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Discrete differential geometry: theory and applications

I. General aspects

by

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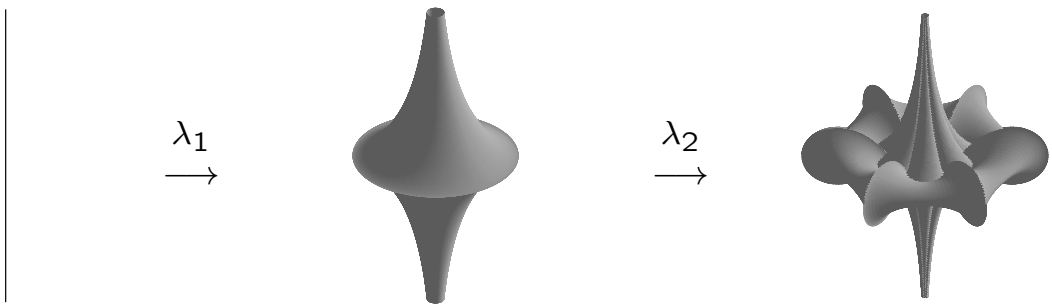
ARC Centre of Excellence for Mathematics and Statistics of Complex Systems

1. The classical Bäcklund transformation (Bianchi 1879, Bäcklund 1883)

Geometric construction of surfaces of constant negative Gaußian curvature:



The Bäcklund parameter λ is the key ingredient in the Bäcklund transformation!

2. Invariance of the sine-Gordon equation (Darboux 1883, Bianchi 1885)

Gauß equation for the angle ω made by the asymptotic lines on pseudospherical surfaces:

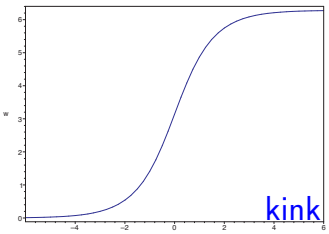
$$\omega_{xy} = \sin\omega$$

Bianchi's algebraic version of the Bäcklund transformation:

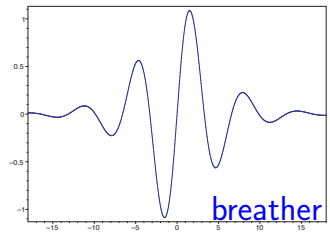
$$\left(\frac{\omega' - \omega}{2}\right)_x = \lambda \sin\left(\frac{\omega' + \omega}{2}\right), \quad \left(\frac{\omega' + \omega}{2}\right)_y = \frac{1}{\lambda}\sin\left(\frac{\omega' - \omega}{2}\right)$$

Generation of non-trivial solutions ω' from 'nothing':

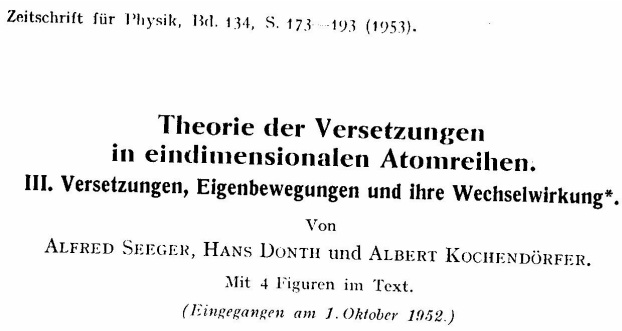
$$\omega = 0 \quad \xrightarrow{\lambda_1}$$



$$\xrightarrow{\lambda_2}$$



3. Nonlinear superposition principle (Bianchi 1892)


Zeitschrift für Physik, Bd. 134, S. 173—193 (1953).

Theorie der Versetzungen in eindimensionalen Atomreihen.
III. Versetzungen, Eigenbewegungen und ihre Wechselwirkung*.

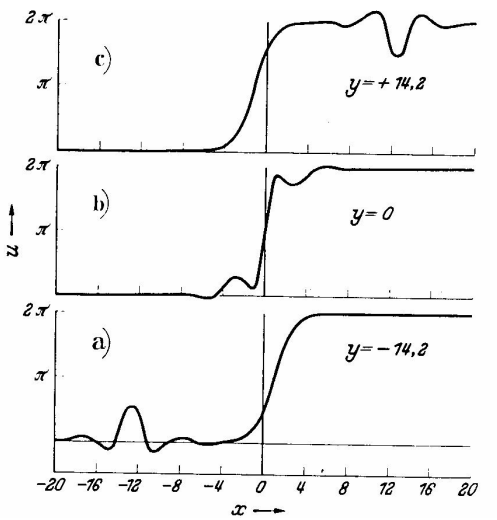
Von

Alfred Seeger, Hans Donth und Albert Kochendörfer.

Mit 4 Figuren im Text.

(Eingegangen am 1. Oktober 1952.)

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} - \frac{\partial^2 u_n}{\partial t^2} = \sin u_n$$



Nonlinear superposition of solutions (Eigenbewegungen) in the context of Frenkel and Kontorova's dislocation theory by means of Bianchi's permutability theorem

$$\tan\left(\frac{\omega_{12} - \omega}{4}\right) = \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}\tan\left(\frac{\omega_1 - \omega_2}{4}\right)$$

4. (Integrable) discrete differential geometry: History

- Sauer (1950): Discrete pseudospherical surfaces
- Wunderlich (1951): Discrete pseudospherical surfaces
- Hirota (1977): Discrete sine-Gordon equation
- Bobenko & Pinkall (1996): Discrete pseudospherical surfaces and the Hirota equation
- Bobenko & Pinkall (1996): Discrete isothermic surfaces, discrete minimal surfaces, discrete surfaces of constant mean curvature
- Cieślińkski, Doliwa, Konopelchenko, Santini, WKS ...

