fig. 122.

Fig. 123.



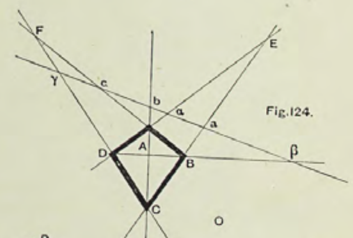


Fig. 124.

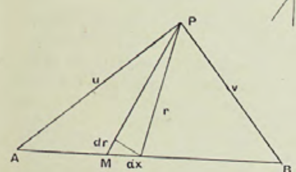


Fig. 125.

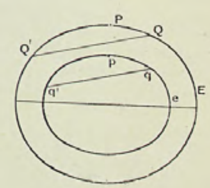


Fig. 126.

viz. these vector parts are the half sum and half difference of the motor A and its polar motor ωA. If we write $\xi = \frac{1}{2}(1+\omega)$, $\eta = \frac{1}{2}(1-\omega)$, then

$$\xi^n = \xi, \quad \eta^n = \eta, \quad \xi\eta = 0,$$

and we have, if $\alpha\ \beta\ \gamma\ \delta$ are rotors through a fixed point,

$$\frac{\xi\alpha + \eta\beta}{\xi\gamma + \eta\delta} = \xi\frac{\alpha}{\gamma} + \eta\frac{\beta}{\delta},$$

whereby the quotient of any two motors is expressed as a biquaternion. For on substituting for $\xi\ \eta$ their values, the expression becomes $q + \omega r$, where q and r are the quaternions

$$\tfrac{1}{2}\left(\frac{\alpha}{\gamma} + \frac{\beta}{\delta}\right), \ \tfrac{1}{2}\left(\frac{\alpha}{\gamma} - \frac{\beta}{\delta}\right).$$

[ii] The co-ordinates of a straight line being $p_{12}\ p_{23}\ p_{31}\ p_{14}\ p_{24}\ p_{34}$, let

$$M = p_{12}\,p_{34} + p_{13}\,p_{42} + p_{14}\,p_{23},$$

and let $\Omega = 0$ be the condition that this line touches the absolute or has no length. If we now write

$$\alpha = a_{12}\,\partial_{p_{12}} + \ldots = \Sigma\, a_{ij}\,\partial_{p_{ij}},$$

then $\alpha\beta M = 0$ is the condition that the lines α, β (i.e. the lines whose equations are $\alpha M = 0$, $\beta M = 0$) should meet. $\alpha\Omega = 0$ is the equation of the polar line of α.

In general there are two straight lines which meet two lines and their polars. For the five equations $\alpha M = 0$, $\beta M = 0$, $\alpha\Omega = 0$, $\beta\Omega = 0$, $M = 0$, determine quadricly the ratios of the p. Now these two lines are polars. If γ be one, we have $\alpha\gamma M = 0$, $\beta\gamma M = 0$, $\alpha\gamma\Omega = 0$, $\beta\gamma\Omega = 0$, $\gamma^2 M = 0$.

Moment of two motors $= \dfrac{\alpha\beta M_2}{\sqrt{\alpha^2\Omega_2 \,.\, \beta^2\Omega_2}}$.

[iii] $(\xi q + \eta r)(\xi A + \eta A) =$ a pure motor if $S\xi qA = 0$, $S\eta rA = 0$, or if we write

$$A = \xi\alpha + \eta\beta,$$

then q must directly operate on α and r on β.

Axes of $\xi\alpha + \eta\beta = \frac{1}{2}(T\alpha \pm T\beta)(\xi U\alpha \pm \eta U\beta)$,

$$\xi\alpha + \eta\beta = \frac{\alpha+\beta}{2} + \omega\,\frac{\alpha-\beta}{2}.$$

This is a rotor if $S\,.\,\overline{\alpha+\beta}\,.\,\overline{\alpha-\beta} = 0$, or if $\alpha^2 - \beta^2 = 0$. *Hence the sum of two right and left vectors of equal length is a pure rotor.*

Or we may say the axes of A are $(T\xi A \pm T\eta A)(\xi U\xi A \pm \eta U\eta A)$.

