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On the Local Stability of Dynamical Systems in the Saddle-point Sense

by Shozo Murata

1. Introduction

In this paper we consider some sufficient conditions under which a 2n-square matrix J is s·p stable. For n=1 it is well known that the matrix J is s·p stable if and only if det J<0. In this case the matrix (det J) is stable, i.e., all the characteristic values of the matrix (det J) have negative real parts. From this result we expect that under some conditions a similar result can be carried to the 2n-square matrix case by replacing the stability of J with the stability of the n-square matrix AB-CD, where $J=\begin{bmatrix} A & D \\ C & B \end{bmatrix}$.

Two lemmas are given which relate the eigenvalues of 2n-square matrices and they are used to prove the local stability of dynamical systems. One of the results obtained here is a special case of the theorem of Kurz [2] but it shows an additional property on the eigenvalues of the matrix.

2. Preliminaries

In what follows all the matrices considered have real entries. For any matrix M we denote by M^T the transpose of M. Following classical terminology, we call an n-square matrix A stable if all its eigenvalues have negative real parts. In order to state our results easily, we call the 2n-square matrix J s·p stable if n eigenvalues have negative real parts and the remaining n gigenvalues have positive real parts.

As is well known, if A and C are n-square matrices and AC=CA, then

$$\det\begin{bmatrix} A & D \\ C & B \end{bmatrix}=\det[AB-CD]$$

for any n-square matrix B and D. Thus when $A+B=\alpha I_n$,

$$\det\begin{bmatrix} A-\lambda I_n & D \\ C & B-\lambda I_n \end{bmatrix}=\det[AB-CD-(\alpha+\lambda)I_n],$$

where α is an arbitrary real number, λ is an arbitrary complex number, and I_n is an n-unit matrix. This is a result which we will need in what follows. Using this result we derive a criterion for determining the local stability of a non-linear system from its linearized equations for it at the equilibrium point.

3. Lemmas and Theorems

We consider some sufficient conditions that make 2n-square matrix J s·p stable. In this case there exist n-square matrices A, B, C and D such that

$$J=\begin{bmatrix} A & D \\ C & B \end{bmatrix}.$$

Lemma 1: *Let* A, B, C *and* D *be n-square matrices. Let* AB=BA, BC=CB, *and* AC=CA. *Then*

$$\det\begin{bmatrix} A-\lambda I_n & D \\ C & B-\lambda I_n \end{bmatrix}=\det\begin{bmatrix} A-\Lambda & D \\ C & B-\Lambda \end{bmatrix}.$$

Here the notation Λ *denotes* $A+B-\lambda In$.

PROOF: Let $\tilde{\Lambda}$ be $\lambda I_n-(A+B)/2$. Then

$$\det\begin{bmatrix} A-\lambda I_n & D \\ C & B-\lambda I_n \end{bmatrix}=\det\left[\left(\frac{A-B}{2}-\tilde{\Lambda}\right)\left(\frac{B-A}{2}-\tilde{\Lambda}\right)-CD\right]$$

$$=\det\begin{bmatrix} A-\Lambda & D \\ C & B-\Lambda \end{bmatrix}. \quad \blacksquare$$

THEOREM 1: *Let* $AC=CA$ *and* $A+B=\alpha I_n$ *where* α *is a real number. If* $AB-CD$ *is a stable matrix*[(1)], *then* $J=\begin{bmatrix} A & D \\ C & B \end{bmatrix}$ *is s·p stable.*

PROOF: Let λ be an eigenvalue of J. Then

$$\det\,[J-\lambda I_n]=\det\,[AB-CD-(\alpha-\lambda)\lambda I_n]=0.$$

For the assumption of the stability of the matrix AB-CD ensures that

$$0>R(\alpha\lambda-\lambda^2)\geqq R(\lambda)(\alpha-R(\lambda)),$$

where $R(\lambda)$ is the real part of λ, we obtain

$$R(\lambda)>\max(0,\ \alpha) \text{ or } R(\lambda)<\min(0,\ \alpha).$$

Since conditions of Lemma 1 are trivially verified in this case, therefore,

if λ is an eigenvalue of J and if $R(\lambda)>\max(0,\ \alpha)$, then $\alpha-\lambda$ is also an eigenvalue of J and $R(\alpha-\lambda)<0$. ■

REMARK: By Theorem 1 it is clear that the eigenvalues of J are symmetric about $\alpha/2$ in the complex plane. For n=1 Theorem 1 reduces to the well-known formula for the s·p stability of J: det $J<0$.

COROLLARY 1: *If an n-square matrix* A *is a real symmetric non-singular matrix, then* $J=\begin{bmatrix} A & \delta_1 I_n \\ \delta_2 I_n & -A \end{bmatrix}$ *is s·p stable for all non-negative real number* δ_1 *and* δ_2.

PROOF: Since $-A^2-\delta_1\delta_2 I_n$ is stable by assumption, Corollary 1 is implied by Theorem 1. ■

REMARK: We can establish the same result when $\delta_1\leqq 0$ and $\delta_2\leqq 0$.

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(1) By Mackenzie [3], if a matrix has q.d.d. that is negative, all its characteristic roots have negative real parts. Therefore the condition of Theorem1 is satisfied when $A+B=\alpha I_n$ and AB-CD has q.d.d. that is negative.