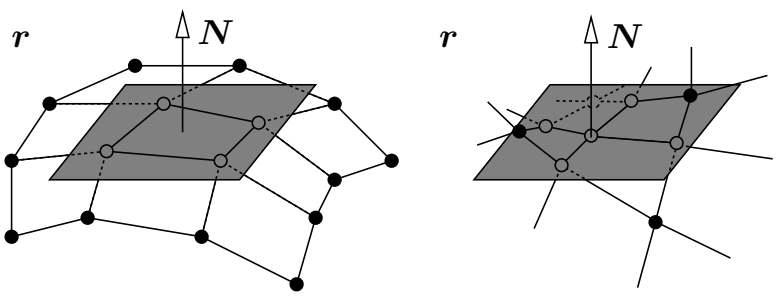
5. Discretisation of parametrised surfaces

In the following, we consider discrete surfaces (discrete nets)

$$\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^3$$

Definition. A discrete net is termed discrete conjugate if all quadrilaterals are planar.

Definition. A discrete net is termed discrete asymptotic if all stars are planar.



(Sauer 19??)

Note: $|\Delta_{12}\boldsymbol{r}, \Delta_1\boldsymbol{r}, \Delta_2\boldsymbol{r}| = 0$ $\qquad$ $\Delta_1\boldsymbol{r} \cdot \Delta_1\boldsymbol{N} = \Delta_2\boldsymbol{r} \cdot \Delta_2\boldsymbol{N} = 0$

Application: Discretisation of surfaces of constant negative Gaussian curvature (pseudospherical surfaces).

Example: An integrable discretisation of the 'Kuehn surface' parametrised in terms of asymptotic coordinates (Bobenko *et al.*)

6. Discrete curvature nets ('curvature lattices')

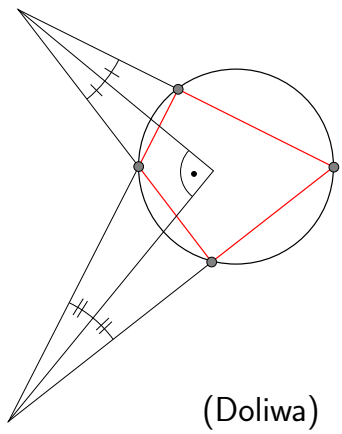
Definition. A lattice of $\mathbb{Z}^2$ combinatorics is termed a discrete curvature net if its quadrilaterals may be inscribed in circles.

(Gregory 1986 (CAD), Bobenko 1996 (DDG)).

Justification:

Classical theory: Generically, pairs of lines of curvature through a point are conjugate and meet orthogonally.

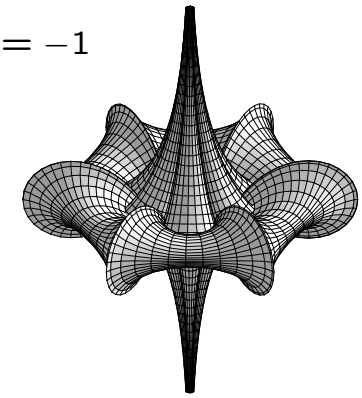
Discrete theory: Quadrilaterals are planar and bisectors of pairs of edges meet orthogonally.



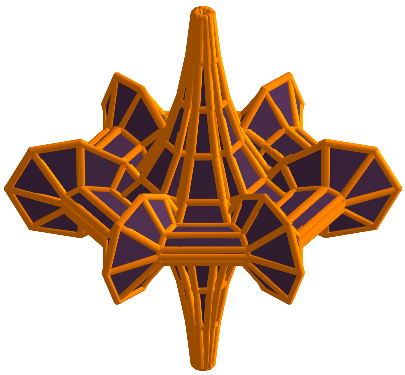
(Doliwa)

7. Discrete pseudospherical surfaces based on lines of curvature (WKS 2003)

$\mathcal{K} = -1$



$\leftrightarrow$

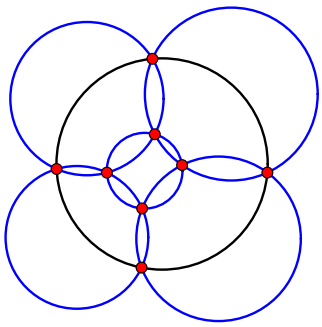


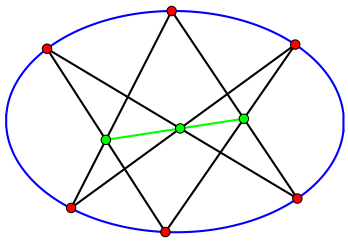
$\updownarrow$ $\updownarrow$

$$\left.\begin{array}{c} \omega_{xx} - \omega_{yy} = \sin\omega \\ \boldsymbol{N}_{xx} - \boldsymbol{N}_{yy} + (\boldsymbol{N}_x^2 - \boldsymbol{N}_y^2)\boldsymbol{N} = \boldsymbol{0} \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \text{discrete sine-Gordon eq} \\ \text{discrete nonlinear } \sigma\text{-model} \end{array}\right.$$

Remark: The above example constitutes a (discrete) Enneper surface, that is, there exists a family of planar (discrete) lines of curvature.

8. The origin of integrability: Incidence theorems





Miquel's theorem

$\downarrow$

discrete Lamé system

(discrete orthogonal coordinate systems!)

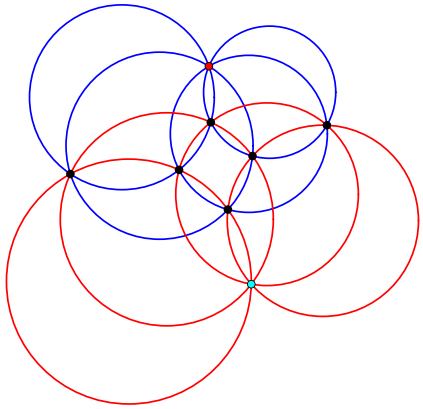
Pascal's theorem

$\downarrow$

discrete CKP equation

9. Integrable 'Clifford lattices' (Konopelchenko & WKS 2002)

A $\mathcal{C}_4$ Clifford configuration



Theorem. The six (black) points P_i (regarded as ordered complex numbers) obey the multiratio condition