Observe that $\omega(\xi A + \eta B) = \xi A - \eta B$. The general expression for rotor right parallel to i is $\xi\iota + \eta\beta$, β being of unit length; the axis is $\xi i + \eta\beta$ as it should be, or $\frac{1}{2}(\iota+\beta) + \frac{1}{2}\omega(\iota - \beta)$: therefore if ρ meets this, $\xi(\iota + k\rho) + \eta(\beta + k\rho)$ is a rotor; or

$$T(\iota + k\rho) = T(\beta + k\rho),$$

$$S\iota\rho = S\beta\rho,$$

$$\therefore S(\iota - \beta)\rho = 0.$$

Hence *ρ is in a plane equally inclined to* ι and β.

The following fragments have some points of interest, and I print them in their incomplete state as they may suggest lines of working to some readers. The portion on curves [v] is headed *Appendix:* it looks at first sight as if it were intended to be an appendix to Frost's *Curve Tracing.*

[iv] *Geometry in a Quadric Space Q_2.*

Space-sections are represented by quadric-surfaces passing through a fixed conic ω_2. The points of this conic represent a *cone* K_2 lying entirely in the space Q_2. The vertex o of the cone is represented by the plane O of the conic ω_2.

A *Plane* represents a quadric surface containing two lines of K_2, since it meets ω_2 in two points.

The complete section of Q_2 by a space of order n passing a times through o is represented by a surface of order $2n-a$, passing $n-a$ times through the conic ω_2.

A surface of order μ passing b times through ω_2 represents a surface in Q_2 of order $2\mu-2b$, having o for a conical point of order $\mu-2b$.

Q_2 is cut by a *surface* of order r passing a times through o in a curve represented by a curve of order $2r-a$ meeting ω in $2r-2a$ points.

And a curve of order ρ meeting ω in b points represents a curve of order $2\rho-b$ passing $\rho-b$ times through o.

Thus e.g. a quadriquadric surface is represented by a surface of the fourth order passing twice through ω_2; and is therefore unicursal.

A Quadric space can contain no surface of odd order, unless its discriminant vanish. In this case the conic ω_2 is replaced by two lines $\omega\omega'$, and a surface passing a, b times through these represents a surface of order $2\mu-a-b$. Thus the quadric space now contains two pencils of planes represented by the planes through these lines ω, ω'. And a [quadri]quadric surface passing through one of them represents a skew cubic surface, passing once through o, meeting each of the planes of its own system in a line, and each of the others in two points. This cubic surface is ruled, then; and the other lines of the representative quadric represent a pencil of conics upon it, passing through o, but otherwise not meeting each other, but each meeting all the lines. Plane sections of the quadric represent skew cubic curves on the surface, namely its space-sections made by spaces passing through o. But the general space-sections are represented by skew cubics, meeting the conics twice and the lines once. We may generally represent the skew cubic surface upon a plane in the following manner. Its space-sections are represented by conics passing through one point a; sections by space of nth order, curves $2n$, n times through a; curves μ, a times through a represent curves of order $2\mu-a$. Thus the point a itself and the lines through it are the only straight lines on the surface; all other lines in the plane represent a doubly infinite system of conics, any two meeting in one point; each of these is met twice by a space-section; and so on.

Cubic Space C_3.

Space-sections are cubic surfaces any three of which meet in three points; i.e. they have common a quintic curve of deficiency 2 and one fixed point, or else a quartic curve of deficiency 0 and two fixed points.

