[Among Clifford's Papers three *Cahiers* have been found relating to the theory of Elliptic functions. The most complete of these is entitled "Algebraic Introduction to Elliptic Functions;" the second is "A Tract on Elliptic Functions;" the third is without any title. In all three, the elliptic functions are treated according to what may be termed the second method of Jacobi; viz. the properties of the theta functions are first investigated and the properties of the elliptic functions are deduced from them. The three cahiers appear to have been intended either as notes for a course of Lectures on Elliptic functions, or as drafts for a treatise. They contain no new results, and perhaps no original methods of investigation. But as the mode of treatment adopted by Clifford is not employed in any English treatise, and as information respecting it would have to be sought by the student in scattered original memoirs, it has been thought advisable to print, in this collection, nearly the whole of the "Algebraic Introduction," and one section of the "Tract." It is unnecessary to say that Clifford is never a mere copyist, and that these fragments possess an independent value of their own, even when they relate to elementary parts of the subject. H. J. S. S.]

ALGEBRAIC INTRODUCTION TO ELLIPTIC FUNCTIONS.

I.

Definitions and Elementary Properties of the Theta Functions.

The geometric series $a+ar+ar^2+\ldots$ may be easily expressed as a sum of exponentials. If $\log a=\alpha$, $\log r=\beta$, the series in fact becomes

$$e^{\alpha}+e^{\alpha+\beta}+e^{\alpha+2\beta}+\ldots$$

in which the general term is $e^{\alpha+n\beta}$; and we may then conveniently write it $\Sigma e^{\alpha+n\beta}$. Here the exponent of the n^{th} term is of the first order in n. It seems natural to extend the conception of this series by considering exponents of the second and higher orders in n, which lead to the series

$$\Sigma e^{\alpha+n\beta+n^2\gamma},\ \Sigma e^{\alpha+n\beta+n^2\gamma+n^3\delta},\ \text{etc.}$$

We shall in fact now occupy ourselves with the series $\Sigma e^{\alpha+n\beta+n^2\gamma}$ which is called a Theta-series.

All these series may be regarded as extending in both directions; i.e. n may have all integer values both positive and negative; but there is an important difference between the cases in which the exponent is of an odd order in n and those in which it is of even order. The geometric series for example may be written

$$\ldots+e^{\alpha-3\beta}+e^{\alpha-2\beta}+e^{\alpha-\beta}+e^{\alpha}+e^{\alpha+\beta}+e^{\alpha+2\beta}+e^{\alpha+3\beta}+\ldots$$

but, as is well known, the series is always convergent towards one end and divergent towards the other, unless $\beta=0$. But the series $\Sigma e^{\alpha+n\beta+n^2\gamma}$ is always convergent towards both ends or divergent towards both ends, according as the real part of γ is negative or positive. And a similar thing is true of all those series in which the exponent is of even order in n.

For simplicity we shall suppose $\alpha=0$, which is equivalent to dividing the whole series by e^{α}. We shall also now use the symbols $2u$, a instead of β and γ. This being so, *the series*

$$\sum_{n=-\infty}^{n=+\infty} e^{n^2a+2nu} = \ldots + e^{9a-6u} + e^{4a-4u} + e^{a-2u} + 1 + e^{a+2u} + e^{4a+4u} + e^{9a+6u} + \ldots$$

$$= 1 + e^{a}(e^{2u}+e^{-2u}) + e^{4a}(e^{4u}+e^{-4u}) + e^{9a}(e^{6u}+e^{-6u}) + \ldots$$

will be called $\Theta(u, a)$.

Another form, differing from this in being multiplied by an exponential, is sometimes useful; namely

$$G(U, A) = \sum_{n=-\infty}^{n=+\infty} e^{(nA+U)^2} = e^{U^2}\,.\,\Sigma e^{n^2A^2+2nAU} = e^{U^2}\,.\,\Theta(AU, A^2).$$

Since $e^{2\pi i}=1$, we shall leave unaltered every term in the series $\Theta(u, a)$ if we increase or diminish u by any multiple of πi; that is to say,

$$\Theta(u+p\pi i, a) = \Theta(u, a).$$

Again, in the summation Σe^{n^2a+2nu} the whole number n takes all integer values + and −; we shall therefore get exactly the same series if we write $n+q$ instead of n. That is

$$\Theta(u, a) = \Sigma e^{(n+q)^2a+2(n+q)u} = e^{q^2a+2qu}\,\Sigma e^{n^2a+2n(u+qa)}.$$

But the Σ here is precisely $\Theta(u+qa)$. Hence we have

$$\Theta(u, a) = e^{q^2a+2qu}\,\Theta(u+qa, a) \text{ or } \Theta(u+qa, a) = e^{-q^2a-2qu}\,\Theta(u, a).$$

These results may be stated as follows. If in the function $\Theta(u, a)$ we increase the argument u by any multiple of πi, the function is unaltered; but if we increase the argument by any multiple qa of a, it is altered by being multiplied into the factor e^{-q^2a-2qu}. The function Θ is accordingly said to have the *period* πi and the *quasi-period* a.

Similar properties belong to the function $G(u, a) = \Sigma e^{(na+u)^2}$. It is obvious that to increase u by qa is the same thing as to increase n by q, and therefore this operation leaves the function unaltered, or

$$G(u+qa, a) = G(u, a),$$

and the function G has the period a. But it also has a quasi-period; for

$$G(u+b, a) = \Sigma e^{(na+u+b)^2} = e^{2ub+b^2}\,\Sigma e^{2nab}\,e^{(na+u)^2}$$

$$= e^{2ub+b^2}\,G(u, a),$$

provided that $e^{2ab}=1$, or, which is the same thing, that ab is a multiple of πi. We shall in future write b for $\frac{\pi i}{a}$, and we then have the theorem

$$G(u+pb, a) = e^{2upb+p^2b^2}\,G(u, a),$$

or the function G has the quasi-period b.

Other forms of these functions are obtained by adding to the argument either half the period, or half the quasi-period, or both. For the Θ function we have by adding half the period πi

$$\Theta(u+\tfrac{1}{2}\pi i, a) = 1 - e^{a}(e^{2u}+e^{-2u}) + e^{4a}(e^{4u}+e^{-4u}) - e^{9a}(e^{6u}+e^{-6u}) + \text{etc.}$$

the terms being alternately + and −, instead of all + as before. This is distinguished as $\Theta'(u, a)$; viz., we have $\Theta(u+\frac{1}{2}\pi i, a) = \Theta'(u, a)$ and moreover

$$\Theta'(u+\tfrac{1}{2}\pi i, a) = \Theta(u, a).$$

Next, adding half the quasi-period, we find

$$\Theta(u+\tfrac{1}{2}a, a) = \Sigma e^{n^2a+2nu+na} = e^{-u-\frac{a}{4}}\,\Sigma e^{(n+\frac{1}{2})^2a+2(n+\frac{1}{2})u}$$

$$= e^{-u-\frac{a}{4}}\{e^{\frac{1}{4}a}(e^{u}+e^{-u}) + e^{\frac{9}{4}a}(e^{3u}+e^{-3u}) + e^{\frac{25}{4}a}(e^{5u}+e^{-5u}) + \ldots\}.$$

The series in the brackets is distinguished as $\Theta_1(u, a)$, so that we have

$$\Theta(u+\tfrac{1}{2}a, a) = e^{-u-\frac{a}{4}}\Theta_1(u, a),\quad \Theta_1(u+\tfrac{1}{2}a, a) = e^{-u-\frac{a}{4}}\Theta(u, a).$$

We now calculate the result of adding $\frac{1}{2}\pi i + \frac{1}{2}a$ to u. It is

$$\Theta(u+\tfrac{1}{2}\pi i+\tfrac{1}{2}a) = e^{-u-\frac{a}{4}}\{e^{\frac{1}{4}a}(e^{u}-e^{-u}) - e^{\frac{9}{4}a}(e^{3u}-e^{-3u}) + e^{\frac{25}{4}a}(e^{5u}-e^{-5u}) - \ldots\}$$

(since $e^{\frac{1}{2}\pi i}=i$, $e^{-\frac{1}{2}\pi i}=-i$). The series in the brackets is distinguished as $\Theta_1'(u, a)$; thus

$$\Theta(u+\tfrac{1}{2}\pi i+\tfrac{1}{2}a) = e^{-u-\frac{a}{4}}\,\Theta_1'(u, a),\quad \Theta_1'(u+\tfrac{1}{2}\pi i+\tfrac{1}{2}a) = ie^{-u-\frac{a}{4}}\,\Theta(u, a).$$

It will be observed that this last function vanishes when $u=0$ and changes sign with u.

For most physical applications it is convenient to convert these series into a trigonometrical form, by the substitution of $+ix$ for u. These then regarded as functions of x will be denoted by a small ϑ instead of a large one. Namely we shall write (q standing for e^{a}),

$$\vartheta(x, a) = \Theta(ix, a) = 1 + 2q\cos 2x + 2q^4\cos 4x + 2q^9\cos 6x + \ldots$$

$$\vartheta'(x, a) = \Theta'(ix, a) = 1 - 2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + \ldots$$

$$\vartheta_1(x, a) = \Theta_1(ix, a) = 2q^{\frac{1}{4}}\cos x + 2q^{\frac{9}{4}}\cos 3x + 2q^{\frac{25}{4}}\cos 5x + \ldots$$

$$i\vartheta_1'(x, a) = \Theta_1'(ix, a) = i\{2q^{\frac{1}{4}}\sin x - 2q^{\frac{9}{4}}\sin 3x + 2q^{\frac{25}{4}}\sin 5x - \ldots\}.$$

The three latter functions differ from ϑ in the matter of the period and quasi-period. ϑ has the period π and the quasi-period ia; all have the period 2π and are multiplied by an exponential factor when the argument is increased by $2ia$. But ϑ_1 and ϑ_1' change sign when the argument is increased by π, and ϑ' and ϑ_1' when it is increased by ia. These changes are indicated in the following table: (m, n any two positive integers),

$$\vartheta(x+m\pi+nai) = \qquad e^{-n^2a+2nxi}\vartheta(x),$$

$$\vartheta'(x+m\pi+nai) = (-)^n\, e^{-n^2a+2nxi}\vartheta'(x),$$

$$\vartheta_1(x+m\pi+nai) = (-)^m\, e^{-n^2a+2nxi}\vartheta_1(x),$$

$$\vartheta_1'(x+m\pi+nai) = (-)^{m+n}\, e^{-n^2a+2nxi}\vartheta_1'(x).$$

The quantities $\frac{1}{2}\pi$, $\frac{1}{2}ai$, $\frac{1}{2}\pi+\frac{1}{2}ai$, are conveniently called *quadrants*. When we increase the arguments of the ϑ by quadrants, they pass into one another according to the following table:—

$\vartheta(x+\frac{1}{2}\pi)=\vartheta' x$	$\vartheta(x+\frac{1}{2}ai)=e^{ix-\frac{a}{4}}.\vartheta_1 x$	$\vartheta(x+\frac{1}{2}\pi+\frac{1}{2}ai)=\ e^{ix-\frac{a}{4}}i\vartheta_1' x$
$\vartheta'(x+\frac{1}{2}\pi)=\vartheta x$	$\vartheta'(x+\frac{1}{2}ai)=e^{ix-\frac{a}{4}}.i\vartheta_1' x$	$\vartheta'(x+\frac{1}{2}\pi+\frac{1}{2}ai)=\ e^{ix-\frac{a}{4}}\vartheta_1 x$
$\vartheta_1(x+\frac{1}{2}\pi)=i\,.\,i\vartheta_1' x$	$\vartheta_1(x+\frac{1}{2}ai)=e^{ix-\frac{a}{4}}.\vartheta x$	$\vartheta_1(x+\frac{1}{2}\pi+\frac{1}{2}ai)=i\,.\,e^{ix-\frac{a}{4}}\vartheta' x$
$i\vartheta_1'(x+\frac{1}{2}\pi)=i\,.\,\vartheta_1 x$	$i\vartheta_1'(x+\frac{1}{2}ai)=e^{ix-\frac{a}{4}}.\vartheta' x$	$i\vartheta_1'(x+\frac{1}{2}\pi+\frac{1}{2}ai)=i\,.\,e^{ix-\frac{a}{4}}\vartheta x$

where it will be observed that the factor i has been restored to ϑ_1' for the sake of symmetry; the rule is then that the addition of $\frac{1}{2}\pi$ multiplies ϑ_1 and $i\vartheta_1'$ by i (besides the transformation) and the addition of $\frac{1}{2}ai$ multiplies all the ϑ by $e^{ix-\frac{a}{4}}$.

Putting these results together we obtain the following table:—(Königsberger)

Increase of argument.	ϑ	ϑ'	ϑ_1	$i\vartheta_1'$	Exponential factor.
$m\pi+nai$	ϑ	$(-)^n\vartheta'$	$(-)^m\vartheta_1$	$(-)^{m+n}i\vartheta_1'$	$e^{-n^2a+2nxi}$
$(m+\frac{1}{2})\pi+nai$	$(-)^n\vartheta'$	ϑ	$i(-)^{m+n}i\vartheta_1'$	$i(-)^m\vartheta_1$	
$m\pi+(n+\frac{1}{2})ai$	ϑ_1	$(-)^n i\vartheta_1'$	$(-)^m\vartheta$	$(-)^{m+n}\vartheta'$	$e^{-(n+\frac{1}{2})^2a+2(n+\frac{1}{2})xi}$
$(m+\frac{1}{2})\pi+(n+\frac{1}{2})ai$	$(-)^n i\vartheta_1'$	ϑ_1	$i(-)^{m+n}\vartheta'$	$i(-)^{m+n}\vartheta$	

II.

Product of two Theta Functions—Differential formulæ—Introduction of the Elliptic functions.

We shall now return for simplicity to the exponential form Θ, and establish two theorems of great importance.

To prove that

(i) $\Theta u\,.\,\Theta v=\Theta(u+v,\,2a)\,\Theta(u-v,\,2a)+\Theta_1(u+v,\,2a)\,\Theta_1(u-v,\,2a)$,

we have $\Theta u=\Sigma e^{n^2a+2nu}$, $\Theta v=\Sigma e^{m^2a+2mv}$, where n, m take all integer values. Let us multiply these series together term by term; the result will be

$$\Sigma\Sigma e^{(m^2+n^2)a+2nu+2mv},$$

where a double summation has to be effected, namely in regard to m and in regard to n. But now if we write $m+n=\mu$, $m-n=\nu$, this gives

$$2(m^2+n^2)=\mu^2+\nu^2,$$

and we find

$$\Theta u\,.\,\Theta v=\Sigma\Sigma e^{\frac{1}{2}(\mu^2+\nu^2)a+\mu(v+u)+\nu(v-u)},$$

where μ, ν must not take all values independently, but must be either both even or both odd, because they are the sum and difference of two numbers. Putting

them first equal to $2p$, $2q$, and then to $2p+1$, $2q+1$ we obtain two parts of the sum, which are respectively

$$\Sigma\Sigma e^{2(p^2+q^2)a+2p(v+u)+2q(v-u)}=\Sigma e^{2p^2a+2p(v+u)}\times\Sigma e^{2q^2a+2q(v-u)}$$

and

$$\Sigma\Sigma e^{2(\overline{p+\frac{1}{2}}^2+\overline{q+\frac{1}{2}}^2)a+2\overline{p+\frac{1}{2}}(v+u)+2\overline{q+\frac{1}{2}}(v-u)}$$

$$=\Sigma e^{2\overline{p+\frac{1}{2}}^2a+2\overline{p+\frac{1}{2}}\,\overline{v+u}}\times\Sigma e^{2\overline{q+\frac{1}{2}}^2a+2\overline{q+\frac{1}{2}}\,\overline{v-u}}.$$

But the products on the right are simply

$$\Theta(u+v,\,2a)\,\Theta(u-v,\,2a)\text{ and }\Theta_1(u+v,\,2a)\,\Theta_1(u-v,\,2a)$$

and therefore

$$\Theta u\,.\,\Theta v=\Theta(u+v,\,2a)\,\Theta(u-v,\,2a)+\Theta_1(u+v,\,2a)\,\Theta_1(u-v,\,2a)$$

as was to be proved.

We shall apply this theorem to obtain formulæ relating to the squares and the products of the Θ. Writing $v=u$, and then $u=0$, we have

$$\Theta^2u=\Theta(2u,\,2a)\,\Theta(0,\,2a)+\Theta_1(2u,\,2a)\,\Theta_1(0,\,2a)\text{ and }\Theta^2 0=\Theta^2(0,\,2a)+\Theta_1^2(0,\,2a),$$

and now adding to u successively $\frac{1}{2}\pi i$, $\frac{1}{2}a$, $\frac{1}{2}\pi i+\frac{1}{2}a$, we get

$$\Theta'^2u=\Theta(2u,\,2a)\,\Theta(0,\,2a)-\Theta_1(2u,\,2a)\,\Theta_1(0,\,2a),\qquad \Theta'^2 0=\Theta^2(0,\,2a)-\Theta_1^2(0,\,2a),$$

$$\Theta_1^2u=\Theta_1(2u,\,2a)\,\Theta(0,\,2a)+\Theta(2u,\,2a)\,\Theta_1(0,\,2a),\qquad \Theta_1^2 0=2\Theta(0,\,2a)\,\Theta_1(0,\,2a),$$

$$\Theta_1'^2u=\Theta_1(2u,\,2a)\,\Theta(0,\,2a)-\Theta(2u,\,2a)\,\Theta_1(0,\,2a).$$

From these we obtain the following important equations (Θ_1^2 written for $\Theta_1^2(0)$, etc.)

$$\Theta_1^2.\,\Theta'^2u+\Theta^2\,.\,\Theta_1'^2u=\Theta'^2.\,\Theta_1^2u,$$

$$\Theta_1^2.\,\Theta_1'^2u+\Theta^2\,.\,\Theta'^2u=\Theta'^2.\,\Theta^2u,$$

$$\Theta'^2.\,\Theta'^2u+\Theta_1^2.\,\Theta_1^2u=\Theta^2\,.\,\Theta^2u.$$

And, either by putting $u=0$ in the last, or directly from the equation on the right, we get

$$\Theta'^4+\Theta_1^4=\Theta^4.$$

It is convenient to write k for $\frac{\Theta_1^2}{\Theta^2}$; if $k^2+k'^2=1$, the last equation shews us that $k'=\frac{\Theta'^2}{\Theta^2}$.

To obtain expressions for the products of the Θ, substitute $u+\frac{1}{2}\pi i$ for v in the formula above; thus we find

$$\Theta u\,.\,\Theta' u=\Theta'(2u,\,2a)\,\Theta'(0,\,2a),\text{ since }\Theta_1'(0,\,2a)=0.$$

Next put $v=u+\frac{1}{2}a$; then

$$e^{-u-\frac{a}{4}}\Theta u\,.\,\Theta_1 u=\Theta(2u+\tfrac{1}{2}a,\,2a)\,\Theta(\tfrac{1}{2}a,\,2a)+\Theta_1(2u+\tfrac{1}{2}a,\,2a)\,\Theta_1(\tfrac{1}{2}a,\,2a).$$

Lastly put $v=u+\frac{1}{2}i\pi+\frac{1}{2}a$; thus

$$e^{-u-\frac{a}{4}}\Theta u\,.\,\Theta_1' u=\Theta'(2u+\tfrac{1}{2}a,\,2a)\,\Theta'(\tfrac{1}{2}a,\,2a)-\Theta_1'(2u+\tfrac{1}{2}a,\,2a)\,\Theta_1'(\tfrac{1}{2}a,\,2a).$$

From these three formulæ we may derive three others by increasing the argument in each by $\frac{1}{2}a$ or $\frac{1}{2}\pi i$. Thus we obtain

$$\Theta_1 u\,.\,\Theta_1' u = \Theta_1'(2u, 2a)\,\Theta'(0, 2a),$$

$$e^{-u-\frac{a}{4}}\,\Theta' u\,.\,\Theta_1' u = \Theta(2u+\tfrac{1}{2}a, 2a)\,\Theta(\tfrac{1}{2}a, 2a) - \Theta_1(2u+\tfrac{1}{2}a, 2a)\,\Theta_1(\tfrac{1}{2}a, 2a),$$

$$e^{-u-\frac{a}{4}}\,\Theta' u\,.\,\Theta_1 u = \Theta'(2u+\tfrac{1}{2}a, 2a)\,\Theta'(\tfrac{1}{2}a, 2a) + \Theta_1'(2u+\tfrac{1}{2}a, 2a)\,\Theta_1'(\tfrac{1}{2}a, 2a).$$

But we have clearly $\Theta(\frac{1}{2}a, 2a) = \Theta_1(\frac{1}{2}a, 2a)$ and $\Theta'(\frac{1}{2}a, 2a) = -\Theta_1'(\frac{1}{2}a, 2a)$; whence, putting $u=0$ in the first, second and sixth of the formulæ just written down, we get $\Theta\Theta' = \Theta'^2(0, 2a)$; $\Theta\Theta_1 = 2e^{\frac{1}{4}a}\,.\,\Theta^2(\frac{1}{2}a, 2a)$; $\Theta'\Theta_1 = 2e^{\frac{1}{4}a}\,.\,\Theta'^2(\frac{1}{2}a, 2a)$.

Precisely as the formula for $\Theta u\,.\,\Theta v$ was proved, the following may be demonstrated :—

(i) $\Theta u\,.\,\dot\Theta v - \dot\Theta u\,.\,\Theta v = 2\dot\Theta(v-u, 2a)\,\Theta(v+u, 2a) + 2\dot\Theta_1(v-u, 2a)\,\Theta_1(v+u, 2a)$

and from this, by giving special values to u and v, we may derive the following:

$$\dot\Theta u\,.\,\Theta' u - \Theta u\,.\,\dot\Theta' u = 2\dot\Theta_1'(0, 2a)\,\Theta_1'(2u, 2a), = \frac{2\dot\Theta_1'(0, 2a)}{\Theta'(0, 2a)}\,\Theta_1 u\,.\,\Theta_1' u.$$

For the transformation of this formula it is necessary to consider the values of the fluxions of Θ for special values of the argument. We have clearly

$$\dot\Theta(u+\tfrac{1}{2}\pi i) = \dot\Theta' u,\quad \dot\Theta'(u+\tfrac{1}{2}\pi i) = \dot\Theta u,\quad \dot\Theta_1(u+\tfrac{1}{2}\pi i) = i\,.\,\dot\Theta_1'(u),\quad \dot\Theta_1'(u+\tfrac{1}{2}\pi i) = i\,.\,\dot\Theta' u,$$

and therefore $\dot\Theta_1(\frac{1}{2}\pi i) = i\,.\,\dot\Theta_1'$, the others all vanishing for $u=0$. But from

$$\Theta(u+\tfrac{1}{2}a) = e^{-u-\frac{a}{4}}\,\Theta_1 u$$

we get $$\dot\Theta(u+\tfrac{1}{2}a) = -e^{-u-\frac{a}{4}}\,\Theta_1 u + e^{-u-\frac{a}{4}}\,\dot\Theta_1 u,$$

and therefore $$\dot\Theta(\tfrac{1}{2}a) = -e^{-\frac{a}{4}}\,\Theta_1 + e^{-\frac{a}{4}}\,\dot\Theta_1 = e^{-\frac{a}{4}}(\dot\Theta_1 - \Theta_1).$$

To avoid such complications it will be convenient to use the symbol $\partial_0 f(u)$ instead of $\dot f(0)$; viz. ∂_0 means the result of putting $u=0$ in the fluxion of the function which follows it. This being so, we have

$$\dot\Theta(u+\tfrac{1}{2}a)\,\Theta' u - \dot\Theta' u\,.\,\Theta(u+\tfrac{1}{2}a) = e^{-u-\frac{a}{4}}\{\dot\Theta_1 u\,.\,\Theta u - \dot\Theta' u\,.\,\Theta_1 u - \Theta' u\,.\,\Theta_1 u\}.$$

But by the formula it is also equal to

$$2\dot\Theta'(\tfrac{1}{2}a, 2a)\,\Theta'(2u+\tfrac{1}{2}a, 2a) + 2\dot\Theta_1'(\tfrac{1}{2}a, 2a)\,\Theta_1'(2u+\tfrac{1}{2}a, 2a).$$

Moreover we have

$$e^{-u-\frac{a}{4}}\,\Theta' u\,.\,\Theta_1 u = \Theta'(\tfrac{1}{2}a, 2a)\,\Theta'(2u+\tfrac{1}{2}a, 2a) + \Theta_1'(\tfrac{1}{2}a, 2a)\,\Theta_1'(2u+\tfrac{1}{2}a, 2a).$$

Now

$$\Theta'(u+\tfrac{1}{2}a, 2a) = e^{-u}\,\Theta_1'(u-\tfrac{1}{2}a, 2a) \text{ or } e^{\frac{1}{2}u}\,\Theta'(u+\tfrac{1}{2}a, 2a) = e^{-\frac{1}{2}u}\,\Theta_1'(u-\tfrac{1}{2}a, 2a);$$

$$\therefore\ \dot\Theta'(u+\tfrac{1}{2}a, 2a) = -e^{-u}\,\Theta_1'(u-\tfrac{1}{2}a, 2a) + e^{-u}\,\dot\Theta_1'(u-\tfrac{1}{2}a, 2a);$$

$$\therefore\ \dot\Theta'(\tfrac{1}{2}a, 2a) = \dot\Theta_1'(-\tfrac{1}{2}a, 2a) - \Theta_1'(-\tfrac{1}{2}a, 2a) = \dot\Theta_1'(\tfrac{1}{2}a, 2a) + \Theta_1'(\tfrac{1}{2}a, 2a);$$

$$\therefore\ \text{also } \Theta'(\tfrac{1}{2}a, 2a) + 2\dot\Theta'(\tfrac{1}{2}a, 2a) = \Theta_1'(\tfrac{1}{2}a, 2a) + 2\dot\Theta_1'(\tfrac{1}{2}a, 2a) = 2\partial_0\,.\,e^{\frac{1}{2}u}\,\Theta'(u+\tfrac{1}{2}a, 2a).$$

Hence finally

$$e^{-u-\frac{a}{4}}\{\dot\Theta_1 u\,.\,\Theta' u - \dot\Theta' u\,.\,\Theta_1 u\} = 2\partial_0 e^{\frac{1}{2}u}\,\Theta'(u+\tfrac{1}{2}a, 2a)\{\Theta'(2u+\tfrac{1}{2}a, 2a) + \Theta_1'(2u+\tfrac{1}{2}a, 2a)\},$$

$$\therefore\ \dot\Theta_1 u\,.\,\Theta' u - \dot\Theta' u\,.\,\Theta_1 u = \frac{2\partial_0 e^{\frac{1}{2}u}\,\Theta'(u+\tfrac{1}{2}a, 2a)}{\Theta'(\tfrac{1}{2}a, 2a)}\,\Theta u\,.\,\Theta_1' u.$$

Similarly, we find by putting $u+\frac{1}{2}\pi i$, $v+\frac{1}{2}\pi i$ in the original formula

$$\Theta' u\,.\,\dot\Theta' v - \dot\Theta' u\,.\,\Theta' v = 2\dot\Theta(v-u, 2a)\,\Theta(v+u, 2a) - 2\dot\Theta_1(v-u, 2a)\,\Theta_1(v+u, 2a).$$

Now write $v = u+\frac{1}{2}a$; we have in the first place

$$\Theta' u\,.\,\dot\Theta'(u+\tfrac{1}{2}a) - \dot\Theta' u\,.\,\Theta'(u+\tfrac{1}{2}a) = e^{-u-\frac{a}{4}}\{\Theta' u\,.\,\dot\Theta_1' u - \dot\Theta' u\,.\,\Theta_1' u - \Theta' u\,.\,\Theta_1' u\},$$

but also $$= 2\dot\Theta(\tfrac{1}{2}a, 2a)\,\Theta(2u+\tfrac{1}{2}a, 2a) - 2\dot\Theta_1(\tfrac{1}{2}a, 2a)\,\Theta_1(2u+\tfrac{1}{2}a, 2a).$$

Moreover we have

$$e^{-u-\frac{a}{4}}\,\Theta' u\,.\,\Theta_1' u = \Theta(\tfrac{1}{2}a, 2a)\,\Theta(2u+\tfrac{1}{2}a, 2a) - \Theta_1(\tfrac{1}{2}a, 2a)\,\Theta_1(2u+\tfrac{1}{2}a, 2a).$$

Now

$$\Theta(u+\tfrac{1}{2}a, 2a) = e^{-u}\,\Theta_1(u-\tfrac{1}{2}a, 2a) \text{ or } e^{\frac{1}{2}u}\,\Theta(u+\tfrac{1}{2}a, 2a) = e^{-\frac{1}{2}u}\,\Theta_1(u-\tfrac{1}{2}a, 2a),$$

whence

$$\tfrac{1}{2}e^{\frac{1}{2}u}\,\Theta(u+\tfrac{1}{2}a, 2a) + e^{\frac{1}{2}u}\,\dot\Theta(u+\tfrac{1}{2}a, 2a) = -\tfrac{1}{2}e^{-\frac{1}{2}a}\,\Theta_1(u-\tfrac{1}{2}a, 2a) + e^{-\frac{1}{2}u}\,\dot\Theta_1(u-\tfrac{1}{2}a, 2a),$$

or $(u=0)$

$$\Theta(\tfrac{1}{2}a, 2a) + 2\dot\Theta(\tfrac{1}{2}a, 2a) = -\Theta_1(\tfrac{1}{2}a, 2a) - 2\dot\Theta_1(\tfrac{1}{2}a, 2a) = \partial_0 e^{\frac{1}{2}u}\,\Theta(u+\tfrac{1}{2}a, 2a).$$

Hence finally

$$e^{-u-\frac{a}{4}}\{\Theta' u\,.\,\dot\Theta_1' u - \dot\Theta' u\,.\,\Theta_1' u\} = \partial_0 e^{\frac{1}{2}u}\,\Theta(u+\tfrac{1}{2}a, 2a)\{\Theta(2u+\tfrac{1}{2}a, 2a) + \Theta_1(2u+\tfrac{1}{2}a, 2a)\},$$

and $$\Theta' u\,.\,\dot\Theta_1' u - \dot\Theta' u\,.\,\Theta_1' u = \frac{\partial_0 e^{\frac{1}{2}u}\,\Theta(u+\tfrac{1}{2}a, 2a)}{\Theta(\tfrac{1}{2}a, 2a)}\,.\,\Theta u\,.\,\Theta_1 u.$$

The three formulæ thus arrived at may be written as follows:—

$$\Theta' u\,.\,\dot\Theta u - \dot\Theta' u\,.\,\Theta u = \alpha\,.\,\Theta_1 u\,.\,\Theta_1' u,$$

$$\Theta' u\,.\,\dot\Theta_1 u - \dot\Theta' u\,.\,\Theta_1 u = \beta\,.\,\Theta_1' u\,.\,\Theta u,$$

$$\Theta' u\,.\,\dot\Theta_1' u - \dot\Theta' u\,.\,\Theta_1' u = \gamma\,.\,\Theta u\,.\,\Theta_1 u,$$

and the quantities α, β, γ may now be determined in terms of Θ, Θ', Θ_1 and $\dot\Theta_1'$ by substituting particular values of u. Thus putting $u = \frac{1}{2}a+\frac{1}{2}\pi i$, $\frac{1}{2}\pi i$, and 0 successively, we find

$$\Theta_1\dot\Theta_1' = -\alpha\Theta\Theta',$$

$$\Theta\dot\Theta_1' = \beta\Theta'\Theta_1,$$

$$\Theta'\dot\Theta_1' = \gamma\Theta\Theta_1.$$

If we now write

$$fu = \frac{\Theta}{\Theta_1}\,.\,\frac{\Theta_1' u}{\Theta' u},\qquad gu = \frac{\Theta'}{\Theta_1}\,.\,\frac{\Theta_1 u}{\Theta' u},\qquad hu = \frac{\Theta'}{\Theta}\,.\,\frac{\Theta u}{\Theta' u},$$

(the multipliers in the two latter being so chosen as to make them $=1$ for $u=0$, and in the former to simplify the following formulæ, which are derived from those on p. 447, viz.:—

$$1+fu^2=gu^2, \quad 1+k^2fu^2=hu^2, \quad k'^2+k^2gu^2=hu^2),$$

then we have for the fluxions of these functions

$$\dot{f}u=\lambda gu\,.\,hu, \quad \dot{g}u=\lambda hu\,.\,fu, \quad \dot{h}u=k^2\lambda fu\,.\,gu, \quad \text{where } \lambda=\frac{\Theta\dot{\Theta}_1'}{\Theta_1\Theta'}.$$

By substitution from the three equations connecting the squares of f, g, h we obtain

$$\dot{f}u=\lambda\sqrt{(1+fu^2)(1+k^2fu^2)}, \qquad \lambda u=\int_0^f\frac{df}{\sqrt{(1+f^2)(1+k^2f^2)}},$$

$$\dot{g}u=\lambda\sqrt{(gu^2-1)(k'^2+k^2gu^2)}, \qquad \lambda u=\int_1^g\frac{dg}{\sqrt{(g^2-1)(k'^2+k^2g^2)}},$$

$$\dot{h}u=\lambda\sqrt{(1+h^2)(k'^2+h^2)}, \qquad \lambda u=\int_1^h\frac{dh}{\sqrt{(1+h^2)(k'^2+h^2)}},$$

when $u=\frac{1}{2}\pi i$, $fu=i$; writing ix for f in the integral,

$$\tfrac{1}{2}\pi\lambda=\int_0^1\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=K, \qquad \lambda=\frac{2K}{\pi}.$$

To suit the trigonometric form, we may write $if(u)=z(u)$; then $z(\frac{1}{2}\pi)=1$, which is the same thing as saying that

$$\frac{\lambda\pi}{2}=\int_0^1\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}=K; \text{ then } \lambda=\frac{2K}{\pi},$$

which gives λ as an explicit function of k.

In passing now to the trigonometric form, we shall divide the argument by λ, because $f\left(\frac{x}{\lambda}\right)$ becomes 1 when $x=0$. Writing then $u=\frac{ix}{\lambda}=\frac{i\pi x}{2K}$, we shall have in Jacobi's notation,

$$f\left(\frac{i\pi x}{2K}\right)=i\sin\operatorname{am}x=\frac{i}{\sqrt{k}}\frac{\vartheta_1'\left(\frac{\pi x}{2K}\right)}{\vartheta'\left(\frac{\pi x}{2K}\right)},$$

$$g\left(\frac{i\pi x}{2K}\right)=\cos\operatorname{am}x=\sqrt{\frac{k'}{k}}\frac{\vartheta_1\left(\frac{\pi x}{2K}\right)}{\vartheta'\left(\frac{\pi x}{2K}\right)},$$

$$h\left(\frac{i\pi x}{2K}\right)=\Delta\operatorname{am}x=\sqrt{k'}\frac{\vartheta\left(\frac{\pi x}{2K}\right)}{\vartheta'\left(\frac{\pi x}{2K}\right)}.$$

III.

The Addition-Theorem.

Starting from the formula

$$\theta u\,.\,\theta v=\theta\overline{u+v,2a}\,.\,\theta\overline{u-v,2a}+\theta_1\overline{u+v,2a}\,.\,\theta_1\overline{u-v,2a}\;\ldots\ldots\ldots(1),$$

we derive by writing $u+\frac{1}{2}a$ for u and $v+\frac{1}{2}a+\frac{1}{2}\iota\pi$ for v,

$$\theta_1u\,.\,\theta_1'v=\theta_1'\overline{u+v,2a}\,.\,\theta'\overline{u-v,2a}-\theta'\overline{u+v,2a}\,.\,\theta_1'\overline{u-v,2a}\;\ldots\ldots\ldots(2),$$

and so $\theta_1'u\,.\,\theta_1v=\theta_1'\overline{u+v,2a}\,.\,\theta'\overline{u-v,2a}+\theta'\overline{u+v,2a}\,.\,\theta_1'\overline{u-v,2a}\;\ldots\ldots\ldots(3).$

Write $v=0$ in this formula; then

$$\theta_1'u\,.\,\theta_1=2\theta'\overline{u,2a}\,.\,\theta_1'\overline{u,2a}\;\ldots\ldots\ldots(4).$$

So from $\theta_1u\,.\,\theta_1v=\theta_1\overline{u+v,2a}\,.\,\overline{\theta u-v,2a}+\overline{\theta u+v,2a}\,.\,\theta_1\overline{u-v,2a}\;\ldots\ldots\ldots(5),$

we get $\theta_1u\,.\,\theta_1=2\theta_1\overline{u,2a}\,.\,\theta\overline{u,2a}$, and $\theta_1^2=2\theta_1(0,2a)\,.\,\theta(0,2a)\;\ldots\ldots(6, 7).$

Substituting in (2), (3), we get

$$\theta_1'\overline{u+v,2a}\,.\,\theta'\overline{u-v,2a}-\theta'\overline{u+v,2a}\,.\,\theta_1'\overline{u-v,2a}=\theta_1u\,.\,\theta_1'v$$

$$=\frac{4\theta_1\overline{u,2a}\,.\,\theta\overline{u,2a}\,.\,\theta'\overline{v,2a}\,.\,\theta_1'\overline{v,2a}}{(\theta_1^2=)\,2\theta_1\overline{0,2a}\,.\,\theta\overline{0,2a}},$$

or omitting the $2a$ throughout

$$\theta\,.\,\theta_1(\theta_1'\overline{u+v}\,.\,\theta'\overline{u-v}-\theta'\overline{u+v}\,.\,\theta_1'\overline{u-v})=2\theta_1u\,.\,\theta u\,.\,\theta'v\,.\,\theta_1'v,$$

$$\theta\,.\,\theta_1(\theta_1'\overline{u+v}\,.\,\theta'\overline{u-v}+\theta'\overline{u+v}\,.\,\theta_1'\overline{u-v})=2\theta_1v\,.\,\theta v\,.\,\theta'u\,.\,\theta_1'u.$$

Divide throughout by $\theta_1^2\,.\,\theta'\overline{u+v}\,.\,\theta'\overline{u-v}$; thus

$$f(u+v)+f(u-v)=\frac{2\theta_1v\,.\,\theta v\,.\,\theta'u\,.\,\theta_1'u}{\theta_1^2\,.\,\theta'\overline{u+v}\,.\,\theta'\overline{u-v}}=2fu\,.\,gv\,.\,hv\,\frac{\theta'u^2\,.\,\theta'v^2}{\theta'^2\,.\,\theta'\overline{u+v}\,.\,\theta'\overline{u-v}}\;\ldots\ldots(11).$$

To find the value of the dexter side we proceed as follows. We have

$$\theta'x\,.\,\theta y=\theta'\overline{x+y,2a}\,.\,\theta'\overline{x-y,2a}-\theta_1'\overline{x+y,2a}\,.\,\theta_1'\overline{x-y,2a}\;\ldots\ldots(12),$$

$$\theta x\,.\,\theta'y=\theta'\overline{x+y,2a}\,.\,\theta'\overline{x-y,2a}+\theta_1'\overline{x+y,2a}\,.\,\theta_1'\overline{x-y,2a}\;\ldots\ldots(13),$$

therefore $\theta x\,.\,\theta'x=\theta'\overline{2x,2a}\,.\,\theta'\overline{0,2a}\;\ldots\ldots\ldots\ldots(14).$

Multiply together (12) and (13), taking account of (14) on the left, and omit the $2a$ throughout; thus

$$\theta'^2\,.\,\theta'2x\,.\,\theta'2y=\theta'\overline{x+y}^2\,.\,\theta'\overline{x-y}^2-\theta_1'\overline{x+y}^2\,.\,\theta_1'\overline{x-y}^2,$$

or writing $2x=u+v,\ 2y=u-v,\ x+y=u,\ x-y=v,$

$$\theta'^2\,.\,\theta'\overline{u+v}\,.\,\theta'\overline{u-v}=\theta'u^2\,.\,\theta'v^2-\theta_1'u^2\,.\,\theta_1'v^2,$$

and therefore $$\frac{\theta'^2 . \theta'\overline{u+v} . \theta'\overline{u-v}}{\theta' u^2 . \theta' v^2} = 1 - k^2 fu^2 . fv^2;$$

whence $$f(u+v) + f(u-v) = \frac{2fu . gv . hv}{1 - k^2 fu^2 . fv^2};$$

and by interchanging u with v
$$f(u+v) - f(u-v) = \frac{2fv . gu . hu}{1 - k^2 fu^2 . fv^2};$$

therefore $$f(u \pm v) = \frac{fu . gv . hv \pm fv . gu . hu}{1 - k^2 fu^2 . fv^2}.$$

IV.

Elliptic Functions of the second kind.

