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Solutions to discrete Painlevé systems arising from two types of orthogonal polynomials

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Abstract. We consider the relation between discrete Painlevé systems and orthogonal polynomials associated with the Christoffel transformation. We construct a method to obtain the particular solutions to discrete Painlevé systems by using orthogonal polynomials and their kernel polynomials. In particular, we treat the cases of the Hermite polynomials and the discrete q-Hermite II polynomials as examples.

1 Introduction

Some discrete Painlevé systems have been found in the studies of random matrices [1, 6, 8]. As one such example, let us consider the partition function of the Gaussian Unitary Ensemble of an $n\times n$ random matrix:

$$Z_n^{(2)}=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Delta(t_1,\cdots,t_n)^2\prod_{i=1}^{n}e^{\eta(t_i)}\mathrm{d}t_i,\quad \eta(t_i)=\sum_{m=0}^{\infty}z_m{t_i}^m, \tag{1.1}$$

where $\Delta(t_1,\cdots,t_n)$ is Vandermonde's determinant. Note that throughout this paper we assume

$$\prod_{i=1}^{n}f(i)=1,\quad \prod_{i=0}^{n-1}f(i)=1,\quad \Delta(t_1,\cdots,t_n)=1, \tag{1.2}$$

for an arbitrary function $f(i)$ when $n=0$. Here we choose

$$\eta(t_i)=-g_1{t_i}^2-g_2{t_i}^4\quad (g_2>0). \tag{1.3}$$

Setting

$$R_n=\frac{Z_{n+1}^{(2)}Z_{n-1}^{(2)}}{\left(Z_n^{(2)}\right)^2}, \tag{1.4}$$

we obtain the following difference equation[1, 10]:

$$R_{n+1}+R_n+R_{n-1}=\frac{n}{4g_2}\frac{1}{R_n}-\frac{g_1}{2g_2}. \tag{1.5}$$

Equation (1.5) is referred to as a discrete Painlevé I equation, denoted by d-$\mathrm{P_I}$, and has the space of initial condition of type $E_6^{(1)}$. Such relations between discrete Painlevé systems and random matrices are well known.

Now, we introduce a q-version of a partition function, using (1.1) as our reference. We consider $\psi_n^{l,m}$ $(l,n\in\mathbb{Z}_{\geq 0},\ m\in\mathbb{Z},\ a\in\mathbb{C},\ c_1\in\mathbb{R}_{>0})$ given as

$$\psi_n^{l,m}=\frac{q^{n(n-1)(2l-1)/2}}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Delta(t_1,\cdots,t_n)^2\prod_{i=1}^{n}\frac{\prod_{j=0}^{l-1}(q^jt_i-q^ma)}{E_{q^2}\left({c_1}^2{t_i}^2\right)}\,\mathrm{d}_qt_i. \tag{1.6}$$

The definitions of the q-definite integral $\int_{-\infty}^{\infty}\mathrm{d}_qt$ and the q-exponential function $E_q(t)$ appearing here are given at the end of this section. As in the case of (1.1), we can obtain a solution to a discrete Painlevé equation expressible in terms of $\psi_n^{l,m}$. Specifically, we have the following:

Lemma 1.1 *A q-analogue of the Painlevé IV equation corresponding to the surface of type $A_4^{(1)}$ (q-$\mathrm{P_{IV}}$) [9] :*

$$(X_{n+1}X_n-1)(X_{n-1}X_n-1)=q^{-N+2n-m-1}a_0{a_1}^{3/2}{a_2}^2\frac{(X_n+q^{N-m}{a_1}^{1/2})(X_n+q^{-N+m}{a_1}^{-1/2})}{X_n+q^{-N+n-m}{a_1}^{1/2}a_2}\,, \tag{1.7}$$

has the following solution:

$$X_n=i\,\frac{(1-q^{n+1})q^n}{c_1}\,\frac{\psi_{n+1}^{0,0}\psi_n^{1,-m}}{\psi_n^{0,0}\psi_{n+1}^{1,-m}}\,. \tag{1.8}$$

Here

$${a_0}^{1/2}=-iqa^{-1}{c_1}^{-1},\quad {a_0}^{1/2}{a_1}^{1/2}=q^{-N},\quad a_2=q^{2N+2},\quad a\neq 0. \tag{1.9}$$

