XII.

ON THE GENERAL THEORY OF ANHARMONICS*.

1. THE theory of Anharmonics on the straight line may be stated in the following symmetrical form:—

(i) There is an identical relation connecting the distances of four points, 1, 2, 3, 4, on a right line, viz.,

$$12 . 34 + 13 . 42 + 14 . 23 \equiv 0.$$

(ii) The ratios of the three terms in this identity are not altered by projection or integral linear transformation. There is a corresponding theory of four straight lines meeting in a point.

2. In applying this theory to geometry of two dimensions, we meet with this third proposition:—

(iii) If a straight line meet four fixed straight lines, so that the distances of the points which they determine on it satisfy a relation of the form

$$\lambda . 12 . 34 + \mu . 13 . 42 + \nu . 14 . 23 = 0,$$

then the envelope of the line is of the second class, touching the four given lines. There is, of course, a correlative proposition on the locus of a point subtended by four points in a given manner. The propositions (i), (ii), (iii), each including its converse and correlative propositions, may be regarded as constituting the entire theory of anharmonics in geometry of one dimension, and its application to geometry of two dimensions.

* [From the *Proceedings of the London Mathematical Society*, Vol. II. No. 9, pp. 3—6.]

3. I proceed to state the theory of anharmonics in geometry of two dimensions.

(i) There are six identical relations connecting the areas of triangles formed by six points, 1, 2, 3, 4, 5, 6, in a plane, viz.,

$$123 . 456 + 124 . 563 + 125 . 634 + 126 . 345 \equiv 0,$$

with five others obtained from this by permutation.

(ii) The ratios of the terms in these identities are not altered by projection or integral linear transformation. Under each of these propositions are included *three* correlatives, to explain which I must introduce three new definitions:

DEF. 1st. The *projector* of a plane triangle is the square of the area divided by the continued product of the sides.

DEF. 2nd. In a solid angle considered as determined by three concurrent straight lines, the sine of the angle between the first line and the plane of the other two, multiplied by the sine of the angle between the other two lines, is a symmetrical quantity in respect of the three lines, and is called the *sine* of the three lines.

DEF. 3rd. In a solid angle considered as determined by three planes, the sine of the angle between the first plane and the intersection of the other two, multiplied by the sine of the angle between the other two planes, is a symmetrical quantity in respect of the three planes, and is called the *sine* of the three planes.

4. It is convenient for several purposes to use the word "distance" as including all these notions: thus I shall speak of the *distance* of two lines in a plane, meaning the sine of the angle between them;

the distance of three points is the area of their triangle;
" " of three straight lines in a plane, the projector of their triangle;
" " of three planes, the sine of the planes;
" " of three concurrent lines in space, the sine of the lines.

And I shall have occasion afterwards to define the distance of four points and of four planes in space. By means of these definitions, the propositions (i), (ii) may be interpreted in four different ways, corresponding to the four aspects of bi-dimensional extension: the symbol 123 being always understood to mean the *distance* of the three things considered.

5. The propositions (i), (ii), including their converse and correlative propositions, constitute the entire theory of anharmonics in geometry of two dimensions. To apply this theory to geometry of three dimensions, I state the following proposition, which has only *one* correlative:

(iii) If a plane meet six fixed planes so that the distances of the lines they determine upon it satisfy a relation of the form

$$\lambda . 123 . 456 + \mu . 124 . 563 + \nu . 156 . 234 + \ldots = 0,$$

then the envelope of the plane is of the second class, touching the six given planes.

6. The theory of anharmonics in three dimensions is so entirely analogous to the two former theories, that it wants no further discussion. The distance of four points is the volume of their tetrahedron, and the distance of four planes is the cube of the volume divided by the product of the areas of the faces.

7. Two finite lines 11′, 22′, measured on the same straight line, are said to be *harmonically situate* when

$$12 . 1'2' + 12' . 1'2 = 0;$$

and when this is the case, if the two pairs of points be represented by the equations $U=0$, $V=0$, there is an invariant relation connecting the quadrics U, V which may be denoted by

$$\square (U, V) = 0;$$

the notation indicating that if we expand the discriminant of $\lambda U + \mu V$, or

$$\square (\lambda U + \mu V),$$

then the coefficient of $\lambda\mu$ in the expansion vanishes.

There is a similar relation between two angles having a common vertex; when this relation holds, I say that the two covertical angles are harmonically situate. I also speak of an harmonic pair of angles, or of an harmonic pair of finite lines, or lengths; meaning in this case covertical angles, or collinear lengths.

8. This being so, it is known that two lengths (I use the word *length* to denote a pair of points), anyhow placed in a plane, determine a conic passing through their ends, which conic is the locus of points which the lengths subtend harmonically, that is, in a pair of harmonic angles. I call this the harmonic conic of the two lengths. This (with the correlative propositions) completes the theory of harmonics in one dimension, and its application to two dimensions. I now come to consider harmonics in two dimensions.

9. If the harmonic conic of two lengths 11′, 22′ divides harmonically a third length 33′, then the relation between the three lengths is symmetrical, and I say that the three lengths are *harmonically situate* in the plane. The following relation subsists among their distances, viz.,

$$123 . 1'2'3' + 1'23 . 12'3' + 12'3 . 1'23' + 123' , 1'2'3 = 0.$$

And if each length, or point-pair, be considered as a degenerate conic, so that the equations to the three point-pairs are U, V, W $=0$, then, when the three lengths are harmonically situate, there is an invariant relation connecting the quadrics U, V, W, which may be denoted by

$$\square (U, V, W) = 0;$$

$\square (U, V, W)$ denoting the coefficient of $\lambda\mu\nu$ in the expansion of

Discriminant of $(\lambda U + \mu V + \nu W)$.

It is obvious that a similar relation may subsist among three angles in a plane, three pairs of lines through a point in space, or three pairs of planes through a point in space.

10. Three lengths anyhow placed in space determine a quadric surface passing through their ends, which surface is the locus of points which the lengths subtend harmonically, that is,

in a triad of harmonic angles. I call this the harmonic quadric of the three lengths.

11. If the harmonic quadric of three lengths 11′, 22′, 33′ divides harmonically a fourth length 44′, then the relation between the four lengths is symmetrical, and I say that the four lengths are *harmonically situate* in space. The following relation subsists among their distances, viz.,

$$\Sigma \,.\, 1234 \,.\, 1'2'3'4' = 0 \text{ (eight terms).}$$

And if each length, or point-pair, be considered as a degenerate quadric, so that the equations to the four point-pairs are $U=0$, $V=0$, $W=0$, $T=0$, then, when the four lengths are harmonically situate, there is an invariant relation connecting the quadrics U, V, W, T, which may be denoted by

$$\square\,(U,\ V,\ W,\ T)=0;$$

$\square\,(U, V, W, T)$ denoting the coefficient of $\lambda\mu\nu\rho$ in the expansion of

$$\text{Discriminant of } (\lambda U + \mu V + \nu W + \rho T).$$

It is obvious that a similar relation may subsist among four pairs of planes.

12. It only remains to explain the meaning of the conditions

$$\square\,(U,\ V,\ W)=0,$$
$$\square\,(U,\ V,\ W,\ T)=0,$$

when the quadrics do *not* break up into factors. Two conics U, V determine an harmonic conic F, locus of points which they subtend in a pair of harmonic angles. If a triangle self-conjugate of F can be inscribed in W, then the relation between UVW is symmetrical, and $\square\,(U, V, W)=0$. The conics may then be spoken of as three mutually harmonic conics; a similar relation may hold between three covertical cones. Thus, in fact, three quadric *surfaces* U, V, W determine an harmonic quadric F, locus of points which they subtend in three harmonic cones. If a tetrahedron self-conjugate of F can be inscribed in T, then the relation between U, V, W, T is symmetrical, and $\square\,(U, V, W, T)=0$.

XIII.

ON A GENERALIZATION OF THE THEORY OF POLARS*.

(The present Note establishes the idea of the polar curve of a curve of given class in respect of a curve of given order, the class being less than the order; and of the polar curve of a curve of given order in respect of a curve of given class, the order being less than the class. It also deals with a certain invariant of two curves, such that the order of one is equal to the class of the other; and with certain other invariants and contravariants arising out of the theory of polars. I desire to present these ideas by themselves to the Society, because they seem likely to be useful for other purposes than that to which I propose to apply them subsequently, viz., the extension of Grassmann's Geometric Analysis.)

1. Let B_n be a curve of the nth order, and c_m a curve of the mth class. Let the equations of the curves be

$$B_n \equiv (A, B, C, D \ldots\ldots \between x\, y\, z)^n$$
$$c_m \equiv (a, b, c, d \ldots\ldots\ldots\ldots \between \xi\, \eta\, \zeta)^m$$

in point and line coordinates respectively.

In c_m write $\dfrac{\delta}{dx}, \dfrac{\delta}{dy}, \dfrac{\delta}{dz}$ in place of ξ, η, ζ respectively, and operate on B_n with the symbol thus formed. I denote the result by merely writing c_m as an operator before B_n; thus

$$c_m B_n \equiv \left(a, b, c \ldots\ldots \between \frac{\delta}{dx}, \frac{\delta}{dy}, \frac{\delta}{dz}\right)^m . (A, B, C, \ldots\ldots \between x, y, z)^n;$$

* [From the *Proceedings of the London Mathematical Society*, Vol. II. No. 16, pp. 116—118.]

then we find

(i) If m is less than n, $c_m B_n$ is a covariant, which I call the *polar curve* of c_m in respect of B_n. It is given in the point coordinates x, y, z, and is of order $n-m$.

(ii) If m is equal to n, $c_n B_n$ is an invariant of the two curves. When this invariant vanishes, I say that the curves are *harmonic* of each other.

(iii) If m is greater than n, $c_m B_n = 0$ always.

2. It is important to shew that these new meanings of the words *polar* and *harmonic* include the old meanings. Now the mth polar of a point (x, y, z) whose tangential equation is $x\xi + y\eta + z\zeta = 0$, say the point P, is

$$\left(x\frac{\delta}{dx} + y\frac{\delta}{dy} + z\frac{\delta}{dz}\right)^m . B_n = 0;$$

which will be denoted in our present notation by $p^m B_n = 0$. But this is, according to the new definition, the polar of m times the point p; the point p taken m times being of course a particular case of a curve of the mth class. Again, the two conics

$$(a, b, c, f, g, h \between \xi, \eta, \zeta)^2, \qquad (A, B, C, F, G, H \between x, y, z)^2$$

have been called *harmonic* when

$$Aa + Bb + Cc + 2Ff + 2Gg + 2Hh = 0,$$

the invariant being obviously the result of turning the first into an operator and applying it to the second.

3. Returning to the curves c_m, B_n, we may convert B_n into an operator by writing in it $\frac{\delta}{d\xi}, \frac{\delta}{d\eta}, \frac{\delta}{d\zeta}$ in place of x, y, z respectively. The result of the operation on c_m may be denoted by $B_n c_m$. As before, if $n > m$, the result is zero; if $n = m$, it is the harmonic invariant; if $n < m$, it is a curve of class $m - n$, which may be called the polar curve of B_n in respect of c_m.

4. Another definition of polar curves may be given. Suppose that the curves g_m, h_n together make up a curve which is harmonic of B_{m+n}. It is convenient to say that the curves g_m, h_n

are *complementary*. Then it is clear that *all the curves complementary to a given curve are harmonic of its polar*. This may be regarded as a generalization of the theorem: All the points conjugate to a fixed point in regard to a conic, lie on the polar of the fixed point.

5. Consider a curve of even order B_{2n}. There is an invariant of the order $\frac{1}{2}(n+1)(n+2)$ in the coefficients, which vanishes when the nth differentials are in involution; this invariant is a symmetrical determinant. Its evectant is a contravariant of the order $2n$ in the variables, and $\frac{1}{2}(n^2 + 3n)$ in the coefficients, which may be called b_{2n}. The curves B_{2n}, b_{2n} are so related that *if X_n be the polar of y_n in respect of B_{2n}, then the y_n is the polar of X_n in respect of b_{2n}*. That is to say, if $X_n = y_n B_{2n}$, then $X_n b_{2n} = I . y_n$ where I is the invariant just defined. For example, if we consider a series of conics c_2 and their polars C_2 in regard to a given quartic curve Q_4, then there exists a curve q_4 of the fourth class, such that in respect to it the conics c_2 are the polars of C_2.

6. Two curves whose order and class are different may be made susceptible of the harmonic relation by taking each a proper number of times. Thus, curves B_6, c_8 have an invariant $(B_6)^4 (c_8)^3$, of the order 4 in the coefficients of B_6 and of the order 3 in the coefficients of c_8*. It is to be observed that the equation $B_{mn}(c_m)^n = 0$, is the most general relation of the nth order that can subsist among the coefficients of c_m.

7. The following remarks relate to theorems in Dr Henrici's paper "On certain formulæ concerning the Theory of Discriminants†."

* Thus we may have an invariant which vanishes when a curve is harmonic to itself. Let U_m be the curve, u_n its reciprocal: then $U_m^n u_n^m$ is the invariant in question, or a power of it. For a conic, it is the discriminant; for a cubic, the invariant T of the sixth order. See Salmon's *Higher Algebra*, 1st ed., note to p. 67.

If the curve U_m has no node, the class $= m(m-1)$, and the invariant is $(U_m)^{m-1} u_{m(m-1)}$. If however U_m has a node d, d^2 is part of the reciprocal, and the invariant is $d^2 u_{m(m-1)-2} (U_m)^n$. Cusps may be similarly dealt with.

† [*Proceedings of the London Mathematical Society*, Vol. II. Nos. 15, 16.]

If the polar of C_m in respect of B_n has a node, C_m is harmonic of a curve of order $\xi m(n-m-1)^2$, which is Dr Henrici's curve $S^{(m)}$. In fact when C_m is a point taken m times, the point is *on* the curve $S^{(m)}$; this is Dr Henrici's theorem.

In general, $x^m y^n B_{m+n+1}$ denotes a straight line. If it vanishes identically, x is a node on the nth polar of y, and y is a node on the mth polar of x. In this case $x^m y^{n-1} B_{m+n-1}$ and $x^{m-1} y^n B_{m+n-1}$ are conics having nodes at y and x respectively*. In the relation

$$x^m y^n B_{m+n+1} \equiv 0$$

write $x+\delta x$ for x and $y+\delta y$ for y. This is equivalent to supposing δx and δy to be points on the tangents at x and y to the curves $S^{(m)}$ $S^{(n)}$ which are the loci of those points respectively. Then we have

$$(my\delta x + nx\delta y)\, x^{m-1} y^{n-1} B_{m+n+1} \equiv 0;$$

operate on this with y; we know that

$$n\delta y \,.\, x^m y^n B_{m+n+1} = 0,$$

it follows that

$$m\delta x \,.\, x^{m-1} y^{n+1} B_{m+n+1} = 0;$$

that is to say, the tangent at x to $S^{(m)}$ is the line-polar of y in respect of the $(m-1)$th polar of x. This is another of Dr Henrici's theorems. I have added this proof as an example of the readiness with which the operative notation lends itself to such investigations.

* Viz., these are the pairs of tangents at the two nodes. It is observable that the tangents at x to the nth polar of y, the tangent to the curve $S^{(m)}$, and the line xy, form a harmonic pencil.

XIV.

ON SYZYGETIC RELATIONS AMONG THE POWERS OF LINEAR QUANTICS*.

In his *Géométrie de Direction* (Paris, 1869), M. Paul Serret makes very beautiful use of a principle which he states nearly as follows (p. 138):

"*In order that a system of points (in a plane) may be so related that every curve of order m passing through all but one of them must pass through the remaining one, it is necessary and sufficient that the m*th *powers of their distances from an arbitrary line should satisfy a linear homogeneous relation*

$$\lambda_1 P_1^{\,m} + \lambda_2 P_2^{\,m} + \lambda_3 P_3^{\,m} + \ldots\ldots \equiv 0\dagger."$$

There is, of course, an analogous theorem for surfaces, and in fact M. Serret combines the two enunciations into one; he states also the correlative theorems concerning a system of lines or planes such that every curve or surface touching all but one of them, touches also the remaining one. For the sake of clearness I have here stated in full only one of these four theorems.

By the use of Professor Sylvester's method of Contravariant Differentiation I have arrived at certain extensions of these theorems, which I now proceed to explain:—

Theorem I. *In order that a system of N points in a plane should all lie on a curve of order n, it is sufficient that the p*th

* [From the *Proceedings of the London Mathematical Society*, Vol. III. No. 21, pp. 9—12.]

† In the *Bulletin des Sciences Mathématiques et Astronomiques*, January, 1870, M. Darboux observes that this theorem, for the special case $m=2$, had been given by Hesse, *Vier Vorlesungen aus der analytischen Geometrie*, Leipzig, 1866.

powers of their distances from an arbitrary line should satisfy a linear homogeneous relation; the number N being given by the formula

$$N = \tfrac{1}{2}\alpha n(n+3) + \tfrac{1}{2}(\beta+1)(\beta+2),$$

where α is the quotient and β the remainder of the division of p by n, so that $p = \alpha n + \beta$, and $\beta < n$.

Theorem II. *In order that a system of N points in space should all lie on a surface of order n, it is sufficient that the p^{th} powers of their distances from an arbitrary plane should satisfy a linear homogeneous relation; the number N being given by the formula*

$$N = \tfrac{1}{6}\alpha n(n^2+6n+11) + \tfrac{1}{6}(\beta+1)(\beta+2)(\beta+3),$$

where as before

$$p = \alpha n + \beta,\ \ \beta < n.$$

To render the nature of these theorems somewhat more clear, I add the following tables of the values of N for given values of p and n:—

TABLE A.—CURVES.

Values of p.	2	3	4	5	6	7	8	9	10	11	12
Line	5	7	9	11	13	15	17	19	21	23	25
Conic...............	6	8	11	13	16	18	21	23	26	28	31
Cubic...............		10	12	15	19	21	24	28	30	33	37
Quartic			15	17	20	24	29	31	34	38	43
Quintic				21	23	26	30	35	41	43	46
Sextic					28	30	33	37	42	48	55
Septic						36	38	41	45	50	56
Octavic							45	47	50	54	59

TABLE B.—SURFACES.

Values of p.	2	3	4	5	6	7	8	9	10	11	12
Plane	7	10	13	16	19	22	25	28	31	34	37
Quadric............	10	13	19	22	28	31	37	40	46	49	55
Cubic		20	23	29	39	42	48	58	61	67	77
Quartic			35	38	44	54	69	72	78	88	103
Quintic				56	59	65	75	90	111	114	120
Sextic					84	87	93	103	118	139	167
Septic						120	123	129	139	154	175
Octavic							165	168	174	184	199

Here, for example, in the first table opposite the word Cubic and under the power 5 we find the number 15; the theorem corresponding to this is—

If 15 points are such that every quintic through 14 of them passes through the remaining one, all these points must lie on a cubic curve.

Now if we take 15 points arbitrarily on a cubic curve, it is not in general true that the fifth powers of their distances from an arbitrary line satisfy a linear homogeneous relation. That this may be the case, the 15 points must be intersections of the cubic with a quintic; and these are not arbitrary points, but 14 of them being given, the 15th is determined, by a theorem of Jacobi and Plücker. The theorem immediately derived from the table, then, must be completed by this statement; the points are not only all on a cubic, but they are intersections of a cubic and a quintic.

It is to be understood also that if we take a number N of points lying between any two adjacent numbers in the same vertical column of the table, then the same theorem is true about N that is true about the greater of these numbers. Thus we are informed by the first table that a syzygy among the 4th powers of the distances of 12 points makes them lie on a cubic, and that a similar syzygy for 15 points makes them lie on a quartic; this latter theorem is true for the intermediate numbers 13 and 14. It is not however *all* that is true in either of these cases; the 14 points are points of intersection of two quartics, and the 13 points are (I believe) points on a cubic such that no twelve of them are intersections of the cubic with a quartic. I wish particularly to draw attention to these intermediate cases, where it appears that more is true than can be proved by the method to be presently explained.

Method of Demonstration. Let the tangential equation of a point be

$$0 = \alpha\xi + \beta\eta + \gamma\zeta\ (\equiv p,\ \text{say})$$

and let the equation of a curve of the n^{th} order be

$$0 = (*\between x, y, z)^n\ (\equiv B_n,\ \text{say})$$

then I say that

$$(*\between \frac{\delta}{\delta\xi}, \frac{\delta}{\delta\eta}, \frac{\delta}{\delta\zeta})^n . (\alpha\xi+\beta\eta+\gamma\zeta)^n = (*\between \alpha, \beta, \gamma)^n \lfloor n ;$$

that is to say, *if we operate with* B_n *on the* n^{th} *power of* p, *we shall obtain the result of substituting the coordinates of* p *for* x, y, z *in* B . If, then, this result vanishes, the point p is on the curve B_n.

I will now prove that if the 12th powers of the *nil-facta* in the tangential equations of 43 points are connected by a linear syzygy, the 43 points are on a quartic curve. We can draw a quartic B_4 through 14 of the points; operate with B_4 on the syzygy, then these 14 points are cleared away, and there remains a syzygy between the 8th powers of the remaining 29 points. We have therefore now to prove that these 29 points are on a quartic. Draw a curve C_4 through 14 of them, and operate on the new syzygy with C_4. This clears away 14 more points, and we are left with a syzygy among the 4th powers of 15 points. But then by Serret's theorem these lie on a quartic. Hence, *any* 15 of the original 43 points are on the same quartic; therefore all the 43 are on the same quartic.

To prove that if the cubes of 13 points in space are connected by a syzygy they lie on a quadric surface, operate with the plane through three of them; we are then left with a syzygy among the squares of 10 points, and Serret's theorem again applies.

The application of this method to the remaining cases will now be easy.

*XV.

ON SYZYGETIC RELATIONS CONNECTING THE POWERS OF LINEAR QUANTICS.

I THINK the first treatment of this subject is to be found in some very interesting articles of M. Paul Serret's; *Nouvelles Annales*, t. IV. (1865), pp. 145, 193, and 433. M. Serret's attention was confined to the *squares* of linear quantics; and in regard to these he establishes such propositions as the following:—If the squares of the characteristics of the equations of six lines satisfy a syzygetic relation, the six lines touch a conic section. That is to say, if $P_1=0$, $P_2=0$, ... $P_6=0$ are the equations of the lines, and if we have an identical relation

$$\lambda_1P_1^2+\lambda_2P_2^2+\lambda_3P_3^2+\lambda_4P_4^2+\lambda_5P_5^2+\lambda_6P_6^2\equiv 0 ;$$

or, as he finds it convenient to write

$$\Sigma_1^6\lambda P^2\equiv 0$$

where the λs are numerical coefficients, then the lines P_1 P_2 ... P_6 touch the same conic. Another of his propositions is that if eight planes P_1, P_2, ... P_8 satisfy an identical relation

$$\Sigma_1^8\lambda P^2\equiv 0,$$

then the eight planes are such that any quadric surface touching seven of them touches also the eighth. These propositions are arrived at by a somewhat circuitous path, though the steps severally are elegant. From the latter M. Serret obtains a very beautiful and immediate proof of Hesse's theorem that two tetrahedra self-conjugate to the same quadric are such that every quadric touching seven of their faces touches also the eighth. Namely, the equation of the first quadric may be written in either of the forms

$$\lambda_1P_1^2+\lambda_2P_2^2+\lambda_3P_3^2+\lambda_4P_4^2=0,\ \lambda_5P_5^2+\lambda_6P_6^2+\lambda_7P_7^2+\lambda_8P_8^2=0,$$

and these two being identical to a factor *près*, we have a syzygetic relation among the eight squares, from which by the second of the above propositions the theorem in question at once follows.

Having, by an application of Prof. Sylvester's most powerful method of contravariant differentiation, succeeded in extending these propositions to higher powers of linear quantics, and to curves and surfaces of any order, I found as a particular result that two quadrilaterals of the same system totally inscribed in a cubic are such that every curve of the third class touching seven of their sides touches also the eighth. Doubtful of this proposition, I communicated it to the Mathematical Society, and was subsequently informed by Mr Cotterill that the eight lines in question touch the same conic. This is equivalent to the analytic theorem, "if the cubes of eight linear quantics are syzygetic, the squares of any six of them are syzygetic." The proof of this by contravariant differentiation and the statement of a series of analogous propositions occupy the following notes.

I.

If in the tangential equation of a curve

$$c_p \equiv (\xi, \eta, \zeta)^p = 0$$

we write

$$\frac{d}{dx},\ \frac{d}{dy},\ \frac{d}{dz} \text{ for } \xi, \eta, \zeta,$$

and operate upon

$$(lx + my + nz)^p,$$

we shall get $\underline{|p}$ multiplied by the result of substituting l, m, n for ξ, η, ζ in c_p; that is to say

$$\left(\frac{d}{dx},\ \frac{d}{dy},\ \frac{d}{dz}\right)^p . (lx + my + nz)^p = \underline{|p} . (l, m, n)^p.$$

For shortness, denote $(lx + my + nz)$ by Q, and let c_p mean also the differential operator

$$\left(\frac{d}{dx},\ \frac{d}{dy},\ \frac{d}{dz}\right)^p.$$

Then *if the operator* c_p *reduces* Q^p *to zero, the line* Q *touches the curve* c_p, *and conversely.*

Suppose now that there are several lines $Q_{(1)}$, $Q_{(2)}$, ... and that there is an identical relation,

$$\Sigma\lambda Q^p \equiv 0,$$

connecting the p^{th} powers of the quantities Q. Let also c_p be a curve touching all but one of the lines, so that the operator c_p reduces to zero all but one of the quantities Q^p. The expression $\Sigma\lambda Q^p$ being identically zero, the result of operating upon it with c_p must be zero, or we have

$$\Sigma\lambda c_p Q^p = 0.$$

But of the terms $\lambda_1 c_p Q_{(1)}{}^p$, $\lambda_2 c_p Q_{(2)}{}^p$, ... we know that all vanish but one; it follows that this last one also vanishes, or the curve c_p touches the remaining line. We may therefore enunciate the proposition:—*If there are* n *lines* $Q_1, Q_2, \dots Q_n = 0$, *and if there is an identical relation*

$$\Sigma_1^n \lambda Q^p \equiv 0,$$

then every curve c_p *of class* p *which touches* $n-1$ *of the lines will also touch the* n^{th}.

It will be sufficient merely to enunciate the obviously corresponding proposition in three dimensions:—

If there are n *planes* $Q_1, Q_2, \dots Q_n = 0$, *and if there is an identical relation*

$$\Sigma_1^n \lambda Q^p = 0,$$

then every surface c_p *of class* p *which touches* $n-1$ *of the planes will also touch the* n^{th}.

In solid geometry, however, as usual, the analogy branches off into two distinct directions, and we are led to consider a somewhat different theory.

Let the number of straight lines which can be drawn through a fixed point and in a fixed plane to touch a given surface be called the *rank* of the surface (viz. this is both the class of a general plane section and the order of a general tangent cone), then that relation between the six coordinates of a line which expresses that the line touches the surface will be of a degree equal to the rank of the surface. I shall denote the expression equated to zero in this equation by a Greek letter whose suffix indicates the rank; thus, for example, $\beta_2 = 0$ is the rank-equation* of a quadric surface.

The six coordinates being a, b, c, f, g, h, where $af + bg + ch = 0$, it is very easy to prove that if in β_n an expression of the n^{th} degree in these coordinates, we substitute for a, b, c, f, g, h respectively

$$\frac{d}{df}, \frac{d}{dg}, \frac{d}{dh}, \frac{d}{da}, \frac{d}{db}, \frac{d}{dc},$$

we shall get an invariant symbol of operation. I shall use β_n to mean not only the function of the coordinates, but also this operator obtained by the substitution just defined. This being so, if we call the condition that the line $(abcfgh)$ shall meet a given line σ or $(lmnpqr)$, the *equation* of the line σ (namely the equation is

$$\sigma \equiv pa + qb + rc + lf + mg + nh = 0),$$

then *the condition that σ shall touch the surface β_n is*

$$\beta_n \sigma^n = 0.$$

From this it follows that

If there are n *straight lines $\sigma_{(1)}, \sigma_{(2)}, \ldots \sigma_{(n)}$, and if there is an identical relation*

$$\Sigma_1^n \, l\sigma^p = 0,$$

then every surface β_p of rank p *which touches* n − 1 *of the lines will also touch the* nth.

II.

At this point I digress somewhat to consider the interpretation of what I have elsewhere called the harmonic invariant of two curves or surfaces, the order of one being equal to the class of the other. First, in the case of two conics, the point-equation of the first being $B_2 = 0$, and the line-equation of the second $c_2 = 0$, the harmonic invariant is $c_2 B_2$, which is commonly called the invariant Θ. Suppose that this vanishes; then if B_2 can be written in the form

$$X^2 + Y^2 + Z^2$$

* [The expression "line-equation" would have been the more natural one, but a confusion might arise between this line-equation of a surface, and the line-equation of a plane curve.—C.]

(so that the lines XYZ form a self-conjugate triangle), since the operator c_2 reduces this to zero we see that if the conic c_2 touch two of the lines it must touch also the third. Similarly, if B_2 can be written in the form

$$X^2 + Y^2 + Z^2 + W^2$$

(so that $XYZW$ form a self-conjugate quadrilateral), if c_2 touches three of the lines it must touch also the fourth. Hence $c_2 B_2 = 0$ is the condition both (1) that c_2 shall be inscribed in an infinite number of *triangles* self-conjugate to B_2, and (2) that c_2 shall be inscribed in an infinite number of *quadrilaterals* self-conjugate to B_2. These are known interpretations; the latter, given first I think by Dr Salmon under a slightly different form (*Conics*, [§ 375]), was reduced to this more simple and natural shape by Professor Cremona (*Educational Times*, Reprint)*. Next, in the case of two quadric surfaces $c_2 = 0$ and $B_2 = 0$, if $c_2 B_2 = 0$, and B_2 can be expressed in either of the forms

$$\Sigma_1^4 . X^2, \ \Sigma_1^5 . X^2, \ \Sigma_1^6 . X^2,$$

we see at once that if c_2 touch all but one of the planes X it must touch also that other. Hence $c_2 B_2$ is the condition that c_2 shall be inscribed (1) in an infinity of *tetrahedra* self-conjugate to B_2, (2) in an infinity of *pentahedra* self-conjugate to B_2, (3) in an infinity of *hexahedra* self-conjugate to B_2. The terms *conjugate hexahedron*, *conjugate pentahedron*, are introduced by M. Serret, and seem likely to be of considerable use.

Passing now to the rank-equations of the two quadrics, which I shall write $\beta_2 = 0$, $\gamma_2 = 0$, I observe that if β_2 can be thrown into the form $\Sigma_1^6 \sigma^2$,
then the 6 lines σ are such that each is conjugate to all the rest; or the lines are the edges of a self-conjugate tetrahedron. If then the harmonic invariant $\gamma_2 \beta_2$ (Dr Salmon's invariant T) vanishes, and γ_2 touches five of these lines, it will touch the sixth; or γ_2 can touch the edges of an infinite number of tetrahedra self-conjugate to β. We have not yet studied the properties relative to β_2 of a system of lines such that β_2 may be expressed in the form $\Sigma_1^p \sigma^2$ when p is greater than 6; yet it is

* [Cf. Vol. IV. p. 109; Vol. IX. pp. 62, 74.]

obvious that such systems will give new interpretations of the invariant T.

The general extension of this method of interpretation is now perfectly easy. *If* B_n *is a curve of* n^{th} *order harmonic of* c_n *a curve of* n^{th} *class, and if* B_n *can be written in the form* $\Sigma_1^p . X^n$, *then if* c_n *touch* $p-1$ *of the lines* X *it will touch also the* p^{th}. Similarly for surfaces in regard to lines and planes. The converse proposition is

The curve or surface $\Sigma_1^p X^n = 0$ *is harmonic of every curve or surface* c_n *of the* n^{th} *class which touches all the lines or planes* X.

I forbear to state the correlative propositions in which lines and planes are replaced by points.

III.

Let us return to the original question.

If there is an identical relation

$$\Sigma_1^8 \lambda P^3 \equiv 0$$

between the cubes of the 8 *linear quantics* P, *there shall also be an identical relation between the squares of any* 6 *of them.*

For we know that we can find a linear differential operator which shall reduce any two of the quantics to zero; namely, let h be the point of intersection of the lines $P_{(7)}$, $P_{(8)}$, having for coordinates a, b, c, then the operator

$$a\frac{d}{dx} + b\frac{d}{dy} + c\frac{d}{dz}$$

being also denoted by h, we have $hP_{(7)} = 0$, $hP_{(8)} = 0$; and so if we operate with h on the given syzygy we obtain therefrom

$$3\Sigma_1^6 \lambda . hP . P^2 \equiv 0,$$

a syzygy connecting the squares of the quantics $P_{(1)}, P_{(2)} \dots P_{(6)}$.

If there is an identical relation

$$\Sigma_1^{11} \lambda P^4 = 0$$

between the fourth powers of the 11 *linear quantics* P, *there shall also be an identical relation between the squares of any* 6 *of them.*

Let c_2 be the conic section touching the five lines $P_{(7)} \dots P_{(11)}$; then $c_2 P_{(7)}^2 = 0, \dots c_2 P_{(11)}^2 = 0$. If then we operate with c_2 on the given syzygy we shall obtain

$$12\Sigma_1^6 \lambda . c^2P^2 . P^2 \equiv 0,$$

a syzygy connecting the squares of the quantics $P_{(1)}, \dots P_{(6)}$; provided all the $c_2 P^2$ do not vanish. If however all these vanish, all the lines touch the conic c_2, and there is again a syzygy connecting the squares of any 6 of them. Using these two demonstrations as samples, we are enabled to construct the following table.

Powers $p=$	$n=2$	3	4	5	6	7	8	9	10	11
2	6									
3	8	10								
4	11	12	15							
5	13	15	17	21						
6	16	19	20	23	28					
7	18	21	24	26	30	36				
8	21	24	29	30	33	38	45			
9	23	28	31	35	37	41	47	55		
10	26	30	34	41	42	45	50	57	66	
11	28	33	38	42	48	50	54	60	68	78

*

* [This is in effect the Table A of paper XIV., where it is explained that the number in the Table is

$$N = \frac{\alpha}{2} n(n+3) + \frac{1}{2}(\beta+1)(\beta+2),$$

where α is the quotient and β the remainder of the division of p by n, so that $p = \alpha n + \beta$, and $\beta < n$.

It may be remarked that so long as p is not greater than n, that is down to the bar in each column of the Table, the Number

$$= \tfrac{1}{2} n(n+3) + \tfrac{1}{2}(p-n+2)(p-n+1),$$

and that the several columns are then continued as follows:

Col. $n=2$.	$n=3$.	&c.
6		
8	10	
$11 = 6+5$	12	
$13 = 8+5$	15	
$16 = 11+5$	$19 = 10+9$	
$18 = 13+5$	$21 = 12+9$	
$21 = 16+5$	$24 = 15+9$	
&c.	&c.	

It is easy to see that this is in fact an equivalent construction of the table. C.]

*XVI.

[ON THE THEORY OF DISTANCES.]

[PRELIMINARY*.]

I EXPLAIN in the first place the notation employed, which is an extension of the Geometric Analysis of GRASSMANN, explained by him in the "Ausdehnungslehre" and in Crelle's Journal, and founded in part on a remark of Leibnitz.

GRASSMANN employs single large letters, as A, B, C, to represent straight lines in a plane, and single small letters, as a, b, c, to represent points. When two large letters come together, as AB, the notation is taken to mean the point of intersection of the lines A, B. So when two small letters come together, as ab, the notation is taken to mean the line joining the points a, b. The equation $ABC=0$ means that the lines A, B, C meet in a point; the equation $abc=0$, that the points a, b, c lie in a line; and the equation aB or $Ba=0$, that the point a lies on the line B. No signification is given to the separated symbols ABC, abc, aB, except as equated to zero. The main principle of the application of this method is that the order of an equation in any letter contained is measured by the number of times that letter occurs; a remark which will be further explained in the sequel.

I now explain the extensions of this notation which I have found it convenient to make.

* [The *Preliminary* matter forms the substance of notes given to Prof. Henrici by the Author at the British Association Meeting in the year 1869. The paper itself (pp. 134—157), without a title, appears to have been written subsequently. The title I have given to the two communications has been taken from that of a paper Prof. Clifford read at the above meeting, an abstract of which is given below (p. 164). I have employed ι to represent $\sqrt{-1}$.]

A curve may be given by its points, or by its tangents; that is to say, we may know its equation in point-coordinates (x,y,z), the degree of the equation being the *order* of the curve; or we may know its equation in the contravariant, tangential, or line-coordinates (ξ, η, ζ), the degree of the equation being then the *class* of the curve. This being so, I denote by a large letter a quantic in (x,y,z), and I write the order of the quantic in the form of a suffix. Thus C_3 denotes a cubic in (x,y,z), and $C_3=0$ is the equation to a curve of the third order. Next, I use a small letter with a suffix to denote a quantic in (ξ, η, ζ), the order of the quantic being denoted by the suffix. Thus b_3 denotes a cubic in (ξ, η, ζ), and $b_3=0$ is the equation to a curve of the third class. When there is no suffix, the suffix 1 is to be understood; in this respect the notation coincides with that of GRASSMANN.

When two large letters come together, each is raised to the power denoted by the suffix of the other, as $A_m{}^n B_n{}^m$. The symbol then denotes a quantic which, equated to zero, gives the equation in tangential coordinates of the mn intersections of the curves A_m, B_n. Similarly $a_m{}^n b_n{}^m=0$ is the equation of the mn common tangents of the curves a_m, b_n. The reason of the indices is now apparent; such equation being of the degree n in the coefficients of the first curve, and of the degree m in those of the second.

When three large letters or three small letters come together, each is raised to a power denoted by the product of the suffixes of the other two; the symbol then denotes the resultant of the three quantics.

When a small letter comes before a large one, as $b_m C_n$, the notation is taken to mean the result of changing (ξ, η, ζ) in b_m into $(\partial_x, \partial_y, \partial_z)$ and performing the operation thus indicated on C_n. So, finally, when a large letter comes before a small one, as $B_m c_n$, the notation is taken to mean the result of changing (x, y, z) in B_m into $(\partial_\xi, \partial_\eta, \partial_\zeta)$ and performing the operation thus indicated on c_n.

In the use of these symbols to investigate the relations of geometrical magnitudes, it is to be observed that the absolute values of (xyz) or $(\xi\eta\zeta)$ or the coefficients of a quantic are not given, but only their ratios; and consequently that the symbols defined above can have no special value but zero. If however we form a fraction such that every letter mentioned occurs an equal number of times in the numerator and denominator, this *will* have a definite numerical value, being a function of the known ratios aforesaid; and may accordingly represent a geometrical magnitude. This theory of *characteristics* is due to Prof. SYLVESTER.

It is further to be observed that the metric properties of figures in plane geometry depend upon the circular points at infinity, which I denote by i, j; and those of figures in spherical geometry upon the imaginary circle at infinity, which I denote by O_2 or o_2 according as it is given by points or tangents. The points i, j in the one case, and the circle O_2 in the other, have received the name of "the Absolute" from Prof. CAYLEY, to whom this theory is due *.

FORMULÆ FOR A PLANE CONIC.