By easy calculation we may establish the theorem
$$\ddot\theta u . \theta v - \dot\theta u . \dot\theta v = 2\left\{\begin{array}{l}\dot\Theta\overline{u+v} . \dot\Theta\overline{u-v} + \Theta\overline{u+v} . \ddot\Theta\overline{u-v} \\ + \dot\Theta_1\overline{u+v} . \dot\Theta_1\overline{u-v} + \Theta_1\overline{u+v} . \ddot\Theta_1\overline{u-v},\end{array}\right.$$

and thence putting $u = v$, and observing that $\dot\Theta = \dot\Theta_1 = 0$,
$$\ddot\theta u . \theta u - \dot\theta u^2 = 2\{\ddot\Theta . \Theta(2u) + \ddot\Theta_1 . \Theta_1(2u)\},$$
$$\ddot\theta' u . \theta' u - \dot\theta' u^2 = 2\{\ddot\Theta . \Theta(2u) - \ddot\Theta_1 . \Theta_1(2u)\}.$$

We are therefore entitled to assume
$$\ddot\theta' u . \theta' u - \dot\theta' u^2 = \alpha\theta u^2 + \beta\theta' u^2,$$

and in fact, writing successively $u = \frac{1}{2}a$ and $u = \frac{1}{2}\pi i + \frac{1}{2}a$, we find
$$-\dot\theta_1'^2 = \alpha\theta_1^2, \quad \ddot\theta_1 . \theta_1 = \beta\theta_1^2,$$

so that $$\theta_1^2(\ddot\theta' u . \theta' u - \dot\theta' u^2) = -\dot\theta_1'^2 . \theta u^2 + \ddot\theta_1 . \theta_1 . \theta' u^2;$$

therefore
$$\partial_u \frac{\dot\theta' u}{\theta' u} = -\frac{\dot\theta_1'^2}{\theta_1^2}\frac{\theta u^2}{\theta' u^2} + \frac{\ddot\theta_1}{\theta_1} = \lambda^2 hu^2 + \mu \qquad \left(\lambda = i\frac{\theta\dot\theta_1'}{\theta_1\theta'}, \quad \mu = \frac{\ddot\theta_1}{\theta_1}.\right)$$

Suppose then that $\frac{\dot\theta' u}{\theta' u} = \lambda Z(\lambda u)$; we shall have
$$\partial_x Z(x) = \mathrm{dn}^2 x + \frac{\mu}{\lambda^2}, \quad \partial_y Z(\mathrm{sn}^{-1} y) = \sqrt{\frac{1 - k^2y^2}{1 - y^2}} + \frac{\mu}{\lambda^2}\frac{1}{\sqrt{(1 - y^2 . 1 - k^2y^2)}},$$

integrating from 0 to 1, since $Z(K) = 0$, we have $0 = E + \frac{\mu K}{\lambda^2}$,
$$\therefore Z(x) = \int_0^x \mathrm{dn}^2 x dx - \frac{E}{K}x, \text{ (Jacobi's definition).}$$

Whence also $$\int_0^y \sqrt{\frac{1 - k^2y^2}{1 - y^2}} dy = Z(\mathrm{sn}^{-1} y) + \frac{E}{K}\mathrm{sn}^{-1} y.$$

V.

Product of Four Theta functions. Smith's reconstruction of Jacobi's method.

Consider four integers n_1, n_2, n_3, n_4, and let $n_1 + n_2 + n_3 + n_4 = s$, $\nu_1 = s - 2n_1$, $\nu_2 = s - 2n_2$, $\nu_3 = s - 2n_3$, $\nu_4 = s - 2n_4$. Then $\nu_1 + \nu_2 + \nu_3 + \nu_4 = 2s$, and $2s - 2\nu_1 = 4n_1$, etc.

Multiply now together the four θ-series.
$$\theta x = \Sigma e\,|\,n_1^2 a + 2n_1 x\,|, \quad \theta y = \Sigma e\,|\,n_2^2 a + 2n_2 y\,|,$$
$$\theta z = \Sigma e\,|\,n_3^2 a + 2n_3 z\,|, \quad \theta w = \Sigma e\,|\,n_4^2 a + 2n_4 w\,|,$$

and we get a quadruply infinite series in which the exponent of the general term is
$$a\Sigma n^2 + 2(n_1 x + n_2 y + n_3 z + n_4 w).$$

Now $\Sigma\nu^2 = 4\Sigma n^2$, and if we write $\sigma = x + y + z + w$,

then $$2(n_1 x + n_2 y + n_3 z + n_4 w) = \nu_1\xi + \nu_2\eta + \nu_3\zeta + \nu_4\omega,$$

where $$2\xi = \sigma - 2x, \ 2\eta = \sigma - 2y, \ 2\zeta = \sigma - 2z, \ 2\omega = \sigma - 2w.$$

So the exponent becomes
$$\tfrac{1}{4}a\Sigma\nu^2 + (\nu_1\xi + \nu_2\eta + \nu_3\zeta + \nu_4\omega).$$

Here if s is odd, the ν are all odd, and if s is even, they are all even. Thus the ν and $\frac{1}{2}\Sigma\nu$ must be either all odd or all even together.

First let them be even, and $= 2p_1, 2p_2, 2p_3, 2p_4$ respectively. Then
$$p_1 + p_2 + p_3 + p_4$$
must be even. Substituting, the exponent becomes
$$a\Sigma p^2 + 2(p_1\xi + p_2\eta + p_3\zeta + p_4\omega),$$
and this has to be summed under the condition that Σp is even. Call the exponent P, twice the sum required is
$$\Sigma^4 e\,|\,P\,| + \Sigma^4(-)^{\Sigma p} e\,|\,P\,|, \text{ or } \theta\xi . \theta\eta . \theta\zeta . \theta\omega + \theta'\xi . \theta'\eta . \theta'\zeta . \theta'\omega.$$

Next let the ν be odd and equal to $2p_h + 1$ $(h = 1, 2, 3, 4)$. Then $\frac{1}{2}\Sigma\nu = \Sigma p + 2$ must be odd, or Σp must be odd. Hence if P is the exponent
$$a\Sigma(p + \tfrac{1}{2})^2 + 2(\overline{p_1 + \tfrac{1}{2}}\xi + \ldots),$$
twice the sum is
$$\Sigma^4 e\,|\,P\,| - \Sigma^4(-)^{\Sigma p} e\,|\,P\,|, \text{ or } \theta_1\xi . \theta_1\eta . \theta_1\zeta . \theta_1\omega - \theta_1'\xi . \theta_1'\eta . \theta_1'\zeta . \theta_1'\omega.$$

Hence the Theorem
$$2\theta x . \theta y . \theta z . \theta w = \theta\xi . \theta\eta . \theta\zeta . \theta\omega + \theta'\xi . \theta'\eta . \theta'\zeta . \theta'\omega + \theta_1\xi . \theta_1\eta . \theta_1\zeta . \theta_1\omega - \theta_1'\xi . \theta_1'\eta . \theta_1'\zeta . \theta_1'\omega$$
$$= \Sigma(-)^{\alpha\beta}\,\theta_\beta^\alpha(\xi) . \theta_\beta^\alpha(\eta) . \theta_\beta^\alpha(\zeta) . \theta_\beta^\alpha(\omega).$$

To generalize this, add to the arguments $x\ y\ z\ w$ the quadrants $\begin{smallmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{smallmatrix}$

respectively whose sum shall be $\frac{p}{q}$, (p, q must be even); and let $\frac{2\alpha}{2\beta}=\frac{p-2\alpha}{q-2\beta}$; then we shall find that $\xi\ \eta\ \zeta\ \omega$ will be increased by $\frac{\alpha_1}{\beta_1}$ etc.; and so

$$2\theta^{a_1}_{b_1}(x).\theta^{a_2}_{b_2}(y).\theta^{a_3}_{b_3}(z).\theta^{a_4}_{b_4}(w)=\Sigma\,(-)^{g\,(f+\frac{1}{2}p)}\,\theta^{\alpha_1+f}_{\beta_1+g}(\xi).\theta^{\alpha_2+f}_{\beta_2+g}(\eta).\theta^{\alpha_3+f}_{\beta_3+g}(\zeta).\theta^{\alpha_4+f}_{\beta_4+g}(\omega),$$

$$(f,\ g=0,\ 1.$$

Make $x\ y\ z\ w$ all $=0$ in the original theorem; then

$$\theta^4=\theta_1{}^4+\theta'^4.$$

Next write for them $x\ x\ 0\ 0$, and $\begin{matrix} a & a & 0 & 0 \\ b & b & 0 & 0\end{matrix}$ for the a and b. Then

$$\theta^{a}_{b}x^2\,.\,\theta^2=\theta^{a+1}_{b}x^2\,.\,\theta'^2+(-)^a\theta^{a}_{b+1}x^2\,.\,\theta_1{}^2,$$

that is

$$\begin{aligned}\theta x^2\ .\ \theta^2&=\theta' x^2\ .\ \theta'^2+\theta_1 x^2\ .\ \theta_1{}^2\\ \theta' x^2\ .\ \theta^2&=\theta x^2\ .\ \theta'^2-\theta_1' x^2\ .\ \theta_1{}^2\\ \theta_1 x^2\ .\ \theta^2&=\theta_1' x^2\ .\ \theta'^2+\theta x^2\ .\ \theta_1{}^2\\ \theta_1' x^2\ .\ \theta^2&=\theta_1 x^2\ .\ \theta'^2-\theta' x^2\ .\ \theta_1{}^2\end{aligned}\qquad\text{or}\qquad\left\{\begin{aligned}hu^2&=k'^2+k^2gu^2,\\ 1&=hu^2-k^2fu^2,\\ gu^2&=k'^2fu^2+k^2hu^2,\\ fu^2&=gu^2-1.\end{aligned}\right.$$

Next writing for the $x,\ x-y,\ x+y,\ 0,\ 0$ we find for the $\xi,\ y,\ -y,\ x,\ x$, and we get such formulæ as

$$\theta\overline{x-y}\,.\,\theta_1\overline{x+y}\,.\,\theta\,.\,\theta_1=\theta_1 y\,.\,\theta y\,.\,\theta_1 x\,.\,\theta x+\theta_1' y\,.\,\theta' y\,.\,\theta_1' x\,.\,\theta' x.$$

Differentiate in regard to y and then put $y=0$; we get

$$\theta\theta_1\,(\theta x\,.\,\dot\theta_1 x-\dot\theta x\,.\,\theta_1 x)=+\,\dot\theta_1'\theta\,.\,\theta_1' x\,.\,\theta' x,\ \text{etc.}$$

The important formulæ of this kind are the three

$$\theta'\overline{x-y}\,.\,\theta_1\overline{x+y}\ .\,\theta'\theta_1=\theta' y\,.\,\theta_1 y\,.\,\theta' x\,.\,\theta_1 x-\theta y\,.\,\theta_1' y\,.\,\theta x\,.\,\theta_1' x,$$

$$\theta'\overline{x-y}\,.\,\theta_1'\overline{x+y}\,.\,\theta\theta_1=\theta_1 y\,.\,\theta y\,.\,\theta_1' x\,.\,\theta' x+\theta_1' y\,.\,\theta' y\,.\,\theta_1 x\,.\,\theta x,$$

$$\theta'\overline{x-y}\,.\,\theta\overline{x+y}\ .\,\theta'\theta=\theta' y\,.\,\theta y\,.\,\theta' x\,.\,\theta x-\theta_1' y\,.\,\theta_1 y\,.\,\theta_1' x\,.\,\theta_1 x\,;$$

and the fluxional equations are

$$\left.\begin{aligned}(\dot\theta_1' x.\theta' x-\theta_1' x.\dot\theta' x)\,\theta\theta_1&=\dot\theta_1'\theta'.\theta x.\theta_1 x\\ (\dot\theta_1 x.\theta' x-\theta_1 x.\dot\theta' x)\,\theta'\theta_1&=\dot\theta_1'\theta\,.\,\theta_1' x\,.\,\theta x\\ (\dot\theta x.\theta' x-\theta x\,.\dot\theta' x)\,\theta\theta'&=\dot\theta_1'\theta_1\,.\,\theta_1 x\,.\theta_1' x\end{aligned}\right\}\text{giving}\ \begin{aligned}\dot fu&=\lambda gu\,.\,hu=\lambda\sqrt{\overline{1+f^2}\,.\,\overline{1+k^2f^2}}\\ \dot gu&=\lambda hu\,.\,fu=\lambda\sqrt{\overline{k'^2+k^2g^2}\,.\,\overline{g^2-1}}\\ \dot hu&=\lambda k^2fu\,.\,gu=\lambda\sqrt{\overline{h^2-1}\,.\,\overline{h^2-k'^2}}\end{aligned}$$

whence λ is given as a function of k by $\frac{1}{2}\pi\lambda=\int_0^1\frac{dx}{\sqrt{\overline{1-x^2}\,.\,\overline{1-k^2x^2}}}=K.$

In the circular form $f\left(\frac{ix}{\lambda}\right)=i\ \mathrm{sn}\,x,\ g\left(\frac{ix}{\lambda}\right)=\mathrm{cn}\,x,\ h\left(\frac{ix}{\lambda}\right)=\mathrm{dn}\,x$; and then

$$\left.\begin{aligned}\dot{\mathrm{sn}}\,x&=\mathrm{cn}\,x\ \mathrm{dn}\,x=\sqrt{(1-\mathrm{sn}^2x)(1-k^2\mathrm{sn}^2x)}\\ \dot{\mathrm{cn}}\,x&=-\,\mathrm{dn}\,x\ \mathrm{sn}\,x=-\sqrt{(1-\mathrm{cn}^2x)(k'^2+k^2\mathrm{cn}^2x)}\\ \dot{\mathrm{dn}}\,x&=-\,k^2\,\mathrm{sn}\,x\ \mathrm{cn}\,x=-\sqrt{(1-\mathrm{dn}^2x)(\mathrm{dn}^2x-k'^2)}\end{aligned}\right\}\dot{\mathrm{tn}}\,x=\sqrt{(1+\mathrm{tn}^2x)(1+k'^2\mathrm{tn}^2x)}.$$

The Addition-Theorem.

We have found expressions for the product of two different θ functions of $x+y,\ x-y$; the following value for $\theta^a_b(x+y)\,\theta^a_b(x-y)$ is used subsequently for functions of the second kind, as well as here for the addition-theorem. Writing for the xab the values $\begin{matrix}x-y & x+y & 0 & 0\\ a & a & 1 & 1\\ b & b & 0 & 0\end{matrix}$ we have for the $\xi\alpha\beta$, $\begin{matrix}y & -y & x & x\\ 1 & 1 & a & a\\ 0 & 0 & b & b\end{matrix}$ and so

$$2\theta^a_b(x+y)\,.\,\theta^a_b(x-y)\,\theta'^2$$

$$=\theta' y^2\,.\,\theta^a_b x^2+\theta y^2\,.\,\theta^{a+1}_b x^2+(-)^a\theta_1' y^2\,.\,\theta^a_{b+1}x^2-(-)^a\theta_1 y^2.\theta^{a+1}_{b+1}x^2\ldots\ldots(B).$$

But now writing for the xab, $\begin{matrix}x-y & x+y & 0 & 0\\ a & a & 1 & 1\\ b+1 & b+1 & 1 & 1\end{matrix}$ we get for the $\xi\alpha\beta$, $\begin{matrix}y & -y & x & x\\ 1 & 1 & a & a\\ 1 & 1 & b+1 & b+1\end{matrix}$

so that

$$0=+\,\theta_1' y^2\,.\,\theta^a_{b+1}x^2+\theta_1 y^2\,.\,\theta^{a+1}_{b+1}x^2+(-)^a\theta' y^2\,.\,\theta^a_b x^2-(-)^a\theta y^2\,.\,\theta^{a+1}_b x^2\ldots\ldots(B')\,;$$

then $B\pm(-)^aB'$ gives

$$\theta^a_b(x+y)\,.\,\theta^a_b(x-y)\,\theta'^2=\theta' y^2\,.\,\theta^a_b x^2+(-)^a\theta_1' y^2\,.\,\theta^a_{b+1}x^2$$

$$=\theta y^2\,.\,\theta^{a+1}_b x^2-(-)^a\theta_1 y^2\,.\,\theta^{a+1}_{b+1}x^2.$$

For example, writing $a,\ b=1,\ 0$ we have the important formula

$$\theta'(x+y)\,.\,\theta'(x-y)\,\theta'^2=\theta' y^2\,.\,\theta' x^2-\theta_1' y^2\,.\,\theta_1' x^2,\ \text{or}\ \frac{\theta'\overline{x+y}\,.\,\theta'\overline{x-y}\,\div\theta'^2}{\theta' x^2\,.\,\theta' y^2}=1-k^2fx^2\,.\,fy^2.$$

Dividing by this equation the three formulæ

$$\theta'\overline{x-y}\,.\,\theta_1'\overline{x+y}\,.\,\theta\,.\,\theta_1=\theta_1 y\,.\,\theta y\,.\,\theta_1' x\,.\,\theta' x+\theta_1' y\,.\,\theta' y\,.\,\theta_1 x\,.\,\theta x,$$

$$\theta'\overline{x-y}\,.\,\theta_1\overline{x+y}\,.\,\theta_1\,.\,\theta'=\theta_1 y\,.\,\theta' y\,.\,\theta_1 x\,.\,\theta' x+\theta_1' y\,.\,\theta y\,.\,\theta_1' x\,.\,\theta x,$$

$$\theta'\overline{x-y}\,.\,\theta\overline{x+y}\,.\,\theta'\,.\,\theta=\theta' y\,.\,\theta y\,.\,\theta' x\,.\,\theta x+\theta_1' y\,.\,\theta_1 y\,.\,\theta_1' x\,.\,\theta_1 x,$$

we find from the first

$$\frac{\theta_1'\overline{x+y}}{\theta'\overline{x+y}}\cdot\frac{\theta\theta_1}{\theta'^2}=\frac{\theta_1 y\,.\,\theta y\,.\,\theta_1' x\,.\,\theta' x+\theta_1' y\,.\,\theta' y\,.\,\theta_1 x\,.\,\theta x}{\theta' x^2\,.\,\theta' y^2-\theta_1' x^2\,.\,\theta_1' y^2},$$

or

$$f(x+y)=\frac{fx\,.\,gy\,.\,hy+fy\,.\,gx\,.\,hx}{1-k^2fx^2\,.\,fy^2},$$

and so from the second and third

$$g\,(x+y)=\frac{gx\,.\,gy+fx\,.\,fy\,.\,hx\,.\,hy}{1-k^2fx^2\,.\,fy^2},$$

$$h\,(x+y)=\frac{hx\,.\,hy+k^2fx\,.\,fy\,.\,gx\,.\,gy}{1-k^2fx^2\,.\,fy^2}.$$

Functions of the Second Kind.

Differentiate the equation

$$\theta'(x+y)\,.\,\theta'(x-y)\,.\,\theta'^2=\theta'y^2\,.\,\theta'x^2-\theta_1'y^2\,.\,\theta_1'x^2$$

twice with respect to y; we obtain successively

$$(\dot\theta'\overline{x+y}\,.\,\theta'\overline{x-y}-\theta'\overline{x+y}\,.\,\dot\theta'\overline{x-y})\,\theta'^2=2\theta'y\,.\,\dot\theta'y\,.\,\theta'x^2-2\theta_1'y\,.\,\dot\theta_1'y\,.\,\theta_1'x^2,$$

$$(\ddot\theta'\overline{x+y}\,.\,\theta'\overline{x-y}+\theta'\overline{x+y}\,.\,\ddot\theta'\overline{x-y}-2\dot\theta'\overline{x+y}\,.\,\dot\theta'\overline{x-y})\,\theta'^2$$
$$=2\,(\theta'y\,.\,\ddot\theta'y+\dot\theta'y^2)\,\theta'x^2-2\,(\theta_1'y\,.\,\ddot\theta_1'y+\dot\theta_1'y^2)\,\theta_1'x^2.$$

In this put $y=0$; then

$$(\ddot\theta'x\,.\,\theta'x-\dot\theta'x^2)\,\theta'^2=\theta'\ddot\theta'\,.\,\theta'x^2-\dot\theta_1'^2\,.\,\theta_1'x^2,$$

$$\text{or } \partial_x\frac{\dot\theta'x}{\theta'x}=\frac{\ddot\theta'}{\theta'}-\frac{\dot\theta_1'^2}{\theta'^2}\,.\,\frac{\theta_1'x^2}{\theta'x^2}=\mu-\lambda^2k^2fx^2,\ \text{if } \mu=\frac{\ddot\theta'}{\theta'},\ \lambda=\frac{\dot\theta_1'\theta}{\theta'\theta_1}.$$

Herein writing $x=\frac{iu}{\lambda}$, we have $\partial_u=\frac{i}{\lambda}\partial_x$, and

$$\partial_u\frac{\partial_u\theta'\left(\frac{iu}{\lambda}\right)}{\theta'\left(\frac{iu}{\lambda}\right)}=-\frac{\mu}{\lambda^2}-k^2\,\mathrm{sn}^2\,u.$$

Integrating from 0 to u, we get

$$\partial_u\log\theta'\left(\frac{iu}{\lambda}\right)=-\frac{\mu}{\lambda^2}u-\int_0^u k^2\,\mathrm{sn}^2\,u\,du.$$

If $y=\mathrm{sn}\,u$, $dy=\sqrt{(1-y^2)(1-k^2y^2)}\,.\,du$, and we have

$$\int_0^u k^2\,\mathrm{sn}^2\,u\,du=\int_0^y\frac{k^2y^2dy}{\sqrt{Y}}=\int_0^y\frac{dy}{\sqrt{Y}}-\int_0^y\frac{\overline{1-k^2y^2}\,.\,dy}{\sqrt{Y}},$$

$$=u-\int_0^y\sqrt{\frac{1-k^2y^2}{1-y^2}}\,,dy.$$

Put here $y=1$, $u=K$, $\frac{iu}{\lambda}=\frac{1}{2}i\pi$, then since $\dot\theta'(\frac{1}{2}i\pi)=0$, we have $\left(\lambda=\frac{2K}{\pi}\right)$

$$0=-\frac{\mu}{\lambda^2}K-K+E,\ \text{or}-\frac{\mu}{\lambda^2}=1-\frac{E}{K},$$

if $$E=\int_0^1\sqrt{\frac{1-k^2y^2}{1-y^2}}\,dy=\int_0^{\frac{\pi}{2}}\sqrt{(1-k^2\sin^2\phi)}\,.\,d\phi,$$

$$\int_0^y\sqrt{\frac{1-k^2y^2}{1-y^2}}\,dy=u-\int_0^u k^2\,\mathrm{sn}^2\,u\,du=\frac{E}{K}u+\partial_u\log\theta'\left(\frac{iu}{\lambda}\right)$$

$$=\frac{E}{K}\,\mathrm{sn}^{-1}\,y+\frac{i\pi}{2K}\frac{\dot\theta'\left(\frac{i\pi\,\mathrm{sn}^{-1}\,y}{2K}\right)}{\theta'\left(\frac{i\pi\,\mathrm{sn}^{-1}\,y}{2K}\right)}.$$

Functions of the Third Kind.

In the equation

$$\frac{\theta'\overline{x+y}\,.\,\theta'\overline{x-y}\,.\,\theta'^2}{\theta'x^2\,.\,\theta'y^2}=1-k^2fx^2\,.\,fy^2$$

take logarithmic fluxion in respect of y; this is

$$\partial_y\log\theta'\overline{x+y}-\partial_y\log\theta'\overline{x-y}-2\partial_y\log\theta'y=\frac{-2k^2fx^2\,.\,\lambda gy\,.\,hy\,.\,fy}{1-k^2fx^2\,.\,fy^2}.$$

Integrate in respect of x from 0 to x; thus

$$\tfrac{1}{2}\log\frac{\theta'\overline{x-y}}{\theta'\overline{x+y}}+\frac{\dot\theta'y}{\theta'y}x=\lambda fy\,.\,gy\,.\,hy\int_0^x\frac{k^2fx^2dx}{1-k^2fy^2\,.\,fx^2};$$

$$\therefore\quad\tfrac{1}{2}\log\frac{\theta'\left(\frac{i\pi}{2K}\overline{u-v}\right)}{\theta'\left(\frac{i\pi}{2K}\overline{u+v}\right)}+u\frac{\dot\theta'\left(\frac{i\pi v}{2K}\right)}{\theta'\left(\frac{i\pi v}{2K}\right)}=\int_0^u\frac{\mathrm{sn}\,v\,\mathrm{cn}\,v\,\mathrm{dn}\,v\,k^2\,\mathrm{sn}^2\,u\,du}{1-k^2\,\mathrm{sn}^2\,v\,\mathrm{sn}^2\,u}.$$

VI.

Abelian form of the Addition-Theorem.

Consider the curve

$$y^2=x(1-x)(1-k^2x)$$

If we write $$x=\mathrm{sn}^2\,u,$$

then $$y=\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u,$$

and every point on the curve may be denoted by its parameter u.

Let us now cut the curve by the straight line

$$\xi x+\eta y=1;$$

the abscissæ of the point of contact are given by the equation

$$0=(1-\xi x)^2-\eta^2x(1-x)(1-k^2x),\ \text{say } \phi(x)=0.$$

If we vary ξ, η in this equation, we shall also vary the roots of it. Let x now signify any one of the roots, then we shall have

$$\phi'x\,dx+2x(1-\xi x)\,d\xi-2\eta x(1-x)(1-k^2x)\,d\eta=0.$$

Now since $\phi x=0$, $\eta\sqrt{x(1-x)(1-k^2x)}=1-\xi x$,

$$\therefore\ \frac{dx}{\sqrt{x(1-x)(1-k^2x)}}+2\frac{\eta d\xi-(1-\xi x)\,\eta d\eta}{\phi'x}=0.$$

Sum this equation for the three roots $x_1\ x_2\ x_3=\mathrm{sn}^2u$, sn^2v, sn^2w; the sum of the second terms vanishes, and the first terms are $2du\ 2dv\ 2dw$. Hence integrating from $\xi=0$ $\eta=0$, which gives $x_1\ x_2\ x_3=\infty$ and therefore $u=v=w=iK$, we have $u+v+w=3iK$ when the three points are in a line; or if $u-iK=u_1$, $v-iK=v_1$, $w-iK=w_1$, then $u+v+w=0$.

But eliminating $\xi\ \eta$ between the three equations

$$\begin{matrix} 1-\xi x_1-\eta\sqrt{(x_1\,.\,\overline{1-x_1}\,.\,\overline{1-k^2x_1})}=0 \\ 1-\xi x_2-\eta\sqrt{(x_2\,.\,\overline{1-x_2}\,.\,\overline{1-k^2x_2})}=0 \\ 1-\xi x_3-\eta\sqrt{(x_3\,.\,\overline{1-x_3}\,.\,\overline{1-k^2x_3})}=0 \end{matrix} \quad \text{whence} \quad \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}=0,$$

and substituting $\operatorname{sn}^2u\ \operatorname{sn}^2v\ \operatorname{sn}^2w$ for $x_1\ x_2\ x_3$, we get the addition-theorem in the form

$$\begin{vmatrix} 1 & \operatorname{sn}^2u & \operatorname{sn}u\operatorname{cn}u\operatorname{dn}u \\ 1 & \operatorname{sn}^2v & \operatorname{sn}v\operatorname{cn}v\operatorname{dn}v \\ 1 & \operatorname{sn}^2w & \operatorname{sn}w\operatorname{cn}w\operatorname{dn}w \end{vmatrix}=0$$

when $u+v+w\equiv iK'$ (mod. $2K$, $2iK'$).

But observing that

$$\operatorname{sn}(u+iK')=\frac{1}{k\operatorname{sn}u},\ \operatorname{cn}(u+iK')=-i\frac{\operatorname{dn}u}{k\operatorname{sn}u},\ \operatorname{dn}(u+iK')=-i\frac{\operatorname{cn}u}{\operatorname{sn}u},$$

we have

$$x=\operatorname{sn}^2(u_1+iK')=\frac{1}{k^2\operatorname{sn}^2u}$$

$$y=\operatorname{sn}(u_1+iK')\,.\operatorname{cn}(u_1+iK')\operatorname{dn}(u_1+iK')=-\frac{\operatorname{cn}u\operatorname{dn}u}{k^2\operatorname{sn}^3u},$$

and the addition-theorem becomes

$$\begin{vmatrix} \operatorname{sn}u & \operatorname{sn}^3u & \operatorname{cn}u\operatorname{dn}u \\ \operatorname{sn}v & \operatorname{sn}^3v & \operatorname{cn}v\operatorname{dn}v \\ \operatorname{sn}w & \operatorname{sn}^3w & \operatorname{cn}w\operatorname{dn}w \end{vmatrix}=0$$

when $u+v+w\equiv 0$.

To verify this observe that

$$\operatorname{sn}u\operatorname{cn}v\operatorname{dn}v-\operatorname{sn}v\operatorname{cn}u\operatorname{dn}u=\operatorname{sn}(u-v)(1-k^2\operatorname{sn}^2u\operatorname{sn}^2v).$$

We have therefore to prove that

$$\operatorname{sn}^3u\,.\operatorname{sn}\overline{v-w}+\operatorname{sn}^3v\,.\operatorname{sn}\overline{w-u}+\operatorname{sn}^3w\,.\operatorname{sn}\overline{u-v}$$
$$=k^2\operatorname{sn}^2u\operatorname{sn}^2v\operatorname{sn}^2w\,(\operatorname{sn}u\,.\operatorname{sn}\overline{v-w}+\operatorname{sn}v\,.\operatorname{sn}\overline{w-u}+\operatorname{sn}w\operatorname{sn}\overline{u-v}).$$

Examples of use of the elliptic parameter.

1. The inflexions of the cubic (cusps of 3rd class curve) are given by $3u\equiv 0$ (mod. $2K$, $2iK'$) or $u=\frac{2}{3}aK+\frac{2}{3}a'iK'$, ($a$, $a'=0, 1, 2$). Hence there are nine of them, and the line joining any two passes through a third. If v is point of contact of tangent from an inflexion, $2v+u\equiv 0$, and thus the points of contact of the *three* tangents from u are in one straight line.

2. If six points $a\ b\ c\ f\ g\ h$ lie on a conic, $a+b+c+f+g+h=0$. This may be proved in the same way as the corresponding property for a straight line, or

thus: if $a+f+x=0$, $b+g+y=0$, $c+h+z=0$, we know that $x+y+z=0$, and therefore...(Clebsch). Hence if a conic have six-pointic contact at v, $6v\equiv 0$; this shews that the v are points of contact of tangents from an inflexion, and there are 27 of them.

3. The points of contact of four tangents from u are $-\frac{1}{2}u$, $-\frac{1}{2}u+K$, $-\frac{1}{2}u+iK'$, $-\frac{1}{2}u+K+iK'$. Hence the line joining two of them meets the line joining the other two on the cubic. [Theory of *corresponding points.*]

4. Grassmann's construction; $xaA\,.\,xbB\,.\,xcC=0$ is equation to a cubic circumscribing the triangles abc, ABC and passing through the three intersections bcA, caB, abC of their corresponding sides. For BC, CA, AB write α, β, γ;

then $\qquad b+c\equiv\beta+\gamma,\ c+a\equiv\gamma+\alpha,\ a+b\equiv\alpha+\beta,$

whence we find $\qquad a-\alpha\equiv b-\beta\equiv c-\gamma\equiv K$ or iK' or $K+iK'$.

Thus given abc there is one triangle $\alpha\beta\gamma$ on the same branch.

VII.

The Theta functions expressed as infinite products.

To expand the product of n factors

$$\Pi^n(1+r^sx)=(1+x)(1+rx)(1+r^2x)\ldots(1+r^{n-1}x)$$

in powers of x, assume that the expansion is

$$P=1+p_1x+p_2x^2+\ldots+p_nx^n.$$

Now if in the product we change x into rx, it becomes multiplied by $1+r^nx$ and divided by $1+x$. But if in P we change x into rx, it becomes

$$Q=1+p_1rx+p_2r^2x^2+\ldots+p_nr^nx^n.$$

Consequently we must have $P\,.\,(1+r^nx)=Q\,.\,(1+x)$. But

$$P(1+r^nx)=1+(p_1+r^n)\,x+(p_2+r^np_1)\,x^2+\ldots+(p_n+r^np_{n-1})\,x^n,$$
$$Q(1+x)=1+(p_1r+1)\,x+(p_2r^2+p_1r)x^2+\ldots+(p_nr^n+p_{n-1}r^{n-1})\,x^n.$$

Equating coefficients of like powers, we find

$$p_1+r^n=p_1r+1, \quad \text{or } p_1=\frac{1-r^n}{1-r},$$

$$p_2+r^np_1=p_2r^2+p_1r, \text{ or } p_2=p_1\,.\,\frac{1-r^{n-1}}{1-r^2}\,.\,r,$$

$$p_3+r^np_2=p_3r^3+p_2r^2, \text{ or } p_3=p_2\,.\,\frac{1-r^{n-2}}{1-r^3}\,.\,r^2,$$

etc. = etc.

$$p_n+r^np_{n-1}=p_nr^n+p_{n-1}r^{n-1}, \text{ or } p_n=p_{n-1}\,.\,\frac{1-r}{1-r^n}\,.\,r^{n-1}.$$

Let the product $(1-r)(1-r^2)\ldots(1-r^n)$ be called $\mathfrak{R}(n)$. Then our result may be written

$$\frac{\Pi^n(y-r^s x)}{\mathfrak{R}(n)}=\frac{y^n}{\mathfrak{R}(n)}+\frac{y^{n-1}}{\mathfrak{R}(n-1)}\cdot\frac{x}{\mathfrak{R}(1)}\cdot r^0+\frac{y^{n-2}}{\mathfrak{R}(n-2)}\cdot\frac{x^2}{\mathfrak{R}(2)}\cdot r^1$$
$$+\ldots+\frac{x^n}{\mathfrak{R}(n)}r^{\frac{1}{2}n(n-1)}.$$

As r approaches the limit 1, the fraction $\frac{1-r^p}{1-r^q}$ approaches the limit $\frac{p}{q}$. Hence the series just obtained passes into the binomial theorem.

To expand the product of the two factorials

$$\Pi^n(1+xr^{2s-1})=(1+xr)(1+xr^3)(1+xr^5)+\ldots(1+xr^{2n-1})$$
$$\Pi^n(1+x^{-1}r^{2s-1})=(1+x^{-1}r)(1+x^{-1}r^3)(1+x^{-1}r^5)\ldots(1+x^{-1}r^{2n-1})$$

in positive and negative powers of x, we proceed as follows. Let each factor $1+x^{-1}r^{2s-1}$ of the second be replaced by $x^{-1}r^{2s-1}(1+xr^{1-2s})$, to which it is equal; then the factorial becomes

$$\Pi^n(1+x^{-1}r^{2s-1})=x^{-n}r^{nn}(1+xr^{-1})(1+xr^{-3})(1+xr^{-5})\ldots(1+xr^{1-2n}),$$

and in this form it is seen to be a continuation of the former factorial backwards, for negative powers of r. To put them both together therefore we must begin with the last factor of the second, $(1+xr^{1-2n})$. Let this be called $1+y$, then $x^{-n}r^{nn}=y^{-n}r^{n(1-n)}$, and we have

$$\Pi^n.(1+xr^{2s-1})(1+x^{-1}r^{2s-1})=y^{-n}r^{n(1-n)}(1+y)(1+yr^2)(1+yr^4)\ldots(1+yr^{2(n-1)})$$
$$=y^{-n}r^{n(1-n)}\Pi^{2n}(1+yr^{2s}).$$

But by our previous result

$$\frac{\Pi^{2n}(1+yr^{2s})}{\mathfrak{R}^2[2n]}=\frac{1}{\mathfrak{R}^2[2n]}+\frac{1}{\mathfrak{R}^2[2n-1]}\cdot\frac{y}{\mathfrak{R}^2[1]}$$
$$+\frac{1}{\mathfrak{R}^2[2n-2]}\cdot\frac{y^2}{\mathfrak{R}^2[2]}r^2+\ldots+\frac{y^{2n}}{\mathfrak{R}^2[2n]}r^{2n(2n-1)};$$

$$\therefore\frac{\Pi^n(1+xr^{2s-1})(1+x^{-1}r^{2s-1})}{\mathfrak{R}^2[2n]}=\frac{x^{-n}}{\mathfrak{R}^2[2n]}r^{nn}+\frac{x^{1-n}}{\mathfrak{R}^2[2n-1]\mathfrak{R}^2[1]}\cdot r^{(n-1)^2}$$
$$+\frac{x^{2-n}}{\mathfrak{R}^2[2n-2]\mathfrak{R}^2[2]}r^{(n-2)^2}+\ldots+\frac{1}{\mathfrak{R}^2[n].\mathfrak{R}^2[n]}+\ldots+\frac{x^{n-1}}{\mathfrak{R}^2[2n-1]\mathfrak{R}^2[1]}\cdot r^{(n-1)^2}$$
$$+\frac{x^n}{\mathfrak{R}^2[2n]}r^{n^2};$$

therefore

$$\frac{\mathfrak{R}^2[n].\mathfrak{R}^2[n]}{\mathfrak{R}^2[2n]}\Pi^n(1+xr^{2s-1})\Pi^n(1+x^{-1}r^{2s-1})=1+r(x+x^{-1}).\frac{1-r^{2n}}{1-r^{2n+2}}$$
$$+r^4(x^2+x^{-2}).\frac{\overline{1-r^{2n}}.\overline{1-r^{2n-2}}}{1-r^{2n+2}.1-r^{2n+4}}+\ldots$$
$$+r^{ss}(x^s+x^{-s}).\frac{1-r^{2n}.1-r^{2n-2}\ldots1-r^{2n-2s+2}}{1-r^{2n+2}.1-r^{2n+4}\ldots1-r^{2n+2s}}\ldots+r^{nn}(x^n+x^{-n})\frac{\mathfrak{R}^2[n]\mathfrak{R}^2[n]}{\mathfrak{R}^2[2n]}.$$

Suppose r less than 1, and let n increase indefinitely. Then $\frac{\mathfrak{R}^2[n]\mathfrak{R}^2[n]}{\mathfrak{R}^2[2n]}=\frac{1-r^2.1-r^4\ldots1-r^{2n}}{1-r^{2n+2}.1-r^{2n+4}\ldots1-r^{4n}}$; now each of the factors $\frac{1-r^{2s}}{1-r^{2n+s}}$ approaches the value $1-r^{2s}$; therefore the limiting value of the expression is $\mathfrak{R}^2[\infty]$. On the right hand side the factors $\frac{1-r^{2n-2s}}{1-r^{2n+2s+2}}$ approach the limit 1; therefore we have

$$\mathfrak{R}^2[\infty]\Pi_0^\infty(1+xr^{2s+1})(1+x^{-1}r^{2s+1})=1+r(x+x^{-1})+r^4(x^2+x^{-2})$$
$$+r^9(x^3+x^{-3})+\text{etc.}$$

Write q for $\mathfrak{R}$ and $e^{2i\phi}$ for x; the formula becomes

$$q^2[\infty].\Pi_0^\infty(1+2q^{2s+1}\cos2\phi+q^{4s+2})=1+2q\cos2\phi+2q^4\cos4\phi+2q^9\cos6\phi+\ldots$$
$$=\theta(y,\log q)\text{ if }\phi=iy.$$

Either putting $\phi+\frac{\pi}{2}$ for ϕ or changing the sign of q, we have

$$q^2[\infty].\Pi_0^\infty(1-2q^{2s-1}\cos2\phi+q^{4s-2})=1-2q\cos2\phi+2q^4\cos4\phi-2q^9\cos6\phi+\ldots$$
$$=\theta'(y,\log q).$$

In the original formula write rx for x; it becomes

$$\mathfrak{R}^2[\infty].\Pi_1^\infty(1+xr^{2s})(1+x^{-1}r^{2s-2})=1+r^2x+x^{-1}+r^6x^2+r^2x^{-2}+r^{12}x^3+r^6x^{-3}+\ldots$$
$$=x^{-\frac{1}{2}}r^{-\frac{1}{4}}\{r^{\frac{1}{4}}(x^{\frac{1}{2}}+x^{-\frac{1}{2}})+r^{\frac{9}{4}}(x^{\frac{3}{2}}+x^{-\frac{3}{2}})+\ldots\}.$$

Here also write q for $\mathfrak{R}$ and $e^{2i\phi}$ for x, then we find

$$q^2[\infty].2q^{\frac{1}{4}}\cos\phi\Pi_0^\infty(1+2q^{2s+2}\cos2\phi+q^{4s+4})$$
$$=2q^{\frac{1}{4}}\cos\phi+2q^{\frac{9}{4}}\cos3\phi+2q^{\frac{25}{4}}\cos5\phi+\ldots$$

In this if we write $\phi+\frac{\pi}{2}$ for ϕ, it becomes

$$q^2[\infty].2q^{\frac{1}{4}}\sin\phi\Pi_0^\infty(1-2q^{2s+2}\cos2\phi+q^{4s+4})=2q^{\frac{1}{4}}\sin\phi-2q^{\frac{9}{4}}\sin3\phi$$
$$+2q^{\frac{25}{4}}\sin5\phi-\ldots$$

Now observing that $q=\epsilon^a$, and that the factor

$$1+2q^{2s+1}\cos2\phi+q^{4s+2}=(1+q^{2s+1}e^{2i\phi})(1+q^{2s+1}e^{-2i\phi}),$$

which becomes

$$(1+e^{(2s+1)a+2i\phi})(1+e^{(2s+1)a-2i\phi})$$
$$=\frac{(e^{-i\phi-(s+\frac{1}{2})a}+e^{i\phi+\overline{s+\frac{1}{2}}.a})(e^{i\phi-(s+\frac{1}{2})a}+e^{-i\phi+\overline{s+\frac{1}{2}}.a})}{e^{-(2s+1)a}}$$
$$=4e^{(2s+1)a}\cos(\phi+\overline{s+\tfrac{1}{2}}ia)\cos(\phi-\overline{s+\tfrac{1}{2}}.ia).$$

Put $\phi=0$ in this; we get

$$1+2q^{2s+1}+q^{4s+2}=4e^{(2s+1)a}\cos^2\overline{s+\tfrac{1}{2}}\,.\,ia,$$

and then by division

$$\frac{1+2q^{2s+1}\cos 2\phi+q^{4s+2}}{1+2q^{2s+1}+q^{4s+2}}=\frac{\cos(\phi+\overline{s+\tfrac{1}{2}}\,.\,ia)\cos(\phi-\overline{s+\tfrac{1}{2}}\,.\,ia)}{\cos^2 s+\tfrac{1}{2}\,.\,ia}.$$

Hence $$\theta(i\phi)=\theta\,.\,\Pi_0^\infty\frac{\cos(\phi+\overline{s+\tfrac{1}{2}}ia)\cos(\phi-\overline{s+\tfrac{1}{2}}\,.\,ia)}{\cos^2 s+\tfrac{1}{2}\,.\,ia},$$

so $$\theta'(i\phi)=\theta\,.\,\Pi_0^\infty\frac{\sin(\phi+\overline{s+\tfrac{1}{2}}\,.\,ia)\sin(\phi-\overline{s+\tfrac{1}{2}}\,.\,ia)}{\cos^2\overline{s+\tfrac{1}{2}}\,.\,ia},$$

whence $$\theta'=\theta\,.\,\Pi_0^\infty\tan^2\overline{s+\tfrac{1}{2}}\,.\,ia,$$

and therefore

$$\theta'(i\phi)=\theta'\,.\,\Pi_0^\infty\frac{\sin(\phi+\overline{s+\tfrac{1}{2}}\,.\,ia)\,.\,\sin(\phi-\overline{s+\tfrac{1}{2}}\,.\,ia)}{\sin^2 s+\tfrac{1}{2}\,.\,ia}.$$

Similarly

$$1+2e^{(2s+2)a}\cos 2\phi+e^{(4s+4)a}=(1+e^{\overline{2s+2}a+2i\phi})(1+e^{\overline{2s+2}a-2i\phi})$$
$$=4e^{\overline{2s+2}a}\cos(\overline{s+1}\,.\,ai+\phi)\cos(s+1\,.\,ai-\phi),$$

and therefore

$$\theta_1(i\phi)=\theta_1\,.\,\Pi_1^\infty\frac{\cos(\phi+s\,.\,ia)\cos(\phi-s\,.\,ia)}{\cos^2 s\,.\,ia};$$

so also

$$\theta_1'(i\phi)=i\theta_1\,.\,\Pi_1^\infty\frac{\sin(\phi+s\,.\,ia)\sin(\phi-s\,.\,ia)}{\cos^2 s\,.\,ia}.$$

Returning to the factorial expression for θx, which may be written in the form

$$\theta x=\theta\,.\,\Pi_0^\infty\frac{\cos i\,.\,(x+\overline{s+\tfrac{1}{2}}\,.\,a)\,.\cos i(x-\overline{s+\tfrac{1}{2}}\,.\,a)}{\cos^2 i\,.\,s+\tfrac{1}{2}\,.\,a},$$

we observe that

$$\cos x=\Pi_0^\infty\left(1-\frac{x^2}{(t+\tfrac{1}{2})^2\pi^2}\right),\ (t\text{ an integer}),\ =\Pi_0^\infty\left(1+\frac{x}{(t+\tfrac{1}{2})\pi}\right)\left(1-\frac{x}{(t+\tfrac{1}{2})\pi}\right).$$

Hence

$$\frac{\cos i.(x+\overline{s+\tfrac{1}{2}}.a)}{\cos i.(\overline{s+\tfrac{1}{2}\cdot a})}=\Pi_0^\infty\frac{1+\dfrac{ix+\overline{s+\tfrac{1}{2}}\,.\,ai}{(t+\tfrac{1}{2})\pi}}{1+\dfrac{(s+\tfrac{1}{2})\,.\,ai}{t+\tfrac{1}{2}\pi}}\cdot\Pi_0^\infty\frac{1-\dfrac{ix+\overline{s+\tfrac{1}{2}}\,.\,ai}{(t+\tfrac{1}{2})\pi}}{1-\dfrac{(s+\tfrac{1}{2})\,.\,ai}{(t+\tfrac{1}{2})\pi}}$$

$$=\Pi_0^\infty\frac{ix+\overline{s+\tfrac{1}{2}}\,.\,ai+\overline{t+\tfrac{1}{2}}\pi}{s+\tfrac{1}{2}\,.\,ai+\overline{t+\tfrac{1}{2}}\pi}\cdot\Pi_0^\infty\frac{ix+\overline{s+\tfrac{1}{2}}\,.\,ai-(t+\tfrac{1}{2})\pi}{(s+\tfrac{1}{2})ai-(t+\tfrac{1}{2})\pi}$$

$$=\Pi_0^\infty\left(1+\frac{x}{(s+\tfrac{1}{2})a+(t+\tfrac{1}{2})\pi i}\right)\Pi_0^\infty\left(1+\frac{x}{(s+\tfrac{1}{2})a-(t+\tfrac{1}{2})\pi i}\right).$$

Let now $(\bar s,\bar t)$ denote $\pm(s+\tfrac{1}{2})a\pm(t+\tfrac{1}{2})\pi i$, and let

$$\Pi\left(1+\frac{x}{(\bar s,\bar t)}\right)=\Pi_0^\infty\Pi_0^\infty\left(1-\frac{x^2}{\{(s+\tfrac{1}{2})a+(t+\tfrac{1}{2})\pi i\}^2}\right)\cdot\left(1-\frac{x^2}{\{(s+\tfrac{1}{2})a-(t+\tfrac{1}{2})\pi i\}^2}\right),$$

it being understood that the infinite values of t are infinitely greater than the infinite values of s; then we shall have

$$\theta x=\theta\,.\,\Pi\left(1+\frac{x}{(\bar s,\bar t)}\right).$$

VIII.