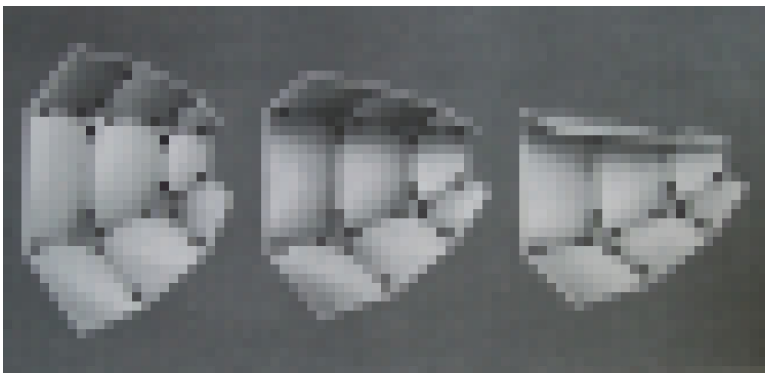
$$\frac{(P_1-P_2)(P_3-P_4)(P_5-P_6)}{(P_2-P_3)(P_4-P_5)(P_6-P_1)}=-1$$

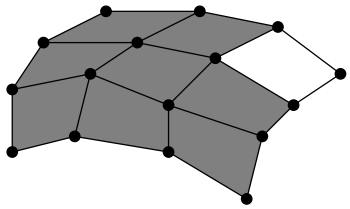
Idea: Extend the Clifford configuration to a 'Clifford lattice' of fcc type.

Result: The above lattice equation constitutes an integrable discretisation of the Schwarzian KP equation!

10. Finite isometric deformation of polyhedral surfaces

Illustration: An elementary but open problem [cf. Cauchy's theorem]





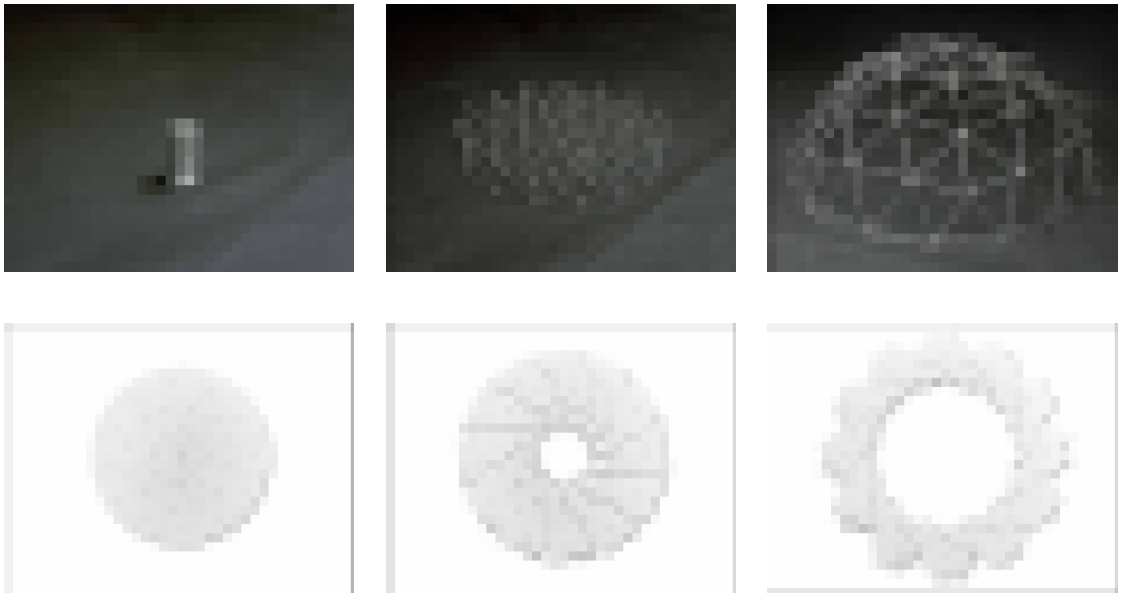
Sauer & Graf, Mathematische Annalen (1930)

Example: Discrete Voss surfaces $\rightarrow$ discrete sine-Gordon equation (Hirota 1977)

$$\sin\left(\frac{\omega_{12}-\omega_2-\omega_1+\omega}{4}\right)=\tan\frac{\mu}{2}\tan\frac{\nu}{2}\sin\left(\frac{\omega_{12}+\omega_2+\omega_1+\omega}{4}\right)$$

Note: The classical analogue was shown to be 'integrable' by Bianchi in 1892!

11. Possible application: Movable structures in architecture



Florian Rist, Bewegliche Systeme in der Architektur, Lehrstuhl für Tragwerksplanung, TU München (2005).

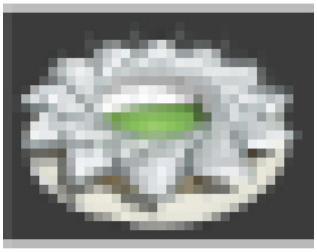
Taut-Preis 2006 - Architekturpreis des Beauftragten der Bundesregierung für Kultur und Medien und der Bundesarchitektenkammer

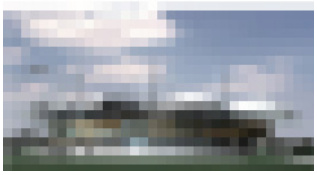
VDI/VDE Preis 2005

Quotes from the eulogy:

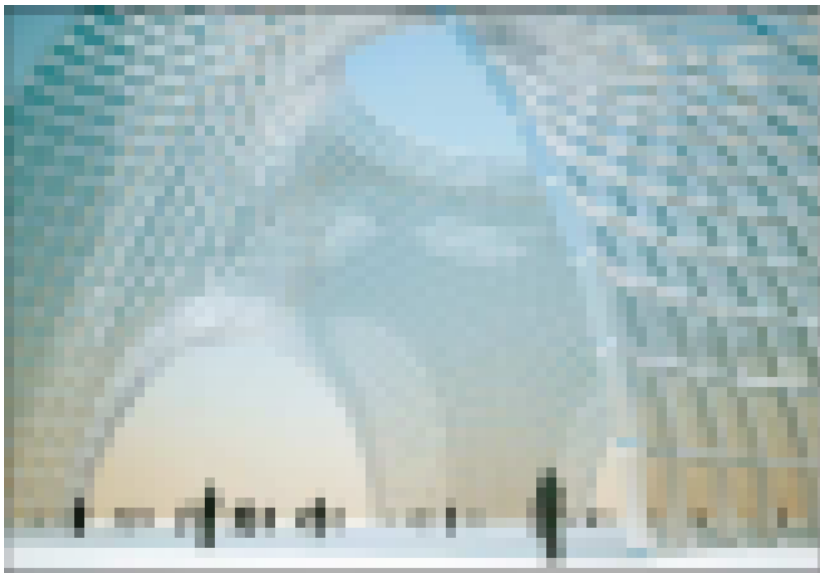
"One of the most fascinating of areas in architecture is the consideration of flexible structures which requires knowledge of kinematics and kinetics."