Expressions are obtained below for the *distance* of a point from a conic given tangentially, and for the *distance* of a line from a conic given by its points. Two different geometrical definitions are obtained for each of these; their ratio is a quantity which I have called the *distance of the curve from the absolute.*

The asymptotes of the conic are denoted by P, Q; a pair of foci, viz. either the two real or the two imaginary foci, are denoted by p, q; the conic is called C_2 or c_2.

DISTANCE OF THE POINT a FROM c_2.

Let any straight line B be drawn through the point a, meeting the conic in l, m; let also the tangents from a to the conic be L, M.

* [$\overline{ab}$ = distance between points a and b, ab = line joining a and b.]

First Dist. $a, c_2 = \sin^2 LM . ap^2 . aq^2$

$$= \frac{a^2C_2 . (aij)^2}{\overline{ai}^2c_2 . \overline{aj}^2c_2} \cdot \frac{\overline{ai}^2c_2 . \overline{aj}^2c_2}{(aij)^4 . (\overline{ij}^2c_2)^2} = \frac{a^2C_2}{(aij)^2 . (\overline{ij}^2c_2)^2}.$$

Second Dist. $a, c_2 = al . am . \sin BP . \sin BQ$

$$= \frac{a^2C_2 . (Bi . Bj)}{(aij)^2 . (B\overline{ij})^2C_2} \cdot \frac{(B\overline{ij})^2C_2}{(Bi . Bj) . \sqrt{(i^2C_2 . j^2C_2)}}$$

$$= \frac{a^2C_2}{(aij)^2 . \sqrt{(i^2C_2 . j^2C_2)}}.$$

The ratio of these two is

$$\frac{\sin^2 LM . ap^2 . aq^2}{al . am . \sin BP . \sin BQ} = \frac{\sqrt{(i^2C_2 . j^2C_2)}}{(\overline{ij}^2c_2)^2} = pq^2.$$

DISTANCE OF THE LINE A FROM C_2.

Let any point b be taken on the line A, the tangents from b to the conic being LM; also let A meet the conic in the points l, m.

First Dist. $A, C_2 = lm^2 . \sin^2 AP . \sin^2 AQ$

$$= \frac{A^2c_2 . Ai . Aj}{(A\overline{ij})^2C_2} \cdot \frac{(A\overline{ij})^2C_2}{(Ai . Aj)^2 . i^2C_2 . j^2C_2}$$

$$= \frac{A^2c_2}{Ai . Aj . i^2C_2 . j^2C_2}.$$

Second Dist. $A, C_2 = \sin AL \sin AM . bp . bq$

$$= \frac{A^2c_2 . (bij)^2}{Ai . Aj \sqrt{(\overline{bi}^2c_2 . \overline{bj}^2c_2)}} \cdot \frac{\sqrt{(\overline{bi}^2c_2 . \overline{bj}^2c_2)}}{(bij)^2 . \overline{ij}^2c_2}$$

$$= \frac{A^2c_2}{Ai . Aj . \overline{ij}^2c_2}.$$

The ratio of these two is

$$\frac{lm^2 . \sin^2 AP . \sin^2 AQ}{\sin AL \sin AM . bp . bq} = \frac{\overline{ij}^2c_2}{i^2C_2 . j^2C_2} = \sin^2 PQ.$$

In the case of a sphero-conic we obtain analogous expressions for the distance of a point or line (great circle) from the conic, but the value depends on the pair of foci or cyclic planes selected; the ratio of such different values is however the same for all points and lines. Moreover, the ratio of the two distances of a point or of a line is a quantity independent of the point or line, but I have as yet obtained no geometrical definition of it. For this reason I have not treated separately the formulæ for a sphero-conic, which are of course like the preceding included in the general formulæ of curves*.

I.

All magnitudes which are concerned in plane geometry may be expressed in terms of three, which on this account are of primary importance. These are the distance of two points, the distance of a point from a line, and the sine of the angle between two lines. We obtain the most simple expressions for these magnitudes by employing rectangular Cartesian coordinates for the points, and the coordinates of Dr Booth for the lines; but it is convenient from the first to make these expressions homogeneous by the introduction of a third coordinate which may be put $=1$ or -1 in the two cases respectively. Thus if a_1, a_2, a_3 are the coordinates of the point a, $\frac{a_2}{a_3}$ and $\frac{a_1}{a_3}$ are its distances from the axes; and if A_1, A_2, A_3 are the coordinates of a line A, $-\frac{A_3}{A_1}$ and $-\frac{A_3}{A_2}$ are the intercepts it cuts off from the axes. This being so, the expressions for our primary magnitudes are

$$\text{Dist. } ab = \frac{\sqrt{\{(a_1b_3 - a_3b_1)^2 + (a_2b_3 - a_3b_2)^2\}}}{a_3b_3},$$

$$\text{Dist. } aB = \frac{a_1B_1 + a_2B_2 + a_3B_3}{a_3\sqrt{(B_1^2 + B_2^2)}},$$

$$\sin AB = \frac{A_1B_2 - A_2B_1}{\sqrt{(A_1^2 + A_2^2)}\sqrt{(B_1^2 + B_2^2)}},$$

* [Cf. however V., p. 152 *infra*, of this paper.]

and these are clearly not invariant in regard to the points and lines. Let us now ask what is the locus of the points b which are at zero distance from a. We find that it consists of the two straight lines

$$a_3b_1 + \iota a_3b_2 - (a_1 + \iota a_2)\, b_3 = 0,$$
$$a_3b_1 - \iota a_3b_2 - (a_1 - \iota a_2)\, b_3 = 0.$$

Each of these lines passes through the point a; thus we learn that *the lines of null-length are straight lines, and two of them pass through every point of the plane.*

If the point a moves about, each of the lines of null-length remains parallel to the same direction, or, which is the same thing, passes through a fixed point at an infinite distance. Let these two points be called i, j; their coordinates may be taken to be

$$i_1 : i_2 : i_3 = \frac{1}{\sqrt{2}} : \frac{\iota}{\sqrt{2}} : 0,$$

$$j_1 : j_2 : j_3 = \frac{\iota}{\sqrt{2}} : \frac{1}{\sqrt{2}} : 0,$$

so that the line $i\,j$ has coordinates (0, 0, 1). Then we have

$$abi = \frac{1}{\sqrt{2}} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 1 & \iota & 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \{(a_2b_3 - a_3b_2) - \iota\,(a_1b_3 - a_3b_1)\},$$

$$abj = \frac{1}{\sqrt{2}} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \iota & 1 & 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \{\iota\,(a_2b_3 - a_3b_2) - (a_1b_3 - a_3b_1)\},$$

$$aij = a_3, \quad bij = b_3,$$

$$abi \,.\, abj = \frac{\iota}{2} \cdot \{(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2\}$$

and therefore

$$\kappa \,.\, \text{Dist. } ab = \frac{\sqrt{(abi \,.\, abj)}}{aij \,.\, bij}, \quad \text{where } \kappa^2 = \frac{\iota}{2}.$$

Thus we learn that *all the lines of null-length pass through one or other of the points i, j*; and *the distance ab may be expressed in terms of the invariants of a, b, i, j* to a numerical factor *près*. This factor κ is the same for all distances, depending only upon the absolute value given to the coordinates of i, j. To reduce κ to the value unity, we have only to multiply these coordinates throughout by $\kappa^{-\frac{1}{2}}$.

Similar expressions may now be found for the other two primary magnitudes. We have in fact

$$Ai = \frac{1}{\sqrt{2}}(A_1 + \iota A_2), \quad Aj = \frac{1}{\sqrt{2}}(\iota . A_1 + A_2),$$

$$Ai . Aj = \frac{\iota}{2} . (A_1^2 + A_2^2),$$

and thence

$$\left[\frac{1}{\kappa}\right] \text{Dist. } aB = \frac{aB}{a\overline{ij} . \sqrt{(Bi . Bj)}}$$

$$-2\iota . \sin AB = \frac{AB\overline{ij}}{\sqrt{(Ai . Aj . Bi . Bj)}}.$$

(The last expression may be simplified as follows:

We have

$$AB\overline{ij} = Ai . Bj - Aj . Bi$$

by a well-known theorem of determinants; but also

$$Ai . Bj + Aj . Bi$$
$$= \tfrac{1}{2}\{(A_1 + \iota A_2)(\iota B_1 + B_2) + (\iota A_1 + A_2)(B_1 + \iota B_2)\}$$
$$= \iota (A_1 B_1 + A_2 B_2)$$

and therefore

$$2 \cos AB = 2 \frac{A_1 B_1 + A_2 B_2}{\sqrt{(A_1^2 + A_2^2)}\sqrt{(B_1^2 + B_2^2)}} = \frac{Ai . Bj + Aj . Bi}{\sqrt{(Ai . Aj . Bi . Bj)}}.$$

If therefore we write θ for the angle AB, we have

$$\cos\theta + \iota \sin\theta = \epsilon^{\theta\iota} = \frac{Aj . Bi}{\sqrt{(Ai . Aj . Bi . Bj)}}$$

$$\cos\theta - \iota \sin\theta = \epsilon^{-\theta\iota} = \frac{Ai . Bj}{\sqrt{(Ai . Aj . Bi . Bj)}},$$

and thence

$$\epsilon^{2\theta\iota} = \frac{Aj . Bi}{Ai . Bj}.$$

This last is an anharmonic ratio in which the lines AB divide the segment $\overline{ij}$, and so is an absolute invariant.)

By analogy and for convenience of expression we shall call the sine of the angle between two lines the *distance* of the lines; we may then derive from these formulæ the following theorems:—

If a line pass through either of the points i, j, it is at an infinite distance from all other lines and points, and the distance between any two points on it is zero.

If a point lie upon the line $\overline{ij}$, it is at an infinite distance from all other lines and points, and the distance between any two lines through it is zero.

II.

A conic may be given as a curve of the second order C_2 or as a curve of the second class c_2; but either of these expressions may be derived from the other by means of the formula

$$\begin{vmatrix} axC_2, & bxC_2 \\ ayC_2, & byC_2 \end{vmatrix} = 2\overline{ab}\,\overline{xy}\,c_2$$

and its reciprocal

$$\begin{vmatrix} AXc_2, & BXc_2 \\ AYc_2, & BYc_2 \end{vmatrix} = 2\overline{AB}\,\overline{XY}\,C_2.$$

Namely, if we make a, b identical with x, y, and A, B with X, Y, we have

$$2\overline{xy}^2 c_2 = x^2 C_2 . y^2 C_2 - (xyC_2)^2$$

and

$$2\overline{XY}^2 C_2 = X^2 c_2 . Y^2 c_2 - (XYc_2)^2.$$

The discriminant $C_2 c_2$ may be given in a similar manner. Namely, we have

$$3\overline{xC_2}\;\overline{yC_2}\;\overline{zC_2} = 8xyz . C_2 c_2.$$

The relations of a conic to the points ij are most conveniently expressed in terms of the asymptotes and the foci. The asymptotes P, Q are tangents at the points where the conic is met by the line ij. The equation to the pair of tangents at the points where a line X meets the conic C_2 must be of the form

$$C_2 - \lambda X^2 = 0.$$

Equating to zero the discriminant of this, we find that

$$C_2c_2 - 3\lambda X^2c_2 = 0,$$

whence the equation to the tangents is

$$3C_2 . X^2c_2 - X^2 . C_2c_2 = 0.$$

Substituting herein for X the line ij, we obtain for the asymptotes the expression

$$P . Q = 3C_2 . \overline{ij}^2c_2 - \overline{ij}^2 . C_2c_2.$$

Thus the product $P . Q$ is of the third order in the coefficients of the conic, whether it is given by its point- or by its line-equation.

To determine the angle between the asymptotes, we observe that the point PQ is the centre of the conic, which is the pole of the line ij, and is therefore represented by $\overline{ij}c_2$. Consequently $PQij$ must be proportional to some power of $\overline{ij}^2c_2$. If we regard C_2 as initially given, we shall in fact have

$$(PQij)^2 = \kappa (\overline{ij}^2c_2)^3,$$

each side being of the sixth order in the coefficients. If however c_2 is initially given, the formula becomes

$$(PQij)^2 = \kappa' (\overline{ij}^2c_2)^3 . C_2c_2.$$

Next we find by direct substitution

$$Pi . Qi = 3i^2C_2 . \overline{ij}^2c_2,$$
$$Pj . Qj = 3j^2C_2 . \overline{ij}^2c_2,$$

and therefore

$$\sin^2 PQ = \frac{\overline{ij}^2c_2}{i^2C_2 . j^2C_2} \text{ to a factor } près.$$

But we have also

$$\overline{ij}^2c_2 = i^2C_2 . j^2C_2 - (ijC_2)^2;$$

now it is clear that ijC_2 vanishes when the points ij are conjugate in regard to the conic, or when the asymptotes are at right angles, that is, when $\sin PQ = 1$. It follows therefore that the factor in the last equation is unity, and that

$$\cos PQ = \frac{ijC_2}{\sqrt{(i^2C_2 . j^2C_2)}}.$$

We find in this way that the κ of our formula is $= -36$, so that we may write

$$(PQij)^2 = -36 (\overline{ij}^2c_2)^3.$$

This constant might also have been determined by means of the particular case in which the conic breaks up into two straight lines, in which case these lines are themselves the asymptotes. If $C_2 = X . Y$, $-4c_2 = \overline{XY}^2$, and $-4\overline{ij}^2c_2 = \overline{XYij}^2$; by these formulæ the values just obtained may be compared with the known values of $\sin XY$, $\cos XY$.

If the conic c_2 touches the four lines joining p, q to i, j, we must have

$$\kappa p . q = c_2 + \lambda . i . j.$$

But the equation

$$0 = \text{Disct.} (c_2 + \lambda . i . j) = C_2c_2 + \tfrac{3\lambda}{2} ijC_2 - \tfrac{3}{4}\lambda^2\overline{ij}^2c_2$$

gives two values for λ, one belonging to $p . q$ and the other to $p' . q'$. Hence, eliminating λ, we may write

$$p . q . p' . q' = C_2c_2 . i^2 . j^2 - \tfrac{3}{2} ijC_2 . i . j . c_2 - \tfrac{3}{4}\overline{ij}^2c_2 . c_2^2,$$

a result of the third order in the coefficients of c_2.

If the conic touch the line ij, that is, if $\overline{ij}^2c_2 = 0$, one of the foci, say q, coincides with the point of contact, and two others, p' and q', coincide with the points i and j respectively; we have therefore in this case $qij = 0$, $p'q'i = 0$, $p'q'j = 0$. If the conic pass through the point i, which happens when $i^2C_2 = 0$, the foci coincide two and two, p with q', suppose, and q with p'; we have then $pqi = 0$, $p'q'i = 0$. Hence the product

$$pqi . pqj . p'q'i . p'q'j$$

will vanish in three cases; (1) when $i^2C_2 = 0$, (2) when $j^2C_2 = 0$,

(3) when $\overline{\ddot{ij}}^2 c_2 = 0$; and it is easy to see that it cannot vanish in any other case. Consequently we must have

$$pq\dot{i} \,.\, pq\dot{j} \,.\, p'q'\dot{i} \,.\, p'q'\dot{j} = \kappa\,(\dot{i}^2 C_2 \,.\, \dot{j}^2 C_2)^x\,(\overline{\ddot{ij}}^2 c_2)^y,$$

where a comparison of dimensions gives us the equations

$$4x + y = 6,\quad 2x + 2y = 6,\quad x = 1,\quad y = 2.$$

We have also by direct substitution

$$p\ddot{ij} \,.\, q\ddot{ij} \,.\, p'\ddot{ij} \,.\, q'\ddot{ij} = \overline{\ddot{ij}}^4\,(p \,.\, q \,.\, p' \,.\, q') = -\tfrac{3}{4}\,(\overline{\ddot{ij}}^2 c_2)^3,$$

and hence the following expression for the product of the squares of the distances pq, $p'q'$,

$$\overline{pq}^2 \,.\, \overline{p'q'}^2 = \kappa'\,\frac{(\dot{i}^2 C_2 \,.\, \dot{j}^2 C_2)^x\,(\overline{\ddot{ij}}^2 c_2)^y}{(\overline{\ddot{ij}}^2 c_2)^6}.$$

Since $\overline{pq}^2 + \overline{p'q'}^2 = 0$, this product is the same thing as $-pq^4$. But we get another equation between x and y by supposing the conic c_2 to break up into two points u, v, which are then themselves the foci. In that case $-4C_2 = \overline{uv}^2$, $\overline{\ddot{ij}}^2 c_2 = u\ddot{ij} \,.\, v\ddot{ij}$, and the expression for the fourth power of the distance is

$$\overline{uv}^4 = \kappa^4\,\frac{(uv\dot{i} \,.\, uv\dot{j})^2}{(u\ddot{ij} \,.\, v\ddot{ij})^4} = \frac{\dot{i}^2 C_2 \,.\, \dot{j}^2 C_2}{(\overline{\ddot{ij}}^2 c_2)^4},$$

whence $x = 1$, $y = 2$, and the general formula is

$$\overline{pq}^2 = \frac{\sqrt{(\dot{i}^2 C_2 \,.\, \dot{j}^2 C_2)}}{(\overline{\ddot{ij}}^2 c_2)^2}.$$

In this it is clear that c_2 is primarily given; if C_2 is given, the process of calculating it back from the coefficients of c_2 introduces the factor $C_2 c_2$, and we have

$$\overline{pq}^2 = \frac{\sqrt{(\dot{i}^2 C_2 \,.\, \dot{j}^2 C_2)}\;C_2 c_2}{(\overline{\ddot{ij}}^2 c_2)^2}.$$

With the aid of this formula and the angle between the asymptotes we may now determine the axes of the conic.

If h and k be the axes, we have

$$\overline{pq}^2 = h^2 - k^2,$$

$$\cos(PQ) = \frac{h^2 + k^2}{h^2 - k^2},$$

and therefore

$$h^2 + k^2 = \overline{pq}^2 \cos PQ = \frac{\sqrt{(\dot{i}^2 C_2 \,.\, \dot{j}^2 C_2)} \,.\, C_2 c_2}{(\overline{\ddot{ij}}^2 c_2)^2} \cdot \frac{\ddot{ij} C_2}{\sqrt{(\dot{i}^2 C_2 \,.\, \dot{j}^2 C_2)}}$$

$$= \frac{\ddot{ij} C_2 \,.\, C_2 c_2}{(\overline{\ddot{ij}}^2 c_2)^2}.$$

Again we have

$$\sin^2 PQ = 1 - \left(\frac{h^2 + k^2}{h^2 - k^2}\right)^2 = -\frac{4h^2k^2}{(h^2 - k^2)^2},$$

and therefore

$$-4h^2k^2 = \overline{pq}^4 \sin^2 PQ$$

$$= \frac{\dot{i}^2 C_2 \,.\, \dot{j}^2 C_2 \,.\, (C_2 c_2)^2}{(\overline{\ddot{ij}}^2 c_2)^4} \cdot \frac{\overline{\ddot{ij}}^2 c_2}{\dot{i}^2 C_2 \,.\, \dot{j}^2 C_2}$$

$$= \frac{(C_2 c_2)^2}{(\overline{\ddot{ij}}^2 c_2)^3}.$$

These last formulæ are of course the well-known ones.

III.

We go on to consider the relations between a point and a conic; and in particular to determine the angle between the tangents from the point to the conic (fig. 12).

The tangents LM from x to the conic C_2 are given by the equation

$$LM = x^2 C_2 \,.\, C_2 - (xC_2)^2,$$

we have also

$$-4\sin^2 LM = \frac{(LM\ddot{ij})^2}{L\dot{i} \,.\, L\dot{j} \,.\, M\dot{i} \,.\, M\dot{j}}.$$

The numerator of this fraction vanishes when the intersection of L and M is on the line $\ddot{ij}$. If these tangents are distinct, the intersection is x; if they coincide, that is, if x is on the conic, or if the conic breaks up, the intersection is indeterminate. Hence $(LM\ddot{ij})^2$ must vanish whenever $x\ddot{ij}$ or $x^2 C_2$, or $C_2 c_2$, vanishes; and in no other case. But $(LM\ddot{ij})^2$ is of the fourth order in the coefficients of x and of the conic; therefore

$$(LM\ddot{ij})^2 = \kappa x^2 C_2 \,.\, (x\ddot{ij})^2 \,.\, C_2 c_2.$$

The denominator may be expressed by direct operation on LM; we should find in fact

$$Li\,.\,Lj\,.\,Mi\,.\,Mj = \{x^2C_2\,.\,i^2C_2 - (xiC_2)^2\}\,\{x^2C_2\,.\,j^2C_2 - (xjC_2)^2\}$$
$$= \overline{xi}^2c_2\,.\,\overline{xj}^2c_2,$$

and thus finally

$$\sin^2 LM = \kappa'\,\frac{x^2C_2\,.\,(xij)^2\,.\,C_2c_2}{\overline{xi}^2c_2\,.\,\overline{xj}^2c_2}.$$

But the denominator may be put into another form which is more useful. Let us assume that the absolute values of the coefficients of p, q, p', q' are so chosen that $pi = p'i$, $qi = q'i$, $pj = q'j$, $qj = p'j$. We have by direct substitution

$$pix\,.\,qix\,.\,p'ix\,.\,q'ix = -\tfrac{3}{4}\,\overline{ij}^2c_2\,.\,(\overline{xi}^2c_2)^2,$$

or $$(pix\,.\,qix)^2 = -\tfrac{3}{4}\,\overline{ij}^2c_2\,(\overline{xi}^2c_2)^2.$$

Hence we have

$$xpi\,.\,xqi\,.\,xpj\,.\,xqj = -\tfrac{3}{4}\,\overline{ij}^2c_2\,.\,\overline{xi}^2c_2\,.\,\overline{xj}^2c_2,$$

and consequently, since

$$\overline{xp}^2\,.\,\overline{xq}^2 = \kappa\,\frac{xpi\,.\,xqi\,.\,xpj\,.\,xqj}{(xij)^4\,(pij\,.\,qij)^2},$$

and $$(pij\,.\,qij)^2 = -\tfrac{3}{4}\,(\overline{ij}^2c_2)^3,$$

it follows that

$$xp^2\,.\,xq^2 = \kappa\,\frac{\overline{xi}^2c_2\,.\,\overline{xj}^2c_2}{(xij)^4\,(\overline{ij}^2c_2)^2} = xp'^2\,.\,xq'^2.$$

Combining these two results, we find

$$xp^2\,.\,xq^2\,.\,\sin^2 LM = \kappa''\,\frac{x^2C_2\,.\,C_2c_2}{(xij)^2\,.\,(\overline{ij}^2c_2)^2},$$

a result which may be further simplified by help of the formula for the distance between the foci. Namely, we have

$$\frac{xp^2\,.\,xq^2\,.\,\sin^2 LM}{pq^2} = \frac{x^2C_2}{(xij)^2\sqrt{(i^2C_2\,.\,j^2C_2)}} \text{ to a factor } \textit{près}.$$

We shall call this quantity the *distance** of the point from the conic; it vanishes when the point is on the conic, and is infinite if either the point or the conic has contact with the absolute.

* [Second distance; cf. p. 133.]

It is to be noted that the conic is given as a curve of the second order, in the form C_2; if it were given in the form c_2, the formula preceding would become

$$xp^2\,.\,xq^2\,.\,\sin^2 LM = \kappa''\,\frac{x^2C_2}{(xij)^2\,.\,(\overline{ij}^2c_2)^2},\text{*}$$

and this quantity might be taken as the distance of the point from a conic given as a curve of the second class. If the conic breaks up into two lines, the former expression becomes the product of the perpendicular distances of the point from the two lines; if the conic breaks up into two points, the latter expression becomes four times the squared area which they include with the given point. The former expression, however, in which the conic is given as of the second order, admits of a further interpretation, to which we now proceed.

Through the point x (fig. 13) let a line X be drawn, meeting the conic in l, m. For the product of the segments xl, xm we have the formula

$$xl\,.\,xm = \kappa^2\,\frac{\sqrt{(xli\,.\,xlj\,.\,xmi\,.\,xmj)}}{(xij)^2\,.\,lij\,.\,mij}.$$

The numerator of this expression clearly vanishes if x is on the conic, when one of the lines xl, xm becomes indeterminate; otherwise each of these lines is simply the line X. We must have therefore

$$xli\,.\,xlj\,.\,xmi\,.\,xmj = (x^2C_2\,.\,Xi\,.\,Xj)^2 \text{ to a factor } \textit{près}.$$

For the denominator we observe that if U is any arbitrary line,

$$lU\,.\,mU = \overline{XU}^2C_2,$$

and taking the co-ordinates of U for the current line co-ordinates, this gives the tangential equation to l, m. From this we get

$$lij\,.\,mij = \overline{Xij}^2C_2,$$

and our expression is therefore transformed into

$$xl\,.\,xm = \frac{x^2C_2\,.\,Xi\,.\,Xj}{(xij)^2\,.\,\overline{Xij}^2C_2}.$$

* [First distance; cf. p. 133.]

The denominator of this will clearly vanish if X is parallel to one or other of the asymptotes P, Q, or if $XPij \,.\, XQij = 0$. Let us, therefore, now seek the product of the distances of X from the asymptotes.

We have

$$-4 \sin XP \,.\, \sin XQ = \frac{XPij \,.\, XQij}{(Xi \,.\, Xj)\sqrt{(Pi \,.\, Pj \,.\, Qi \,.\, Qj)}}.$$

But we find by operating with $(Xij)^2$ upon $P \,.\, Q$ that

$$XPij \,.\, XQij = 3\overline{Xij}^2 C_2 \,.\, \overline{ij}^2 c_2,$$

and moreover

$$Pi \,.\, Pj \,.\, Qi \,.\, Qj = 9 i^2 C_2 \,.\, j^2 C_2 \,.\, (\overline{ij}^2 c_2)^2.$$

Hence we have

$$-4 \sin XP \,.\, \sin XQ = \frac{\overline{Xij}^2 C_2}{3 Xi \,.\, Xj \sqrt{(i^2 C_2 \,.\, j^2 C_2)}}.$$

Multiplying this by $xl \,.\, xm$, the line X disappears from the result and we find

$$xl \,.\, xm \,.\, \sin XP \,.\, \sin XQ = \frac{x^2 C_2}{(xij)^2 \sqrt{(i^2 C_2 \,.\, j^2 C_2)}} \text{ to a factor.}$$

But the expression on the right is the same that we previously obtained for the distance of the point from the conic. Hence the quantity $xl \,.\, xm \,.\, \sin XP \sin XQ$ must be proportional to the quantity $\frac{xp^2 \,.\, xq^2}{pq^2} \sin^2 LM$. To determine the constant factor, suppose the conic to break up into a pair of points; these may be taken to be the points p, q, and the asymptotes will both coincide with the line pq (fig. 14). Here it is clear that $xl.xm.\sin XP \sin XQ = xl^2.\sin^2 XP =$ squared distance of x from line pq; while $\frac{xp \,.\, xq \sin pxq}{pq} = \frac{\text{twice area } pxq}{pq} =$ distance of x from pq. Thus the factor is unity, and we have always

$$xl \,.\, xm \,.\, \sin XP \,.\, \sin XQ = \frac{xp^2 \,.\, xq^2}{pq^2} \sin^2 LM.^*$$

There is no difficulty in investigating the correlative formulæ. First to find the length of the chord cut off a line X

* [Second distance, p. 133.]

by the conic; let lm be the points of intersection, then we have

$$lm^2 = \left[\frac{1}{\kappa^2}\right] \frac{lmi \,.\, lmj}{(lij)^2 \,.\, (mij)^2}.$$

The numerator vanishes if the line X pass through either of the points ij, or if l, m coincide, that is to say, if X touch the conic c_2, or if the conic break up into a pair of points. Moreover, since we have

$$l \,.\, m = X^2 c_2 \,.\, c_2 - (X c_2)^2,$$

the expression $(lmi)^2$ must be of the fourth order in the coefficients of X and of the conic, and therefore

$$(lmi)^2 = (Xi)^2 \,.\, X^2 c_2 \,.\, C_2 c_2 \text{ to a factor.}$$

Again, we have by direct substitution

$$lij \,.\, mij = X^2 c_2 \,.\, \overline{ij}^2 c_2 - (X\overline{ij} c_2)^2 = \overline{Xij}^2 C_2,$$

and thence

$$lm^2 = \frac{Xi \,.\, Xj \,.\, X^2 c_2 \,.\, C_2 c_2}{(\overline{Xij}^2 C_2)^2} \text{ to a factor.}$$

But we have already found that

$$\sin XP \,.\, \sin XQ = \frac{\overline{Xij}^2 C_2}{Xi \,.\, Xj \sqrt{(i^2 C_2 \,.\, j^2 C_2)}},$$

consequently

[i] $$lm^2 \,.\, \sin^2 XP \,.\, \sin^2 XQ = \frac{X^2 c_2 \,.\, C_2 c_2}{Xi \,.\, Xj \,.\, i^2 C_2 \,.\, j^2 C_2}.^*$$

Lastly, c_2 being primarily given, we have

$$\sin^2 PQ = \frac{\overline{ij}^2 c_2 \,.\, C_2 c_2}{i^2 C_2 \,.\, j^2 C_2},$$

and so

[ii] $$lm^2 \frac{\sin^2 XP \,.\, \sin^2 XQ}{\sin^2 PQ} = \frac{X^2 c_2}{Xi \,.\, Xj \,.\, \overline{ij}^2 c_2}.$$

Let now x be a variable point on the line X (fig. 15), and draw the tangents L, M from x to the conic. Then

$$\sin XL \,.\, \sin XM = \frac{XLij \,.\, XMij}{Xi \,.\, Xj \sqrt{(Li \,.\, Lj \,.\, Mi \,.\, Mj)}}.$$

* [First distance of line X from Conic C_2; cf. p. 133.]

In this fraction the numerator vanishes when X touches the conic, and when x is on the line ij. It must be of the first order in the coefficients of c_2 and of the second in those of x. Hence we have, to a factor *près*,

$$XLij \,.\, XMij = X^2 c_2 \,.\, (xij)^2.$$

Moreover by a previous formula

$$Li \,.\, Lj \,.\, Mi \,.\, Mj = \overline{xi}^2 c_2 \,.\, \overline{xj}^2 c_2.$$

Thus

$$\sin XL \,.\, \sin XM = \frac{X^2 c_2 \,.\, (xij)^2}{Xi \,.\, Xj \sqrt{(\overline{xi}^2 c_2 \,.\, \overline{xj}^2 c_2)}}.$$

But, also by a previous formula,

$$xp \,.\, xq = \frac{\sqrt{(\overline{xi}^2 c_2 \,.\, \overline{xj}^2 c_2)}}{(xij)^2 \,.\, \overline{ij}^2 c_2};$$

therefore

$$\text{[iii]} \quad xp \,.\, xq \sin XL \sin XM = \frac{X^2 c_2}{Xi \,.\, Xj \,.\, \overline{ij}^2 c_2}$$

$$= lm^2 \frac{\sin^2 XP \,.\, \sin^2 XQ}{\sin^2 PQ}.^*$$

To verify this, suppose the conic to break up into two lines P, Q (fig. 16), which are themselves the asymptotes; the two foci will then coincide at the point PQ, and the tangents LM will pass through the same point. Then $xp \,.\, xq \,.\, \sin XL \,.\, \sin XM = xp^2 \sin^2 XL =$ squared perpendicular from PQ on X. And

$$\frac{lm}{\sin PQ} \,.\, \sin XP \sin XQ = lp \,.\, mp \,.\, \frac{lm}{\sin PQ} \,.\, \frac{\sin XP}{lp} \,.\, \frac{\sin XQ}{mp}$$

$$= \frac{lp \,.\, mp \,.\, \sin PQ}{lm} = \frac{2 \text{ area } lmp}{lm}$$

$$= \text{perpendicular from } PQ \text{ on } X.$$

This quantity†, of which three expressions are given in our last equation, may be called the *distance* of the line X from the conic c_2.

* [Second distance of line X from Conic c_2.]

† [i.e. iii. supra.]

IV.

We now consider a curve C_n of the n^{th} order, which may also be given as a curve $c_{n(n-1)}$ of class $n(n-1)$. To this number $n(n-1)$ there are no reductions in virtue of any singularities that C_n may have; its nodes will enter as double factors and its cusps as triple factors in $c_{n(n-1)}$. This being so, we may write

$$\text{Disct.}_\lambda\, (x + \lambda y)^n C_n = \frac{\{\underline{|n}\}^{2(n-1)}}{\underline{|n(n-1)}} \,.\, \overline{xy}^{n(n-1)} c_{n(n-1)}.$$

Conversely, if we are given a curve c_m of class m, this is also a curve $C_{m(m-1)}$ of order $m(m-1)$; but each double tangent is now a double factor and each stationary tangent a triple factor in $C_{m(m-1)}$. We may gain shortness without introducing confusion, if when C_n is primarily given, we denote $n(n-1)$ by m; and if when c_m is primarily given, we denote $m(m-1)$ by n. Thus in the latter case we shall have

$$\text{Disct.}_\lambda\, (X + \lambda Y)^m c_m = \frac{\{\underline{|m}\}^{2(m-1)}}{\underline{|n}} \,.\, \overline{XY}^n C_n.$$

The curve c_m has m^2 foci, which are the intersections of the m tangents from the point i with the m tangents from the point j. But for every tangent from i there is *one* tangent from j which meets it in a real point; thus there are m real foci. The foci may be arranged in various ways into m sets, such that no two points of the same set are collinear with i or j. By joining the points of any one set with i and j we obtain all the tangents. The real foci constitute a set; and from them we may pass to any other set by successively substituting for each pair pq their *antipoints* $p'q'$, that is to say, the remaining intersections of pi, pj with qi, qj. Now for every such substitution the following equations hold good:

$$p'q'^2 = -pq^2,$$

$$xp' \,.\, xq' = xp \,.\, xq,$$

where x is any point in the plane. Hence if we form the product Πpq^2 of the squared distances of the real foci from one another, this product can differ only in sign from a similar product formed with any other set; and the same is true of Πxp. The number of possible sets is clearly the same as the number of terms in a determinant of the m^{th} order, viz.: $\underline{|m}$. If we then raise Πpq^2 to the power $\underline{|m}$, we must have some power of $\Pi\Pi pq^2$, the product of the squared distances of all pairs of foci from one another, excepting of course those pairs which are collinear with i or j. But there are $m^2(m-1)^2$ such pairs; thus if k is the power in question

$$\underline{|m}\,.\,m(m-1) = km^2(m-1)^2,$$

or

$$k = \frac{\underline{|m}}{m(m-1)},$$

whence

$$(\Pi pq^2)^{m(m-1)} = \Pi\Pi pq^2.$$

In a similar way we find

$$(\Pi xp)^m = \Pi\Pi xp.$$

Now

$$\Pi\Pi pq^2 = \Pi\Pi\,\frac{pqi\,.\,pqj}{pij^2\,.\,qij^2} = \frac{\Pi\Pi pqi\,.\,pqj}{(\Pi pij)^{4m^2(m-1)^2}} \text{ [to factor } près\text{]}.$$

To obtain the equation of the foci we may proceed as follows. The equation of the tangents from i, j to c_m is $I_m = 0$, $J_m = 0$,

where

$$x^m I_m = \overline{jx}^m c_m,\quad x^m J_m = \overline{jx}^m c_m,$$

and if we then form the equation of the points of intersection of I_m, J_m (which may be written $(XIJ)_{mm} = 0$) it is of the order m in each of them and must contain the equation of the foci. But also it must contain the factor $\overline{ij}^m c_m$, and this to the degree $\overline{m-1}$; for the factor is involved as the condition that I_m should pass through a double point of J_m. Thus the equation of the foci is of the order $2m - \overline{m-1} = m+1$ in c_m and $m^2 - m(m-1) = m$ in i and j. In fact one term in it is a numerical multiple of $\overline{ij}^m c_m\,.\,(c_m)^m$.

As in the case of the conic, the product $\Pi\Pi pqi\,.\,pqj$ will vanish when $i^n C_n = 0$, or when $j^n C_n = 0$, or when $\overline{ij}^m c_m = 0$.

Thus

$$\Pi\Pi pqi\,.\,pqj = \kappa\,(i^n C_n\,.\,j^n C_n)^x\,(\overline{ij}^m c_m)^y.$$

Now the left-hand side is of order $2(m-1)^2$ in the foci and besides of order $\frac{1}{2}m^2(m-1)^2$ in i and j; that is, of order $2(m+1)(m-1)^2$ in c_m, and $2m(m-1)^2 + \frac{1}{2}m^2(m-1)^2 = \frac{1}{2}m(m+4)(m-1)^2$ in i and j. First regard c_m as given; then C_n is of order $2(m-1)$ in the coefficient of c_m, and we have

$$2(m+1)(m-1)^2 = 4x(m-1) + y,$$

$$\tfrac{1}{2}m(m+4)(m-1)^2 = xm(m-1) + my,$$

whence $x = \frac{1}{2}m(m-1)$, $y = 2(m-1)^2$, and consequently

$$\Pi\Pi pqi\,.\,pqj = \kappa\,(i^n C_n\,.\,j^n C_n)^{\frac{1}{2}m(m-1)}\,(\overline{ij}^m c_m)^{2(m-1)^2}.$$

Next we have

$$\Pi\Pi pij = (\overline{ij}^m c_m)^{m+1} \text{ to a factor},$$

and therefore

$$\Pi\Pi pq^2 = \frac{\Pi\Pi pqi\,.\,pqj}{(\Pi\Pi pij)^{2(m-1)^2}} = \kappa\,\frac{(i^n C_n\,.\,j^n C_n)^{\frac{1}{2}m(m-1)}}{(\overline{ij}^m c_m)^{2m(m-1)^2}},$$

but

$$(\Pi pq^2)^{m(m-1)} = \Pi\Pi pq^2;$$

therefore

$$\Pi pq^2 = \frac{(i^n C_n\,.\,j^n C_n)^{\frac{1}{2}}}{(\overline{ij}^m c_m)^{2(m-1)}}.$$

We now proceed to determine Πxp, where x is any point in the plane. We have

$$\Pi\Pi xp^2 = \frac{\Pi\Pi xpi\,.\,xpj}{(xij)^{2m^2}\,\Pi\Pi(pij)^2} = \frac{(\overline{xi}^m c_m\,.\,\overline{xj}^m c_m)^m\,(\overline{ij}^m c_m)^2}{(xij)^{2m^2}\,(\overline{ij}^m c_m)^{2(m+1)}}$$

$$= \frac{(\overline{xi}^m c_m\,.\,\overline{xj}^m c_m)^m}{(xij)^{2m^2}\,(\overline{ij}^m c_m)^{2m}},$$

and therefore

$$\Pi xp^2 = \frac{\overline{xi}^m c_m\,.\,\overline{xj}^m c_m}{(xij)^{2m}\,(\overline{ij}^m c_m)^2}.$$

The curve C_n has n asymptotes, which are the tangents at the points where it is met by the line ij. If a point x lie on one of the asymptotes, its first polar xC_n must meet C_n on the line ij. The condition for this is of the order n in xC_n, $n-1$ in C_n, and $n(n-1)$ in ij; that is, of the order n in x, $2n-1$ in C_n, and $n(n-1)$ in ij. It must also be of the form

$$Ax^n C_n + x^{n-2} B_{n-2}\,.\,(xij)^2 = 0,$$

since the n asymptotes form a curve of the n^{th} order touching C_n where it is met by ij. If A vanishes, the line ij becomes a double factor; now this can only happen when $\overline{ij}^m c_m = 0$, and a comparison of dimensions shews that A differs from this only by a numerical factor. We may therefore write for the equation of the asymptotes

$$\overline{ij}^m c_m \, . \, C_n + B_{n-2} \, . \, \overline{ij}^2 \equiv \Pi P.$$

We may now find the product of the sines of the angles between them, $\Pi \sin PQ$, and of the angles they make with a line X, $\Pi \sin XP$. Namely, we have

$$\Pi \sin^2 PQ = \frac{\Pi \, (PQij)^2}{(\Pi Pi \, . \, \Pi Pj)^{n-1}} \quad \text{[to factor } \textit{près}\text{]}.$$

Now

$$(\Pi PQij)^2 = (\overline{ij}^m c_m)^{2n-1},$$

and therefore

$$\Pi \sin^2 PQ = \frac{(\overline{ij}^m c_m)^{2n-1}}{(\overline{ij}^m c_m)^{2(n-1)} \, (i^2 C_n \, . \, j^2 C_n)^{n-1}} = \frac{\overline{ij}^m c_m}{(i^2 C_n \, . \, j^2 C_n)^{n-1}}.$$

In the next place

$$\Pi \sin^2 XP = \frac{\Pi \, (XPij)^2}{(Xi \, . \, Xj)^n \, . \, \Pi Pi \, . \, \Pi Pj} = \frac{\{(Xij)^n C_n\}^2}{(Xi \, . \, Xj)^n \, . \, i^n C_n \, . \, j^n C_n}.$$

Let us now suppose a variable line X to be drawn through the fixed point x, meeting the curve C_n in the points $l, m, n, \ldots$ then

$$\Pi xl^2 = \frac{\Pi xli \, . \, \Pi xlj}{(xij)^{2n} \, \Pi \, (lij)^2} = \frac{(x^n C_n)^2 \, . \, (Xi)^n \, . \, (Xj)^n}{(xij)^{2n} \, . \, \{(Xij)^n C_n\}^2},$$

or

$$\Pi xl = \frac{x^n C_n \, . \, (Xi \, . \, Xj)^{\frac{n}{2}}}{(xij)^n \, . \, (Xij)^n C_n}.$$

Hence

$$\Pi xl \, . \, \Pi \sin XP = \frac{x^n C_n}{(xij)^n \, . \, \sqrt{(i^n C_n \, . \, j^n C_n)}};$$

this product is therefore independent of the position of the line X, and may be called the *distance* of the point x from the curve C_n.