Cayley's Theory of Doubly-infinite factorials.

A product like $\Pi\left(1+\frac{x}{(\bar s,\bar t)}\right)$ containing a doubly infinite number of factors depending on two variable integers s, t, is not fully defined until we have fixed upon the relations of the infinite values of s and t. Let these be regarded as coordinates of a point in a plane; then we may suppose that the product is formed first with those values of s t which lie in a certain closed curve surrounding the origin, and that then this curve is allowed to expand without limit, remaining always similar to itself and similarly situated in regard to the origin. The doubly-infinite product so obtained will depend in general upon the shape of this curve.

If we suppose the curve to have the origin for a centre, i.e. that every line through the origin is bisected by it, then *the value of the product will be determined with the exception of a factor* e^{Ax^2}. For suppose two curves to be drawn having the origin for centre, and let Π, Π' be the products belonging to them. Then

$$\log\Pi-\log\Pi'=\Sigma\log\left(1+\frac{x}{(\bar s,\bar t)}\right)=-x\Sigma\frac{1}{(\bar s,\bar t)}+\frac{x^2}{2}\Sigma\frac{1}{(\bar s,\bar t)^2}-\ldots$$

the summation extending over those values of s, t which correspond to points lying between the two curves. Now $\Sigma\frac{1}{(\bar s,\bar t)}=0$ because of the symmetry; and let A be the limit of $\Sigma\frac{1}{(\bar s,\bar t)^2}$, in comparison with which all subsequent terms must vanish, because s, t become infinite when the curves are increased without limit. Therefore

$$\log\Pi-\log\Pi'=Ax^2,$$

or $$\Pi=e^{Ax^2}\,.\,\Pi'.$$

Now we have shewn that when $\frac{s_\infty}{t_\infty}=0$,

$$\theta x=\theta\,.\,\Pi\left(1+\frac{x}{(\bar s,\bar t)}\right)=\theta\,.\,\Pi\left(1+\frac{x}{(s+\tfrac{1}{2})a+(t+\tfrac{1}{2})\pi i}\right).$$

But

$$\frac{x}{(s+\frac{1}{2})a+(t+\frac{1}{2})\pi i}=\frac{\frac{x\pi i}{a}}{(s+\frac{1}{2})\pi i-(t+\frac{1}{2})\frac{\pi^2}{a}},$$

$$\therefore \theta x=\theta\,.\,\Pi\left(1+\frac{x}{(s,\,t)}\right)=\theta\,.\,\Pi\frac{\frac{x\pi i}{a}}{(s+\frac{1}{2})\pi i+(t+\frac{1}{2})\frac{\pi^2}{a}}$$

$$=\frac{\theta\,.\,e^{Ax^2}\,.\,\theta\left(\frac{x\pi i}{a},\ \frac{\pi^2}{a}\right)}{\theta\left(0,\ \frac{\pi^2}{a}\right)},$$

since, in order that the Π may represent $\theta\left(\frac{x\pi i}{a},\ \frac{\pi^2}{a}\right)$, we must have $\frac{t_\infty}{s_\infty}=0$; the former arrangement regarded the plane as an infinite rectangle, infinitely longer in the direction of t than in that of s; the present one makes it infinitely longer in the direction of s than in that of t.

We must now determine A so that $e^{Ax^2}\theta\left(\frac{x\pi i}{a},\ \frac{\pi^2}{a}\right)$ may be unaltered when x is increased by πi. We have

$$e^{A(x+\pi i)^2}\theta\left(\frac{x\pi i}{a}-\frac{\pi^2}{a},\ \frac{\pi^2}{a}\right)=e^{2Ax\pi i-A\pi^2-\frac{\pi^2}{a}+\frac{2x\pi i}{a}}\,.\,e^{Ax^2}\,\theta\left(\frac{x\pi i}{a},\ \frac{\pi^2}{a}\right),$$

whence $A=-\frac{1}{a}$, and the formula may be written

$$\theta\left(0,\ \frac{\pi^2}{a}\right).\,\theta x=\theta\,.\,e^{-\frac{x^2}{a}}\,\theta\left(\frac{x\pi i}{a},\ \frac{\pi^2}{a}\right)=\theta\,.\,\Sigma e^{-\frac{1}{a}(x+n\pi i)^2}.$$

Observe that the formula may also be written

$$\theta\,.\,\theta\left(\frac{x\pi i}{a},\ \frac{\pi^2}{a}\right)=e^{\frac{x^2}{a}}\,\theta\left(0,\ \frac{\pi^2}{a}\right).\,\theta x=\theta\left(0,\ \frac{\pi^2}{a}\right)\Sigma e^{\frac{1}{a}(x+na)^2}.$$

Similarly for the other θ, we have

$$\theta' x=\theta'\,.\,\Pi\frac{\sin i\,.(x+\overline{s+\frac{1}{2}}\,.\,a)\sin i\,.(x-\overline{s+\frac{1}{2}}\,.\,a)}{\sin^2 i\cdot\overline{s+\frac{1}{2}}\,.\,a},\ \text{and}\ \sin x=x\Pi\left(1-\frac{x^2}{t^2\pi^2}\right);$$

$$\therefore \frac{\sin i\,.(x+\overline{s+\frac{1}{2}}\,.\,a)}{\sin i\,.\,\overline{s+\frac{1}{2}}a}=\frac{x+\overline{s+\frac{1}{2}}\,.\,a}{(s+\frac{1}{2})a}\,\Pi\left(\frac{1+i\frac{x+\overline{s+\frac{1}{2}}\,.\,a}{t\pi}}{1+i\frac{\overline{s+\frac{1}{2}}\,.\,a}{t\pi}}\right)$$

$$=\left(1+\frac{x}{\overline{s+\frac{1}{2}}\,.\,a}\right)\Pi\left(1+\frac{x}{\overline{s+\frac{1}{2}}.a+t\pi i}\right),$$

$$\therefore \theta' x=\theta'\,.\,\Pi\left\{1+\frac{x}{(s,\,t)}\right\}\qquad\left(\frac{s_\infty}{t_\infty}=0\right).$$

Hence also

$$\theta_1 x=\theta_1\,.\,\Pi\left\{1+\frac{x}{(s,\,t)}\right\},\qquad\left(\frac{s_\infty}{t_\infty}=0\right),$$

$$\theta'_1 x=\theta'_1\,.\,\Pi\left\{1+\frac{x}{(s,\,t)}\right\}.\,x.\qquad ,,$$

From these formulæ we may derive the transformations

$$*\ \theta_1^0\left(0,\ \frac{\pi^2}{a}\right).\,\theta' x=\theta'\,.\,e^{-\frac{x^2}{a}}\,\theta_1\left(\frac{x\pi i}{a},\ \frac{\pi^2}{a}\right)=\theta'.\,\Sigma e^{-\frac{1}{a}(x+\overline{n+\frac{1}{2}}\,.\,\pi i)^2},$$

$$\theta'\left(0,\ \frac{\pi^2}{a}\right).\,\theta_1 x=\theta_1\,.\,e^{-\frac{x^2}{a}}\,\theta'\left(\frac{x\pi i}{a},\ \frac{\pi^2}{a}\right)=\theta_1.\,\Sigma(-)^n e^{-\frac{1}{a}(x+n\pi i)^2},$$

$$i\theta'_1\left(0,\ \frac{\pi^2}{a}\right).\,\theta'_1 x=\theta'_1.\,e^{-\frac{x^2}{a}}\,\theta'_1\left(\frac{x\pi i}{a},\ \frac{\pi^2}{a}\right)=\theta'_1.\,\Sigma(-)^n e^{-\frac{1}{a}(x+\overline{n+\frac{1}{2}}\,.\,\pi i)^2},$$

and, expressing these in terms of $f\ g\ h$, we find $\left\{f'u \text{ for } f\left(u,\ \frac{\pi^2}{a}\right)\right\}$

$$f'\left(\frac{x\pi i}{a}\right)=\frac{ifx}{gx},\ g'\left(\frac{x\pi i}{a}\right)=\frac{1}{gx},\ h'\left(\frac{x\pi i}{a}\right)=\frac{hx}{gx}.$$

Whence

$$h'^2+k^2f'^2=g'^2=1+f'^2,\ \text{or}\ h'^2=1+(1-k^2)f'^2=1+k'^2f'^2,$$

from which it follows that k' is the k of the transformed functions.

To express the same result in terms of the elliptic functions we have only to observe that the transformation changes K into K' and vice versa; so that we have, if $x=\frac{\pi ui}{2K}$, $\frac{x\pi i}{a}=-\frac{\pi^2 u}{2Ka}=\frac{\pi u}{2K'}$ and consequently

$$\operatorname{sn}(ix)=i\frac{\operatorname{sn}(x,\,k')}{\operatorname{cn}(x,\,k')}=i\operatorname{tn}(x,\,k'),\qquad \operatorname{cn}(ix)=\frac{1}{\operatorname{cn}(x,\,k')},\qquad \operatorname{dn}(ix)=\frac{\operatorname{dn}(x,\,k')}{\operatorname{cn}(x,\,k')}.$$

The transformation may also be represented as follows. In virtue of the equation $\operatorname{sn}^2 x+\operatorname{cn}^2 x=1$, we are at liberty to write

$$\operatorname{sn} x=i\tan\phi,\quad \operatorname{cn} x=\sec\phi,\quad \operatorname{dn} x=\surd(1+k^2\tan^2\phi),$$

and then

$$\operatorname{cn} x\operatorname{dn} x\,dx=i\sec^2\phi\,d\phi,$$

$$dx=\frac{id\phi\sec\phi}{\surd(1+k^2\tan^2\phi)}=\frac{id\phi}{\surd(\cos^2\phi+k^2\sin^2\phi)}$$

$$=\frac{id\phi}{\surd(1-k'^2\sin^2\phi)},$$

whence

$$\phi=-am\,(ix),$$

$$\operatorname{sn}(ix,\,k')=-\sin\phi=-\frac{\tan\phi}{\sec\phi}=i\frac{\operatorname{sn} x}{\operatorname{cn} x},\quad \operatorname{cn}(ix,\,k')=\cos\phi=\frac{1}{\operatorname{cn} x},$$

$$\operatorname{dn}(ix,\,k')=\surd(1-k'^2\sin^2\phi)=\frac{\operatorname{dn} x}{\operatorname{cn} x},$$

which formulæ are equivalent to the former. Observe that the transformation gives

$$\int_0^1\frac{du}{\surd(1-u^2)(1-k'^2u^2)}=K'.$$

* [? θ_1].

IX.

Problem of linear transformations.

The transformation just considered amounts to an interchange of the period and quasi-period; for the function $\theta\left(\frac{x\pi i}{a}, \frac{\pi^2}{a}\right)$ has the period a and the quasi-period πi. If $\alpha\ \beta\ \gamma\ \delta$ are whole numbers, the problem *to express the θ-function which has the period $\alpha\pi i+\beta a$ and the quasi-period $\gamma\pi i+\delta a$ in terms of the θ-function with period πi and quasi-period a is called the* Transformation-Problem. If we write $\alpha\pi i+\beta a=\omega$, $\gamma\pi i+\delta a=\omega'$, the θ-function is $\theta\left(\frac{x\pi i}{\omega}, \frac{\omega'\pi i}{\omega}\right)$. Now if $\alpha\delta-\beta\gamma=-1$, [in which case the transformation is said to be linear] we shall find $-\pi i=\delta\omega-\beta\omega'$, $-a=\alpha\omega'-\gamma\omega$; and on the plane of complex numbers the area of the parallelogram included by $\omega\ \omega'$ is the same as that included by πi and a. The quotient $\theta\left(\frac{x\pi i}{\omega}, \frac{\omega'\pi i}{\omega}\right) : \theta\left(0, \frac{\omega'\pi i}{\omega}\right)$ is then equal to the doubly infinite product $\Pi\left(1+\frac{x}{(s+\frac{1}{2})\,\omega+(t+\frac{1}{2})\,\omega'}\right)$, $\frac{t_\infty}{s_\infty}=0$, that is when the plane is regarded as an infinite parallelogram whose sides are parallel to $\omega\,\omega'$, but the former side infinitely greater than the latter. Hence we must have

$$\frac{\theta\left(\frac{x\pi i}{\omega}, \frac{\omega'\pi i}{\omega}\right)}{\theta\left(0, \frac{\omega'\pi i}{\omega}\right)}=e^{Ax^2}\frac{\theta_q^p(x)}{\theta_q^p} \text{ if } pq=0, \text{ or } =e^{Ax^2}\frac{\theta_1'(x)}{\theta_1'} \text{ if } p=1,\ q=1.$$

To determine A, observe that the left-hand side is unaltered when x is increased by ω, which is $=\alpha\pi i+\beta a$. Consequently

$$e^{A(x+\alpha\pi i+\beta a)^2}\,e^{q\alpha\pi i-\beta^2 a-2\beta x}\,\theta_q^p x=e^{Ax^2}\,\theta_q^p x,$$

therefore

$$A(\alpha\pi i+\beta a)=\beta, \quad \therefore\ A=\frac{\beta}{\omega},$$

$$A(\alpha\pi i+\beta a)^2=-q\alpha\pi i+\beta^2 a+2m\pi i,$$

or

$$\beta(\alpha\pi i+\beta a)=-q\alpha\pi i+\beta^2 a+2m\pi i.$$

Hence $\quad \alpha q=\alpha\beta$ mod. 2 but $q\equiv\beta+\delta+1$, $\ \therefore\ \alpha\delta=\alpha$ mod. 2.

Now

$$(s+\tfrac{1}{2})\,\omega+(t+\tfrac{1}{2})\,\omega'=(s+\tfrac{1}{2})(\alpha\pi i+\beta a)+(t+\tfrac{1}{2})(\gamma\pi i+\delta a)$$
$$=(\alpha s+\gamma t+\tfrac{1}{2}\,\overline{\alpha+\gamma})\,\pi i+(\beta s+\delta t+\tfrac{1}{2}\cdot\overline{\beta+\delta})\,a.$$

Hence the formulæ must be

$$\frac{\theta\left(\frac{x\pi i}{\omega}, \frac{\omega'\pi i}{\omega}\right)}{\theta\left(0, \frac{\omega'\pi i}{\omega}\right)}=e^{\frac{\beta x^2}{\omega}}\frac{\theta_1'(x+\frac{1}{2}\omega+\frac{1}{2}\omega')}{\theta_1'(\frac{1}{2}\omega+\frac{1}{2}\omega')},$$

or generally since if $p,\ q=0$ or 1,

$$\left(s+\frac{p}{2}\right)\omega+\left(t+\frac{q}{2}\right)\omega'=\left(s+\frac{p}{2}\right)(\alpha\pi i+\beta a)+\left(t+\frac{q}{2}\right)(\gamma\pi i+\delta a)$$
$$=(\alpha s+\gamma t+\tfrac{1}{2}\,\overline{p\alpha+q\gamma})\,\pi i+(\beta s+\delta t+\tfrac{1}{2}\,\overline{p\beta+q\delta})\,a,$$

$$\frac{\theta_q^p\left(\frac{x\pi i}{\omega}, \frac{\omega'\pi i}{\omega}\right)}{\theta_q^p\left(0, \frac{\omega'\pi i}{\omega}\right)}=e^{\frac{\beta x^2}{\omega}}\frac{\theta_{q+1}^{p+1}(x+\frac{1}{2}\omega+\frac{1}{2}\omega')}{\theta_{q+1}^{p+1}(\frac{1}{2}\omega+\frac{1}{2}\omega')}.$$

There are six cases of this. Since $\beta\gamma-\alpha\delta=1$, the four numbers cannot be all even or all odd. If $\beta\gamma$ is odd, α or δ or both may be even; if $\alpha\delta$ is odd, β or γ or both may be even. Besides the case already considered, the only one of interest is $\alpha=\gamma=\delta=1$, $\beta=0$ or $\omega=\pi i$, $\omega'=\pi i+a$; in this we have

$$\theta(x, a+\pi i)=\Sigma e^{n^2(a+\pi i)+2nx}=\theta'(x, a),$$

$$\theta'(x, a+\pi i)=\Sigma e^{n^2\pi i+n^2(a+\pi i)+2nx}=\theta(x, a),$$

$$\theta_1(x, a+\pi i)=\Sigma e^{(n+\frac{1}{2})^2(a+\pi i)+2(n+\frac{1}{2})x}=i\theta_1(x, a),$$

and

$$\theta_1'(x, a+\pi i)=\Sigma e^{n^2\pi i+(n+\frac{1}{2})^2\overline{a+\pi i}+2\overline{n+\frac{1}{2}}\cdot x}=i\theta_1'(x, a).$$

Hence

$$\mathrm{sn}'x=k'\frac{\mathrm{sn}\,x}{\mathrm{dn}\,x}, \quad \mathrm{cn}'x=\frac{\mathrm{cn}\,x}{\mathrm{dn}\,x}, \quad \mathrm{dn}'x=\frac{1}{\mathrm{dn}\,x}.$$

So if λ be the new modulus,

$$\mathrm{dn}'^2x+\lambda^2\,\mathrm{sn}'^2x=\frac{1+\lambda^2k'^2\,\mathrm{sn}^2x}{\mathrm{dn}^2x}=1, \text{ or } \lambda^2=-\frac{k^2}{k'^2},\ \lambda=\frac{ik}{k'}.$$

Also

$$\mathrm{sn}'x=k'\frac{\mathrm{cn}\,x\,\mathrm{dn}^2x+k^2\,\mathrm{cn}\,x\,\mathrm{sn}^2x}{\mathrm{dn}^2x}=k'\,\mathrm{cn}'x\,\mathrm{dn}'x,$$

whence

$$\mathrm{sn}'x=\mathrm{sn}\left(k'x, \frac{ik}{k'}\right)=k'\frac{\mathrm{sn}\,x}{\mathrm{dn}\,x}, \quad \mathrm{cn}\left(k'x, \frac{ik}{k'}\right)=\frac{\mathrm{cn}\,x}{\mathrm{dn}\,x}, \quad \mathrm{dn}\left(k'x, \frac{ik}{k'}\right)=\frac{1}{\mathrm{dn}\,x}.$$

X.

General Problem of Transformation.

Jacobi's Theorem for product of n θ-functions.

Consider the product of the n functions

$$\theta(x+u_s, a)=\Sigma e^{m_s^2 a+2m_s(x+u_s)}, \qquad (s=1,\ 2,\ \dots n)$$

we shall have

$$\Pi\theta(x+u_s)=\Sigma^n e^{a\Sigma m^2+2x\Sigma m+2\Sigma mu},$$

the summation being so taken that the numbers $m_1\ m_2\ \dots m_n$ take all values from $-\infty$ to $+\infty$. For any particular set of values of the m, let their sum Σm be divided by n; let β be the quotient and α the remainder, so that α is less than n and $\Sigma m = n\beta + \alpha$. Then let $m_s - \beta$ be denoted by μ_s; we shall have $\Sigma\mu = \Sigma m - n\beta = \alpha$, or $\Sigma\mu$ is positive and less than n. This being so, the numbers $\mu_s + \beta$ will take all values from $-\infty$ to $+\infty$, provided that (1) the μ take all values consistent with $0 < \Sigma\mu < n$, and (2) β take all values from $-\infty$ to $+\infty$. Now we have for any given set of values of the m,

$$\Sigma m = \alpha + n\beta,$$

$$\Sigma m^2 = \Sigma(\mu+\beta)^2 = \Sigma\mu^2 + 2\beta(\alpha) + n\beta^2,$$

$$\Sigma mu = \Sigma\mu u + \beta\Sigma u.$$

The exponent of the general term becomes therefore

$$a(\Sigma\mu^2 + 2\alpha\beta + n\beta^2) + 2x(\alpha + n\beta) + 2\Sigma\mu u + 2\beta\Sigma u$$

and we have

$$\Pi\theta(x+u_s) = \sum_{\alpha=0}^{\alpha=n-1} P_\alpha e^{2\alpha x} \sum_{\beta=-\infty}^{\beta=+\infty} e^{\beta^2 . na + 2\beta(nx + a\alpha + \Sigma u)}$$

$$= \sum_{\alpha=0}^{\alpha=n-1} P_\alpha e^{2\alpha x} \theta(nx + a\alpha + \Sigma u, na),$$

where $P_\alpha = \Sigma e^{a\Sigma\mu^2 + 2\Sigma\mu u}$, the summation extended over all those values of μ which make $\Sigma\mu = \alpha$. The values of the P_α may be determined as follows. Write in the formula $x + \frac{h\pi i}{n}$ for x; then we find

$$\Pi\theta\left(x + u_s + \frac{h\pi i}{n}\right) = \sum_{\alpha=0}^{\alpha=n-1} P_\alpha e^{2\alpha x} e^{2\alpha\frac{h\pi i}{n}} \theta(nx + a\alpha + \Sigma u, na).$$

By giving to h the values $0, 1, 2 \ldots n-1$ we obtain n equations between the n quantities $\theta(nx + a\alpha + \Sigma u, na)$ for values of α from 0 to $n-1$. Solving these by means of the known properties of the n^{th} roots of unity, we find

$$nP_\alpha e^{2\alpha x} \theta(nx + a\alpha + \Sigma u, na) = \sum_{h=0}^{h=n-1} e^{-2\alpha\frac{h\pi i}{n}} \prod_{s=1}^{s=n} \theta\left(x + u_s + \frac{h\pi i}{n}\right),$$

this determines the P_α when we put $x = 0$.

Suppose now that n is a prime number, and that $u_s = \frac{s\pi i}{n}$. Then $\Pi\theta(x + u_s)$ is unaltered when we increase x by $\frac{h\pi i}{n}$; moreover we have $\Sigma u = \frac{1}{2}(n-1)\pi i$. Therefore, since $\Sigma e^{-2\alpha\frac{h\pi i}{n}} = 0$ unless $\alpha = 0$, and then $= n$, it follows that

$$P_0 \theta(nx, na) = \Pi\theta\left(x + \frac{s\pi i}{n}\right).$$

Putting $x = 0$, we convert the equation into

$$\frac{\theta(nx, na)}{\theta(0, na)} = \frac{\Pi\theta\left(x + \frac{s\pi i}{n}\right)}{\Pi\theta\left(\frac{s\pi i}{n}\right)}.$$

Write $x + \frac{1}{2}\pi i$ for x, and remember that n is odd; thus

$$\frac{\theta'(nx, na)}{\theta(0, na)} = \frac{\Pi\theta'\left(x + \frac{s\pi i}{n}\right)}{\Pi\theta\left(\frac{s\pi i}{n}\right)} \text{ whence } \frac{\theta'(0, na)}{\theta(0, na)} = \frac{\Pi\theta'\left(\frac{s\pi i}{n}\right)}{\Pi\theta\left(\frac{s\pi i}{n}\right)}.$$

Again, writing successively $x + \frac{1}{2}a$, $x + \frac{1}{2}\pi i + \frac{1}{2}a$ for x, we find

$$\frac{\theta_1(nx, na)}{\theta_1(0, na)} = \frac{\Pi\theta_1\left(x + \frac{s\pi i}{n}\right)}{\Pi\theta_1\left(\frac{s\pi i}{n}\right)},$$

$$\frac{\theta'_1(nx, na)}{\theta(0, na)} = \frac{\Pi\theta'_1\left(x + \frac{s\pi i}{n}\right)}{\Pi\theta\left(\frac{s\pi i}{n}\right)},$$

whence

$$\frac{\theta'_1(0, na)}{\theta(0, na)} = \frac{\theta'_1 \prod_{s=1}^{s=n-1} \theta'_1\left(\frac{s\pi i}{n}\right)}{\Pi\theta\left(\frac{s\pi i}{n}\right)},$$

$$\therefore \operatorname{sn}\left(\frac{x}{M}, \lambda\right) = \frac{\prod_{s=0}^{s=n-1} \operatorname{sn}\left(x + \frac{2sK}{n}\right)}{\prod_{s=0}^{s=n-1} \operatorname{sn}\left(K + \frac{2sK}{n}\right)},$$

where

$$\frac{1}{M} = \frac{\prod_{s=1}^{s=n-1} \operatorname{sn}\left(\frac{2sK}{n}\right)}{\Pi \operatorname{sn}\left(K + \frac{2sK}{n}\right)},$$

$$\operatorname{cn}\left(\frac{x}{M}, \lambda\right) = \frac{\Pi \operatorname{cn}\left(x + \frac{2sK}{n}\right)}{\Pi \operatorname{cn}\left(\frac{2sK}{n}\right)},$$

$$\operatorname{dn}\left(\frac{x}{M}, \lambda\right) = \frac{\Pi \operatorname{dn}\left(x + \frac{2sK}{n}\right)}{\Pi \operatorname{dn}\left(\frac{2sK}{n}\right)}.$$

XI.

Schröter's Theorem for product of two θ-functions.

We propose to find the value of the product

$$\theta(x, pa) \,.\, \theta(y, qa) = \Sigma e \,|\, m^2pa + 2mx \,| \,.\, \Sigma e \,|\, n^2qa + 2ny \,|$$
$$= \Sigma\Sigma e \,|\, \overline{m^2p + n^2q} \,.\, a + 2mx + 2ny \,|.$$

Let $n-m$ when divided by $p+q$ give s for quotient and μ for remainder, so that $n-m = s(p+q)+\mu$, $(\mu < p+q)$.

If then we make

$$m = t - qs,$$

we must have

$$n = t + ps + \mu,$$

and the numbers m, n will take all integer values from $-\infty$ to $+\infty$ if μ takes all positive integer values $< p+q$, and s, t take all integer values from $+\infty$ to $-\infty$. Making this substitution, the exponent of the general term becomes

$$(t-qs)^2 pa + (t+ps+\mu)^2 qa + 2(t-qs)x + 2(t+ps+\mu)y$$
$$= \mu^2 qa + 2\mu y + (t^2 \,.\, \overline{p+q} \,.\, a + 2t \,.\, \overline{x+y+\mu qa}) + (s^2 \,.\, pq \,.\, \overline{p+q} \,.\, a + 2s \,.\, \overline{py - qx + \mu pqa}).$$

Consequently we have

$$\theta(x, pa)\, \theta(y, qa)$$
$$= \sum_{\mu=0}^{\mu=p+q-1} e^{\mu^2 qa + 2\mu y}\, \theta(x+y+\mu qa,\ \overline{p+q} \,.\, a) \,.\, \theta(py - qx + \mu pqa,\ pq \,.\, \overline{p+q} \,.\, a)$$
$$= \sum_{\mu=0}^{\mu=p+q-1} e^{\mu^2 pa + 2\mu x}\, \theta(x+y+\mu pa,\ \overline{p+q} \,.\, a)\, \theta(qx - py + \mu pqa,\ pq \,.\, \overline{p+q} \,.\, a).$$

For x, y write $y+x$, $ny-x$, and for p, q, 1 and n respectively, in the second formula; then

$$\theta(x+y, a)\, \theta(ny-x, na)$$
$$= \sum_{\mu=0}^{\mu=n} e^{\mu^2 a + 2\mu \,.\, \overline{x+y}}\, \theta(\overline{n+1} \,.\, y + \mu a,\ \overline{n+1} a)\, \theta(\overline{n+1} \,.\, x + \mu na,\ n \,.\, \overline{n+1}\, a).$$

Assuming now that n is odd, write $x + \frac{1}{2}\pi i$ for x; then

$$\theta'(x+y, a)\, \theta'(ny-x, na)$$
$$= \Sigma e^{\mu^2 a + 2\mu \,.\, \overline{x+y}} (-)^\mu\, \theta(\overline{n+1} \,.\, y + \mu a,\ \overline{n+1}\, a)\, \theta(\overline{n+1}\, x + \mu na,\ n \,.\, \overline{n+1}\, a).$$

When we add together these formulæ, only those terms remain on the right for which μ is even; we may therefore write 2μ instead of μ, and then

$$\theta\overline{x+y, a} \,.\, \theta\overline{ny-x, na} + \theta'\overline{x+y, a} \,.\, \theta'\overline{ny-x, na}$$
$$= 2 \sum_{\mu=0}^{\mu=\frac{1}{2}(n-1)} e^{4\mu^2 a + 4\mu\overline{x+y}}\, \theta(\overline{n+1} \,.\, y + 2\mu a,\ \overline{n+1}\, a)\, \theta(n+1 \,.\, x + 2\mu na,\ n \,.\, \overline{n+1}\, a).$$

Now $\theta x + \theta' x = 2\theta(2x, 4a)$, $\theta x - \theta' x = 2\theta_1(2x, 4a)$, by direct addition of series, and hence

$$\theta(x, a)\, \theta(y, b) + \theta'(x, a)\, \theta'(y, b) = 2\theta(2x, 4a)\, \theta(2y, 4b) + 2\theta_1(2x, 4a)\, \theta_1(2y, 4b).$$

Transforming by this formula the left-hand of the equation last arrived at, and then writing $\frac{1}{2}x$, $\frac{1}{2}y$, $\frac{1}{4}a$ instead of x, y, a, we find

$$\theta\overline{x+y, a}\ \theta\overline{ny-x, na} + \theta_1\overline{x+y, a} \,.\, \theta_1\overline{ny-x, na}$$
$$= \Sigma e^{\mu^2 a + 2\mu\overline{x+y}}\, \theta\left(\frac{n+1}{2} y + \frac{\mu}{2} a,\ \frac{n+1}{4} a\right) \theta\left(\frac{n+1}{2} x + \tfrac{1}{2}\mu na,\ \frac{n \,.\, \overline{n+1}}{4} a\right).$$

Assuming further that $n+1$ is divisible by 4, let $n = 4m-1$, so that

$$\theta(x+y, a)\, \theta(ny-x, na) + \theta_1(x+y, a)\, \theta_1(ny-x, na)$$
$$= \Sigma e^{\mu^2 a + 2\mu\overline{x+y}}\, \theta(2my + \tfrac{1}{2}\mu a,\ ma)\, \theta(2mx + \tfrac{1}{2}\mu na,\ mna).$$

Now writing $x + \frac{1}{2}\pi i$ for x, adding the two formulæ, and putting 2μ for μ, since only those terms remain in which μ is even, we get

$$\left.\begin{aligned} &\theta(x+y, a)\, \theta(\overline{4m-1} \,.\, y - x,\ \overline{4m-1}\, a) \\ &+ \theta'(x+y, a)\, \theta'(\overline{4m-1} \,.\, y - x,\ \overline{4m-1}\, a) \\ &+ \theta_1(x+y, a)\, \theta_1(\overline{4m-1} \,.\, y - x,\ \overline{4m-1}\, a) \\ &- \theta'_1(x+y, a)\, \theta'_1(\overline{4m-1} \,.\, y - x,\ \overline{4m-1}\, a) \end{aligned}\right\}$$
$$= 2 \sum_{0}^{m-1} e^{4\mu^2 a + 2\mu\overline{x+y}}\, \theta(2my + \mu a,\ ma)\, \theta(2nx + \mu na,\ mna).$$

In the case $m=1$, $n=3$, $x=y=0$, we find, writing Θ for $\theta(0, 3a)$

$$\theta_1\Theta_1 + \theta'\Theta' = \theta\Theta, \text{ or } \sqrt{kl} + \sqrt{k'l'} = 1,$$

which is the modular equation in a form given by Jacobi. It is worth observing that we have more generally

$$\theta_1 x\Theta_1 x + \theta' x\Theta' x - \theta_1' x\Theta_1' x = \theta x\Theta x.$$

XII.

Rosenhain's functions, and integrals of the third kind.

[Unfinished.]

If we write $R = \sqrt{x(1-x)(1-k^2x)(1-l^2x)(1-m^2x)}$, and if

$$u = \int_0^{x_1} \frac{\alpha + \beta x}{R}\, dx \pm \int_0^{x_2} \frac{\alpha + \beta x}{R}\, dx,$$
$$v = \int_0^{x_1} \frac{\alpha' + \beta' x}{R}\, dx \pm \int_0^{x_2} \frac{\alpha' + \beta' x}{R}\, dx,$$

then x_1 x_2 are the roots of a quadratic equation $L + 2Mx + Nx^2 = 0$, whose coefficients LMN can be rationally expressed in terms of [double] θ-functions of u, v. Such a [double θ-] function is defined by the equation

$$\theta(u, v) = \sum_m \sum_n e \mid m^2 a + 2mnb + n^2 c + 2mu + 2nv \mid$$

and the most general form is

$$\theta^{pq}_{rs}(u, v) = \sum\sum_{mn} e \left| \left(m + \frac{r}{2}\right)^2 a + 2\left(m + \frac{r}{2}\right)\left(n + \frac{s}{2}\right) b + \left(n + \frac{s}{2}\right)^2 c \right.$$
$$\left. + 2\left(m + \frac{r}{2}\right)(u + \tfrac{1}{2}p\pi i) + 2\left(n + \frac{s}{2}\right)(v + \tfrac{1}{2}q\pi i) \right|,$$

where $p\ q\ r\ s$ are either 1 or 0. This is Jacobi's form of the inversion problem for the integrals $\int \frac{\alpha + \beta x}{R}\, dx$.

We now consider the particular case $l = m$, in which these integrals are reduced to elliptic integrals. We then have

$$R = (1 - l^2 x)\sqrt{x(1-x)(1-k^2 x)},$$

and as the quantities $\alpha\beta\ \alpha'\beta'$ are arbitrary we will make $2\alpha = 1$, $2\beta = -l^2$, so that $2(\alpha + \beta x) = 1 - l^2 x$. The first of our two equations then becomes

$$u = \int_0^{x_1} \frac{\frac{1}{2}dx}{\sqrt{x(1-x)(1-k^2x)}} \pm \int_0^{x_2} \frac{\frac{1}{2}dx}{\sqrt{x(1-x)(1-k^2x)}} = u_1 + u_2,$$

where $$x_1 = \mathrm{sn}^2 u_1, \quad x_2 = \mathrm{sn}^2 u_2.$$

To determine conveniently $\alpha'\ \beta'$, observe that we have

$$\tfrac{1}{2}\log\frac{\theta'\overline{x-y}}{\theta'\overline{x+y}} + \frac{\theta'y}{\theta'y}\,x = \lambda fy \,.\, gy \,.\, hy \int_0^x \frac{k^2 fx^2 dx}{1 - k^2 fy^2 \,.\, fx^2},$$

or $$\tfrac{1}{2}\log\frac{\theta'\overline{x-y}}{\theta'\overline{x+y}} = \int_0^x \frac{-\dfrac{\theta'y}{\theta'y} + k^2\left(fy^2\dfrac{\theta'y}{\theta'y} + \lambda fy \,.\, gy \,.\, hy\right) fx^2}{1 - k^2 fy^2 \,.\, fx^2}\, dx,$$

which becomes, if we make $x = \frac{i\pi u_1}{2K} = \lambda u$, $y = \frac{i\pi a}{2K} = \lambda a$, $Za = \frac{\theta'\left(\frac{i\pi a}{2K}\right)}{\theta'\left(\frac{i\pi a}{2K}\right)}$,

$$\tfrac{1}{2}\log\frac{\theta'\frac{i\pi}{2K}(u_1 - a)}{\theta'\frac{i\pi}{2K}(u_1 + a)} = \int_0^{u_1} \frac{-Za + k^2(\mathrm{sn}^2 a Za + \mathrm{sn}\, a\, \mathrm{cn}\, a\, \mathrm{dn}\, a)\,\mathrm{sn}^2 u_1}{1 - k^2 \mathrm{sn}^2 a\, \mathrm{sn}^2 u_1}\, du_1.$$

Hence if we write $k^2 \mathrm{sn}^2 a = l^2$, $2\alpha' = -Za$, $2\beta' = k^2(\mathrm{sn}^2 a Za + \mathrm{sn}\, a\, \mathrm{cn}\, a\, \mathrm{dn}\, a)$ we shall have from the second equation

$$v = \tfrac{1}{2}\log\frac{\theta'\frac{i\pi}{2K}(u_1 - a)}{\theta'\frac{i\pi}{2K}(u_1 + a)} + \tfrac{1}{2}\log\frac{\theta'\frac{i\pi}{2K}(u_2 - a)}{\theta'\frac{i\pi}{2K}(u_2 + a)},$$

whence $$e^{2v} = \frac{\theta'\frac{i\pi}{2K}(u_1 - a)\, \theta'\frac{i\pi}{2K}(u_2 - a)}{\theta'\frac{i\pi}{2K}(u_1 + a)\, \theta'\frac{i\pi}{2K}(u_2 + a)}.$$

Now since $u = u_1 + u_2$, [we have, transforming the expressions for

$$\vartheta' \,.\, \vartheta' u \,.\, \vartheta'\overline{u_1 + a} \,.\, \vartheta'\overline{u_2 + a}, \quad \vartheta_1' \,.\, \vartheta_1' u \,.\, \vartheta_1'\overline{u_1 + a} \,.\, \vartheta_1'\overline{u_2 + a},$$

of which the second vanishes, by the formula for the multiplication of four theta-functions, and adding the results]

$$\tfrac{1}{2}\vartheta' \,.\, \vartheta' u \,.\, \vartheta'\overline{u_1 + a} \,.\, \vartheta'\overline{u_2 + a} = \vartheta'\overline{u + a} \,.\, \vartheta' a \,.\, \vartheta' u_1 \,.\, \vartheta' u_2 + \vartheta_1'\overline{u + a} \,.\, \vartheta_1' a \,.\, \vartheta_1' u_1 \,.\, \vartheta_1' u_2,$$

whence $$e^{2v} = \frac{\vartheta'\overline{u - a} \,.\, \vartheta' a \,.\, \vartheta' u_1 \,.\, \vartheta' u_2 - \vartheta_1'\overline{u - a} \,.\, \vartheta_1' a \,.\, \vartheta_1' u_1 \,.\, \vartheta_1' u_2}{\vartheta'\overline{u + a} \,.\, \vartheta' a \,.\, \vartheta' u_1 \,.\, \vartheta' u_2 + \vartheta_1'\overline{u + a} \,.\, \vartheta_1' a \,.\, \vartheta_1' u_1 \,.\, \vartheta_1' u_2},$$

and $$\frac{\vartheta_1' u_1 \,.\, \vartheta_1' u_2 \,.\, \vartheta_1' a}{\vartheta' u_1 \,.\, \vartheta' u_2 \,.\, \vartheta' a} = \frac{e^{-v}\vartheta'(u - a) - e^{v}\vartheta'(u + a)}{e^{-v}\vartheta_1'(u - a) + e^{v}\vartheta_1'(u + a)} = k^{\frac{3}{2}}\, \mathrm{sn}\, a \sqrt{x_1 x_2}.$$ *

* [Here the MS. ends.]