Below, we investigate the solutions to d-$\mathrm{P_I}$ and q-$\mathrm{P_{IV}}$ from the viewpoint of orthogonal polynomials. First, however, we define orthogonal polynomials:

Definition 1.1 *A polynomial sequence $(P_n(t))_{n=0}^{\infty}$ which satisfies the following conditions is called an orthogonal polynomial sequence over the field $\mathscr{K}$, and each term $P_n(t)$ is called an orthogonal polynomial over the field $\mathscr{K}$.*

- *$deg(P_n(t))=n$.*
- *There exists a linear functional $\mathscr{L}:\mathscr{K}(t)\to\mathscr{K}$ which holds the orthogonal condition:*

$$\mathscr{L}[t^kP_n(t)]=h_n\delta_{n,k}\quad (n\geq k), \tag{1.10}$$

where $\delta_{n,k}$ is Kronecker's symbol. Here, h_n is called the normalization factor and $\mu_n=\mathscr{L}[t^n]$ $(n=0,1,\dots)$ is called the moment sequence.

An orthogonal polynomial sequence whose coefficient of leading term is 1 is especially called a monic orthogonal polynomial sequence (MOPS).

First, we reconsider the solution to d-$\mathrm{P_I}$ appearing in (1.4). Let $(P_n(t))_{n=0}^{\infty}$ be MOPS defined as

$$\mathscr{L}[P_n(t)P_k(t)]=\int_{-\infty}^{\infty}P_n(t)P_k(t)e^{-g_1t^2-g_2t^4}\mathrm{d}t=h_n\delta_{n,k}. \tag{1.11}$$

The normalization factor is given as

$$h_n=\frac{Z_{n+1}^{(2)}}{Z_n^{(2)}}\,, \tag{1.12}$$

and then we can rewrite (1.4) as (cf. [1])

$$R_n=\frac{h_n}{h_{n-1}}\,. \tag{1.13}$$

We next reconsider the solution to q-$\mathrm{P_{IV}}$ appearing in (1.8). The Hankel determinant expression of $\psi_n^{l,m}$ is given by the following lemma:

Lemma 1.2 *$\psi_n^{l,m}$, given in (1.6), can be expressed as*

$$\psi_n^{l,m}=q^{n(n-1)(2l-1)/2}\begin{vmatrix} H_{l,m,0} & H_{l,m,1} & \cdots & H_{l,m,n-1}\\ H_{l,m,1} & H_{l,m,2} & \cdots & H_{l,m,n}\\ \vdots & \vdots & \ddots & \vdots\\ H_{l,m,n-1} & H_{l,m,n} & \cdots & H_{l,m,2n-2}\end{vmatrix}. \tag{1.14}$$

Here, the entries are given as

$$H_{l,m,k}=\int_{-\infty}^{\infty}t^k\,\frac{\prod_{j=0}^{l-1}(q^jt-q^ma)}{E_{q^2}({c_1}^2t^2)}\,\mathrm{d}_qt. \tag{1.15}$$

Letting $(P_n^{l,m})_{n=0}^{\infty}$ be MOPS defined as

$$\int_{-\infty}^{\infty} P_n^{l,m}(t) P_k^{l,m}(t)\ \frac{\prod_{j=0}^{l-1}(q^j t - q^m a)}{E_{q^2}\left(c_1{}^2 t^2\right)}\,\mathrm{d}_q t = h_n^{l,m}\delta_{n,k}, \tag{1.16}$$

we can regard $(H_{l,m,n})_{n=0}^{\infty}$ as a moment sequence. Therefore, by using the normalization factor

$$h_n^{l,m} = q^{(1-2l)n}\,\frac{\psi_{n+1}^{l,m}}{\psi_n^{l,m}}\,, \tag{1.17}$$

the solution to q-$\mathrm{P_{IV}}$ can be rewritten as

$$X_n = i\ \frac{1-q^{n+1}}{c_1 q^n}\ \frac{h_n^{0,0}}{h_n^{1,-m}}\,. \tag{1.18}$$

Note that $P_n^{0,m}(t)$ defined as (1.16) is referred to as the discrete q-Hermite II polynomial (cf. [7]) and $P_n^{1,m}(t)$ is said to the kernel polynomial of $P_n^{0,m}(t)$ given by the Christoffel transformation. The definitions of the kernel polynomial and the Christoffel transformation will be given in the next section.