If we draw to the curve from the point x the tangents $L, M, N, \ldots$ we shall have

$$\Pi \sin^2 LM = \frac{\Pi \, (LMij)^2}{(\Pi Li \, . \, \Pi Lj)^{m-1}} = \frac{x^n C_n \, . \, (xij)^n}{(\overline{xi}^m c_m \, . \, \overline{xj}^m c_m)^{m-1}};$$

$$\therefore \, (\Pi xp^2)^{m-1} \, . \, \Pi \sin^2 LM = \frac{x^n C_n}{(xij)^n \, . \, (\overline{ij}^2 c_m)^{2(m-1)}},$$

and finally

$$\frac{(\Pi xp^2)^{m-1}}{\Pi pq^2} \, . \, \Pi \sin^2 LM = \frac{x^n C_n}{(xij)^n \, . \, \sqrt{(i^n C_n \, . \, j^n C_n)}} \quad \text{[to a factor]}$$

$$= \Pi xl \, . \, \Pi \sin XP \quad \text{to a factor,}$$

and by supposing the curve to break up into m points the factor is easily determined to be unity.

Considering now the line X as fixed and the point x as variable, we have

$$\Pi \sin^2 XL = \frac{\Pi \, (XLij)^2}{(Xi \, . \, Xj)^m \, \Pi Li \, . \, \Pi Lj} = \frac{(X^m c_m)^2 \, (xij)^{2m}}{(Xi \, . \, Xj)^m \, . \, \overline{xi}^m c_m \, . \, \overline{xj}^m c_m};$$

$$\therefore \, \Pi \sin XL \, . \, \Pi xp = \frac{X^m c_m}{(Xi \, . \, Xj)^{\frac{m}{2}} \, . \, \overline{ij}^m c_m}.$$

But also

$$\Pi lm^2 = \frac{\Pi lmi \, . \, \Pi lmj}{(\Pi lij)^{2(n-1)}} = \frac{X^m c_m \, . \, (Xi \, . \, Xj)^{\frac{m}{2}}}{\{(Xij)^n C_n\}^{2(n-1)}},$$

and therefore

$$(\Pi \sin^2 XP)^{n-1} \, . \, \Pi lm^2 = \frac{X^m c_m}{(Xi \, . \, Xj)^{\frac{m}{2}} \, (i^n C_n \, . \, j^n C_n)^{n-1}},$$

whence

$$\frac{(\Pi \sin^2 XP)^{n-1}}{\Pi \sin^2 PQ} \, \Pi lm^2 = \frac{X^m c_m}{(Xi \, . \, Xj)^{\frac{m}{2}} \, . \, \overline{ij}^m c_m}$$

$$= \Pi \sin XL \, . \, \Pi xp, \quad \text{to a factor,}$$

which, as before, the special case of n lines shews to be unity. We shall call the quantity for which three expressions are here given the *distance* of the line X from the curve c_m.

V.

The elliptic geometry of two dimensions as Dr Klein calls it, or, which is the same thing, geometry on the sphere in which two opposite points are regarded as identical, differs from plane geometry in that instead of the two points *ij* we have the proper conic O_2 or o_2. Lines touching this conic, and points lying on it, are at an infinite distance from all other lines and points; distances measured on them are zero. Using the ordinary co-ordinates we may write

$$O_2 = x_1^2 + x_2^2 + x_3^2,$$
$$o_2 = \xi_1^2 + \xi_2^2 + \xi_3^2,$$
$$a = a_1\xi_1 + a_2\xi_2 + a_3\xi_3,$$
$$A = A_1x_1 + A_2x_2 + A_3x_3,$$

and then

$$\sin^2 ab = \frac{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}$$

$$= \frac{2\overline{ab}^2 o_2}{a^2O_2 \,.\, b^2O_2} \text{ [when } O_2 \text{ given]},$$

$$\sin^2 AB = \frac{(A_2B_3 - A_3B_2)^2 + (A_3B_1 - A_1B_3)^2 + (A_1B_2 - A_2B_1)^2}{(A_1^2 + A_2^2 + A_3^2)(B_1^2 + B_2^2 + B_3^2)}$$

$$= \frac{2\overline{AB}^2 O_2}{A^2o_2 \,.\, B^2o_2}.$$

The distance of a point a from a line B may be derived from these two as follows (fig. 17): through a draw a variable line A meeting B in b; then we have

$$\sin^2 ab = \sin^2 aAB = \frac{2(aB)^2 \,.\, A^2o_2}{a^2O_2 \,.\, \overline{AB}^2O_2},$$

and

$$\sin^2 AB = \frac{2\overline{AB}^2O_2}{A^2o_2 \,.\, B^2o_2};$$

$$\therefore \sin^2 ab \,.\, \sin^2 AB = \frac{4(aB)^2}{a^2O_2 \,.\, B^2o_2};$$

II.

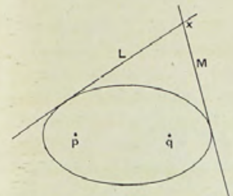


Fig. 12.

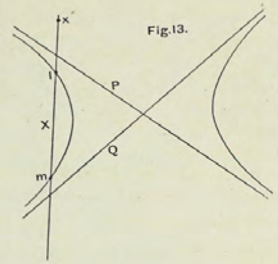


Fig. 13.

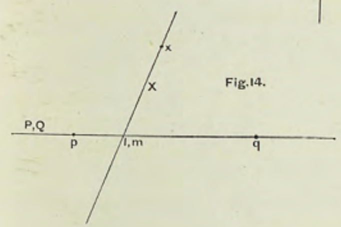


Fig. 14.

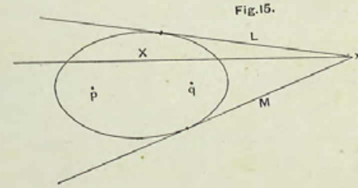


Fig. 15.

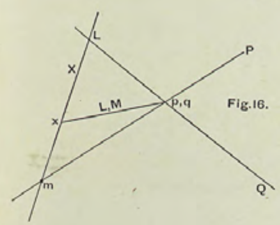


Fig. 16.

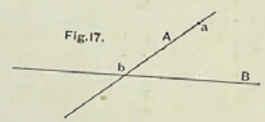


Fig. 17.

face p. 152.

a result independent of the position of the line A, which when A is at right angles to B becomes $\sin^2 aB$.

A conic C_2 or c_2 has with the absolute O_2 or o_2 four common points and four common tangents. The lines joining the common points form three pairs of common chords, P, Q; P', Q'; P'', Q''; the intersections of the common tangents form three pairs of foci, p, q; p', q'; p'', q''. The theories of the foci and the common chords are identical, and it will be sufficient to consider one of them; we shall choose for this purpose the foci.

We obtain a pair of foci by determining λ so that

$$c_2 + \lambda o_2$$

shall break up into factors; the condition for this is

$$0 = 6 \text{ Disct. } (c_2 + \lambda o_2) = C_2 c_2 + 3\lambda C_2 o_2 + 3\lambda^2 c_2 O_2 + \lambda^3 O_2 o_2,$$

and substituting here for λ, $-\frac{c_2}{o_2}$, we obtain the equation of the three pairs of foci:—

$$0 = C_2 c_2 \,.\, o_2^3 - 3 C_2 o_2 \,.\, o_2^2 \,.\, c_2 + 3 c_2 O_2 \,.\, o_2 \,.\, c_2^2 - O_2 o_2 \,.\, c_2^3.$$

If $c_2 + \lambda o_2$ break up into factors p, q, its reciprocal will be $-\frac{1}{4}\overline{pq}^2$. Thus we have

$$-\tfrac{1}{4}\overline{pq}^2 = C_2 + 2\lambda (co)_2 + \lambda^2 O_2,$$

where $(co)_2$ is the locus of points at which c_2 subtends a right angle. Consequently

$$-\tfrac{1}{4}\overline{pq}^2 o_2 = C_2 o_2 + 2\lambda c_2 O_2 + \lambda^2 O_2 o_2 = 2\partial_\lambda \text{ Disct. } (c_2 + \lambda o_2).$$

Moreover

$$pqO_2 = c_2 O_2 + \lambda O_2 o_2 = \partial_\lambda^2 \text{ Disct. } (c_2 + \lambda o_2),$$

but

$$p^2 O_2 \,.\, q^2 O_2 - (pqO_2)^2 = \tfrac{1}{3}\overline{pq}^2 o_2 \,.\, O_2 o_2,$$

hence

$$\begin{aligned} 3p^2 O_2 \,.\, q^2 O_2 &= 3(c_2 O_2 + \lambda O_2 o_2)^2 - 4(C_2 o_2 + 2\lambda c_2 O_2 + \lambda^2 O_2 o_2) O_2 o_2 \\ &= 3\overline{c_2 O_2}^2 - 4 C_2 o_2 \,.\, O_2 o_2 - 2\lambda c_2 O_2 \,.\, O_2 o_2 - \lambda^2 \overline{O_2 o_2}^2 \\ &= 3(\overline{c_2 O_2}^2 - C_2 o_2 . O_2 o_2) - (C_2 o_2 + 2\lambda c_2 O_2 + \lambda^2 O_2 o_2) O_2 o_2. \end{aligned}$$

To simplify these formulæ, let us write

$$6\text{ Disct. }(c_2+\lambda o_2)=f=a\lambda^3+3b\lambda^2+3c\lambda+d$$
$$\tfrac{1}{3}\partial_\lambda f=x=a\lambda^2+2b\lambda+c$$
$$y=3(b^2-ac)-ax;$$

then whenever λ has a value which makes f vanish, so that $c_2+\lambda o_2$ breaks up into the factors p, q, we must have

$$-\tfrac{1}{4}\overline{pq}^2 o_2=x,$$
$$3p^2O_2\,.\,q^2O_2=y.$$

We shall now prove that if $x_1\, x_2\, x_3$ are the values of x, and $y_1\, y_2\, y_3$ the values of y, corresponding to the three values of λ given by $f=0$, then

$$x_1^2y_1=x_2^2y_2=x_3^2y_3=-R_f,\text{ the discriminant of } f.$$

We get an equation for determining x by eliminating λ between $f=0$ and $\frac{1}{3}\partial_\lambda f-x=0$. Namely, the resultant of these two equations is

$$\begin{vmatrix} a & 3b & 3c & d & . \\ . & a & 3b & 3c & d \\ a & 2b & c-x & . & . \\ . & a & 2b & c-x & . \\ . & . & a & 2b & c-x \end{vmatrix} \begin{aligned} &=-a^2x^3+3(b^2-ac)ax^2+aR_f,\\ &=a(R_f+x^2y),\end{aligned}$$

which vanishes if

$$x^2y=-R_f;$$

this equation is therefore true for each of the corresponding pairs of values of x and y. Substituting for these their values, we have

$$(\overline{pq}^2o_2)^2\,.\,p^2O_2\,.\,q^2O_2=(\overline{p'q'}^2o_2)^2\,.\,p'^2O_2\,.\,q'^2O_2$$
$$=(\overline{p''q''}^2o_2)^2\,.\,p'^2O_2\,.\,q''^2O_2=-\tfrac{16}{3}R_f.$$

Now R_f is the osculant of c_2 and o_2, that is, the invariant whose vanishing is the condition that the two conics shall touch.

It follows further from the cubic equation for x that the product of its three values is $\dfrac{R_f}{a}$. Hence we have

$$\overline{pq}^2o_2\,.\,\overline{p'q'}^2o_2\,.\,\overline{p''q''}^2o_2=-64\frac{R_f}{a},$$

and therefore

$$p^2O_2\,.\,q^2O_2\,.\,p'^2O_2\,.\,q'^2O_2\,.\,p''^2O_2\,.\,q''^2O_2=-\frac{a^2R_f}{27};$$

whence by division

$$\sin^2 pq\,.\,\sin^2 p'q'\,.\,\sin^2 p''q''=+\frac{12^3\,[2^3]}{6^3}=+[64],^*$$

since $$\sin^2 pq=\frac{[2]\,\overline{pq}^2o_2}{p^2O_2\,.\,q^2O_2}\,.\,\frac{O_2o_2}{6}\text{ when } o_2 \text{ is given.}$$

From the same equation we learn that the sum of the reciprocals of the x is zero. Now

$$-\frac{R_f}{x^3}=\frac{y}{x}=[-2\,O_2o_2]\,(\sin pq)^{-2};$$

therefore

$$(\sin pq)^{-\frac{2}{3}}+(\sin p'q')^{-\frac{2}{3}}+(\sin p''q'')^{-\frac{2}{3}}=0.$$
$$[\text{Also } (\sin pq)^{-2}+(\sin p'q')^{-2}+(\sin p''q'')^{-2}=\tfrac{3}{4}.]^*$$

We have thus two equations connecting the quantities $\sin pq$, $\sin p'q'$, $\sin p''q''$, by which when one is given the other two may be determined.

Now let x be any point of the plane (or sphere). Then we have

$$\sin^2 xp\,.\,\sin^2 xq=\frac{\overline{xp}^2o_2\,.\,\overline{xq}^2o_2}{(x^2O_2)^2\,.\,p^2O_2\,.\,q^2O_2}\left[\left(\frac{O_2o_2}{6}\right)^2\right].$$

But the numerator of this fraction is clearly the result of operating with x^4 on the common tangents of pq and o_2, that is of $c_2+\lambda o_2$ and o_2, a result which is clearly independent of λ. Hence we have

$$\overline{xp}^2o_2\,.\,\overline{xq}^2o_2=\overline{xp'}^2o_2\,.\,\overline{xq'}^2o_2=\overline{xp''}^2o_2\,.\,\overline{xq''}^2o_2.$$

Moreover, since

$$(\overline{pq}^2o_2)^2\,p^2O_2\,.\,q^2O_2=-\tfrac{16}{3}R_f,$$

* [The introduced factor, cf. p. 152, brings the results into accordance with Prof. Cayley's equations. See Note to this paper, p. 159 (3).]

it follows that

$$\sin^4 pq = \tfrac{1}{36}\left(\frac{\overline{pq}^2 o_2 \,.\, O_2 o_2}{p^2 O_2 \,.\, q^2 O_2}\right)^2 = -\tfrac{4}{27}\,\frac{R_f (O_2 o_2)^2}{(p^2 O_2 \,.\, q^2 O_2)^3},$$

and consequently

$$\frac{\sin xp \,.\, \sin xq}{(\sin pq)^{\frac{2}{3}}} = \frac{\sin xp' \,.\, \sin xq'}{(\sin p'q')^{\frac{2}{3}}} = \frac{\sin xp'' \,.\, \sin xq''}{(\sin p''q'')^{\frac{2}{3}}}$$

$$\left[= \frac{\sqrt{\overline{xp}^2 o_2 \,.\, \overline{xq}^2 o_2}}{x^2 O_2}\,\frac{1}{\sqrt[3]{16}\,\sqrt{3}}\,\frac{(O_2 o_2)^{\frac{2}{3}}}{\sqrt[6]{-R_f}}\right]$$

$$= \left[\frac{1}{\sqrt[3]{16}\,\sqrt{3}}\right]\frac{\sqrt{x^4 (co)_4}\,(O_2 o_2)^{\frac{2}{3}}}{x^2 O_2 \sqrt[6]{-R_f}}.$$

The three equations which we have just proved establish the theory of three pairs of antipoints on a sphere; viz. the three pairs of intersections of four tangents to the absolute. They take the place of the *two* equations which we have already used in regard to the two pairs of antipoints on a plane; namely,

$$pq^2 = -p'q'^2,$$

and

$$xp \,.\, xq = xp' \,.\, xq'.$$

We now proceed to use them in connection with the theory of the conic c_2.

From the point x let the tangents L, M be drawn to the conic; then

$$\sin^2 LM = [2]\,\frac{\overline{LM}^2 O_2}{L^2 o_2 \,.\, M^2 o_2} = \frac{x^2 O_2 \,.\, x^2 C_2}{\overline{xp}^2 o_2 \,.\, \overline{xq}^2 o_2}\ \text{[to factor } \textit{près}\text{]},$$

where pq are any set of foci. But also

$$\frac{\sin^2 xp \,.\, \sin^2 xq}{(\sin^2 pq)^{\frac{2}{3}}} = \left[\frac{1}{12\sqrt[3]{4}}\right].\,\frac{\overline{xp}^2 o_2 \,.\, \overline{xq}^2 o_2 (O_2 o_2)^{\frac{4}{3}}}{(x^2 O_2)^2 \sqrt[3]{-R_f}},$$

and therefore

$$\frac{\sin^2 xp \,.\, \sin^2 xq}{(\sin^2 pq)^{\frac{2}{3}}}\,\sin^2 LM = \left[\frac{1}{12\sqrt[3]{4}}\right]\frac{x^2 C_2 \,.\, (O_2 o_2)^{\frac{4}{3}}}{x^2 O_2 \,.\, \sqrt[3]{-R_f}}$$

$$= \left[\frac{1}{12\sqrt[3]{4}}\right].\,\frac{x^2 C_2 \,.\, (O_2 o_2)^2 (C_2 c_2)^{\frac{2}{3}}}{x^2 O_2 \,.\, \sqrt[3]{-R_F}}.$$

Again, if we draw through x a variable line X, meeting the conic in l, m, we shall have

$$\sin^2 xl \,.\, \sin^2 xm = \frac{\overline{xl}^2 o_2 \,.\, \overline{xm}^2 o_2}{(x^2 O_2)^2 \,.\, l^2 O_2 \,.\, m^2 O_2} = \frac{(x^2 C_2)^2 \,.\, (X^2 o_2)^2}{(x^2 O_2)^2 \,.\, X^4 (CO)_4}$$

$$= \frac{(x^2 C_2)^2 \,.\, (X^2 o_2)^2}{(x^2 O_2)^2 \,.\, \overline{XP}^2 O_2 \,.\, \overline{XQ}^2 O_2},$$

and

$$\frac{\sin^2 XP \,.\, \sin^2 XQ}{(\sin^2 PQ)^{\frac{2}{3}}} = \left[\frac{1}{6^{\frac{4}{3}}}\,\frac{\overline{XP}^2 O_2 \,.\, \overline{XQ}^2 O_2\, (O_2 o_2)^{\frac{4}{3}}}{(X^2 o_2)^2 \{P^2 o_2 \,.\, Q^2 o_2 (\overline{PQ}^2 O_2)^2\}^{\frac{1}{3}}}\right]$$

$$= \tfrac{3}{16}\,\frac{\overline{XP}^2 O_2 \,.\, \overline{XQ}^2 O_2 \,.\, (O_2 o_2)^{\frac{2}{3}}}{(X^2 o_2)^2 \sqrt[3]{R_F}},$$

where R_F signifies the osculant of C_2 and O_2; $R_F = (C_2 c_2)^2 (O_2 o_2)^2 R_f$.

Therefore

$$\sin xl \,.\, \sin xm\,\frac{\sin XP \sin XQ}{(\sin PQ)^{\frac{2}{3}}} = \left[\frac{1}{\sqrt[6]{12^3 \,.\, 4}}\right]\frac{x^2 C_2 \,.\, (O_2 o_2)^{\frac{2}{3}}}{x^2 O_2 \,.\, \sqrt[6]{-R_F}}.$$

[The MS. here ends abruptly. In the results of the last two pages I have introduced a few additions, all, or nearly all, of which agree with results obtained by Prof. Henrici, who most kindly gave me his valuable help in revising the proof sheets.

The subject of V. has been treated from a different point of view by Prof. Cayley, who has allowed me to append the following note.

The results obtained in No. V. of the paper On the Theory of Distances may be worked out in greater detail, and in some measure in a more complete form.

Using line-coordinates, we have $x^2+y^2+z^2=0$, the conic called the absolute; a conic $(a, b, c, f, g, h \between x, y, z)^2=0$, which will be called simply the conic; and a point (x) the equation of which is $lx+my+nz=0$. The common tangents of the conic and the absolute intersect in pairs in six points $p, q : p', q' : p'', q''$, which are the foci of the conic; or if we regard the four lines simply as any four tangents of the absolute, then the six points are a system of foci; and we obtain in the first instance formulæ relating to such a system, alone or in connection with the point x: afterwards, taking them to be the foci of the conic, we further consider the two tangents L, M from the point (x) to the conic; and an arbitrary line X through the point (x).

1. The coordinates of a tangent of the absolute are (x_1, y_1, z_1), where these are any values such that $x_1^2+y_1^2+z_1^2=0$; and we consider the four tangents

$$(x_1, y_1, z_1),\quad (x_2, y_2, z_2),\quad (x_3, y_3, z_3),\quad (x_4, y_4, z_4).$$

Writing for a moment

$$\lambda,\ \mu,\ \nu = y_1z_4 - y_4z_1,\quad z_1x_4 - z_4x_1,\quad x_1y_4 - x_4y_1,$$

$$\lambda',\ \mu',\ \nu' = y_2z_3 - y_3z_2,\quad z_2x_3 - z_3x_2,\quad x_2y_3 - x_3y_2,$$

we have
$$\lambda x + \mu y + \nu z = 0,\quad \lambda' x + \mu' y + \nu' z = 0$$

for the equations of the points p and q respectively; and the expression for the distance is given by

$$\cos^2 pq = \frac{(\lambda\lambda' + \mu\mu' + \nu\nu')^2}{(\lambda^2+\mu^2+\nu^2)(\lambda'^2+\mu'^2+\nu'^2)};$$

and if for shortness we write

$$12 = x_1x_2 + y_1y_2 + z_1z_2,\ \&c.,$$

then the values of λ, μ, ν give

$$\lambda^2 + \mu^2 + \nu^2 = -(14)^2,$$

$$\lambda'^2 + \mu'^2 + \nu'^2 = -(23)^2,$$

$$\lambda\lambda' + \mu\mu' + \nu\nu' = -31\,.\,24 + 12\,.\,34.$$

I write
$$\sqrt{23\,.\,14} = \mathrm{f},\quad \sqrt{31\,.\,24} = \mathrm{g},\quad \sqrt{12\,.\,34} = \mathrm{h},$$

and I say that we have $\mathrm{f}+\mathrm{g}+\mathrm{h}=0$. The formula becomes

$$\cos^2 pq = \frac{(\mathrm{g}^2-\mathrm{h}^2)^2}{\mathrm{f}^4},$$

and we thence have

$$\sin^2 pq = \frac{(\mathrm{f}^2-\mathrm{g}^2+\mathrm{h}^2)(\mathrm{f}^2+\mathrm{g}^2-\mathrm{h}^2)}{\mathrm{f}^4} = \frac{-2\mathrm{fh}\,.\,-2\mathrm{fg}}{\mathrm{f}^4},\quad = \frac{4\mathrm{gh}}{\mathrm{f}^2};$$

and consequently for the three pairs of foci respectively

$$\sin^2 pq = \frac{4\mathrm{gh}}{\mathrm{f}^2};\ \sin^2 p'q' = \frac{4\mathrm{hf}}{\mathrm{g}^2};\ \sin^2 p''q'' = \frac{4\mathrm{fg}}{\mathrm{h}^2}. \qquad *$$

2. The assumed relation $\mathrm{f}+\mathrm{g}+\mathrm{h}=0$ is obtained from the equation

$$\begin{vmatrix} x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \\ x_4, & y_4, & z_4, & w_4 \end{vmatrix}^2 = 0,$$

which is identically true if w_1, w_2, w_3, w_4 are each $=0$; attending to the equations $x_1^2+y_1^2+z_1^2=0$, &c., this is

$$\begin{vmatrix} . & (12)^2, & (13)^2, & (14)^2 \\ (21)^2, & . & (23)^2, & (24)^2 \\ (31)^2, & (32)^2, & . & (34)^2 \\ (41)^2, & (42)^2, & (43)^2, & . \end{vmatrix} = 0,$$

which is in fact the rationalised form of $\mathrm{f}+\mathrm{g}+\mathrm{h}=0$.

* Results are marked with an asterisk.

3. The foregoing values give

$$\left.\begin{aligned} &\sin^2 pq\,.\,\sin^2 p'q'\,.\,\sin^2 p''q'' = 64, \\ &\sin^{-\frac{2}{3}} pq + \sin^{-\frac{2}{3}} p'q' + \sin^{-\frac{2}{3}} p''q'' = \frac{\mathrm{f}+\mathrm{g}+\mathrm{h}}{(4\mathrm{fgh})^{\frac{2}{3}}},\ = 0 \\ &\sin^{-2} pq + \sin^{-2} p'q' + \sin^{-2} p''q'' = \frac{\mathrm{f}^3+\mathrm{g}^3+\mathrm{h}^3}{4\mathrm{fgh}},\ = \tfrac{3}{4}. \end{aligned}\right\} \qquad *$$

4. Considering now, in connection with the foci, the point (x) determined by the equation $lx+my+nz=0$, we have

$$\cos^2 xp = \frac{(l\lambda + m\mu + n\nu)^2}{(l^2+m^2+n^2)(\lambda^2+\mu^2+\nu^2)},$$

λ, μ, ν as before, and therefore

$$\lambda^2+\mu^2+\nu^2 = -(14)^2.$$

Moreover

$$(l\lambda+m\mu+n\nu)^2 = \begin{vmatrix} l, & m, & n \\ x_1, & y_1, & z_1 \\ x_4, & y_4, & z_4 \end{vmatrix}^2,\ = \begin{vmatrix} l^2+m^2+n^2, & 01, & 04 \\ 01 & , \ . & , 14 \\ 04 & , 14, & . \end{vmatrix},$$

(if for shortness $\quad 01 = lx_1 + my_1 + nz_1$, &c.)

$$= -(l^2+m^2+n^2)(14)^2 + 2\,.\,01\,.\,04\,.\,14.$$

The formula thus is

$$\cos^2 xp = \frac{-(l^2+m^2+n^2)(14)^2 + 2\,.\,01\,.\,04\,.\,14}{-(l^2+m^2+n^2)(14)^2};$$

or passing to $\sin^2 xp$, and then writing down the analogous value of $\sin^2 xq$, we have

$$\sin^2 xp = \frac{2\,.\,01\,.\,04}{14\,.\,l^2+m^2+n^2};\ \sin^2 xq = \frac{2\,.\,02\,.\,03}{23\,.\,l^2+m^2+n^2};$$

and in like manner for the other two pairs of foci

$$\sin^2 xp' = \frac{2\,.\,02\,.\,04}{24\,.\,l^2+m^2+n^2};\ \sin^2 xq' = \frac{2\,.\,03\,.\,01}{31\,.\,l^2+m^2+n^2},$$

$$\sin^2 xp'' = \frac{2\,.\,03\,.\,04}{34\,.\,l^2+m^2+n^2};\ \sin^2 xq'' = \frac{2\,.\,01\,.\,02}{12\,.\,l^2+m^2+n^2}.$$

5. These formulæ give

$$\sin^2 xp\,\sin^2 xq\,\sin^{-\frac{4}{3}} pq = \sin^2 xp'\,\sin^2 xq'\,\sin^{-\frac{4}{3}} p'q' = \sin^2 xp''\,\sin^2 xq''\,\sin^{-\frac{4}{3}} p''q''$$

$$= 4\,.\,01\,.\,02\,.\,03\,.\,04\,.\,(4\mathrm{fgh})^{-\frac{2}{3}}\,.\,(l^2+m^2+n^2)^{-2};$$

or as this may also be written

$$= 4^{\frac{1}{3}}(01\,.\,02\,.\,03\,.\,04)(12\,.\,13\,.\,14\,.\,23\,.\,24\,.\,34)^{-\frac{1}{3}}(l^2+m^2+n^2)^{-2}, \qquad *$$

where it will be recollected that 01 denotes $lx_1+my_1+nz_1$, &c., and 12 denotes $x_1x_2+y_1y_2+z_1z_2$, &c. ...

6. Taking now (x_1, y_1, z_1), &c., as the common tangents of the absolute and the conic, or say as the roots of the equations

$$x^2+y^2+z^2=0, \quad (a, b, c, f, g, h \between x, y, z)^2=0,$$

the expression on the right hand side, quà symmetrical function, homogeneous of the degree zero in the roots, and also homogeneous of the degree zero in the coefficients l, m, n, will be expressible as an absolute invariant of the two quadric functions and of the linear function $lx+my+nz$: and I say that the value is

$$= -4^{-\frac{2}{3}} . \square . \Omega^{-\frac{1}{3}} (l^2+m^2+n^2)^{-2},$$

$\square$ being the Resultant of the three functions, and Ω the Tactinvariant of the two quadric functions, as presently appearing. It is to be observed that $l^2+m^2+n^2$ is in fact the Reciprocant of $lx+my+nz$ and $x^2+y^2+z^2$, viz. the Reciprocant of $(a, \ldots \between x, y, z)^2$ and $lx+my+nz$ is

$$(bc-f^2, ca-g^2, ab-h^2, gh-af, hf-bg, fg-ch \between l, m, n)^2,$$

and for the quadric function $x^2+y^2+z^2$ this becomes $=l^2+m^2+n^2$.

7. Considering the three functions

$$x^2+y^2+z^2,$$
$$(a, b, c, f, g, h \between x, y, z)^2,$$
$$lx+my+nz,$$

it will be sufficient as regards the resultant to write down those terms which are independent of f, g, h; these are at once obtained by writing f, g, h each $=0$, and the resultant then presents itself as the norm of $l\sqrt{b-c}+m\sqrt{c-a}+n\sqrt{a-b}$; and we thus obtain (attending only to the terms in question)

$$\square = l^4(b-c)^2+m^4(c-a)^2+n^4(a-b)^2$$
$$-2m^2n^2(c-a)(a-b)-2n^2l^2(a-b)(b-c)-2l^2m^2(b-c)(c-a).$$

The Resultant is at once expressed in terms of the roots (x_1, y_1, z_1), &c., by the formula

$$\square = C(lx_1+my_1+nz_1)(lx_2+my_2+nz_2)(lx_3+my_3+nz_3)(lx_4+my_4+nz_4),$$

or according to the foregoing notation

$$\square = C . 01 . 02 . 03 . 04,$$

where, and in what follows, C *is written to denote an essentially indeterminate constant*, having (it may be) different values in different equations.

8. Moreover writing as usual

$$\mathrm{K} = abc-af^2-bg^2-ch^2+2fgh,$$
$$\Theta = bc-f^2+ca-g^2+ab-h^2,$$
$$\Theta' = a+b+c,$$
$$\mathrm{K}' = 1,$$

the Tactinvariant is taken to be

$$\Omega = 27\mathrm{K}^2\mathrm{K}'^2+4\mathrm{K}\Theta'^3+4\mathrm{K}'\Theta^3-18\mathrm{K}\mathrm{K}'\Theta\Theta'-\Theta^2\Theta'^2$$

(which for f, g, h each $=0$, reduces itself to

$$\Omega = -(b-c)^2(c-a)^2(a-b)^2).$$

The Tactinvariant vanishes if, and only if, a pair of roots (x_1, y_1, z_1), (x_2, y_2, z_2) become identical, say $x_1 : y_1 : z_1 = x_2 : y_2 : z_2$. But we have $(y_1z_2-y_2z_1)^2 = (y_1^2+z_1^2)(y_2^2+z_2^2)-(y_1y_2+z_1z_2)^2, =0$, if $x_1x_2+y_1y_2+z_1z_2=0$, that is if $12=0$; and similarly $(z_1x_2-z_2x_1)^2=0$, and $(x_1y_2-x_2y_1)^2=0$, if $12=0$. And we are thus led to the equation

$$\Omega = C.12.13.14.23.24.34.$$

9. The combination $\square^3\Omega^{-1}$ contains the roots homogeneously in the degree zero, and it will therefore have a determinate value, which is in fact found by the process which I present as a verification. The result is

$$\square^3\Omega^{-1} = -64\,(01.02.03.04)^3\,(12.13.14.23.24.34)^{-1}.$$

In verification, take the function $(a, \ldots \between x, y, z)^2$ to be $=x^2+\omega y^2+\omega^2z^2$, ω an imaginary cube root of unity: the roots may be taken to be $(1, \omega^2, \omega)$, $(1, -\omega^2, \omega)$, $(1, \omega^2, -\omega)$, $(1, -\omega^2, -\omega)$. Attending only to the terms in l, we have

$$\square = -3l^4;\ \Omega=27;\ 01.02.03.04=l^4;\ 12.13.14.23.24.14$$

(is a product of factors such as $1-\omega+\omega^2, =-2\omega$, and is) $=-64$; or the equation becomes $(-3l^4)^3(27)^{-1} = -64l^{12}(64)^{-1}$, which is right. We have thus

$$-4\square\,\Omega^{-\frac{1}{3}} = (01.02.03.04)\,(12.13.14.23.24.34)^{-\frac{1}{3}},$$

and hence the foregoing value of $\sin^2 xp \sin^2 xq \sin^{-\frac{4}{3}} pq$: say

$$\sin^2 xp \sin^2 xq \sin^{-\frac{4}{3}} pq = -4^{-\frac{2}{3}}\square\,\Omega^{-\frac{1}{3}}(l^2+m^2+n^2)^{-2}.$$

10. From the point (x) we draw to the conic tangents L, M: taking their coordinates to be (x_1, y_1, z_1), (x_2, y_2, z_2), these are the roots of

$$lx+my+nz=0,$$
$$(a, \ldots \between x, y, z)^2=0,$$

and we have

$$\sin^2 LM = \frac{(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)-(x_1x_2+y_1y_2+z_1z_2)^2}{(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)}.$$

11. We have $x_1^2+y_1^2+z_1^2=0$, or $x_2^2+y_2^2+z_2^2=0$, as the condition in order that the resultant $\square$ may vanish, and consequently

$$\square = C(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2).$$

It is easy to see that the function in the numerator will vanish if $l^2+m^2+n^2=0$, or if the Reciprocant $(bc-f^2, \ldots \between l, m, n)^2$ of the function $(a, \ldots \between x, y, z)^2$ and $lx+my+nz$ is $=0$: or calling this reciprocant F we have

$$(l^2+m^2+n^2)F = C\{(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)-(x_1x_2+y_1y_2+z_1z_2)^2\}.$$

The values of C in these two equations have a determinate ratio, and we find

$$\frac{(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)-(x_1x_2+y_1y_2+z_1z_2)^2}{(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)} = \frac{-4F.(l^2+m^2+n^2)}{\square}.$$

In verification, assume $(a,\ldots\between x, y, z)^2 = x^2 + \omega y^2 + \omega^2 z^2$ as before, $lx+my+nz = x-z$; the roots (x_1, y_1, z_1) and (x_2, y_2, z_2) may be taken to be $(1, 1, 1)$, $(1, -1, 1)$: we have $\square = \{l^2(b-c) - n^2(a-b)\}^2, = (2\omega - 1 - \omega^2)^2, = 9\omega^2$; $F = bcl^2 + abn^2, = b(a+c), = \omega(1+\omega^2), = -\omega^2$, and the equation is

$$\frac{8}{9} = \frac{-4 \,.\, -\omega^2 \,.\, 2}{9\omega^2}.$$

Hence we have

$$\sin^2 LM = -4 \,.\, F \square^{-1} \,.\, (l^2+m^2+n^2), \qquad *$$

and consequently also

$$\sin^2 LM \sin^2 xp \sin^2 xq \sin^{-\frac{4}{3}} pq = 4^{\frac{1}{3}} \Omega^{-\frac{1}{3}} F \,.\, (l^2+m^2+n^2)^{-1}, \qquad *$$

which is Clifford's formula, p. 156.

12. We take through the point (x) an arbitrary line X, coordinates (α, β, γ): these coordinates satisfy therefore the equation $\alpha l + \beta m + \gamma n = 0$.