ON ELLIPTIC FUNCTIONS.

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*On the multiplication of infinite series**.

The properties of the θ-functions are most easily investigated by multiplying together the series which expand them and rearranging the terms. We shall now therefore examine some of the conditions under which such rearrangement is justified.

A singly infinite series consisting entirely of positive terms, if it converges, must converge independently of the order of the terms. For let P be the sum of the series, if the terms are arranged in a certain order; and let Q_n be the sum of n terms, when they are arranged in any other order. Then Q_n cannot exceed P, and therefore must have some limit when n is increased indefinitely. Let Q be this limit, and let P_n be the sum of n terms of the former arrangement. Then P_n cannot exceed Q, because the terms are all positive. Hence $P=Q$, because neither of them can exceed the other.

If the series consist of positive and negative terms, its sum will be independent of the order if the positive and negative parts converge separately. Let P and $-Q$ be the sums, P_m and $-Q_n$ the sums of m and n terms, of the positive and negative parts respectively. Then $P-P_m$ and $Q-Q_n$ can be made as small as we like by taking m and n large enough. Hence $P-Q-P_m+Q_n$ can be made as small as we like, and therefore $P-Q$ is the sum of the compound series, whatever the order of the terms.

(The proof applies to *any* two convergent series P and Q, provided that the order of the terms in each is preserved in mixing them up.)

Similarly, in a series of complex terms, if the positive and negative real and imaginary parts converge separately, no change can be made in the sum by altering the order of the terms.

This will be the case if the series converges when we substitute for each term $p+iq$ its modulus $\sqrt{(p^2+q^2)}$; for this is at least as great as p or q, and hence the series of the real and imaginary parts of the positive and negative terms must converge to a sum less than that of the moduli.

* [The *Tract on Elliptic Functions*, cf. p. 442, consists of 16 pages of MS. with the following headings:—Definition of the Theta-series (3 pp.); the four Theta-functions (3 pp.); on the Multiplication of Infinite Series (6 pp.); Reciprocal Sets of Numbers (2 pp.); the Multiplication of Four Theta-functions (2 pp.).]

If we multiply together two series—for example the series for e^x and e^y, we get a result of this kind:

$$\begin{array}{cccccccccccc}
1 & + & x & + & \frac{x^2}{\Pi 2} & + & \frac{x^3}{\Pi 3} & + & \frac{x^4}{\Pi 4} & + \dots \\
+ & & + & & + & & + & & + & \\
y & + & xy & + & \frac{x^2y}{\Pi 2} & + & \frac{x^3y}{\Pi 3} & + & \frac{x^4y}{\Pi 4} & + \dots \\
+ & & + & & + & & + & & + & \\
\frac{y^2}{\Pi 2} & + & \frac{xy^2}{\Pi 2} & + & \frac{x^2y^2}{\Pi 2 . \Pi 2} & + & \frac{x^3y^2}{\Pi 3 . \Pi 2} & + & \frac{x^4y^2}{\Pi 4 . \Pi 2} & + \dots \\
+ & & + & & + & & + & & + & \\
\frac{y^3}{\Pi 3} & + & \frac{xy^3}{\Pi 3} & + & \frac{x^2y^3}{\Pi 2 . \Pi 3} & + & \frac{x^3y^3}{\Pi 3 . \Pi 3} & + & \frac{x^4y^3}{\Pi 4 . \Pi 3} & + \dots \\
+ & & + & & + & & + & & + & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

This may be called a *doubly infinite* series; the general term is $\frac{x^m y^n}{\Pi m . \Pi n}$, containing *two* variable integers m, n, and we obtain the series by giving to each of these all positive integral values, zero included. In summing this series, the terms may be taken in various orders. We may first take all the terms in the first horizontal line, then all those in the second, and so on; this presents the series in the singly infinite form

$$e^x + ye^x + \tfrac{1}{2}y^2e^x + \dots$$

the sum of which we know to be e^xe^y. Or we may take the left-hand column first, then the second column, and so on; this presents the series in the form

$$e^y + xe^y + \tfrac{1}{2}x^2e^y + \dots$$

whose sum is again e^ye^x. The former process approaches the quarter of an infinite plane on which the series is spread out as an infinitely long horizontal rectangle whose breadth is increased without limit; the latter as an infinitely long vertical rectangle whose breadth is increased without limit.

We may also reduce the series to a singly infinite one in the following way. Namely, it is equal to

$$1+(x+y)+\left(\frac{x^2}{\Pi 2}+xy+\frac{y^2}{\Pi 2}\right)+\left(\frac{x^3}{\Pi 3}+\frac{x^2y}{\Pi 2}+\frac{xy^2}{\Pi 2}+\frac{x^3}{\Pi 3}\right)+\dots$$

$$=1+x+y+\frac{(x+y)^2}{\Pi 2}+\frac{(x+y)^3}{\Pi 3}+\dots=e^{x+y}$$

since, by the binomial theorem for a positive integer exponent,

$$\frac{(x+y)^n}{\Pi n}=\frac{x^n}{\Pi n}+\frac{x^{n-1}y}{\Pi(n-1)}+\frac{x^{n-2}y^2}{\Pi(n-2).\Pi 2}+\dots+\frac{y^n}{\Pi n}=\Sigma\frac{x^ay^b}{\Pi a.\Pi b}. \qquad (a+b=n)$$

Now this mode of summing the doubly infinite series takes together the terms on an oblique line joining corresponding terms of the top row and left-hand column. It approaches the quarter of an infinite plane as a triangle in the shape of half a square whose size is indefinitely increased.

We may sum this doubly infinite series in yet another way which leads to a useful result. Let $x = uv$, $y = \frac{u}{v}$; and let moreover

$$fu = 1 + u + \frac{u^2}{(\Pi 2)^2} + \frac{u^3}{(\Pi 3)^2} + \frac{u^4}{(\Pi 4)^2} + \dots$$

$$f_k u = (\partial_u)^k fu = \frac{1}{\Pi k} + \frac{u}{\Pi(k+1)} + \frac{u^2}{\Pi 2 \,.\, \Pi(k+2)} + \dots$$

$$= \Sigma \frac{u^n}{\Pi k \,.\, \Pi(n+k)}.$$

Then we shall find for the product $e^{uv} \,.\, e^{\frac{u}{v}}$ the doubly infinite series

$$\begin{array}{ccccccccc} 1 & + & uv & + & \frac{u^2v^2}{\Pi 2} & + & \frac{u^3v^3}{\Pi 3} & + \dots \\ + & & + & & + & & + & \\ uv^{-1} & + & u^2 & + & \frac{u^3v}{\Pi 2} & + & \frac{u^4v^2}{\Pi 3} & + \dots \\ + & & + & & + & & + & \\ \frac{u^2v^{-2}}{\Pi 2} & + & \frac{u^3v^{-1}}{\Pi 2} & + & \frac{u^4}{\Pi 2 \,.\, \Pi 2} & + & \frac{u^5v}{\Pi 2 \,.\, \Pi 3} & + \dots \\ + & & + & & + & & + & \\ \frac{u^3v^{-3}}{\Pi 3} & + & \frac{u^4v^{-2}}{\Pi 3} & + & \frac{u^5v^{-1}}{\Pi 3 \,.\, \Pi 2} & + & \frac{u^6}{\Pi 3 \,.\, \Pi 3} & + \dots \\ + & & + & & + & & + & \\ \vdots & & \vdots & & \vdots & & \vdots & \end{array}$$

And if we now sum first all the terms on the middle diagonal going downwards to the right from the term 1, then those in the two lines parallel to this on either side, and so on, we shall obtain

$$e\,|\,u(v+v^{-1})\,| = f(u^2) + (v+v^{-1})\,uf_1(u^2) + (v^2+v^{-2})\,u^2f_2(u^2) + (v^3+v^{-3})\,u^3f_3(u^2) + \dots$$

In this formula write ix for u, and $ie^{i\theta}$ for v, so that

$$u(v+v^{-1}) = -2x\sin\theta, \quad u^k(v^k+v^{-k}) = (+i)^k\, 2x^k \cos k\left(\theta + \frac{\pi}{2}\right),$$

then $\quad e\,|-2ix\sin\theta\,| = f(-x^2) - 2ixf_1(-x^2) \,.\, \sin\theta - 2x^2f_2(-x^2)\cos 2\theta - \dots$

$$= f(-x^2) + 2\Sigma_1^\infty (i)^n \,.\, x^n f_n(-x^2) \,.\, \cos n(\theta + \tfrac{1}{2}\pi),$$

so that

$$\cos(2x\sin\theta) = f(-x^2) - 2x^2f_2(-x^2)\cos 2\theta + 2x^4f_4(-x^2)\cos 4\theta - \dots$$

$$\sin(2x\sin\theta) = 2xf_1(-x^2)\sin\theta - 2x^3f_3(-x^2)\sin 3\theta + 2x^5f_5(-x^2)\sin 5\theta - \dots$$

The function $x^nf_n(-x^2)$ is called Bessel's function of the n^{th} order, and is generally denoted by $J_n(2x)$.

This process of summing the doubly infinite series approaches the infinite area as an infinitely long figure parallel to the middle diagonal, whose breadth is indefinitely increased.

Thus we have considered four ways of approaching the infinite plane, and each of them consists in taking an area of a certain shape and then allowing it to expand indefinitely.

If the two numbers m, n which determine the place of any term in the series are allowed to take negative as well as positive values, the doubly infinite series will cover the whole plane instead of only a quarter of it. The process of summing the series will still consist in taking an area of a certain shape, and allowing it to expand indefinitely while it remains similar to itself and similarly situated in regard to the origin.

The question is, does the sum of the series depend upon the shape of this area? We may shew very easily that it does *not* so depend when the terms are all real and positive. For let P be the sum of the series when it is summed in one way; then the sum of no portion of the series, however selected, can exceed P. If then P' be an approximation to P made by taking a large number of steps towards the summation that way, Q' a similar approximation to Q, the sum obtained by another arrangement; then P' cannot exceed Q, nor Q', P. And since $P-P'$, $Q-Q'$ can be made as small as we like by proceeding sufficiently with the summation, it follows that P cannot exceed Q, nor Q, P, and consequently $P=Q$.

Hence as before the sum is independent of the mode of summation if the series converges when we substitute for each term its modulus.

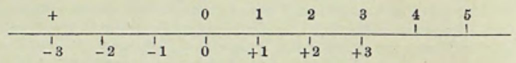
Similar reasoning applies to multiply infinite series, or, as a particular case, to products of any number of singly infinite series.

Now in the case of the product just considered, and in that of the products of θ-functions to be presently treated, it is plain that this condition is satisfied. The transformations of this section, therefore, are all of them valid.

NOTES OF LECTURES ON QUATERNIONS*.

We will define the signs + and − as indicating steps, whereby any magnitude may be increased or diminished, or by means of which we may move from one point of a progression to another.

Let the quantity which has the progressive values be measured along a line; but we shall suppose at first that the numbers stand for amounts of any kind of quantity, not necessarily length.



Then we should have a zero value and successive values marked off, *above* the line, by the numbers 1 2 3 etc. Numbers standing alone mean amounts of the given quantity, while the sign + or − before them means a step is to be taken through successive values of that quantity so as to lead to a different value from that from which we set out.

Thus the equation

$$2+3=5$$

means that if we start with 2 things represented by 2 places on the line and step through 3 following places we get to the place indicated by 5.

But there is another way of considering this equation. Instead of marking off on our line a scale of numbers merely it may be a scale of steps as by the positive and negative numbers below the line.

And then our equation would be

$$+2+3=+5.$$

Or whatever number we set out from and take two steps to the right and then three again to the right it is the same as taking 5 at once to the right. Here are two interpretations of the equation.

Again consider this form of the same truth

$$5-3=2.$$

This may mean first—

Starting with a number 5 denoted by its place on the line, and taking three steps backward, to the left, we get to the number 2. Or it may mean, if we consider the scale as one of steps,

* [I am indebted to Miss E. Watson for these *Notes* of lectures delivered at University College, towards the close of 1877.]

A step 5 to the right and then a step 3 to the left leads to the same number as if starting with the same we took a step 2 to the right.

Now let us combine two forward steps,

$$2+3+4=9=2+7.$$

This means that if we start at the place 2 and take 3 steps to the right and then 4 to the right we get to the place 9; or this is the same as starting with the place 2 and taking a step 7 forward.

But if this is written

$$+2+3+4=+9,$$

we should interpret it thus:

Starting with any number and making the steps to the right represented by 2, 3, 4, is the same as making the step 9 to the right.

Further

$$2=9-4-3$$

may mean either that starting at the place 9 and taking 4 steps to the left and then 3 again to the left we get to the place: or it may mean that starting anywhere, making a step 9 to the right and then steps 4 and 3 to the left is the same as making the step 2 to the right.

On the whole then we have two modes of interpreting these operations of addition and subtraction.

The corresponding forms of the last written equation are

$$2+3+4=9,$$

$$+2+3+4=+9.$$

The common interpretation would be 2 things and 3 things and 4 things all treated in the same way make 9 things.

I. The first equation consists of a number with two steps performed on it, leading to a number.

II. The second means altogether steps, which lead to a step performed on *any* number.

Whenever as a result we get a negative expression, it means a step of *decrease* of a quantity or of direction.

We now go on to the symbols of Multiplication.

Again every equation has two distinct interpretations, as well as the common arithmetical one:

$$2\times 3=6,$$

may be read '6 is the product of 2 and 3', in which case numbers are treated in just the same way.

But we shall leave this meaning out of account and read the equation either as

I. Twice three are 6. I perform an operation on three things by the operator 2; or,

II. I take any number, double and triple it. This is equivalent to the operation of sextupling it.

Now let us consider the multiplication of steps instead of things.

In the equation

$$2 \times (+3) = +6$$

the last term on each side is a step, the first is an operator and the equation means by means of doubling I can turn a step 3 to the right into a step 6 to the right.

We notice in passing that in Multiplication the operation to be performed comes first; in addition it comes after the thing or step operated on.

Now what operator is required to turn the step -3 to the left into the step $+6$ to the right?

First we reverse the step by an operator which we will call r, $\{r(-3) = +3\}$: thus it becomes $+3$. Now double it, and the whole operation is written,

$$2r(-3) = +6,$$

so the required operator is $2r$, which means reverse and *then* multiply by 2.

But we may change the order of the process, viz. double and then reverse and we get the same result

$$r2(-3) = +6.$$

Now we will construct an equation analogous to this but which shall consist entirely of operators. Here we have two steps.

Let $k3$ mean triple without reversing. And let us suppose any step taken, tripled, reversed, doubled and reversed again. The two reversals will clearly destroy each other and give "no reversal" or k, and we shall have our step sextupled without reversal. This may be written as an equation like the last. And in the same way we have two others in which the direction is reversed:

I.	$r2(-3) = +6,$	$r2(+3) = -6,$
II.	$r2(r3) = k6,$	$r2(k3) = r6:$

and we are led to assign a new meaning to the symbols $+$, $-$; we may use them instead of k and r respectively.

Thus their meaning is extended from that of indicators of steps to operators on steps. Unless this extended meaning is borne in mind, many equations would be unmeaning, e.g. the familiar one,

$$-2(-3) = +6.$$

For there is no meaning in Multiplying one step by another.

We may assign two reasonable interpretations. Either both $-$ signs mean reverse, or the second is a step and the first means reverse.

In every equation of multiplication the last factor on each side may mean a step and then all the rest must be operations, or they may all be operations. It will be necessary to examine in the particular cases whether both meanings are allowable or whether only one may be given.

Let us now take lengths to stand for quantities either commensurable or incommensurable. If they are incommensurable (that is if they are among the values of a continuous quantity) the only way to represent them is on a certain scale.

Quantities of length are generally added by placing them end to end, treating them simultaneously in exactly the same way.

But we shall always suppose one length given and the other numbers with their signs to mean operations to be performed on it. Addition shall be represented by a step to the right along the line of the given length and subtraction a step to the left along the same line.

With the meaning of a quantity so far developed the product of two quantities represented by lines will be given by the rectangle on these lines. Then any product of higher degree than 3 will be unmeaning because it will be of higher dimensions in space than 3.

We learn however that before Descartes, this linear way of representing quantities was the only one used for the solution of equations. Vieta, in his treatise on Algebra imagines space of 9 dimensions in order to explain his equations.

The different orders began with Linear, Planum, Plano-planum, and went down to, solido-solido-solidum.

The equation

$$x^3 + ax^2 + bx + c$$

was interpreted as the sum of a number of solids, viz.

cube of $x + a$ linear $x^2 + b$ planum $x + c$ solidum,

and the equation was actually solved by cutting up a cube.

Just as in the first treatise on Algebra introduced into Europe from Arabia, by Cardan*, the equation $x^2 + 2x = 15$ was solved by a construction in a plane [see Fig. 60] which gives the value

$$x = 3.$$

Descartes first gave another meaning to the product of two quantities. He arrived at this by letting numbers stand for the *ratios of quantities*. The length of any line would then be the operation which is necessary to convert the unit of length into that line.

And the product of two quantities becomes the *ratio* compounded of the ratios which they bear to the unit.

With such a change of meaning we can write x^n without supposing a figure of n dimensions. It will mean simply the nth power of the ratio of the line x to the unit. This is clearly equivalent to our second way of looking at multiplication. But we may also use the first way as well. ab may now mean, not the rectangle on ab, but either

I. The line which bears the same ratio to b which a does to 1.

II. Where a and b are both operations; *the ratio* compounded of the ratios of a and b to the unit.

Just as before, the only choice we have is with the last number on either side, which may be either a *quantity* or an *operation*.

* [*de Arithmetica*, lib. x. cap. v.; cf. also Chasles, *Aperçu historique*, pp. 489, 541.]

Or if instead of a scale of quantities we have a scale of steps,

$$(a)(-b) = -c$$

may mean

I. Take a step b to the left and increase it in the ratio a to 1 and you get a step c to the left, or

II. Take any step, multiply it by the ratio b, reverse it, multiply it by the ratio a,—then the result is the step multiplied by c and reversed.

As yet we have considered only scalar quantities; which are either steps of addition or subtraction, i. e. steps of position on a straight line, or operations performed on those steps. These last may be equally considered as ratios of steps. We extended the meaning of quantity from simple number to these two: steps and operations. We were led to 'steps' by the appearance of negative quantity. But there is still another unexplained symbol, namely the square root of a negative quantity. Descartes' method takes no account of such a quantity.

We again define the symbol + by the equation

$$OP = OM + MP,$$

so that + now means a step in a plane [Fig. 61].

This equation holds as a definition whenever a step can be made in two instalments.

Whatever may be the angles made between the component steps we shall write [Fig. 62]

$$ac = ab + bc.$$

If OM and MP are to stand for the ratios of these lines to a unit line, we must have a unit in the direction of OM and another in that of MP. Then if x and y are ratios,

$$OP = x \text{ times } OI + y \text{ times } OI'.$$

x and y are scalars, ratios of steps to the unit steps. x is positive whenever P is on the right of O and y is positive whenever P is above OX.

Thus two scalar numbers x and y must be assigned in order to determine OP.

We will lay down a rule for adding two steps. Place the beginning of the second at the end of the first, then the line joining the beginning of the first to the end of the second will be their sum.

In giving this rule we make an assumption, viz. that the step PR is the same as OQ. Assuming this,

$$OR = OP + OQ.$$

But we have also

$$OR = ON + NR.$$

And ON is the sum of the horizontal parts of OP and OQ and NR of their vertical parts. We may express this by the equations—

if $\quad OP = x \text{ times } OI + y \text{ times } OI'$; and $OQ = x' \text{ times } OI + y' \text{ times } OI'$;

$$OP + OQ = \overline{x + x'} \text{ times } OI + \overline{y + y'} \text{ times } OI'.$$

So that if (xy) are the pair of numbers required to describe OP and $(x'y')$ the pair required to describe OQ we must add corresponding numbers from these two pairs, if we want to get the pair required to determine $OP + OQ$, or

$$(x, y) + (x', y') = (x + x', y + y'),$$

the reason of this from the figure is that OQL is the same triangle as PRS only in a different position.

We have considered

1. Quantities. 2. Steps of those quantities. 3. Ratios of steps of quantities.

numbers, lengths on a line, lengths on a plane, lengths in space to be considered hereafter.

We have seen that with the extended meanings given to + and =,

$$ab + bc = ac,$$

and extending this result to vectors in space

$$ab + bc + cd + de + ef = af.$$

Or adopting a notation used by Hamilton,

$$a + ab = b, \qquad b - a = ab,$$

(point + line = point, point − point = line;

or in Hamilton's terms,

vehend + vector = vectum, vectum − vehend = vector.

He calls the operation symbolized by the first equation "ordinal synthesis." It is a putting together of a line and a point—and the result is a point. The second operation he calls "ordinal analysis." And he enunciates this theorem: If we start with the result of a synthesis and perform on it the corresponding analysis we shall get the instrument of synthesis.)

Using this notation, the equation for addition of vectors in space becomes

$$b - a + c - b + \ldots + f - e = f - a,$$

which is an identity in this form.

It is convenient to represent steps by points in this way when we have to consider steps beginning at the same point.

We have here [Fig. 63],

$$oa + ob = oa + ac = oc = 2of,$$

that is in the symbolical point form

$$a - o + b - o = 2(f - o),$$

and hence

$$a + b = 2f;$$

f may then be called the 'mean' of the two points a, b.

Generally, if we have to express of drawn to any point of ab in terms of oa, ob, we have, [Fig. 64],

$$of = oa + af = oa + \frac{m}{l + m} \cdot ab,$$

where $l : m$ is the ratio in which f divides ab. That is

$$of = oa + \frac{m}{l+m}(ob - oa),$$

and

$$(l+m)\,of = (l+m)\,oa + m \, . \, ob - m \, . \, oa = l \, . \, oa + m \, . \, ob,$$

and hence

$$of = \frac{l \, . \, oa + m \, . \, ob}{l+m}.$$

Expressing this in terms of the points

$$(l+m)(f-o) = l(a-o) + m(b-o),$$

$$\therefore \; (l+m) f = la + mb \quad \text{.............................(A).}$$

From this we are led to consider points in a plane as having masses. The expression for f shews that it is the centre of gravity (or centre of mass) of the masses l and m placed at b and a respectively. The point f divides ab in the inverse ratio of the masses.

It may also be expressed in terms of steps, transposing in the last equation

$$l(f-a) + m(f-b) = 0,$$

that is

$$l \, . \, af + m \, . \, bf = 0.$$

We may make use of this result to prove the vector form of equation (A).

We have of course

$$of = oa + af \text{ and } of = ob + bf.$$

Hence

$$(l+m)\,of = l \, . \, oa + m \, . \, ob + (l \, . \, af + m \, . \, bf) = 0, \text{ as just found,}$$

or as before

$$(l+m)\,of = l \, . \, oa + m \, . \, ob.$$

If we want to extend the rule for finding the mean of two points to a greater number of points, the sought point is called their mid-centre and is defined by this equation.

If m is mid-centre and $abc...f$ etc. the points,

$$ma + mb + mc + \ldots + mf = 0,$$

that is

$$a - m + b - m + \ldots + f - m = 0,$$

or

$$a + b + c + \ldots + f = nm,$$

where n = number of points.

Then

$$m = \frac{a+b+c\ldots+f}{n}.$$

For example, the mid-centre of four points a, b, c, d is by this rule

$$\frac{a+b+c+d}{4}.$$

Clearly it lies at the point B in the figure. [Fig. 65].

In the case of an odd number of points, $2n+1$, we should first find the mid-centre B of $2n$ of them and then joining it to the last divide this line in the ratio $1 : 2n$.

We can now find the resultant of n steps. It is n times the step from the mid-centre of the beginnings to the mid-centre of the ends.

This is obvious in the case of two steps.

We have then, [Fig. 66],

$$ab + cd = 2fg,$$

that is

$$(b+d) - (c+a) = 2g - 2f,$$

which follows from the definition of the mean of two points.

If there are four steps and p is the mid-centre of the a's and q of the b's, we have [Fig. 67]

$$\begin{aligned} & a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 \\ &= b_1 + b_2 + b_3 + b_4 - (a_1 + a_2 + a_3 + a_4) \\ &= \quad 4q \quad - \quad 4p \\ &= \quad 4(q-p), \end{aligned}$$

and this is the resultant.

This construction is sometimes more convenient than the tandem arrangement. For the mid-centres lie always within the area covered by the steps. The construction can always be carried out provided the steps are given.

If we want to add together certain multiples l, m, n, r of steps we attribute masses l, m, n, r to the points a and also to the points b. Then we should have,

$$lb_1 + mb_2 + nb_3 + rb_4 - (la_1 + ma_2 + na_3 + ra_4) = (l+m+n+r)(q-p),$$

as may be easily seen, by joining each of the points b_1 etc., a_1 etc. to the mid-centre of the set.

We will now apply the new interpretation of the signs +, −, = to the description of motions.

Let p be a point moving uniformly along a line. Then if r is the space traversed in $1''$, rt = space gone over in t''.

We must find an expression for the step necessary to get from o to the position of p at any time t. Suppose p goes over ab [Fig. 68] in $1''$.

Then we have,

$$op = oa + t \, . \, ab,$$

$$\rho = \alpha + t\beta.$$

This is the equation of uniform motion of a point on a line.

We may also find the equation of uniform circular motion.

Let the arc gone over in $1''$ be an [Fig. 69], then the arc described in $t'' = t \, . \, an$. If this is represented by the arc ap, the circular measure of the angle aop is nt. We have

$$\overline{om} = \overline{oa} \, . \cos nt \; ; \; \overline{mp} = \overline{ob} \, . \sin nt.$$

Hence

$$op = \overline{om} + \overline{mp} = \overline{oa} \,.\, \cos nt + \overline{ob} \,.\, \sin nt,$$

or

$$\rho = \alpha \cos nt + \beta \sin nt.$$

This is the step from the centre to the position of p at any time t.

Suppose now this uniform plane circular motion projected on an inclined plane. The curve becomes an ellipse, the parts of any radius vector to which from the centre are always proportional to the corrresponding ones of the circle [Fig. 70],

$$\frac{o'm'}{o'a'} = \frac{om}{oa} = \cos nt,$$

$$\frac{m'p'}{o'b'} = \frac{mp}{ob} = \sin nt.$$

Hence we have the same form of equation for harmonic motion on an ellipse as for uniform motion on a circle, viz.

$$\rho = \alpha \cos nt + \beta \sin nt.$$

But here α and β are not at right angles to each other.

The rate of change of op is the same as the velocity of the point p, as we see at once.

If $op = r$,

$$\dot{r} = \dot{p} - \dot{o},$$

$$= \dot{p}, \text{ since we suppose } o \text{ fixed.}$$

If we extend this to a step $\overline{op}$ in space, the rate of change of $\overline{op}$ is in this case a step,

i.e. if $\rho = \alpha + \beta t$, $\dot{\rho} = \beta$, $(ab = \beta = pq)$ [Fig. 71].

We have found, putting $oa = \alpha$ (of length a),

$$\rho = \alpha \cos nt + \beta \sin nt,$$

and we can at once obtain from this an expression for the velocity of p at any instant.

Since the angle described in $1'' = n$,

the arc ,, $= na$,

this then is the numerical value of the velocity. The direction is in the tangent line at p [Fig. 72].

Hence if we draw a radius oq perpendicular to op, we have, since $oq = \alpha$,

$$\dot{\rho} = n \,.\, \overline{oq}.$$

But $\overline{oq}$ is what op becomes when nt is increased by $\frac{\pi}{2}$,

$$\therefore \; \dot{\rho} = n\alpha \cos\left(nt + \frac{\pi}{2}\right) + n\beta \sin\left(nt + \frac{\pi}{2}\right)$$

$$= -n\alpha \sin nt + n\beta \cos nt.$$

The rule then for finding the vector-velocity from the position or vector-radius is—multiply this last vector by n and increase its angle by $\frac{\pi}{2}$.

We have gone on the supposition that α and β are at right angles, and equal to each other; but the result is true independently of this, since

$$\partial_t \cos nt, \; \partial_t \sin nt$$

always equal $-n \sin nt$, $n \cos nt$ respectively.

If then we take the elliptic projection of the circular motion, we may still conclude that velocity of $p = n \,.\, \overline{oq}$ if $\overline{oq}$ is parallel to the tangent at p [Fig. 73]. For the velocity at p is always parallel to the tangent at p.

Now consider the velocity of the point when it arrives at q. For this point

$$\rho = \alpha \cos\left(nt + \frac{\pi}{2}\right) + \beta \sin\left(nt + \frac{\pi}{2}\right).$$

Hence, multiplying by n and increasing angle by $\frac{\pi}{2}$,

(for q)
$$\dot{\rho} = n\alpha \cos(nt + \pi) + n\beta \sin(nt + \pi)$$
$$= -n\alpha \cos nt - n\beta \sin nt$$
$$= -n \,.\, \overline{op} = n \,.\, \overline{po}.$$

But this velocity must be in the direction qs of the tangent at q. Hence we may conclude, if we draw oq parallel to the tangent at p, then op will be parallel to the tangent at q.

The characteristic property of conjugate diameters is proved of the projections of diameters of a circle at right angles to each other.

We have seen that

$$op = \overline{oa} \,.\, \cos nt + \overline{ob} \,.\, \sin nt, \quad \text{and also } op = om + mp,$$

$$oq = -\overline{oa} \,.\, \sin nt + \overline{ob} \,.\, \cos nt, \text{ and} \quad oq = on + nq.$$

Hence, we have

$$\frac{om}{oa} = \cos nt, \qquad \frac{mp}{ob} = \sin nt,$$

$$\frac{nq}{ob} = \cos nt, \qquad \frac{on}{oa} = -\sin nt,$$

and from these equations

$$\frac{om}{oa} = \frac{nq}{ob}, \qquad \frac{on}{oa} = -\frac{mp}{ob},$$

a result which is thus interpreted:

We have two pairs of semi-conjugate diameters, oa and ob; and op and oq. The projection of op on oa is to oa as the projection of oq on ob to ob. And the projection of oq on oa is to oa as the reversed projection of op on ob to ob.

The first of these equalities is the same as

$$\frac{om}{oa} = \frac{or}{ob}.$$

Since $\dot{\rho}$ is a vector the expression for it may be treated in the same way as that for ρ. If we take n an improper fraction so that $oq' = n . oq$, the extremity of the vector $\dot{\rho}$ will trace out an ellipse similar and similarly situated to the first. Generally, the curve which describes the way in which the path of a moving point is gone over is called the hodograph (of that path). In the case of uniform rectilinear motion $\rho = \alpha + \beta t$, $\dot{\rho} = \beta$, and the hodograph reduces to a point.

For uniform circular motion it is a circle with dimensions n times those of the first circle; and for harmonic motion in an ellipse, n times that ellipse.

We will now consider the velocity of q as it moves round its ellipse.

We have

$$\overline{oq} = \dot{\rho} = n\alpha \cos(nt + \tfrac{1}{2}\pi) + n\beta \sin(nt + \tfrac{1}{2}\pi).$$

The process for finding its velocity is the same as in the case of op. Hence

$$\begin{aligned} \ddot{\rho} &= n^2\alpha \cos(nt + \pi) + n^2\beta \sin(nt + \pi) \\ &= -n^2 . \overline{op} \\ &= n^2 . \overline{po}. \end{aligned}$$

Thus we learn that in harmonic motion in an ellipse the acceleration of the moving point is directed towards the centre of the ellipse and is proportional to the distance from the centre.

We will now consider a new kind of motion of which the equation is

$$\rho = \alpha e^{nt} + \beta e^{-nt}.$$

These exponential functions are quantities which are equally multiplied in equal times.

Consider quantities, s, defined by this property.

We may see at once that their rate of change at any instant is proportional to the value of the quantity at that instant.

We have by def. $s_2 = ks$, and moving the interval $t_2 - t$, along the axis [Fig. 74], we still have $s_4 = ks_3$. Hence we infer the rates of change of the two quantities at the beginning and end of the interval are to each other as $1 : k$, or

$$\dot{s}_2 = k\dot{s}_1.$$

Hence we have the proportion

$$\dot{s}_2 : \dot{s}_1 :: s_2 : s_1,$$

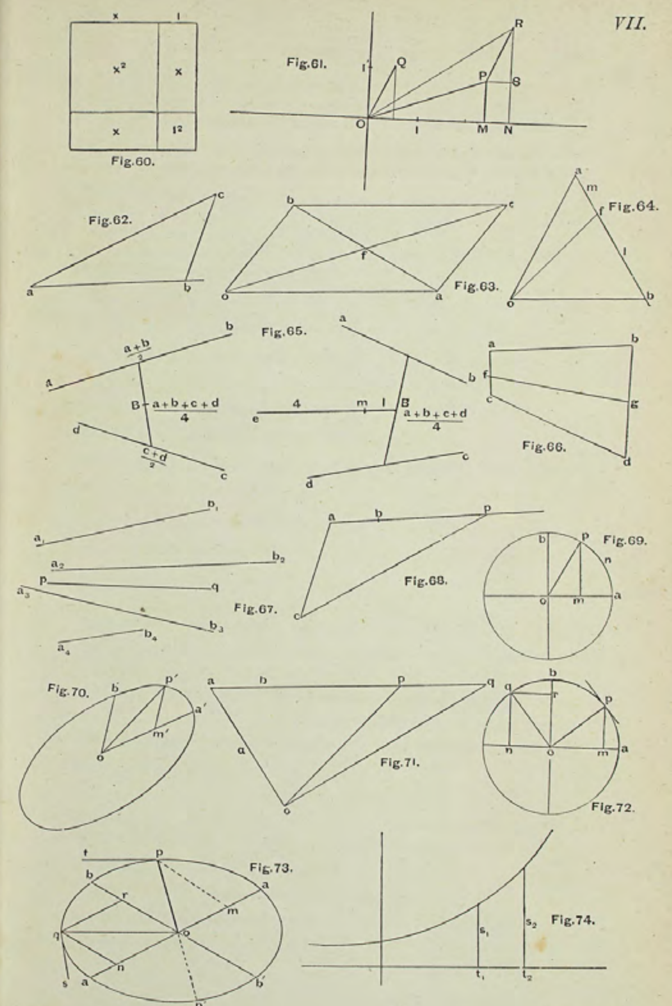
and therefore

$$\dot{s} = ps.$$

Calling p the logarithmic rate, the last equation declares that s is a quantity which increases at uniform logarithmic rate.

If we change our unit of time from $1''$ to n'', $\dot{s}$ becomes $n\dot{s}$, and therefore p becomes np.

VII.



face p. 488.

Having given that the quantity varies at logarithmic rate p, we are required to find P the number by which it is multiplied in $1''$. In n seconds s is multiplied by P^n; and n must be a commensurable number, whole or fractional.

If it is a fraction, e.g. $\frac{1}{2}$, we must give one value only to $\sqrt{P}$, because at half a second after the time at which we begin, s will have a perfectly definite value.

If n is incommensurable, it may be defined by this physical property.—Let P^n mean the multiplier which s gets in this incommensurable time. Let us take a quantity $=1$, which increases at logarithmic rate 1. Then at the end of $1''$ it will have reached a certain definite value. Call it e. Then e is the result of making unity grow at logarithmic rate 1 for $1''$.

Then, I say, the result of making unity grow at logarithmic rate 1 for $t''=e^t$.

For since in $1''$ it is multiplied by e, in t seconds it will be multiplied by e^t.

Now change the unit of time to n seconds. We see then that the result of making unity grow at logarithmic rate 1 for nt'' of old units $=e^{nt}$; therefore in the new unit e^{nt} results from making 1 grow at logarithmic rate n for t''.

The rate of change of e^{nt} is then ne^{nt} $(\dot{s}=ps)$.

If the logarithmic rate is negative the quantity decreases. Rate of change of $e^{-nt}=-ne^{-nt}$.

We wrote the equation of the curve $\rho=\alpha e^{nt}+\beta e^{-nt}$, where $\alpha=oa$, $\beta=ob$.

But $\rho=op=om+mp$. [Fig. 75].

Hence $\frac{om}{oa}=e^{nt}$, $\frac{mp}{ob}=e^{-nt}$.

Hence $om\,.\,mp=oa\,.\,ob=\text{const.}$

This is the relation between the vector coordinates of p, and hence its locus is a hyperbola. We see that the axes of α and β are asymptotes to the curve. For by taking nt large enough, that is the length of $om=e^{nt}$ very large, we can make e^{-nt} or the length of mp as small as we like. And by taking nt small enough we can make e^{-nt} or mp as large and e^{nt} or om as small as we like.

For the flux of ρ we have

$$\dot{\rho}=n\alpha e^{nt}-n\beta e^{-nt},$$

a similar curve to that for ρ; but it is turned round the α axis so that the β coordinates are negative. It is shewn by the dotted line in the figure.

We see that the tangent at p is parallel to oq,

$$oq=om+mq=om-mp.$$

And the velocity which is measured in this direction is n times oq.

The figure is drawn for $n=\frac{1}{2}$.

The locus of q is the conjugate hyperbola, and op and oq are semi-conjugate diameters. In the ellipse both of them meet the curve, but in the hyperbola one meets the conjugate branch.

We have,

$$\rho = \alpha e^{nt} + \beta e^{-nt} = (\alpha+\beta)\tfrac{1}{2}(e^{nt}+e^{-nt}) + (\alpha-\beta)\tfrac{1}{2}(e^{nt}-e^{-nt}).$$

We will call

$\tfrac{1}{2}(e^{nt}+e^{-nt})$, the hyperbolic cosine of nt, $hc\,.\,nt$,

and $\tfrac{1}{2}(e^{nt}-e^{-nt})$, „ „ sine of nt, $hs\,.\,nt$.

If we also write

$$\alpha+\beta=\gamma, \quad \alpha-\beta=\delta,$$

the last formula becomes

$$\rho = \gamma hc\,.\,nt + \delta hs\,.\,nt.$$

We thus have an expression for ρ in terms of two semi-conjugate diameters, analogous to the one we found for the ellipse.

γ and δ are clearly the values of op and oq for $t=0$.

The corresponding expression for the flux is

$$\dot{\rho} = n(\gamma hs\,.\,nt + \delta hc\,.\,nt),$$

since each of the functions $hs\,.\,nt$, $hc\,.\,nt$ has for its flux n times the other. Or we may see that this follows from the first expression found for $\dot{\rho}$.

If we take any two semi-conjugate diameters, OC and OD, and also two others OP and OQ [Fig. 76], we have,

$$\rho = OP = OM + MP$$
$$= \gamma hc\,.\,nt + \delta hs\,.\,nt.$$

Hence,

$$\frac{OM}{OC} = hc\,.\,nt\,; \quad \frac{MP}{OD} = hs\,.\,nt,$$

and

$$\frac{\dot{\rho}}{n} = OQ = ON + NQ$$
$$= \gamma hs\,.\,nt + \delta hc\,.\,nt.$$

Hence,

$$\frac{ON}{OC} = hs\,.\,nt\,; \quad \frac{NQ}{OD} = hc\,.\,nt.$$

From these we find

$$\frac{OM}{OC} = \frac{NQ}{OD} \text{ and } \frac{ON}{OC} = \frac{MP}{OD},$$

for any two pairs of conjugate diameters.

These equations declare that,

The projection of OP on OC is to OC as the projection of OQ on OD is to OD; and the projection of OQ on OC is to OC, as the projection of OP on OD is to OD.

From the equation for $\dot{\rho}$, we see that the hodograph is a curve of n times the dimensions of the conjugate hyperbola, and is similarly situated to that hyperbola.

For the acceleration, we find,

$$\ddot{\rho} = n^2(\alpha e^{nt} + \beta e^{-nt})$$
$$= n^2\rho,$$

by repeating the operation by which $\dot{\rho}$ was found.

Thus the acceleration of a point P moving on a hyperbola is in the direction of OP, but away from the centre.

In the ellipse the acceleration is directed towards the centre.

Velocities and accelerations resemble steps in having magnitude and direction. All three belong to the class of "vectors," which are characterised by these two properties, and also by a third, not pointed out by Hamilton, namely, that they have no definite position.

Hence we must consider vectors as steps, not of points, but of a rigid body. And similarly, velocities and accelerations must be regarded as velocities and accelerations, of translation, of a rigid body.

Thus the uniform circular motion which we have discussed, is to be considered as the revolution of a rigid body, round a centre, so that all its particles describe circles of the same size.

Any enclosed area may be represented by a line drawn perpendicular to its plane, and of length proportional to its magnitude, so that there are as many linear units in the line, as there are square units in the area.

Now the area is not completely defined until the direction in which we go round it is known. This direction may be either clock-wise or the reverse, and the corresponding vectors would be drawn in opposite directions, upwards in the one case, downwards in the other.

The rule for drawing the representative vector is this :—It must be drawn so that when we look back along it the area appears to be gone round counter clock-wise.

For the area shewn in the figure [Fig. 77] it must therefore be drawn upwards.

If we have a figure of 8 [Fig. 78] bounding two areas which are gone round in opposite directions, looked at from one side, the true value of the whole area is the difference of the two parts. It is represented by the difference of the lengths of the vectors whose lengths correspond to their magnitudes; for these are drawn in opposite directions. (We will call the rotation opposite to that of the clock-hands, positive rotation.) This method for the addition of areas was given by Möbius.