"[Rist's work] also demonstrates that a systematic inclusion of areas such as geometry and physics has enormous potential for architectural design."





12. Real application: Design of freeform structures



Double glazing with support structure of 'constant width'
(Pottmann (Vienna), Bobenko *et al.*)

13. Parallel discrete surfaces (WKS 2006)

Problem: Is it possible to introduce a concept of parallelism for polyhedral surfaces which allows one to formulate analogues of classical theorems for parallel surfaces such as Bonnet's theorem?

Fact: In the smooth case, curvature coordinates (x, y) are preserved by the map

$$\boldsymbol{r} \mapsto \boldsymbol{r}^{\parallel} = \boldsymbol{r} + c\boldsymbol{N}.$$

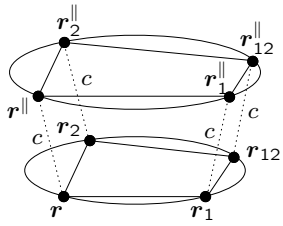
It is therefore natural to consider curvature lattices.

Remark: $\boldsymbol{r}^{\parallel}(x, y, z = c)$ constitutes a particular orthogonal coordinate system for any given surface $\boldsymbol{r}(x, y)$.

In the discrete case, one may generate a parallel discrete surface by translating the vertices along normals defined on the vertices by a fixed distance c, say.

However, how does one choose the normals?

Definition. A discrete surface $\Sigma^{\parallel}$ is considered parallel and at a distance c to a discrete surface Σ if the 'vertical' quadrilaterals constitute isosceles trapezoids of edge length c.



Remark: Any infinite sequence of distance parameters

$$\cdots < c_{-2} < c_{-1} < c_0 = 0 < c_1 < c_2 < \cdots$$

generates a lattice of $\mathbb{Z}^3$ combinatorics which has the property that all 'horizontal' and 'vertical' quadrilaterals may be inscribed in circles. The latter is the defining property for discrete orthogonal coordinate systems (Bobenko 1999).

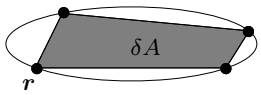
It is evident that any discrete surface $\Sigma^{\parallel}$ which is parallel to a discrete surface Σ admits the representation

$$\boldsymbol{r}^{\parallel} = \boldsymbol{r} + c\boldsymbol{N}$$

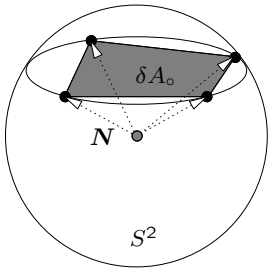
but what is the geometric meaning of $\boldsymbol{N}$?

14. A discrete Gauß map (Konopelchenko & WKS 1998)

Consider a standard discrete curvature net, that is a discrete surface Σ whose quadrilaterals are embedded in circles.



Choose a point $\boldsymbol{N}(0,0)$ on the unit sphere S^2. Then, there exists a unique discrete surface $\Sigma_\circ$ with vertices on S^2 whose edges are parallel to those of Σ.

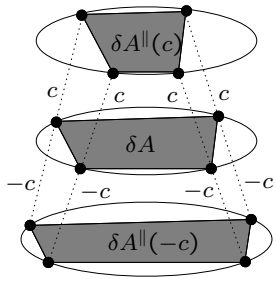


We call the discrete surface $\boldsymbol{N} : \mathbb{Z}^2 \to S^2$ a spherical representation or discrete Gauß map.

Note: Any discrete curvature net admits a two-parameter family of spherical representations labelled by $\boldsymbol{N}(0,0)$.

15. Discrete Gaußian and mean curvatures (WKS 2003)

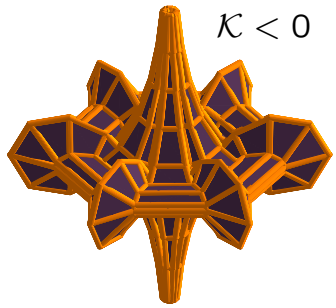
Definition. The discrete Gaußian and mean curvatures of a discrete surface Σ with respect to an associated spherical representation $\Sigma_\circ$ are defined by



$$\boxed{\mathcal{K} = \frac{\delta A_\circ}{\delta A}, \quad \mathcal{H} = \frac{\delta A^{\parallel}(-c) - \delta A^{\parallel}(c)}{4c\delta A}}$$

The latter is independent of c!

Theorem. Discrete surfaces of (i) constant Gaußian curvature, (ii) constant mean curvature and (iii) discrete minimal surfaces defined by



(i) $\mathcal{K} = \text{const}$, (ii) $\mathcal{H} = \text{const}$, (iii) $\mathcal{H} = 0$

are integrable!

In the cases (i) with $\mathcal{K} > 0$, (ii) and (iii), one retrieves precisely those discrete surfaces defined earlier by Bobenko & Pinkall (1996)!