The solution to d-$\mathrm{P_I}$, (1.13), is given by the single orthogonal polynomial, while that to q-$\mathrm{P_{IV}}$, (1.18), is expressed by the two different orthogonal polynomials. From this viewpoint, the types of solutions to d-$\mathrm{P_I}$ and q-$\mathrm{P_{IV}}$ are different. In the past the solutions to discrete Painlevé systems expressed in terms of normalization factors of one type of orthogonal polynomial has been studied[1, 6, 8, 11], but as far as I know, there is no study about one expressed in terms of normalization factors of two types of orthogonal polynomial. The purpose of this paper is to construct the method to give the solutions expressed in terms of normalization factors of two types of orthogonal polynomials. We note here that solutions to the Painlevé equations expressed in terms of normalization factors of two types of orthogonal polynomial is studied in [2, 3].

This paper is organized as follows. In Section 2, we consider the compatibility conditions of an orthogonal polynomial and its kernel polynomial. In Section 3, we demonstrate with examples that from the compatibility condition given in Section 2 we can obtain the solution to the discrete Painlevé systems.

Throughout this paper, we assume $0<|q|<1$ and the expression "α is a constant" means $\mathrm{d}\alpha/\mathrm{d}t=0$, where t is the independent variable of the orthogonal polynomial. We use the following conventions of q-analysis[4, 7].
q-Shifted factorials:

$$(a;q)_\infty = \prod_{i=1}^{\infty}(1-aq^{i-1}),\quad (a;q)_\lambda = \frac{(a;q)_\infty}{(aq^\lambda;q)_\infty}\,,\quad (a_1,\cdots,a_s;q)_\lambda = \prod_{j=1}^{s}(a_j;q)_\lambda,\quad (\lambda\in\mathbb{C}). \tag{1.19}$$

q-Exponential function:

$$E_q(z) = \sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q;q)_n}\,z^n = (-z;q)_\infty. \tag{1.20}$$

q-Definite integral:

$$\int_{-\infty}^{\infty} f(t)\mathrm{d}_q t = (1-q)\sum_{n=-\infty}^{\infty}\left(f(q^n)+f(-q^n)\right)q^n. \tag{1.21}$$

Basic hypergeometric series:

$${}_s\varphi_r\left(\begin{matrix}a_1,\cdots,a_s\\ b_1,\cdots,b_r\end{matrix};q,z\right) = \sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_s;q)_n}{(b_1,\cdots,b_r;q)_n(q;q)_n}\left[(-1)^n q^{n(n-1)/2}\right]^{1+r-s} z^n. \tag{1.22}$$

2 The compatibility conditions associated with the Christoffel transformation

In this section, we consider the compatibility conditions of an orthogonal polynomial and its kernel polynomial.

Let $(P_n)_{n=0}^{\infty} = (P_n(t))_{n=0}^{\infty}$ and $(\hat{P}_n)_{n=0}^{\infty} = (\hat{P}_n(t))_{n=0}^{\infty}$ be MOPSs with linear functionals $\mathscr{L}$ and $\hat{\mathscr{L}}$ over $\mathbb{C}$ given as

$$\mathscr{L}[t^k P_n(t)] = h_n \delta_{n,k} \qquad (n \geq k), \tag{2.1}$$

$$\hat{\mathscr{L}}[t^k \hat{P}_n(t)] = \mathscr{L}[(t-c_0)t^k \hat{P}_n(t)] = \hat{h}_n \delta_{n,k} \qquad (n \geq k,\ c_0 \in \mathbb{C}), \tag{2.2}$$

respectively. We refer to the transformation from P_n to $\hat{P}_n$ as the Christoffel transformation and $\hat{P}_n$ as the kernel polynomial. On the other hand, we also refer to the transformation from $\hat{P}_n$ to P_n as the Geronimus transformation. We have the following relations from the orthogonal conditions:

$$(t-c_0)\hat{P}_n = P_{n+1} + \frac{\hat{h}_n}{h_n} P_n, \quad P_n = \hat{P}_n + \frac{h_n}{\hat{h}_{n-1}} \hat{P}_{n-1}, \tag{2.3}$$

and then we obtain the following three-term recurrence relations from the compatibility conditions of equations above:

$$tP_n = P_{n+1} + \left(\frac{\hat{h}_n}{h_n} + \frac{h_n}{\hat{h}_{n-1}} + c_0 \right) P_n + \frac{h_n}{h_{n-1}} P_{n-1}, \tag{2.4}$$

$$t\hat{P}_n = \hat{P}_{n+1} + \left(\frac{h_{n+1}}{\hat{h}_n} + \frac{\hat{h}_n}{h_n} + c_0 \right) \hat{P}_n + \frac{\hat{h}_n}{\hat{h}_{n-1}} \hat{P}_{n-1}. \tag{2.5}$$