THEOREM 2: *Let* λi (i=1, 2, ⋯, n) *be eigenvalues of* $J=\begin{bmatrix}\delta_1 I_n & D\\ C & \delta_2 I\end{bmatrix}$. *If* δ_1 *and* δ_2 *are real numbers and if* $-CD$ *is stable, then there exist* n *eigenvalues* λi $i \in \{1, 2, \cdots\cdots, 2n\}$ *such that* $R(\lambda i) > max(\delta_1, \delta_2)$ *and* $R(\lambda i) < min(\delta_1, \delta_2)$ *for the remaining* n *eigenvalues.*

PROOF: Let λ be an eigenvalue of J. Then

$$\det[J-\lambda I_{2n}] = \det[-CD-(\lambda-\delta_1)(\delta_2-\lambda)I_n] = 0.$$

For the assumption of the stability of the matrix $-CD$ ensures that

$$0 > R((\lambda-\delta_1)(\delta_2-\lambda)) \geqq (\delta_1-R(\lambda))(R(\lambda)-\delta_2),$$

we obtain

$$R(\lambda) > max(\delta_1, \delta_2) \text{ or } R(\lambda) < \min(\delta_1, \delta_2).$$

From Lemma 1 it is clear that $\delta_1+\delta_2-\lambda$ is also an eigenvalue of J for every eigen value λ of J. ■

REMARK: In other words, Theorem 2 can be stated as follows. The eigenvalues of J are symmetric about $(\delta_1+\delta_2)/2$ in the complex plane. If a 2n-square matrix J satisfies all the conditions of Theorem 2 and if $\delta_1 \delta_2 \leqq 0$, then J is s·p stable.

Lemma 2: *Let* $Z=XY$, *where* X *is symmetric and* $Y+Y^T$ *is positive definite. Let* z_+, z_0, z_- *denote the number of positive, vanishing, and negative real parts of the eigenvalues of* Z, *and let* x_+, x_0, x_- *denote the number of positive, vanishing, and negative real parts of the eigenvalues of* X. *Then* $z_+=x_+$, $z_0=x_0$, $z_-=x_-$.

PROOF: See Theorem 5 in Wielandt [4]. ■

REMARK: If $Y+Y^+$ is positive definite then $-Y$ is stable. It would be natural to expect the converse to hold. That this is not always the case can be seen in the matrix $Y=\begin{bmatrix}1 & -4\\ 0 & 2\end{bmatrix}$. In this case $Y+Y^T$ is not positive definite, but s·p stable matrix.

We shall now show the sufficient conditions that make a 2n-square matrix J s·p stable, where $J=\begin{bmatrix}\delta I_n-A & D\\ C & A^T\end{bmatrix}$ and δ is a real number.

THEOREM 3: *Let* C *and* D *positive definite symmetric and let* $\delta \geqq 0$.

If one of the following holds, then J *is a s·p stable matrix.*

(i) $A=\theta I_n \qquad \theta(\delta-\theta)\leqq 0$

(ii) K^δ *is positive definite, where* $K^\delta=\begin{bmatrix} D & \frac{\delta}{2}I_n \\ \frac{\delta}{2}I_n & C \end{bmatrix}$

(iii) $-(\delta I_n-A)C^{-1}-C^{-1}(\delta I_n-A^T)$ *and* $A^TD^{-1}+D^{-1}A$ *are positive definite.*

(iv) $-(\delta I_n-A)C^{-1}-C^{-1}(\delta I_n-A^T)$ *and* $A^TD^{-1}+D^{-1}A$ *are negative definite.*

PROOF: (i) Since -CD is a stable n-square matrix by Lemma 2, and since $\delta I_n-A=(\delta-\theta)I_n$, the theorem follows from Theorem 2. (ii),(iii) and (iv): For proofs see Benhabib=Nishimura [1]. ∎

REMARK: Corollary 1 gives an interesting information about the positive definiteness of the K^δ matrix. If n-square matrix -C and D are positive definite and if C=-D then K^δ is a s·p stable matrix for any real number δ.

4. An Example of Application

Consider the following optimal control problem:

$$\text{Max}\int_0^\infty f(x,u)e^{-\delta t}dt$$

subject to

$$u=\frac{dx}{dt}=\dot{x},\ x_{(0)}=x_0$$

where $f: R^n\times R^n\to R$ is a twice continuously differentiable function. Let $H=\max_u H^\circ$, where the modified Hamiltonian function H° is defined by

$$H^\circ=f(x,\ u)+\lambda u.$$

Assuming an interior solution, we can define $H=H(x,\ \lambda)$ by the implicit function theorem when

$$\det\left[\frac{\partial^2H^0}{\partial u_i\partial u_j}\right]\neq 0.$$

Applying the Maximum Principle, the optimal trajectry must satisfy the following:

$$(1)\quad \begin{aligned} \dot{\lambda}_j &= \delta\lambda_j - \frac{\partial H(x,\lambda)}{\partial x_j} & j=1,\cdots\cdots,n \\ \dot{x}_j &= \frac{\partial H(x,\lambda)}{\partial \lambda_j} & j=1,\cdots\cdots,n \end{aligned}$$

The Jacobian matrix of equation (1) will be

$$J(\delta)=\begin{bmatrix} \delta I - H_{x\lambda} & -H_{xx} \\ H_{\lambda\lambda} & H_{\lambda x} \end{bmatrix}$$

where

$$H_{xx}=\left[\frac{\partial^2 H}{\partial x_i \partial x_j}\right] \qquad H_{\lambda\lambda}=\left[\frac{\partial^2 H}{\partial \lambda_i \lambda \partial_j}\right]$$

$$H_{x\lambda}=\left[\frac{\partial^2 H}{\partial x_i \partial \lambda_j}\right] \qquad H_{\lambda x}=H^T{}_{x\lambda}.$$

By Theorem 3·(i), the 2n-square matrix $J(\delta)$ is s·p stable if $-H_{xx}$ and $H_{\lambda\lambda}$ are positive definite and $H_{x\lambda}=\theta I_n$ $(\theta(\delta-\theta)\leqq 0)$. This shows us a new result on the local saddle point stability of the equilibrium point of system (1).

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