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[v] *On the general shapes of Algebraic Curves.*

Those who have studied the preceding chapters of this work must still be convinced that, even with the facility which such study has given them, the operation of tracing a curve from its equation is one which requires some time and much close attention for its successful accomplishment. They will readily admit that if, in order to know the chief varieties of form among curves of the first four or five orders, it were necessary to trace one after another the curves represented by all varieties of equations of those degrees, by such processes as have already been explained; this work would be enormous, and beyond the reach of any but a lifelong study. Accordingly, it has never been attempted in the case of any degree exceeding the second. In the enumeration of lines of the third order, either the equation has been reduced by some preliminary discussion (to be presently mentioned) to a very special form, or its consideration has been evaded altogether. Of quartic curves the enumeration, so far as I know, has not yet been systematically attempted; but such general results as have been arrived at were obtained by considerations and methods other than those which the equation itself suggests.

To the explanation of such considerations and methods the present appendix is devoted. The study of them presents many direct advantages to the mathematician, besides the indirect ones of the help they afford him in the tracing of curves and the investigation of their properties. These methods are exceedingly simple and easy of application; they partake more of the nature of a manual craft than of a purely intellectual occupation, and may so be used as a rest from severer studies; and—as we can only imagine things of which we have seen the like—by appealing directly to the senses they extend those powers of concrete realization which the growing complication of modern analysis renders daily more desirable.

They consist of rules for transformation, by the aid of which the figures of a vast variety of curves may be obtained from a few simple types. These rules will be explained under three heads. In the first section is considered Projection, a process by which no alteration is made in the order, the class, nor in any other purely descriptive property of a curve. In the second section those modifications of form are described which leave the *order* of a curve unaltered. In the third, mention is made of those changes which exercise no effect upon the class. In the two latter sections a process will be described which may be called the *composition* of curves; by which a curve of any order or class may be built up out of the simplest elements.

Section 1. *Projection.*

The shadow of a hyperbola may be an ellipse, if it is properly held; and conversely, an ellipse may be so placed as to have a hyperbola for its shadow. More generally, the shadow of a curve may have more, or fewer, infinite branches than the curve itself. Now the fewer infinite branches a curve has, the easier it is to get a general idea of its shape; thus, to return to our first example, the shape of an ellipse is easier to grasp and remember than that of a hyperbola. The object of the method of Projection is to substitute for any given curve that shadow of it which has the least possible number of infinite branches; and thus to classify the enormous variety of shapes as the shadow of a comparatively small number of simple forms.

[vi] *The One-Two Correspondence of Two Planes.*

1. I consider two planes X and Ξ, the points of which are related in such a way that to every point x of X correspond in general two points ξ and ξ' of Ξ, while to every point ξ of Ξ corresponds in general one and only one point x of X.

2. Let the correspondent of a line L in X be a curve Γ_m in Ξ. I shall now prove that the correspondent of a line Λ in Ξ is a curve C_m in X; the curves C_m and Γ_m being both of the same order m. The curves Γ_m must be such that two of them intersect in two points (variable), and that only one can be drawn through two assigned points. Γ_m meets Λ in m points, and to each of these corresponds *one* point in X, m is therefore the number of intersections of L and the correspondent of Λ, which consequently is of order m as asserted.

3. The two representatives on Ξ of a point on X have between themselves a (1, 1) correspondence; let this be of order μ. Then the entire correspondent of C_m is not merely the line Λ but besides a curve Δ_μ which is unicursal and passes through a set of Cremona's principal points. The intersections of C_m, C'_m are represented then by (1) the point $\Lambda\Lambda'$; (2) the point Δ_μ, Δ'_μ; (3) the 2μ points Λ, Δ'_μ and Λ', Δ_μ—in all $2+2\mu$ points. So then two curves, C_m, C'_m intersect in $\overline{\mu+1}$ variable points. They are unicursal, as corresponding uniquely to straight lines. And through two arbitrary points ab four of them can be drawn; viz., the representatives of the lines $\alpha\beta$, $\alpha\beta'$, $\alpha'\beta$, $\alpha'\beta'$.