16. A discrete Bonnet theorem

- Observation: If $\mathcal{K} = \pm 1/\rho^2$, $\rho = \text{const}$ then

$$(c^2 \mp \rho^2)\mathcal{K}^{\parallel} + 2c\mathcal{H}^{\parallel} + 1 = 0 \quad \text{(discrete linear Weingarten surfaces)}$$

Theorem (discrete Bonnet). A discrete surface of positive constant Gaußian curvature $\mathcal{K} = 1/\rho^2$ admits two discrete parallel surfaces of mean curvature $\pm 1/(2\rho)$.

- Observation: The dilation factor $Q = \dfrac{\delta A^{\parallel}}{\delta A}$ obeys

$$Q = 1 - 2c\mathcal{H} + c^2\mathcal{K}.$$

This is a discrete version of Steiner's classical formula!

Discrete differential geometry: theory and applications

II. Discrete shell membrane theory

by

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ARC Centre of Excellence for Mathematics and Statistics of Complex Systems

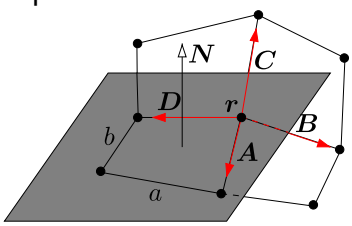
0. Graphical statics

Taylor (18??): 'practical draughtsman'
Rankine (1858): *Applied Mechanics*
Maxwell (1864): stresses in frameworks, diagram of forces, reciprocal figures
Culmann (1866): *Graphische Statik*
Jenkin (1869): *Bridges* in Encyclopedia Britannica
Cremona (1872): *Le Figure Reciproche Nella Statica Grafica*
Koechling (1889): one of two chief engineers with Gustave Eiffel
'Artifical Intelligence' (1970s): reciprocal figures = polyhedral scenes
Konopelchenko and Schief (2002): reciprocal figures and discrete integrable systems

geometric compatibility = algebraic integrability

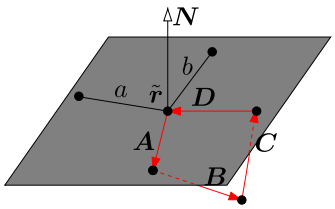
1. Reciprocal-parallel lattices (Sauer 1930s, 1950, Wunderlich 1951)

planar faces



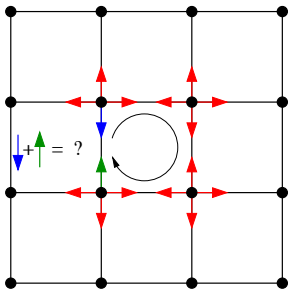
discrete conjugate net

planar stars



discrete asymptotic net

Equilibrium condition: $\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C} + \boldsymbol{D} = \boldsymbol{0}$



Theorem. A discrete conjugate net $\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^3$ is in equilibrium if and only if there exists a reciprocal-parallel discrete (asymptotic) net $\tilde{\boldsymbol{r}} : \mathbb{Z}^2 \to \mathbb{R}^3$.

Remark. If $\tilde{\boldsymbol{r}}$ exists then it is (essentially) unique!

2. Discrete pseudo-spherical surfaces (Sauer 1950, Wunderlich 1951)

Physical constraint:

$$|\boldsymbol{A}| = |\boldsymbol{C}|, \quad |\boldsymbol{B}| = |\boldsymbol{D}|$$

Geometric implication:

reciprocal-parallel quadrilaterals = skew parallelograms

Result: The reciprocal-parallel net is both discrete asymptotic and discrete Tchebychev.

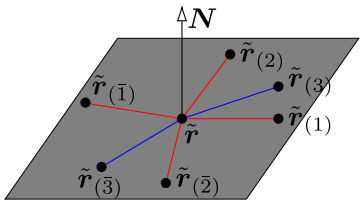
Classical Theorem. A surface constitutes a pseudo-spherical surface if and only if the asymptotic lines form a Tchebychev net on the surface.

Thus, the reciprocal parallel nets constitute a discretisation of classical surfaces of constant negative Gaußian curvature.

(Bobenko *et al.*)

3. Three-dimensional asymptotic lattices

Question: Is it possible to construct three-dimensional asymptotic lattices $\tilde{\boldsymbol{r}} : \mathbb{Z}^3 \to \mathbb{R}^3$?



Answer: Yes, by virtue of Moebius' theorem!

Algebraic description:

$$\tilde{\boldsymbol{r}}_{(i)} - \tilde{\boldsymbol{r}} = \alpha_i \boldsymbol{N}_{(i)} \times \boldsymbol{N}, \quad i = 1, 2, 3$$

Compatibility condition $\tilde{\boldsymbol{r}}_{(ik)} = \tilde{\boldsymbol{r}}_{(ki)}$:

$$\mathcal{N}_{(ik)} - \mathcal{N} = a_{ik}(\mathcal{N}_{(i)} - \mathcal{N}_{(k)}), \quad i \neq k$$

for some scaled normal $\mathcal{N} \sim \boldsymbol{N}$.

Compatibility condition $\mathcal{N}_{(ikl)} = \mathcal{N}_{(ilk)} = \cdots$:

$$\boxed{\tau\tau_{(123)} + \tau_{(1)}\tau_{(23)} + \tau_{(2)}\tau_{(13)} + \tau_{(3)}\tau_{(12)} = 0,} \quad a_{ik} = \pm\frac{\tau_{(i)}\tau_{(k)}}{\tau\tau_{(ik)}}$$

Discrete BKP equation!

4. The discrete analogue of the classical Bäcklund transformation (Wunderlich 1951)

Fact: It is consistent to demand that all quadrilaterals constitute skew-parallelograms.

Conclusion: The map

$$\tilde{\boldsymbol{r}}(n_3 = m) \mapsto \tilde{\boldsymbol{r}}(n_3 = m + 1), \quad m \in \mathbb{Z}$$

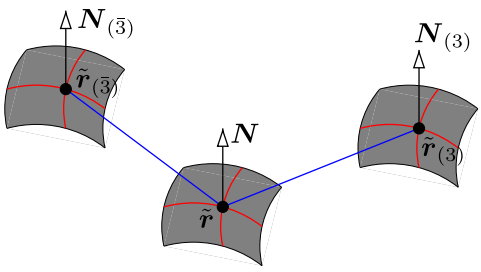
constitutes a discrete analogue of the classical Bäcklund transformation for pseudo-spherical surfaces.