We define the constants α_n, β_n, $\hat{\alpha}_n$ and $\hat{\beta}_n$ as

$$tP_n = P_{n+1} + \alpha_n P_n + \beta_n P_{n-1}, \quad t\hat{P}_n = \hat{P}_{n+1} + \hat{\alpha}_n \hat{P}_n + \hat{\beta}_n \hat{P}_{n-1}. \tag{2.6}$$

From (2.4) and (2.5), we obtain the following:

$$\alpha_n = \frac{\hat{h}_n}{h_n} + \frac{h_n}{\hat{h}_{n-1}} + c_0, \quad \beta_n = \frac{h_n}{h_{n-1}}, \quad \hat{\alpha}_n = \frac{h_{n+1}}{\hat{h}_n} + \frac{\hat{h}_n}{h_n} + c_0, \quad \hat{\beta}_n = \frac{\hat{h}_n}{\hat{h}_{n-1}}. \tag{2.7}$$

Setting

$$x_n = \frac{h_n}{\hat{h}_n}, \quad y_n = \frac{\hat{h}_n}{h_{n+1}}. \tag{2.8}$$

we obtain the following equations from (2.7):

$$x_n = -\frac{1}{\beta_n x_{n-1} - \alpha_n + c_0}, \tag{2.9}$$

$$y_n = -\frac{1}{\hat{\beta}_n y_{n-1} - \hat{\alpha}_n + c_0}. \tag{2.10}$$

When we give an orthogonal polynomial P_n such that both α_n and β_n are rational functions of n (or q^n), we can regard (2.9) as a discrete Riccati equation. Similarly, when we give an orthogonal polynomial $\hat{P}_n$, (2.10) can be also regarded as a discrete Riccati equation. Therefore we find that the compatibility conditions of an orthogonal polynomial and its kernel polynomial can be related to the discrete Painlevé equation through the discrete Riccati equation. In the next section, we demonstrate this point with examples in the case where both α_n and β_n (or, $\hat{\alpha}_n$ and $\hat{\beta}_n$) are rational functions of n and in the case where both α_n and β_n (or, $\hat{\alpha}_n$ and $\hat{\beta}_n$) are rational functions of q^n.

3 Relation between the compatibility conditions and discrete Painlevé systems

In this section, we show that from (2.9) and (2.10) we can obtain the solutions to discrete Painlevé systems. We demonstrate the construction by taking two examples. The first example is the Hermite polynomials in the case where both α_n and β_n (or, $\hat{\alpha}_n$ and $\hat{\beta}_n$) are rational functions of n. The second one is the discrete q-Hermite II polynomials in the case where both α_n and β_n (or, $\hat{\alpha}_n$ and $\hat{\beta}_n$) are rational functions of q^n.

3.1 Example I: The case where $(P_n)_{n=0}^{\infty}$ are the Hermite polynomials

We define P_n as

$$P_n(t) = \frac{H_n(c_2 t + c_1)}{{c_2}^n}\,, \quad (c_1 \in \mathbb{C},\ c_2 \in \mathbb{C}^*), \tag{3.1}$$

where $H_n(t)$ is the Hermite polynomial:

$$H_n(t) = (-1)^n e^{t^2/2}\, \frac{\mathrm{d}^n}{\mathrm{d}t^n}\,(e^{-t^2/2}). \tag{3.2}$$

The linear functionals are given as

$$\mathscr{L}[f(t)] = \int_{-c_2\infty}^{c_2\infty} f(t) e^{-{c_2}^2 t^2/2 - c_1 c_2 t} \mathrm{d}t, \quad \hat{\mathscr{L}}[f(t)] = \int_{-c_2\infty}^{c_2\infty} (t-c_0) f(t) e^{-{c_2}^2 t^2/2 - c_1 c_2 t} \mathrm{d}t. \tag{3.3}$$