4. The curves Γ_m pass through certain fixed points. To single points correspond lines in X, to double ones conics, &c., exactly as in Cremona's theory. These curves are all unicursal. But to common points of the C_m may correspond curves which are not unicursal.

5. The case $m=2$ is of little interest and easily studied. The Γ_2 are conics through two fixed points, with a linear relation; their Jacobian is then a conic K_2 and a line Λ, both passing through the two fixed points. μ is equal to 2, and the Δ_2 are conics through the fixed points and the pole of Λ in respect of K_2. The C_2 are conics through a fixed point correspondent of Λ having double contact with a fixed conic, the correspondent of K_2. To the fixed points on Ξ correspond the tangents from the point to the conic on X.

6. The case $m=3$ gives us two systems: (1) cubics with a node and three points fixed, satisfying besides a linear relation; (2) cubics with seven fixed points. The first case is obtained by quadric transformation from the case $m=2$, whereby we see that the Jacobian is a cubic of the system (not satisfying the linear relation) and the straight lines joining the node to the fixed points. $\mu=3$, and the Δ_3 are cubics through the fixed points and another. The C_3 are cubics having triple contact with a fixed conic and a node at a certain fixed point.

The second system clearly inverts into itself. The Jacobian is a sextic S_6 with seven nodes at the fixed points. The correspondent of this is a curve, N_{12}.

[vii] *On the* (1, n) *Correspondence of Two Planes.*

1. I consider planes X, Ξ so related that to every point x on the first plane correspond in general n points, $\xi^1, \xi^2 \ldots \xi^n$, on the second, while to every point ξ of the second plane corresponds in general only one point x of the first.

2. Let the correspondent of a line L in X be a curve Γ_m in Ξ. I shall now prove that the correspondent of a line Λ in Ξ is a curve C_m in X; the curves C_m and Γ_m being both of the same order m. The curves Γ_m must be such that two of them intersect in n variable points, and that only one can in general be drawn through two assigned points. Γ_m meets Λ in m points, and to each of these corresponds *one* point in X; m is therefore the number of intersections of L and the correspondent of Λ, which consequently is of order m as asserted.

3. Two representatives on Ξ of the same point on X have between them a $(n-1, n-1)$ correspondence; let this be of order μ. Then the entire correspondent of C_m is not merely the line Λ, but besides a curve Δ_μ. Two curves Δ_μ, Δ'_μ intersect in $n-1$ variable points. The intersections of C_m, C'_m are represented then by (1) the point $\Lambda\Lambda'$; (2) the $\overline{n-1}$ points Δ_μ, Δ'_μ; (3) the 2μ points Λ, Δ'_μ and Λ', Δ_μ—in all $n+2\mu$ points. Thus the number of variable intersections of two curves C_m, C'_m is $1+2\frac{\mu}{n}$, and consequently (except when $n=2$), μ must be a multiple of n. The curves C_m are unicursal, as corresponding uniquely to straight lines. And through two arbitrary points a, b there may be drawn n^2 of them; viz., those corresponding to the n^2 lines joining the n correspondents of a, $\alpha^1, \alpha^2 \ldots \alpha^n$ to the n correspondents of b, $\beta^1, \beta^2 \ldots \beta^n$.

[viii] *Geometry on a Cubic Surface.*

1. A system of values of the coordinates $X^{(1)}, X^{(2)}, X^{(3)}$ in general determines one point on the surface. There are however six systems of values which determine not points but loci (viz. one-half of a double-sixer). These systems are to be denoted by $a^{(1)}, a^{(2)} \ldots a^{(6)}$, and called the Extension-Absolute (as distinguished from the Measure-Absolute, which is *another* set of six values).

2. Let $K_2^{(1)}, K_2^{(2)}, K_2^{(3)}$ be quadric functions of the X which are satisfied by the systems $a^{(i)}, a^{(j)}\,a^{(k)}$. In general, a system of ratios of the K_2 determines one point on the surface; but the system $K_2^{(1)}=0, K_2^{(2)}=0, K_2^{(3)}=0$ has a meaning, viz. the loci $a^{(i)}, a^{(j)}, a^{(k)}$.