We may take a more complex plane curve [Fig. 79]. It is easy to see that the two areas in the centre are to be counted as 0. The lower one for example is seen to be formed by the overlapping parts of a positive and a negative area, and the superposed parts destroy one another.

The addition of areas in space corresponds to the addition of vectors which are not in the same straight line.

If we want to represent an area on a curved surface, for example, a cone, we break it up into small areas, nearly plane, and draw from each of these its representative vector in the manner before described. These lines will not be parallel, and must be added together according to one of the rules for finding the resultant of vectors in space. Their resultant represents the area. It represents not the actual size of the area, but in a certain physical sense, the area of the contour. Hence every closed surface may be considered as a quantity having a certain magnitude and direction. This point of view was first taken by Hayward*.

If A is any plane area, its projection on a plane making an angle θ with its plane is $A \cos\theta = A'$.

We know that for any plane area there is a set of planes on which the projection is a maximum, viz. all parallel planes, and another set on which the projection is a minimum, viz. planes perpendicular to the first.

All this holds good for areas, not plane, which are represented in the way we have described.

There is a certain set of planes on which the projection is a maximum. Call it A. On all other planes making an angle θ with the maximum planes, the projection is $A \cos\theta$. And there is a plane on which the projection is zero.

Take as an example of a non-plane area, two triangles and a parallelogram in different planes. [Fig. 80.]

Let ab, bc, cd be the vectors representing the areas 1, 2, 3, then ad represents the whole area.

If now we project the areas 1, 2, 3 on a plane D, it is clear that the new representative vectors are the projections of the lines 1, 2, 3 on a line perpendicular to the plane D.

For the angle between two representative vectors = the angle between their planes. This being so, the projection of the compound area (1, 2, 3) on any plane is represented by the projection of ad, on a normal D to that plane.

Hence all planes perpendicular to ad are maximum planes for the compound area, and its projection on them is A, the projection on any plane making an angle θ with them is $A\cos\theta$. And the projection on planes parallel to ad is a minimum. If then the maximum planes are known, and the maximum projection, we find the line representing the compound area by this rule:—Draw a line perpendicular to the maximum planes, of length A.

Consider first two triangles in a plane, having a common side and vertex; we can prove that their sum is the triangle having the same side, with its base equal to the (vector-) sum of their bases.

If the two triangles are [Fig. 81] OAB, OAC, it is to be shewn that

$$OAD = OAB + OAC.$$

* [*Proc. of L. Mathematical Society*, Vol. IV. pp. 289—91, 417.]

If we complete the parallelogram with sides AC, AB, the triangle

$$CAD = DAB,$$

for their areas are equal and the motion round both is positive.

Taking account of the way in which we go round their contours,

$$OAD = OAC + OCD + CAD$$

$$= OAC + OCD + DAB.$$

Now, $$OAB = OCD + DAB,$$

(for, drawing perpendiculars to a line at right angles to AB, the areas are,

$$OAB = \tfrac{1}{2}ln \,.\, AB,$$

$$DAB = \tfrac{1}{2}mn \,.\, AB,$$

$$OCD = \tfrac{1}{2}lm \,.\, AB.$$

Hence $$OAB = OCD + DAB.)$$

substituting, this gives

$$OAD = OAB + OAC.$$

If we agree to call the common side OA the height of the triangles, we may write the last result thus:—The sum of two triangles with the same height, is a triangle having the same height and with its base equal to the sum of their bases.

We may now extend this to triangles in space, which are represented by vectors.

Take two triangles standing up from the plane of the paper, and first suppose their common "height" perpendicular to the plane.

We have to shew that the line representing OAD [Fig. 82] is the sum of the lines representing OAC and OAB.

This is at once seen by looking at the space-figure. We may represent OAC by a vector perpendicular to AC, numerically equal in length to AC. Let this be AC'. Then OAB and OAD will be represented in the same way by AB', AD'. These representative vectors must be all drawn in the plane of AB and AC, that is, the plane of the paper. For each of the triangles is perpendicular to this plane. The new vectors are the sides and diagonals of the same parallelogram turned through a right angle, and we see that $AC' + AB' = AD'$.

This proves the proposition.

We may easily pass to the general case where the height OA is not supposed perpendicular to the plane of AB and AC.

If we draw through O a plane perpendicular to OA it cuts the plane of each triangle in a line perpendicular to OA. Take OAD [Fig. 83] for example; then

$$\text{area } 2OAD = OP \times AD = OA \,.\, \frac{OP}{OA} \,.\, AD$$

$$= OA \,.\, AD \,.\, \cos\theta = OA \,.\, OF.$$

Similarly, if OG, OH are the projections on the plane perpendicular to OA of the bases AC, AB, their areas will be $OA \,.\, OG$, $OA \,.\, OH$, respectively. Now the lines OG, OH, OF are the sides and diagonals of a parallelogram; hence by the addition of vectors, $OF = OG + OH$. And since we may represent the triangles by lines drawn from O in the plane perpendicular to all their planes, at right angles to the plane of each, and equal in length to OG, OH, OF, these representative vectors form the same parallelogram turned through a right angle. If they are OF', OG', OH' we still have

$$OF' = OG' + OH',$$

and hence for the corresponding areas

$$OAD = OAB + OAC.$$

In the ancient geometry the product of two lines was the rectangle contained by them. We will now extend this representation of a product, and say that the product of two lines inclined at any angle is the area of the parallelogram contained by them. Thus the product of two vectors is the vector perpendicular to their plane and proportional in length to the area they enclose. Thus we make the definition $OA \,.\, OB = 2 \,.$ triangle OAB.

Remembering the convention about signs of areas, we see that OAB is positive.

By interchange of letters, the formula gives,

$$OB \,.\, OA = 2 \,.\, \text{triangle } OBA.$$

Now OBA is $-OAB$.

Hence we learn that the product of two vectors is altered in sign when the order of multiplication is reversed.

We may now interpret the proposition about the sum of two triangles in space, by this formula.

Their areas will be

$$2AOD = AO \,.\, AD,$$
$$2AOB = AO \,.\, AB,$$
$$2AOC = AO \,.\, AC,$$

and since $\quad AOD = AOB + AOC,$

$$AO \,.\, AD \,\{= AO\,(AB + AC)\} = AO \,.\, AB + AO \,.\, AC.$$

Hence we conclude that vector multiplication is distributive.

Using shorter symbols the theorem may be written

(1) $\quad V\alpha\,(\beta + \gamma) = V\alpha\beta + V\alpha\gamma.$

We have seen that ($OA \,.\, OB = - OB \,.\, OA$, or)

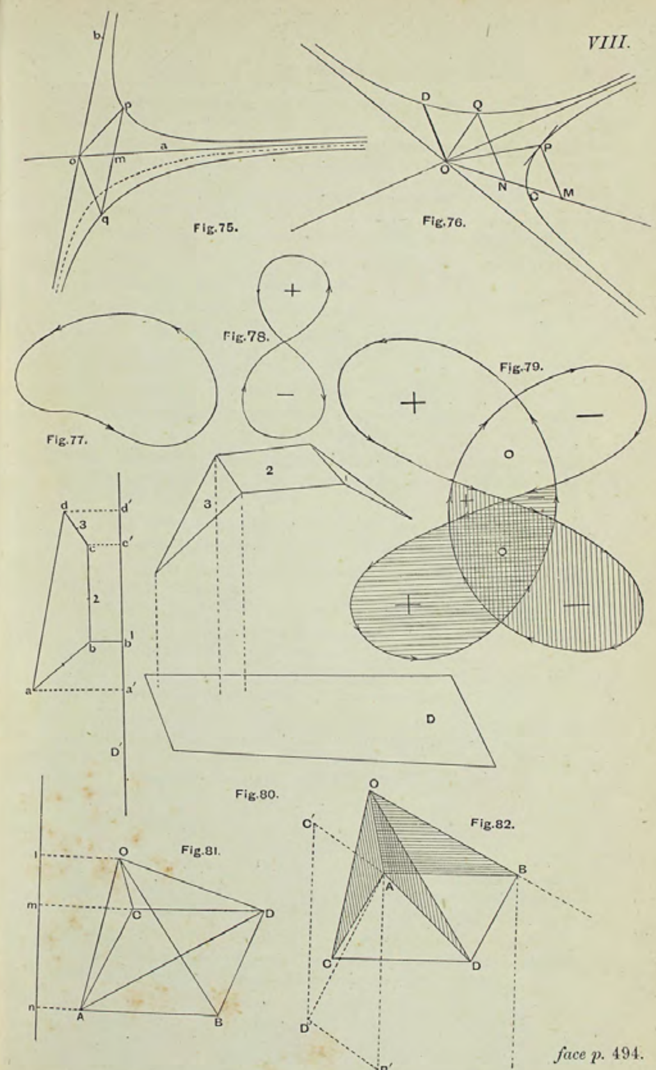
(2) $\quad V\alpha\beta = - V\beta\alpha.$

This shews that vector multiplication is not commutative.

If we change the order in each term of equation (1), we change the sign of each term and get,

(3) $\quad V(\beta + \gamma)\,\alpha = V\beta\alpha + V\gamma\alpha.$

VIII.



This however is not the same equation as (1). The signs of its terms are opposite to those of the terms in (1).

Grassmann called the vector-product of two lines their "outer product," because it has no existence unless one is outside the other.

The following proof of equation (2) was given by Sylvester. It shews on what geometrical fact the rule depends.

If *OA* and *OB* coincide they will enclose no parallelogram, and we shall have,

$$V\alpha\alpha = 0. \tag{4}$$

This being true, try it on $V(\alpha+\beta)(\alpha+\beta)$.

This product is broken up into four parts,

$$0 = V\alpha(\alpha+\beta) + V\beta(\alpha+\beta) = V\alpha\alpha + V\alpha\beta + V\beta\alpha + V\beta\beta.$$

The first and last terms vanish, and hence

$$0 = V\alpha\beta + V\beta\alpha.$$

If we had assumed (2), equation (4) would follow at once from it, for we must have

$$V\alpha^2 = -V(\alpha^2).$$

This kind of multiplication has been called "polar multiplication." We see then that any kind of multiplication which is distributive and where the square of any quantity $=0$, must be polar.

We may now use the theorem about the geometrical representation of products by areas, to investigate the rotation of rigid bodies.

Let the angular velocity of a rigid body about the axis ab be ω. Any particle p of the body describes a circle round ab. At this moment p is moving perpendicular to the plane of the paper and its velocity (arc described in 1″) is, $\omega . mp$.

If we measure off on ab a length ab proportional to ω, the velocity of p is twice the area of the triangle abp. [Fig. 84.]

It is necessary however to take into account the direction of motion of p, and to give a different sign to the triangle which represents the velocity.

Suppose ab measured so that, looking back from b to a, the motion of p appears to be counter-clockwise; p must be moving at this moment back from the paper. We make this convention because the sign of abp is negative, and therefore its representative vector must be measured downwards, or in the negative direction.

We have found, then,

$$\dot{\rho} = \omega . mp = 2 . abp = V\overline{ab} . \overline{ap} = V\omega\rho.$$

Similarly for every point in the space occupied by the body there is a certain vector representing the velocity of a particle at that point, and which is expressed in terms of its position-vector ρ. The aggregate of these vectors forms a velocity system.

We might have also an acceleration system; that is a definite acceleration belonging to every point of space. Both of these are vector-systems.

Now if we have to compound two vector-systems we shall compound the two vectors belonging to the systems respectively at every point of space. As an application of this we will compound the velocity systems corresponding to two rotations about a fixed axis. These rotations may be called "spins." They are completely defined when we know the rotation axis and the angular velocity.

The problem then is to find the resultant of two spins with angular velocities ω_1, ω_2, and with axes passing through a fixed point a.

We have $$\dot{\rho}_1 = V\omega_1\rho, \quad \dot{\rho}_2 = V\omega_2\rho$$

for any point of space which has the position-vector ρ.

Since the velocities are represented by the areas abp, acp we have for the resultant velocity ($adp = abp + acp$) [Fig. 85]

$$\dot{\rho} = \dot{\rho}_1 + \dot{\rho}_2 = V\omega_1\rho + V\omega_2\rho$$
$$= V(\omega_1 + \omega_2)\rho.$$

Hence a velocity system compounded of two spins is a velocity system with a spin which has the sum of the component angular velocities.

This way of writing the theorem of distributive vector multiplication is only a short-hand for the geometrical proof of the same proposition.

This expression for the resultant does not mean that the body has first a little turn about one axis and then about the other. The spin of the body is made up of two spins but it does not get those spins actually. We may easily extend this result to find the resultant of any number of angular velocity systems with axes passing through a fixed point.

We shall be led to consider another kind of product of two vectors:—the scalar product.

In the ancient geometry the product of these lines was represented by the rectangular parallelepiped which they contained.

We will extend this to the parallelepiped contained by any three lines meeting in a point, and we may then make the definition

$$OA \, . \, OB \, . \, OC = 6 \times \text{tetrahedron } OABC.$$

Now if we draw OM perpendicular to OAB and proportional in length to $2OAB$ [Fig. 86], we have

$$OA \, . \, OB \, . \, OC = \overline{OM} \, . \, \overline{OC}.$$

Now $$6 \, . \, OABC = OM \, . \, ON = OM \, . \, OC \cos\phi.$$

Then, if the lengths of OA, OB, OC, OM are a, b, c, d we have

$$OA \, . \, OB \, . \, OC = c \, . \, d \, . \cos\phi.$$

We must now take account of signs.

The volume of $OABC$ shall be reckoned positive when the three points A, B, C, looked at from O are gone round counter-clockwise. The volume in the figure then is negative. It has been seen that the product of two vectors OA and OB is a directed quantity for it is an area, considered as to size and aspect, and may therefore be represented by a vector. Its magnitude is $ab \sin\theta$

and it is perpendicular to the plane of OA and OB. It is the vector-product. But now the product of three vectors has also been reduced to the product of two, viz. OM and OC. And since this is a volume, it can only have quantity not direction, and it is called the scalar product of two vectors. In the scalar product we consider one of the vectors OM as the area, the other as a vector. While in the vector-product both are considered as vectors.

This distinction between two kinds of vectors was first made by Maxwell. One kind of vector corresponds to a force, the other to a flow.

The product of two flows or of two forces will be a vector-product. The product of a force and a flow will be a scalar-product.

The point of view we have taken in considering an area as a product of two vectors of the same kind and a volume as a product of three vectors of the same kind, is that of Grassmann, not of Hamilton.

Since the volume represented by $OC \, . \, OM$ is to be negative we must draw OM so that looking back along it the rotation of OA to OB is in the negative direction. It is drawn right then in the figure. If the parallelepiped were rectangular OM would fall on OC, and supposing it of the same length we should have

$$S\alpha^2 = - \text{ squared length of } \alpha.$$

We may see at once that a scalar-product is not altered when we change the order of the factors. For interchanging OC and OM, the sign of ϕ is altered but not that of $\cos\phi$.

Therefore $S\alpha\beta = S\beta\alpha$.

We may define the scalar-product of two vectors geometrically as a solid, one of the vectors being an area; or physically as the product of three vectors of the same kind. Similarly the vector-product may be defined geometrically as an area considered in aspect and size or physically as the product of two vectors of the same kind.

Since the scalar-product of two vectors is one vector multiplied by the projection of the other upon its direction, we can at once deduce the distributive law for scalar multiplication, or

$$S\alpha(\beta + \gamma) = S\alpha\beta + S\alpha\gamma,$$

since the projection of $\beta + \gamma$ on α is the same as the sum of the projections on it of β and γ. [Fig. 87.]

Illustration of a scalar product in physics. The rate of doing work by a body moving with a velocity σ and acted on by a force $\varpi = S\sigma\varpi$.

Another illustration. The kinetic energy of a rotating body $= S$. spin $\times$ moment of momentum. Each of these last quantities is a vector.

Again, if a plate is bent out of its plane or deformed in its plane,

$$\text{potential energy} = S \, . \, \text{strain} \times \text{stress}.$$

Recalling the double interpretation of the equality $2 \times 3 = 6$; viz. 1st, 2 and 3 multiplied together give 6; 2nd. 3 (operated on) by 2 gives 6; we will make a corresponding second interpretation of the product of two vectors in a plane, and ask what operation must be performed in order to turn one vector into another.

To change OA into OB I must multiply it first by a number to change its length to that of OB and then turn it through an angle so that it may lie on OB. The whole of the operator required in order to do these two things may be called q.

$$q \cdot OA = OB. \quad \text{[Fig. 88.]}$$

Drawing ab parallel to AB we see that

$$q \cdot Oa = Ob.$$

If we want to apply the same operator q to any other vector OC we must draw another triangle OCD; and if this is similar and of the same sign with OAB,

$$OA : OB :: OC : OD.$$

This proportion would not hold for a triangle $OC'D'$, similar to OAB but of different sign.

We make use of the operator q to prove a proposition about the motion of bars.

If ohk, ode, ogf [Fig. 89] are three similar triangular plates which can move about o; and if the parallelograms df, eh, gk are completed, we can shew that the triangle abc is similar to each of the three triangles which are riveted together at o.

Let q be the operator which turns oh into ok. Then

$$ok = q \cdot oh,$$
$$af = do = q \cdot de,$$
$$fg = q \cdot fo = q \cdot ad,$$
$$gc = ok = q \cdot oh = q \cdot eb.$$

Adding we have

$$ac = q\,(de + ad + eb) = q \cdot ab;$$

therefore the triangle abc is similar to triangle ohk, etc.

We may suppose all the parallelograms to be made of bars joined at their corners.

We may remove all the parts except the triangle ode and the parallelograms eb and da, or all except ohk, hb and kc, or all except ofg, fa and gc.

Then, in each case, abc must still be of the same shape.

The curve described by o in the three cases will be the same. Hence we learn that a curve which can be described with one three-bar motion can be described equally well with two other three-bar motions.

The operator q which we have seen to be the product of two operations is also the sum of two parts.

If [Fig. 90]

$$q \cdot OA = OB, \text{ and } OB = OM + MB,$$

$$q = \frac{OM}{OA} + \frac{MB}{OA}$$

$$= \frac{OM}{OA} + \frac{MB}{OA'} \cdot \frac{OA'}{OA}$$

$$= a + b \cdot \frac{OA'}{OA},$$

where a and b are numbers, with positive or negative signs.

We want a symbol for the operator which turns a step through a right angle in the same plane. If i stands for this operator

$$q = a + bi:$$

i may be regarded as a handle perpendicular to the plane of OA and OB which turns OA through a right angle.

It is at once seen to be a property of i that if it is used twice on a vector it reverses that vector; or

$$i^2 = -1.$$

We can now use the operator i to establish some important results. First however we must find an expression for the rate of change of a product of two numbers, pq. Take their values p_2q_2 at the time t_2, and p_1q_1 at the time t_1, then for the mean rate of change of the product during the interval we have,

$$\frac{p_1q_1 - p_2q_2}{t_1 - t_2} = p_1 \cdot \frac{q_1 - q_2}{t_1 - t_2} + \frac{p_1 - p_2}{t_1 - t_2} \cdot q_2.$$

Making the beginning and end of the interval coincide, we get the actual rate of change of the product; that is

$$\partial_t pq = p\dot{q} + \dot{p}q \quad \text{..............(1).}$$

In getting out this result all we have assumed is that multiplication is distributive. Hence the result is true when p or q or both of them are vectors. Letting q only equal a vector ρ, we have

$$\partial_t (p\rho) = p\dot{\rho} + \dot{p}\rho \quad \text{..............(2).}$$

If both are vectors the result holds good for either the scalar or the vector-product; hence

$$\partial_t V(\alpha\beta) = V\alpha\dot{\beta} + V\dot{\alpha}\beta \quad \text{..............(3),}$$

$$\partial_t S(\alpha\beta) = S\alpha\dot{\beta} + S\dot{\alpha}\beta \quad \text{..............(4).}$$

We may apply equation (2) for one quantity a scalar and the other a vector to the problem of resolving the acceleration of a moving point along the tangent and the normal to its path.

The velocity of p, $\dot{\rho}$, is in the direction of the tangent at p, and its magnitude equals $\dot{s} = v$, where s is the length of the curve up to p, measured from a fixed point a. [Fig. 91.]

Hence $\frac{\dot{\rho}}{\dot{s}}$ is a vector of unit length parallel to pt. Let oc represent it; and denote it by ρ'.

Then we have

$$\dot{\rho}=\dot{s}\rho'=v\rho'.$$

Hence by equation (2) the acceleration of p is

$$\ddot{\rho}=\dot{v}\rho'+v\dot{\rho}'.$$

The first part is the acceleration parallel to the tangent. Its magnitude $=\dot{v}$.

The second term is v multiplied by the velocity of c. Since c is always at the end of a unit vector its path is a circle, of radius 1. Hence the velocity of c is always in a direction perpendicular to oc, and to pt, that is along the normal. Its magnitude = the angular velocity of oc, that is of the tangent pt.

The magnitude then of the normal component of the acceleration is the velocity of the point multiplied by the rate of turning round of the tangent.

But we know that,

$$\frac{\dot{\rho}'}{v}=\frac{\text{rate of turning round of tan } pt}{\text{velocity of } p}$$

$=$ rate of turning round of tangent per unit length of curve

$=$ curvature.

Hence the normal acceleration $=v^2\times$ curvature, curvature being measured by a line drawn along the inner normal to the curve.

Suppose a point or system of points rigidly connected, moving in a plane which also moves over a fixed plane. Each of these is to be a velocity-system consistent with rigidity. The characteristic of such a system is that the distance of any two points remains constant. This is the same as saying that the velocity of any point relatively to any other is perpendicular to the line joining them. For since ab is of constant length, and we suppose a fixed for a moment, b clearly moves perpendicular to ab.

If we compound with this system another velocity-system consistent with rigidity, for the same points a and b, the velocity of b relatively to a is a line perpendicular to ab, suppose it standing up perpendicular to the plane of the paper. Then the resultant of the two velocities of b is again in a direction perpendicular to ab. Hence the resultant of two velocity-systems each consistent with rigidity is a velocity-system consistent with rigidity. If we have any velocity-system of this nature in which some point a is moving with a certain velocity and combine with it a motion of translation of the whole system, equal and opposite in direction to the velocity of a, its velocity is destroyed and there results a spin about a as centre. Conversely, any velocity-system consistent with rigidity is made up of a translation and a spin round some point.

Now let us find an expression for the acceleration of any point P moving in a moving plane. At any moment the motion of P is a spin about some point a which is called the instantaneous centre.

We have then

$$\dot{P}=i\,.\,aP\,.\,\omega,$$

where ω is the angular velocity about a.

In general the instantaneous centre will move about; and then we find from equation (2) for the acceleration of P

$$\ddot{P}=i\,.\,\dot{\overline{aP}}\,.\,\omega+i\,.\,\overline{aP}\,.\,\dot{\omega}$$
$$=i\,(\dot{p}-\dot{a})\,\omega+i\,.\,\overline{aP}\,.\,\dot{\omega}$$
$$=-aP\,.\,\omega^2+i\,.\,\overline{aP}\,.\,\dot{\omega}-i\,.\,\dot{a}\,.\,\omega \quad \text{......................(1)},$$

where we have used the expression for a vector symbolically in terms of points, viz.

$$aP=p-a, \qquad \therefore\ \dot{\overline{aP}}=\dot{p}-\dot{a},$$

and have then substituted for $\dot{P}$ its value, $i\,.\,\overline{aP}\,.\,\omega$.

It is important to notice that $\dot{a}$ does not mean the velocity of the point a which is a point fixed in the plane, but the velocity of the instantaneous centre.

The first term in the expression for $\ddot{P}$ is the normal acceleration of P, supposing it to move uniformly round a. The second term is the tangential acceleration.

The third term is the acceleration of P at right angles to the velocity of the instantaneous centre.

We will apply this result to examine the motion of the instantaneous centre. Let the point P coincide with the instantaneous centre; calling it O, we have

$$\dot{O}=-i\,.\,\dot{a}\,.\,\omega,$$

since $aP=aO$ vanishes.

Here $\ddot{O}$ is the acceleration of the point O, fixed in the moving plane, $\dot{a}$ is the velocity of the instantaneous centre in the fixed plane.

The problem is the same as that of determining the relative motion of the two planes. If we suppose them fixed alternately, all angular velocities of the first with regard to the second will be the opposite of the corresponding velocities of the second with regard to the first.

Hence if O_1 is a point in the other plane (supposed fixed before), we have, since ω is changed in sign,

$$\ddot{O}_1=+i\,.\,\dot{a}_1\,.\,\omega,$$

where $\dot{a}_1$ is now the velocity of the instantaneous centre in the plane which was moving before and is now considered as fixed.

But the acceleration of O in one plane with regard to a point O_1 considered as fixed in the other is the same as the acceleration of O_1 with regard to O considered as fixed, only in the reversed direction; that is

$$\ddot{O}_1=-\ddot{O}\,;$$

$$\therefore\ i\,.\,\dot{a}_1\,.\,\omega=i\,.\,\dot{a}\,.\,\omega,$$

that is

$$\dot{a}_1=\dot{a}.$$

The velocity of the instantaneous centre in the fixed plane is the same as the velocity of the instantaneous centre in the moving plane. Since these velocities are always in the same direction we see that the curves traced out by the instantaneous centre in the fixed plane and the moving plane respectively must touch. Further since the velocities of a and a_1 are equal in magnitude, we have $\dot{s}=\dot{s}_1$, where s, s_1, are the lengths of the two curves, loci of instantaneous centres, measured from any fixed point. Hence the curves must roll on each other without sliding, and the point P will lie on a roulette.

We shall be able to find the curvature of this roulette.

Let an be the normal to the two loci of instantaneous centres. [Fig. 92.]

The normal acceleration of P (along $\overline{aP}$), is the first term of equation (1) + the resolved parts of the other two terms. But the second part has no component along aP. Hence calling the magnitude of the velocity of a, $\dot{s}$, we have

$$\text{normal acceleration of } P = -\overline{aP}\,.\,\omega^2 + \dot{s}\,.\,\omega\cos\theta.$$

Also

$$\text{velocity of } P = \overline{aP}\,.\,\omega.$$

And

$$\text{curvature} = \frac{\text{normal acceleration}}{(\text{velocity})^2}.$$

Hence

$$\text{curvature} = -\frac{1}{aP} + \frac{\dot{s}\,.\,\cos\theta}{\omega\,.\,aP^2}.$$

Now $\frac{\omega}{\dot{s}}$ = rate of turning round of any line (aP) in the moving plane per unit length of the curve of instantaneous centres.

Hence we have

$$(\text{curvature of roulette}) = -\frac{1}{aP} + \frac{\cos\theta}{aP^2}\,\frac{1}{\frac{1}{r}-\frac{1}{r'}}.$$

There are two circles, loci respectively of points which have no normal, and no tangential acceleration. Their intersection is a point having no velocity at all.

We now go on to apply the symbol i to an exponential function.

We defined e^{pt}, as the result of making unity grow at logarithmic rate p for t seconds; and the rate of change of this quantity is got by multiplying itself by p.

Let us now examine what meaning we can give to e^{it}. This is a step OP in the plane, the rate of change of which is got by multiplying it by i, that is by turning it through a right angle.

Suppose OP was originally of length $1 = OA$.

Then since the rate of change of OP is always perpendicular to its direction, P moves on a circle.

We can shew in a special case that e^{it} is a step in the plane making an angle of circular measure t with the fixed unit line OA.

Suppose $t=1$. Then we have

$$e^{i} = 1 + i + \frac{i^2}{2} + \frac{i^3}{2\,.\,3} + \frac{i^4}{2\,.\,3\,.\,4} + \ldots\ldots$$

We can construct the step e^{i} by operating on $OA=1$ by this series.

The series converges rapidly. The first five terms take us up to E. [Fig. 93.] Some point very near this gives the end of the step e^{i}. Now the angle subtended by the arc $AP=57^0$; E is therefore the point on the circle whose arc measured from A = the radius. Since the radius was taken $=1$, the arc $AE=1=t$. Hence for this case $(t=1)$ we have verified the meaning given to e^{it}.

From the figure we have, if now AE is any arc t,

$$OE = OM + ME,$$

or

$$e^{it} = \cos t + i\sin t.$$

Hence from the series for e^{it} we can get the series for the sine and cosine in powers of the angle.

Any step ρ of greater length than the radius, is thus expressed

$$\rho = re^{i\theta},$$

where r is the length of the step; θ the angle it makes with OA.

From equation (2) we can find the radial and transversal velocities and accelerations

$$\dot{\rho} = \dot{r}e^{i\theta} + ir\dot{\theta}\,e^{i\theta},$$

$$\ddot{\rho} = (\ddot{r} - r\dot{\theta}^2)\,e^{i\theta} + i\,(2\dot{r}\dot{\theta} + r\ddot{\theta})\,e^{i\theta}.$$

(radial) (transversal)

We will return to the general consideration of rectangular versors.

In one plane if the lengths of om, mb [Fig. 94] are x, y we have

$$ob = \left(\frac{om}{oa} + \frac{mb}{oa}\,i\right) oa$$

$$= (x + yi)\,oa \text{ for any versor.}$$

A rectangular versor is the operator which turns a vector through a right angle in its plane and stretches it. We may then represent it by a handle perpendicular to the plane in which the turning takes place and of length numerically equal to the ratio in which the length of the vector is to be altered. If the handle is of length 1 the vector is not stretched. We will now consider what is meant by the sum of two rectangular versors.

If we have

$$A\rho + B\rho = C\rho,$$

where A, B and C are rectangular versors, how can we represent C.

Let A and B lie in the same plane. Then we must take the vector operated on (ρ) perpendicular to each of these versors and therefore perpendicular to their plane.

Suppose it is of length 1.

Operating on it by the two versors successively we see that A turns it down into the plane of the paper and increases its length to A, B also turns it down into the same plane and increases its length to B, $A\rho$ is of course perpendicular to A, $B\rho$ perpendicular to B. The result of both operations is then $A\rho+B\rho$ [Fig. 95]; if now we ask what versor would be required in order to bring ρ into this position and to give it this length, the answer clearly is—a versor perpendicular to $A\rho+B\rho$ and equal to it in length; that is C, the diagonal of the parallelogram formed by A, B.

Hence $A+B=C$,

and the sum of two rectangular versors is formed in the same way as the sum of two vectors.

To find the sum of three rectangular versors not in the same plane,

$$A+B+C,$$

we must add $A+B$ by the preceding rule and then add the resulting rectangular versor to C by the same rule. The way in which we take the pairs is indifferent. We see that it is necessary to take the operations in instalments if we are to give any interpretation to successive steps. For if A, B, C are not all in the same plane, there is no vector on which they can operate simultaneously.

Of course the preceding result is independent of the length of ρ being unity. We may multiply each term of the equation by any number and the result will still hold good.

We now go on to consider the product of two rectangular versors. We assume them each of length unity. Let them be A and B inclined at an angle θ. [Fig. 96.]

The vector to be operated on must be placed so that after A has operated on it B will be able to operate on it. It must therefore be in the plane of AB perpendicular to A. After both operations it will be in the position ρ' perpendicular to B and in the same plane. The effect of the product is then a versor not rectangular, which turns ρ through the supplemental angle of θ.

Since BA turns oa into ob, we have,

$$BA=\frac{ob}{oa}=\frac{om}{oa}+\frac{mb}{oa}=(-\cos\theta+I\sin\theta),$$

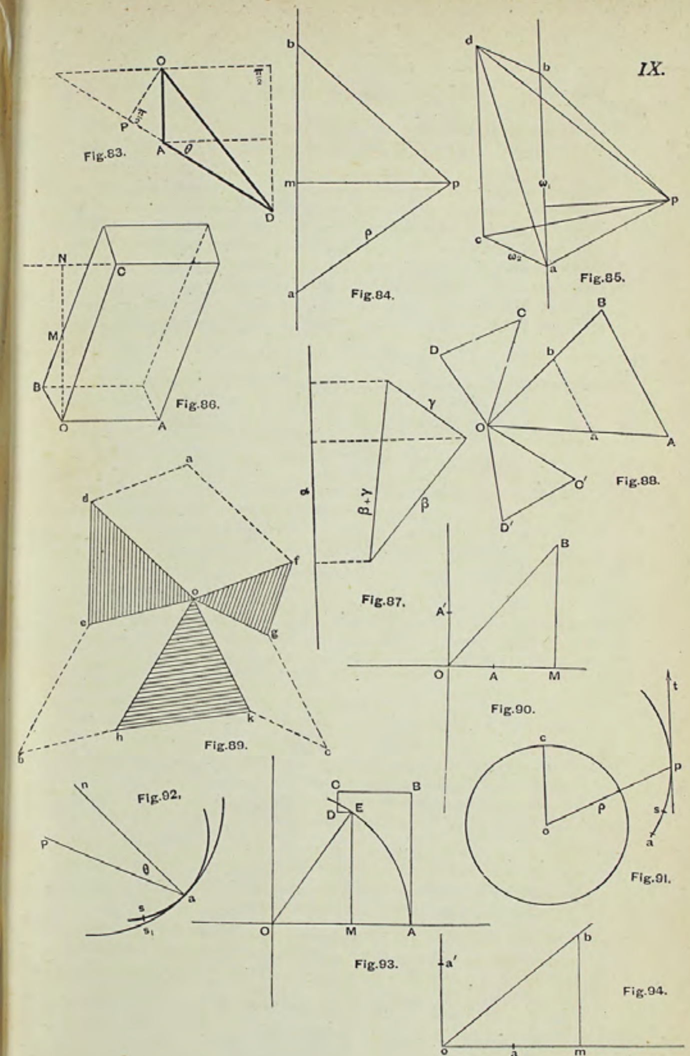
where I is a unit vector perpendicular to the plane of AB. If these versors have any lengths x, y, ob will of course be xy times the length of oa, and we shall have

$$BA=(-\cos\theta+I\sin\theta)\,xy.$$

Now we have seen that $-\cos\theta\,.\,xy$ is the scalar-product of B, A, regarded as vectors. We saw also that $I\sin\theta\,.\,xy$ was their vector-product. We shall now say that these are respectively the scalar and vector *parts* of the product and write

$$BA=S\,.\,BA+V\,.\,BA.$$

IX.



In order that $I \sin\theta \,.\, xy$ may represent the vector part of the product we need only agree to measure I so that looking back along it B is turned into A by a positive rotation.

In the case where the two versors are at right angles to each other the scalar part of the product vanishes and we can at once trace the analogy between the double interpretation allowable here and that which we gave to equations of multiplication in arithmetic.

We said that when two factors are multiplied together, either they may both be operators or the first may be an operator and the second a step. Similarly, when quantities can be measured in any direction in a plane, an equation of multiplication for two directions at right angles to each other may be written in two ways

$$BA = V \,.\, BA, \quad B\alpha = V \,.\, B\alpha.$$

In the first equation both factors are versors; in the second, the first only is a versor, the second is a vector.

We saw that in arithmetic the rule for finding the product was the same for both interpretations, e.g.

$$(-3)(+2) = -6.$$

Rule. Multiply the numbers together and remember that like signs give +, unlike signs −. The interpretation is either—

The product of the operations of first tripling and reversing, and next doubling is equivalent to the operation of sextupling and reversing; or,

The effect of the operation of tripling and reversing on the step +2 is to turn it into the step −6.

Similarly if B and A, or B and α are at right angles to each other, we have the same rule for finding the product in each case, viz. measure off the product of their lengths along a vector at right angles to their plane, and drawn so that looking back along it, we go from B to A, counter-clockwise.

The second factor may be either a vector or a versor, but the first *must* be a rectangular versor; and in both cases the directions must be at right angles to each other.

If however B and A are not at right angles the double interpretation is no longer possible. We must not regard A as a vector α. For a versor can operate only on vectors perpendicular to it. In this case both B and A must be rectangular versors. Then also the scalar part of the product would not vanish. We should have

$$BA = V \,.\, BA + S \,.\, BA.$$

The rule found for the addition of rectangular versors enables us to represent any one in terms of three units at right angles to each other.

If i, j, k, are unit lines at right angles to each other, then any vector or rectangular versor may be represented by

$$xi + yj + zk,$$

where x, y, z are components of its length.

To find the product of two such rectangular versors, first in one plane, take

$$(xi+yj)(ai+bj)=xi(ai+bj)+yj(ai+bj).$$

The right-hand side expresses the effect of performing the two operations successively. This is equal to

$$xai^2+xbij+yaji+ybj^2.$$

Since i or j operating twice reverse the direction of a vector, we have

$$i^2=-1, \qquad j^2=-1.$$

Also $ij=-ji$. For since these two lines are at right angles there is only a vector part in their product, and we have seen that the effect of changing the order of the factors in a vector product is merely to change the sign.

Hence the whole product is equal to

$$-(xa+yb)+(xb-ya)\,ij.$$

Taking now three such vectors i, j, k in space, at right angles to each other, if k is drawn so that looking back along it i is turned by positive rotation into j, we have, since the lengths are all $=1$

$$ij=k=-ji.$$

Similarly $$jk=i=-kj,$$

$$ki=j=-ik.$$

We can then form a multiplication table of versors.

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

If we call $i^2=-1$, etc. $ij=ji$, etc. the primary rules and

$ij=k$, $jk=i$, $ki=j$ the secondary rules,

we may explain these last by means of the primary rules, if we regard i, j, k as binary products of quantities which fulfil the primary rules.

Let $\quad a^2=-1,\ b^2=-1,\ c^2=-1,\ ab=-ba,\ bc=-cb,\ ca=-ac.$

Then suppose $\quad i=bc,\ j=ca,\ k=ab.$

We shall find that i, j, k satisfy the primary and also the secondary rules.

For $$i^2=bc\,bc=-bb\,cc=-1, \text{ etc.}$$

And $$ij=bc\,ca=bc^2a=-ba=ab=k, \text{ etc.}$$

Thus the secondary rules may be considered as bound up in the primary rules, if we agree to let i, j, k stand for these binary products. We may suppose a, b, c are points or stars at an infinite distance in three directions at right angles to each other, then the step ab will clearly be a rectangular versor.

Since any versor not rectangular is the sum of a scalar and a vector part and any vector may be expressed in terms of i, j, k, we are led to consider quantities of the form

$$\omega+xi+yj+zk,$$

which may also be written

$$m(\cos\theta+I\sin\theta),$$

where $$m^2=x^2+y^2+z^2.$$

This is called a *quaternion* since it involves four numbers. It is the operation of turning one vector into another in the plane perpendicular to I, that is to the rectangular versor $xi+yj+zk$. This is its most general form. We may consider however that we have already studied quaternions in which the versors were in the same plane and in the same direction.

If the quaternion q is represented by

$$q=\omega+xi+yj+zk=\omega+\rho,$$

its conjugate is

$$Kq=\omega-xi-yj-zk=\omega-\rho.$$

Hamilton calls the part ρ the vector part of the quaternion. Strictly speaking it can only be a rectangular versor. For a quaternion, and therefore each of its parts, is an operation on a vector. It is convenient however to regard a rectangular versor as a vector. They are subject to the same laws.

The stretching power of Kq and also its plane are the same as those of q, but the angle through which it turns a vector is in the opposite direction to the angle through which q turns it.

If $$q=\frac{OB}{OA}=\frac{OM}{OA}+\frac{MB}{OA} \quad \text{[Fig. 97]}$$

we shall have (by definition)

$$Kq=\frac{OM}{OA}-\frac{MB}{OA}=\frac{OM}{OA}+\frac{MB'}{OA}=\frac{OB'}{OA}.$$

The effect of $q\,.\,Kq$, or of $Kq\,.\,q$ is merely to lengthen the line OA. For the turning successively through equal angles in opposite directions brings OA back to the same direction.

And in fact, the expressions for q and Kq in terms of ω and ρ give $q\,.\,Kq$ a scalar quantity

$$q\,.\,Kq=(\omega+\rho)(\omega-\rho)=\omega^2-\rho^2=\omega^2+x^2+y^2+z^2.$$

The fundamental property of the conjugate of a quaternion is that it is another quaternion such that being multiplied by the first it gives a scalar.

We will now prove a proposition about the multiplication of quaternions.

The conjugate of the product of two quaternions is the product of their conjugates in the reversed order.

For two quaternions r and q,

$$K(qr) = Kr \,.\, Kq.$$

If we consider only the turning part of the quaternion it may be represented by an arc of a great circle on a sphere.

Each arc [Fig. 98] represents the operation of turning the radius to one end of it from the centre of the sphere, through its length to the other end. These are of course rectangular versors. If the two triangles are equal, we see at once that

$$AC = K(qr) = Kr \,.\, Kq \text{..................................} (1).$$

We may see that the use of this equation leads to a right result, if we form the product, $qr \,.\, K(qr)$. This ought to be a scalar, from our first notions of a conjugate.

In fact if the equation (1) is true we have

$$q \,.\, r \,.\, K(qr) = q \,.\, r \,.\, Kr \,.\, Kq = q \,.\, Kq \,.\, r \,.\, Kr,$$

and this is a scalar.

We may at once extend equation (1) to any number of quaternions. We have

$$K(qrs) = Ks \,.\, Kqr = Ks \,.\, Kr \,.\, Kq.$$

And generally for n quaternions a similar equation holds good.

If all the quaternions are vectors (really rectangular versors) we have a particular case of the proposition

$$K(\alpha_1\alpha_2 \ldots \alpha_n) = (-)^n \alpha_n \,.\, \alpha_{n-1} \ldots \alpha_2 \,.\, \alpha_1.$$

This product is in general a quaternion.

The two equations

$$S \,.\, \alpha\beta = S \,.\, \beta\alpha, \quad V \,.\, \alpha\beta = -V \,.\, \beta\alpha,$$

are particular cases of the equation

$$\alpha\beta = K\beta\alpha.$$

If we multiply together a quaternion, a vector and the conjugate of the quaternion we get a vector.