From the three-term recurrence relation:

$$tP_n = P_{n+1} - \frac{c_1}{c_2} P_n + \frac{n}{{c_2}^2} P_{n-1}, \tag{3.4}$$

we obtain

$$\alpha_n = -\frac{c_1}{c_2}\,, \quad \beta_n = \frac{n}{{c_2}^2}\,. \tag{3.5}$$

From (2.9), we obtain the following discrete Riccati equation:

$$x_n = -\frac{{c_2}^2}{n x_{n-1} + c_2(c_1 + c_0 c_2)}\,. \tag{3.6}$$

Let us consider the following difference equation[5, 8, 10]:

$$X_{n+1} + X_{n-1} = \frac{(an+b)X_n + c}{1 - {X_n}^2}\,. \tag{3.7}$$

Equation (3.7) is referred to as a discrete Painlevé II equation, denoted by d-$\mathrm{P_{II}}$, and has the space of initial condition of type $D_5^{(1)}$. d-$\mathrm{P_{II}}$ admits a specialization to the discrete Riccati equation:

$$X_{n+1} = \frac{4X_n - 2an - a - 2b + 4}{4(X_n + 1)}\,, \tag{3.8}$$

with

$$c = -\frac{a}{2}\,. \tag{3.9}$$

Therefore we obtain the following theorem:

Theorem 3.1 d-$\mathrm{P_{II}}$ (3.7) *admits the following solution*:

$$X_n = \frac{2(n+1)}{c_2(c_1 + c_0 c_2)}\, x_n + 1, \quad (n \in \mathbb{Z}_{\geq 0}). \tag{3.10}$$

Here

$$a = \frac{8}{(c_1 + c_0 c_2)^2}\,, \quad b = \frac{12}{(c_1 + c_0 c_2)^2}\,, \quad c = -\frac{4}{(c_1 + c_0 c_2)^2}\,, \quad c_1 + c_0 c_2 \neq 0. \tag{3.11}$$

3.2 Example II: The case where $(\hat{P}_n)_{n=0}^{\infty}$ are the Hermite polynomials

We next consider the case where $\hat{P}_n$ is the Hermite polynomial:

$$\hat{P}_n(t) = \frac{H_n(c_2t+c_1)}{{c_2}^n}, \quad (c_1 \in \mathbb{C},\ c_2 \in \mathbb{C}^*). \tag{3.12}$$

We assume here that c_0 is not a real number. The linear functionals are given as

$$\mathscr{L}[f(t)] = \int_{-c_2\infty}^{c_2\infty} \frac{f(t)}{t-c_0}\, e^{-{c_2}^2t^2/2-c_1c_2t}\mathrm{d}t, \quad \hat{\mathscr{L}}[f(t)] = \int_{-c_2\infty}^{c_2\infty} f(t)e^{-{c_2}^2t^2/2-c_1c_2t}\mathrm{d}t. \tag{3.13}$$

From the three-term recurrence relation:

$$t\hat{P}_n = \hat{P}_{n+1} - \frac{c_1}{c_2}\,\hat{P}_n + \frac{n}{{c_2}^2}\,\hat{P}_{n-1}, \tag{3.14}$$

we obtain

$$\hat{\alpha}_n = -\frac{c_1}{c_2}, \quad \hat{\beta}_n = \frac{n}{{c_2}^2}. \tag{3.15}$$

From (2.10), we obtain the following discrete Riccati equation:

$$y_n = -\frac{{c_2}^2}{ny_{n-1}+c_2(c_1+c_0c_2)}. \tag{3.16}$$

Therefore we obtain the following theorem:

Theorem 3.2 d-$\mathrm{P_{II}}$ (3.7) *admits the following solution*:

$$X_n = \frac{2(n+1)}{c_2(c_1+c_0c_2)}\, y_n + 1, \quad (n \in \mathbb{Z}_{\geq 0}). \tag{3.17}$$

Here

$$a = \frac{8}{(c_1+c_0c_2)^2}, \quad b = \frac{12}{(c_1+c_0c_2)^2}, \quad c = -\frac{4}{(c_1+c_0c_2)^2}, \quad c_1+c_0c_2 \neq 0. \tag{3.18}$$