Three systems of values of the K_2 (viz. the other three points a) now represent loci and not points. But besides these there are three other systems which represent the *lines* $a^{(j)}a^{(k)}, a^{(k)}a^{(i)}, a^{(i)}a^{(j)}$. I call these six value-systems $k^{(1)}...k^{(6)}$. The K-coordinates then are precisely equivalent in their properties to the X, except that they vanish all together on the locus $a^{(i)}, a^{(j)}, a^{(k)}$.

But now the *point* $a^{(i)}$ may be approached in various ways; and it will be found that though the K_2 all vanish at that *point*, yet the ratios in which they vanish depend on the direction in which the *point* is approached. In fact, the vanishing ratios of the K_2 enable us to discriminate between the points of the *locus* $a^{(i)}$. We may then, if we like, consider only their ratios, deprive the system $K_2 = 0$ of its meaning, and so arrive at a new coordinate-system exactly similar to the one with which we started.

In fact, if $A^{(i)}$ denote $a^{(j)}a^{(k)}$, etc. we can find quadric functions of the K_2 respectively equal to $X^{(1)}A^{(i)}A^{(j)}A^{(k)}, X^{(2)}A^{(i)}A^{(j)}A^{(k)}, X^{(3)}A^{(i)}A^{(j)}A^{(k)}$.

Here it is clear that $A^{(i)} = 0$ makes these three simultaneously vanish, but their ratios are those of the X.

3. I now take three cubic functions $C_3^{(1)}\,C_3^{(2)}\,C_3^{(3)}$ of the X, which are all satisfied doubly by $a^{(i)}$ and singly by all the other points a except $a^{(j)}$. A system of ratios of the C_3 will in general determine *one* point of the surface; but the system $C_3^{(1)} = 0, C_3^{(2)} = 0, C_3^{(3)} = 0$ has a meaning; viz. the locus $a^{(i)}$ twice, and the other loci a except $a^{(j)}$ once.

The system of values $a^{(j)}$ of the C_3 now represents a locus and not a point. But there are systems of values which represent (1) the lines joining $a^{(i)}$ to the other a except $a^{(j)}$ and (2) the conic passing through all the a but $a^{(j)}$. I call these six values $c^{(1)}...c^{(6)}$. The C_3-coordinates are thus again equivalent to the former systems, with the exception that they all vanish together on the locus $\Sigma a - a^{(j)}$.

The ratios of the C_3 in the immediate neighbourhood of the point $a^{(i)}$ will be found to acquire *two* distinct value-systems for every direction of approach. The locus $a^{(i)}$ is therefore represented by a quadric equation in the C_3. The remaining loci a are represented by linear equations as before.

4. Similar remarks are to be made on quartic functions $Q_4^{(1)}, Q_4^{(2)}, Q_4^{(3)}$ having nodes at $a^{(i)}, a^{(j)}, a^{(k)}$ and single points at the other three. They remove entirely the pre-existing absolute, and substitute a new one, $q^{(1)}...q^{(6)}$ consisting of Cremona's well-known principal system. So further the quintics $P_5^{(1)}, P_5^{(2)}, P_5^{(3)}$ which have a node at each a, substitute a new absolute representing the six conics each through five a points. Altogether we have $1 + 20 + 30 + 20 + 1 = 72$ equivalent linear systems.

Plane sections of cubic skew cubics	cubics through 6 points a straight lines on plane
skew cubics generally	(1) straight lines
	(2) conics through 3 points a
	(3) cubics with node at a_1, passing through a_2, a_3, a_4, a_5
	(4) quartics with nodes at a_1, a_2, a_3, and single points a_4, a_5, a_6
	(5) quintics with a node at each point a.