Continuum limit:

$$x = \epsilon n_1$$

$$y = \delta n_2$$

$$\epsilon, \delta \to 0$$



(Bäcklund 1882)

5. The equilibrium equations of classical shell membrane theory

Thin shell theory: Well-established branch of structural mechanics

[Lamé and Clapeyron (1831), Lecornu (1880), Beltrami (1882), Love (1888; 1892, 1893)]

Idea (see Novozhilov (1964)): Replace the three-dimensional stress tensor σ_{ik} of elasticity theory defined throughout a thin shell by statically equivalent internal forces T_{ab}, N^a and moments M_{ab} acting on its mid-surface Σ.

$$\left.\begin{array}{ll} \text{Vanishing of total force:} & \mathsf{T}^a{}_{b;a} = h_{ab}\mathsf{N}^a, \quad \mathsf{N}^a{}_{;a} + h_{ab}\mathsf{T}^{ab} = 0 \\ \text{Vanishing of total moment:} & \mathsf{M}^a{}_{b;a} = \mathsf{N}_b, \quad \mathsf{T}_{[ab]} = h_{c[a}\mathsf{M}^c{}_{b]} \end{array}\right\} \text{No external forces}$$

Definition of (shell) membranes: $\mathsf{M}_{ab} = 0$

6. Curvature coordinates

In terms of curvature coordinates:

$$\begin{aligned} \mathrm{I} &:= \quad d\boldsymbol{r}^2 \quad &&= H^2dx^2 + K^2dy^2 \\ \mathrm{II} &:= \quad -d\boldsymbol{r}\cdot d\boldsymbol{N} \quad &&= \kappa_1 H^2dx^2 + \kappa_2 K^2dy^2 \end{aligned}$$

($H, K =$ metric coefficients, $\kappa_i =$ principal curvatures)

In terms of the resultant stress components

$$T_1, T_2, T_{12}, T_{21}, \quad N_1, N_2$$

of the tensors T_{ab}, N^a, the membrane equilibrium equations become

$$\boxed{\begin{array}{rll} (KT_1)_x + (HS)_y + H_yS - K_xT_2 = & 0, & T_{12} = T_{21} = S \\ (HT_2)_y + (KS)_x + K_xS - H_yT_1 = & 0, & N_1 = N_2 = 0 \\ \kappa_1T_1 + \kappa_2T_2 + \bar{p} = & 0 & \end{array}}$$

Note: $S =$ shear stress, $\bar{p} =$ constant 'normal loading'

7. Special (integrable) cases

- Pure shear stresses and no loading ($T_1 = T_2 = \bar{p} = 0$):

$$\boxed{H = K} \qquad \text{(cf. Smyth 2004)}$$

$\rightarrow$ Isothermic surfaces

- No shear ($S = 0$, cf. biological membrane and liquid crystal theory):

$$\boxed{\langle \mathsf{H}, \mathsf{K} \rangle = 0} \qquad \text{(Rogers \& WKS 2003)}$$

$\rightarrow$ Particular class of O surfaces

- No shear and 'homogeneous' stress distribution $T_1 = T_2 = c$:

$$\boxed{\mathcal{H} = \frac{\kappa_1 + \kappa_2}{2} = -\frac{\bar{p}}{2c}} \qquad \text{(Young 1805; Laplace 1806)}$$

$\rightarrow$ Constant mean curvature/minimal surfaces (modelling thin films ('soap bubbles'))

8. The problem

Problem: Can shell membranes be 'discretized' in such a way that (some of) the geometric and algebraic properties (e.g. integrability) are preserved?

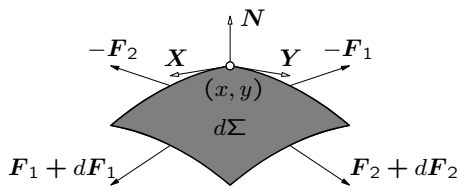
'Answer': Yes, but one should discretize simultaneously the geometry and physics!

(c.f. finite element modelling of plates and shells: 'discrete Kirchhoff techniques')

9. The equilibrium conditions for membranes

$\boldsymbol{F}_1$, $\boldsymbol{F}_2$: resultant internal stresses acting on infinitesimal cross-sections $x = \text{const}$, $y = \text{const}$

Differentials: $d\boldsymbol{r}_1 = \boldsymbol{r}(x + dx, y) - \boldsymbol{r}(x, y)$
$d\boldsymbol{r}_2 = \boldsymbol{r}(x, y + dy) - \boldsymbol{r}(x, y)$



Vanishing total force acting on $d\Sigma$: $\quad d\boldsymbol{F}_1 + d\boldsymbol{F}_2 = \boldsymbol{0}$

Vanishing total moment: $\quad d\boldsymbol{r}_1 \times \boldsymbol{F}_1 + d\boldsymbol{r}_2 \times \boldsymbol{F}_2 = \boldsymbol{0}$

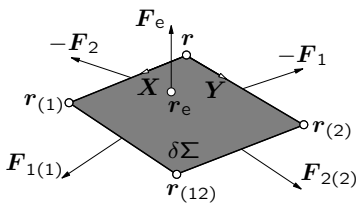
Decomposition into resultant stress components per unit length according to

$$\boldsymbol{F}_1 = (T_1\boldsymbol{X} + T_{12}\boldsymbol{Y} + N_1\boldsymbol{N})K dy, \qquad \boldsymbol{F}_2 = (T_{21}\boldsymbol{X} + T_2\boldsymbol{Y} + N_2\boldsymbol{N})H dx$$

results in the membrane equilibrium equations (for $\bar{p} = 0$).