We have $\sin^2 XL \,.\, \sin^2 XM$

$$= \frac{(\alpha^2+\beta^2+\gamma^2)(x_1^2+y_1^2+z_1^2) - (\alpha x_1 + \beta y_1 + \gamma z_1)^2}{(\alpha^2+\beta^2+\gamma^2)(x_1^2+y_1^2+z_1^2)} \,.\, \frac{(\alpha^2+\beta^2+\gamma^2)(x_2^2+y_2^2+z_2^2) - (\alpha x_2 + \beta y_2 + \gamma z_2)^2}{(\alpha^2+\beta^2+\gamma^2)(x_2^2+y_2^2+z_2^2)}.$$

13. To reduce this expression write for shortness

$$\alpha^2+\beta^2+\gamma^2 = V,$$
$$x_1^2+y_1^2+z_1^2 = \rho_1,$$
$$x_2^2+y_2^2+z_2^2 = \rho_2,$$
$$\alpha x_1 + \beta y_1 + \gamma z_1 = \sigma_1,$$
$$\alpha x_2 + \beta y_2 + \gamma z_2 = \sigma_2,$$
$$x_1 x_2 + y_1 y_2 + z_1 z_2 = \tau.$$

The expression is

$$\frac{V\rho_1 - \sigma_1^2}{V\rho_1} \,.\, \frac{V\rho_2 - \sigma_2^2}{V\rho_2},$$

where the numerator is

$$= V(V\rho_1\rho_2 - \sigma_1^2\rho_2 - \sigma_2^2\rho_1) + \sigma_1^2\sigma_2^2.$$

But from the equations $lx_1 + my_1 + nz_1 = 0$, $lx_2 + my_2 + nz_2 = 0$, $l\alpha + m\beta + n\gamma = 0$, we have

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix} = 0,$$

or squaring and reducing

$$\begin{vmatrix} V, & \sigma_1, & \sigma_2 \\ \sigma_1, & \rho_1, & \tau \\ \sigma_2, & \tau, & \rho_2 \end{vmatrix} = 0,$$

that is $V(\rho_1\rho_2 - \tau^2) + 2\sigma_1\sigma_2\tau - \sigma_1^2\rho_2 - \sigma_2^2\rho_1 = 0,$

and by reason hereof the foregoing numerator becomes

$$V(V\tau^2 - 2\sigma_1\sigma_2\tau) + \sigma_1^2\sigma_2^2, = (V\tau - \sigma_1\sigma_2)^2.$$

We thus have

$$\sin^2 XL \,.\, \sin^2 XM = \frac{(V\tau - \sigma_1\sigma_2)^2}{V^2\rho_1\rho_2},$$

$$= \frac{\{(\alpha^2+\beta^2+\gamma^2)(x_1x_2+y_1y_2+z_1z_2) - (\alpha x_1+\beta y_1+\gamma z_1)(\alpha x_2+\beta y_2+\gamma z_2)\}^2}{(\alpha^2+\beta^2+\gamma^2)^2 \,.\, (x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)}.$$

14. The numerator-function

$$(\alpha^2+\beta^2+\gamma^2)(x_1x_2+y_1y_2+z_1z_2) - (\alpha x_1+\beta y_1+\gamma z_1)(\alpha x_2+\beta y_2+\gamma z_2)$$

is $= (\beta z_1 - \gamma y_1)(\beta z_2 - \gamma y_2) + (\gamma x_1 - \alpha z_1)(\gamma x_2 - \alpha z_2) + (\alpha y_1 - \beta x_1)(\alpha y_2 - \beta x_2)$,

which vanishes if $\alpha : \beta : \gamma = x_1 : y_1 : z_1$ or $= x_2 : y_2 : z_2$, that is if $(a,\ldots\between \alpha, \beta, \gamma)^2 = 0$. Moreover observing that $l : m : n = \beta z_1 - \gamma y_1 : \gamma x_1 - \alpha z_1 : \alpha y_1 - \beta x_1 = \beta z_2 - \gamma y_2 : \gamma x_2 - \alpha z_2 : \alpha y_2 - \beta x_2$, it also vanishes if $l^2+m^2+n^2 = 0$: and we hence have

$$C\{(\alpha^2+\beta^2+\gamma^2)(x_1x_2+y_1y_2+z_1z_2) - (\alpha x_1+\beta y_1+\gamma z_1)(\alpha x_2+\beta y_2+\gamma z_2)\}$$
$$= (l^2+m^2+n^2) \,.\, (a,\ldots\between \alpha, \beta, \gamma)^2$$

(viz. this equation is true when $l\alpha + m\beta + n\gamma = 0$: it is a particular case of a more general formula where α, β, γ are arbitrary, and there are on the right hand side terms containing the factor $l\alpha + m\beta + n\gamma$). And we have as before

$$C(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2) = \square.$$

Squaring each side of the first equation, and dividing by the two sides of the second equation, we obtain a determinate result which is

$$\frac{\{(\alpha^2+\beta^2+\gamma^2)(x_1x_2+y_1y_2+z_1z_2) - (\alpha x_1+\beta y_1+\gamma z_1)(\alpha x_2+\beta y_2+\gamma z_2)\}^2}{(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)}$$
$$= \frac{(l^2+m^2+n^2)^2\{(a,\ldots\between \alpha, \beta, \gamma)^2\}^2}{\square};$$

viz. if to verify we assume as before $lx+my+nz = x-z$; and $(a,\ldots\between x, y, z)^2 = x^2+\omega y^2+\omega^2 z^2$: consequently

$$\gamma = \alpha \text{ and } (a,\ldots\between \alpha, \beta, \gamma)^2 = \alpha^2 + \omega\beta^2 + \omega^2\alpha^2, = -\omega(\alpha^2 - \beta^2):$$

also $(x_1, y_1, z_1), (x_2, y_2, z_2) = (1, 1, 1), (1, -1, 1), \square = 9\omega^2$,

then the equation becomes

$$\frac{\{2\alpha^2+\beta^2-(2\alpha+\beta)(2\alpha-\beta)\}^2}{9} = \frac{4\{\omega(\beta^2-\alpha^2)\}^2}{9\omega^2},$$

which is right.

15. We hence have

$$\sin^2 XL \sin^2 XM = (l^2+m^2+n^2)^2\{(a,\ldots\between \alpha, \beta, \gamma)^2\}^2 \,.\, \square^{-1} \,.\, (\alpha^2+\beta^2+\gamma^2)^{-2}, \qquad *$$

and thence also

$$\sin^2 XL \sin^2 XM \sin^2 xp \sin^2 xq \sin^{-\frac{4}{3}} pq$$
$$= -4^{-\frac{2}{3}} \,.\, \{(a,\ldots\between \alpha, \beta, \gamma)^2\}^2 . \Omega^{-\frac{1}{3}} \,.\, (\alpha^2+\beta^2+\gamma^2)^{-2}, \qquad *$$

which is the reciprocal of Clifford's formula, p. 157.]

ON THE THEORY OF DISTANCES*.

THIS communication relates to the following two theorems on the foci and asymptotes of curves.

Theorem i. $L, M, N, \ldots$ are the m tangents from a point a to a curve C_m of the m^{th} class; B is any line through a, meeting the curve in $m(m-1)$ points; $l, m, n, \ldots P, Q, R, \ldots$ are the $m(m-1)$ asymptotes of the curve, and $p, q, r, \ldots$ are a set of m foci.

$$\frac{\sin^2 LM \,.\, \sin^2 LN \,.\, \sin^2 MN \ldots (\overline{ap}^2 \,.\, \overline{aq}^2 \,.\, \overline{ar}^2 \ldots)^{m-1}}{al \,.\, am \,.\, an \ldots \sin BP \,.\, \sin BQ \,.\, \sin BR \ldots} = \overline{pq}^2 \,.\, \overline{qr}^2 \,.\, \overline{pr}^2 \ldots$$

Theorem ii. $l, m, n, \ldots$ are the n intersections of a line A with a curve C_n of the n^{th} order; b is any point on A from which are drawn the $n(n-1)$ tangents; $L, M, N, \ldots p, q, r, \ldots$ are a set of $n(n-1)$ foci, and $P, Q, R, \ldots$ are the n asymptotes.

$$\frac{\overline{lm}^2 \,.\, \overline{ln}^2 \,.\, \overline{mn}^2 \ldots (\sin^2 AP \,.\, \sin^2 AQ \,.\, \sin^2 AR \ldots)^{n-1}}{\sin AL \,.\, \sin AM \,.\, \sin AN \ldots lp \,.\, lq \,.\, lr \ldots}$$
$$= \sin^2 PQ \,.\, \sin^2 QR \,.\, \sin^2 PR \ldots$$

The numerator and denominator of the fraction on the left-hand side of the Equation in Theorem i. are quantities either of which I call the distance of the point a from the curve C_m. The corresponding quantities in Theorem ii. I call the Distance of the line A from the curve C_n. The reason of this is in the similarity of the analytical expressions for the distance of two geometrical forms in all cases, viz. the distance vanishes when the two forms have contact, and is infinite when either of them has contact with the "absolute." The "absolute" in plane geometry (so called by Professor Cayley) is the two circular points at infinity.

I also consider the modifications undergone by these theorems in the case of spherical curves. The method of investigation employed is an extension of the "geometric analysis" of Grassmann, itself a development of a remark of Leibnitz.

* [Notices and Abstracts... from *Report of the thirty-ninth meeting of the British Association for the Advancement of Science*, held at Exeter, August, 1869, p. 9.]

XVII.

ON A CASE OF EVAPORATION IN THE ORDER OF A RESULTANT*.

A PARTICULAR case of the following Theorem was required in the course of my proof that every rational equation has a root†; but I have thought that the theorem itself (though indeed a mere obvious remark) was worthy of being placed on record, because of the extremely small number of results of this kind that have yet been arrived at, and of their great importance in analysis.

Theorem. Let it be required to eliminate x between two equations homogeneous in x and certain other variables y, z, ..., in which equations, however, x only occurs in virtue of the occurrence of a quantity

$$w = x^{\alpha} y^{\beta} z^{\gamma} \ldots\ldots,$$

where
$$\alpha + \beta + \gamma + \ldots\ldots = \mu;$$

let also m, n be the orders of the equations, and h, k the remainders after division of m, n respectively by μ; then the order of the resultant is

$$= \frac{mn - hk}{\mu}.$$

Demonstration. Suppose that p, q are the quotients of the division of m, n respectively by μ; that is to say, let

$$m = p\mu + h, \quad n = q\mu + k,$$

* [From the *Proceedings of the London Mathematical Society*, Vol. III. Nos. 25, 26, pp. 80—82.]

† [See p. 22, supra.]

then the two equations may be written

$$a_m + a_{m-\mu} \cdot w + a_{m-2\mu} \cdot w^2 + \ldots\ldots + a_h w^p = 0,$$
$$b_n + b_{n-\mu} \cdot w + b_{n-2\mu} \cdot w^2 + \ldots\ldots + b_k w^q = 0,$$

where the suffixes of the several coefficients indicate their orders in the variables $y, z, \ldots\ldots$ Instead of directly eliminating x from these equations, we may eliminate w; and the result of this may be written down at once by Professor Sylvester's dialytic method. It is in fact

$$\begin{vmatrix} a_m & a_{m-\mu} \cdots\cdots\cdots & & \\ \cdot & a_m \cdots\cdots\cdots & & \\ \cdot & \cdot \cdots\cdots\cdots & & \\ & & \cdots\cdots\cdots\cdots & \\ & & \ldots b_k \ldots & \\ & & \ldots b_{k+\mu} & b_k \end{vmatrix}$$

where in the principal diagonal of the determinant the constituent a_m occurs q times, and the constituent b_k occurs p times. The order of the resulting term $(a_m)^q . (b_k)^p$ is $mq + kp$, and this therefore (since the determinant must be homogeneous) is the order of the resultant itself. If we had written the b coefficients before the a coefficients, we should have obtained $np + hq$ as the value of the same quantity. These two values are identical, since by hypothesis

$$q(m-h) = p(n-k) = pq\mu,$$

and therefore $qm + pk = pn + qh = r$, suppose.

Now
$$\begin{aligned} mn &= (p\mu + h)(q\mu + k) \\ &= pq\mu^2 + pk\mu + hq\mu + hk \\ &= \mu r + hk; \\ \therefore\ r &= \frac{mn - hk}{\mu}, \end{aligned}$$

as was to be proved.

The following extension is brought to light by a different method of proving the original theorem.

Let it be required to eliminate $k-1$ variables $x, y, \ldots$ from k equations, homogeneous in these, and certain other variables, in which equations, however, $x, y, \ldots$ only occur in virtue of the occurrence of $k-1$ quantities $u, v, \ldots$ all of the same order μ; let also $m_1, m_2, \ldots m_k$ be the orders of the equations, and

$$m_i = p_i\,\mu + h_i, \quad h_i < \mu;$$

then the order of the resultant is

$$\Pi p\,(\Sigma \frac{h}{p} + \mu).$$

For the equations may be written

$$a_h(u, v, \ldots)^p + \theta a_{h+\mu}(u, v, \ldots)^{p-1} + \ldots + \theta^p a_{h+p\mu} = 0,$$

where the h, p are to be affected successively with the suffixes $1, 2, \ldots k$, and θ may be considered $= 1$. Now these equations may be regarded as having coefficients of the constant order h, but the weight of every coefficient of θ^r equal to $r\mu$. This being so, the degree of the resultant in the uneliminated variables will be the sum of its order and weight calculated on these suppositions. But its order is $h_1 p_2 p_3 \cdots p_k$, or $\frac{h_1}{p_1}\Pi p$, due to the coefficients of the first equation, $\frac{h_2}{p_2}\Pi p$ due to the coefficients of the second, and so on; while its weight is $\mu \Pi p$. Hence the entire order of the resultant is

$$\Pi p\,(\Sigma \frac{h}{p} + \mu),$$

as stated above.

XVIII.

ON A THEOREM RELATING TO POLYHEDRA, ANALOGOUS TO MR COTTERILL'S THEOREM ON PLANE POLYGONS*.

MR COTTERILL'S theorem, presented last year to the Society, is as follows: For every plane polygon of n vertices there is a curve of class $n-3$ touching all the diagonals; the number of diagonals is such as to exactly determine this curve and no more; and when the curve touches the line at infinity, the area of the polygon is zero.

The proof of this depends essentially upon the fact that if we join the vertices of the polygon to any point in its plane, the area of the polygon is equal to the sum of the triangles so formed, taken of course with their proper signs according to the rule of Möbius.

The analogous theorem in space should therefore apply in the first instance to those solids whose volume can be expressed as the sum of tetrahedra, having one vertex at an arbitrary point of space, and the other three at three vertices of the figure; that is to say, it should apply to solids having *triangular faces.*

For such solids I find accordingly that the analogy is very complete and exact. It is convenient to define a plane containing three vertices but not being a face, as a diagonal plane; and

* [From the *Proceedings of the London Mathematical Society*, Vol. IV. No. 51, pp. 178—185. Mr Cotterill's paper is given, in part, in Vol. IV. No. 49.]

a line joining two vertices but not being an edge, as a diagonal line. This being so, the theorems which I shall prove are the following:

For every polyhedron of n *summits having only triangular faces* (Δ*-faced* n*-acron* CAYLEY) *there is a surface of class* n − 4 *touching all the diagonal planes.*

This surface contains all the diagonal lines.

The diagonal planes and lines are so situated, however, that the conditions of touching the planes and containing the lines are precisely sufficient to determine a surface of class n − 4.

When this surface touches the plane at infinity, the volume of the solid is zero.

To apply these propositions to polyhedra having other than triangular faces, we must consider such polygonal faces as *singularities.* Each of them, in fact, may by a small deformation of the polyhedron be resolved into a certain number of triangles; and we may thus regard a quadrangular face, for example, as the special case of two adjacent triangular faces being in one plane. Thus the quadrangular face *abcd* [fig. 18] may be regarded as produced by coplanarity of the triangles *abd, cbd.* The effect of this is also to unite together the two diagonal planes *abc, adc,* and to make the diagonal line *ac* lie in the face. Thus the surface of class $n-4$ must touch the face *abcd*; but it does not in general contain the lines *ac, bd.* It touches the face at their point of intersection. And, in general, it is not necessary to consider the diagonals of a polygonal face as diagonals of the polyhedron, and they do not in fact lie upon the surface d_{n-4}. But *a polygonal face with* m *vertices is a multiple tangent plane of order* m − 3, *and the curve of contact is Mr Cotterill's curve appertaining to the polygon.*

It is interesting to consider from this point of view the correlative propositions. Just as we have regarded a solid with a given number of summits, or *polyacron,* as having normally or in the most general case only triangular faces, while polygonal faces present themselves as singularities, and polyacra possessing them as degenerate forms; so we must regard a *polyhedron,* or solid with a given number of faces, as having normally or in general only three-edged summits (tripleural summits, CAYLEY), while summits having a greater number of edges will present themselves as singularities, and polyhedra possessing them as degenerate. Every solid with plane faces, except the tetrahedron, must have singularities of one kind or the other; just as only loci of the second order are general at the same time of their order and of their class.

The proof of these results is as follows. Let $a, b, c, \ldots l, m, n$ be the summits of a Δ-faced polyacron, and p any point in space; let also $X=0$ be the equation to the plane at infinity, and the result of substituting in X the coordinates of any point, as a, be denoted by aX. Now if fgh is a face, and the summits f, g, h, looked at from p, go round the face clockwise, then the expression $\frac{(pfgh)}{pX.fX.gX.hX}$ represents the volume of the tetrahedron $pfgh$ according to the rule of Möbius. (Here $(pfgh)$ means the determinant formed with the coordinates of p, f, g, h.) Hence, if V be the volume of the whole solid, we have

$$\Sigma \frac{(pfgh)}{pX.fX.gX.hX} = V,$$

the summation being extended over all the faces, and the summits of each so mentioned that every edge occurs twice in two different orders; that is to say, if we have mentioned $(pfgh)$, we must not mention $(pfgk)$, but $(pgfk)$ or $(pfkg)$ or $(pkgf)$. To render this equation homogeneous in all the quantities mentioned, I call to mind that the volume of a tetrahedron is not given absolutely by the formula $\frac{(pfgh)}{pX.fX.gX.hX}$, but only to a factor *près*, depending on the unit of volume employed. If we take as this unit of volume the volume of the fundamental tetrahedron, whose vertices may be denoted by 1, 2, 3, 4, then our equation becomes

$$\Sigma \frac{(pfgh)}{pX.fX.gX.hX} = V.\frac{(1234)}{1X.2X.3X.4X} \ldots\ldots\ldots (1).$$

Here V is a ratio, depending on the positions of the points

$a, b, c, \ldots$ relatively to the plane X, but absolutely independent of the position of p. If, then, we make the equation integral, by multiplying both sides into $pX.1X.2X.3X.4X.\Pi.fX$, we see that the expression

$$\Sigma (pfgh) \frac{\Pi.fX}{fX.gX.hX}$$

must be divisible by pX; because its equivalent on the other side is so divisible, and the equation is an identity so far as p is concerned. The result of the division is of the order $n-4$ in X; or, which is the same thing, *if* X *be regarded as a variable plane, the equation*

$$\Sigma (pfgh) \frac{\Pi.fX}{pX.fX.gX.hX} = 0 \ldots\ldots\ldots\ldots\ldots (2)$$

represents a surface of class n − 4.

Two things are now clear from our previous equation and from the form of this one.

1°. If the equation is satisfied when X is the plane at infinity, then $V=0$; or, *if the surface* (2) *touch the plane at infinity, the volume of the solid is zero.*

2°. The equation (2) is satisfied if $lX=0$, $mX=0$, $nX=0$, where l, m, n are any three vertices not in the same face. Therefore *the surface* (2) *touches all the diagonal planes.*

The investigation, so far, is a mere reproduction of that of Mr Cotterill, with the addition of an extra letter to apply it to three dimensions instead of two. I shall take the liberty of calling the surface thus arrived at the *index-surface* of the polyacron, and shall denote it by the symbol v_{n-4}.

The index-surface contains all the diagonal lines. For let ab be a diagonal line, and c any other summit of the solid; then abc is a diagonal plane. For if it were a face, ab would be an edge, contrary to the supposition. Consequently $n-2$ diagonal planes can be drawn through every diagonal line; now all these are touched by the index-surface. But if more than $n-4$ tangent planes can be drawn through a straight line to a surface of class $n-4$, the line must lie altogether in the surface.

Through an edge of the solid, on the other hand, two faces and $n-4$ diagonal planes can be drawn; which latter are, of course, the tangent planes from it to the surface.

There are certain diagonal planes which it is convenient to consider separately. They are those which contain three edges of the solid; and I shall call them *single planes*. A diagonal plane may, of course, contain three diagonal lines, or two, or one, or none; but if it contains any diagonal line, the condition of touching it is already involved in the condition of containing that line. So that the facts we know about the index-surface may be summed up in saying that it passes through all the diagonal lines and touches all the single planes.

I now go on to prove that in general these conditions are precisely sufficient to determine a surface of class $n-4$. In order to do this, it will be necessary to make use of the researches of Prof. Cayley upon the Δ-faced polyacra, contained in the 1st volume of the 3rd series of the *Manchester Memoirs*, p. 248; particularly of the following passage:—

"An n-acron has n summits, $3n-6$ edges, $2n-4$ faces; and it is easy to see that there are the following three cases only, viz.:

1. The polyacron has at least one tripleural summit.
2. The polyacron, having no tripleural summit, has at least one tetrapleural summit.
3. The polyacron, having no tripleural or tetrapleural summit, has at least twelve pentipleural summits.

In fact, if the polyacron has c tripleural summits, d tetrapleural summits, e pentipleural summits, and so on, then we have

$$n = c + d + e + f + g + h + \&c.,$$

$$6n - 12 = 3c + 4d + 5e + 6f + 7g + 8h + \&c.;$$

and therefore $12 = 3c + 2d + e + 0f - g - 2h - \&c.,$

or $3c + 2d + e = 12 + g + 2h + \&c.;$

whence, if $c = 0$ and $d = 0$, then $e = 12$ at least."

Upon this theorem Prof. Cayley founds a method of deriving all polyacra with $n+1$ summits from those with n summits. If we remove from a polyacron a tripleural summit, as a in the figure [fig. 19], we may derive from it a new polyacron with one summit less by regarding the diagonal plane bcd as a face of the new solid. Conversely, we may add one summit to any polyacron by crowning any one of its faces with a tripleural summit, and then regarding this face as a diagonal plane. This process is called by Prof. Cayley the First Process. In a similar manner, the skew quadrilateral $bced$ [fig. 20] formed by two adjacent faces may be crowned by a tetrapleural summit a, with the edges ab, ac, ae, ad, the faces bcd, cde becoming diagonal planes of the new solid; this is called the Second Process. Again, the skew pentagon $bcfed$ [fig. 21] formed by three adjacent faces may be crowned by a pentipleural summit a, with the edges ab, ac, af, ae, ad, the faces bcd, cde, cef becoming diagonal planes; this is called the Third Process. And it appears from the theorem quoted above, that every $(n+1)$-acron can be made out of an n-acron by one or other of these processes, according as it belongs to the first, second, or third case of the theorem.

I shall now show, then, that if the conditions of containing the diagonal lines and touching the single planes are precisely sufficient to determine the index-surface of an n-acron, then the same thing will be true for any $(n+1)$-acron derived from it by either of these processes. This will prove that the theorem is true for all Δ-faced polyacra, provided we can show that it is true for all pentacra. Now there is only one pentacron, the figure formed of two tetrahedra with a common face, $abcde$ [fig. 22]. This figure has the diagonal line ae and the single plane bcd; and the index-surface is the point v, which is precisely determined as the intersection of these.

In determining the number of conditions involved in passing through a system of lines, we must remember that every intersection of two lines diminishes the number by one, except where three or more lines are in one plane. We have only to deal with the case of three lines in one plane; the number of conditions is then reduced by two for the three intersections.

First Process.—Let D be the number of diagonal lines of the n-acron; then when we pass to the $(n+1)$-acron, the following is the increase in the number of conditions:

The D-lines are on a surface of class $n-3$ instead of $n-4$; this makes an increase	$+D$
There are $n-3$ new diagonals joining the new summit a to all the old summits except b, c, d	$+(n-2)(n-3)$
One or other of these, however, meets each of the old ones at least once; which is all that need be counted, because if any old diagonal meets two new ones a triangle is formed	$-D$
The new diagonals all meet in a point, counting as $\frac{1}{2}(n-3)(n-4)$ intersections......	$-\frac{1}{2}(n-3)(n-4)$
There is a new single plane bcd	$+1$
The total increase is therefore	$\frac{1}{2}(n-1)(n-2)$,

which is the difference between the number of conditions required to determine a surface of class $n-3$ and the number required for class $n-4$.

Second Process.—Let D be the number of old diagonal lines less be; then we have the following increase:

The D lines on surface of higher class.........	$+D$
There are $n-4$ new diagonals........	$+(n-2)(n-4)$
One or other of these, however, meets each of the D lines at least once; and, as before, this is all that need be counted...	$-D$
The new diagonals all meet in a point, counting as $\frac{1}{2}(n-4)(n-5)$ intersections......	$-\frac{1}{2}(n-4)(n-5)$
The edge cd becomes a diagonal line	$+(n-2)$
If, however, we join this edge to the $n-4$ summits different from $bcde$, there is a reduction 1 in the case of each; for either the plane was a single plane, or it contained one or two diagonals.........	$-(n-4)$
The diagonal be is on surface of higher class	$+1$
Total, as before	$\frac{1}{2}(n-1)(n-2)$.

Third Process.—Let D be the number of old diagonal lines, less df, fb, be; then we have the following increase:—

The D lines on surface of higher class.........	$+D$
There are $n-5$ new diagonals........	$+(n-2)(n-5)$
Intersections of these with D lines	$-D$
New diagonals meet in a point	$-\frac{1}{2}(n-5)(n-6)$
The edges cd, ce become diagonal lines	$+2(n-2)$
If we join these to the $n-5$ summits different from $bcdef$, there is a reduction 1 for each plane	$-2(n-5)$
The diagonals df, fb, be on surface of higher class	$+3$
Their intersections with cd, ce, and of these with one another	-3
Total, as before	$\frac{1}{2}(n-1)(n-2)$.

Passing now to the consideration of polygonal faces, I remark first that, by direct application of Mr Cotterill's theorem, we have an expression for the volume of the pyramid standing on a plane polygon. For let p be the vertex of the pyramid, q any point in the plane of the polygon; then we have

$$\Sigma' \frac{(pqab)}{pX \,.\, qX \,.\, aX \,.\, bX} = U \frac{(1234)}{1X \,.\, 2X \,.\, 3X \,.\, 4X},$$

in which U is the ratio of the volume of the pyramid to that of the fundamental tetrahedron, and Σ' refers to a summation going round the m sides of the polygon in order. From this it appears that the expression

$$\frac{1}{qX} \,.\, \Sigma' \{(pqab) \,.\, \Pi' \,.\, cX\}$$

(in which $\Pi' \,.\, cX$ means a product involving all the vertices of the polygon except a and b) is integral, independent of q and of the order $m-3$ in X. If we equate it to zero, we in fact obtain the equation of Mr Cotterill's curve belonging to the polygon.

Now if this polygon form a face P of a polyacron, we obtain, as before, the following expression for the volume of the solid:—

$$\Sigma' . \frac{(pqab)}{pX . qX . aX . bX} + \Sigma . \frac{(pfgh)}{pX . fX . gX . hX}$$

$$= V . \frac{(1234)}{1X . 2X . 3X . 4X} .$$

Thus the equation of the index-surface may be written

$$\frac{\Pi . fX}{pX . qX} . \Sigma' \{(pqab) . \Pi' . cX\} + \frac{1}{pX} . \Sigma \{(pfgh) \, \Pi . aX\} = 0.$$

Here $\Pi . aX$ must in every case contain $m-2$ factors at least belonging to points on the plane P; or it vanishes in the order $m-2$ when the coordinates of P are substituted in it. The term Σ' vanishes as we have seen in the order $m-3$ in the same case. Thus P is a multiple tangent plane of order $m-3$, and the curve of contact is determined by the term Σ'; that is to say, it is Mr Cotterill's curve belonging to the polygon.

Fig. 18.

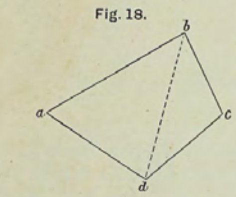


Fig. 19.

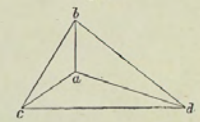


Fig. 20.

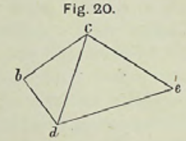


Fig. 21.

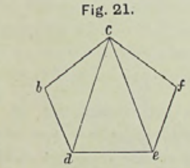
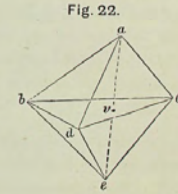


Fig. 22.



XIX.

GEOMETRY ON AN ELLIPSOID*.

THE metric properties of an ellipsoid are entirely determined by the four points in which it is met by the imaginary circle at infinity. I shall start, therefore, by assuming the existence of these four coplanar points o_1, o_2, o_3, o_4, which, taken all together, I call the absolute.

1. To represent the ellipsoid on a plane we require also two fixed points i, j; the plane sections of the ellipsoid are then represented by conics through these points, and the generating lines by lines on the plane through them. In fact, if we take a fixed point a on the ellipsoid E_2, and draw a line through a and a variable point x on the ellipsoid, this line will meet a plane L in one point y, which is the representative of x; the points i, j will then represent the generators through a. If we take the points i, j to be the absolute of the plane L, then all the plane sections will be represented by circles, the lines through i will represent one system of generating lines, and the lines through j the other. We shall have then, in addition, to consider the four points o; and the geometry of the ellipsoid will be merely the geometry of the plane considered in relation to these four points, which are concyclic.

2. We know, then, that the antipoints of the o lie upon three new circles, orthotomic of each other and of the first. These correspond to the principal sections of the ellipsoid. The antipoints themselves represent umbilici, four of which are real;

* [From the *Proceedings of the London Mathematical Society*, Vol. IV. No. 54, pp. 215—217.]

I call these u_1, u_2, u_3, u_4, and we may now take the points u as our absolute instead of the points o.

3. What now are the directions of the lines of curvature at any point x of the ellipsoid? First, the indicatrix at x is represented by the point-circle at its corresponding point y, so that conjugate directions at x correspond to rectangular directions at y. Next, the tangent plane at x meets the plane at infinity in a line, say β. Through the points o can be drawn two conics to touch β, say at p, q. The lines xp, xq are tangents to the lines of curvature at x, since the points p, q are conjugates both of the section of the ellipsoid at infinity and of the imaginary circle. But now let o_1o_2 and o_3o_4 meet β in r, s respectively. Then the involution made by rs and the points where β meets the imaginary circle have p, q for double points. The interpretation of this on the plane is, that the directions through y corresponding to xp, xq make equal angles with the circles yo_1o_2, yo_3o_4. Hence—

The lines of curvature of the ellipsoid are represented by confocal anallagmatics having the u for foci.

Sections made by two conjugate planes of the ellipsoid are represented by orthotomic circles.

4. A straight line γ in space may be denoted by the two points c_1, c_2, where it meets the ellipsoid. The sections drawn through this line will be represented by the series of coaxal circles through c_1, c_2. The sections through δ, the polar of γ, will therefore be represented by the series of coaxal circles through d_1, d_2, the antipoints of c_1, c_2. Thus, a straight line being represented by a pair of points, its polar is represented by their antipoints, as is otherwise obvious.

I denote further the principal sections and the plane at infinity by $XYZU$, which notation will serve also for the circles which represent them. Now, in general, a section of E_2 passing through a fixed point of space is represented by a circle orthotomic of a fixed circle. In particular, the points o_1 o_2, o_3 o_4; o_1 o_3, o_4 o_2; o_1 o_4, o_2 o_3, correspond in this way to the circles XYZ. I want now to find the interpretation on the plane of

the rectangularity of the lines γ and δ. The planes joining them to the point o_1o_2, o_3o_4 are harmonic of the lines o_1o_2, o_3o_4. Hence the circles c_1c_2X, d_1d_2X are harmonic of the circles coaxal with them and passing through o_1o_2, o_3o_4 respectively. This is to be true when X and Y are interchanged: the conditions may finally be written

$$\frac{c_1c_2YZ}{d_1d_2YZ}=\frac{c_1c_2ZX}{d_1d_2ZX}=\frac{c_1c_2XY}{d_1d_2XY}.$$

If now for d_1, d_2 we may substitute c_1, c'_1 where c'_1 is indefinitely near to c_1 *in any direction*, c_1c_2 represents the normal at c_1.

5. A circle P, orthotomic of U, represents a diametral section. Let the pole of this section be called p; p is a point at infinity. We know that it is always possible to find another point q at infinity, which is conjugate to p with respect both to the ellipsoid and to the imaginary circle. We may then endeavour to find the circle Q, of which q is the pole. Further, lines λ, μ can be drawn through p, q respectively, which are at right angles, and also conjugate polars of the ellipsoid. To represent these we must find a pair of points on P which have their antipoints on Q. These circles cut orthogonally; on each of them, then, there is a singly infinite number of point-pairs representing axes of the quadric, viz., the point-pairs determined by diameters of the other circle. That is to say, any circle P, orthotomic of U, being given, there can always be found a point q^0, such that the lines through q^0 determine on P point-pairs representing axes of the quadric.

The determination of q^0 depends on the position of the projecting point a. The generators through a meet the diametral section Q in two points; the remaining generators through these intersect on the representative of q^0.

6. I now proceed to construct Q when P is given. In the first place, Q has to be orthotomic of P and U. Next, if we draw through P and Q two new circles, one of which has o_1 o_2 for harmonics, and the other o_3 o_4, these must be harmonics of

P and Q. But a circle orthotomic of U and having $o_1\, o_2$ for harmonics, must have them for inverse points, and therefore have its centre on $o_1\, o_2$. Hence the line joining the centres of P and Q is cut harmonically by the lines o_1o_2, o_3o_4. Similarly, it is cut harmonically by o_1o_3, o_4o_2, and by o_1o_4, o_2o_3. Hence the centres of P and Q are polar opposites in regard to the quadrangle $o_1\, o_2\, o_3\, o_4$. They are therefore conjugate points in regard to the circle U.

7. Intersections of the ellipsoid by spheres are represented by anallagmatics *passing through* the four points o_1, o_2, o_3, o_4. There are two systems of real circles passing through pairs of them; these represent the circular sections. Sphero-conics are represented by such of these anallagmatics as have $XYZU$ for focal circles. To find the axes of any circle P of the U system we must then draw two such anallagmatics having double contact with P; the point of contact in pairs will represent the axes of the corresponding section.

XX.

PRELIMINARY SKETCH OF BIQUATERNIONS*.

I.

THE *vectors* of Hamilton are quantities having magnitude and direction, but no particular position; the vector AB being regarded as identical with the vector CD when AB is equal and parallel to CD and in the same sense. The translation of a rigid body is an example of such a quantity; for since all particles of the body move through equal distances along parallel straight lines in the same sense, the motion is entirely specified by a straight line of the given length and direction drawn through any point whatever. A couple, again, may be adequately represented by a vector; since the axis of a couple is any line of length proportional to its moment drawn perpendicular from a given face of its plane.

For many purposes, however, it is necessary to consider quantities which have not only magnitude and direction, but *position* also. The rotational velocity of a rigid body is about a certain definite axis, and equal rotations about two parallel axes are not equivalent to one another. A force acting upon a solid has a definite line of action, and equal forces acting along parallel lines differ by a certain couple. The difference between the two kinds of quantities is clearly seen when we consider the geometric calculus which is used for the study of each. In

* [From the *Proceedings of the London Mathematical Society*, Vol. IV. Nos. 64, 65, pp. 381—395.]

studying the motions of a particle or the composition of couples, the only construction required is that of the "force-polygon," and the theory involved is that of the addition of vectors; but in the static or kinematic of solids we require in addition the construction of the "link-polygon," and there is involved the theory of the involution of lines in space, or of the linear complex.

The name *vector* may be conveniently associated with a velocity of *translation*, as the simplest type of the quantity denoted by it. In analogy with this, I propose to use the name *rotor* (short for *rotator*) to mean a quantity having magnitude, direction, and position, of which the simplest type is a velocity of *rotation* about a certain axis. A rotor will be geometrically represented by a length proportional to its magnitude measured upon its axis in a certain sense. The rotor AB will be identical with CD if they are in the same straight line, of the same length, and in the same sense; *i.e.* a vector may move anywise parallel to itself, but a rotor *only* in its own line.

The *addition* of rotors will proceed by the rules which govern the composition of forces and rotations. Here, however, we come upon a very important break in the analogy between rotors and vectors. While the sum of any number of vectors is always a vector, it will only happen in special cases that the sum of a number of rotors is a rotor. In fact, the composition of two forces whose lines of action do not intersect, or of two rotation-velocities whose axes do not intersect, gives rise to a system of forces on the one hand, and the most general velocity of a rigid body on the other. These still more complex quantities have been studied, and the theory of their addition or composition completely worked out, by Dr Ball.

A system of forces may be reduced in one way to a single force P, and a couple G whose plane is perpendicular to the line of action of the force, or *central axis*. Dr Ball speaks of the system of forces as a *wrench* about a certain *screw*; the axis of the screw being the central axis, and the pitch being the ratio $\frac{G}{P}$ of the couple to the force. Similarly the velocity of a

rigid body may be represented in one way only as a rotation-velocity ω about a certain axis combined with a translation-velocity v along that axis. Dr Ball speaks of this velocity as a *twist*-velocity about a certain screw; the axis of the screw being the axis of rotation, and its pitch the ratio $\frac{v}{\omega}$ of the translation to the rotation. A *screw* is here a geometrical form resulting from the combination of an *axis* or straight line given in position with a *pitch* which is a linear magnitude. A *wrench* is the association with this geometrical form of a magnitude whose dimensions are those of a force; a *twist*-velocity the association of a magnitude whose dimensions are those of an angular velocity. The extreme convenience of this nomenclature is well exemplified in the remarkable memoir above referred to.

Just as a vector (translation-velocity or couple) is magnitude associated with direction, and as a rotor (rotation-velocity or force) is magnitude associated with an axis; so this new quantity, which is the sum of two or more rotors (twist-velocity or wrench), is magnitude associated with a screw. Following up the analogy thus indicated, I propose to call this quantity a *motor;* the simplest type of it being the general motion of a rigid body. And we shall say that in general the sum of rotors is a motor, but that in particular cases it may degenerate into a rotor or a vector.

II.

A *quaternion* is the ratio of two vectors, or the operation necessary to make one into the other. Let the vectors be [fig. 23] AB and AC, as they may both be made to start from any arbitrary point A. Then AB is made into AC by turning it round an axis through A perpendicular to the plane BAC until its direction coincides with that of AC, and then magnifying or diminishing it until it is of the same length as AC. The ratio of two vectors then is the combination of an ordinary numerical ratio with a *rotation;* or, as Hamilton expresses it, a quaternion is the product of a tensor and a versor. Since the point A is perfectly arbitrary, this rotation is not about a definite axis;

but is completely specified when its angular magnitude and the direction of its axis are given.

This quaternion* $\frac{AC}{AB} = q$, then, is an operation which, being performed on AB, converts it into AC, so that $q \,.\, AB = AC$. The axis of the quaternion is perpendicular to the plane BAC; and it is clear that the quaternion operating upon any other vector AD in this plane will convert it into a fourth vector AE in the same plane, the angle DAE being equal to BAC and the lengths of the four lines proportionals. But a quaternion can *only* operate upon a vector which is perpendicular to its axis. If AF be any vector not in the plane BAC, the expression $q \,.\, AF$ is absolutely unmeaning. A meaning is indeed subsequently given to an analogous expression *in which the signification of* AF *is different.* But it is very important to remark that so long as AF means a *vector* not perpendicular to the axis of q, the expression $q \,.\, AF$ has no meaning at all.