Any one of Clebsch's 72 nets of skew cubics may be taken for a linear system on the cubic surface. The net cuts twice 6 lines on the cubic; the other half of the double-six forms the absolute. Analytically there are six sets of ratios $X : Y : Z$ which mean not points but these six lines; I call these quasi-points (a, b, c, f, g, h). A *line* $lX + mY + nZ$ which satisfies the condition of passing through one of these say, a, breaks up into the line a and a conic through a', the corresponding line of the double-six. The *line ab* contains these two lines and the line which meets ab', $a'b$. Abstractions made of the absolute, then, the *lines ab*... are the other 15 st. lines of the cubic. The lines a'... are represented by the *conics* through each five of the points.

When the surface has a node, the linear system is that of the plane sections through the node. The points $a\,b\,c\,f\,g\,h$ are then upon one conic.

$$K_2^{(1)} \equiv X^{(2)}X^{(3)} \text{ the point } K_2^{(2)} = 0,\quad K_2^{(3)} \equiv 0 \text{ gives merely } X^{(1)} = 0$$
$$K_2^{(2)} \equiv X^{(3)}X^{(1)} \quad \text{,,} \quad K_2^{(3)} = 0,\quad K_2^{(1)} \equiv 0 \quad \text{,,} \quad X^{(2)} = 0$$
$$K_2^{(3)} \equiv X^{(1)}X^{(2)} \quad \text{,,} \quad K_2^{(1)} = 0,\quad K_2^{(2)} \equiv 0 \quad \text{,,} \quad X^{(3)} = 0$$

[ix] *On the Correspondence between a Doubled Line and a Cubic.*

If I take a fixed point o on a cubic curve C_3, I can draw through it lines which unicursally correspond to the points on a straight line X. I have then for every point x on the line a pair of points c, c' on the cubic. But now I may suppose this line to be doubled, call it XX', and then the two (coincident) points xx' may be held either to correspond as a whole to the pair cc', or x may be held to correspond to c, and x' to c'. I shall now take instead of the doubled *line* XX', a doubled *circle*, so as to have the whole of it within reach. But now from the fixed point o four tangents can be drawn to the cubic; let us assume to begin with that they are all real. Then (calling them $ABCD$) if between A and B the points cc' are real, they must also be real between C and D, but imaginary between B and C and between D and A. Thus the *real* part of the cubic may be represented by two portions of a doubled circle*.

We have in fact substituted for the cubic an anallagmatic quartic on the point of becoming a doubled circle. If two of the tangents are imaginary we

* [There are three figures, (3) is a complete ring, (1) consists of two parts (AB), (CD) of the same ring, and (2) is a part of a ring, semicircular (AD).]

may suppose B and C to coincide, and the curve to be represented by *one* very thin oval; if they are all imaginary, the curve is represented by the entire contour of two indefinitely near circles.

(MS. ends.)

[x] *Syllabus of Lectures**.

General Principle—all the properties of a geometric form depend on its Order. Hence begin by establishing theory of imaginaries, on which that of **History** the order depends. Thus, 1. Historique. 2. Fundamental Hypotheses. Con- **Space** tinuity of Space. Motion without change of size. Infinite extent. Definitions of Line and Plane, perpendicular and parallel. 3. Calculus of Ratios and Position. Primary theorem of number. Deduction of arithmetical rules. Opera- **Quantity** tions. Algebraic calculus of numbers. Primary theorem of continuous quantity. Rules analogous to arithmetical operations. Algebraic calculus of ratios. Calculus of Position in one dimension. 4. Position in two dimensions. Gaussian Formulæ. Discontinuity of the reversor symbol, is made continuous by calculus of position in two dimensions. Idea of functional correspondence, monodromie, **Imaginaries** similarity of the smallest parts. Every equation of nth order has exactly n roots. 5. Position in two dimensions. Cartesian formulæ. Co-ordinates of a point, of a straight line, distance between given points, angle between given lines. **Equations** Equation of straight line, of circle, ellipse, hyperbola, parabola.