10. 'Plated' membranes (WKS 2005, 2010)

'Discrete' (plated) membrane: polyhedral 'membrane' composed of quadrilateral 'plates'



Assumptions:
- Forces $\boldsymbol{F}_i$ act on edges
- 'Constant normal loading' $\boldsymbol{F}_\mathrm{e} = \bar{p}\delta\Sigma\boldsymbol{N}$, $\quad \bar{p} = \text{const}$
- $\boldsymbol{F}_\mathrm{e}$ acts at some 'canonical' point $\boldsymbol{r}_\mathrm{e}$

Equilibrium equations:

$$\boldsymbol{F}_{1(1)} - \boldsymbol{F}_1 + \boldsymbol{F}_{2(2)} - \boldsymbol{F}_2 + \boldsymbol{F}_\mathrm{e} = \boldsymbol{0} \qquad \text{(force)}$$

$$(\boldsymbol{r}_{(12)} + \boldsymbol{r}_{(1)}) \times \boldsymbol{F}_{1(1)} - (\boldsymbol{r}_{(2)} + \boldsymbol{r}) \times \boldsymbol{F}_1 \qquad \text{(moment)}$$

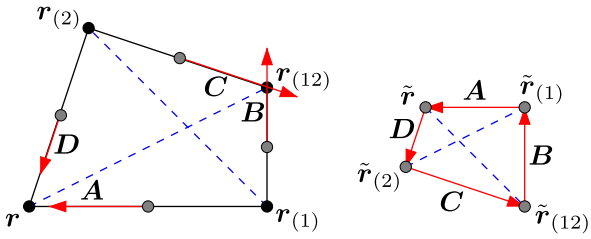
$$+ \quad (\boldsymbol{r}_{(12)} + \boldsymbol{r}_{(2)}) \times \boldsymbol{F}_{2(2)} - (\boldsymbol{r}_{(1)} + \boldsymbol{r}) \times \boldsymbol{F}_2 + 2\boldsymbol{r}_\mathrm{e} \times \boldsymbol{F}_\mathrm{e} = \boldsymbol{0}$$

Claim: Under certain circumstances, plated membranes are governed by integrable difference equations!

11. Pure shear stresses: discrete Koenigs nets (WKS 2010)

'Discrete membrane': $\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^3$ with planar quadrilaterals

Forces: Acting on midpoints and parallel to edges (pure 'shear stresses')



Equilibrium equations:

$$\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C} + \boldsymbol{D} = \boldsymbol{0} \qquad \text{(force)}$$

$$(\boldsymbol{r}_{(1)} + \boldsymbol{r}) \times \boldsymbol{A} + (\boldsymbol{r}_{(12)} + \boldsymbol{r}_{(1)}) \times \boldsymbol{B} \qquad \text{(moment)}$$

$$+ \quad (\boldsymbol{r}_{(2)} + \boldsymbol{r}) \times \boldsymbol{D} + (\boldsymbol{r}_{(12)} + \boldsymbol{r}_{(2)}) \times \boldsymbol{D} = \boldsymbol{0}$$

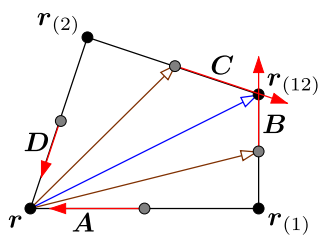
Vanishing total force $\Rightarrow$ There exists a discrete Combescure transform $\tilde{\boldsymbol{r}} : \mathbb{Z}^2 \to \mathbb{R}^3$ of $\boldsymbol{r}$, that is

$$\tilde{\boldsymbol{r}}_{(i)} - \tilde{\boldsymbol{r}} \parallel \boldsymbol{r}_{(i)} - \boldsymbol{r}, \quad i = 1, 2$$

Vanishing total moment $\Rightarrow$ 'Non-corresponding' diagonals are parallel, that is

$$\boldsymbol{r}_{(12)} - \boldsymbol{r} \parallel \tilde{\boldsymbol{r}}_{(1)} - \tilde{\boldsymbol{r}}_{(2)}$$

$$\tilde{\boldsymbol{r}}_{(12)} - \tilde{\boldsymbol{r}} \parallel \boldsymbol{r}_{(1)} - \boldsymbol{r}_{(2)}$$



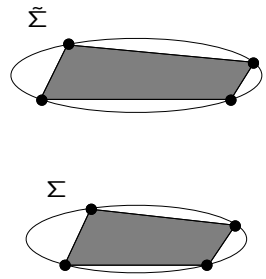
[Proof: $(\boldsymbol{r}_{(12)} - \boldsymbol{r}) \times (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{0}$]

Result: $\boldsymbol{r}$ and $\tilde{\boldsymbol{r}}$ constitute pairs of discrete Koenigs nets (cf. Sauer 1933, Doliwa 2003, Bobenko & Suris 2009)!

Note: Continuous Koenigs nets are classical and governed by hyperbolic equations with equal Laplace-Darboux invariants.

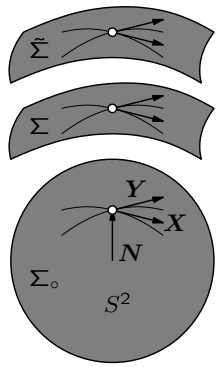
12. The (discrete) Combescure transformation

Definition: Two discrete surfaces Σ and $\tilde{\Sigma}$ are Combescure transforms of each other if corresponding edges are parallel.



Observation: Generically, Combescure transforms must consist of planar quadrilaterals.

Fact: Since the Combescure transformation preserves angles, the circularity condition is preserved.