For the result is a quantity whose square is a scalar; and a quaternion squared, $(\omega + \rho)^2$, does not give a scalar unless either ω vanishes or ρ vanishes. $\{(\omega + \rho)^2 = \omega^2 + 2\omega\rho + \rho^2$. The middle term is a vector$\}$.

Forming now the square of $q \,.\, \rho \,.\, Kq$,

$$q \,.\, \rho \,.\, Kq \,.\, q \,.\, \rho \,.\, Kq = qKq \,.\, q\rho^2 \,.\, Kq = (q \,.\, Kq)^2 \rho^2.$$

Hence $q \,.\, \rho \,.\, Kq$ is a vector.

We will now examine the meaning of $Kq \,.\, r \,.\, q$, where r is any rectangular versor.

If the lines here drawn [Fig. 99] represent great circles on a sphere, we see that AC is $Kq \,.\, r \,.\, q$. That is the result of putting r between a quaternion and its conjugate is to slide r along the great circle of q through twice the length of the arc q; or to make r rotate round the pole of q through twice the angle of q.

In supposing the arcs drawn on a sphere, that is assuming the versors do not stretch, we have really made no restriction. Whatever the stretching powers are, the stretched vectors from the centre come out at the same points of the sphere.

A quaternion is an operator which turns any one vector into another. We will now consider an operator which turns any two vectors α and β into two others γ and δ. Hamilton called this a linear and vector function. We will denote it by ϕ. Then

$$\phi\alpha = \gamma,$$
$$\phi\beta = \delta,$$
$$\phi(l\alpha + m\beta) = l\gamma + m\delta.$$

The last equation expresses what is meant by the function being linear.

The effect of a quaternion, operating on all the vectors in a plane, for example a picture, would be to turn the whole picture through a certain angle and to increase its size, so that the result is a similar picture.

The effect of the function ϕ would be that while all parallel lines remain parallel to one another, their distances as well as their directions are altered. They are all altered in the same proportion. We may express this by saying "All parallel vectors are multiplied by the same quaternion, but the quaternions are different for different vectors."

A circle would become an ellipse. For the equation of a circle may be put into the form

$$\rho = \alpha \cos\theta + \beta \sin\theta \text{..................................} (1),$$

if α and β are equal and at right angles.

Operating on this with ϕ we get

$$\phi\rho = \phi\alpha \cos\theta + \phi\beta \sin\theta,$$

or

$$\phi\rho = \gamma \,.\, \cos\theta + \delta \,.\, \sin\theta \text{..................................} (2),$$

where γ and δ are in general not equal and not perpendicular. But this is the equation of an ellipse.

Putting θ proportional to the time, the first equation is that of uniform motion in a circle; the second that of harmonic motion in an ellipse.

If the lines which correspond to the axes of the circle are at right angles to each other, they give the axes of the ellipse: for parallel lines remain parallel.

The tangents at the ends of one axis must be parallel to the other axis.

The whole change may be represented thus. Take two lines at right angles; lengthen each in a certain proportion. Thus we get an ellipse. Then turn it through a certain angle.

When two directions at right angles remain so we have a pure strain. $\phi\rho$ is then called a pure function. There is no rotation.

For ϕ a pure function we can always get an ellipse such that the function of any semi-diameter is perpendicular to the conjugate diameter and equal to it in length.

We will suppose

$$\phi\alpha = i\beta, \quad \phi\beta = -i\alpha.$$

Then we have, if oq is the semi-conjugate diameter to op,

$$\begin{aligned}\phi(op) &= \phi(om) + \phi(mp)\\ &= \frac{om}{oa}\cdot\phi\alpha + \frac{mp}{ob}\cdot\phi\beta\\ &= \frac{nq}{ob}(-i\beta) - \frac{on}{oa}i\alpha\\ &= -i(nq+on) = -i\,.\,oq \ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}.\end{aligned}$$

The linear function of a vector is best represented physically, as the result of a strain, undergone by a body. Then the function of any vector is the new, that is the strained vector. Or it may represent the displacement of the point at the end of a vector from the origin; and then the new vector will be the old one + the function. If the body undergoes a homogeneous strain, that is one which is the same at every point, the whole change consists of two parts. The body is pulled out in different proportions in three directions at right angles to each other and then turned through a certain angle. The first effect is pure strain, the second is rotation. The pure strain changes a circle into an ellipse, if we consider first a plane sheet, with only two dimensions.

We can now shew that the linear function of a vector may in certain cases be completely represented by an ellipse, called the *strain* ellipse. Any vector namely is in the direction of some diameter of this ellipse and we can find its function at once in this way. Take the conjugate diameter and draw a line perpendicular to it and equal to it in length. This is the function or strained vector.

For the pure strain consists in pulling out two lines at right angles to each other in different proportions and either positively or negatively. That is two diameters of a circle in these rectangular directions are lengthened or shortened, in certain proportions, while their directions are unaltered.

If these extensions are such that two lines oa, ob at right angles to each other have their lengths interchanged, this is represented by the equations

$$\phi(oa) = -i\,.\,ob, \quad \phi(ob) = i\,.\,oa,^{*}$$

where the sign − occurs in the first, because we suppose the rotation from oa to ob to be positive. Each vector is to be simply stretched; not reversed.

Then all we have to do is to construct an ellipse on these lines as axes and the property in question being true for the axes is at once proved for all the diameters.

We have secured that

$$\phi(oa) = -i(ob),$$
$$\phi(ob) = i(oa).$$

Then we have the function of any other diameter [Fig. 100],

$$\begin{aligned}\phi(op) &= \phi(om) + \phi(mp)\\ &= \frac{om}{oa}\cdot\phi(oa) + \frac{mp}{ob}\cdot\phi(ob)\\ &= i\left(-\frac{om}{oa}\cdot ob + \frac{mp}{ob}\cdot oa\right)\\ &= -i(nq+on)\ \{\text{see (B) p. 510}\} = -i\,.\,oq.\end{aligned}$$

Hence pure strain of a body is this:—Every line (diameter of the strain ellipse) is changed into one equal in length to the conjugate diameter and perpendicular to it.

An exactly similar proof applies to the hyperbola where we start with the supposition

$$\phi(oa) = i\,.\,ob \text{ and } \phi(ob) = i\,.\,oa,$$

that is we suppose oa is reversed in direction. Then

$$\phi(op) = i\,.\,oq.$$

We may include both cases in the following theorem:

if $$\phi(oa) = \pm i\,.\,ob \text{ and } \phi(ob) = i\,.\,oa,$$

then $$\phi(op) = i\,.\,oq.$$

The upper sign applies to the hyperbola; the lower to the ellipse.

There is an important difference between the two strains. The elliptic strain is the only one which occurs in nature. Lines are not reversed in strain. Hyperbolic strain is however the same as elliptic strain, followed by turning the body round through an angle of continuation. Hyperbolic strain is an ideal case if we are merely representing a strain by the function ϕ.

If however the linear function of a vector represents the displacement of a point at the end of that vector, both the ellipse and the hyperbola can be used. The displacement of a point may be in the positive or negative direction along the vector to that point.

We shall find that if the displacements of points round a fixed point o are all outwards from, or inwards towards that point we must use an ellipse. If some are outwards and some inwards we must use a hyperbola. Of course all we have to do is to examine whether the two principal displacements along oa, ob are outwards or inwards.

Taking, as before, two rectangular vectors oa and ob such that

$$\phi(oa) = i\,.\,ob, \quad \phi(ob) = i\,.\,oa,$$

then if e and f are the elongations along the two axes we have

$$\phi(oa) = e \,.\, oa, \quad \phi(ob) = f \,.\, ob.$$

Hence writing a, b for oa, ob, respectively,

$$ea = b, \quad fb = a.$$

Hence
$$\frac{e}{f} = \frac{b^2}{a^2}.$$

The squares of the axes are inversely proportional to the elongations.

If both elongations are of the same sign, the squares and axes are of the same sign; this is the case in the ellipse. If the elongations are of different sign so are the squared axes, and this is the case of a hyperbola.

If we add to the stretching of pure strain, a rotation round some axis, we get the whole effect of a homogeneous strain, that is a strain in which all parallel lines remain parallel though their directions and distances are altered. This turns a circle into an ellipse, in every plane which cuts the strained body. Hence a sphere in the body is changed into an ellipsoid.

The whole effect of the strain being to pull the body out in three directions at right angles to each other and to turn it round; the axes of the ellipsoid are in these three directions; they are the shortest and the longest lines and one at right angles to them.

Any three diameters of the sphere at right angles to each other become three conjugate diameters of the ellipsoid. For parallel planes are to remain parallel and we know that the tangent plane at the end of a diameter of the sphere is parallel to the plane containing the other two diameters at right angles to it. This property must also belong to the corresponding diameters of the ellipsoid. Hence the three diameters perpendicular to each other in the sphere are changed into three diameters of the ellipsoid such that the tangent plane at the end of any one is parallel to the plane containing the other two. And this is the definition of conjugate diameters.

The strain ellipsoid in space gives the linear and vector function ϕ of any vector in a similar way to that in which ϕ is given by the ellipse in a plane.

If op, namely, is the original position of a vector, its new position is represented by the conjugate area, $\phi(op)$ is perpendicular to this area and proportional to it in magnitude. Then a vector oq is the strained position of the vector op.

If however we use ϕ to represent the displacement we shall want a surface bearing the same relation to the ellipsoid that the hyperbola bears to the ellipse.

These are got by making the hyperbola and its conjugate hyperbola revolve round the major axis and then squeezing both in one direction perpendicular to this axis. Thus we get hyperboloids of one and two sheets respectively.

(The ellipsoid was got in the same way by making the ellipse rotate round the major axis and then squeezing it in a direction perpendicular to this.)

These surfaces are then got by homogeneous strain from surfaces of revolution; and the plane conjugate to a diameter remains parallel to the tangent plane at the end of it. If the strained body is shortened or lengthened along all the axes we must use the ellipsoid in order to get the function ϕ which represents displacement. If it is shortened along one or two axes and lengthened along the other we must use the hyperboloid. The displacement of any diameter op is represented by the conjugate area oqr which in the figure [Fig. 101] is an ellipse. Some conjugate sections however will cut the hyperboloid in hyperbolas. What meanings can we give to the conjugate area in this case? We must take the area of the ellipse on the same axes in order to keep the same rule for all possible diameters.

This is what is meant by a linear and vector function of a vector in space. Every vector is turned into its linear function which is a vector represented by the area conjugate to the original one. This is pure strain. To get homogeneous strain generally which includes rotation we must first get the linear vector function and then turn it round.

In the preceding investigation all the Geometry we require is given by the properties of the homogeneous strain itself.

If we take an elastic rod of square section and twist and bend it there is a certain strain at every point, which may be represented by a vector. For instance, if, before the twisting and bending, a body is moving along the rod with uniform velocity then the strain at any point is represented in magnitude and direction by the velocity of this body. The rod tends to untwist and unbend itself. The action at any point which balances this tendency is the stress. There is a bending and also a twisting couple which compounded give another couple and this is represented by a vector perpendicular to its plane. Hence at every point the stress and the strain can be represented by vectors. It is found that the stress is a linear vector function of the strain. It is moreover a pure function. There is no turning round.

As another example we may take an elastic plate bent in two different ways. One kind of bending would at any point change the plane surface into a surface bent like a sphere. The other kind tends to bend the plane like a saddle. These two strains have two corresponding stresses. The plate tends to flatten itself again from both kinds of bending.

Here again the stress is a pure function of the strain.

Another instance is furnished by a solid body in which one point is fixed. Rotating it, there is one axis about which the moment of momentum is a maximum. If ω is the vector representing the spin and the momentum is represented by μ, then

$$\mu = \phi(\omega), \quad \text{momentum} = \text{a pure function of spin.}$$

This pure function depends on an ellipsoid called the momentum ellipsoid.

We can only give a short sketch of a part of the theory in which little was done by Hamilton, but which has been worked out by Tait; the part that is which treats of the operator

$$\nabla = i\partial_x + j\partial_y + k\partial_z.$$

Suppose a scalar quantity having various values all over a plane. The height of the plane for instance is such a quantity. At any point in a sloping plane or on the side of a hill there is one direction in which the height increases most rapidly. If we draw a line from any point in the direction of this greatest increase and proportional to the rate of increase, this is called the slope of the height.

If z is the height, slope of $z = i\partial_x z + j\partial_y z$.

We must then set off lines representing the rate of increase in two directions at right angles to each other and take their resultant. This will give the magnitude and direction of the greatest increase of height, that is the slope of z. Suppose for example that going northwards [Fig. 102] we rise one foot in 30, and going eastwards, one in fifty, the resultant of the lines $\frac{1}{30}$ and $\frac{1}{50}$ is the slope.

Now take a scalar quantity u having different values at different points of space. The slope of u is then

$$i\partial_x u + j\partial_y u + k\partial_z u.$$

It is the rate of increase in the direction in which u increases fastest and is found in this way: measure off, in three directions at right angles to each other, lengths which represent rates of increase and find their resultant.

Physical examples of slopes.

If u = potential energy, ∇u = force.

If u = velocity potential, ∇u = velocity.

In these cases ∇ operates on a scalar. Tait investigated the effect of ∇ on a vector.

Let a vector σ have the constituents p, q, r. Then

$$\nabla\sigma = (i\partial_x + j\partial_y + k\partial_z)(ip + jq + kr)$$
$$= -(\partial_x p + \partial_y q + \partial_z r) + i(\partial_y r - \partial_z q) + j(\partial_z p - \partial_x r) + k(\partial_x q - \partial_y p).$$

The quantity in the first part is a scalar; the remaining part is a vector.

The two parts have definite relations to the vector σ. This vector may mean the velocity of a fluid at any point.

Then $-S.\nabla\sigma$ represents the expansion, that is, the rate of change of unit volume. In general there is a certain spin at every point of a fluid. $V.\nabla\sigma$ represents twice this spin.

If σ has a different meaning the interpretations given to the two parts of its slope are of course different. Suppose it means the vector called by Faraday the "electrotonic state" or magnetic induction.

Then a law of electricity is that $S.\nabla\sigma$ equals 0. But $-S.\nabla\sigma$ is the expansion in a moving fluid. Maxwell therefore interprets the law thus:—there

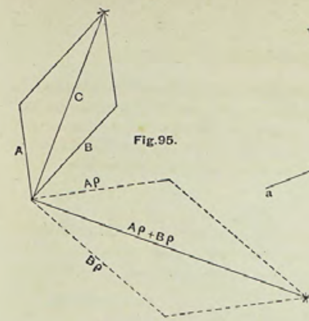


Fig. 95.

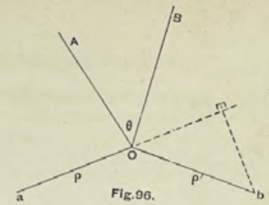


Fig. 96.

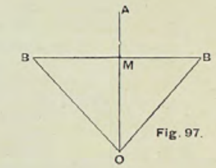


Fig. 97.

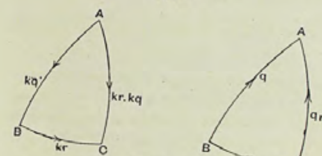


Fig. 98.

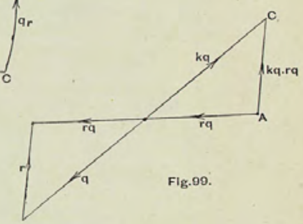


Fig. 99.

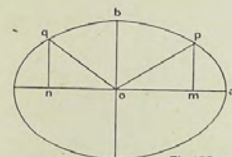


Fig. 100.

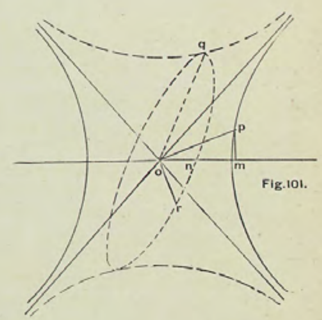


Fig. 101.

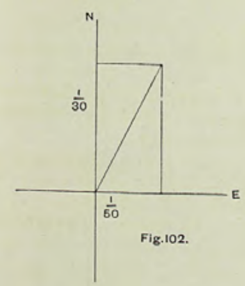


Fig. 102.

is no expansion of the ether in electrical motions. But the analogy is not exact, for the electrotonic state σ represents a momentum, not a velocity. $V.\nabla\sigma$ here represents twice the magnetic induction.

The slope of the slope, $\nabla^2\sigma$ has no meaning for a fluid-motion, but for σ = the electrotonic state it means electromotive force.

The equation

$$\nabla\sigma = S.\nabla\sigma + V.\nabla\sigma$$

is interpreted for σ any vector,

slope of a vector = convergence + curl,

$\nabla^2\sigma$ = the slope of the slope = the concentration.

Laplace's operator

$$-(\partial_x^2 + \partial_y^2 + \partial_z^2) = \nabla^2$$

is the square of

$$i\partial_x + j\partial_y + k\partial_z = \nabla.$$

Hamilton laid great stress on his having thus discovered the square root of this operator, which is of importance in many physical investigations. Its properties come out n a remarkable way from he properties of ∇.

SYLLABUS OF LECTURES ON MOTION*.

Division of the Subject.

The science which teaches how to describe motion accurately, and how to compound different motions together, without considering the circumstances under which motions take place, is called *Kinematic* (κινημα, motion).

The simplest kind of motion is that in which a body without changing its size or shape moves so that all straight lines in the body remain parallel to their original positions; this motion is called a *Translation*. As all parts of the body move alike, we may confine our attention to any one of them, however small; for this reason that part of Kinematic which treats of translations is often called the Kinematic of Particles.

A body which does not change its size or shape during the time considered is called a *rigid* body. That part of Kinematic which treats of motions in which there is no change of size or shape is called the Kinematic of Rigid Bodies.

A change of size or shape, considered without reference to change of position, is called a *strain*. The Kinematic of Strains teaches how to describe strains accurately, and how to compound them together. Bodies which change their size or shape are called *elastic*; and the corresponding branch of Kinematic is called the Kinematic of Elastic Bodies.

The science which teaches under what circumstances particular motions take place is called by one or other of two different names according to the view that is taken of it. If it is regarded as mainly based upon the Law of Force, and if its results are expressed in terms of force, it is called *Dynamic* (δυναμις, force); but if it is regarded as mainly based upon the Law of Energy, and if its results are expressed in terms of energy, it is called *Energetic* (ἐνεργεια). In either case it is divided into two parts; *Static*, which treats of those circumstances under which *rest* or *null motion* is possible, and *Kinetic*, which treats of circumstances under which actual motion takes place. Properly speaking, Static is a particular case of Kinetic which it is convenient to consider separately.

We may also make divisions between the Static and Kinetic of particles, rigid bodies, and elastic bodies; but the Static of particles and of rigid bodies

* [This syllabus appears to me to have been drawn up for lectures at University College, Michaelmas Term, 1873.]

is generally treated as one subject, while the Kinematic and Dynamic or Energetic of elastic bodies are grouped together as the science of *Elasticity*.

These divisions may be represented by the following scheme :—

Science of Motion.

$$\left.\begin{matrix}\text{Kinematic,}\\ \text{Dynamic, or}\\ \text{Energetic, viz.}\end{matrix}\right\}\left\{\begin{matrix}\text{Static,}\\ \text{Kinetic,}\end{matrix}\right\}\text{ of }\left\{\begin{matrix}\text{Particles (Translations),}\\ \text{Rigid Bodies (Twists),}\\ \text{Elastic Bodies (Strains).}\end{matrix}\right.$$

Translations.

Def. If two bodies A and B are in motion, the motion of B is said to be compounded of the motion of B relative to A, and the motion of A.

Prop. Translations represented by the sides of a parallelogram compound together into a translation represented by the diagonal.

Def. A *vector* is a quantity having magnitude and direction. A translation is a particular kind of vector, and the composition of translations is equivalent to their addition as vectors; it satisfies the law

$$\alpha+\beta=\beta+\alpha.$$

Def. Uniform rectilinear motion is that in which equal spaces are traversed in equal times.

Its equation is

$$\rho=\alpha+\beta t.$$

Prop. Two uniform rectilinear motions compound into a uniform rectilinear motion.

Harmonic Motion.

Def. Uniform motion in a circle is that in which equal arcs are traversed in equal times.

Def. If a point P move uniformly in a circle, and a perpendicular PM be always drawn from it to a fixed diameter AA' of the circle, the foot M of the perpendicular will oscillate to and fro in the diameter; this motion of the point M is called a *Simple Harmonic Motion*.

Its equation is

$$\rho=a\cos(nt-\epsilon).$$

Def. The radius of the circle is called the *amplitude* of the S. H. M.

Def. The time which P takes to go once round the circle is called the *period* of the S. H. M.

Def. The circular measure of the arc described by P from the era of reckoning till it came to the positive end of the diameter AA' is called the *epoch* of the S. H. M.

Def. The portion of the whole period which has elapsed since the point M last passed through its middle position in the positive direction is called the *phase* of the S. H. M.

PROP. Two S. H. M. of the same period compound into a S. H. M. of that period.

The construction here made use of for compounding two S. H. M. is exemplified in the Tidal Clock of Sir W. Thomson. The clock has two hands whose lengths are proportional to the solar and lunar tides respectively, while their periods of revolution are made equal to the periods of these tides. A jointed parallelogram is constructed, having the hands of the clock for two sides; the height of that extremity of the parallelogram which is furthest from the centre will then be proportional to the height of the compound tide. For this purpose a series of horizontal strings at equal distances are stretched across the face of the clock, and the height is read off by running the eye along these to a vertical scale of feet in the middle.

DEF. The curve described by a point which has a uniform rectilinear motion compounded with a S. H. M. perpendicular to it is called a *harmonic curve*.

The composition of S. H. M. of different periods in the same line may be represented graphically by the super-position of harmonic curves; i.e. by drawing a curve whose height at any point is the sum of their heights.

PROP. Any S. H. M. may be resolved into two in the same line, differing in phase by a quarter period, and one of them having any given epoch.

PROP. S. H. M. on any number of different lines, having the same period and phase, compound into one having that period and phase.

PROP. Two S. H. M. on different lines, having the same period, but differing in phase by $\frac{1}{4}$, compound into harmonic motion in an ellipse (viz. an orthogonal projection of circular motion).

Its equation is

$$\rho = \alpha \cos(nt - \epsilon) + \beta \sin(nt - \epsilon).$$

PROP. Any number of S. H. M. having the same period compound into harmonic motion in an ellipse.

Two harmonic motions in different directions and with different periods produce a resultant which is best studied by wrapping round a cylinder of suitable size paper on which is traced a harmonic curve. The curve thus drawn on the cylinder may then be constructed in wire, and when turned round the axis of the cylinder will represent to an eye at a sufficient distance the curve of compound harmonic motion for varying values of the difference of phase of the simple motions. The simplest case is that in which the circumference of the cylinder is equal to the length of a wave of the harmonic curve; here the periods are equal, and the curve traced on the cylinder is merely an ellipse. The same result is produced by turning the cylinder round its axis while a pencil moves with simple harmonic motion up and down a generating line.

DEF. A motion which exactly repeats itself in the same place after a certain interval of time is called a *periodic* motion.

The resultant of any number of simple harmonic motions whose periods are commensurable is a periodic motion, its period being the least common multiple of their periods.

Fourier's Theorem. Every rectilinear periodic motion of period P may be resolved into a series of simple harmonic motions whose periods are P, $\frac{1}{2}P$, $\frac{1}{3}P$, etc.

Let $\phi(t)$ be the distance of the moving point from a fixed point on the line at a time t, then the periodicity of the motion is expressed by the fact that $\phi(t+P)=\phi(t)$, whatever t is. And the theorem asserts that in this case the quantities a, b can always be found so as to make true the following equation, where

$$\theta = \frac{2\pi t}{P} \quad \phi(t) = \tfrac{1}{2} b_0 + b_1 \cos\theta + b_2 \cos 2\theta + \dots$$
$$+ a_1 \sin\theta + a_2 \sin 2\theta + \dots$$

The amplitudes and epochs of the several harmonic components may be represented as follows. Let a vertical cylinder revolve about its axis, while a pencil moves up and down one of its generating lines, so as to trace out a curve on the cylinder. If the motion of the pencil is periodic, and has a period equal to that of the cylinder or any exact multiple of it, this curve will return into itself and be a finite curve on the cylinder. Now let the pencil have the given periodic motion which it is required to resolve into harmonic constituents. When the cylinder revolves once in the period P, let the curve described be called C_1; when it revolves twice in that period let the curve be called C_2; when it revolves m times, let this curve be called C_m. And let a circle be drawn on the cylinder whose height is the mean height of the curve C_1; this will be called the mean circle.

If a plane be drawn through the axis of the cylinder, any curve traced on the cylinder may be orthogonally projected on that plane. It is necessary now to define the area between this projection and the line in which the plane is cut by the plane of the mean circle. Let AB be this line [Fig. 104], and let $PMQNP$ be the projection, where PMQ is projected from the near half of the cylinder, and QNP from the further half. Then for the *near* half, the area APM which is *below* AB must be considered negative, and the area MQB which is *above* it, positive. For the *further* half, QNB must be considered negative, and NPA positive. Thus the area is

$$-APM + MBQ - NBQ + APN$$
$$= MPN + MNQ = MPNQ.$$

The same rule is to be applied when the curve cuts itself or the line AB any number of times. Now it is found that for every closed curve traced on a cylinder, there is one plane through the axis such that the area of the projection on it is zero; and that for the plane at right angles to it the area is the greatest possible; while for an intermediate plane the area varies as the sine of the angle which it makes with the zero plane. It is thus possible to draw an ellipse upon the cylinder, the area of whose projection upon any plane whatever through the axis shall be the same as that of a given closed curve. Let the ellipse E_1 have the same projected area as the curve C_1, E_2 half that of the curve C_2, E_m one-mth that of the curve C_m, and so on. If, while the cylinder revolves once on its axis during the period P, the pencil be made to follow the ellipse E_1, always remaining in the same vertical line, it will have a S. H. M. with the period P. If while the cylinder revolves m times during the period P, the pencil be made to follow the ellipse E_m, it will have a S. H. M. with the period $\frac{1}{m}P$. These motions are the harmonic components of the given periodic motion; and that motion may be reproduced by compounding them all together *.

Parabolic Motion.

PROP. If rectilinear motion in which the space passed over from the beginning is proportional to the square of the time occupied, be compounded with rectilinear motion, the resultant will be motion in a parabola.

Its equation is $$\rho = \alpha + \beta t + \gamma t^2.$$

Velocity.

DEF. If a body is in uniform rectilinear motion, and travels v centimetres in every second, the body is said to have at every instant a *velocity* of v centimetres per second, or simply a velocity v.

DEF. If a body undergo a translation whereby a point of it is carried in any manner by any path from A to B in t seconds, the body is said to have a *mean velocity* $\frac{AB}{t}$ in that interval of t seconds.

The two quantities here defined have magnitude and direction; they are *vectors*. A velocity may be expressed in terms of other units than centimetres per second; in feet or miles per second, leagues per hour, etc.; but when expressed as a number of centimetres per second, it is said to be given in *absolute measure*. In uniform rectilinear motion the mean velocity is the same in any interval whatever, and is equal to the instantaneous velocity at any instant; but the latter is a property which the body possesses *at an epoch* or point of time, while the former is a fact relating to its motion during an interval.

DEF. If any rectilinear motion of a point be compounded with a uniform motion of unit velocity at right angles to it, the curve traced out by the point is called the curve of positions for that rectilinear motion.

Lemma. PT is the tangent at a point P of a circle [Fig. 105]. Any angle being proposed, it is always possible to take a point Q on the circle so near to P that the chord of every arc pq included in PQ shall make with the tangent PT an angle less than the proposed angle.

Let C be the centre of the circle; make PCQ less than the proposed angle, and draw CM perpendicular to pq. Then PCM is the angle which the chord pq makes with PT, and it is always less than PCQ, therefore less than the proposed angle. Q.E.D.

* [Cf. *Dynamic*, p. 37, where it is said a proof of Fourier's Theorem will be given in the Appendix.]

DEF. R, P are points on any curve, Q moves from R along the curve towards P; if when any angle is proposed, it is always possible to take Q so near to P that the chord of every arc pq included in PQ shall make with a certain line TP an angle less than the proposed angle; then the curve is said to have TP for a *tangent* at the point P [Fig. 106].

If S is a point on the other side of P and if Q moves from S towards P, there may be another line PT' such that an arc PQ may always be taken in which no chord shall be inclined to PT' so much as by a proposed angle. In this case we may speak of TP as the tangent *up to* P and of PT' as the tangent *on from* P. When TPT' is a straight line, the curve is said to be *elementally straight* or to have the property of *elemental straightness* at the point P; for the more it is magnified, the more will a portion containing P of given length in the magnified figure approach to the straight line TPT' in shape and position. For this, three conditions are necessary; there must be a tangent up to P, a tangent on from P, and these tangents must be in one straight line.

PROP. If the curve of positions of a rectilinear motion has a tangent at a point P, then it is possible to choose an interval ending at the instant corresponding to the point P so that the mean velocity in that interval (and in all intervals included in it) shall differ less than by a given amount from a certain quantity.

Let QP [Fig 107] be a portion of the curve of positions, PT the tangent at P; QN, PM parallel to the (vertical) rectilinear motion considered, and perpendicular to the (horizontal) uniform motion with which it is compounded; QR perpendicular to PM. Since the uniform motion has unit velocity, the number of units of length in NM is equal to the number of seconds in which the body has performed the vertical motion RP, and the mean velocity in the interval NM is therefore $\frac{RP}{NM}$. Now take AB a horizontal line equal to the unit of length, and draw AC, AD parallel to PT, PQ, meeting the vertical line through B in C, D. Then BD represents the mean velocity in the interval NM. Similarly if pq be any arc included in PQ (pm, qn perpendicular to NM), and if we draw Ad parallel to the chord pq, Bd will represent the mean velocity in the interval nm. Now it is possible by hypothesis to choose Q so near to P that the angle QPT, which is equal to CAD, shall be less than any proposed angle; and that the angle which any chord pq makes with PT, which angle is equal to CAd, shall be less than the proposed angle. Therefore it is possible so to choose N that for every interval included in NM the length Cd shall be less than a proposed amount; or so that the mean velocity shall differ from the velocity represented by BC by a quantity less than the proposed amount. Q. E. D. The quantity Bc or $\frac{MP}{TM}$ is then called the *instantaneous velocity* of the rectilinear motion at the instant M.

DEF. Let Q, P be successive positions of a moving point, and let BD represent the mean velocity during an interval included in the passage from Q to P; then if it is always possible to find Q so near to P that for all intervals between Q and P the distance DC from D to a fixed point C shall be less than a proposed length, the point at the instant of arriving at P is said to have an *instantaneou velocity* BC in magnitude and direction [Fig. 108].

Prop. If a moving point has a velocity, the curve described has a tangent in the same direction; and if a length equal to the arc RQ be measured off on a straight line as Q moves, this rectilinear motion will have a velocity whose magnitude is equal to that of Q.

Prop. If each of two motions has a velocity at a certain instant of time, the motion compounded of them has a velocity which is compounded of their velocities by the rule for addition of vectors.

Let AB and AC be the given velocities; complete the parallelogram $ABDC$ [Fig. 109]. Let also AB', AC' be the mean velocities during an interval which ends at the given instant; if the parallelogram $AB'D'C'$ be completed, we know that AD' is the mean velocity of the resultant motion. Now the interval may be so chosen that for it and all shorter ones included in it BB' and CC' are each less than half of any proposed length; and therefore DD', which is their vector-sum, less than the proposed length. Consequently AD is the velocity of the resultant motion at the given instant. Q. E. D.

It is to be noticed that in accordance with our definitions a motion may have one velocity *up to* a certain instant and another velocity *on from* that instant; or, as we may say, an *arrival* and a *departure* velocity. Such motions are for mathematical convenience supposed to take place in the theory of collisions; but it is believed that they do not occur in nature, and that the arrival and departure velocities are always identical. If a point has an arrival and a departure velocity at a given instant and if they are identical, its motion is said to be *elementally uniform;* for if a small portion of the path containing the position of the point at that instant be magnified to a definite length, and the times of traversing different parts of it be preserved in their proportions, then the smaller the portion taken, the nearer will the path approach to a straight line and the motion to uniform motion along it.

Prop. The velocity in the s. h. m.,

$$\rho = a\cos(nt-\epsilon)$$

is

$$\dot{\rho} = -na\sin(nt-\epsilon),$$

(when the position-vector of a point is called ρ, its velocity is denoted by $\dot{\rho}$).

The s. h. m. has a velocity, because its curve of positions has a tangent, being produced by unrolling an ellipse from a cylinder. Now uniform circular motion being compounded of two simple harmonic motions, its velocity is compounded of their velocities by the law of addition of vectors. Thus the velocity of P is compounded of the velocities of M and N; but these velocities are respectively perpendicular to the lines CP, CM, and MP, the vector CP being equal to $CM+MP$ [Fig. 110]. The velocities are therefore proportional to the lengths of these lines, and as the velocity of P is $n\,.\,CP$ along the tangent, the velocities of M and N are $n\,.\,MP$ and $n\,.\,CM$ along MC and CN respectively. But a length $n\,.\,MP = n\,.\,AC\sin PCM$ along MC is equal to $-na\sin(nt-\epsilon)$. Q. E. D.

Fig. 104.

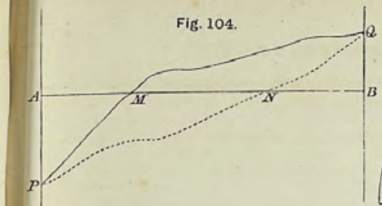
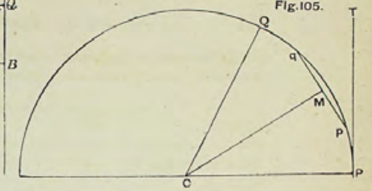


Fig. 105.



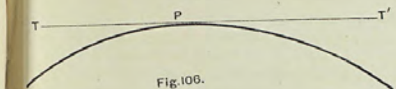
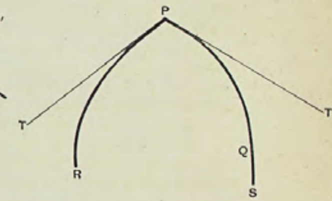


Fig. 106.



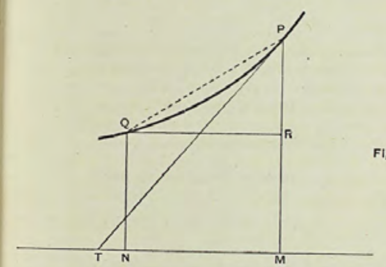
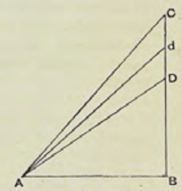


Fig. 107.



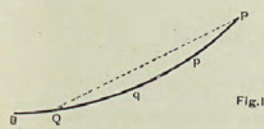


Fig. 108.

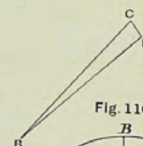


Fig. 109.

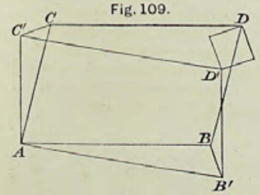
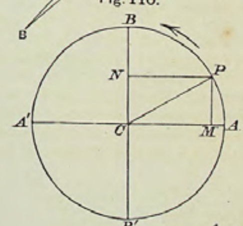


Fig. 110.



PROP. The velocity in the elliptic harmonic motion,

$$\rho = \alpha \cos(nt - \epsilon) + \beta \sin(nt - \epsilon),$$

is

$$\dot{\rho} = -n\alpha \sin(nt - \epsilon) + n\beta \cos(nt - \epsilon)$$

$$= n\left\{\alpha \cos\left(nt - \epsilon + \frac{\pi}{2}\right) + \beta \sin\left(nt - \epsilon + \frac{\pi}{2}\right)\right\},$$

and is therefore proportional to the conjugate diameter.

PROP. The velocity in the parabolic motion

$$\rho = \alpha + \beta t + \gamma t^2$$

is

$$\dot{\rho} = \beta + 2\gamma t.$$

Let t_1, t_2, t_3, t be successive values of t, these quantities being therefore in ascending order of magnitude; ρ_1, ρ_2, ρ_3, ρ the corresponding values of ρ. Then the mean velocity in the interval from t_2 to t_3 is

$$\frac{\rho_3 - \rho_2}{t_3 - t_2} = \beta + \gamma(t_2 + t_3).$$

Since t_2 and t_3 are intermediate between t_1 and t, this vector differs from $\beta + 2\gamma t$ less than $\beta + 2\gamma t_1$ does; that is, less than $2\gamma(t - t_1)$. Now it is possible so to choose t_1 that this shall be shorter than any proposed length γx; that is, it is possible to choose an interval ending at t, so that the mean velocity for every interval included in it differs from $\beta + 2\gamma t$ by less than a proposed amount. The same thing may be shewn for intervals beginning at t. Therefore the motion is elementally uniform and has $\beta + 2\gamma t$ for its velocity. Q. E. D.

PROP. In the motion whose equation is

$$\rho = \alpha t^n$$

(n a positive integer), the velocity is

$$\dot{\rho} = n\alpha t^{n-1}.$$

With the notation of the previous proposition, the mean velocity in the interval from t_2 to t_3 is

$$\frac{\rho_3 - \rho_2}{t_3 - t_2} = \alpha \frac{t_3^n - t_2^n}{t_3 - t_2} = \alpha(t_3^{n-1} + t_3^{n-2} t_2 + \ldots + t_2^{n-1}).$$

Since t_2 and t_3 are intermediate between t_1 and t, this quantity differs from $n\alpha t^{n-1}$ less than $n\alpha t_1^{n-1}$ does; that is, less than $n\alpha(t^{n-1} - t_1^{n-1})$, which by proper choice of t_1 can be made less than an assigned quantity. Whence as before.

LECTURE NOTES*.

1. Geometry is a physical science. It is concerned with the sizes and shapes of objects, and the positions which they may occupy. The doctrines Riemann. which it teaches on these subjects are derived from experience extended by hypotheses so as to become precise. The hypotheses by which geometrical experience is made precise are *three*.

2. *Hypothesis of Continuity.* When two adjacent portions of space differ in any way, e.g. the space occupied by a body and the space not occupied by it, the boundaries of the two portions appear—so far as we can examine—to be identical. By the hypothesis of continuity we assume that they really *are* identical, or that the surface is surface to both these portions and takes up absolutely no room. This definition is from *Arist.* Cat. 6. On continuity see *Boscovitch*, de Continuitatis lege.

Similarly when two adjacent portions of a surface are different, the boundary is by this hypothesis assumed to be common to both; it is called a line, and takes up absolutely no surface-room. And when two adjacent portions of a line are different, their common boundary, taking up no room of any kind upon them, is called a point. From this definition it follows that there is an infinite number of points between two points on a line, and an infinite number of lines between two lines on a surface, and an infinite number of surfaces between two surfaces in space.

3. *Hypothesis of Rigid Motion.* When we move bodies about they do not in general sensibly alter in size or shape. By this second hypothesis we assume that the appearance is accurate, and that a body may move from one position to another without undergoing any the very least alteration in size or shape. Or, that exactly the *same* geometrical relations may exist in two different portions of space.

Leibnitz. 4. *Hypothesis of Infinite Extent.* A surface which is of the same shape all over and of the same shape on both sides is called a plane. A line in a plane which is of the same shape all along and of the same shape on both sides is called straight. If a point travel along a straight line there appears to be no reason for supposing that it would ever come back to the same position from the other direction. The third hypothesis assumes that a straight line is actually of infinite extent in both directions, and that the point might travel on for ever

* [I am indebted to Mr A. B. Kempe for the information that these *Notes* were given to students attending a course of lectures at Trinity College, Cambridge, in the year 1870. The page containing Articles 19, 20, 21, is missing from all the copies I have met with.]

without revisiting any of its former positions. According to this a plane and space itself are infinitely extended in every direction.

5. *Perpendicular and Parallel.* From the first two hypotheses it may be proved that straight lines can only intersect in two points. Those straight lines which meet therefore divide space into four regions, each of these is called an angle. If these are all of the same shape the lines are called perpendicular. By using now the third hypothesis we may shew that two lines meet in only one point. Two lines making equal angles with the same line can then not meet at all. They are called parallel. It appears probable from experience that only one line can be drawn through a fixed point parallel on given line. We may assume either that lines meet only in one point, and that this probable experience is accurately true, or we may assume the third hypothesis; either assumption involves the other.

6. *Quantity and Measuring.* A discrete assemblage of things, as of chairs, say, is estimated by counting the number of them. It is assumed that the number is the same in whatever order they are counted. From this assumption flow the theorems of addition and multiplication $a+b=b+a$, $ab=ba$, $a(b\pm c)=ab\pm ac$. We cannot however count the number of points in a piece of line. Yet we suppose it to have a certain magnitude; we speak of another piece as greater, or less, and greater or less in a certain degree. The degree in which one piece is greater than another is called the *ratio* of the two pieces. The ratio of $A+B$ to C is called the sum of the ratios of A to C and B to C. The ratio of A to C is called the product of the ratios of A to B and B to C. It is assumed that two things may each be broken up into any number of parts and the parts rearranged without altering their ratio. From this assumption flow as with numbers the theorems of addition and multiplication $a+b=b+a$, $ab=ba$, $a(b\pm c)=ab\pm ac$.

7. *Calculus of Ratios.* Every quantity is therefore measured by the ratio which it bears to some fixed quantity, called the unit. But between any two ratios is an infinite number of ratios; it is therefore impossible to tabulate all ratios, or to give them names. A ratio then can only be described approximately, as being very near to the ratio of two numbers, that is, of two quantities which have a common measure. On the assumption that two ratios which are always greater or less than the *same* numerical fractions are equal, it may be shewn, as in Euclid, that similar triangles have proportional sides.