Let us now consider what is the operation necessary to convert one *rotor* into another. There is one straight line which meets at right angles the axes of any two rotors, and part of which constitutes the shortest distance between them. Let AC [fig. 24] be the shortest distance between the rotors AB and CD. Then AB may be converted into CD by a process consisting of three steps. First, turn AB about the axis AC into the position AB', parallel to CD. Then slide it along this axis into the position CD'. Lastly, magnify or diminish it in the ratio of CD' to CD. The first two operations may be regarded as together forming a twist about a screw whose axis is AC and whose pitch is

$$\frac{AC}{\text{circ. meas. of } BAB'}.$$

The ratio of two rotors, then, is the combination of an ordinary

* Professor Cayley, by a very convenient notation, distinguishes $\frac{\lfloor AC}{AB\rvert}$ and $\frac{AC\rvert}{\lfloor AB}$; viz., $AB\frac{\lfloor AC}{AB\rvert} = 1$, but $\frac{AC\rvert}{\lfloor AB} AB = 1$. It should, I think, be a convention that $\frac{X}{Y}$ is *always* to mean $\frac{X\rvert}{\lfloor Y}$, viz., the operation which converts Y into X, or which, coming after the operation Y, is equivalent to the operation X.

numerical ratio with a *twist*. This twist is associated with a perfectly definite screw, and is only specified when its angular magnitude and the screw (involving direction, position, and pitch) are given. We may say also that just as the rotation (versor) involved in a quaternion is the ratio of two directions, so the twist involved in the ratio of two rotors is really the ratio of their axes.

Here again a remark must be made about the range of this operation. Using the expression *tensor-twist* to mean the ratio of two rotors (which is in fact a twist multiplied by a tensor), we may say that a tensor-twist can operate upon any rotor which meets its axis at right angles. Let t denote the operation which converts AB into CD, so that $t = \frac{CD}{AB}$, and $t \,.\, AB = CD$; then if EF be any other rotor which meets AC at right angles, the expression $t \,.\, EF$ will have a definite meaning, viz., it will mean a rotor obtained by sliding EF along a distance equal to AC, turning it about AC as axis through an angle equal to BAB', and altering its length in the ratio $AB : CD$. But if EF be a rotor not meeting AC, or meeting it at any other than a right angle, the expression $t \,.\, EF$ will have no meaning whatever.

We have now defined the ratio of two rotors, and shown that like a quaternion it has a restricted range of operation. The question naturally arises, What now is the operation which converts one *motor* into another? We can answer this question very easily in the case in which the two motors have the same pitch; for in this case their ratio is a tensor-twist whose tensor is the ratio of their magnitudes and whose twist is the ratio of their axes. We are led to this by considering each motor as the sum of two rotors which do not intersect. Let α and β be two such rotors, t a tensor-twist whose axis meets them both at right angles; then $t\alpha$ is a rotor, say γ, and $t\beta$ is another rotor, say δ. If therefore we assume the distributive law, we have

$$t\,(m\alpha + n\beta) = m\gamma + n\delta,$$

or

$$t = \frac{m\gamma + n\delta}{m\alpha + n\beta}.$$

It is a mere translation of known theorems to say that the axis of t meets at right angles the axes of the motors $m\alpha + n\beta$ and $m\gamma + n\delta$, and that one of these axes is converted into the other by the same twist that makes α into γ or β into δ.

The solution of this problem in the general case in which the pitches are different, is not so easy. In the first place, we must remember that every motor consists of a rotor part and a vector part, and that its pitch is determined by the ratio of these two parts. By combining a suitable vector with a motor, therefore, we may make the pitch of it anything we like, without altering the rotor part. Now let it be required to find the operation which will convert a motor A into a motor B. Let B' be a motor having the same rotor part as B, and the same pitch as A; and let $B = B' + \beta$, where β is a vector parallel to the axis of B. Then the ratio $\frac{B}{A} = \frac{B'}{A} + \frac{\beta}{A}$; but $\frac{B'}{A}$ is a tensor-twist, say t, and we may write

$$\frac{B}{A} = t + \frac{\beta}{A},$$

where it now only remains to find an operation which will convert a motor A into a vector β.

In order to do this, we must introduce a symbol whose nature and operation will at first sight appear completely arbitrary, but will be justified in the sequel. *The symbol ω, applied to any motor, changes it into a vector parallel to its axis and proportional to the rotor part of it.* That is to say, it changes rotation about any axis into translation parallel to that axis, and a force into a couple in a plane perpendicular to its line of action. But if the rotation is accompanied by translation or the force by a couple, the symbol takes no account whatever of these accompaniments; and if made to operate directly on a vector, reduces it to zero. It follows from this that if it be made to operate twice upon a motor, it reduces it to zero; or $\omega^2 A = 0$ always. The portion of any expression which involves ω must therefore be treated as an infinitesimal of the first order; all higher orders being uniformly neglected.

Since then $\omega A = \alpha$, a vector, and the ratio $\frac{\beta}{\alpha}$ is a quaternion q so that $q\alpha = \beta$, we may write successively

$$\beta = q\alpha = q\omega A,$$

$$\frac{\beta}{A} = q\omega,$$

and then $$\frac{B}{A} = t + q\omega,$$

or *the ratio of two motors may be expressed as the sum of two parts, one of which is a tensor-twist, and the other is ω multiplied by a quaternion.*

The same ratio may be expressed in another form. Let an arbitrary point O be assumed as the origin; then every motor may be expressed in one way as the sum of a rotor passing through O and a vector. Now the theory of rotors passing through a fixed point is exactly the same as that of vectors in general, and the ratio of any two of them is a tensor-twist whose pitch is zero, or what is the same thing, a quaternion whose axis is constrained to pass through the fixed point. If we use cursive Greek letters (as α, β) in general to represent rotors through the origin, we may distinguish vectors from them by prefixing the symbol ω; thus $\omega\alpha$ denotes a vector parallel and proportional to the rotor α. The ratio $\frac{\beta}{\alpha}$ will then be a quaternion q, which is also the ratio $\frac{\omega\beta}{\omega\alpha}$ *. The general expression for a motor is then $\alpha + \omega\beta$. Let it now be required to find the ratio of two motors $\alpha + \omega\beta$, $\gamma + \omega\delta$; or the value of the expression $$\frac{\gamma + \omega\delta}{\alpha + \omega\beta}.$$

First, let $\frac{\gamma}{\alpha} = q$; then $q(\alpha + \omega\beta) = \gamma + q\omega\beta = \gamma + \omega q\beta$.

The symbol $q\beta$ has at present no geometrical meaning; for in general the rotors α, β, γ will not be coplanar, and cannot

* It follows from this that $\omega q = q\omega$, or ω is commutative with quaternions.

therefore be operated on by the same quaternion q. If however (as in the Calculus of Quaternions) we consider all these quantities as expressed in terms of three rectangular unit rotors through the origin, $\frac{\delta - q\beta}{\alpha}$ will be a perfectly definite quaternion r. The equation

$$r\alpha = \delta - q\beta$$

is, like the equation $q(\alpha + \omega\beta) = \gamma + \omega q\beta$,

at present purely literal and devoid of meaning. Yet if (remembering the properties of the symbol ω) we add ω times the first equation to the second and assume the distributive law, we obtain

$$(q + \omega r)(\alpha + \omega\beta) = \gamma + \omega\delta.$$

In this way the ratio $\frac{\gamma + \omega\delta}{\alpha + \omega\beta}$ is expressed in the form $q + \omega r$, which expression may conveniently be called a *biquaternion**. The final equation, however, is not susceptible of interpretation in the same sense as the equation $q\alpha = \gamma$. The expression $q + \omega r$ does not denote the sum of geometrical operations which can be applied to the motor $\alpha + \omega\beta$ as a whole; and the ratio of two motors is only expressed by a symbol as the sum of two parts, each of which separately has a definite meaning in certain other cases, but not in the case in point. In following sections this difficulty will be partly overcome by showing that the system here sketched is the limit of another in which it does not occur.

The preceding remarks may however explain, and be illustrated by, the following table:—

Geometrical Form	Quantity	Example	Ratio
Sense on st. line	Vector on st. line	Addition or Subtraction	Signed Ratio
Direction in plane	Vector in plane	Complex quantity	Complex Ratio
Direction in space	Vector in space	Translation, Couple	Quaternion
Axis	Rotor	Rotation-Velocity, Force	Twist
Screw	Motor	Twist-Velocity, System of Forces	Biquaternion

* Hamilton's *biquaternion* is a quaternion with complex coefficients; but it is convenient (as Prof. Peirce remarks) to suppose from the beginning that all scalars may be complex. As the word is thus no longer wanted in its old meaning, I have made bold to use it in a new one.

III.

That geometry of three-dimensional space which assumes the Euclidian postulates has been called by Dr Klein the *parabolic* geometry of space, to distinguish it from two other varieties, which assume uniform positive and negative curvature respectively, and which he calls the *elliptic* and *hyperbolic* geometry of space. The investigations which follow involve the postulates of elliptic geometry. As, however, the postulate of uniform positive curvature is not sufficient to define this, it may be worth while to devote a short space to an explanation of its nature.

Space of three dimensions is that the points of which may be associated with systems of values of three variables x, y, z. It is not in general possible, however, so to make this association that to every system of values there shall correspond in general one point, and to every point in general one system of values. When this is the case, the space is called *unicursal*. An *algebraic* space is one in which the position of a point may be uniquely defined by a set of values of periodic algebraic integrals, without exceptions which form a part of the space. Thus, unicursal spaces are a particular case of algebraic. Attending now to unicursal spaces only, we must observe that there are in general exceptions to the unique correspondence of points and value-systems; namely, there are certain points to each of which correspond an infinite number of values of the coordinates satisfying a certain equation or equations; and there are certain value-systems to which correspond, not points, but loci in the space. The assignment of these point-equations and loci-values and of their relations with one another serves to determine the *projective-connection* of the space; and when once these are known, the whole of its projective geometry may be worked out. The point-equations and loci-values may or may not involve imaginary values of the variables or their coefficients; but in all cases they must be taken into account. The points which correspond to real systems of values are called real points; those which correspond to imaginary systems, imaginary points: the study of these latter, which does not strictly belong to that of three-dimensional space, is undertaken only for the sake of the former.

Loci which correspond to linear equations between the coordinates may at present be called *planes*, and their intersections *lines;* this is a purely projective definition, and these loci are not necessarily *flat* planes and *straight* lines in the metrical sense. Points, lines, and planes are included in the name *elements.*

The *metric* geometry of space* is the theory of the projective relations of certain fixed geometrical forms with all other geometrical forms, or of the invariant relations of certain fixed algebraic forms with all other algebraic forms. The word *power* will be explained as much as is wanted in the sequel; meanwhile it may be said that these fixed forms (called all together *the absolute*) are given when we know the points, the lines, and the planes of the absolute, or say the elements of the absolute; and that the power of an element of the absolute in regard to any arbitrary element is infinite. In other words, we *require* in general equations of the absolute in point-, line-, and plane-coordinates respectively.

A unicursal space the points of which may be represented uniquely by value-systems of the coordinates x, y, z, without the exception of any point-equations or loci-values, is called a *linear* space. This is merely a projective definition, and leaves the absolute, therefore the whole of metric geometry, undetermined.

There is a particular determination of the absolute in a linear space which is of the utmost importance. It is that in which the points of the absolute are those of a certain quadric surface, while the lines and planes of the absolute are those which touch this surface; or in which the three equations of

* This theory of metric geometry is due to Prof. Cayley: "Sixth Memoir on Quantics," *Phil. Trans.*, 1859.

the absolute are of the second degree. There are three cases* to be considered, as being the only ones of which observed space can form a part:—

(1) *Elliptic* geometry; all the elements of the absolute are imaginary.

(2) *Hyperbolic* geometry; the absolute contains no real straight lines, and surrounds us. In this case, real points situate on the other side of the surface are called *ideal.*

(3) *Parabolic* geometry; the surface degenerates into an imaginary conic in a real plane. The points of the absolute are points in the (real) plane of this conic; the lines and planes are the imaginary lines and planes which meet and touch the conic respectively.

The *first* of these suppositions will be made in what follows. It may be well here to set down in what it consists.

(1) The space to be considered is such that there is one point of it for every set of values of the coordinates x, y, z, and one set of values for every point, without any exception whatever.

(2) There is a certain quadric surface, called the absolute, all whose points and tangent planes are imaginary. If the line joining two points a, b meet the absolute in i, j, the quantity

$$\frac{ab \,.\, ij}{\sqrt{(ai\,.\,aj\,.\,bi\,.\,bj)}} \equiv \overline{ab},$$

(which is a function of anharmonic ratios, and therefore an invariant,) is called the *power* of the points a, b in regard to one another, or of either in regard to the other. The *distance* of these two points is an angle θ such that

$$\sin\theta = \overline{ab}.$$

Similarly, if through the line of intersection of the planes A, B there be drawn the tangent planes I, J to the absolute,

* On this division see Dr Klein, "Ueber die so-genannte Nicht-Euklidische Geometrie," *Math. Annalen*, Bd. 4. The second case is the geometry of Lobatschewsky and Bolyai.

the power of the planes A, B in regard to one another is the quantity

$$\frac{AB \cdot IJ}{\sqrt{(AI \cdot AJ \cdot BI \cdot BJ)}} = \overline{AB},$$

and the angle between them is an angle ϕ such that

$$\sin\phi = \overline{AB}.$$

(3) If two points are conjugate in regard to the absolute, they are distant a *quadrant* from one another; if two lines or planes are conjugate in regard to the absolute, they are at right angles. Thus all the points at a quadrant distance from a given point are situate on its polar plane in regard to the absolute, and every plane through it cuts this polar plane at right angles. Every line has a polar line in regard to the absolute, such that every point on the polar line is distant a quadrant from every point on the line; and every line which is at right angles to either meets the other. Through an arbitrary point can in general be drawn *one* line perpendicular to a given plane; namely, the line joining the point to the pole of the plane. If, however, the point *is* the pole of the plane, every line through it is perpendicular to the plane. Similarly, from a point not on the polar of a given line can be drawn one and only one perpendicular to the line; namely, the line through the point which meets the given line and its polar.

(4) *In general, two lines can be drawn so that each meets two given lines at right angles, and these are polars of one another.* One line may therefore be converted into another by rotation about two polar axes. These axes are determined as the lines which meet the two given lines and their polars. If we travel continuously along one of these lines and draw perpendiculars on the other, one of these axes determines the shortest distance between the lines, and the other the longest. If then these two are equal, the lines are equidistant along their whole length. Thus *there is a case of exception in which two lines and their polars belong to the same set of generators of a hyperboloid; the lines are then equidistant along their whole length, and meet the same two generators of one system of the absolute.* I shall use the word *parallel* to denote two lines so situated; and they shall be called *right* parallel or *left* parallel according as one is converted into the other by a right-handed or left-handed twist. Through an arbitrary point can be drawn one right parallel and one left parallel to a given line; the angle between them is twice the distance of the point from the line. There are many points of analogy between the *parallels* here defined and those of parabolic geometry. Thus, if a line meet two parallel lines, it makes equal angles with them; and a series of parallel lines meeting a given line constitute a ruled surface of zero curvature. The geometry of this surface is the same as that of a finite parallelogram whose opposite sides are regarded as identical.

(5) A twist-velocity of a rigid body must be regarded as having *two* axes. For a motion of translation along any axis is the same thing as a rotation about the polar axis, and *vice versâ*. Hence a twist-velocity is compounded of rotation-velocities about two polar axes; say these are θ, ϕ. Then the motion may be regarded either as a twist-velocity about a screw whose pitch is $\frac{\phi}{\theta}$ and whose axis is the first axis, or about a screw whose pitch is $\frac{\theta}{\phi}$ and whose axis is the polar axis. In general, then, a motor has two axes, and is expressible in one way only as the sum of two polar rotors. There is, however, one case of exception in which the axes of a motor are indeterminate; that, namely, in which the magnitudes of the two polar rotors are equal*. If a rigid body receive at the same time a rotation about an axis and an equal translation along it, all the points of the body will describe parallel straight lines; and the motion of the body is at the same time a rotation about any one of these lines combined with an equal translation along it. Such a motion may be adequately represented by a line of given length drawn through any point whatever parallel to a given line. A motor of pitch unity, or which is its own polar, may therefore

* This motion is described in another connection by Drs Klein and Lie, *Math. Annalen*, Bd. 4; it is a transformation of the absolute into itself in which two generators remain unaltered.

be regarded as having the nature of a *vector*, and shall in future be denoted by that name. For we may define a vector as a motor whose axes are indeterminate; and the case we are now considering is the only case of such indetermination which occurs in elliptic geometry. Vectors will be called *right* or *left* according as the twist of them is right- or left-handed.

Prop.: *Every motor is the sum of a right and a left vector.* For let A be a motor, and A' the polar motor; then we have $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$. Now $A + A'$ and $A - A'$ are both motors of pitch unity, but one right-handed and the other left-handed.

IV.

A fixed point being chosen as origin, let three lines perpendicular to one another be drawn through it, and let three unit-rotors having these lines as axes be denoted by the symbols i, j, k. Then every rotor through the origin will be denoted by an expression of the form $ix + jy + kz$, where x, y, z are scalar quantities, or the ratios of magnitudes. The symbols i, j, k shall have also another meaning; viz., each shall signify the rotation through a right angle about its axis of any rotor which meets that axis at right angles. When they are performed on rotors passing through the origin, these operations satisfy the equations $i^2 = j^2 = k^2 = ijk = -1$, by the ordinary rules of quaternions; and it is easy to see that the same equations hold good when the operations are performed on rotors not passing through the origin. The compound symbol $ix + jy + kz$ is also to have an analogous secondary meaning; viz., a rectangular rotation about the axis of the rotor which it previously denoted, combined with a tensor $\sqrt{(x^2 + y^2 + z^2)}$. It can operate only on a rotor which meets its axis at right angles. This being so, the ratio of any two rotors through the origin is a *quaternion* of the form $q \equiv w + ix + jy + kz \equiv w + \rho$, say. The axis ρ of this quaternion is perpendicular to the plane of the two rotors. If α be a rotor through the origin and q a quaternion, the product $q\alpha$ can be formed according to the Hamiltonian rules of multiplication, and is in general a quaternion r. In this general case

the equation $q\alpha = r$ can only be interpreted by giving to α its *secondary* meaning; and the translation of this statement into words is as follows:—If a rotor be capable of being successively operated upon by the rectangular versor α and the quaternion q, the final result will be the same as if it had been originally operated upon by the quaternion r. If, however, the axes of q and α are at right angles, the scalar part of r will be wanting, and we may write the equation $q\alpha = \rho$. This equation is now susceptible of a *primary* interpretation; viz., the quaternion q operating on the rotor α produces the rotor ρ; although the *secondary* interpretation does not cease to be true.

With such conventions, the two sides of the equation

$$(q + r)s = qs + rs$$

(in which q, r, s are quaternions) have always the same meaning when both are interpretable; which is what is meant by saying that the distributive law holds good for these symbols.

The ratio of two rotors which do not meet is a twist which in general has perfectly definite axes. But when the rotors are polars of one another, the axes of the twist are indeterminate; for any line meeting both meets them at right angles, and will serve for an axis. It is therefore always possible to find a twist which shall simultaneously convert two given rotors into their polars; and any two rectangular twists with pitch 1 or -1 have a pair of common rotors on which they can operate, and which they convert into one another. Hence we may consider that

All rectangular twists of pitch 1 are equivalent to one another; and all rectangular twists of pitch -1 are equivalent to one another.

The rectangular twist of pitch 1 shall be denoted by the symbol ω; the expression $\omega\alpha$ will denote the rotor polar to α and equal to it in magnitude, obtained from it by a left-handed twist. During the operation of this twist, every point of the rotor describes a straight line; if therefore the twist be continued through two right angles, the rotor will be replaced in its original position, *not* reversed; we have therefore

$$\omega^2 = 1.$$

Every motor can be expressed as the sum of two rotors, one passing through the origin and the other being polar to a rotor through the origin. The general expression for a motor is therefore

$$\alpha + \omega\beta.$$

This will represent a *rotor* if the two rotor constituents intersect, or if each is perpendicular to the polar of the other; or if $S\alpha\beta = 0$.

Let now $$\xi = \frac{1+\omega}{2}, \quad \eta = \frac{1-\omega}{2};$$

then $$\xi^2 = \frac{1+2\omega+\omega^2}{4} = \frac{2+2\omega}{4} = \xi,$$

$$\eta^2 = \frac{1-2\omega+\omega^2}{4} = \frac{2-2\omega}{4} = \eta,$$

$$\xi\eta = \frac{1-\omega^2}{4} = 0.$$

Any motor $\alpha + \omega\beta$ can also be expressed in the form $\xi\gamma + \eta\delta$. It is clear that $\xi\gamma$ is the right vector part of this motor, and that $\eta\delta$ is the left vector part. If we multiply $\xi\gamma + \eta\delta$ by ξ, the result is merely $\xi\gamma$; so the effect of multiplying a motor by ξ is merely to pick out the right vector part of it. The symbols ξ, η are thus in a certain sense *selective* symbols, and are analogous to the S and V of quaternions.

Ratio of two motors.—We can find immediately now the operation which converts a motor $\xi\gamma + \eta\delta$ into a motor $\xi\alpha + \eta\beta$. For if we perform the operation

$$\left(\xi\frac{\alpha}{\gamma} + \eta\frac{\beta}{\delta}\right)(\xi\gamma + \eta\delta),$$

remembering the laws of multiplication of ξ, η, we obtain the result $\xi\alpha + \eta\beta$. If then $\frac{\alpha}{\gamma} = q$, $\frac{\beta}{\delta} = r$, we may write

$$\frac{\xi\alpha + \eta\beta}{\xi\gamma + \eta\delta} = \xi\frac{\alpha}{\gamma} + \eta\frac{\beta}{\delta} = \xi q + \eta r,$$

and the latter may be written in the form

$$\frac{q+r}{2} + \omega \cdot \frac{q-r}{2} = s + \omega t,$$

showing that *the ratio of two motors is a biquaternion.*

The motor $\xi\alpha + \eta\beta$ will be a *rotor* if

$$S(\alpha+\beta)(\alpha-\beta) = 0,$$

or if $$T\alpha = T\beta;$$

and it is easy to see from this that the biquaternion $\xi q + \eta r$ will be a *twist*, or the ratio of two rotors, if $Tq = Tr$.

V.

1. *Position-Rotor of a Point.*—The coordinates of a point in regard to a quadrantal tetrahedron 1234 being x_1, x_2, x_3, x_4, the equation to the absolute is $\Sigma x^2 = 0$. The rotor from the origin (the point 4) to the point x is represented by

$$i_1\frac{x_1}{x_4} + i_2\frac{x_2}{x_4} + i_3\frac{x_3}{x_4}, \text{ or } \Sigma i_k\frac{x_k}{x_4}\ (k = 1, 2, 3),$$

where i_1, i_2, i_3 are rotors along the edges of the tetrahedron from the origin to the middle points of the edges. The tensor of this rotor is the tangent of the angular distance from the origin to the point it represents. For if

$$\rho = i_1\frac{x_1}{x_4} + i_2\frac{x_2}{x_4} + i_3\frac{x_3}{x_4},$$

$$(T\rho)^2 = \frac{x_1^2 + x_2^2 + x_3^2}{x_4^2} = \tan^2 \widehat{ox}, \text{ where } o \text{ is the origin.}$$

The angular distance from the origin to a point has an infinite number of values, which differ by multiples of π. If therefore a rotor be considered to have this angular distance as its length, the rotor of a point can only be defined by such an equation as $\breve{\rho} \equiv \breve{\alpha}$ (mod. $\breve{\pi}_a$). To obviate this indetermination, there is required a one-valued unicursal function having the period π; the tangent of the angular distance is hereby completely singled out.

2. *Equation of a Straight Line.*—Let OM [fig. 25] be the perpendicular from the origin O upon the straight line MP;

and let ON be a line perpendicular to OM in the plane MOP. Then from the triangle MOP we have

$$\frac{\tan OM}{\tan OP} = \cos MOP;$$

or if $\quad OM = \alpha, \ OP = \rho, \ ON = \beta, \ T\alpha = T\rho \cos MOP;$

so that α is the component of ρ in the direction OM, and we have $\rho = \alpha + \beta x$, where x is some scalar.

By varying x, then, we get all the points in the line MP. But if α_1 is *any* particular value of ρ, the equation may just as well be written $\quad \rho = \alpha_1 + \beta x,$

where now α_1 is not necessarily perpendicular to β.

This form may be reduced to the preceding as follows:

To find the perpendicular from O, put $\delta T\rho = 0$; this gives

$$S\alpha_1\beta + \beta^2 x = 0,$$

and the equation becomes

$$\rho = \alpha_1 - \beta S\frac{\alpha_1}{\beta} - \beta x,$$

where $\alpha_1 - \beta S\frac{\alpha_1}{\beta} = \alpha$ of the former equation.

3. *Rotor along Straight Line whose Equation is given.*

Let OR [fig. 26] be the rotor through the origin which has right parallelism with MP. Then $\angle NOR = OM$. Let OK be perpendicular to ON and OM, and of such length that

$$\frac{\tan OK}{\tan ON} = \tan NOR.$$

Then, if $\quad \gamma = OK, \ OR = \beta + \gamma.$

Now $\frac{T\gamma}{T\beta} = T\alpha$, and $U\gamma = U\alpha\beta$, since γ is perpendicular to α and β. Hence $\gamma = \alpha\beta$; and if R be a rotor along MP, m a scalar,

$$\text{right vector of } R = \xi R = m\xi(\beta + \gamma) = m\xi(\beta + \alpha\beta),$$

so $\quad \text{left vector of } R = \eta R = m\eta(\beta - \gamma) = m\eta(\beta - \alpha\beta);$

therefore $\quad R = m(\beta + \omega\alpha\beta).$

Now if R have the same length as β, we have

$$\beta^2 = R^2 = m^2(\beta^2 + \overline{\alpha\beta}^2) = m^2\beta^2(1 - \alpha^2);$$

therefore $\quad R = \frac{\beta + \omega\alpha\beta}{\sqrt{(1 - \alpha^2)}}.$

Conversely, equation to axis of rotor $\gamma + \omega\delta$ is

$$\rho = \frac{\delta}{\gamma} + \gamma x.$$

This finds the rotor in the case in which $\rho = \alpha + \beta x$, where $S\alpha\beta = 0$. But in the general case we have only to write the equation in the form

$$\rho = \alpha - \beta S\frac{\alpha}{\beta} + \beta x,$$

whence $\quad R = \frac{\beta + \omega\left(\alpha - \beta S\frac{\alpha}{\beta}\right)\beta}{\sqrt{\left(1 - \alpha^2 - \beta^2 S^2\frac{\alpha}{\beta} + 2S\alpha\beta S\frac{\alpha}{\beta}\right)}}$

$$= \frac{\beta + \omega V\alpha\beta}{\sqrt{\left(1 + S\alpha\beta S\frac{\alpha}{\beta} - \alpha^2\right)}}.$$

4. *Rotor* ab *joining Points whose Position-Rotors are* α, β.

The equation of this rotor is

$$\rho = \alpha + (\beta - \alpha)x,$$

whence $\quad mR = \beta - \alpha + \omega V\alpha\beta.$

Now if a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4 are the coordinates of the points, we have

$$(TR)^2 = \tan^2 ab = \frac{\Sigma(a_h b_k - a_k b_h)^2}{(\Sigma a_h b_h)^2} = -\frac{(\alpha - \beta)^2 + (V\alpha\beta)^2}{(1 - S\alpha\beta)^2},$$

therefore $\quad R = \frac{\beta - \alpha + \omega V\alpha\beta}{1 - S\alpha\beta}.$

Cor.—If ρ [fig. 27] be the rotor of a variable point on a curve, $d\lambda$ a rotor along the tangent of length equal to the arc of the curve between ρ and $\rho + d\rho$, we have

$$d\lambda = \frac{d\rho + \omega V\rho d\rho}{1 - \rho^2}.$$

5. *Rotor parallel to β through Point whose Position-Rotor is α.*

The general equation to a line through the point α is $\rho = \alpha + \lambda x$, where λ is any rotor through the origin. A rotor along this line is $\lambda + \omega V\alpha\lambda$; if this is right parallel to β, we have

$$\xi (\lambda + V\alpha\lambda) = \xi\beta, \qquad (\xi\omega = \xi)$$

or

$$\lambda + V\alpha\lambda = \beta.$$

Operating by $S\alpha$, we have, since $S . \alpha V\alpha\lambda = 0$,

$$S\alpha\lambda = S\alpha\beta,$$

whence, by addition, $\lambda + \alpha\lambda = \beta + S\alpha\beta$,

and $\lambda = (1+\alpha)^{-1} (\beta + S\alpha\beta) = \beta - (1+\alpha)^{-1} V\alpha\beta.$

The rotor required is

$$\lambda + \omega V\alpha\lambda, \text{ or } \lambda + \omega (\beta - \lambda).$$

This becomes, then,

$$\beta - (1+\alpha)^{-1} V\alpha\beta + \omega (1+\alpha)^{-1} V\alpha\beta = \beta - 2\eta (1+\alpha)^{-1} V\alpha\beta.$$

Instead of operating by $S\alpha$ on the equation

$$\lambda + V\alpha\lambda = \beta,$$

we might have operated with $V\alpha$, and got

$$V\alpha\lambda + \alpha V\alpha\lambda = V\alpha\beta, \text{ since } V . \alpha V\alpha\lambda = \alpha V\alpha\lambda,$$

therefore $V\alpha\lambda = (1+\alpha)^{-1} V\alpha\beta,$

and $\lambda = \beta - V\alpha\lambda = \beta - (1+\alpha)^{-1} V\alpha\beta.$

Similarly, we have for the rotor *left* parallel to β,

$$\lambda = \beta + (1-\alpha)^{-1} V\alpha\beta,$$

and the rotor is

$$\lambda + \omega (\lambda - \beta) = \beta + (1-\alpha)^{-1} V\alpha\beta + \omega (1-\alpha)^{-1} V\alpha\beta$$
$$= \beta + 2\xi (1-\alpha)^{-1} V\alpha\beta.$$

XXI.

GRAPHIC REPRESENTATION OF THE HARMONIC COMPONENTS OF A PERIODIC MOTION*.

FOURIER'S theorem asserts that any motion having the period P may be decomposed into simple harmonic motions having periods P, $\frac{1}{2}P$, $\frac{1}{3}P$, &c.; and assigns the amplitudes and phases of these motions by means of definite integrals. In fact, if $\phi(x + 2\pi) = \phi(x)$ for all values of x,

then $\phi(x) = \frac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + \ldots + b_m \cos mx + \ldots$
$$+ a_1 \sin x + a_2 \sin 2x + \ldots + a_m \sin mx + \ldots,$$

where
$$\pi b_m = \int_{-\pi}^{+\pi} \phi(\alpha) \cos m\alpha \, d\alpha,$$

$$\pi a_m = \int_{-\pi}^{+\pi} \phi(\alpha) \sin m\alpha \, d\alpha;$$

and this is made applicable to the general case of periodic motion by putting $\frac{x}{2\pi} = \frac{t}{P}$, where t is the time elapsed since the era of reckoning.

The terms $b_m \cos mx + a_m \sin mx$ constitute a simple harmonic motion of period $\frac{P}{m}$; the object of the present communication is to represent this motion by a graphical construction.

If a right circular cylinder be made to revolve uniformly about its axis, while a pencil in contact with its surface has

* [From the *Proceedings of the London Mathematical Society*, Vol. v. No. 67, pp. 11—14.]

a rectilinear motion parallel to the axis, the pencil will trace out upon the cylinder a curve representing its motion. In particular, if this motion is simple harmonic and of a period equal to that of the revolution of the cylinder, the curve traced out will be an ellipse. The amplitude of the motion will be $r \cot\theta$, where r is the radius of the cylinder and θ the inclination of its axis to the plane of the ellipse; the phase at epoch is determined by the orientation of the major axis. This ellipse may thus be regarded as a graphical representation of the simple harmonic motion.

Now let the pencil have any arbitrary motion whose period is P. For convenience let us suppose that the axis of the cylinder is vertical. Then, if the cylinder be made to turn once round in the time P, a curve C_1 will be traced on it, representing the arbitrary periodic motion. Next let the cylinder turn round twice in the period P; a curve C_2 will be traced on it. And generally when the cylinder turns round m times in the time P, a curve C_m will be traced on it, going m times round the cylinder. All these curves C will be closed curves, because the motion is periodic.

At this point I call to mind Mr Hayward's extension of the meaning of "area," whereby it is made to have direction as well as magnitude*. Any closed contour, not necessarily plane, being given, the area of its projection on a plane is found to be a maximum when the plane has a certain aspect. The magnitude of this maximum area, considered as having this particular aspect, is called the area of the contour; and the area of the projection on any other plane is proportional to the cosine of the angle which it makes with the maximum plane. If, therefore, we know the area of the projection of a contour on any three planes at right angles, we can find the area of the projection on any other plane.

Now I say that it is possible to draw on the cylinder an ellipse which shall have the same area in magnitude and direction as the curve C_1. For the projection of this curve on a

* [*Proceedings of the London Mathematical Society*, Vol. IV. No. 59, pp. 289—291.]

plane perpendicular to the axis (*i.e.* a horizontal plane) is merely the circular section of the cylinder, which is the same as the projection of any ellipse traced on it. If therefore we cut the cylinder by a plane parallel to the maximum plane of the contour C_1, the elliptic section E_1 will have the same area as that contour in magnitude and direction.

The contour C_2 goes twice round the cylinder; therefore its projection on a plane perpendicular to the axis is the circle described twice in the same direction, and its area is twice that of the projection of any ellipse. If therefore we cut the cylinder by a plane parallel to the maximum plane of C_2, we shall obtain an ellipse E_2 whose area is half that of the contour C_2 and parallel to it. Similarly, the contour C_m goes m times round the cylinder, and a plane parallel to its maximum plane will determine an ellipse E_m whose area is $\frac{1}{m}$th of that of C_m.

Now first let a circle be drawn on the cylinder whose height is the mean height of C_1. On this circle is the middle point of all the component oscillations.

Next, while the cylinder goes round once in the period P, let the pencil follow the ellipse E_1; it will then have a simple harmonic motion of period P, which is, in fact, the first or fundamental component. Then, while the cylinder goes round twice in the time P let the pencil follow the ellipse E_2: the resulting simple harmonic motion of period $\frac{P}{2}$ is the second component. Generally, while the cylinder goes round m times in the time P, let the pencil follow the ellipse E_m; this simple harmonic motion is the mth component.

The demonstration of this result is very simple. The values of a_m and b_m may be written as follows:

$$\pi b_m = \int_{-\pi}^{+\pi} \phi x \cos mx\,dx = \frac{1}{m}\int_{x=-\pi}^{x=\pi} \phi x\,d(\sin mx),$$

$$\pi a_m = \int_{-\pi}^{+\pi} \phi x \sin mx\,dx = -\frac{1}{m}\int_{x=-\pi}^{x=\pi} \phi a\,d(\cos mx).$$

Suppose now that $\phi\alpha$ is set up vertically at P [fig. 28], when $PCA = m\alpha$, then $d\cos m\alpha$ is the element of CA, and $d\sin m\alpha$ is the element of CB; so that the differentials under the integral signs are respectively elements of the areas projected on vertical planes through AA' and BB'. If in these integrals we write $b_m \cos\alpha + a_m \sin\alpha$ in place of $\phi\alpha$, we get the areas of the corresponding projections of the ellipse E_m; these are πb_m and πa_m respectively. Thus the area of C_m projected on three planes at right angles is m times that of the ellipse E_m; or the areas of the two curves have the same aspect and are in the ratio $m : 1$; which was to be proved.

XXII.

ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS*.

THE following communication is an attempt to apply Jacobi's geometrical representation of the addition-theorem in elliptic functions to the theory of their transformation. For this purpose I use the said representation in the following form.

Consider two circles, one of which is wholly within the other, but which are not concentric; as in the figure [fig. 29]. The points of the outer circle may be uniquely represented by a parameter x, such that if 0 and ∞ are the points represented by these values respectively, and $0t$ is the tangent at the former, x is proportional to the ratio of the sines of the angles which $0x$ makes with the lines $0t$ and 0∞. Let the angle $x0t = \phi$; then, if we make $x = i\tan\phi$ $(i = \sqrt{-1})$, the values 1, -1 of the parameter will belong to the circular points at infinity. Let then k^{-1}, $-k^{-1}$ be the values belonging to the imaginary points of intersection of the two circles. Through the points 0, x let tangents be drawn to the inner circle, meeting the outer circle in c, ξ; these being so chosen that, when x moves continuously to 0, ξ will move continuously to c. Then Jacobi's theorem is that, if

$$x = \text{sn}(u, k),\quad \xi = \text{sn}(r, k),\quad c = \text{sn}(\gamma, k),\quad \text{then } r = u + \gamma.$$

The extension to any two conics, made by Prof. Cayley, may be put into the same form. The representation of each point

* [From the *Proceedings of the London Mathematical Society*, Vol. VII. Nos. 90, 91, pp. 29—38.]

of a conic by a parameter is determinate when we know the parameters of any three points. Now the four intersections of two conics U, V may be divided into pairs in three ways, and will so determine three involutions upon the conic U. Let one of these be chosen, and let the parameters 0 and ∞ be assigned to its united points. Then, if the value 1 be assigned to one of the intersections, -1 will belong to another of them; and the remaining two will have parameters equal in magnitude but of contrary signs. Call these $\pm k^{-1}$, and draw the tangents as before; then Jacobi's theorem remains true, except that we must write $\pm \text{sn}^{-1} \xi = \text{sn}^{-1} x + \text{sn}^{-1} c$; the sign of $\text{sn}^{-1} \xi$ depending on the reality of the common tangents.

The proof of it depends on the symmetrical (2, 2) correspondence, considered by Euler, in relation to the addition of elliptic integrals. Given that the relation between the points x and ξ is that the line $x\xi$ touches the conic V, it is clear that to one position of x correspond two positions of ξ; and to one position of ξ, two positions of x; or, the points have a (2, 2) correspondence. Hence the equation connecting the quantities x, ξ must be of the second order in each; and it must obviously be symmetrical. Let the equation be

$$(ax^2 + 2bx + c)\,\xi^2 + 2\,(bx^2 + 2b'x + c')\,\xi + (cx^2 + 2c'x + c'') = 0;$$

or, which is the same thing,

$$(a\xi^2 + 2b\xi + c)\,x^2 + 2\,(b\xi^2 + 2b'\xi + c')\,x + (c\xi^2 + 2c'\xi + c'') = 0.$$

The values of x which make the two corresponding values of ξ coincide are given by the equation

$$X = (ax^2 + 2bx + c)\,(cx^2 + 2c'x + c'') - (bx^2 + 2b'x + c')^2 = 0,$$

and similarly the values of ξ which make the two corresponding values of x coincide are given by $\Xi = 0$, where Ξ is the same function of ξ that X is of x. Now, by differentiating the original equation, we easily find $dx : \sqrt{X} = d\xi : \sqrt{\Xi}$.

The roots of the equation $X = 0$ are clearly the parameters of the points of intersection of the two conics; for these are the only points on U from which two coincident tangents can be drawn to V. If, then, the parameters of these points have

been made equal to ± 1, $\pm k^{-1}$, X must be proportional to $(1 - x^2)(1 - k^2x^2)$, and the differential equation becomes

$$\frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}} = \frac{d\xi}{\sqrt{(1 - \xi^2)(1 - k^2\xi^2)}};$$

whence, since, when $x = 0$, $\xi = c$,

$$\pm \text{sn}^{-1} x + \text{sn}^{-1} c = \text{sn}^{-1} \xi,$$

which is the theorem in question.