3.3 Example III: The case where $(P_n)_{n=0}^{\infty}$ are the discrete q-Hermite II polynomials

We define P_n as

$$P_n(t) = \frac{h_n^{\mathrm{II}}(c_1t;q)}{{c_1}^n}, \quad (c_1 > 0), \tag{3.19}$$

where $h_n^{\mathrm{II}}(t;q)$ is the discrete q-Hermite II polynomial:

$$h_n^{\mathrm{II}}(t;q) = t^n {}_2\varphi_1\left(\begin{matrix} q^{-n}, q^{-n+1} \\ 0 \end{matrix}; q^2, -\frac{q^2}{t^2}\right). \tag{3.20}$$

The linear functionals, the three-term recurrence relation and the discrete Riccati equation are given by

$$\mathscr{L}[f(t)] = \int_{-\infty}^{\infty} \frac{f(t)}{(-{c_1}^2t^2;q^2)_\infty}\, \mathrm{d}_qt, \quad \hat{\mathscr{L}}[f(t)] = \int_{-\infty}^{\infty} \frac{(t-c_0)f(t)}{(-{c_1}^2t^2;q^2)_\infty}\, \mathrm{d}_qt, \tag{3.21}$$

$$tP_n = P_{n+1} + q^{-2n+1}(1-q^n){c_1}^{-2}P_{n-1}, \quad x_n = -\frac{1}{q^{-2n+1}(1-q^n){c_1}^{-2}x_{n-1}+c_0}. \tag{3.22}$$

Theorem 3.3 q-$\mathrm{P_{IV}}$ (1.7) *admits the following solution*:

$$X_n = i\,\frac{1-q^{n+1}}{c_1q^n}\, x_n. \tag{3.23}$$

Here

$${a_0}^{1/2} = -iq^{-m+1}{c_0}^{-1}{c_1}^{-1}, \quad {a_0}^{1/2}{a_1}^{1/2} = q^{-N}, \quad a_2 = q^{2N+2}, \quad c_0 \neq 0. \tag{3.24}$$

We find that the solution to q-$\mathrm{P_{IV}}$ given in Theorem 3.3 coincides with one given in (1.18).

3.4 Example IV: The case where $(\hat{P}_n)_{n=0}^{\infty}$ are the discrete q-Hermite II polynomials

We consider the case where $\hat{P}_n$ is the discrete q-Hermite II polynomial:

$$\hat{P}_n(t) = \frac{h_n^{\mathrm{II}}(c_1 t; q)}{{c_1}^n}, \quad (c_1 > 0). \tag{3.25}$$

We assume that $c_0 \neq q^a$ for ${}^\forall a \in \mathbb{Z}$. Then we have the linear functionals, the three-term recurrence relation and the discrete Riccati equation as

$$\mathscr{L}[f(t)] = \int_{-\infty}^{\infty} \frac{f(t)}{(t-c_0)(-{c_1}^2 t^2; q^2)_\infty} \mathrm{d}_q t, \quad \hat{\mathscr{L}}[f(t)] = \int_{-\infty}^{\infty} \frac{f(t)}{(-{c_1}^2 t^2; q^2)_\infty} \mathrm{d}_q t, \tag{3.26}$$

$$t\hat{P}_n = \hat{P}_{n+1} + q^{-2n+1}(1-q^n){c_1}^{-2}\hat{P}_{n-1}, \quad y_n = -\frac{1}{q^{-2n+1}(1-q^n){c_1}^{-2} y_{n-1} + c_0}. \tag{3.27}$$

Theorem 3.4 q-$\mathrm{P_{IV}}$ (1.7) *admits the following solution*:

$$X_n = i\, \frac{1-q^{n+1}}{c_1 q^n}\, y_n. \tag{3.28}$$

Here

$${a_0}^{1/2} = -iq^{-m+1}{c_0}^{-1}{c_1}^{-1}, \quad {a_0}^{1/2}{a_1}^{1/2} = q^{-N}, \quad a_2 = q^{2N+2}, \quad c_0 \neq 0. \tag{3.29}$$

4 Concluding remarks

In this paper, we constructed the method to give the solutions to discrete Painlevé systems expressed in terms of normalization factors of two types of orthogonal polynomials and also presented some examples.

It seems that the solutions of various discrete Painlevé systems can be constructed by using the method in this paper. One interesting project is to make a list of discrete Painlevé systems related with orthogonal polynomials given in [7] by this method.

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