8. *Analysis of Position on a Straight line.* On a straight line take a fixed point o, and a fixed length oa in a given direction from o; then the position of a point p on the line is known if we know the ratio of op to oa and the side of o on which p is, and *vice versa*. Let μ denote the ratio, then $op=\mu\,.\,oa$. μ may be regarded as a direction to perform the following operation: change the value of oa in the ratio of 1 to μ. The equation asserts that by performing this upon oa we attain the value of op (op and oa may be regarded as quantities of motion). Subtraction will mean motion in the contrary direction. Hence $-$ may be regarded as an abbreviation for $-+$ or *reversed addition*.

9. *Vectors. Ratios of Vectors.* The quantity of motion which carries a point from the position a to the position b is called the *vector ab*. The vector ab is said to be *equal* to the vector cd when b is on the same side of a that d is of c,

and at the same distance from it. *Addition* of vectors is then defined by the equation $ab+bc=ac$, or $ab+bc+ca=0$. The *ratio* of two vectors is that operation which changes the second into the first. The operation consists of two parts; a *tensor* or stretching part which merely alters the length of the vector or the quantity of motion, and a *versor* or turning part, which either *preserves* the direction of motion or *reverses* it. Let μ be the ratio of the quantities of motion in the vectors ab and cd; then if ab is in the same direction as cd we shall have $\frac{ab}{cd}=+\mu$; but if they are in different directions $\frac{ab}{cd}=-\mu$. Ratios of vectors are added and compounded or multiplied by the same rules as the ratios of magnitude (6). In the ordinary language of algebra ratios of quantities are called *signless numbers*, while ratios of vectors are called *numbers with signs to them*. The theorems $\alpha+\beta=\beta+\alpha$, $\alpha\beta=\beta\alpha$, $\alpha(\beta\pm\gamma)=\alpha\beta\pm\alpha\gamma$, are still true when $\alpha\beta\gamma$ are numbers with signs to them.

Analysis of Position on a Plane.

10. *Gauss's Plane of Numbers.* By the operation -1 a vector has its direction *suddenly* changed into the opposite one. We may however conceive the same result brought about by the continuous rotation of the vector in a plane through two right angles. The operation may then be *halved;* that is to say, the vector may be turned through a right angle. This operation of turning a vector through a right angle is denoted by the letter i. Thus if aa', bb' are two equal lines bisecting each other at right angles at the point o, $oa'=-oa$, and $ob'=-ob$; moreover $ob=i\,.\,oa$, $oa'=i\,.\,ob$, $ob'=i\,.\,oa'=-i\,.\,oa$, and $oa=i\,.\,ob'$. From this definition it appears that $i^2=-1$.

11. *Vectors in a Plane. Complex Numbers.* The *Equality* of vectors in a plane is thus defined: $ab=cd$ means that the line ab is parallel to cd and in the same direction, and that the length ab is equal to the length cd. Addition of vectors is then defined by the equation $ab+bc=ac$, or, as before, $ab+bc+ca=0$. The *ratio* of two vectors is that operation which changes the second into the first. But when we have defined the addition and composition of ratios as in (6) we may shew that the ratio of any two vectors is the sum of two ratios, one of which is a *signed number* (9) and the other is the product of i by a signed number. For let ab and ac be two vectors; then draw bm perpendicular to ac and ad parallel to bm so that in length $ad=ac$. Then

$$\frac{ab}{ac}=\frac{am+mb}{ac}=\frac{am}{ac}+i\,.\,\frac{mb}{ad}\ (\text{since } ad=i\,.\,ac).$$

Now $\frac{am}{ac}$ and $\frac{mb}{ad}$ are both signed numbers, or ratios of vectors on the same straight line; denote them by x and y. Then we have proved that the ratio of any two vectors on a plane is of the form $x+iy$, where x and y are signed numbers. The expression $x+iy$ is called a *Complex Number*, and may be denoted by a single letter z.

12. *Modulus and Argument.* The ratio of the lengths ab and ac (a signless number, 9) is called the *modulus* of the complex number which is the ratio of

these vectors. The angle bac is called the *argument* of the same complex number. Thus if r, ϕ are the modulus and argument of z,

we have
$$z=x+iy=r(\cos\phi+i\sin\phi),$$
$$r^2=x^2+y^2,\quad \tan\phi=\frac{y}{x}.$$

13. *Addition.* To add together two complex numbers, we must add separately the parts which are not multiplied by i and the parts which are so multiplied. Thus $Oa=Om+ma$, and

$$ab=ap+pb;\ \text{now } Oa+ab=Ob=On+nb=\overline{Om+ap}+\overline{ma+pb}.$$

Hence, on substituting for these vectors their ratios to the fixed vector OI, we obtain the rule enunciated. From the fact that the opposite sides of parallelograms are equivalent vectors flows the theorem

$$Oa+ab=Oc+cb=ab+Oa,\ \text{or } z+w=w+z.$$

14. *Multiplication.* To compound the ratios of OI to Oa, and of OI to Ob is to find a vector which bears the same relation to Ob that Oa does to OI. Let Oc be such a vector, then length $Oc=\frac{Oa\,.\,Ob}{OI}$, and angle $I\hat{O}c=I\hat{O}a+I\hat{O}b$. Hence to multiply two complex numbers, multiply their moduli and add their arguments. This rule shews that $zw=wz$. By altering the triangle Oab in a certain ratio (mod. z) and turning it through a certain angle (arg. z) we may shew that $z(u\pm v)=zu\pm zv$. $\left(u=\frac{Oa}{OI},\ v=\frac{ab}{OI}\right)$.

15. *Expansion of* F (x+y). By means of the three laws of addition and multiplication which we have now proved true for complex numbers we may Newton.
shew that the Binomial Theorem with a positive integral exponent is true for these numbers; that is, that $\frac{(x+y)^n}{\underline{|n}}=\Sigma\frac{x^a}{\underline{|a}}\,.\,\frac{y^b}{\underline{|b}}$, where a, b take all values consistent with the equation $a+b=n$. Now if $F(x)=ax^n+bx^{n-1}+\ldots+kx+l$, by applying this theorem to each term we may shew that

$$F(x+y)=F(x)+yF'(x)+\frac{y^2}{\underline{|2}}\,.\,F''(x)+\ldots$$ Taylor.

where $F'(x)$, $F''(x)$... are rational integral functions of x; we shall return to consider the method of deriving these from $F(x)$.

16. *Transformation* z = F (x). *Similarity of smallest parts.* Let the complex number x be as before the ratio of the vector Ox to the vector OI in a certain plane, and let z be the ratio of the vector $O'z$ to the vector $O'I'$ in *another* plane. Then by the equation $z=F(a)$ (a rational integral function) a is determined as soon as x is known. Let x receive a small change and become $x'=x+y$, and let z consequently become $z'=z+v$. Then by (15) $u=y\,.\,F'(x)+$terms containing y^2. Suppose y so small that it may be neglected in comparison with 1, and therefore y^2 may be neglected in comparison with y. In that case u is got from y by increasing it in the ratio mod. $F'(x):1$ and turning it through the angle arg. $F'(x)$.

Gauss. Thus the direction of zz' makes a constant angle with the direction of xx', and if we take three points $xx'x''$ very near to each other, the triangle formed by their corresponding points $zz'z''$ will be similar to $xx'x''$. So if any picture be drawn in the first plane, while x describes the lines of this picture z will describe the lines of a distorted copy; but the two pictures will be *similar in their smallest parts*, and any two lines in the one will cut at the same angle as the corresponding lines in the other.

Argand. 17. *The Equation* F (x)=0 *has* n *roots*. By properly choosing x I say that we can make z come to the origin. For if not, there is some position of z which is the *nearest* to the origin that it can possibly have. Consider this position, and the corresponding position of x; and let λ be the corresponding value of $F'(x)$. If x move in any direction, z moves in a direction making an angle arg. λ with this; if therefore x move in a direction making an angle arg. λ with the line joining z to the origin, z will move straight towards the origin, contrary to the supposition that it could not get any nearer. Hence there is some value of x for which $F(x)=0$. Let a be this value, then we know that $F(x)=(x-a)F_1(x)$. But now as before $F_1(x)$ must have a root, say β; then $F(x)=(x-a)(x-\beta)F_2(x)$, and so on. Thus finally $F(x)$ has as many roots as dimensions.

Cauchy. 18. *Number of Roots in a given Closed Contour*. The argument of $F(x)$ is then the sum of the arguments of $\overline{x-a}, \overline{x-\beta}, \ldots$. Now if x describe any closed contour containing no roots, these arguments may increase or decrease but will ultimately resume their original value. But if a is within the contour the argument of $x-a$ will increase by 2π. Hence the number of roots of $F(x)$ within the contour is the number of (2π)s added to its argument, when

[*a page is missing.*]

22. *Position on a Plane. Cartesian formulæ*. In the method of Descartes the position of a point is defined by its distances from two fixed straight lines X and Y called the *axes*; the distance from each axis is measured parallel to the other, as pn, pm in Fig. [111]. The signed numbers $\frac{np}{OI}$, $\frac{mp}{OI}$, are denoted by x and y, and called the *coordinates* of the point. The position of a straight line is defined by its distance p from the origin, measured perpendicular to the line, and by the angles α, β which this perpendicular makes with Y and X respectively. These are connected by the equation $\alpha+\beta=\omega$ (the angle between X and Y) which may also be written

$$\cos^2\alpha+\cos^2\beta-2\cos\alpha\cos\beta\cos\omega=\sin^2\omega.$$

23. *Expressions for the simplest geometric magnitudes.*

(1) Distance of points $(x_1\, y_1)\ (x_2\, y_2)$

$$\delta^2=(x_1-x_2)^2+(y_1-y_2)^2+2(x_1-x_2)(y_1-y_2)\cos\omega.$$

(2) Area of triangle $(x_1\, y_1)\ (x_2\, y_2)\ (x_3\, y_3)$

$$\frac{2\Delta}{\sin\omega}=x_2y_3-x_3y_2+x_3y_1-x_1y_3+x_1y_2-x_2y_1=\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

(3) Perpendicular distance of point (x, y) from line (p, α, β)

$$=+p-x\cos\alpha-y\cos\beta.$$

(4) Angle between the lines $(p\,\alpha\,\beta)\ (p'\alpha'\beta')$; $\theta=\alpha-\alpha'=\beta'-\beta$, or

$$\cos\theta\sin^2\omega=\cos\alpha\cos\alpha'+\cos\beta\cos\beta'-(\cos\alpha\cos\beta'+\cos\alpha'\cos\beta)\cos\omega$$

$$\sin\theta\sin^2\omega=\cos\alpha\cos\beta'-\cos\alpha'\cos\beta.$$

24. *Equations*. If we know that $x\cos\alpha+y\cos\beta-p=0$, we know that the point (x, y) is on the line $(p\alpha\beta)$, thus all the values of x, y which satisfy this equation represent all the points on that line. The equation itself may then be said to represent all the points on the line; or, (less accurately) to represent the line. In general, the condition that (x, y) may be on a known curve is called the *equation of the curve*.

$$\left(\text{In the following } \omega \text{ is taken } =\frac{\pi}{2}\right).$$

If a point is on a

Circle, its distance from a fixed point (the centre) is constant,

say distance of (x, y) from (a, b) is equal to r,

$$\therefore (x-a)^2+(y-b)^2=r^2.$$

Parabola, its distance from a fixed point (focus) = dist. from fixed line (directrix),

say distance of (x, y) from (a, o) = dist. from $\left(a, \pi, -\frac{\pi}{2}\right)$,

$$\therefore (x-a)^2+y^2=(a+x)^2, \qquad \text{or } y^2=4ax.$$

Ellipse / *Hyperbola* } its distance from a fixed point $=e$ times dist. from fixed line, say dist. of (x, y) from $(ae, o)=e$ times dist. from $\left(\frac{a}{e}, o, \frac{\pi}{2}\right)$.

$$\therefore (x-ae)^2+y^2=e^2\left(\frac{a}{e}-x\right)^2, \text{ or } \frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1.$$

For the ellipse e is less than 1, and we write $a^2(1-e^2)=b^2$,

" hyperbola e is greater than 1, " " $a^2(e^2-1)=b^2$.

25. *Reduction of Equation of the First Order*. The equation $lx+my+n=0$ will be reduced to the form $x\cos\alpha+y\cos\beta-p=0$ if we multiply it by a quantity R, provided that $R^2(l^2+m^2-2lm\cos\omega)=\sin^2\omega$. From the value of R thus indicated we derive the formula,

$$\cos\alpha=\frac{l\sin\omega}{\sqrt{(l^2+m^2-2lm\cos\omega)}},\ \cos\beta=\frac{m\sin\omega}{\sqrt{(l^2+m^2-2lm\cos\omega)}},\ p=\frac{-n\sin\omega}{\sqrt{(l^2+m^2-2lm\cos\omega)}}$$

whence by substitution

perpendicular from $x'y'$ on $lx+my+n=0$ is $\frac{lx'+my'+n'}{\sqrt{(l^2+m^2-2lm\cos\omega)}}\sin\omega$;

angle between lines $lx+my+n=0,\ l'x+m'y+n'=0$,

$$\cos\theta=\frac{ll'+mm'-(lm'+l'm)\cos\omega}{\sqrt{(l^2+m^2-2lm\cos\omega)}\sqrt{(l'^2+m'^2-2l'm'\cos\omega)}}$$

$$\sin\theta=\frac{(lm'-l'm)\sin\omega}{\sqrt{(l^2+m^2-2lm\cos\omega)}\sqrt{(l'^2+m'^2-2l'm'\cos\omega)}}.$$

26. *Grassmann Notation for points and lines.* A point is denoted by a single small letter, and a line by a single large letter. The symbol aB or Ba is taken to mean the result of substituting the coordinates of the point a in the equation of the line B. The symbol ab stands for the line joining the points a, b, and AB for the point of intersection of the lines A, B. Accordingly abc stands for the determinant formed with the coordinates of the points a, b, c, and ABC for the determinant formed with the coefficients of the lines A, B, C. $abc=0$ means that the three points are in a line, and $ABC=0$ means that the three lines meet in a point. Let a mean the point $(x_1 y_1)$, b the point $(x_2 y_2)$ and c the point $(x_3 y_3)$; also let A be the line $lx+my+n=0$, B the line $l'x+m'y+n'=0$, and C the line $l''x+m''y+n''=0$. Then $aB=Ba=l'x_1+m'y_1+n'$.

$$abc=\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \quad ABC=\begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix} \quad \text{Equation of } ab \text{ is } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$$

Coordinates of AB are given by

$$\frac{x}{mn'-m'n}=\frac{y}{nl'-n'l}=\frac{z}{lm'-l'm}.$$

[I have not met with any further *Notes*.]

ANALYSIS OF LOBATSCHEWSKY.

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*Propositions common to the two theories**.

1. A straight line fits itself in all its positions: i.e. if we turn the surface containing it about two points of the line, the line does not move.

2. Two straight lines cannot cut in two points.

3. A straight line may be produced indefinitely.

4. Two lines perpendicular to the same line and in the same plane cannot meet each other.

5. One straight line must cut another if it has points on both sides of it.

6. Angles (plane or dihedral) vertically opposite are equal.

7. Straight lines making equal angles with a third straight line cannot meet.

8. In a triangle equal angles are opposed by equal sides, and conversely.

9. Greater angle opposed to greater side. In right-angled triangle, hypothenuse is greater than either of the other sides, and the two angles next it are acute.

10. Equalities of triangles.

11. A line perpendicular to two other lines not in the same plane with it is perpendicular to every other line in their plane.

12. The intersection of a sphere and a plane is a circle.

13. A line perpendicular to the intersection of two perpendicular planes, lying in one of them, is perpendicular to the other.

14. In a spherical triangle equal sides are opposed by equal angles, and conversely.

15. Two spherical triangles are equal when they have equal sides containing equal angles, or equal angles adjacent to equal sides.

* [*i.e.* to Euclid's and Lobatschewsky's.]

THE POLAR THEORY OF CUBICS.

1. Consider a net of conics B_2 (viz., $B_2^{(1)}\ B_2^{(2)}\ldots$). All the conics b_2 which are harmonic of all these form a tangential net. The locus of points x whose polars in respect of the entire net B_2 meet in a point is a curve of the third degree H_3. So also the envelope of lines X whose poles in respect of the entire net b_2 are in a line is a curve of the third class h_3. These curves are defined by the equations

$$(xB_2\,.\,xB_2'\,.\,xB_2'')=x^3H_3,$$
$$(Xb_2\,.\,Xb_2'\,.\,Xb_2'')=X^3h_3.$$

Let y be the point of intersection of the polars of x, and Y the line on which the poles of X lie. Then y is clearly a point on H_3, and Y is a tangent to h_3. x, y are called corresponding points of H_3, and X, Y are corresponding tangents of h_3.

2. The points x, y and the lines X, Y satisfy the equations

$$xyB_2=0, \qquad XYb_2=0,$$

where B_2, b_2 are any conics of their respective nets. These shew that the point-pair xy belongs to the net b_2 and that the line-pair XY belongs to the net B_2; for each net includes all conics that are harmonic of all conics of the other. Again, since the whole net B_2 is harmonic of the points xy, and since *one* conic of the net can be drawn through any two points, there must be one conic containing the line xy as half of it; and this line must consequently be a tangent to h_3. So likewise the point XY must be on H_3. We have then these theorems:—

If a conic b_2 *breaks up into two points* x, y, *these are points on* H_3, *and the line* xy *is a tangent to* h_3.

If a conic B_2 *breaks up into two lines* XY, *these are tangents to* h_3 *and the point* XY *is on* H_3.

3. Let a be a point subtended in involution by $b_2\ b_2'\ b_2''$. Then $\lambda\mu\nu$ can be so chosen that

$$a\,(\lambda b_2+\mu b_2'+\nu b_2'')\equiv 0,$$

which means that a is one of the two points into which $\lambda b_2+\mu b_2'+\nu b_2''$ breaks up; or that a is a point on H_3. Thus *all points on* H_3 *are subtended in involution by the net* b_2.

In a similar way we should prove that *all tangents to* h_3 *are cut in involution by the net* B_2. Or *we may define* H_3 *as the locus of points subtended in involution by the* b_2, *and* h_3 *as the envelope of lines cut in involution by the* B_2.

4. Let x be the intersection of XY, and y its corresponding point. Let also $x'y'$ be a pair of corresponding points very near to xy. Then $x'y'$ is a point-pair belonging to the net b_2, and XY is a line-pair belonging to the net B_2. But every conic of the net b_2 is harmonic of every conic of the net B_2. Therefore xx', X, xy', Y is a harmonic pencil. Now xx' is the tangent to H_3 at x, and xy' is the same as xy. Hence

The three tangents to h_3 *from a point* x *on* H_3, *together with the tangent to* H_3 *at* x, *form a harmonic pencil.*

The three points in which H_3 *cuts a tangent* X *to* h_3, *together with the point of contact of* X, *form a harmonic range.*

5. There are three cubics $C_3\ C_3'\ C_3''$ which have H_3 for their hessian, and three curves of the 3rd class $c_3\ c_3'\ c_3''$ which have h_3 for their hessian. These correspond in pairs, so that we have

$$C_3c_3=0,\ C_3'c_3'=0,\ C_3''c_3''=0.$$

I attend only to the pair $C_3\ c_3$.

We have now also

$$C_3h_3=24S,\ c_3H_3=24s,\ H_3h_3=T=t,$$

where S, T are the fundamental invariants of C_3, and s, t of c_3.

The conics B_2 are the first polars of C_3, and the b_2 are the first polars of c_3. In fact, x being any point and X any line,

$$xC_3=B_2,\ B_2c_3\equiv 0,\ \ B_2h_3=x,$$
$$Xc_3=b_2,\ \ b_2C_3\equiv 0,\ \ b_2H_3=X.$$

Suppose B_2 and b_2 to break up; then we have the theorems:

The mixed polar in regard to h_3 *of two of its corresponding tangents is a point on* H_3.

The mixed polar in regard to H_3 *of two of its corresponding points is a tangent to* h_3. (Cayley.)

We may here take the mixed polars in either case with regard to any syzygetic cubic.

The equations $B_2h_3=x$, &c., are virtually given by Salmon. *Conics*, 5th ed., p. 349.

6. The condition that xC_3 shall touch X is of the second order in x, X and the coefficients of C_3. Denote it by $(\overline{xC_3}^2,\ X^2)$, then if $X=yz$, we have

$$(\overline{xC_3}^2,\ X^2)=xy^2C_3\,.\,xz^2C_3-\overline{xyzC_3}|^2.$$

Regard X as fixed; then x describes a conic, the (second) polar envelope of X, which

(*a*) is the locus of points whose first polars touch X,

Cayley. (*b*) is the envelope of second polars of points on X,

Steiner. Cremona. (*c*) is the locus of poles of X in regard to first polars of its points,

Cayley. (*d*) breaks up when X touches h_3.

Salmon, l. c. Let $\overline{xC_3}^2$ denote the tangential form of the first polar of x; then $\overline{xC_3}^2 H_3 = x^2 C_3 . S$; or the first polar in regard to the cubic of a point on the cubic is harmonic of the first polar of that point in regard to the hessian.

The condition that X shall pass through an intersection of xC_3, x^2C_3 (that is, through a double conjugate of x) is, of course,

$$xC_3(x^2C_3)^2X^2 = 0.$$

X being fixed, this is a quintic curve whose equation may also be written

$$E_2C_3 + \lambda L^2 H_3 = 0,$$

where E_2 is the second polar envelope of X as above. This equation indicates the properties of the curve.

7. Operating with the mixed concomitant of (6') [?] on h_3 we obtain a new contravariant q_5 of the fifth order in the coefficients of C_3, which is thus seen to be the envelope of a line whose polar envelope is harmonic of its first polar in respect of h_3.

8. Two point-cubics 1 2 3 and 1′2′3′ are said to be harmonic of one another when

$$11' . 22' . 33' + 12' . 23' . 31' + 13' . 21' . 32' + 13' . 22' . 31'$$
$$+ 12' . 21' . 33' + 11' . 23' . 32' = 0,$$

or when

$$\Sigma 11' . 22' . 33' = 0.$$

If 1 2 3 is harmonic of 1′2′3′ and 1 coincides with 1′, it must have the same harmonic in regard to 2 3, 2′3′.

If three of the six points coincide, a fourth coincides with them.

Every point-cubic is harmonic of itself.

The envelope of a line cut harmonically by two cubic curves is of the third class. This envelope is the same for any two cubics through the nine points of intersection.

Two syzygetic cubics cut every line harmonically. To a *net* of cubics will thus correspond a net of curves of the third class, one for every pencil belonging to the net.

The condition that X may be cut harmonically by C_3 and $xC_3 . x^2C_3$ is the vanishing of the mixed concomitant $a_x c_x^2 (abu)^2 (bcu)$.

If ϵ_3 is the harmonic envelope of the pencil $\lambda C_3 + \mu D_3$, the tangents to ϵ_3 from a point of intersection are the three flexes at that point.

ON PFAFFIANS.

I.

Consider $2n$ alternate units $\alpha_1 \alpha_2 \ldots \alpha_{2n}$, and form a linear function of their binary products, viz.

$$\Sigma p_{hk} \alpha_h \alpha_k \qquad [h, k = 1, 2 \ldots 2n.$$

Since to every term in this $p_{hk}\alpha_h\alpha_k$ there corresponds a term $p_{kh}\alpha_k\alpha_h$, the complete coefficient of $\alpha_h\alpha_k$ is $p_{hk} - p_{kh}$, and the whole expression involves only these differences; hence it is convenient to make $p_{kh} = -p_{hk}$, and then

$$\Sigma p_{hk}\alpha_h\alpha_k = 2\Sigma' p_{hk}\alpha_h\alpha_k,$$

where the Σ' signifies that the summation is to be extended to those values only for which $k > h$.

The n^{th} power of $\Sigma' p_{hk}\alpha_h\alpha_k$ is $\Pi\alpha = \alpha_1\alpha_2 \ldots \alpha_n$ multiplied by a numerical constant and a function of the p which is called a *Pfaffian of the* n[th] *order;* viz. it is

$$\frac{1}{N} \Sigma \pm p_{12} p_{34} \cdots p_{2n-1, 2n},$$

where the suffixes are to be permuted in all possible ways, and signs prefixed according to the rule for determinants. Each term occurs $N = \Pi n$ times and accordingly the result is divided by this number. The equation may then be written

$$(\Sigma' p_{hk}\alpha_h\alpha_k)^n = \Pi_n . P_n . (\alpha)^n.$$

The Pfaffian of the first order is p_{12}, or generally any constant. For the second order we have

$$(p_{12}\alpha_1\alpha_2 + p_{23}\alpha_2\alpha_3 + p_{13}\alpha_1\alpha_3 + p_{14}\alpha_1\alpha_4 + p_{24}\alpha_2\alpha_4 + p_{34}\alpha_3\alpha_4)^2$$
$$= 2\,(p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23})\,\alpha_1\alpha_2\alpha_3\alpha_4,$$

so that

$$P_2 = p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23}.$$

It is sometimes convenient to denote this by P_{1234}, and so for higher orders.

Observe that these P are the concomitants of a linear complex in n dimensions.

This being so, we have for any n units $a_1 a_2 \ldots a_n$

$$(\Sigma' p_{hk} a_h a_k)^2 = 2 \Sigma P_{hklm} a_h a_k a_l a_m,$$

$$(\Sigma' p_{hk} a_h a_k)^3 = 6 \Sigma P_{hklmnr} a_h a_k a_l a_m a_n a_r,$$

and generally

$$(\Sigma' p_{hk} a_h a_k)^m = \Pi_m \, . \, \Sigma P_m (a)^m. \qquad [2m < n.$$

We may also form products such as $(\Sigma' p_{hk} a_h a_k)(\Sigma' q_{hk} a_h a_k) \ldots$; these give rise to what may be called mixed Pfaffians, the relation of which to the preceding may be easily recognized.

II.

Let there be two sets of n variables $a_1 a_2 \ldots a_n$ and $\beta_1 \beta_2 \ldots \beta_n$. If we form the product of the n linear functions,

$$\varpi_1 = \Sigma p_{h1} a_h, \ \varpi_2 = \Sigma p_{h2} a_h, \ \ldots \varpi_n = \Sigma p_{hn} a_h,$$

it is known that $\Pi \varpi = |p| \, . \, \Pi a$. Hence if we now form the lineo-linear function $\Sigma \varpi_h \beta_h = \Sigma p_{hk} a_h \beta_k$, we shall find

$$(\Sigma p_{hk} a_h \beta_k)^n = (\Sigma \varpi_h \beta_h)^n = (-)^{\frac{1}{2}n(n-1)} \Pi n \, . \, |p| \, . \, \Pi a \, . \, \Pi \beta,$$

that is, the nth power of a lineo-linear function of two sets of n units is the product of the units into a numerical multiple of the determinant formed by the coefficients.

Now suppose that $p_{hk} = -p_{kh}$, $p_{hh} = 0$; then

$$\Sigma p_{hk} a_h \beta_k = \Sigma' p_{hk} (a_h \beta_k - a_k \beta_h) \qquad [h < k \text{ under } \Sigma';$$

or we have a linear function of the binary determinants formed with the a and the β. Under these circumstances the determinant $|p|$ is skew symmetrical; or *the nth power of a linear function of the binary determinants of the a, β is* $\Pi n \, . \, \Pi a \, . \, \Pi \beta \times$ *by a skew symmetrical determinant.*

These binary determinants are symmetrical in regard to the a and the β; for $a_h \beta_k - a_k \beta_h = a_h \beta_k + \beta_h a_k = (h, k)$ suppose. We may easily prove the following theorems, viz.,

$$(h, k)^2 = -2 a_h a_k \beta_h \beta_k,$$

$$(h, k)(h, l) = +a_h \beta_h (a_k \beta_l + a_l \beta_k).$$

Hence with $n = 3$ we have

$$-\tfrac{1}{2}\{p_{12}(1, 2) + p_{13}(1, 3) + p_{23}(2, 3)\}^2 = (p_{12} a_1 a_2 + p_{13} a_1 a_3 + p_{23} a_2 a_3) \times (p_{12} \beta_1 \beta_2 + p_{13} \beta_1 \beta_3 + p_{23} \beta_2 \beta_3).$$

But with $n \equiv\!> 4$,

$$-\tfrac{1}{2}\{\Sigma' p_{hk}(h, k)\}^2 = (\Sigma' p_{hk} a_h a_k) \, . \, (\Sigma' p_{hk} \beta_h \beta_k) - 2 \Sigma P_{hklm} (a_h a_k \beta_l \beta_m + \ldots 6 \text{ terms}).$$

Here the coefficient of $2P_{hklm}$ is worthy of attention. Let us write

$$\left| \begin{matrix} a & a & \beta & \beta \\ h & k & l & m \end{matrix} \right| = (h, k, l, m)$$

$$= a_h a_k \beta_l \beta_m + a_l a_m \beta_h \beta_k + a_h a_m \beta_k \beta_l + a_k a_l \beta_h \beta_m + a_h a_l \beta_m \beta_k + a_m a_k \beta_h \beta_l,$$

then

$$(h, k, l, m)^2 = 6 a_h a_k a_l a_m \beta_h \beta_k \beta_l \beta_m$$

$$(h, k, l, m)(h, k, l, n) = a_h a_k a_l \beta_h \beta_k \beta_l (a_m \beta_n + a_n \beta_m) \qquad \{\text{say } [m, n]$$

$$(h, k, l, m)(h, k, n, r) = a_h a_k \beta_h \beta_k (lmnr)$$

$$(h, k, l, m)(h, n, r, s) = a_h \beta_h [klm, nrs],$$

where $[klm, nrs] = \Sigma (a_n a_l a_m \beta_k \beta_r \beta_s + a_k a_r a_s \beta_n \beta_l \beta_m)$, the klm being permuted among themselves and the nrs among themselves, according to the rule of signs.

In the same way we find

$$\tfrac{1}{8}\left(\Sigma' P_{hk} \, . \left| \begin{matrix} a & \beta \\ h & k \end{matrix} \right.\right)^4 = (\Sigma' p_{hk} a_h a_k)^2 \, . \, (\Sigma' p_{h'k'} \beta_{h'} \beta_{k'})^2 - 2 (\Sigma' p_{hk} a_h a_k)(\Sigma' p_{h'k'} \beta_{h'} \beta_{k'})$$
$$\times \Sigma' P_{lml'm'} \, . \left| \begin{matrix} a & a & \beta & \beta \\ l & m & l' & m' \end{matrix} \right. + 4 \Sigma' P_{hklmh'k'l'm'} \, . \left| \begin{matrix} a & a & a & a & \beta & \beta & \beta & \beta \\ h & k & l & m & h' & k' & l' & m' \end{matrix} \right.,$$

$$\tfrac{1}{24}\left(\Sigma' p_{hk} \, . \left| \begin{matrix} a & \beta \\ h & k \end{matrix} \right.\right)^6 = (\Sigma' p_{hk} a_h a_k)^3 \, . \, (\Sigma' p_{h'k'} \beta_{h'} \beta_{k'})^3 - 3 (\Sigma' p_{hk} a_h a_k)^2 (\Sigma' p_{h'k'} \beta_{h'} \beta_{k'})^2$$
$$\times \Sigma' P_{lml'm'} \, . \left| \begin{matrix} a & a & \beta & \beta \\ l & m & l' & m \end{matrix} \right. + 9 (\Sigma' p_{hk} a_h a_k)(\Sigma' p_{h'k'} \beta_{h'} \beta_{k'}) \, . \, \Sigma' P_8 (a^4 \beta^4) - 27 \Sigma' P_{12} (a^6 \beta^6).$$

To write the general formula more conveniently let us abbreviate thus:

$$p_{a^2} = \Sigma' p_{hk} a_h a_k, \quad p_{a\beta} = \Sigma' p_{hk} \, . \left| \begin{matrix} a & \beta \\ h & k \end{matrix} \right|; \text{ then}$$

$$(p_{a\beta})^{2n} = A_1 p_{a^2}{}^n p_{\beta^2}{}^n - A_2 p_a{}^{n-1} \, . \, p_\beta{}^{n-1} \, . \, \Sigma P_4 (a^2 \beta^2) + A_3 p_a{}^{n-2} p_\beta{}^{n-2} \Sigma P_8 (a^4 \beta^4) - \text{etc.}$$

where the A are numerical multipliers.

ANALYSIS OF CREMONA'S TRANSFORMATIONS.

Quadric transformation [3].

		1	1
1	1		1
1	1	1	

$\mu(\alpha\beta\gamma)$ gives $\overline{2\mu-\Sigma\alpha}\,(\mu-\beta-\gamma, \ldots)$.

Jacobian is three lines bc, ca, ab.

[Cr. 3.] Cubic transformation [41]=[3](001).

	a_2	a	b	c	d
2	1	1	1	1	1
1	1	1			
1	1		1		
1	1			1	
1	1				1

$\mu(\alpha_2\alpha\beta\gamma\delta)$ gives $\overline{3\mu-2\alpha_2-\Sigma\alpha}\,\begin{pmatrix}2\mu-\alpha_2-\Sigma\alpha, \\ \mu-\alpha_2-\alpha, \ldots\end{pmatrix}$

Jacobian is conic a_2abcd and lines a_2a, a_2b, a_2c, a_2d.

[41](2, 0111)=[3].

[Cr. 4.] 1. Quartic transformation [601]=[3](001)(002).

	a_3	a	b	c	d	e	f
3	2	1	1	1	1	1	1
1	1	1					
1	1		1				
1	1			1			
1	1				1		
1	1					1	
1	1						1

$\mu(\alpha_3\alpha\beta\gamma\delta\epsilon\zeta)$ gives $\overline{4\mu-3\alpha_3-\Sigma\alpha}\,\begin{pmatrix}3\mu-2\alpha_3-\Sigma\alpha, \\ \mu-\alpha_3-\alpha\ldots\end{pmatrix}$

Jacobian is cubic $a_3^2abcdef$ and lines a_3a, a_3b, ...

[601](3, 011111)=[3].

[Cr. 4.] 2. Quartic transformation [330]=[3](000).

	a_2	b_2	c_2	a	b	c
2	1	1	1		1	1
2	1	1	1	1		1
2	1	1	1	1	1	
1		1	1			
1	1		1			
1	1	1				

$\mu(\alpha_2\beta_2\gamma_2\alpha\beta\gamma)$ gives $\overline{4\mu-2\Sigma\alpha_2-\Sigma\alpha}\,\begin{pmatrix}2\mu-\Sigma\alpha_2+\alpha-\Sigma\alpha\ldots \\ \mu-\Sigma\alpha+\alpha\ldots\end{pmatrix}$

Jacobian is conics $a_2b_2c_2bc$, ... and lines b_2c_2, c_2a_2, a_2b_2.

[330](222, 011)=[3].

Quintic Transformations.

[Cr.] 5. 1. [8001]=[3](001)(002)(003).

	a_4	a	b	c	d	e	f	g	h
4	3	1	1	1	1	1	1	1	1
1	1	1							
1	1		1						
1	1			1					
1	1				1				
1	1					1			
1	1						1		
1	1							1	
1	1								1

$\mu(\alpha_4\alpha\beta\gamma\delta\epsilon\zeta\eta\theta)$ gives $\overline{5\mu-4\alpha_4-\Sigma\alpha}\,\begin{pmatrix}4\mu-3\alpha_4-\Sigma\alpha \\ \mu-\alpha_4-\alpha, \ldots\end{pmatrix}$

Jacobian is quartic $a_4^3abcdefgh$ and the lines a_4a, ...

In De Jonquières' transformation of the $\overline{n+1}|^{th}$ order $[2n, \ldots 1]\,\mu(\alpha_n, \alpha\beta\gamma\ldots)$ gives $\overline{(n+1)\mu-n\alpha_n-\Sigma\alpha}\,(n\mu-\overline{n-1}\alpha_n-\Sigma\alpha, \mu-\alpha_n-\alpha, \ldots)$, and the Jacobian is $\overline{n}|^{ic}\,\alpha_n^{n-1}\alpha\beta\gamma\ldots$ and the $2n$ lines $\alpha_n\alpha$, $\alpha_n\beta$, ...

[8001](4, 01111111)=[3].

[Cr.] 5. 2. [3310]=[3](001)(001).

	a_3	a_2	b_2	c_2	a	b	c
3	2	1	1	1	1	1	1
2	1	1	1	1	1		
2	1	1	1	1		1	
2	1	1	1	1			1
1	1	1					
1	1		1				
1	1			1			

$\mu(\alpha_3\alpha_2\beta_2\gamma_2\alpha\beta\gamma)$ gives $\overline{5\mu-3\alpha_3-2\Sigma\alpha_2-\Sigma\alpha}\,\begin{pmatrix}3\mu-2\alpha_3-\Sigma\alpha_2-\Sigma\alpha \\ 2\mu-\alpha_3-\Sigma\alpha_2-\alpha, \ldots \\ \mu-\alpha_3-\alpha_2, \ldots\end{pmatrix}$

Jacobian is cubic $a_3^2a_2b_2c_2abc$

3 conics $a_3a_2b_2c_2a$, ...

3 lines a_3a, ...

[3310](3, 222, 011)=[3].

[Cr.] 5. 3. [0600]=[3](000)(111).

	a_2	b_2	c_2	d_2	e_2	f_2
2		1	1	1	1	1
2	1		1	1	1	1
2	1	1		1	1	1
2	1	1	1		1	1
2	1	1	1	1		1
2	1	1	1	1	1	

$\mu(\alpha_2\beta_2\gamma_2\delta_2\epsilon_2\zeta_2)$ gives $\overline{5\mu-2\Sigma\alpha_2}\,(2\mu-\Sigma\alpha_2+\alpha_2, \ldots)$.

Jacobian is six conics $b_2c_2d_2e_2f_2$, &c.

[0600](022222)=[0600]

[0600](002222)=[04040000]=Cr. 9. 4.

Sextic Transformations.

[Cr.] 6. 1. [10, 0001]=[3](001)(002)(003)(004). See 5. 1.

[Cr.] 6. 2. [14200]=[3](000)(011).

	a_3	b_3	a_2	b_2	c_2	d_2	a
3	1	2	1	1	1	1	1
3	2	1	1	1	1	1	1
2	1	1		1	1	1	
2	1	1	1		1	1	
2	1	1	1	1		1	
2	1	1	1	1	1		
1	1	1					

$\mu(\alpha_3\beta_3\alpha_2\beta_2\gamma_2\delta_2\alpha)$ gives $\overline{6\mu-3\Sigma\alpha_3-2\Sigma\alpha_2-\alpha}$
$\begin{pmatrix}3\mu-\alpha_3-2\beta_3-\Sigma\alpha_2-\alpha, \ldots \\ 2\mu-\alpha_3-\beta_3-\Sigma\alpha_2+\alpha_2, \ldots \\ \mu-\alpha_3-\beta_3\end{pmatrix}$

Jacobian is 2 cubics
$a_3^2b_3a_2b_2c_2d_2a$; $a_3b_3^2a_2b_2c_2d_2a$

4 conics $a_3b_3b_2c_2d_2$, etc.

1 line a_3b_3.

[14200](33, 2222, 0)=[3].

[Cr.] 6. 3, 4. [41300] = [3](001)(000); [34010] = [3](000)(002).

	a_3	b_3	c_3	a_2	a	b	c	d
4	2	2	2	1	1	1	1	1
2	1	1	1	1	1			
2	1	1	1	1		1		
2	1	1	1	1			1	
2	1	1	1	1				1
1		1	1					
1	1		1					
1	1	1						

$\mu\,(a_3\beta_3\gamma_3a_2\alpha\beta\gamma\delta)$ gives $\overline{6\mu-3\Sigma a_3-2a_2-\Sigma a}$
$\left(\begin{array}{l}4\mu-2\Sigma a_3-a_2-\Sigma a,\\ 2\mu-\Sigma a_3-a_2-a,\ \ldots\\ \mu-\beta_3-\gamma_3,\ldots\end{array}\right)$

$\mu\,(a_4a_2\beta_2\gamma_2\delta_2a\beta\gamma)$ gives $\overline{6\mu-4a_4-2\Sigma a_2-\Sigma a}$
$\left(\begin{array}{l}3\mu-2a_4-\Sigma a_2-\beta-\gamma,\ \ldots\\ 2\mu-a_4-\Sigma a_2,\\ \mu-a_4-a_2\end{array}\right)$

Jacobian of 6. 3 is the quartic $a_3{}^2b_3{}^2c_3{}^2a_2abcd$, 4 conics $a_3b_3c_3a_2a$, etc. and three lines b_3c_3, c_3a_3, a_3b_3.

Jacobian of 6. 4 is 3 cubics $a_4{}^2a_2b_2c_2d_2bc$, etc., conic $a_4a_2b_2c_2d_2$, and 4 lines a_4a, a_4b, ...

[41300].

Septic Transformations.

[Cr.] 7. 1. [12,00001] = [3](001)(002)(003)(004)(005). See 5. 1.

[Cr.] 7. 2. [232100] = [3](000)(001).

	a_4	a_3	b_3	a_2	b_2	c_2	a	b
4	2	2	2	1	1	1	1	1
3	2	1	1	1	1	1	1	
3	2	1	1	1	1	1		1
2	1	1	1		1	1		
2	1	1	1	1		1		
2	1	1	1	1	1			
1	1	1						
1	1		1					

[Cr.] 7. 3. [034000] = [3](001)(001)(111).

	a_3	b_3	c_3	d_3	a_2	b_2	c_2
3	2	1	1	1	1	1	1
3	1	2	1	1	1	1	1
3	1	1	2	1	1	1	1
3	1	1	1	2	1	1	1
2	1	1	1	1	1		
2	1	1	1	1		1	
2	1	1	1	1			1

[Cr.] 7. 4, 5. [503100] = [3](001)(002)(001); [350010] = [3](001)(001)(003).

	a_4	a_3	b_3	c_3	a	b	c	d	e
5	3	2	2	2	1	1	1	1	1
2	1	1	1	1	1				
2	1	1	1	1		1			
2	1	1	1	1			1		
2	1	1	1	1				1	
2	1	1	1	1					1
1	1	1							
1	1		1						
1	1			1					

Octavic Transformations.