If we change V into $U + \sigma V$, and allow σ to vary, this varying conic will always have the same intersections with U, and therefore k will be constant; but c will depend upon the value of σ. It is clear that c^2 is given uniquely when σ is given, but, when c is given, there are two values of σ. When $c = 0$, the two conics must touch at 0, and therefore must coincide; thus both values of σ vanish. When $c = \infty$, the conic $U + \sigma V$ becomes a pair of lines intersecting on the line $0\,\infty$; let α and β be the values of σ which belong to these, and which are, of course, roots of the equation $\square\,(U + \sigma V) = 0$.

Then we must have $c^2 = \dfrac{m\sigma^2}{(\sigma - \alpha)(\sigma - \beta)}$,

where m is an undetermined constant.

Suppose, now, that a polygon is inscribed in U by the following process; a tangent is drawn from x to a conic σ_1, meeting U again in x_1; then from x_1 to a conic σ_2, meeting U in x_2, and so on; finally, let x_{n-1} be joined to x. If $c_1, c_2 \ldots c_{n-1}$ be the constants belonging to these conics respectively, we shall have

$$\pm \text{sn}^{-1} x_1 = \text{sn}^{-1} x + \text{sn}^{-1} c_1,$$
$$\pm \text{sn}^{-1} x_2 = \text{sn}^{-1} x_1 + \text{sn}^{-1} c_2;$$
$$\&c. \qquad \&c.$$
$$\pm \text{sn}^{-1} x_{n-1} = \text{sn}^{-1} x_{n-2} + \text{sn}^{-1} c_{n-1};$$

whence, by addition, with proper changes of sign,

$$\pm \text{sn}^{-1} x_{n-1} = \text{sn}^{-1} x + \text{sn}^{-1} c_1 \pm \text{sn}^{-1} c_2 \pm \ldots \pm \text{sn}^{-1} c_{n-1}$$
$$= \text{sn}^{-1} x + \text{sn}^{-1} c_n, \text{ suppose.}$$

From this equation it appears that the last side of the polygon will always touch the same conic of the series $U+\sigma V$, wherever the starting-point x is taken. Or, if a polygon be inscribed in U, and move so that all but one of its sides touch conics of the series $U+\sigma V$, then the remaining side will also touch a conic of the series. This theorem of Poncelet's is proved in this way by Jacobi. It is to be noticed that the signs of the quantities c depend upon which tangent is drawn from the corresponding point x; the two tangents belong in a definite way to the two tangents from 0. The final value of c_n being thus determined, one of the two conics belonging to it is singled out by the sign given to $\text{sn}^{-1} x_{n-1}$. In fact, the whole series of conics $U+\sigma V$ is divided into three parts by the three line-pairs it contains. For two conics in the same part of the series the signs of $\text{sn}^{-1}\xi$ are certainly the same; for conics in different parts they may be different.

And it appears that the conics mentioned in the theorem may belong to any part of the series if the signs be properly chosen in each of the equations. It is remarked by Jacobi that with these restrictions the position of x_{n-1} does not depend upon the order in which the conics are taken.

Now suppose two such polygons to be drawn with the same system of conics, x being continuously moved to ξ, and at the same time x_1 to ξ_1, &c. We shall then have the equations

$$\pm\,\text{sn}^{-1} x_1 = \text{sn}^{-1} x + \text{sn}^{-1} c_1,$$

$$\pm\,\text{sn}^{-1} \xi_1 = \text{sn}^{-1} \xi + \text{sn}^{-1} c_1,$$

the signs being the same in both. Consequently

$$\pm\,(\text{sn}^{-1} x_1 - \text{sn}^{-1}\xi_1) = \text{sn}^{-1} x - \text{sn}^{-1}\xi,$$

or the lines $x_1\xi_1$, $x\xi$ touch the same conic of the series $U+\sigma V$.

Proceeding in this way, we may show that

$$\pm\,(\text{sn}^{-1} x_r - \text{sn}^{-1}\xi_r) = \text{sn}^{-1} x - \text{sn}^{-1}\xi;$$

whence it appears that the lines joining corresponding vertices of the two polygons all touch the same conic of the series when the n conics touched by the sides belong to the same part; but if they belong to different parts, each joining line touches one

of two conics which harmonically divide the pairs of lines $U+\alpha V$, $U+\beta V$.

Attending now only to the first case, it will be convenient to re-state the two theorems together, as follows:—

If a polygon be inscribed in a conic U *so that all its sides but one touch conics of the series* $\text{U}+\sigma\text{V}$, *the remaining side will also touch a conic of the series.*—(Poncelet's Theorem.)

When all these conics can pass continuously into one another without breaking up into two straight lines, the lines joining corresponding vertices of two such polygons will all touch a conic of the series.

Let us now consider the particular case in which all the sides of the moving polygon touch the same conic. Here the second theorem is true without restriction; the lines joining corresponding vertices of two such polygons will always touch one conic passing through the intersections of the other two. In this case also *the vertices of the variable polygon determine upon the conic* U *an involution of the* n^{th} *order;* that is to say, if the parameters of the vertices of one polygon be determined by an equation $p_n = 0$ of the n^{th} order, and those of another polygon by an equation $q_n = 0$, then the vertices of any third polygon will be determined by an equation $p_n - yq_n = 0$, where y is a variable quantity, which we may call the parameter of the polygon. The relation between y, the parameter of the polygon, and x, the parameter of any one of its vertices, is $y = p_n : q_n$, where p_n, q_n are rational integral functions of the n^{th} order in x.

Suppose, then, the relation between U and V to be such that a polygon of n sides may be inscribed in U and circumscribed to V. Let $x, x_1 \ldots x_{n-1}$ be the vertices of such a polygon; then, if $x = \text{sn}\,u$, we must have $x_1 = \text{sn}\,(u+\gamma)$, $x_2 = \text{sn}\,(u+2\gamma)$, $x_3 = \text{sn}\,(u+3\gamma) \ldots x_{n-1} = \text{sn}\,\{u+(n-1)\gamma\}$, and consequently $x = \text{sn}\,(u+n\gamma)$. Therefore $n\gamma$ is a period of the elliptic function; and the number of conics of the series $U-\sigma V$, which can be inscribed in n-gons inscribed in U is equal to the number of periods whose n^{th} parts are not congruent, that is, for n a prime number it is $n+1$.

Now let another polygon be drawn having a vertex at ξ, and let η be its parameter. Then the lines $x\xi$, $x_1\xi_1$ ... &c. will all touch a conic W. Let this conic be held fixed, and the two polygons moved so that the lines joining corresponding vertices always touch W. Then to every value of y will belong two values of η, and *vice versâ*, and this (2, 2) correspondence is symmetrical. Hence *a symmetrical* (2, 2) *correspondence between individual vertices implies a symmetrical* (2, 2) *correspondence between the polygons.*

Now, the parameter of every polygon is determined when we know the parameters of three polygons. Let the parameters of the polygons which have a vertex at 0, 1, and ∞ be made equal to 0, 1, and ∞ respectively; this amounts to saying that $p_n = 0$ has a root 0, $q_n = 0$ has a root ∞, and $p_n - q_n = 0$ has a root 1. It is clear, then, from the symmetry of the figure, that y must be an odd function of x, so that $p_n + q_n = 0$ will have a root -1. This amounts to saying that p_n is x multiplied by a rational integral function of x^2, and q_n (which is really only of the order $n-1$) is another rational integral function of x^2. This being so, let $\pm\lambda^{-1}$ be the parameter of those polygons which have vertices at the remaining two points of intersection of the conics. Then the quantities y and η are connected by a symmetrical (2, 2) correspondence such that the values of y which give equal values for η are ± 1, $\pm\lambda^{-1}$. Therefore, if $y = \operatorname{sn}(u', \lambda)$, we must have $\eta = \operatorname{sn}(u' + \delta, \lambda)$, where δ is a constant.

We have arranged that y is divisible by x or $\operatorname{sn} u$, by making 0 the parameter of that polygon which has a vertex at the point 0. It appears thus that y must also be divisible by $x_1, x_2 \ldots x_{n-1}$, since, when any one of these is zero, y vanishes. We may write, therefore, $y = mxx_1 \ldots x_{n-1}$, where m is a constant, since y is only infinite when one or other of the x is infinite. The products $x_1 x_{n-1}$, $x_2 x_{n-2}$, &c., are given rationally as ratios of quadratic functions of x by the original equation of (2, 2) correspondence. To determine m, we have

$$y = m \operatorname{sn} u \operatorname{sn}(u+\gamma) \operatorname{sn}(u+2\gamma) \ldots \operatorname{sn}\{u+(n-1)\gamma\} \; (n\gamma = aK + biK');$$

but, since $y = 1$, when $x = \operatorname{sn} u = 1$, or $u = K$, we have also

$$1 = m \operatorname{sn} K \operatorname{sn}(K+\gamma) \operatorname{sn}(K+2\gamma) \ldots \operatorname{sn}\{K+(n-1)\gamma\},$$

and therefore $y = \dfrac{\operatorname{sn} u \operatorname{sn}(u+\gamma) \ldots \operatorname{sn}\{u+(n-1)\gamma\}}{\operatorname{sn} K \operatorname{sn}(K+\gamma) \ldots \operatorname{sn}\{K+(n-1)\gamma\}}$;

the denominator may, of course, be simplified, as in Jacobi's formula, but for present purposes it may be left as it stands. Now λ^{-1} is the value of y when $x = k^{-1}$, or when $u = K + iK'$,

$$\frac{\lambda}{k} = \frac{\operatorname{sn} K \operatorname{sn}(K+\gamma) \ldots \operatorname{sn}\{K+(n-1)\gamma\}}{\operatorname{sn}(K+iK') \operatorname{sn}(K+iK'+\gamma) \ldots \operatorname{sn}\{K+iK'+(n-1)\gamma\}},$$

which, again, may be easily reduced.

If we now assume that $u = Mu'$, where $y = \operatorname{sn}(u', \lambda)$ and M is a constant, we may determine M by observing that it is the value of $x : y$ when both of them are zero. Namely we have

$$M = \frac{\operatorname{sn}(K+\gamma) \operatorname{sn}(K+2\gamma) \ldots \operatorname{sn}\{K+(n-1)\gamma\}}{\operatorname{sn}\gamma \operatorname{sn} 2\gamma \ldots \operatorname{sn}(n-1)\gamma}.$$

And thus, with the exception of the assumption just made, the theory of transformation is established by means of that of the in-and-circumscribed polygon.

Let us now endeavour to generalize the theory of the in-and-circumscribed polygon. In the first place, we may observe that every involution of the third order gives rise to an elliptic transformation; for two triangles inscribed in a conic are always circumscribed to the same conic*. For convenience, we will now consider the involution as determined on the conic V; namely, the triangles form groups of three tangents which are in involution. And we may now generalize our theorem as follows:—

If a complete n-*gram move with its sides touching a conic so*

* Every substitution $y = \dfrac{U}{V}$, where U, V are cubic functions of x, has an elliptic differential which it transforms. The cubic forms U, V are first polars of two points in regard to a single quartic form F (Gundelfinger, *Math. Annalen*). Let $X = 0$ give the four points whose first polars in regard to F have a square factor; then $dx : \sqrt{X}$ is the elliptic differential required. We have $X = jF + iH$, where H is the Hessian of F, and i, j the quadrinvariant and cubinvariant. [Cf. however p. 221, infra.]

that they form groups of n *tangents in involution, the locus of the* $\frac{1}{2}n(n-1)$ *vertices is a curve of order* $n-1$.

For, consider the number of points which the curve has in common with any one tangent of the conic. It determines uniquely in the involution the group of n tangents to which it belongs, and can have no other point on the locus of intersections except those $n-1$ in which it meets the other $n-1$ tangents of this group. Therefore &c.

If, in a curve of order $n-1$, *it is possible to inscribe one complete* n*-gram whose sides all touch a conic, then it is possible to inscribe a singly infinite number, and the sides determine upon the conic an involution of the* n^{th} *order.*

Let $A, B, C, \ldots N=0$ be the equations of the sides of the complete n-gram, then the equation of the curve may be written

$$\frac{\alpha}{A}+\frac{\beta}{B}+\frac{\gamma}{C}+\ldots+\frac{\nu}{N}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $\alpha, \beta, \gamma \ldots \nu$ are constants. But since A is a tangent to the conic, its equation may be written in the form $X+aY+a^2Z=0$, where a is the parameter of the tangent and X, Y, Z three fixed lines. The condition for three tangents to meet in a point is

$$0=\begin{vmatrix} 1, & x, & x^2 \\ 1, & y, & y^2 \\ 1, & z, & z^2 \end{vmatrix}=(y-z)(z-x)(x-y) \ldots\ldots\ldots(2),$$

where x, y, z are their parameters. Hence the condition that the tangents whose parameters are x, y shall meet on the curve (1) is

$$\frac{\alpha}{(a-x)(a-y)}+\frac{\beta}{(b-x)(b-y)}+\ldots+\frac{\nu}{(n-x)(n-y)}=0 \ldots(3);$$

or, which is the same thing,

$$\Sigma\frac{\alpha}{a-x}=\Sigma\frac{\alpha}{a-y},$$

and therefore, if tangents xy and also xz meet on the curve, it follows that yz will meet on the curve. Starting, then, from any arbitrary tangent x, we can find the $n-1$ points in which

this meets the curve, and draw from them $n-1$ other tangents; the intersection of any two of these will lie on the curve, by what we have just proved. That is to say, we can inscribe a complete n-gram which shall have for one side any arbitrary tangent of the conic.

Now, suppose that x is given, then, regarding the equation (3) as determining $n-1$ values of y, we can find the product of the roots. Namely, it is

$$=\Sigma\frac{\alpha\,.\,\Pi a}{a(a-x)} : \Sigma\frac{\alpha}{a-x}, \quad \Pi a=a\,.\,b\,.\,c\ldots n.$$

If we multiply this product by x, we obtain the product of the parameters of all the tangents forming a complete n-gram; let this be called λ; then, observing that

$$\frac{\alpha x}{a(a-x)}=\frac{\alpha}{a}-\frac{\alpha}{a-x},$$

we shall find $$\left(1-\frac{\lambda}{\Pi a}\right)\Sigma\frac{\alpha}{a-x}=\Sigma\frac{\alpha}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Now this is an equation of the n^{th} order in x, the roots of which are the parameters of the sides of a complete n-gram; and λ is the product of these roots. Since λ is linearly involved, the equation shows that these groups of n tangents form an involution of the n^{th} order, and that λ is proportional to the parameter of such a group in the involution when the groups containing the tangents 0, ∞ are made to have the parameters 0, ∞ respectively. It appears also that the sum of the roots, sum of their products in pairs, &c., are each given as linear functions of λ, and might each be used as parameters of the involution.

We shall now endeavour to find an expression for $ABC\ldots N$.

Let $1-\lambda(\Pi a)^{-1} : \Sigma\alpha a^{-1}$ be called θ, and let $\Pi_a(x-a)$ mean the product $(x-a)(x-b)\ldots(x-n)$, then the equation for x may be put into the form

$$\Pi_a(x-a)+\theta\Pi_a(x-a)\,.\,\Sigma\alpha(x-a)^{-1}=0 \quad \ldots\ldots\ldots(5),$$

where the first term is of order n, and the second of order

$n-1$ in x. By a slight change of notation, let the n roots be called $x_1, x_2, \ldots x_n$, and let $\Pi_x(y-x)$ mean

$$(y-x_1)(y-x_2)\ldots(y-x_n).$$

Then we have

$$(x-x_1)(x-x_2)\ldots(x-x_n) = \Pi_a(x-a) + \theta\Pi_a(x-a)\,\Sigma\alpha(x-a)^{-1};$$

and therefore

$$\Pi_x(y-x) = \Pi_a(y-a) + \theta\Pi_a(y-a)\,\Sigma\alpha(y-a)^{-1}.$$

Multiplying together two such equations, we obtain

$$\begin{aligned}\Pi_x(y-x)\,.\,\Pi_x(z-x) = {} & \\ & \Pi_a(y-a)\,.\,\Pi_a(z-a) \\ & + \theta\Pi_a(y-a)\,.\,\Pi_a(z-a)\,.\,\{\Sigma\alpha(y-a)^{-1} + \Sigma\alpha(z-a)^{-1}\} \\ & + \theta^2\Pi_a(y-a)\,.\,\Pi_a(z-a)\,.\,\Sigma\alpha(y-a)^{-1}\,.\,\Sigma\alpha(z-a)^{-1}.\end{aligned}$$

Now, if we examine this equation, we shall find that the left-hand member is $A'B'C'\ldots N'$, where $A'=0$, $B'=0\ldots$ are the tangents which make up the n-gram belonging to the parameter θ; the first term on the right is $ABC\ldots N$, the n-gram for $\theta=0$, and the last term is $\theta^2 Z_1 Z_2 \ldots Z_n$, the n-gram for $\theta=\infty$, which has the line $Z=0$ in it. Thus we may write the equation

$$\Pi A' = \Pi A + \theta P_n + \theta^2 \Pi Z \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

and it only remains to find the nature of the curve P_n of the n^{th} order. We may see from its equation,

$$0 = \Pi_a(y-a)\,\Pi_a(z-a)\,\{\Sigma\alpha(y-a)^{-1} + \Sigma\alpha(z-a)^{-1}\},$$

that it passes through the points of contact and all the intersections of the n tangents ΠA; and then it is clear, from the symmetry, that it must pass through the points of contact and all the intersections of the tangents ΠZ. But perhaps the simplest way is to consider the envelope of the n-gram $\Pi A'$, which we know must consist of the conic K_2 once, and the locus of the nodes C_{n-1} twice; thus we shall have

$$4\Pi A\,.\,\Pi Z - P_n^2 = K_2 C_{n-1}^2$$

to a factor *près*, and this equation gives at once the intersections of P_n with K_2, and with the n-gram.

We may now state the following propositions:—

Given any two in-and-circumscribed n-*grams* ΠX *and* ΠZ, *there exists always a curve* P_n *of order* n *which passes through their* n(n − 1) *vertices and their* 2n *points of contact with the conic.*

The equation of any other n-*gram may be written in the form*

$$0 = \Pi X + \lambda P_n + \lambda^2 \Pi Z.$$

The relation between λ, *the parameter of the* n-*gram, and* x *the parameter of one of its sides, is*

$$(\Pi a - \lambda)\,\Sigma\alpha(a-x)^{-1} = \Pi a\,.\,\Sigma\alpha a^{-1},$$

and λ *is the product of the roots of this equation.*

I have here taken ΠX, $(X_1 X_2 \ldots X_n)$ for the first n-gram, corresponding to $\lambda=0$, instead of ΠA, corresponding to $\lambda = \Pi a$ or $\theta = 0$.

We may show, conversely, that if the envelope of

$$0 = P + \lambda Q + \lambda^2 R,$$

where $P=0$, $Q=0$, $R=0$ are three curves of the n^{th} order, is $4PR - Q^2 = K_2 C^2$, C being of order $n-1$; then P and R are each an assemblage of n straight lines. For the curve P has nodes on all its intersections with C, since $4PR = Q^2 + K_2 C^2$; that is, $\frac{1}{2}n(n-1)$ nodes, so that it must consist of n straight lines.

[This point of view immediately suggests the extension of the whole theory to quadric surfaces. If the envelope of $P + \theta Q + \phi R + \theta\phi S$ is $PS - QR = K_2 C^2$, where P, Q, R, S are of order n, and C of order $n-1$, each of the surfaces P, Q, R, S will meet the quadric K_2 in two curves of the n^{th} order, and therefore will have $\frac{1}{2}n^2$ contacts with it; and similarly will meet C_{n-1} in two curves of order $\frac{1}{2}n(n-1)$, which intersect in $\frac{1}{4}n^2(n-1)$ points; these are not contacts, but nodes on the surface P. We thus get a theory of surfaces of order n, having $\frac{1}{2}n^2$ contacts with a quadric surface, and $\frac{1}{4}n^2(n-1)$ nodes on a fixed surface of order $n-1$. It appears that n must be even, and of course the variable surface is subject to other conditions.

Thus, in the case of a cone doubly tangent to the quadric, and having its vertex on a fixed plane, it has also to pass through two fixed points on the plane and four on the quadric.

The application of the identity $4PR - Q^2 = K_2C^2$ to surfaces only reproduces the theory of the plane conic.}

At any point of intersection of the two n-grams,

$$0 = \Pi X + xP_n + x^2\,\Pi Z,$$
$$0 = \Pi X + yP_n + y^2\,\Pi Z,$$

we shall have $xy \,:\, -x-y \,:\, 1 = \Pi X \,:\, P_n \,:\, \Pi Z.$

Consequently, any symmetrical (m, m) correspondence between the two n-grams expresses that they intersect on a curve of order mn, namely $(\Pi X,\ -P_n,\ \Pi Z)^m$, if the equation of the (m, m) correspondence is $(xy,\ x+y,\ 1)^m$.

If we substitute the values $xy : -x-y : 1$ for $\Pi X : P_n : \Pi Z$, in the equation of a third n-gram $0 = \Pi X + zP_n + z^2\,\Pi Z$, we shall get simply $(z-x)(z-y) = 0$. Consider, then, m different n-grams ΠA_1, $\Pi A_2 \ldots \Pi A_m$, and form the curve of order $n(m-1)$

$$\frac{\beta_1}{\Pi A_1} + \frac{\beta_2}{\Pi A_2} + \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

If the n-grams x, y meet on this curve we shall have

$$\Sigma \frac{\beta}{(a-x)(a-y)} = 0\,;$$

or, what is the same thing,

$$\Sigma \frac{\beta}{a-x} = \Sigma \frac{\beta}{a-y},$$

where $a_1, a_2 \ldots a_m$ are the parameters of the n-grams. It follows, as before, that a singly infinite number of groups of n-grams can be totally inscribed in the curve (7); a group being totally inscribed when all the intersections of any two n-grams of the group are on the curve.

So far we have dealt only with totally inscribed n-grams; and, as this case is represented only by the triangle when the curve of inscription is a conic, it might seem that there should

be more general theorems corresponding to the inscription of polygons of a greater number of sides in a conic. But, in fact, the case of total inscription is the general case, and all others are cases of decomposition of the curve C_{n-1}. Consider, for example, a hexagon inscribed in a conic. If we produce all the sides, we shall get nine more intersections; three of these lie on a straight line, and the other six on a conic which circumscribes two triangles. These two conics and the straight line make up the curve C_5 of the fifth order. Again, let eight lines A, B, C, D, F, G, H, K (fig. 30) touch a conic, and let the twelve points marked o in the figure touch a cubic; then the octagram may be moved round the conic so as to keep these twelve points on the cubic. The points marked × will lie on a fixed straight line, and a second cubic will pass through the intersections of A, B, C, D among themselves, and of F, G, H, K among themselves. These two cubics and the straight line make up the curve C_7 of the seventh order; and it is easy to see that there is an analogous case for any even number of lines. In order that a porismatic polygon may be inscribed in a curve, it is necessary that either the order or the curve or the number of sides should be even.

XXIII.

NOTES ON THE COMMUNICATION ENTITLED "ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS."*

SOME of the following notes† would have been incorporated in the paper by the process of revision for the press, if that had not been kindly performed for me during an enforced absence from the English climate. As regards all but one of them, I am glad of the opportunity which has thus been afforded me of extension and correction. But it is a matter of great regret to me that I discovered too late the priority of M. Darboux in the principal theorem of the second part of the paper; viz., the porismatic character of a polygram circumscribed to a conic and totally inscribed in a curve of order one less than the number of sides. In one of the notes to a book which it is almost inexcusable in a geometer not to have read, marked, learned, and inwardly digested‡, M. Darboux has stated and proved the theorem, and has followed it by further investigations of the highest interest and importance. The method even of my investigation is the same as that of M. Darboux (as indeed was inevitable from the nature of the subject), namely, the representation of a point in a plane by means of the parameters of the tangents drawn from it to a fixed conic. It is not the first time

* [From the *Proceedings of the London Mathematical Society*, Vol. VII. Nos. 102, 103, pp. 225—233.]

† These remarks apply also to certain developments which I have since thought it better to communicate under separate titles.

‡ *Sur une classe remarquable de courbes et de surfaces algébriques.* Paris, Gauthier-Villars, 1873. Note II., p. 183.

that I have had the honour of following, however distantly, in the footsteps of that eminent geometer; but on other occasions it has been my good fortune to discover the fact in time.

Completion of the Geometric Proof of the Transformation-Formulæ.

In my former paper one point was assumed as given by the analytical theory of transformation, namely, that the new argument u' is equal to the old argument u divided by a constant M. Having now found a simple proof of this, I will take the liberty of re-stating in outline, for the sake of clearness, the whole demonstration; availing myself of a remark of M. Darboux.

It is proved by Jacobi's method that if $x_1\, x_2 \ldots x_n$ are parameters of the points of contact of the n sides of a polygon circumscribed to a conic U and inscribed in a conic V, and if ± 1, $\pm k^{-1}$ are the parameters of the points of contact of the common tangents of the two conics, then

$$x_1 = \text{sn}\, u,\ x_2 = \text{sn}\,(u+\gamma),\ x_3 = \text{sn}\,(u+2\gamma), \ldots\ x_n = \text{sn}\,\{u+(n-1)\,\gamma\},$$

where $n\gamma = 4K + 4iK'$, and the modulus of the elliptic function is k.

This being so, an infinite number of in-and-circumscribed polygons can be drawn.

If $p_n = 0$ be the equation in x whose roots are $x_1\, x_2 \ldots x_n$, and $q_n = 0$ an equation in x whose roots are the parameters of the sides of another such polygon; then $p_n - yq_n = 0$ will have for its roots the parameters of the sides of an in-and-circumscribed polygon, whatever value be given to y.

For the locus of the $\frac{1}{2}n\,(n-1)$ intersections of the tangents at the n points $p_n - yq_n = 0$, when y is made to vary, is a curve of order $n-1$, which has $2n$ points in common with the conic V, and therefore contains that conic entirely*.

The quantity y being now regarded as the parameter of a varying polygon, let p_n be chosen to represent that polygon

* This is in substance the remark of M. Darboux referred to. *Op. cit.*, p. 190.

which has the parameter of one of its sides equal to zero, and q_n to represent that which has the parameter of one of its sides infinite. Then y, which is $p_n : q_n$, will be an odd function of x, because of the symmetry of the figure and the fact that $q_n = 0$ (having one root infinite) is only of degree $n - 1$.

If then p_n and q_n be affected with such constant multipliers that $y = 1$ when $x = 1$, we must have $y = -1$ when $x = -1$. And we may suppose that $y = \pm\lambda^{-1}$ when $x = \pm k^{-1}$.

Now the intersections of corresponding sides of two in-and-circumscribed polygons lie on a conic touching the common tangents of U and V.

For the parameters being respectively sn u, sn $(u + \gamma)$, &c., and sn v, sn $(v + \gamma)$, &c., the common difference of the arguments is $u - v$.

If we suppose the two polygons to vary subject to this condition, their parameters y and η will be connected by a (2, 2) correspondence, such that the values of y which make the two corresponding values of η coincide are $\pm 1, \pm\lambda^{-1}$.

Therefore, if $y = \text{sn}\,(u', \lambda)$, we must have $\eta = \text{sn}\,(u' + c', \lambda)$, where c' is a constant. But the relation between corresponding sides x, ξ, of these polygons is $x = \text{sn}\,(u, k)$, $\xi = \text{sn}\,(u + c, k)$, since they intersect on the fixed conic V.

Hence the quantities u, u' are so related that a constant difference between two values of u implies a constant difference between the corresponding values of u'. Hence* (by Euclid's definition of proportion) a varying difference between two values of u implies a *proportional* difference between the corresponding values of u'. But $u' = 0$ when $u = 0$; therefore the two quantities are proportional, and $u = Mu'$ where M is constant.

Now the relation between the parameters of two consecutive sides of a polygon being of the second degree in each, the products x_2x_n, x_3x_{n-1}, &c., are given as ratios of quadratic functions of x_1. Hence the product $x_1x_2 \ldots x_n$, regarded as a function of

* This is Archimedes' proof that a body which passes over equal spaces in equal times will pass over proportional spaces in unequal times.

x_1, is a rational fraction whose numerator is of order n, and whose denominator is of order $n - 1$. But y is a rational fraction whose numerator and denominator are of just these orders; and y vanishes whenever one of the quantities $x_1, x_2 \ldots x_n$ vanishes, and becomes infinite when one of them becomes infinite. Therefore

$$y = mx_1x_2x_3 \ldots\ldots x_n,$$

where m is a constant; that is to say,

$$\text{sn}\left(\frac{u}{M}, \lambda\right) = m\,\text{sn}\,u\,\text{sn}\,(u + \gamma)\,\text{sn}\,(u + 2\gamma) \ldots\ldots \text{sn}\,\{u + (n - 1)\gamma\},$$

when $n\gamma = 4K + 4iK'$. We determine m by remarking that $y = 1$, when $x = 1$, or when $u = K$; and then λ, by remarking that $y = \lambda^{-1}$, when $x = k^{-1}$, or when $u = K + iK'$. Finally M is determined by differentiating the equation and making $u = 0$.

(*Cayley's Theorem.*)—*Every Cubic Transformation has an Elliptic Differential which it transforms.*

This theorem was given by Prof. Cayley in the *Philosophical Magazine*, Vol. 15 (Fourth Series), p. 363. I here reproduce his investigation, slightly altered to suit the generalization which follows. On the very beautiful solution of the complete question (Given the elliptic differential, to find the transformation) by Hermite (*Crelle*, vol. 60, p. 304) and Clebsch (*Theorie der binären alg. Formen*, p. 405), I hope to say something at another time*.

Let U, V be any two cubic functions of x, and consider the transformation $y = \dfrac{U}{V}$.

Suppose that

$$\text{Disct.}\,(U - Vy) = A + By + Cy^2 + Dy^3 + Ey^4,$$

where, of course, $A = \text{Disct.}\,U$, $E = \text{Disct.}\,V$. And let y_1, y_2, y_3, y_4

* The foot-note in my previous paper gave an erroneous expression for X. The article there referred to (Gundelfinger, *Math. Annalen*, Vol. VII., p. 452) is a simplification of the method and results of Clebsch in regard to the typical representation of two cubics.

be the roots of the equation Disct. $(U - Vy) = 0$. Then each of the cubics $U - Vy_1$, $U - Vy_2$, $U - Vy_3$, $U - Vy_4$ has a square factor, because its discriminant vanishes. Now, if $U - Vy_1$ has a square factor $(x - \alpha)^2$, then $x - \alpha$ divides $U' - V'y_1$; that is, for the value $x = \alpha$ we have at the same time

$$\begin{matrix} U - Vy_1 = 0 \\ U' - V'y_1 = 0 \end{matrix} \qquad \left\{U', V' = \frac{dU}{dx}, \frac{dV}{dx}\right\},$$

and therefore also $VU' - V'U = 0$; that is to say, those four linear factors which are squared in the cubics $U - Vy_1$, &c., occur as single factors in $VU' - V'U$. It follows that

$$A(U - Vy_1)(U - Vy_2)(U - Vy_3)(U - Vy_4) = (VU' - V'U)^2 . X,$$

where X represents the product of the single factors of the four cubics. Or, which is the same thing,

$$AU^4 + BU^3V + CU^2V^2 + DUV^3 + EV^4 = (VU' - V'U)^2 X.$$

Now, since $y = \frac{U}{V}$, we have $dy = \frac{VU' - V'U}{V^2} dx$. Hence, if we transform the differential $\frac{dy}{\sqrt{\text{Disct.}(U - Vy)}}$ by the substitution $y = \frac{U}{V}$, we get

$$\frac{VU' - V'U}{V^2} \cdot \frac{V^2}{(VU' - V'U)\sqrt{X}} dx \text{ or } \frac{dx}{\sqrt{X}}.$$

That is, we have
$$\frac{dy}{\sqrt{\text{Disct.}(U - Vy)}} = \frac{dx}{\sqrt{X}},$$

where y is connected with x by the equation $y = \frac{U}{V}$, which is the theorem in question.

New Stand-Point for the Algebraic Transformation-Theory.

We may generalize this result by applying an analogous treatment to transformations of any order. The problem is considered in the following form: Given a transformation $y = U : V$, it is required to find—

(1) What are the necessary and sufficient conditions to be satisfied by the functions U, V, in order that the transformation $y = U : V$ may be able to transform an elliptic differential;

(2) These conditions being supposed satisfied, what is the differential which can thus be transformed.

We will consider first the case in which U and V are of odd order, say $2m + 1$, or, to speak more correctly, the case in which $U - Vy$ is of order $2m + 1$ in x.

The necessary conditions may at once be derived from consideration of the varying in-and-circumscribed polygon the parameters of whose sides are the roots of the equation $U - Vy = 0$.

Starting with any one side of the polygon, which touches what we may call the inner conic, we find its intersections with the outer conic, and then from these draw new tangents to the inner conic. Proceeding in this way symmetrically on both sides of the original tangent, we find at last that the two tangents to the inner conic meet on the same point of the outer conic. We must clearly end with two tangents, and not with one, because the polygon has an odd number of sides.

We might, however, start with a vertex of the polygon on the outer conic, draw two tangents to the inner conic, then from their intersection with the outer conic two more tangents, and so on: at last we shall reach a pair of vertices such that the line joining them touches the inner conic. In this case we must end with two vertices on one side, not with one vertex on two sides, for the same reason as before, that we have supposed the polygon to have an odd number of sides.

Now suppose that in the first mode of construction we start with a common tangent to the two conics; then its two intersections with the outer conic will coincide, and consequently the tangents from them to the inner conic coincide also. We may, however, go on with the construction; and, after drawing $m + 1$ successive tangents, we shall have an exceptional case of an in-and-circumscribed polygon, in which the side first drawn (the common tangent of the two conics) counts singly, and each of the m others counts doubly, so that the polygon has altogether $2m+1$ sides. But the last pair of tangents being coincident, must be regarded as intersecting on the inner conic; and therefore their point of contact must be an intersection of the two conics.

So that we cannot by the second mode of construction get a degenerate polygon of an odd number of sides different from those just considered. If we start with a vertex at a point of intersection of the two conics, the tangents drawn from this to the inner conic will of course coincide, and so therefore will the points in which they meet the outer conic again, and we may continue the process; but we only have a right to stop when the line joining the two last coincident vertices on the outer conic (*i. e.*, the tangent at that point to the outer conic) touches the inner conic; that is to say, when we come upon a common tangent of the two conics.

What actually happens may be illustrated by the case of an in-and-circumscribed pentagon. Let ab be a common tangent of the two conics, touching the inner conic at a, and the outer at b. From b draw the other tangent bc to the inner conic, meeting the outer conic again at c; then from c draw the other tangent cd to the inner conic meeting the outer conic again at d. Then, *if pentagons can be drawn inscribed to the outer conic, and circumscribed to the inner, the point* d *will be an intersection of the two conics.* And the pentagon whose sides are dc, cb, bab, bc, cd is a degenerate case of an in-and-circumscribed pentagon; the side bab being single, and the sides bc, cd, each of them double.

Observe that, if d were not an intersection of the two conics, we should still have an improper solution of the problem, to find five points on the outer conic such that the line joining every successive two shall touch the inner conic. But if we started from d as an intersection of the conics, and then found the points c, b, a as before, except that ba is now not a tangent to the outer conic at b, we should not have found a solution of that problem, but of this other—To find five tangents to the inner conic, so that the intersection of every successive two shall be upon the outer conic.

For the purpose of our present investigation, the result may be stated thus: an in-and-circumscribed polygon of an odd number of sides can only have two sides coincident, when one of its sides is a common tangent of the two conics, and all the others coincide in pairs.

Or, the equation $U - Vy = 0$ can only have coincident roots, when one represents a common tangent, and the others coincide in pairs: so that $U - Vy$ has one single factor, and m square factors. Moreover, there are four values of y, and four only, which bring this about. Now the discriminant of $U - Vy$ is of the order $4m$. Hence *we must have*

$$\text{Disct.}\,(U - Vy) = Y^m, \text{ where } Y = (1, y)^4,$$

and if $y_1y_2y_3y_4$ *are the roots of the equation* $(1, y)^4 = 0$, *then each of the quantities* $U - Vy_1$, &c., *has one single factor and* m *square factors.*

These conditions are necessary; we shall shew that they are sufficient, by solving the second part of the problem.

In the first place, we have

$$(1, 0)^4 . (U - Vy_1)(U - Vy_2)(U - Vy_3)(U - Vy_4) = (U, V)^4,$$

and therefore, as before,

$$(U, V)^4 = (VU' - V'U)^2 X,$$

where $(VU' - V'U)^2$ is the product of all the square factors, and X of the four single factors. From this it follows immediately that

$$\frac{dy}{\sqrt{Y}} = \frac{dx}{\sqrt{X}},$$

if $y = \dfrac{U}{V}$; which is the transformation required. The result may thus be stated:—

The necessary and sufficient conditions that the substitution $y = U : V$ shall be able to transform an elliptic differential, $U - Vy$ being of order $2m+1$ in x, are that Disct. $(U - Vy)$ shall be a perfect mth power, and that those forms $U - Vy$ which have a square factor at all shall have m square factors. This being so, the differential $\dfrac{dy}{\sqrt[m2]{\text{Disct.}\,(U - Vy)}}$ will be transformed by the given substitution into $\dfrac{dx}{\sqrt{X}}$, where X is the product of the four single factors.

It may be observed that, if those forms $U - Vy$ which have a square factor at all have m square factors, it will follow that Disct. $(U - Vy)$ is a perfect mth power; but the converse is not true.

Passing now to the case of a transformation of even order, we enquire, as before, in what cases the in-and-circumscribed polygon can have two sides coincident. If we start with a tangent to the inner conic, and from its intersections with the outer conic draw two more tangents, and so on; there cannot be an in-and-circumscribed polygon of an even number of sides, unless we come to a pair of intersections such that the line joining them touches the inner conic. Suppose then that the first side is a common tangent, so that its two intersections with the outer conic coincide, and that we draw another tangent from this point, another from its second point of intersection, and so on; we must finally come to a point on the outer conic where the tangent touches the inner conic; that is, we must come to another common tangent. In the case of a quadrilateral, for example, let ab be a common tangent, touching the inner conic at a and the outer at b. From b draw the other tangent to the inner conic, meeting the outer again at c; then cd must be also a common tangent, touching the outer conic at c and the inner at d. Thus the sides of the degenerate quadrilateral are bab, bc, cdc, cb, the sides bab, cdc counting singly, and bc double. And in general the two common tangents will count singly, and all the rest double. It is manifest that there are only two degenerate polygons of this kind, each containing two of the four common tangents.