6. Projection and Linear Transformation. Passage to homogeneous form. Line at infinity. Projection equivalent to Linear Transformation. Order of curve. Number of points in which it is met by *any* line whatever. General **Invariance** notion of invariant. Invariants of points and lines. Grassmann notation for these. Aronhold notation for invariants in general. Duality, contravariant symbols, contravariant differentiation.

Correspondence 7. Correspondence in one dimension. (1,1) correspondence, anharmonics, harmonics, involution. Harmonics and involution of higher orders.

9. Plücker's equations, the Deficiency. Correspondence on a curve.

10. General theory of Polars. Special application to conics. Conjugirte Kerncurven.

[xi] *Geometric Analysis.*

1. The calculus of Ratios, and of onefold Position.
2. The calculus of twofold Position: (1) the Cartesian; (2) Gauss' plane of numbers.
3. The simpler Cartesian formulæ for geometrical magnitudes.
4. Equation in general. Forms of equation of straight line.
5. Equations of circle and conic sections.

* These are evidently the notes of the lectures printed above (pp. 524—530), and I give them as they show what were the subjects of the missing articles on p. 528.

6. Equation of a curve of any order.
7. Passage to homogeneous co-ordinates. The co-ordinates of a geometrical form in general.
8. The Grassmann notation.
9. The imaginary in geometry.
10. Systematic geometry of one dimension. Harmonics and Anharmonics.
11. The Polar Theory of conics.
12. The Bitangent circles of conics [pp. 543—5].
13. General Theory of the circle: powers of circles.
14. Theory of anallagmatic curves [pp. 546—555].
15. Extension of the Grassmann notation. General theory of distance.
16. Plücker's Equations. The Deficiency.
17. Polar theory of curves of the nth order.
18. General theorems relating to cubics.
19. The Polar theory of cubics.
20. Passage from the extended Grassmann notation to the symbolic form of covariants.

[xii]

The following are the 'heads' of two of Clifford's lectures on Quaternions (pp. 478—515). They will show how little of *written* prepared matter (L. and E. Vol. I. p. 8) he took to his lectures.

Whole numbers; scale of them. Steps of addition and subtraction; sum of any number of steps independent of their order.

Multiplication of numbers; in $2\times3=6$, 2 is an *operator*. Other interpretation; 2 and 3 are both operators; then 6 is one also.

Multiplication of steps; $2\times(-3)=-6$; only one interpretation as yet.

Retention and reversal of step; symbols k, r;

$$k2\times(-3)=-6,\ r2\times(-3)=+6.$$

Product of operations on steps; $k2\times r3=r6$, $r2\times r3=k6$. Product independent of order of factors.

Analogy with multiplication of steps; $+$ and $-$ used for k and r; double meaning of $+$ and $-$.

Quantities; all continuous quantities *must* be specified by lines or angles, and angles are conveniently specified by lines. Scale of quantities on straight line.

Ratios of quantities. Ratio as *operator*. Product of ratios. Double meaning of equation $ab=c$.

Ancient ideas about product of quantities. Arabic solution of quadratic equations. Cardan and Tartaglia. Vieta's scale of dimensions. Introduction of ratios by Descartes. His non-recognition of negative quantities. Geometrical view of product.

Addition and subtraction of quantities. Steps on straight line. Ratio of steps. Signed or *scalar* numbers. Hamilton's view of algebra as science of pure time. Reference to geometry and kinematics implied in all the ordinary algebra.

Steps in plane and space. Addition, subtraction. Multiplication by scalars.

Applications. Theory of mass-centre. Equations of uniform and parabolic motion. Mass-centre of number of falling bodies.

Flux of a vector. Flux of product of vector and scalar. Hodograph. Acceleration. Curvature. Tangential and normal acceleration.

Ratio of steps in one plane. Tensor × versor. Scalar + rectangular versor.

Distributive, associative, commutative laws.

Complex numbers. Exponential. Meaning of $e^{i\theta} = \cos\theta + i\sin\theta$.

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