[Cr.] 8. 2. [3230100] = [3](001)(001)(002).

	a_5	a_3	b_3	c_3	a_2	b_2	a	b	c
5	3	2	2	2	1	1	1	1	1
3	2	1	1	1	1	1	1		
3	2	1	1	1	1	1		1	
3	2	1	1	1	1	1			1
2	1	1	1	1		1			
2	1	1	1	1	1				
1	1	1							
1	1		1						
1	1			1					

[Cr.] 8. 3. [1322000] = [3](001)(001)(011).

	a_4	b_4	a_3	b_3	a_2	b_2	c_2	a
4	2	2	2		1	1	1	1
4	2	2		2	1	1	1	1
3	2		1	1	1	1	1	
3		2	1	1	1	1	1	
2	1	1	1	1	1			
2	1	1	1	1		1		
2	1	1	1	1			1	
1	1	1						

[Cr.] 8. 4. [0070000] = [3](001)(001)(111)(222) = [7. 3](222).

	a_3	b_3	c_3	d_3	e_3	f_3	g_3
3	2	1	1	1	1	1	1
3	1	2	1	1	1	1	1
3	1	1	2	1	1	1	1
3	1	1	1	2	1	1	1
3	1	1	1	1	2	1	1
3	1	1	1	1	1	2	1
3	1	1	1	1	1	1	2

[Cr.] 8. 5. [3600010] = [3](000)(002)(004) = 6. 4(004).

[Cr.] 6. [6013000] = [3](001)(002)(000) = 4. 1(000).

	a_6	a_2	b_2	c_2	d_2	e_2	f_2	a	b	c
4	3	1	1	1	1	1	1		1	1
4	3	1	1	1	1	1	1	1		1
4	3	1	1	1	1	1	1	1	1	
3	2	1	1	1	1	1	1			
1	1	1								
1	1		1							
1	1			1						
1	1				1					
1	1					1				
1	1						1			

[Cr.] 8.7. [0520100]=[3](000)(111)(002)=5.4)(002).

8. [2051000]=[3](001)(000)(112)=6.3)(112).

	a_3	a_3	b_3	a_2	b_2	c_2	d_2	e_2
4	2	2	2	1	1	1	1	1
3	2	1	1		1	1	1	1
3	2	1	1	1		1	1	1
3	2	1	1	1	1		1	1
3	2	1	1	1	1	1		1
3	2	1	1	1	1	1	1	
1	1	1						
1	1		1					

[I have printed the above *Analysis* as it is given in a Note-Book, adding only the "Cr.," herein copying Prof. Cayley, who, in his paper "*on the Rational transformation between two spaces*" (Proceedings of L. Math. Society, Vol. III., pp. 127—180), writes:—"Prof. Clifford calculated in this way the following table, shewing how any transformation of an order not exceeding 8 can be expressed by means of a series of quadric transformations; the symbols Cr. 3, &c., refer to the order and number of Cremona's tables." There is further reference to Prof. Clifford's work in connexion with this subject in § 68 and § 69. Prof. Cremona's Memoir, *Sulle trasformazioni Geometriche delle figure piane*, is in the Mem. di Bologna, t. ii., 1863 and t. v., 1865. I have compared the two lists, and have been able to verify the results given above when they are given also in Prof. Cayley's list. I add Cr. 8.1.

[14,000001]=[3](001)(002)(003)(004](005)(006).

and Cr. 8.9 (due to Cayley, see Proc. l. c. p. 143),

=[3303000]=[3](000)(000)(000).

I may refer also to papers by Mr S. Roberts, *On Prof. Cremona's Transformation between two planes and Tables relating thereto* (Proc. L. M. S., Vol. IV. pp. 121—139), and by Mr T. Cotterill, *On a correspondence of Points, such that a curve of the* nth *order in one plane corresponds to a curve of the* 4nth *in another plane, with three multiple points of the order* n *on the line of intersection of the planes, and three other multiple points of the order* 2n. To this last paper are appended some combined observations, due to Profs. Cayley and Clifford (Proc. L. M. S., Vol. II., p. 123.)]

BITANGENT CIRCLES OF A CONIC.

1. If two conics have double contact, any point on the chord of contact has the same polar in respect of them.

For the polar of O, a point on AB, must pass through C where the tangents at A and B intersect, and also through P the harmonic of O in respect of A, B. [Fig. 112.]

COR. If one of the conics is a circle, and the point O at infinity, CP is a diameter of both conics which bisects chords at right angles to itself; i.e. an axis. Hence

If a circle have double contact with a conic, its centre must lie on one of the axes and the chord of contact must be perpendicular to that axis.

2. Through three points to draw a conic having double contact with a given conic.

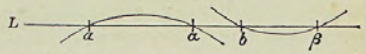
Let O, P, Q be the points: let OP meet the given conic in AB, OQ in CD. Let X, Y be the points which divide harmonically both AB and OP, X', Y' those which divide harmonically both OQ and CD. Then XY', XX', YY', YX' are the four positions of the chord of contact. For suppose X to be a point on the chord, then by the last proposition it must have the same polar in respect of both conics; and therefore the same conjugate in respect of AB and OP. [Fig. 113.]

We see thus that the conics, having double contact with the given one, which pass through OP divide themselves into two systems; viz., those whose chords of contact pass through X and those whose chords of contact pass through Y. In the particular case in which O, P are the points at infinity on a circle, X and Y become the points at infinity on the axis of the conic, and so

Circles having double contact with a given conic divide themselves into two systems, and according as the chord of contact is parallel to one axis or the other; and two circles of each system can be drawn through a given point.

3. The locus of a point whose powers in respect of two given conics are in a fixed ratio is a conic passing through their points of intersection.

Let L be an arbitrary line which cuts the two conics in $a\alpha$ and $b\beta$ respectively.



It is required to find a point x on L such that

$$xa \,.\, x\alpha = m \,.\, xb \,.\, x\beta.$$

Let two points p, q be so related that

$$\frac{pa}{pb} = m\frac{q\beta}{q\alpha},$$

then these two points uniquely determine each other, and therefore there are two positions at which they coincide, or two positions of the point x; that is, two points of the locus are on the arbitrary line L.

Therefore the locus is of the second order, and it passes through the intersections of the conics because two quantities which are in a fixed ratio must vanish simultaneously.

It follows conversely that if three conics pass through the same four points, the ratio of the powers of any point on one in respect of the other two is constant. For let U, V, W be the conics, and consider a point m on W. The powers of m in respect of U, V are in a certain ratio. Now the locus of a point whose powers in respect of U, V are in that ratio is a conic through the intersections of U, V and of course also through m. Thus it has five points in common with W and must therefore coincide with it.

In particular, let W be made up of the two chords L, M of U and V. Then the power of any point on U in respect of V is in a constant ratio to the product of its distances from the lines L, M. [Fig. 114.]

If we suppose L, M to coincide, then U and V will have double contact; and we learn that in this case the power of any point on U in respect of V is in a constant ratio to the square of its distance from L. [Fig. 115.]

Now the power of any point in respect of a circle is the squared tangent from the point to the circle: hence we see that

The tangent from any point of a conic to a bitangent circle is in a constant ratio to the perpendicular on the chord of contact.

4. If we suppose the point on the curve to go to infinity in the direction CP, the ratio becomes ultimately $\frac{CM}{CP}$, and is therefore the same for all circles of this system. For circles of the other system it is $\frac{PM}{CP}$. These two ratios may be called ϵ and ϵ'. [Fig. 116.]

5. The sum or difference of the tangents drawn from any point of a conic to two bitangent circles of the same system is constant.

Let A, B be the circles, L, L' their chords of contact. Then

$$PM = \epsilon \,.\, PA,$$

and
$$PN = \epsilon \,.\, PB;$$

$$\therefore\quad MN = \epsilon\,(PA \pm PB)$$

according as P is between L and L' or outside of them. [Fig. 117.]

(The radical axis of the circles is clearly midway between their chords of contact.

Since the four points of contact are symmetrically situated in respect of the axis, a circle will pass through them having its centre on the axis. This is called the circle of contact.)

6. The product of the tangents drawn from any point of a conic to two bitangent circles of the same system is equal to the square of the tangent drawn from the same point to their circle of contact.

For the conic, the circle of contact, and the pair of chords of contact, make three conics through the same four points: and therefore the power of any point on the conic in respect of the circle of contact bears a constant ratio to the product of its distances from the chords of contact, and therefore a constant ratio to the product of the tangents drawn to the two circles. By considering the points at infinity on the conic we see that this ratio must be one of equality.

OF POWER-COORDINATES IN GENERAL.

The equation of a circle contains *three* disposable constants; the equation of any circle may therefore be put into the form

$$aX+bY+cZ+dW=0 \quad \text{..............(1)},$$

where $X, Y, Z, W=0$ are the equations of four circles. And any equation of the first order in $XYZW$ represents a circle.

The main object of the following paper is to discuss the general equation of the *second* order in $XYZW$. Now if we write

$$\left.\begin{aligned} X &\equiv \overline{x-\alpha_1}^2+\overline{y-\beta_1}^2-r_1^2 \\ Y &\equiv \overline{x-\alpha_2}^2+\overline{y-\beta_2}^2-r_2^2 \\ Z &\equiv \overline{x-\alpha_3}^2+\overline{y-\beta_3}^2-r_3^2 \\ W &\equiv \overline{x-\alpha_4}^2+\overline{y-\beta_4}^2-r_4^2 \end{aligned}\right\} \quad \text{..............(2)},$$

then the equation (1) has an obvious geometrical meaning; viz. X means the squared tangent from the point (xy) to the first circle, or, as it is called, the *power* of that point in respect of the circle; and we have learned that if the squared tangents drawn from a point to four fixed circles satisfy an equation of the first degree, the locus of that point is a circle. We thus come to regard the quantities $XYZW$ as a sort of coordinates; and a set of values of the coordinates may represent a point.

But a point is determined by *two* coordinates; and therefore in order that a set of values of these *four* quantities may represent a point, they must satisfy two equations identically. I shall now prove that one of these equations is non-homogeneous and of the first degree, while the other is homogeneous and of the second degree.

In fact, if we write A for the determinant

$$\begin{vmatrix} 1 & \beta_2 & \gamma_2 \\ 1 & \beta_3 & \gamma_3 \\ 1 & \beta_4 & \gamma_4 \end{vmatrix},$$

and B, C, D for the three similar determinants that can be formed from the other triads of the four circles, we shall have

$$AX+BY+CZ+DW\equiv(1234) \quad \text{..............(3)},$$

where (1234) denotes the determinant

$$\begin{vmatrix} 1 & \alpha_1 & \beta_1 & \alpha_1^2+\beta_1^2-r_1^2 \\ 1 & \alpha_2 & \beta_2 & \alpha_2^2+\beta_2^2-r_2^2 \\ 1 & \alpha_3 & \beta_3 & \alpha_3^2+\beta_3^2-r_3^2 \\ 1 & \alpha_4 & \beta_4 & \alpha_4^2+\beta_4^2-r_4^2 \end{vmatrix}.$$

This is the non-homogeneous equation of the first degree.

If we choose to concern ourselves only with the ratios of the quantities $X\ Y\ Z\ W$, then the equation

$$AX+BY+CZ+DW=0$$

will represent the line (or, as we may say, circle) at infinity. I shall call the form $AX+BY+CZ+DW$ the *first absolute*, and denote it by the symbol ∞.

Again, we have in the equations (2) $X\ Y\ Z\ W$ exhibited as linear functions of x^2+y^2, x, y, 1. It is therefore possible, by solving the equations, to exhibit x^2+y^2, x, y, 1 as proportional to linear functions of X, Y, Z, W. That is to say, we must have

$$\frac{x^2+y^2}{P}=\frac{x}{Q}=\frac{y}{R}=\frac{1}{S},$$

where P, Q, R, S are four linear functions of X, Y, Z, W. But

$$(x^2+y^2)\,.\,1=(x)^2+(y)^2,$$

therefore

$$PS=Q^2+R^2,$$

a homogeneous equation of the second degree in X, Y, Z, W which must be identically satisfied if X, Y, Z, W are the coordinates of a point. The form $PS-Q^2-R^2$ I shall call the *Second Absolute*, and denote by the symbol ϕ.

The question now naturally arises; supposing that the coordinates $XYZW$ do not satisfy the equation $\phi=0$, and therefore do not represent a *point*, what do they represent? This question I proceed to answer.

Let δ be the distance between the centres of two circles, r and r_1 their radii. The quantity $\delta^2-r^2-r_1^2$ I call the *power of one circle in respect of the other.* If one of them becomes a point, so that $r_1^2=0$, the power is δ^2-r^2, which is the squared tangent from that point to the other circle; so that this definition agrees with the previous use of the word *power*. The power is also equal to $-2rr_1\cos\theta$, where θ is the angle of intersection of the circles; for, in the figure, ACB is this angle (namely the angle through which (A) must be turned about the point C, in order that its concavity may coincide with the concavity of (B)), and we have at once

$$\delta^2=r^2+r_1^2-2rr_1\cos\theta,$$

which establishes the equivalence in question. [Fig. 118.]

Let us now examine the effect of supposing X, Y, Z, W to denote the powers of a *circle* in respect of the four fixed circles.

First, the equation (3) is still satisfied; for the coordinates $XYZW$ of the *circle* are got from the coordinates $X_1\ Y_1\ Z_1\ W_1$ of its *centre* by subtracting r^2 from each of them; we have then only to prove that

$$(A+B+C+D)\,r^2\equiv 0,$$

which is obvious, for $A+B+C+D$ is the coefficient of (x^2+y^2) on the left-hand of the equation, and the right-hand is constant.

Next, what is meant by an equation of the first order? Let us write, a little more generally,

$$a_1X = a_1(x^2+y^2)+2b_1x+2c_1y+d_1,$$

and let the equation of our new circle be

$$0 = a(x^2+y^2)+2bx+2cy+d,$$

then we shall have

$$\delta^2 = \left(\frac{b}{a}-\frac{b_1}{a_1}\right)^2+\left(\frac{c}{a}-\frac{c_1}{a_1}\right)^2,$$

$$r^2 = \frac{b^2+c^2-ad}{a^2},$$

$$r_1^2 = \frac{b_1^2+c_1^2-a_1d_1}{a_1^2},$$

and therefore

$$\delta^2 - r^2 - r_1^2 = \frac{ad_1+a_1d-2bb_1-2cc_1}{aa_1} \quad\ldots\ldots\ldots\ldots (4).$$

Now if the power of two circles vanishes, they cut at right angles; for the power is $-2rr_1\cos\theta$, and $\cos\theta=0$ means that θ is a right angle. We see thus that the condition for two circles to cut at right angles is

$$ad_1+a_1d-2bb_1-2cc_1=0,$$

which is linear in the coefficients of each. And further, any linear relation among the coefficients of a circle expresses that it cuts some fixed circle at right angles. For the relation

$$la+mb+nc+sd=0$$

expresses that the circle

$$a(x^2+y^2)+2bx+2cy+d=0$$

cuts at right angles the circle

$$s(x^2+y^2)-mx-ny+l=0.$$

It follows that *if the coordinates of a circle satisfy an equation of the first degree, the circle cuts at right angles a certain fixed circle.* For the equation

$$lX+mY+nZ+sW=0$$

implies a linear relation among $a\ b\ c\ d$, since $X\ Y\ Z\ W$ are proportional to linear functions of these quantities by the equation (3). And this circle is precisely the one that we before represented by this equation, viz. when we considered therein X, Y, Z, W as functions of x, y determined by equations (2). For the circle is cut orthogonally by all circles whose coordinates satisfy the equation; but if these circles become points, they must be points on the circle which they cut orthogonally.

These results I sum up as follows:

The coordinates of a circle are four quantities proportional to its powers in respect of four fixed circles not having the same radical centre.

The coordinates of all circles which cut at right angles a given circle C satisfy a homogeneous equation of the first degree, which is called the equation of the circle C.

The coordinates of all *points* satisfy a homogeneous equation of the second order.

The Orthogonal System.

The following theorem includes about one-third of the metrical properties of points, lines, and circles.

If there are five circles, 1 2 3 4 5, and five other circles, 1′ 2′ 3′ 4′ 5′, and if we form the determinant whose constituents are the powers of the first set of circles in respect of the second set, arranged as in a multiplication table so as to be represented in the umbral notation by the symbol $\begin{matrix}1&2&3&4&5\\1'&2'&3'&4'&5'\end{matrix}$, this determinant vanishes identically.

For if we multiply together the two matrices

$$\begin{matrix}1, & 2b_1, & 2c_1, & d_1\\1, & 2b_2, & 2c_2, & d_2\\1, & 2b_3, & 2c_3, & d_3\\1, & 2b_4, & 2c_4, & d_4\\1, & 2b_5, & 2c_5, & d_5\end{matrix} \qquad \begin{matrix}d_1', & -b_1', & -c_1', & 1\\d_2', & -b_2', & -c_2', & 1\\d_3', & -b_3', & -c_3', & 1\\d_4', & -b_4', & -c_4', & 1\\d_5', & -b_5', & -c_5', & 1\end{matrix}$$

we shall form a determinant which, by the theory of [matrices], vanishes identically; while its constituents are of the type

$$d'+d-2bb'-2cc',$$

which, as we have already seen, means the power of the circles

$$x^2+y^2+2bx+2cy+d=0,$$
$$x^2+y^2+2b'x+2c'y+d'=0.$$

Reserving for the present the further discussion of this theorem and of a similar one, I proceed to consider a particular case. Let us take for the first four circles of each set the fundamental circles $X\ Y\ Z\ W$, and let the coordinates of the circles 5 and 5′ be represented by x, y, z, w and x', y', z', w'. Then the identity

$$\begin{matrix}X & Y & Z & W & 5\\X & Y & Z & W & 5'\end{matrix} = 0$$

becomes

$$\begin{vmatrix}-2r_1^2, & (XY), & (XZ), & (XW), & x\\(XY), & -2r_2^2, & (YZ), & (YW), & y\\(XZ), & (YZ), & -2r_3^2, & (ZW), & z\\(XW), & (YW), & (ZW), & -2r_4^2, & w\\x', & y', & z', & w', & (5\,5')\end{vmatrix} = 0.$$

By this equation the power of two circles is expressed in terms of their coordinates. Let the expression be denoted thus:

$$(5\,5') = (*\between x, y, z, w \between x', y', z', w'),$$

then clearly the *equation* of the circle whose *coordinates* are $x\, y\, z\, w$ is

$$0=(*\between x, y, z, w \between X, Y, Z, W).$$

And further, if we suppose the circles 5, 5′ to coincide in a circle of radius r, we have

$$-2r^2=(*\between x, y, z, w)^2,$$

which gives the radius of a circle in terms of its coordinates. It follows that the Second Absolute is

$$\Phi=(*\between X\, Y\, Z\, W)^2.$$

By dividing the first four rows and columns of the determinant by r_1, r_2, r_3, r_4 respectively, we may reduce this to the simpler form

$$\begin{vmatrix} 1 & \cos XY & \cos XZ & \cos XW & \frac{X}{r_1} \\ \cos XY & 1 & \cos YZ & \cos YW & \frac{Y}{r_2} \\ \cos XZ & \cos YZ & 1 & \cos ZW & \frac{Z}{r_3} \\ \cos XW & \cos YW & \cos ZW & 1 & \frac{W}{r_4} \\ \frac{X}{r_1} & \frac{Y}{r_2} & \frac{Z}{r_3} & \frac{W}{r_4} & 0 \end{vmatrix}$$

which it is worth stating is an identical relation connecting the equations of any four circles.

It is obvious that all these formulæ will be immensely simplified if we take for our fundamental circles four circles cutting each other orthogonally. In this case the quantities (XY), (XZ)...&c. are all zero; the Second Absolute becomes

$$\Phi\equiv\frac{X^2}{r_1^2}+\frac{Y^2}{r_2^2}+\frac{Z^2}{r_3^2}+\frac{W^2}{r_4^2},$$

the radius of $(x\, y\, z\, w)$ is

$$\tfrac{1}{4}\left(\frac{x^2}{r_1^2}+\frac{y^2}{r_2^2}+\frac{z^2}{r_3^2}+\frac{w^2}{r_4^2}\right);$$

the power of $(x\, y\, z\, w)$ in respect of $(x'\, y'\, z'\, w')$ is

$$-\tfrac{1}{2}\left(\frac{xx'}{r_1^2}+\frac{yy'}{r_2^2}+\frac{zz'}{r_3^2}+\frac{ww'}{r_4^2}\right),$$

and consequently the equation of $(x\, y\, z\, w)$ is

$$\frac{x}{r_1^2}X+\frac{y}{r_2^2}Y+\frac{z}{r_3^2}Z+\frac{w}{r_4^2}W=0.$$

I shall simplify these expressions still further by taking as coordinates of a circle not the powers $X\, Y\, Z\, W$ themselves, but the powers each divided by the radius of the corresponding fundamental circle, viz. the quantities

$$\frac{X}{r_1},\ \frac{Y}{r_2},\ \frac{Z}{r_3},\ \frac{W}{r_4}.$$

Calling these quantities $X_1\ Y_1\ Z_1\ W_1$, we find for the absolute merely

$$X_1^2+Y_1^2+Z_1^2+W_1^2,$$

and the equation of $(l\, m\, n\, s)$ is

$$lX_1+mY_1+nZ_1+sW_1=0.$$

This step, it will be seen, is precisely analogous to the step from *areal* to *trilinear* coordinates in the geometry of straight lines. It must be remembered that the only speciality which has been introduced is that our four fundamental circles form an orthogonal system. One of them is therefore imaginary.

The radius of the circle

$$lX_1+mY_1+nZ_1+sW_1=0$$

vanishes when

$$l^2+m^2+n^2+s^2=0.$$

It thus appears that a point (or circle of no radius) is a circle which satisfies the analytical condition of touching the Second Absolute. This remark will be found useful in the sequel.

Equation of an Anallagmatic.

The general equation of the second order in power-coordinates represents an *anallagmatic* curve of the fourth order, i.e. a quartic curve having a double point at each of the circular points at infinity. For if in the expression

$$(*\between X, Y, Z, W)^2$$

we substitute for X, Y, Z, W their values in terms of $x^2+y^2, x, y, 1$, it becomes

$$a(x^2+y^2)^2+L(x^2+y^2)+U,$$

where L is of the first degree in x, y, and U of the second. Now this equated to zero is the general equation of an anallagmatic quartic; and it contains one constant less than the former, which is thus (if we bear in mind the existence of the Second Absolute) just sufficiently general to represent all such anallagmatic curves.

I say, "if we bear in mind the existence of the Second Absolute" for this reason. The equation $\Phi=0$ is satisfied identically by the coordinates of every *point*. Now if we take $\Theta=0$ for an equation of the second order in X, Y, Z, W, the equation

$$\Theta+\lambda\Phi=0$$

must represent exactly the same curve as $\Theta=0$; for the equation is satisfied by the coordinates of all *points* which satisfy this latter equation. Out of the nine apparently arbitrary constants, therefore, in the expression Θ, one is at our disposal independently of the determination of the curve; and the real number of constants is therefore eight.

In fact, if we regard $X\, Y\, Z\, W$ as the coordinates of a point in space, then the Second Absolute represents a quadric surface; and every anallagmatic curve is represented in the first instance by another quadric surface, but in the second instance (viz., when we remember the remarks just made) by the curve of intersection of this surface with the Second Absolute. Now a quadri-quadric curve depends upon eight constants only, and so fitly represents a general anallagmatic curve.

Bitangent Circles.

Consider therefore any quadric Θ and the second absolute Φ. We know that by a linear transformation it is possible in one and only one way to reduce these simultaneously to the canonical form. That is to say, there is precisely one set of four circles $X\ Y\ Z\ W$ such that when we express Θ and Φ in terms of them they take the forms

$$\Phi \equiv X^2 + Y^2 + Z^2 + W^2,$$

$$\Theta \equiv \frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} + \frac{W^2}{d^2}.$$

The first of these indicates that the circles $X\ Y\ Z\ W$ are mutually orthogonal. They may be called the *principal circles* of the curve. (Moutard.)

The quantities a^2, b^2, c^2, d^2, are not necessarily positive; they are written in this form for the sake of subsequent convenience.

It is now possible, with the aid of the second absolute, to express the equation of the curve in terms of any three of the four principal circles. In fact, eliminating W between the equations $\Theta = 0$, $\Phi = 0$, we have

$$d^2\Theta - \Phi \equiv \frac{X^2}{a^2}(d^2 - a^2) + \frac{Y^2}{b^2}(d^2 - b^2) + \frac{Z^2}{c^2}(d^2 - c^2) = 0,$$

an equation which, if X, Y, Z were trilinear coordinates, would represent a conic referred to a self-conjugate triangle.

A circle which satisfies the analytical condition of touching this curve is a bitangent circle of the anallagmatic. For if

$$x \equiv lX + mY + nZ,$$
$$y \equiv l'X + m'Y + n'Z,$$
$$u \equiv l_1X + m_1Y + n_1Z,$$

where x and y are two such tangents, then we know the equation of the curve can be reduced to the form

$$xy = ku^2,$$

where it is clear that, for instance, the circle x touches the curve at the two points where it meets u. All these circles are orthotomic of W; for since XYZ are orthotomic of W, any circle $lX + mY + nZ$ is so. Hence we may enunciate the following propositions:—

The bitangent circles of an anallagmatic curve arrange themselves into four systems, all the circles of each system cutting orthogonally one of the principal circles.

The four points of contact of two circles x, y, of the same system lie on a circle u (circle of contact), and the product of the tangents drawn from any point of the curve to the circles x, y is in a constant ratio to the square of the tangent drawn to the circle u.

Two bitangent circles of a given system can be drawn through an arbitrary point.

If two bitangent circles be drawn through a point on u, their circle of contact is coaxal with x, y.

Further, let x, y, z be any three bitangent circles of the system W, where

$$z \equiv l''X + m''Y + n''Z,$$

then we know from the theory of conics that the equation of the anallagmatic can be written in the form

$$\alpha\sqrt{x} + \beta\sqrt{y} + \gamma\sqrt{z} = 0,$$

where $\alpha\ \beta\ \gamma$ are constants. Hence

The tangents drawn from any point of the curve to three bitangent circles of the same system satisfy a linear relation.

A focus of the curve is a bitangent circle of evanescent radius. Since the two quadrics

$$\frac{X^2}{a^2}(d^2 - a^2) + \frac{Y^2}{b^2}(d^2 - b^2) + \frac{Z^2}{c^2}(d^2 - c^2) = 0,$$

$$X^2 \qquad + Y^2 \qquad + Z^2 \qquad = 0,$$

have just *four* common tangents, and since a circle of evanescent radius is one satisfying the analytical condition of touching the absolute, it follows that:

There are four foci on each principal circle.

The distances of any point on the curve from three foci of the same system are connected by a linear relation.

Any two foci of the same system have a contact circle, such that the product of the distances of any point of the curve from the two foci is proportional to the power *of the point in respect of this circle.*

Confocal Curves.

The equation

$$\frac{X^2}{a^2+\theta} + \frac{Y^2}{b^2+\theta} + \frac{Z^2}{c^2+\theta} + \frac{W^2}{d^2+\theta} = 0$$

(in which θ is a variable parameter) represents a series of confocal anallagmatics: for it is readily observed that the equations for determining the foci involve only the *differences* of the quantities a^2, b^2, c^2, d^2. In fact, the foci are common bitangents of the curve and of the absolute; now the equation above written obviously represents a curve touching all the bitangents common to

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} + \frac{W^2}{d^2} = 0,$$

and

$$X^2 + Y^2 + Z^2 + W^2 = 0.$$

Two confocals cut at right angles. For let $X_1Y_1Z_1W_1$ be a point of intersection. The bitangent circles of the W-system at that point are

$$\frac{XX_1}{a^2}(d^2-a^2)+\frac{YY_1}{b^2}(d^2-b^2)+\frac{ZZ_1}{c^2}(d^2-c^2)=0,$$

$$\frac{XX_1}{a^2+\theta}(d^2-a^2)+\frac{YY_1}{b^2+\theta}(d^2-b^2)+\frac{ZZ_1}{c^2+\theta}(d^2-c^2)=0,$$

which cut at right angles if

$$\frac{X_1^2}{a^2(a^2+\theta)}(d^2-a^2)^2+\frac{Y_1^2}{b^2(b^2+\theta)}(d^2-b^2)^2+\frac{Z_1^2}{c^2(c^2+\theta)}(d-c^2)^2=0.$$

But this equation follows readily from the three

$$\frac{X_1^2}{a^2}+\frac{Y_1^2}{b^2}+\frac{Z_1^2}{c^2}+\frac{W_1^2}{d^2}=0 \quad\text{..........(A)},$$

$$\frac{X_1^2}{a^2+\theta}+\frac{Y_1^2}{b^2+\theta}+\frac{Z_1^2}{c^2+\theta}+\frac{W_1^2}{d^2+\theta}=0 \quad\text{..........(B)},$$

$$X_1^2+Y_1^2+Z_1^2+W_1^2=0 \quad\text{..........(C)}.$$

It is worth while to write down the ratios of $X_1Y_1Z_1W_1$ which are deducible from these. They are

$$kX_1=\frac{aa'}{\sqrt{(\overline{a^2-b^2}\,.\,\overline{a^2-c^2}\,.\,\overline{a^2-d^2})}},$$

$$kY_1=\frac{bb'}{\sqrt{(\overline{b^2-c^2}\,.\,\overline{b^2-d^2}\,.\,\overline{b^2-a^2})}},$$

and so on, where a'^2 has been written for shortness instead of $a^2+\theta$.

The conditions that a circle

$$lX+mY+nZ+sW,$$

shall be a bitangent of the system W to the curve (A) are $s=0$ and

$$\frac{l^2a^2}{d^2-a^2}+\frac{m^2b^2}{d^2-b^2}+\frac{n^2c^2}{d^2-c^2}=0.$$

Call this $U=0$, and let

$$V\equiv\frac{l^2}{d^2-a^2}+\frac{m^2}{d^2-b^2}+\frac{n^2}{d^2-c^2},$$

then the equation $U=0$ can be written in either of the forms

$$\frac{U-Va^2}{a^2}=\frac{U-Vb^2}{b^2}=\frac{U-Vc^2}{c^2}.$$

I shall now prove that the quantities $U-Va^2$, $U-Vb^2$, $U-Vc^2$ are proportional to the products of the powers of $(lmns)$ in respect of pairs of foci of the systems $X\ Y\ Z$ respectively.

In fact, if we write

$$P^2=(a^2-b^2)(c^2-d^2),$$
$$Q^2=(a^2-c^2)(d^2-b^2),$$
$$R^2=(a^2-d^2)(b^2-c^2),$$

then it is easily seen that the coordinates of the sixteen foci are

four of the X-system	0,	$\pm P$,	$\pm Q$,	$\pm R$,
four of the Y-system	$\pm P$,	0,	$\pm R$,	$\pm Q$,
four of the Z-system	$\pm Q$,	$\pm R$,	0,	$\pm P$,
four of the W-system	$\pm R$,	$\pm Q$,	$\pm P$,	0.

Now, for instance,

$$U-Va^2\equiv\frac{m^2(b^2-a^2)}{d^2-b^2}+\frac{n^2(c^2-a^2)}{d^2-c^2},$$

$$\equiv\frac{(mP+nQ)(mP-nQ)}{(d^2-b^2)(d^2-c^2)},$$

$$\equiv\frac{(-mP-nQ)(-mP+nQ)}{(d^2-b^2)(d^2-c^2)},$$

which verifies the assertion.

From these equations we deduce the following propositions:—

Any bitangent circle of the W-system is so related to the four foci of the X-system that the product of its powers in respect of a certain two of them is equal to the product of its powers in respect of the other two.

These products in respect of the X- Y- and Z-systems have constant ratios for all bitangent circles of the W-system.

[The following Notes, "Theory of Powers," preceded the above paper in the Note-book, but appear to have no connection with it. They are very fragmentary and, in places, apparently inaccurate, but it has been thought desirable to print them almost as they were left by the writer. It is not easy to see how the equations in 1 and 4 are got, nor how the other equation in I. contains a linear relation between the powers of a point with respect to a, &c.]

Let a, b, c be three points in a right line: the rectangle or product $ab\,.\,ac$ is called the power of the point a in respect of the point-pair bc.

1. Let a, α; b, β; c, γ be pairs of points dividing harmonically the length XY, then (m, n, p, o being the middle points respectively)

$$a\alpha+XY=[2]\,m\,.\,o,$$
$$b\beta+XY=[2]\,n\,.\,o,$$
$$c\gamma+XY=[2]\,p\,.\,o,$$

multiply by $\overline{n-p}$, $\overline{p-m}$, $\overline{m-n}$ and add; then

$$\overline{n-p}\,.\,a\alpha+\overline{p-m}\,.\,b\beta+\overline{m-n}\,.\,c\gamma=0.$$

Now the origin is arbitrary, and the distances $\overline{n-p}$, $\overline{p-m}$, $\overline{m-n}$ are constant: hence

The powers of any point whatever, in respect of three pairs in involution, satisfy a certain linear relation.

2. Let a, b correspond uniquely to α, β respectively, and let X, Y be the double points of the correspondence.

Then because any two corresponding points make the same anharmonic ratio with the double points, $(a\alpha XY) = (b\beta XY)$, or

$$\frac{a\alpha \,.\, XY}{aX \,.\, \alpha Y} = \frac{b\beta \,.\, XY}{bX \,.\, \beta Y}.$$

Also because the anharmonic ratio of four points = anharmonic ratio of corresponding points,

$$(abXY) = (\alpha\beta XY),$$

or

$$\frac{ab \,.\, XY}{aY \,.\, bX} = \frac{\alpha\beta \,.\, XY}{\alpha Y \,.\, \beta X}.$$

Multiply these equations together; then

$$\frac{a\alpha \,.\, ab}{aX \,.\, bY} = \frac{b\beta \,.\, \alpha\beta}{\beta X \,.\, \beta Y}, \text{ or } \frac{a\alpha \,.\, ab}{\beta\alpha \,.\, \beta b} = \frac{aX \,.\, aY}{\beta X \,.\, \beta Y};$$

that is, if two points be taken one from each system of an unique correspondence, the ratio of their powers in respect of the points corresponding to them is equal to the ratio of the powers in respect of the double points.

If A, B, C are three lines through a point, $\sin AB \,.\, \sin AC$ is called the power of A in respect of the pair BC.

It will follow in a similar manner that the corresponding proposition is true about an unique correspondence of lines.

3. Let now pOp', qOq' be two chords of a conic, meeting in O. Let the lines qq', qp, $p'q'$, $p'p$ be called a, α, b, β respectively; it is clear that the directions a, b correspond uniquely to α, β and that the double lines are the asymptotes. Now

$$\frac{Oq}{Op} = \frac{\sin Opq}{\sin Oqp} = \frac{\sin b\beta}{\sin ab},$$

and

$$\frac{Oq'}{Op'} = \frac{\sin Op'q'}{\sin Oq'p'} = \frac{\sin \alpha\beta}{\sin a\alpha};$$

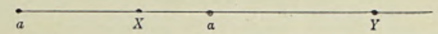
$$\therefore \quad \frac{Oq \,.\, Oq'}{Op \,.\, Op'} = \frac{\sin \beta b \,.\, \sin \beta\alpha}{\sin ab \,.\, \sin a\alpha} = \frac{\sin \beta X \,.\, \sin \beta Y}{\sin aX \,.\, \sin aY},$$

by the previous proposition. [Fig. 119.]

Whence the ratio of the product of segments of chords drawn through any point is independent of the position of the point, and depends only on the direction of the chords relative to the asymptotes. Also the product of segments of any chord drawn through a point, divided by square of parallel semidiameter, is independent of the direction of the chord: this I call the *power* of the point in respect of the conic. Further

The square of any semidiameter is inversely proportional to the products of the sines of the angles it makes with the asymptotes.

4. Properties of Harmonic Section.



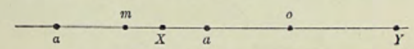
If a, α are harmonic of X, Y, we have by definition

$$\frac{aX}{aY} = -\frac{\alpha X}{\alpha Y}, \text{ or } aX \,.\, \alpha Y + \alpha X \,.\, aY = 0;$$

$$\therefore aX \,.\, aY + \alpha X \,.\, \alpha Y = (a\alpha + \alpha X)\,aY + (\alpha a + aX)\,\alpha Y$$
$$= a\alpha\,(aY - \alpha Y) = a\alpha^2.$$

Therefore if two points are harmonic in respect of X, Y the sum of their powers in respect of X, Y is equal to the square of their distance.

Hence the sum of squares of reciprocals of two conjugate diameters of a conic varies as squared sine of angle between them.



This proposition may also be got from the one concerning the middle points. Thus

$$a\alpha + XY = 2mo,$$

whatever be the origin. Take a and α successively for origin, then

$$aX \,.\, aY = 2am \,.\, ao = a\alpha \,.\, ao,$$

$$\alpha X \,.\, \alpha Y = 2\alpha m \,.\, \alpha o = -a\alpha \,.\, \alpha o;$$

$$\therefore \; aX \,.\, aY + \alpha X \,.\, \alpha Y = a\alpha\,(ao - \alpha o) = a\alpha^2,$$

also

$$aX \,.\, aY - \alpha X \,.\, \alpha Y = -a\alpha^2 \,.\, ao \,.\, \alpha o = -a\alpha^2 \,.\, oX^2$$
$$= -\tfrac{1}{4} \,.\, a\alpha^2 \,.\, XY^2:$$

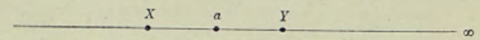
that is, product of powers of a, α in respect of XY = squared distance of a, α × squared distance of X, Y. Hence product of conjugate diameters into sine of angle between them is constant.

Further, by division,

$$\frac{1}{aX \,.\, aY} + \frac{1}{\alpha X \,.\, \alpha Y} = -\frac{4}{XY^2},$$

therefore sum of squares of conjugate diameters is constant.

Theory of the Linear Relations.



XY is divided harmonically by a and ∞, m is any other point. Then

$$mX \,.\, mY = ma^2 - aX^2 = ma^2 + aX \,.\, aY,$$

or

$$mX \,.\, mY \,.\, \overline{a\infty}^2 = \overline{ma}^2 \,.\, X\infty \,.\, Y\infty + aX \,.\, aY \,.\, \overline{m\infty}^2 ;$$

the equation is now homogeneous in each point, and therefore projective. For ∞ we may write a, and a, a will be any two harmonics of X, Y. Then

$$\overline{aa}^2 \,.\, mX \,.\, mY = \overline{ma}^2 \,.\, aX \,.\, aY + \overline{ma}^2 \,.\, aX \,.\, aY \text{..................} (1).$$

Let m be at infinity, then

$$aa^2 = aX \,.\, aY + aX \,.\, aY.$$

Applying (1) to the three points m, $m+n$, n, we have

$$\tfrac{1}{2}aa^2 (mX \,.\, nY + nX \,.\, mY) = ma \,.\, na \,.\, aX \,.\, aY + ma \,.\, na \,.\, aX \,.\, aY.$$

Hence if m, n are conjugate of X, Y, say $m, n = b, \beta$, then

$$\frac{ab \,.\, a\beta}{ab \,.\, a\beta} = \frac{aX \,.\, aY}{aX \,.\, aY}.$$

But if on the other hand m, n coincide with X, Y respectively,

$$-\tfrac{1}{2}aa^2 \,.\, XY^2 = 2aX \,.\, aY \,.\, aX \,.\, aY,$$

or

$$aa^2 \,.\, XY^2 = -4aX \,.\, aY \,.\, aX \,.\, aY.$$

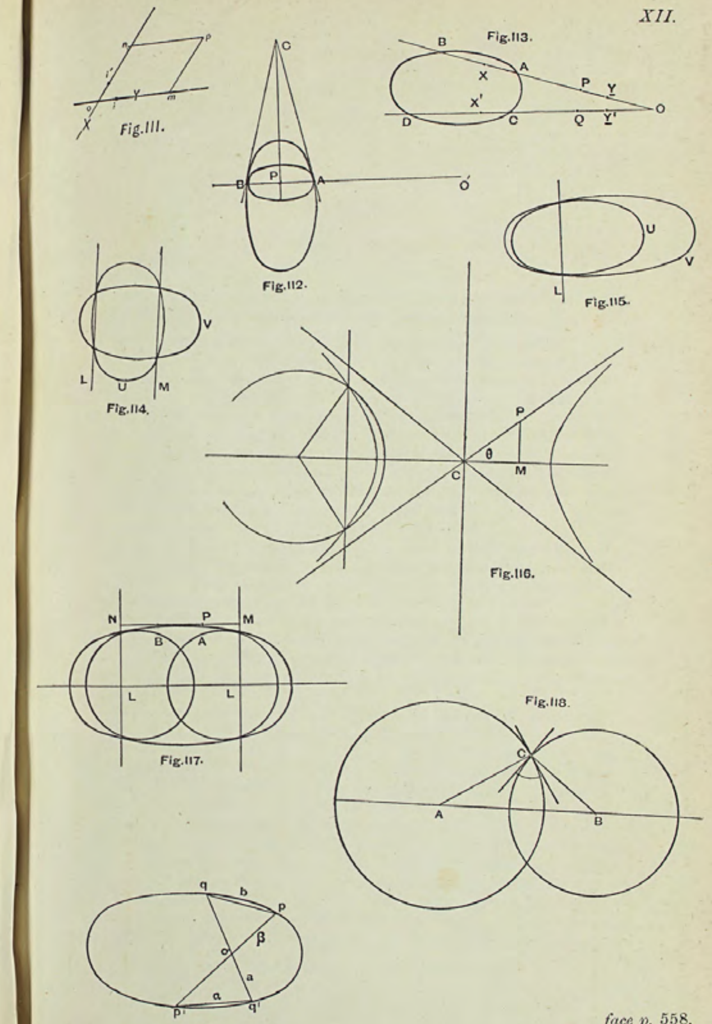


Fig. 111. Fig. 112. Fig. 113. Fig. 114. Fig. 115. Fig. 116. Fig. 117. Fig. 118. Fig. 119.