The second construction also gives us two degenerate polygons, but of quite a different character. Starting with a point on the outer conic, we draw two tangents to the inner, and from their new intersections with the outer, two more, and so on;

we must at last come to two tangents which meet on the outer conic. If then our starting-point is a point of intersection, so that the two tangents coincide all through the process, we must come to a pair whose intersection, that is, their point of contact with the inner conic, is on the outer conic; or, which is the same thing, we must come to another intersection of the conics. To use again the quadrilateral as an illustration, the tangents to the inner conic at two points of intersection α and γ must meet on the outer conic at β, and the sides of the quadrilateral are then $\alpha\beta$, $\beta\gamma$, $\gamma\beta$, $\beta\alpha$, so that *all* of them count double. And generally, in degenerate polygons of this kind, all the sides count double. There are clearly two such degenerate polygons, each having two points of intersection for vertices.

To sum up, then, there are four degenerate polygons of even order $2m$; two of them have each two common tangents as sides, and two of them have each two points of intersection as vertices. The former have the common tangents as single sides, and all the rest double; the latter have all their sides double.

It follows that, if the substitution $y = U : V$ is capable of transforming an elliptic differential, $U - Vy$ being of order $2m$ in x, there are only four values of y which make $U - Vy$ have a square factor; two of these make it have m square factors, and the other two make it have $m - 1$ square factors and two single factors. Consequently the former two are m-fold roots, and the latter two $(m - 1)$-fold roots, of the equation Disct. $(U - Vy) = 0$. That is to say, we have

$$\text{Disct.}\,(U - Vy) = Y^{m-1}\,.\,(y - y_1)(y - y_2),$$

where $$Y = k(y - y_1)(y - y_2)(y - y_3)(y - y_4) = (1, y)^4, \text{ say.}$$

Hence, as before, $$(U, V)^4 = (VU' - V'U)^2\,.\,X,$$

where X is the product of the four single factors due to y_3 and y_4. From these equations it follows directly that

$$\frac{dy}{\sqrt{Y}} = \frac{dx}{\sqrt{X}},$$

which is the transformation required.

In regard to these conditions it is to be observed that in general they imply a special constitution of the quantics U, V, as well as a special relation of them to each other. This consideration, however, does not come in until the sixth order of transformation is reached. Thus, in the case of the quartic transformation, the only condition is that U, V shall be simultaneously reducible to the canonical form; which being so, we may find linear combinations of them such that one is the Hessian of the other, thus falling back upon Hermite's very elegant form. In the quintic transformation U may be taken arbitrarily, but the involution $U - Vy$ is then completely determined. In the sextic transformation, however, U and V must each be the product of three quartics in involution (viz., the same involution in the two cases); so that a certain invariant of each must vanish. (Salmon's *Higher Algebra*, p. 210 and Appendix; see Clebsch, *Alg. Formen*, p. 298.)

*ADDITIONS TO PAPER ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS.

1. Completion of doctrine of in-and-circumscribed polygon so as to make it a formal proof of the transformation theory. In p. [210] the quantities u, u' are proportional because constant difference in the values of u means constant difference in the values of u', and they vanish together.

2. Porismatic representation of spherical harmonics as sums of sectorial harmonics. "Nodal curve" of order n may be expressed as a sum of n^{th} powers, $n+1$ in number in a singly infinite number of ways.

3. Polyhedra whose faces osculate a skew-cubic. Special case noticed by Frahm. Involution of one variable: curve and ruled surface. Involution of two variables, surface of order $n-2$; (2,2,2) correspondence for quadric surfaces; cases of degeneration: Δ-faced polyacra. Application of Cotterill's theorem; vanishing area and volume of the porismatic polygon and polyacron.

4. Multiplication. Sylvester's theory of derived points on cubic. Do. for quadriquadric. Scrolls. Arithmetical Theorem.

* [These mere heads of an intended paper are printed as they were found: no complete paper seems to embody them. Perhaps 1 was worked up into * XXIV., and 2 seems to be connected with XXV.]

Fig. 23.

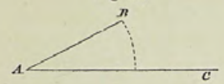


Fig. 24.

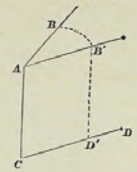


Fig. 25.

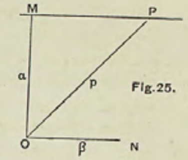


Fig. 26.

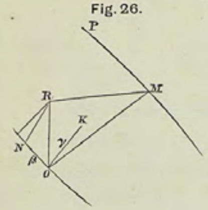


Fig. 27.

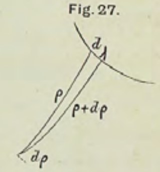


Fig. 28.

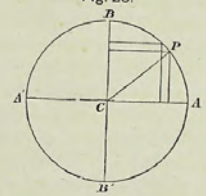


Fig. 29.

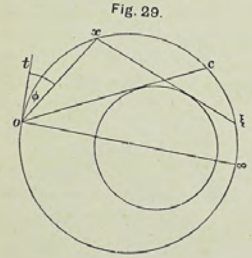
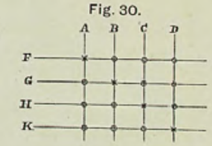


Fig. 30.



*XXIV.

ON IN-AND-CIRCUMSCRIBED POLYHEDRA.

THE extension of the theory of the in-and-circumscribed polygon, made in my paper "On the Transformation of Elliptic Functions," was suggested by a particular case of it studied by Dr. Lüroth in the *Math. Annalen*, I. [pp. 37—53]. It had been remarked by Clebsch (*Crelle*, [Bd. 59]) that not every curve of the fourth order can have its equation expressed as the sum of five fourth-powers; but that for this to be possible a certain invariant (the determinant of the six second derived forms) must vanish. Dr. Lüroth proved that in this case the proposed reduction can be effected in a singly infinite number of ways, and that the lines represented by equating the fourth-powers to zero all touch the same conic. All these pentagrams, moreover, are totally inscribed in a covariant quartic of the original curve (locus of points whose covariant cubics are equianharmonic, so that their Hessians break up into three straight lines). We thus have a varying pentagram circumscribed to the conic and totally inscribed in a quartic; and it was this pentagram which suggested the theorems in my paper above referred to.

The starting-point of the present communication is a remark by Dr. Frahm on a question allied to the foregoing (*Math. Ann.* VII. p. 635). It had been assumed by Dr. Salmon that the equations of three quadric surfaces may simultaneously be reduced to the form of the sum of five squares. Now Hesse established a connection between the theory of three quadric surfaces and that of a plane quartic curve; namely, if $u, v, w = 0$ are the three surfaces, then the equation

$$\text{Disct.}\ (\lambda u + \mu v + \nu w) = 0$$

is of the fourth order in λ, μ, ν, and taking these as co-ordinates of a point in a plane, the equation is that of a quartic curve which corresponds point for point with the locus of vertices of the cones $\lambda u + \mu v + \nu w = 0$. Dr. Frahm remarked that if the three quadrics could be simultaneously reduced to Salmon's canonical form, then this quartic curve is totally circumscribed to a pentagram, and is therefore not the general quartic, but the special form pointed out by Lüroth; so that the problem of effecting this reduction is porismatic—it can either not be solved at all or be solved in a singly infinite number of ways. In the latter case, then, we have a singly infinite number of pentaplanes in regard to which the reduction can be effected; and each of these is totally inscribed in a curve of order 6 and deficiency 3, locus of the vertices of the cones

$$\lambda u + \mu v + \nu w = 0.$$

On considering the envelope of these pentaplanes, I found it to be a twisted cubic. The road to further generalizations was now clearly open.

I.

The equation of any osculating plane to a twisted cubic may be written

$$X + 3\theta Y + 3\theta^2 Z + \theta^3 W = 0,$$

where $X, Y, Z, W = 0$ are four fixed planes, and θ a parameter determining the particular osculating plane. The developable generated by tangent lines to the cubic is

$$(XZ - Y^2)(YW - Z^2) - 4(XW - YZ)^2 = 0 \quad \text{.........(1)},$$

and from this we see that Y passes through the tangent line of X and the point of contact of W, Z through the tangent line of W and the point of contact of X; or we may say that XY

and ZW are two tangent lines, YZ their chord of contact. The equations to the cubic itself are

$$\left\| \begin{matrix} X, & Y, & Z \\ Y, & Z, & W \end{matrix} \right\| = 0.$$

The co-ordinates of the point of intersection of three planes x, y, z are

$$\left\| \begin{matrix} 1, & 3x, & 3x^2, & x^3 \\ 1, & 3y, & 3y^2, & y^3 \\ 1, & 3z, & 3z^2, & z^3 \end{matrix} \right\| = 3xyz : -yz - zx - xy : x + y + z : -3,$$

and if we substitute these in the function $X + 3aY + 3a^2Z + a^3W$ belonging to any fourth plane, we get $3(x-a)(y-a)(z-a)$.

If a variable group of n *osculating planes of a twisted cubic form an involution of the* nth *order, the locus of their lines of intersection is a ruled surface* R *of order* $2(n-1)$, *and the locus of their points of intersection is a curve* γ *the order of which is* $\frac{1}{2}(n-1)(n-2)$, *and which is a triple curve on the ruled surface.*

Consider the sections of this curve and surface by a fixed osculating plane L of the cubic. There is one group of the involution to which it belongs, and it meets the curve only where it meets the lines of intersection of the remaining $n-1$ planes of this group; that is, in $\frac{1}{2}(n-1)(n-2)$ points. This therefore is the order of the curve. All other osculating planes meet the plane L in lines which touch a fixed conic in that plane; in fact it meets the developable (1) in this conic and in its tangent line taken twice. The variable group of n planes in involution determines upon L a variable group of n tangents in involution; and the locus of their intersections is a curve of order $n-1$, by what we have already proved. Besides this curve, the plane L meets the ruled surface in $n-1$ straight lines, in which it meets the $n-1$ other planes of the group which contains it; so that the order of the whole intersection is $2(n-1)$, which is therefore the order of the surface. That the curve is a triple curve on the surface is clear from the fact that through any point of it there may be drawn three straight lines in the surface, these being symmetrically related and not in the same plane.

By means of the in-and-circumscribed polygon determined upon the plane L we may find the equation of the ruled surface. Since the parameters of the several osculating planes of the cubic may be taken as parameters of the tangents to the conic in the plane L in which they are cut by that plane, it follows that the condition for two osculating planes x, y to belong to the same group is the same as the condition for the two tangents x, y to belong to the same group; that is to say, it is a condition of the form

$$\Sigma_i \frac{\alpha_i}{(x-a_i)(y-a_i)} = 0 \text{ or } \Sigma_i \frac{\alpha_i}{x-a_i} = \Sigma_i \frac{\alpha_i}{y-a_i} \quad \text{........(2)},$$

where $a_1, a_2, \ldots a_n$ are parameters of some one group, and $\alpha_1, \alpha_2, \ldots \alpha_n$ are constants.

When this condition is fulfilled, the line of intersection of the planes x, y meets the curve γ. Now through any point on the surface R there can be drawn a line which is the intersection of two osculating planes of the cubic, and these two planes will satisfy the condition (2). If therefore the point of intersection of xyz lies on the surface R, the condition (2) must be satisfied either for yz or for zx or for xy. That is, we must have

$$\Sigma \frac{\alpha_i}{(y-a_i)(z-a_i)} \cdot \Sigma \frac{\alpha_i}{(z-a_i)(x-a_i)} \cdot \Sigma \frac{\alpha_i}{(x-a_i)(y-a_i)} = 0 \quad \text{.....(3)}.$$

It remains to translate this equation into the ordinary point-co-ordinates. Let $A_1, A_2, \ldots A_n = 0$ be the equations to the n planes whose parameters are $a_1, a_2, \ldots a_n$; that is, let

$$A_i = X + 3a_i Y + 3a_i^2 Z + a_i^3 W;$$

then the result of substituting in A_i the co-ordinates of the point of intersection of x, y, z is $3(x-a_i)(y-a_i)(z-a_i)$. If then we multiply together terms of like suffixes in the factors of (3), we get in the product the sum of n terms

$$9\Sigma_i \frac{\alpha_i^3}{A_i^2}.$$

Next, the equation of the plane passing through the tangent line of A_i and the point of contact of A_j is

$$(iij) = X + 2a_i Y + a_i^2 Z + a_j (Y + 2a_i Z + a_i^2 W) = 0,$$

and when we substitute in this the co-ordinates of intersection of xyz, we obtain

$$(x-a_j)(y-a_i)(z-a_i) + (y-a_j)(z-a_i)(x-a_i) + (z-a_j)(x-a_i)(y-a_i).$$

Selecting then from (3) the products of two like suffixes by one unlike, we get the sum of $n(n-1)$ terms

$$\Sigma_{ij} \frac{\alpha_i^2 \alpha_j \,.\, (iij)}{A_i^2 \,.\, A_j}.$$

Lastly the equation of the plane passing through the points of contact of A_i, A_j, A_k is

$$(ijk) = X + (a_i + a_j + a_k) Y + (a_j a_k + a_k a_i + a_i a_j) Z + a_i a_j a_k W = 0,$$

and when we substitute in this the co-ordinates of intersection of x, y, z, we get $\frac{1}{2} \Sigma (x-a_i)(y-a_j)(z-a_k)$, where the x, y, z are to be permuted in all possible ways. Thus the products in (3) of three unlike suffixes give us the $\frac{1}{6} n(n-1)(n-2)$ terms

$$\tfrac{1}{2} \Sigma_{ijk} \frac{\alpha_i \alpha_j \alpha_k \,.\, (ijk)}{A_i A_j A_k},$$

and the equation to the surface R is therefore

$$0 = 18\Sigma_i \frac{\alpha_i^3}{A_i^2} + 2\Sigma_{ij} \frac{\alpha_i^2 \alpha_j \,.\, (iij)}{A_i^2 A_j} + \Sigma_{ijk} \frac{\alpha_i \alpha_j \alpha_k \,.\, (ijk)}{A_i A_j A_k}.$$

The equation shews that when cleared of fractions it is as it ought to be of the order $2(n-1)$.

XXV.

ON A CANONICAL FORM OF SPHERICAL HARMONICS*.

THE canonical form in question is an expression of the general harmonic of order n as the sum of a certain number of sectorial harmonics, this number being, when n is even,

$$\frac{5n-10}{2},$$

and when n is odd,

$$\frac{5n-9}{2}.$$

Laplace's operator,

$$\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2},$$

may be obtained from the tangential equation of the imaginary circle $\xi^2+\eta^2+\zeta^2=0$, by substituting $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ for ξ, η, ζ. If, therefore, a form $U\equiv(x, y, z)^n$ is reduced to zero by this operation, it follows from Prof. Sylvester's theory of contravariants that the curve $U=0$ is connected by certain invariant relations with the imaginary circle. I find that U can be

* [Notices and Abstracts...from *Report of the Forty-first Meeting of the British Association for the Advancement of Science*, held at Edinburgh, August, 1871, p. 10. A discussion followed the reading of Prof. Clifford's paper, and a result was that in the same Report, pp. 25, 26, is printed a communication by Sir W. Thomson, *On the General Canonical Form of a Spherical Harmonic of the nth order*. In this Sir W. Thomson answers the question, Can canonical forms not be found in which the nodal conic of each constituent is not resolvable into circular cones and planes?]

expressed in the form

$$U\equiv A^n+B^n+C^n+\dots$$

where $A=0$, $B=0$, ... are great circles touching the imaginary circle, the number of terms being as above. Now if $L=0$, $M=0$ be two such great circles meeting in a real point a, and if ϕ be a longitude and θ latitude referred to a as pole, it is easy to see that

$$L^n+M^n=l\sin^n\theta\sin n\phi+m\sin^n\theta\cos n\phi,$$

a sum of two sectorial harmonics, which is the proposed reduction.

When n is less than 5, exceptions of interest occur. For $n=3$, if we take a, b, corresponding points on the Hessian of the *nodal curve* $U=0$ (Thomson and Tait, *Treatise on Natural Philosophy*, § 780 [first edition]), and if we call ϕ_1, ϕ_2 the longitudes, θ_1, θ_2 the latitudes referred to these poles, we have

$$U\equiv l\sin^3\theta_1\sin 3\phi_1+m\sin^3\theta_1\cos 3\phi_1$$
$$+n\sin^3\theta_2\sin 3\phi_2+s\sin^3\theta_2\cos 3\phi_2.$$

For $n=4$, the nodal curve is of the species first noticed by Clebsch, of which many most beautiful properties have been pointed out by Dr. Lüroth. The form U is expressible as the sum of *five* fourth powers; so that if we take a, b real points of intersection of two pairs of them, c a real point on the fifth, calling ϕ_1, ϕ_2, ϕ_3, θ_1, θ_2, θ_3 longitudes and latitudes referred to them, we have

$$U\equiv l\sin^4\theta_1\sin 4\phi_1+m\sin^4\theta_1\cos 4\phi_1$$
$$+p\sin^4\theta_2\sin 4\phi_2+q\sin^4\theta_2\cos 4\phi_2$$
$$+r\sin^4\theta_3\,\epsilon^{4\iota\phi_3}.$$

XXVI.

ON THE FREE MOTION UNDER NO FORCES OF A RIGID SYSTEM IN AN N-FOLD HOMALOID. (Provisional Notice.)*

THE problem of the rotation under no forces of a rigid body about a fixed point in ordinary three-dimensional space is the same as the problem of free motion under no forces; for the motion about the centre of inertia takes place as if it were a fixed point. But it is also the same thing as the problem of the free motion of a rigid system on the surface of a sphere, or in elliptic space of two dimensions†. So also the problem of the free motion of a solid in elliptic space of three dimensions is the same as that of the free motion, or motion about a fixed point, in parabolic or homaloidal space of four dimensions. And, in general, the problem of free motion in elliptic space of n dimensions is identical with that of free motion, or motion about a fixed point, in parabolic space of $n+1$ dimensions.

The form of the problem which is considered in what follows is that which deals with the motion about a fixed point in parabolic space of n dimensions.

* [From the *Proceedings of the London Mathematical Society*, Vol. VII., Nos. 92, 93, pp. 67—70.]

† According to Dr Klein's nomenclature, a space every point of which can be uniquely represented by a set of values of n variables is called elliptic, parabolic, or hyperbolic, when its curvature is uniform and positive, zero, or negative. The geometry of the sphere becomes elliptic when opposite points are regarded as identical.

I.

Let the co-ordinates of a point, referred to a rectangular system, be $x_1, x_2, \ldots x_n$. If this point belongs to a rigid system in motion, its velocity is given by the equations

$$\dot{x}_h = \Sigma p_{hk} x_k \quad (h, k = 1, 2, \ldots n) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $p_{hk} = -p_{kh}$, $p_{hh} = 0$. The $\frac{1}{2}n(n-1)$ quantities p are of the nature of rotational velocities of the rigid system. It may be observed that if the vector from the origin to the point x be represented in terms of n unit vectors $\iota_1 \iota_2 \ldots \iota_n$, satisfying the equations $\iota_h \iota_k = -\iota_k \iota_h$, $\iota_h^2 = -1$, then the velocity of the rigid body may be represented in terms of the $\frac{1}{2}n(n-1)$ products $\iota_h \iota_k$; namely, we may write

$$\rho = \Sigma \iota_h x_h, \quad -2p = \Sigma p_{hk} \iota_h \iota_k,$$

and then the equation (1) may be put into the form

$$\dot{\rho} = Vp\rho.$$

If dm be the element of mass at the point x, its kinetic energy is

$$\tfrac{1}{2}\Sigma \dot{x}^2 dm = \tfrac{1}{2} dm \{\Sigma p_{hk}^2 (x_h^2 + x_k^2) + 2\Sigma p_{hl} p_{kl} x_h x_k\}.$$

Let then
$$\int \dot{x}_h^2 dm = \alpha_h, \quad \int x_h x_k \, dm = \beta_{hk},$$

the integrations extending over the whole rigid system; then, if T be the kinetic energy of the system,

$$2T = \Sigma (\alpha_h + \alpha_k) p_{hk}^2 + 2\Sigma \beta_{hk} p_{hl} p_{kl} \quad \ldots\ldots\ldots\ldots\ldots (2).$$

Write now
$$q_{hk} = \frac{\delta T}{\delta p_{hk}}, \quad -2q = \Sigma q_{hk} \iota_h \iota_k,$$

then q is the *momentum* of the system; it is a linear function of the velocity, or $q = \phi(p)$, and twice the kinetic energy is the scalar part of the product of velocity and momentum, $2T = Spq$. The equations of motion are $\dot{q} = f$, where f is the system of applied forces, or in the present case of no forces $\dot{q} = 0$; viz., this is equivalent to $\frac{1}{2}n(n-1)$ equations. From this we get the first integrals, $q = \text{constant}$, and (since $0 = Sp\dot{q} = 2\dot{T}$) $T = \text{constant}$.

But these equations are inconvenient, because the α and β are variable, depending upon the position of the body. We must therefore follow Euler in referring the motion to axes moving with the body, and coinciding with the principal axes at the fixed point. This will make all the β vanish; and we shall have $$q_{hk} = (\alpha_h + \alpha_k)\, p_{hk}.$$

From equation (1) we obtain

$$\dot{q}_{hk} = \int dm\,(\ddot{x}_h x_k - x_h \ddot{x}_k) = \Sigma_l \int dm\,\{\dot{x}_l(p_{hl}x_k - p_{kl}x_h) + x_l(\dot{p}_{hl}x_k - \dot{p}_{kl}x_h)\}.$$

Assuming that $\int dm x_h x_k = 0$ when h and k are different, and remembering that $\dot{q}_{hk} = 0$, we may write this equation in the form $$(\alpha_h + \alpha_k)\,\dot{p}_{hk} + (\alpha_h - \alpha_k)\,\Sigma_l p_{hl} p_{lk} = 0 \;\dots\dots\dots\dots\dots (3).$$

Here $\dot{p}_{hk}$ means the rate of change of that component of velocity which coincides at the moment with one of the principal components; it must be distinguished from the rate of change of the principal component, which we shall call $(\dot{p})_{hk}$. In general, if the system of axes has the rotational velocities r_{hk}, we shall have $$\dot{p}_{hk} = (\dot{p})_{hk} + \Sigma(p_{hl}r_{kl} - p_{kl}r_{hl});$$

but in the present case the r are equal to the p, because the axes move with the body, and so $\dot{p}_{hk} = (\dot{p})_{hk}$. Thus in the equations (3), which are analogous to Euler's equations for three dimensions, the symbols $\dot{p}_{hk}$ may be understood to mean the rates of change of the principal components of rotation.

If the symbol V_2 is regarded as selecting all the binary products of $\iota_1 \iota_2 \dots \iota_n$ out of any rational integral function of them, the last equations may be written

$$\dot{q} = V_2 pq;$$

and it may be observed that we have also

$$q = \int dm V_2 \rho\dot{\rho}.$$

II.

These equations may be integrated by means of Θ-functions of $n - 2, = s$, arguments, one of which is a linear function of the time. It is most convenient to use these in the form employed by Göpel for the case $s = 2$. Let $u_1, u_2, \dots u_s$ be the arguments, and let $B_{11}, B_{12}, \dots B_{1s}, \dots B_{s1}, \dots B_{ss}$ be s^2 constant quantities, and let

$$U_h = (u_h + 2m_1 B_{1h} + 2m_2 B_{2h} + \dots + 2m_s B_{sh})^2 = (u_h + 2\Sigma m_k B_{kh})^2;$$

then we shall write

$$G(u_1, u_2, \dots u_s) = \Sigma_m e^{\Sigma U_h},$$

where the whole numbers m are to take independently all values from $-\infty$ to $+\infty$. It is clear that the function G is unaltered if we simultaneously increase $u_1, u_2, \dots u_s$ by equimultiples of the quantities $2B_{h1}, 2B_{h2}, \dots 2B_{hs}$, for this is only increasing m_h by an integer. Moreover, if we determine s^2 quantities A so that $$4\Sigma_k A_{hk} B_{hk} = \pi i,\quad \Sigma_k A_{hk} B_{lk} = 0,$$

then the function $e^{-\Sigma uu} G$ is unaltered if we simultaneously increase $u_1, u_2, \dots u_s$ by equimultiples of $4A_{h1}, 4A_{h2}, \dots 4A_{hs}$. In what follows we shall write for shortness $G(u + X_h)$ instead of $G(u_1 + X_{h1}, u_2 + X_{h2}, \dots u_n + X_{hn})$, omitting always the last suffix, and mentioning only one argument u.

A linear function of the A, B with coefficients 0 or 1 will be called a *quadrant*; there are clearly 2^{2s} quadrants, if zero be included among them. Let X and Y be two quadrants, A_h the difference between those parts of them which involve the quantities A_{hk}, then the function

$$e^{-\Sigma A_u} \frac{G(u + X)}{G(u + Y)} = Al_{X,Y}(u)$$

is $2s$-periodic in the arguments u, the periods being $4A$, $4B$. It is convenient to speak of the *distance* of two quadrants X and Y, meaning the number of coefficients of the A and B which must be changed from 0 to 1, or *vice versâ* to make one of them into the other. This distance may be any of the numbers

1, 2, ... $2s$; and accordingly there are $2s$ really distinct $2s$-periodic functions.

It is, however, possible to form a group of $n, = s+2$, quadrants $X_1, X_2, \ldots X_n$ having such a relation to the quadrant O that if we write $Al_{hk}(u)$ for $e^{-\Sigma Au}\, G(u+X_h+X_k) : Gu$, the $\frac{1}{2}n(n-1)$ functions $Al_{hk}(u)$ satisfy the equations

$$\delta_u Al_{hk}(u) = \Sigma c_l\, Al_{hl}(u) \,.\, Al_{kl}(u) \;\ldots\ldots\ldots\ldots\ldots(4),$$

where the δ_u applies to any one of the arguments; but the values of the c will depend upon which argument is taken. In the case of the hyperelliptic functions, the four quadrants X may be taken to be the quantities A_1, A_2, B_1, B_2.

It appears therefore that the equations (3) may be integrated if we write $p_{hk} = \lambda_{hk}\, Al_{hk}(u)$, where $u_1 = at + e$.

III.

From the $\frac{1}{2}n(n-1)$ equations (3) let us pick out $n-1$, namely

$$-(\alpha_1+\alpha_h)\,\dot{p}_{1h} = (\alpha_1-\alpha_h)\,\Sigma p_{hk}\, p_{1k};$$

if we write $(\alpha_1+\alpha_h)\,p_{1h} = (\alpha_1-\alpha_h)\,\xi_h$, these equations become

$$-\dot{\xi}_h = \Sigma p_{hk}\, \xi_k.$$

But these are the equations for the velocity of a fixed point ξ relative to the moving axes in $n-1$ dimensions. The rest of the equations (3), if we write in them $p_{1h} = 0$, become the equations for the component rotations in $n-1$ dimensions. Thus the solution for the rotational velocities *and* the position of a point fixed in space, for $n-1$ dimensions, are obtained by diminishing the number of periods in the solution for n dimensions; it consists accordingly of the Al functions expressing the rotation-velocities, and of Rosenhain's combination of Θ-functions and exponentials, expressing the position.

XXVII.

ON THE CANONICAL FORM AND DISSECTION OF A RIEMANN'S SURFACE*.

THE object of this Note is to assist students of the theory of complex functions, by proving the chief propositions about Riemann's surfaces in a concise and elementary manner. To this end I assume only certain results of Puiseux, which are put together at the outset.

I.

Puiseux's Theory of an n-valued Function.

If two variables s and z are connected by an equation of the form $f(s, z), = (s, 1)^n (z, 1)^m, = 0$, each is said to be an algebraic function of the other. Regarding z as a complex quantity $x+iy$, we represent its value by the point whose co-ordinates are x, y, on a certain plane. To every point in this plane belongs one value of z, and consequently, in general, n values of s, which are the roots of the equation $f=0$. The points of the plane may be divided into those at which the n values of s are distinct, and those at which two or more of them are equal. The latter points are finite in number, and correspond to the roots of the equation which is got by equating to zero the discriminant of f in regard to s. If the roots of this equation are distinct, there are $2(n-1)m$ such points, because the discriminant of the

* [From the *Proceedings of the London Mathematical Society*, Vol. VIII., No. 122, pp. 292—304.]

equation of the nth order in s is of degree $2(n-1)$ in the coefficients, and these coefficients are of the order m in z. But a point at which r values of s become equal corresponds to an $(r-1)$-fold root of the discriminant-equation.

Let us now consider an arbitrary point O of the plane [fig. 31], corresponding to a value z_0 of z, which is not a root of the discriminant-equation. Then the equation $f(s, z_0)=0$ will give n different values for s, which we may call $s_1, s_2 \dots s_n$. If we move along any path from the point O to another point P of the plane, the value of z will change continuously, and each of the quantities $s_1, s_2 \dots s_n$ will also change continuously. If therefore the path OP does not go through a point where two values become equal, these n quantities will be distinct all the way, and each of the n values of s at P will belong to a definite one of the values of s at O. But if the path goes through such a point, two or more of the n quantities will become equal and then diverge again, so that it will be impossible after that to distinguish them so as to say which of these belongs to a particular one of the values at the point O. We cannot always avoid this difficulty by going *round* the point, for it is found that the values at P to which the values at O correspond may depend upon the path OP, so that the correspondence is different for a path which goes to the right of the point and for a path which goes to the left of it. When this is the case, the point is called a branch-point. Suppose that, when we go from O to A, the two values p and q of s at O approach one another and become equal at A; then it is found that the value at P which represents p when we go along the path OBP may represent q when we go along OCP, and *vice versâ*. So that, if we travel along $OBPCO$, round the point A and back to O, the values p and q will change continuously into one another. If more than two values are equal at A, the corresponding values at O may be cyclically interchanged by a path going round A. We shall assume, however, that only two values become equal at each branch-point; and, moreover, that no branch-point is at an infinite distance*.

* Roots of the discriminant-equation which are not branch-points correspond to double points on the curve $f(s, z)=0$. Such points behave, in regard to the function s, like two coincident branch-points belonging to the same pair of values, and they have no influence on the connection of the different values of s.

A path going along any line from O to very near A, then round A in a very small circle, and then back to O along the same line, will be called a *loop*.

If we start from O and go round any closed curve not including any branch-points, the n values of s at O will be restored in the same order. For the path may be gradually shrunk into a point without crossing any branch-points, so that no two of the n values can become confused at any point of it. The same thing is true if the closed path includes *all* the branch-points. Suppose it a large circle through O; then it may be gradually increased till it coincide with the tangent at O, then curved over on the other side, and shrunk up into a point; and during the whole process the n values will be distinct at every point of the path.

We shall now go on to shew that this n-valued function, which we have spread out upon a single plane, may be represented as a *one*-valued function on a surface consisting of n infinite plane sheets, supposed to lie indefinitely near together, and to cross into one another along certain lines. This surface is called a RIEMANN'S surface; we shall demonstrate its existence at the same time that we shew how to construct it in the most convenient form.

II.

Construction of the Riemann's Surface.—Lüroth's Theorem.

Draw loops from O [fig. 32] to all the branch-points, and let the first, A, interchange the values p and q. If we go round all the loops successively, starting with the value p at O, we must, as we have seen, come back to that value; but this may happen before we have used all the loops. Let B be the first branch-point after going round which the value p is restored. Draw a line from A to B cutting all the loops which alter p, but none of the others. Then, if we go round any of the branch-points between A and B without crossing the line AB or going round any other branch-points, we shall not alter the value p.

Suppose that A interchanges pq, B interchanges ps, and that the branch-points between A and B are 1, 2, 3, 4, interchanging respectively qr, rs, hk, pl. The value q must in fact be changed into the value p through a longer or shorter series of values; the loops interchanging hk and pl are put in as examples. Now if we go round 4 by the dotted loop passing round outside A, the effect is the same as going in succession round A, 1, 2, 3, 4, 3, 2, 1, A. By the time we have gone round A, 1, 2, 3, we cannot have the value p, for that is first restored by B; and we cannot have the value l, for then 4 would restore the value p. Hence we have some value which is not altered by the loop to 4; and consequently, when we retrace our path, we shall come back to the value p.

Next, let us draw a loop to B which passes within the line AB, but goes round all the included branch-points, as in the figure. The effect of this loop will be to change q into p; for it is the same thing as going round 1, 2, 3, B, 3, 2, 1. Now the effect of 1, 2, 3, B is to change q into p, and this p is not altered in coming back because all the branch-points which alter p are outside the line AB.

Suppose then that all the branch-points of this group which alter p are connected with O by loops going round A, so that they no longer alter p; and that B is connected with O by the loop just described, so that no branch-points are contained in the triangle AOB.

Starting now from this new loop OB, with the value p, let us go round all the loops as before from left to right. We know that when all the loops have been gone round, ending with OA, the value p must be restored. If it is not restored before we have gone round OA, we must draw a line BA cutting all the loops which change the value p but none of the others. But if the value p is restored before we have gone round OA, say after going round OC; then we must draw a new loop to C, going round all the branch-points between A and C except those which change the value p. This new loop will, by our previous

reasoning, change p into q. Hence, if the value p is restored before we have gone round OA, we can make a new loop OC which changes p into q; and this comes next to OB. To those branch-points whose loops have been cut by this new loop we must draw new loops going round to the right of C, so as not to cut OC. The figure comes then into this form [fig. 33], containing

(1) Loops to the left of OA which do not change the value of p, like the dotted loop $O4$ in the previous figure;

(2) Three consecutive loops OA, OB, OC which change p into q;

(3) Loops to the right of OC which may or may not change p.

If now we start with the loop OC and proceed to the right, the value p must be restored *before* we have gone round OA; for, starting with OA and going all round, we must restore the value p in the end. Let p then be restored by OD; and draw a line CD cutting all those loops which change p, but none of the others. Replace the loops which change p by new ones going round between B and C; and replace OD by a new loop going outside all the branch-points whose loops do not alter p. The figure now consists of these elements:

(1) Two triangles AOB, COD, containing no branch-points, and such that the loops OA, OB, OC, OD interchange p and q;

(2) Loops between OB and OC which do not change p;

(3) Unknown loops between OD and OA.

About these unknown loops we may make three suppositions.

First, suppose that none of them change p. Then the value p cannot be altered by any closed curve starting from O and returning to it which does not cut either of the lines AB, CD.

Secondly, suppose that some of these loops change p, but that, when we start with the loop OD and go round to the right, the value p is first restored by OA or OC. (It is clear that it cannot be first restored by OB, because the two loops OA, OB, taken together, make no change in any value; nor by any loop between OB and OC, for none of them change p.) Then we must join D with A by a line cutting all the loops which change p, but no others; and B with C by a line cutting none of the loops between OB and OC. In that case the value p cannot be altered by any closed curve starting from O and returning to it which does not cut either of the lines BC, DA.

Thirdly, suppose that the value p is restored *before* we come to OA, say at OE. Then we must proceed as before, finding a new line EF which shall have the properties of AB or CD. The figure will then consist of three triangles AOB, COD, EOF, containing no branch-points, and such that the loops OA, OB, OC, OD, OE, OF interchange p and q; loops between OB and OC, and between OD and OE, which do not change p; and unknown loops between OF and OA.

It is clear that this process must ultimately stop, and then we shall be left with a finite number of lines such that, if we start from O, follow any continuous path, and come back again, without crossing any of these lines, we shall not alter the value p. The lines are either AB, CD, EF, &c., or else they are BC, DE, &c.; in either case the loops OA, OB, ... interchange p and q.

It follows that, if we take an infinite plane sheet and cut it through along these lines, we may consider a single value of the function s to be attached to every point of the sheet in such a way that this value varies continuously when we move about continuously in the sheet; but there will be different values on the two sides of any cut—namely, we must attach to every point P of the sheet that value of s which changes continuously into p when we go from P to O without crossing any of the cuts. There is only one such value; for if two different paths from O to P gave different values at P, it would be possible to change the value p by means of a closed curve returning to O; and this we have proved not to be the case.

When the lines cut through are AB, CD, ..., the triangles AOB, COD, ... contain no branch-points; but when the lines are BC, DE, ..., the triangles BOC, DOE do in general contain branch-points. We may, however, draw new loops to C, E, ... so as to exclude these branch-points, and the new loops will still change p into q. For no closed curve going round B and C

so as not to cut BC can change the value p, by what we have already proved; but the loop OB changes p into q, therefore OC must change q into p.

We shall assume then that the cuts are AB, CD, ..., and that the triangles AOB, COD, ... contain no branch-points.

Now let us deal with the value q at O in the same way as we have dealt with the value p. It is first to be observed that a path going round one or more of the lines AB makes no change in *any* value at O; so that, if we agree never to cross these lines, we may leave the branch-points A, B, ... entirely out of consideration.

This being so, let us take a loop which changes q into some other value, say r. There must be such a loop, if the function is more than two-valued; for otherwise the values p, q would form a two-valued algebraic function of z, and the expression $f(s, z)$ would have a factor of the second degree in s.

Starting then with this loop, we may proceed in exactly the same way as before, and draw lines $A'B'$, $C'D'$, ... such that a closed curve, starting from O and coming back to it without cutting any of these lines or any of the previously drawn lines, will not alter the value q. Moreover, we shall have drawn loops OA', OB', ..., each of which changes q into r, and such that the triangles $A'OB'$, $C'OD'$, ... contain no branch-points. And since our previous triangles AOB, COD, ... contained no branch-points, it will not have been necessary to cut through them in drawing the new lines $A'B'$, $C'D'$, ...

We shall now speak of the first set of lines AB, CD,... as the lines (pq), and of the second set as the lines (qr).

Let us take two infinite plane sheets, cut them both through along the lines (pq), but only the second one along the lines (qr). To every point of the *first* sheet we will suppose attached that value of s which is arrived at by continuous change of the value p at O; and to every point of the *second*, that value which is arrived at by continuous change of the value q at O.

In each sheet there will be a finite difference in the values on the two sides of each of the cuts (pq); but the value on one side in the upper sheet will be equal to the value on the other side in the lower sheet. At the cut AB, for example, the value continuous with p on the side next to O is equal to the value continuous with q on the side remote from O; because a path taken round A or B from O and back again changes the value p continuously into the value q.

Thus, if we take p, q to denote values at the cut continuous with p, q at O, they will be situated as in the figure [fig. 34], which represents a section across AB perpendicular to the two sheets. If then we make the two sheets cross one another along the lines p, q, as here represented [fig. 35], then these two values will be continuously distributed on the double-sheeted surface so formed.

We may now continue the process with the value r. We must first find a loop which changes r into some other value, say t, and then proceed as before, taking care not to cross the lines qr. (We may cross the lines pq as often as we please, provided that we have not previously crossed the lines qr; for these lines can have no effect upon r unless it has been previously changed into q.) Thus we shall draw lines rt such that the value r cannot be altered by a closed curve not cutting the lines qr or rt, and having their extremities joined to O by loops which change r into t. If we take, then, a third sheet, cut it through along the lines qr and rt, and then join it crosswise to our second sheet along the lines qr; the three values pqr may be continuously distributed on this three-sheeted surface.

By proceeding in this way it is clear that we shall construct an n-sheeted surface, the sheets of which are connected chainwise by cross lines, so that the first is connected only with the second, the second with the third, and so on; but there is no direct connection except between consecutive sheets. And the n values of the function may now be attached to the points of this surface, so that one value only belongs to each point, and that in moving this point about on the surface the value belonging to it always changes continuously. Thus, if we start from a given point of the surface (on a given sheet), and travel by any path so as to come back to the same point (on the same sheet), we shall in all cases return to the former value of the function s.

The theorem that the Riemann's surface may be so constructed that the sheets are only connected *chainwise*—*i. e.*, so that there are no cross-lines except between consecutive sheets—is due to Dr. Lüroth.

III.

Clebsch's Theorem.

All the links between successive sheets except the last may be made to consist of one cross-line only.

First, we shall prove that, if there are two or more lines (pq), one of them may be converted into a line (qr).

The original position of the two lines (pq) and the line (qr) is drawn in fig. [36]. If we move the line qr, keeping, of course, its ends fixed, the effect is to interchange the sheets QR in the area over which it moves; so that, by passing it over the line (pq) on the right, we change this into a line (pr). The position is then as in fig. [37]. If now we pass the remaining line (pq) over this line (pr), we change it into a line (qr); thus we are left with two lines (qr) and one line (pq). [Fig. 38.]

In this way we may convert all but one of the lines (pq) into lines (qr). Then we may convert all but one of the lines (qr) into lines (rs); and so on. Then the first $n-1$ sheets will be connected chainwise by one cross-line each, and the last two by all the remaining cross-lines.

The Riemann's surface is now said to be in its canonical form.

The process of transformation may be made clearer by looking at a section of the three sheets by a plane perpendicular to them cutting the lines pq, qr, pq [figs. 39, 40, 41].

IV.

Transformation of the Riemann's Surface.

The Riemann's surface now consists of n infinite plane sheets, such that the sheet 1 is connected with 2 by a single cross-line, 2 with 3 by another cross-line, and so on; but $(n-1)$

with (n) by a number of cross-lines which we shall call $p+1$. Thus the whole number of cross-lines is $n-2+p+1=n+p-1$. If w is the number of branch-points, this is twice the number of cross-lines, or $w=2(n+p-1)$. Hence $p=\frac{1}{2}w-n+1$.

Let now this n-fold plane be inverted in regard to any point outside it, so that it becomes an n-fold sphere passing through the point. Any two successive sheets of the sphere will be connected by one cross-line, except the two outside sheets, which are connected by $p+1$ cross-lines.

To every point of this n-sheeted spherical surface will correspond one value of the function s, namely, that which belongs to the corresponding point upon the n-fold plane. As for the centre of inversion, it is to be regarded as n distinct points upon the several sheets, corresponding to the n values of s when $z=\infty$.

We shall now prove that this n-fold spherical surface can be transformed without tearing into the surface of a body with p holes in it.

First, suppose we have only two sheets, connected by a single cross-line which joins the branch-points AB. Let the figure [42] represent a section by the plane which bisects AB at right angles.

Suppose each hemisphere of the inner sheet to be moved across the plane of the great circle containing AB (indicated by the dotted line in the figure), so that the points m, n change places. In this process the two hemispheres will have to penetrate and cross each other; but this may be supposed to take place without altering the continuity of either. Each point may be supposed to move on a straight line perpendicular to the dotted plane, till it coincides with what was its reflexion in regard to that plane. The effect on the cross-line will be to change it from the form drawn in fig. [42] to that drawn in fig. [43]; instead of the two sheets crossing along the line, each of them will be doubled under it. The result is that, if we now look down on the double sphere from a point vertically over the line AB, we shall see a spherical shell with a hole in it, in the form of a slit along the line AB [fig. 44]. Conceive the spherical shell to be made of india-rubber or some more elastic substance;

then by mere stretching, without tearing, the slit may be opened out until the shell takes the form of a flat plate; that is, of a body with *no* holes in it.

Next, consider a two-sheeted spherical surface with $p+1$ cross-lines, and suppose them all arranged along the same great circle; which may obviously be done by stretching, without tearing, the surface. Let this great circle be the one represented by the dotted line in figs. [42] and [43]. Then we may apply to the inner sheet the same process as before; viz., we may interchange the two hemispheres into which the sheet is divided by the dotted plane. The effect is to convert all the cross-lines into slits or holes in a spherical shell; and we have supposed that there are $p+1$ of these. One of the slits may be stretched out in the same way as AB was before, so as to convert the spherical shell into a flat plate; but in this flat plate there will remain p holes. A double sphere with $p+1$ crossing lines is thus converted, without tearing, into the surface of a body with p holes in it.

Lastly, suppose that the inner sheet of this two-sheeted sphere is connected by one cross-line with a third inside sheet, the third sheet by one cross-line with a fourth inside it, and so on, until there are n sheets. Let the inner sheet of all be reflected in regard to the plane of the great circle through its crossing line, so that it makes with the sheet next to it a spherical shell with one hole in it. Then, without tearing, the inner sheet may be shrunk up until it merely covers over this hole. The same process may now be applied to shrink up the second sheet into the third, and so on, until we are left with only the two outside sheets connected by $p+1$ cross-lines. These, however, as we have seen, may be converted, without tearing, into the surface of a body with p holes in it. Hence the proposition follows, that *an n-sheeted Riemann's surface with w branch-points may be transformed, without tearing, into the surface of a body with p, $=\frac{1}{2}w-n+1$, holes in it.*

V.

The Number of Irreducible Circuits.

A closed curve drawn on a surface is called a *circuit*. If it is possible to move a circuit continuously on the surface until it shrinks up into a point, the circuit is called *reducible*; otherwise it is *irreducible*. In general there is a finite number of irreducible circuits on a closed surface which are *independent*, that is, no one of which can be made by continuous motion to coincide with a path made out of the others. All other irreducible circuits can then be expressed as compounds of these independent ones. For example, on the surface of a ring (*i.e.*, of a body with one hole through it) there are two independent irreducible circuits; one *round the hole*, as *abc* [fig. 45], and one *through the hole*, as *ade*. If a circuit goes neither round the hole nor through the hole, it can be shrunk up into a point. If it cannot be so shrunk up, it must go a certain number of times round or through the hole or both, that is, it may be made up of circuits like *abc* and *ade*.

In the same way we may see that, on the surface of a body having p holes through it, there are $2p$ independent irreducible circuits; one *round* each hole, and one *through* each hole. For simplicity consider the case $p = 3$. We suppose the body in the form to which we reduced the Riemann's surface, namely, that of a flat plate, represented by figs. [46] and [47], in which A, B, C are the holes. The circuits through each hole are so drawn as to connect the hole directly with the outer rim, like the circuit which is drawn through the hole A. A circuit passing through *two* holes, as B, C [fig. 46], may be moved continuously till it consists of two circuits going through the two holes separately. Similarly, a circuit round two or more holes, as B, C [fig. 47], may be pinched at various points until it is made up of circuits round the separate holes. Such a circuit as *abcd* [fig. 46] may be moved into the form *abcd* [fig. 47], in which it consists of two circuits going through the hole A, but in opposite directions. On this account it may be called a *nugatory* circuit.

VI.

The Canonical Dissection.

Suppose now that it is desired to cut through the Riemann's surface in such a way that it shall still hang together, but that it shall no longer be possible to draw an irreducible circuit upon it. This we may do if we successively prevent the different kinds of irreducible circuits considered in the last section. To prevent the possibility of going *round* any hole, we must cut the surface along a circuit which goes *through* the hole. To prevent the passage *through* a hole, we must cut through a circuit which goes *round* a hole.

Let us make sections a_1, a_2, a_3 [figs. 48, 49] round the holes, and b_1, b_2, b_3 through the holes. Then we shall have prevented the drawing of any irreducible circuits except nugatory ones, like *abcd* in the previous figures. To prevent these also, we may cut the surface along the line c_1 which goes from p to q, that is, from a point on b_2 to a point on b_3, and along the line c_2 which goes from q to r, that is, from a point on b_3 to a point on b_1. We must not cut from r to p also, for then we should divide the surface into two separate parts. We may now open out the upper and under portions of the surface in fig. [48], until it assumes the form of fig. [49]. It then becomes obvious that all our cuts form a continuous line, which is now the boundary of the surface, and is made up of the pieces (beginning at p and going round to the right) $c_1 b_3 a_3 b_3 c_2 b_1 a_1 b_1 a_1 b_1 c_2 b_3 a_3 b_3 c_1$. Moreover, it is a matter of intuition that no irreducible contour can now be drawn on the surface.

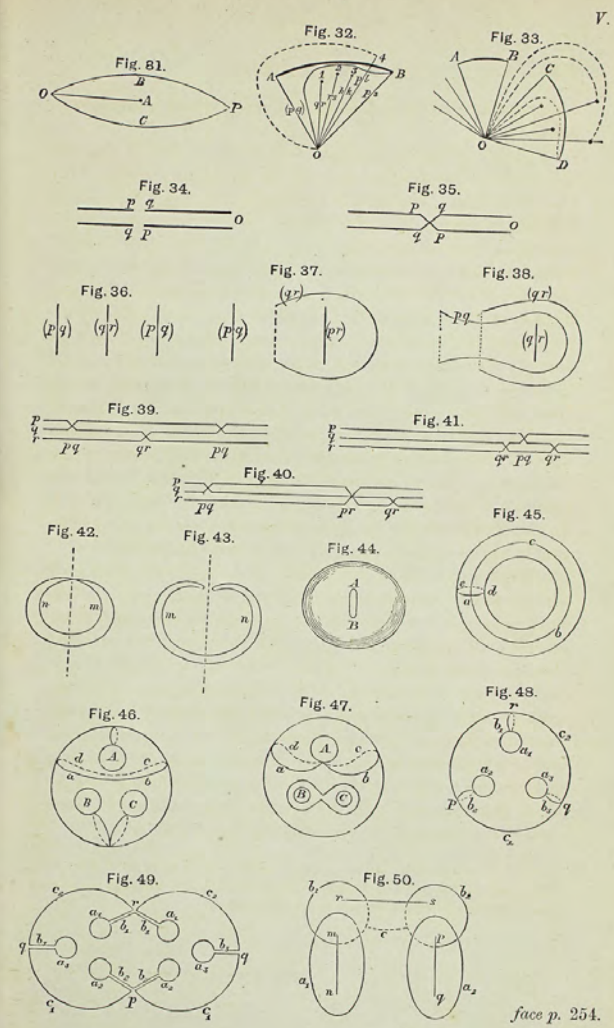
This system of cuts is called a *canonical dissection* of the surface. In the general case it consists of p cuts a going round the holes, p cuts b going through them, and $p - 1$ cuts c joining b_2 to b_3, b_3 to b_4, ... b_p to b_1, but not b_1 to b_2. The cuts c may, if we like, join the a-cuts together, or generally they may join the systems (ab) together, a *system* meaning an a-cut and a b-cut belonging to the same hole. In fact, the c-cuts are only of importance as completing the single boundary of the surface, and so enabling us to see that no irreducible circuit is any longer possible.

It only remains to translate this result so that it may be applicable to the original form of the Riemann's surface, viz., an n-fold plane. We shall do this in the case $p=2$, which will sufficiently explain the general case. We have now two sheets connected by three cross-lines mn, pq, rs [fig. 50]. One of these must be chosen to represent the outer rim of our flat plate; the other two will then correspond to the holes in it. Let mn, pq represent the holes, and rs the outer rim; lines in the upper sheet shall be drawn in full, and lines in the lower sheet shall be dotted. Then we must first make cuts a_1, a_2, which go round the holes mn, pq; these may lie entirely in the upper sheet. Next we must make cuts b_1, b_2, which connect the holes respectively with the outer rim rs. These cuts lie partly in the upper sheet, where they intersect the cuts a, and partly in the under sheet. Lastly, we must connect the system $a_1 b_1$ with the system $a_2 b_2$ by the cut c; this is drawn in the figure from b_1 to b_2 in the under sheet. It is impossible to draw an irreducible circuit on the two-fold plane when it is thus dissected*.

In general, we have proved that in the n-sheeted Riemann's surface which represents the function s determined by the equation $f(s, z)=0$, there are $p+1$ cross-lines such that if one be taken to represent the rim, and the rest holes, of a flat plate, the surface may be dissected into one on which no irreducible contour is possible by the following process:—Cut the surface along curves a each of which goes round one of the cross-lines taken to represent holes, on one of the sheets of the surface which cross at that line. Connect each of these lines with the one taken to represent the rim by a cut b along a closed curve which crosses each of the two cross-lines once. Then connect the systems (ab) chainwise by $p-1$ cuts c.

* It is to be understood that a circuit is *reducible* when all parts of it can be continuously moved away to infinity without crossing any branch-point; because in this theory infinity counts as a single point.

V.



face p. 254.

XXVIII.

REMARKS ON THE CHEMICO-ALGEBRAICAL THEORY.

(Extract from a letter to Mr Sylvester*.)

"THE new Journal [see foot-note] I look forward to with the greatest interest: it will be the only English periodical in which one will have room to print formulæ, except the *Philosophical Transactions*. I had designed for you a series of papers on the application of Grassmann's methods, but there is only one of them fit for printing yet†. It is an *explanation* of the laws of quaternions and of my biquaternions by resolving the units into factors having simpler laws of multiplication; a determination of the compounding systems for space of any number of dimensions; and a proof that the resulting algebra is a *compound* (in Peirce's sense) of quaternion algebras. It thus appears that quaternions are the last word of geometry in regard to complex algebras. Another of them was to be about the very thing you speak of, which was communicated to the British Association at Bristol, *not* Bradford. There is no question of reclamation, because the whole thing is really no more than a translation into other language of your own theories published years ago in the *Cambridge Mathematical Journal*. I have a strong impression that you will find there the analogy of covariants and invariants to compound radicals and saturated molecules.

I consider forms which are linear in a certain number of sets of k variables each. To fix the ideas, suppose $k=2$ and that I have altogether 6 sets of 2 variables each, namely

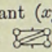
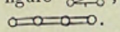
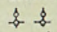
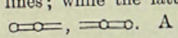
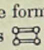
$$x_1x_2,\ y_1y_2,\ z_1z_2,\ u_1u_2,\ v_1v_2,\ w_1w_2.$$

* [From the *American Journal of Mathematics, Pure and Applied*, Vol. I. No. 2, pp. 126—128.]

† [This is XXX. of the present volume.]

Suppose the forms are

$$(xyzu),\ (yzvw),\ (xv),\ (uw);$$

viz. $(xyzu)$ means an expression separately linear and homogeneous in the x, the y, the z, and the u, and so for the rest. I observe that in these four forms each set of variables occurs twice. This being so, there is one invariant of the four forms, which is invariant in regard to *independent* transformations of the six sets of variables. This you knew thirty years ago. All I add is: *to obtain this invariant*, regard the *variables as alternate numbers*, and *simply multiply all the forms together*. By *alternate* numbers I mean those whose multiplication is polar $(xy = -yx)$ and whose squares are zero. The product of the forms will then be equal to the invariant in question multiplied by the product of all the variables. The quartic forms may be represented by the symbol -ǫ-, the quadratics by -o-. Thus the invariant $(xyzu)$ $(yzvw)$ (xv) (uw) will be represented by the figure ; whereas, $(xyzu)$ $(yzvw)$ (xu) (vw) is this form . The former is clearly the product of the two quartic covariants  got by cutting it across the dotted lines; while the latter is the product of the quadri-covariants . A *bond* between two forms means a set of variables common to them. Of course, we may regard two or more of the forms as identical and so form invariants of a single form; thus  is the discriminant of a cubic*...Of course, the main thing is to pass from this system of separate variables to that in which the same variables occur to higher orders in the same form, or back again—what you call 'unravelment'....

The part of the theory which astonished me most is its application to *intergredient* variables when the number in a set is greater than 3,—such as the six co-ordinates of a line in the case of quaternary forms. When the original variables are regarded as alternate numbers, these intergredients are simply their binary products. Thus by simply multiplying the linear forms representing two planes, we get one intergredient form representing their line of intersection. And so generally, whatever be the number of variables in a set, the intergredient variables are merely their products so many together. With

this understanding, the product of a set of forms in which the variables are regarded as alternate numbers is the *only* invariant or covariant of the forms which possess certain definite characters of invariance.

The ordinary theory of symmetrical forms seems to me to bear the same relation to this one (of forms linear in several sets of variables) that a boulder does to a crystal—all the angles rounded off so that you can't see through it so clearly...." †

† [Dr Sylvester has appended several interesting notes, from which a few extracts are given here. "I think Prof. Clifford overstates the obligations which he alleges to my previous papers. At all events he has more than reconquered his title to the merit of the first conception by the completeness he has imparted to it...In a word he has found the universal pass-key to the *quantification of the graphs*...'All that Prof. Clifford adds' is the very pith and marrow of the matter which before was wanting." Dr Sylvester further remarks, "'I will take the example of this figure [cf. * in text] to illustrate Prof. Clifford's rule for finding the *algebraical content* of the graph. Let the bonds be called $\begin{smallmatrix} x & & t & \\ y & z & u & v \end{smallmatrix}$. Then there will be four forms corresponding to the four apices or atoms, viz.

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)\,(x_1, x_2)\,(y_1, y_2)\,(z_1, z_2),$$
$$(b_1, b_2, \ldots\ldots\ldots\ldots\ldots\ldots b_8)\,(z_1, z_2)\,(t_1, t_2)\,(u_1, u_2),$$
$$(c_1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots c_8)\,(t_1, t_2)\,(u_1, u_2)\,(v_1, v_2),$$
$$(d_1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots d_8)\,(v_1, v_2)\,(x_1, x_2)\,(y_1, y_2),$$

where all the x, y, z, t, u, v letters are to be regarded as *polar elements*. [Dr Sylvester objects to the term *alternate numbers* in this connexion.] Take the polar product of these forms; the coefficient of

$$x_1 . x_2 . y_1 . y_2 . z_1 . z_2 . t_1 . t_2 . u_1 . u_2 . v_1 . v_2$$

will be an invariant of three lineo-lineo-linear forms.

If we make the values identical for the same index, whatever the letter which it affects, it becomes an invariant of a single lineo-lineo-linear form; and finally if we make the coefficients of $x_1y_1z_2$, $y_1z_1x_2$, $z_1x_1y_2$ all alike, and again the coefficients of $x_2y_2z_1$, $y_2z_2x_1$, $z_2x_2y_1$ all alike, and identify the letters x, y, z, the form becomes a binary cubic and the invariant becomes its discriminant. We know *à priori* by my permutation-sum test that the algebraical content above indicated will not vanish because

$$\Sigma\,(a-b)^2\,(a-d)^2\,(a-c)\,(b-d)$$

is not zero, whereas the algebraical content of the figure formed by turning round one of each pair of the doubled lines into the position of the two diagonals respectively *will* vanish because the permutation-sum of

$$\Sigma\,(a-b)\,(b-c)\,(c-d)\,(d-a)\,(a-c)\,(b-d)$$

is zero."]

*XXIX.

NOTES ON QUANTICS OF ALTERNATE NUMBERS, USED AS A MEANS FOR DETERMINING THE INVARIANTS AND COVARIANTS OF QUANTICS IN GENERAL*.

THE term *alternate numbers* means a set (λ_1, λ_2) or sets of numbers which satisfy the following relations:

$$\lambda_1^2 = 0,\ \ \lambda_2^2 = 0,\ \ \lambda_1\lambda_2 + \lambda_2\lambda_1 = 0 \ldots\ldots\ldots\ldots(1);$$

to which it is usual to add

$$\lambda_1\lambda_2 = 1.$$

The above set is binary, but there may be ternary, &c. sets, or sets consisting of any number of letters $\lambda_1, \lambda_2, \ldots$

By a *Quadratic Form* is here meant an expression—lineo-linear in two sets of such numbers regarded as variables, say λ_1, λ_2; μ_1, μ_2, such as

$$a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2 \ldots\ldots\ldots\ldots(2).$$

This may be also denoted, for shortness, by the symbol $a\,|\,12$, or even by 12, where the 1, 2 refer to the two sets of variables λ, μ.

* [From the *Proceedings of the London Mathematical Society*, Vol. x. No. 148, pp. 124—129. "This is the substance of some fragments found amongst the papers of the late Professor Clifford. The only published explanation of the method with which I am acquainted, is contained in a letter to Professor Sylvester (see XXVIII.): 'I consider forms which are linear in a certain number of sets of *k* variables each....The product of the forms will then be equal to the invariant in question multiplied by the product of all the variables.'" SP.]

In this case there is one invariant only which allows of the variables being separately transformed, namely, the discriminant, which is got by squaring the form. We have, in fact, by the properties of alternate numbers,

$$(a\,|\,12)^2 = -2\,(a_{11}a_{22} - a_{12}a_{21}) = -2D, \text{ suppose}\ldots\ldots(3).$$

But if the variables are transformed by the same substitution, there is a universal covariant, $\lambda_1\mu_2 - \lambda_2\mu_1$, which may be denoted by (12), or by (21); for $\lambda_1\mu_2 - \lambda_2\mu_1 = \mu_1\lambda_2 - \mu_2\lambda_1$, by the property of alternate numbers.

If we replace the μ's by x, an ordinary, not an alternate, number, we get a linear function of the λ's, whose square vanishes in virtue of the relations (1). Thus we have $(a\,|\,1x)^2=0$, and consequently such a product as $(a\,|\,1x)(a\,|\,1y)$ must be divisible by $(xy) = x_1y_2 - x_2y_1$ since it vanishes when x and y represent the same point; in fact, if we write the expressions in full, thus,

$$(a\,|\,1x) = (a_{11}x_1 + a_{12}x_2)\,\lambda_1 + (a_{21}x_1 + a_{22}x_2)\,\lambda_2,$$
$$(a\,|\,1y) = (a_{11}y_1 + a_{12}y_2)\,\lambda_1 + (a_{21}y_1 + a_{22}y_2)\,\lambda_2,$$

actual multiplication gives

$$(a\,|\,1x)\,(a\,|\,1y) = \begin{matrix} a_{11}x_1 + a_{12}x_2, & a_{21}x_1 + a_{22}x_2 \\ a_{11}y_1 + a_{12}y_2, & a_{21}y_1 + a_{22}y_2, \end{matrix} = \begin{matrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{matrix} \times \begin{matrix} x_1, & x_2 \\ y_1, & y_2. \end{matrix}$$

But $a_{11}a_{22} - a_{12}a_{21}$ may be regarded as the discriminant of any quadratic form having a's for its coefficients, say any form $(a\,|\,1\vartheta)$. And, since it was shown, by equation (3), that the square of such a form is equal to minus twice its discriminant, it follows that the above equation may be written thus,

$$-2\,(a\,|\,1x)\,(a\,|\,1y) = (a\,|\,1\vartheta)^2\,(xy) \ldots\ldots\ldots\ldots(4).$$

Note.—The transformation

$$y_1, y_2 = \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} (x_1, x_2)$$

may also be written

$$a_{11}x_1y_1 + a_{21}x_1y_2 + a_{12}x_2y_1 + a_{22}x_2y_2 = 0 = (a\,|\,yx),$$

where the y's are arbitrary functions satisfying identically

$$y_1y_1 + y_2y_2 = 0.$$

In like manner, a second transformation

$$z_1, z_2 = \begin{vmatrix} b_{11}, & b_{12} \\ b_{21}, & b_{22} \end{vmatrix} (y_1, y_2)$$

may be represented by $(b \mid zy) = 0$; and the result of the two by

$$(b \mid zy)(a \mid yx) = 0.$$

There are one or two other formulæ which it will be convenient to notice before proceeding further. If we multiply the form $(a \mid 12)$ by (12), we get

$$(12)(a \mid 12) = a_{21} - a_{12} = s \text{ (suppose)} \ldots\ldots\ldots\ldots (5).$$

Also, if we suppose the coefficients to remain unaltered, viz., if $(a \mid 21)$ represents $a_{11}\mu_1\lambda_1 + a_{12}\mu_1\lambda_2 + \ldots$, then

$$(a \mid 12) + (a \mid 21) = -s\,(12) \ldots\ldots\ldots\ldots (6),$$

or $$2\,(a \mid 12) = (a \mid 12) - (a \mid 21) - s\,(12),$$

the quantity s vanishing when the form is symmetrical, i.e. when $a_{12} = a_{21}$. Again, if the coefficients be supposed to remain unaltered, so that

$$(a \mid 13) = a_{11}\lambda_1\nu_1 + a_{12}\lambda_1\nu_2 + \ldots,$$

then it will be found that

$$\left.\begin{aligned}(12)(a \mid 23) &= -(a \mid 13)\\ (13)(a \mid 12) &= -(a \mid 32)\end{aligned}\right\} \ldots\ldots\ldots\ldots (7).$$

Such multiplication is in fact tantamount to a substitution of variables, from 2 to 1 (i.e. from μ to λ), or from 1 to 3 (i.e. from λ to ν). This theorem, as will readily be seen, is not restricted to quadric forms; but if a, b be any two different forms, and if 1 be a set of variables in a and not in b, and 2 a set of variables in b and not in a, then, making the same supposition as in b, viz., that the constants a, b remain the same on both sides of the equation, we shall find that

$$(12)(a \mid 1\ldots)(b \mid 2\ldots) = (a \mid 2\ldots)(b \mid 2\ldots) = (a \mid 1\ldots)(b \mid 1\ldots).$$

With reference to the universal covariant $(12) = \lambda_1\mu_2 - \lambda_2\mu_1$, it may be remarked that we may consider such covariants for

more than two sets of variables; and we shall then obtain the following formulæ*:

$$(12)(23) = -(13), \quad (12)(23)(34) = (14), \ldots \quad \ldots\ldots (8),$$

of which the following is a consequence,

$$(12)(23)(31) = 0 \ldots\ldots\ldots\ldots (9).$$

It is clear that quadratic forms can only be combined in a *chain*†, which, when open, gives a quadratic covariant; when closed, an invariant. We can now show that a closed chain of $2n$ sides is equal to $\pm 2D^n$, while a chain of an odd number of sides vanishes.

Take a chain of four sides, (12), (23), (34), (41); then, since

$$(12)(23)(34)(41) = (12)(23) \times (34)(41);$$

and since, by (3) and (4),

$$2\,(12)(23) = (a \mid 28)^2 (31) = -2D\,(31),$$

$$2\,(34)(41) = (a \mid 48)^2 (13) = -2D\,(13),$$

$$(31)(13) = 2,$$

it follows that

$$(12)(23)(34)(41) = 2D^2 \ldots\ldots\ldots\ldots (10).$$

In like manner any chain of an even number of sides may be resolved into a power of the discriminant multiplied by a chain of determinants of the alternate variables. The product of these latter is ± 2, according as the number of terms is even or odd. Hence, generally, a chain of $2n$ sides is $2(-D)^n$.

* [This and other parts of the present paper may be compared with Spottiswoode "On Determinants of Alternate Numbers," *Proceedings of the London Mathematical Society*, Vol. VII., p. 100. Sp.]

† [I have, in this paragraph, retained the language of the original MS., although the term *chain* is not here explained. The author appears to have had in his mind a theory which he propounded verbally at the meeting of the British Association at Bristol, and to which, in his latter days, he attached great importance and devoted much time. Some indications of it will be found in the *American Journal of Pure and Applied Mathematics*, Vol. I. p. 127 [cf. XXVIII.]. Several notes relating to the subject have been found amongst his papers, but as they are almost exclusively memoranda without explanation, it is still uncertain whether they can be published. Sp.]

A chain of $2n+1$ sides may, by analogous processes, be reduced to the product of a determinant of alternate variables by a form containing those variables, and must therefore vanish for symmetrical forms. For unsymmetrical it contains s as a factor. In this case the fundamental formula must be modified as follows:

$$(a\,|\,x1)\,(a\,|\,1y) = (a\,|\,xy)\,s - (xy)\,D;$$

or, with alternate numbers in the place of x, y,

$$(a\,|\,12)\,(a\,|\,23) = (a\,|\,13)\,s - (13)\,D.$$

Multiplying now into $(a\,|\,31)$, we get

$$(a\,|\,12)\,(a\,|\,23)\,(a\,|\,31) = (a\,|\,13)\,(a\,|\,31)\,s + sD$$

in virtue of equation (5).

But $\quad (a\,|\,13)\,(a\,|\,31) = (a_{11}\lambda_1\mu_1 + \ldots)\,(a_{11}\mu_1\lambda_1 + \ldots)$

$$= (2a_{11}a_{22} - a^2_{12} - a^2_{21}) = 2D - s^2.$$

Hence $\quad (a\,|\,12)\,(a\,|\,23)\,(a\,|\,31) = (3D - s^2)\,s \;\ldots\ldots\ldots(11).$

Again,

$$\begin{aligned}(a\,|\,12)\,(a\,|\,23)\,(a\,|\,34) &= (a\,|\,13)\cdot(a\,|\,14)\,s - (13)\,(a\,|\,34)\,D\\ &= (a\,|\,14)\,s^2 - (14)\,sD + (a\,|\,14)\,D\\ &= (a\,|\,14)\,(D + s^2) - (14)\,sD;\end{aligned}$$

whence also

$$(a\,|\,12)\,(a\,|\,23)\,(a\,|\,34)\,(a\,|\,41) = (2D - s^2)\,(D + s^2) + s^2D \ldots(12).$$

Suppose, in general, that

$$(a\,|\,12)\,(a\,|\,23)\ldots(a\,|\,lm) = A_m\,(a\,|\,1m) - B_m\,(1m);$$

then

$$\begin{aligned}(a\,|\,12)\,(a\,|\,23)\ldots(a\,|\,mn) &= A_m\,(a\,|\,1m)\,(amn) + B_m\,(a\,|\,1n)\\ &= A_m\,\{(a\,|\,1n)\,s - (1n)\,D\} + B_m\,(a\,|\,1n)\\ &= (sA_m + B_m)\,(a\,|\,1n) - A_mD\,(1n).\end{aligned}$$

Hence $\quad A_{m+1} = sA_m + B_m, \quad B_{m+1} = A_mD;$

and consequently

$$A_{m+1} = sA_m + DA_{m-1} \;\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

The form $\lambda_1\mu_1 + 0\lambda_1\mu_2 + s\lambda_2\mu_1 + D\lambda_2\mu_2$ is a form having s and D for invariants; and it may therefore be taken as the

canonical form for a bipartite quantic such as we have been here considering. Again, another useful form is the following,

$$\sqrt{(s^2 - 4D)}\,(\lambda_1\mu_2 + \lambda_2\mu_1) + s\,(\lambda\mu).$$

We may, however, combine not only two quadratic forms having like coefficients, say, two forms a; but we may also combine two having different coefficients, say, two forms a and b. Two such forms give rise to an invariant, namely, their product. In fact, we have

$$-(a\,|\,12)\,(b\,|\,12) = a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11} = D_{ab}, \text{ suppose};$$

and from (5), $\quad (a\,|\,12)\,(b\,|\,12) + (a\,|\,21)\,(b\,|\,12) = -s_as_b,$

or $\quad (a\,|\,21)\,(b\,|\,12) = D_{ab} - s_as_b \;\ldots\ldots\ldots\ldots\ldots\ldots(14).$

The product of $(a\,|\,12) = a_{11}\lambda_1\mu_1 + \ldots$ and $(b\,|\,13) = b_{11}\lambda_1\nu_1 + \ldots$ gives a covariant; namely,

$$\begin{aligned}(a\,|\,12)\,(b\,|\,13) = {} & a_{11}\mu_1 + a_{12}\mu_2, \quad a_{21}\mu_1 + a_{22}\mu_2\\ & b_{11}\nu_1 + b_{12}\nu_2, \quad b_{21}\nu_1 + b_{22}\nu_2\\ = {} & \begin{vmatrix} a_{11}b_{21} - a_{21}b_{11}, & a_{11}b_{22} - a_{21}b_{12}\\ a_{12}b_{21} - a_{22}b_{11}, & a_{12}b_{22} - a_{22}b_{12}\end{vmatrix}(\mu_1, \mu_2)\,(\nu_1, \nu_2)\\ = {} & \vartheta_{ab}\,|\,23, \text{ suppose} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).\end{aligned}$$

If, in the covariant, we make the forms a and b coincident, it becomes

$$(a\,|\,12)\,(a\,|\,13) = D_a(23), \quad (a\,|\,12)\,(a\,|\,23) = s_a\,(a\,|\,13) - D_a(13);$$

and, multiplying this by, or into, b, we get

$$(a\,|\,12)\,(a\,|\,13)\,(b\,|\,43) = (b\,|\,43)\,(a\,|\,12)\,(a\,|\,13) = -D_a\,(b\,|\,42),$$

$$(a\,|\,12)\,(a\,|\,13)\,(b\,|\,34) = (b\,|\,43)\,(a\,|\,12)\,(a\,|\,13) = -D_a\,(b\,|\,24),$$

and $\quad (a\,|\,12)\,(a\,|\,13)\,(b\,|\,23) = D_as_b,$

$$(a\,|\,12)\,(a\,|\,13)\,(b\,|\,32) = D_as_b \ldots\ldots\ldots\ldots\ldots (16).$$

The remaining form to be investigated is

$$(a\,|\,12)\,(b\,|\,13)\,(a\,|\,43).$$

The value of this is

$$\begin{array}{cccc}(a_{11}b_{21}-a_{21}b_{11})\mu_1+(a_{12}b_{21}-a_{22}b_{11})\mu_2, & (a_{11}b_{22}-a_{21}b_{12})\mu_1+(a_{12}b_{22}-a_{22}b_{12})\mu_2 \\ a_{11}\rho_1 + \qquad a_{21}\rho_2, & a_{12}\rho_1 + \qquad a_{22}\rho_2\end{array}$$

$$\begin{aligned}
&= (-a_{21}a_{12}b_{11}+a_{11}a_{21}b_{12}+a_{11}a_{12}b_{21}-a_{11}a_{11}b_{22})\,\mu_1\rho_1 \\
&\quad +(-a_{21}a_{22}b_{11}+a_{21}a_{21}b_{12}+a_{11}a_{22}b_{21}-a_{11}a_{21}b_{22})\,\mu_1\rho_2 \\
&\quad +(\ a_{12}a_{12}b_{21}-a_{22}a_{12}b_{21}-a_{11}a_{12}b_{22}+a_{11}a_{22}b_{12})\,\mu_2\rho_1 \\
&\quad +(\ a_{22}a_{12}b_{21}-a_{22}a_{22}b_{11}-a_{21}a_{12}b_{22}+a_{21}a_{22}b_{12})\,\mu_2\rho_2 \\
&= (-a_{11}D_{ab}+b_{11}D_{aa})\,\mu_1\rho_1 \\
&\quad +(-a_{21}D_{ab}+b_{21}D_{aa})\,\mu_1\rho_2 \\
&\quad +(-a_{12}D_{ab}+b_{12}D_{aa})\,\mu_2\rho_1 \\
&\quad +(-a_{22}D_{ab}+b_{22}D_{aa})\,\mu_2\rho_2 \\
&= D_{ab}(a\,|\,42)-D_{aa}(b\,|\,42) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (17).
\end{aligned}$$

Multiplying this into $(b\,|\,45)$, we obtain

$$\begin{aligned}&(a\,|\,12)\,(b\,|\,13)\,(a\,|\,43)\,(b\,|\,45) \\ &\qquad = D_{ab}(a\,|\,42)\,(b\,|\,45)-D_{aa}(b\,|\,42)\,(b\,|\,45)\,;\end{aligned}$$

that is, referring to equation (15), and dropping the suffix ab,

$$(\vartheta\,|\,23)\,(\vartheta\,|\,35)=D_{ab}(\vartheta\,|\,25)-D_{aa}\,.\,D_{bb}(25).$$

But $\qquad (\vartheta\,|\,23)\,(\vartheta\,|\,35)=s\,(\vartheta_\vartheta\,|\,25)-D_\vartheta(25).$

Hence $\qquad s_\vartheta=D_{ab},\quad D_\vartheta=D_{aa}\,.\,D_{bb} \ldots\ldots\ldots\ldots\ldots (18),$

as may be easily verified.

If, however, we multiply the equation

$$(a\,|\,12)+(a\,|\,21)=-s\,(12) \text{ by } (a\,|\,12),$$

we obtain

$$(a\,|\,12)\,(a\,|\,21)=2D_a-s_a^2\,;$$

hence $\qquad (\vartheta\,|\,12)\,(\vartheta\,|\,21)=2D_\vartheta-s_\vartheta^2=2D_aD_b-D_{ab}^2 \ldots\ldots (19),$

and consequently

$$\left.\begin{aligned}\{(\vartheta\,|\,12)-(\vartheta\,|\,21)\}^2&=-4D_aD_b-2\,(2D_aD_b-D_{ab}^2) \\ &=-2\,(4D_aD_b-D_{ab}^2)\end{aligned}\right\}\ldots.(20).$$

The formula $s_\vartheta=D_{ab}$ is important as showing that ϑ is not made symmetrical by making a and b symmetrical. Hence, in passing from these invariants and covariants to those of symmetrical forms, we are obliged to use the symmetrised form of ϑ, namely $(\vartheta\,|\,12)-(\vartheta\,|\,21)$. Thus, to adapt equation (17) to symmetrical forms, we have

$$\begin{aligned}(\vartheta\,|\,23)\,(a\,|\,43)&=D_{ab}(a\,|\,42)-D_{aa}(b\,|\,42), \\ (\vartheta\,|\,23)\,(a\,|\,43)&=(a\,|\,13)\,(b\,|\,12)\,(a\,|\,43) \\ &=D_a(14)\,(b\,|\,12)=-D_a(b\,|\,42).\end{aligned}$$

In general, we write $\bar{a}$ for the mean value of a, i.e., if

$$2\,(\bar{a}\,|\,12)=(a\,|\,12)-(a\,|\,21),$$

we shall have $\qquad (\bar{a}\,|\,12)=(a\,|\,12)+\tfrac{1}{2}s\,(12).$

Hence, since $(12)^2=-2$,

$$(\bar{a}\,|\,12)^2=(a\,|\,12)^2+s^2-\tfrac{1}{2}s^2 \ldots\ldots\ldots\ldots (21),$$

and $\qquad \bar{D}=D-\tfrac{1}{4}s^2,$

where $\bar{D}$ is the discriminant of the symmetrical function.

Similarly

$$\begin{aligned}\vartheta_{\bar{a}\bar{b}}=(\bar{a}\,|\,12)\,(\bar{b}\,|\,13)&=\{(a\,|\,12)+\tfrac{1}{2}s_a(12)\}\,\{(b\,|\,13)+\tfrac{1}{2}s_b(13)\} \\ &=\vartheta_{ab}-\tfrac{1}{2}\{s_a(b\,|\,23)+s_b(a\,|\,23)\}+\tfrac{1}{4}s_as_b(23),\end{aligned}$$

and therefore

$$\left.\begin{aligned}2\bar{\vartheta}_{\bar{a}\bar{b}}&=(\vartheta_{ab}\,|\,23)-(\vartheta_{ab}\,|\,32)-s_as_b(23) \\ &=2\,(\vartheta_{ab}\,|\,23)-s_\vartheta(23)-s_as_b(23)\end{aligned}\right\}\ldots\ldots\ldots\; (22).$$

If we multiply this into $(a\,|\,43)$, we have

$$\begin{aligned}2\,(\vartheta_{\bar{a}\bar{b}}\,|\,23)\,(a\,|\,43)&=2\,(\vartheta\,|\,23)\,(a\,|\,43)-D_{ab}(a\,|\,42)-s_as_b(a\,|\,42) \\ &=D_{ab}(a\,|\,42)-2D_aD_b(b\,|\,42)-s_as_b(a\,|\,42).\end{aligned}$$

Putting $s_a=0$, $s_b=0$, we get the formula for symmetric functions.