Classical analogue for lines of curvature:

$$\begin{aligned} \boldsymbol{r}_x &= H\boldsymbol{X}, & \boldsymbol{r}_y &= K\boldsymbol{Y} & &\text{(surface)} \\ \boldsymbol{N}_x &= H_\circ\boldsymbol{X}, & \boldsymbol{N}_y &= K_\circ\boldsymbol{Y} & &\text{(Gauß map)} \\ \tilde{\boldsymbol{r}}_x &= \tilde{H}\boldsymbol{X}, & \tilde{\boldsymbol{r}}_y &= \tilde{K}\boldsymbol{Y}, & &\text{(Combescure transform)} \end{aligned}$$

13. Pure shear stresses: discrete isothermic membranes

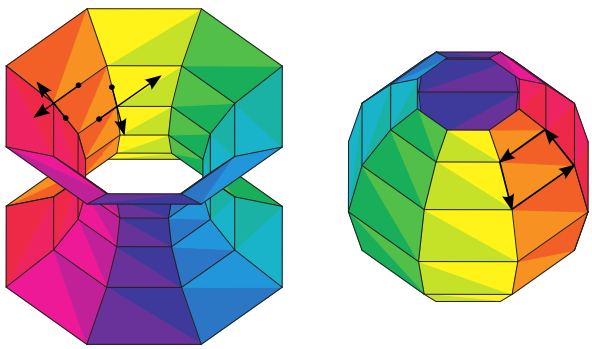
'Discrete membrane': $\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^3$ with quadrilaterals inscribed in circles

Forces: Acting on midpoints and parallel to edges (pure 'shear stresses')

Result: The discrete lines of curvature form discrete Koenigs nets.

$\to$ The discrete membranes are discrete isothermic in the sense of Bobenko & Pinkall (1996).

Example:



discrete catenoid discrete sphere

14. Classical isothermic surfaces

Definition: A surface is isothermic if the curvature coordinates are conformal, that is

$$\begin{aligned} \mathrm{I} &:= \quad d\boldsymbol{r}^2 \quad &= H^2(dx^2 + dy^2) \\ \mathrm{II} &:= -d\boldsymbol{r}\cdot d\boldsymbol{N} &= H^2(\kappa_1 dx^2 + \kappa_2 dy^2) \end{aligned}$$

Fact: A surface is isothermic if and only if the lines of curvature form a Koenigs net. The same is true in the discrete case.

Fact: Isothermic surfaces come in pairs, that is Σ and its Christoffel transform $\tilde{\Sigma}$ related by

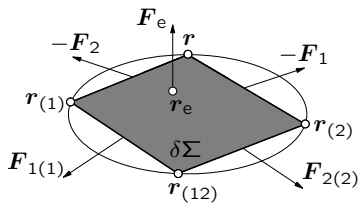
$$\tilde{\boldsymbol{r}}_x = \frac{\boldsymbol{r}_x}{|\boldsymbol{r}_x|^2}, \quad \tilde{\boldsymbol{r}}_y = -\frac{\boldsymbol{r}_y}{|\boldsymbol{r}_y|^2}.$$

Example: $\Sigma =$ minimal surface $\mapsto \tilde{\Sigma} =$ sphere

Conclusion: The discrete Christoffel transform encodes the diagrams of forces!

15. 'Shear-free' membranes (WKS 2005, 2010)

'Discrete membrane': $\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^3$
with quadrilaterals inscribed in circles



Assumptions:
- $\boldsymbol{F}_i \perp$ edges ('$S = 0$')
- 'Constant normal loading' $\boldsymbol{F}_\mathrm{e} = \bar{p}\delta\Sigma\boldsymbol{N}, \quad \bar{p} =$ const
- $\boldsymbol{F}_i$ homogeneously distributed along edges
- $\boldsymbol{F}_\mathrm{e}$ acts on some 'canonical' point $\boldsymbol{r}_\mathrm{e}$ (tbd)

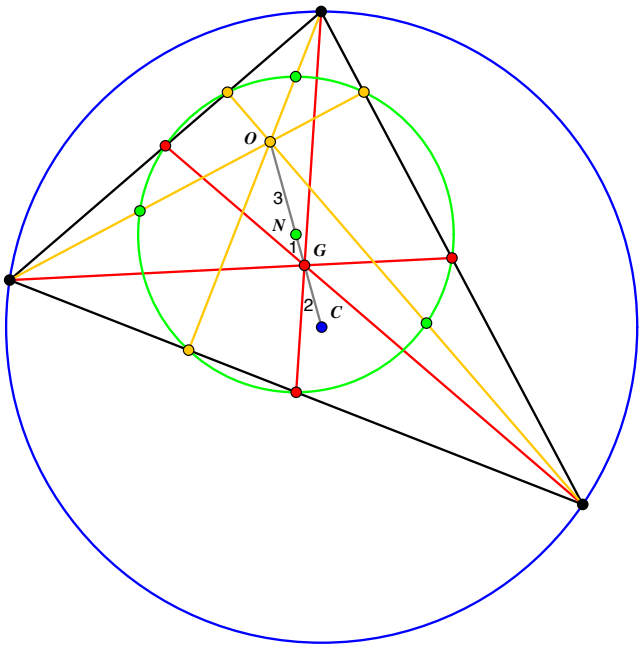
Equilibrium equations:

$$\boldsymbol{F}_{1(1)} - \boldsymbol{F}_1 + \boldsymbol{F}_{2(2)} - \boldsymbol{F}_2 + \boldsymbol{F}_\mathrm{e} = \boldsymbol{0} \qquad \text{(force)}$$

$$\begin{aligned} & (\boldsymbol{r}_{(12)} + \boldsymbol{r}_{(1)}) \times \boldsymbol{F}_{1(1)} - (\boldsymbol{r}_{(2)} + \boldsymbol{r}) \times \boldsymbol{F}_1 \qquad \text{(moment)} \\ + \quad & (\boldsymbol{r}_{(12)} + \boldsymbol{r}_{(2)}) \times \boldsymbol{F}_{2(2)} - (\boldsymbol{r}_{(1)} + \boldsymbol{r}) \times \boldsymbol{F}_2 + 2\boldsymbol{r}_\mathrm{e} \times \boldsymbol{F}_\mathrm{e} = \boldsymbol{0} \end{aligned}$$

Result: The 'canonical' choice (for integrability) is $\boldsymbol{r}_\mathrm{e} =$ quasi-nine-point centre (!) but this is a long story ...

16. The Euler line and the nine-point circle



circumcentre C
centroid G
nine-point centre N
orthocentre O

Euler line:

$\overline{CG} : \overline{GN} : \overline{NO} = 2 : 1 : 3$

There exist canonical analogues of these objects for quadrilaterals!

Discrete differential geometry: theory and applications

III. Deformation of polyhedral surfaces

by

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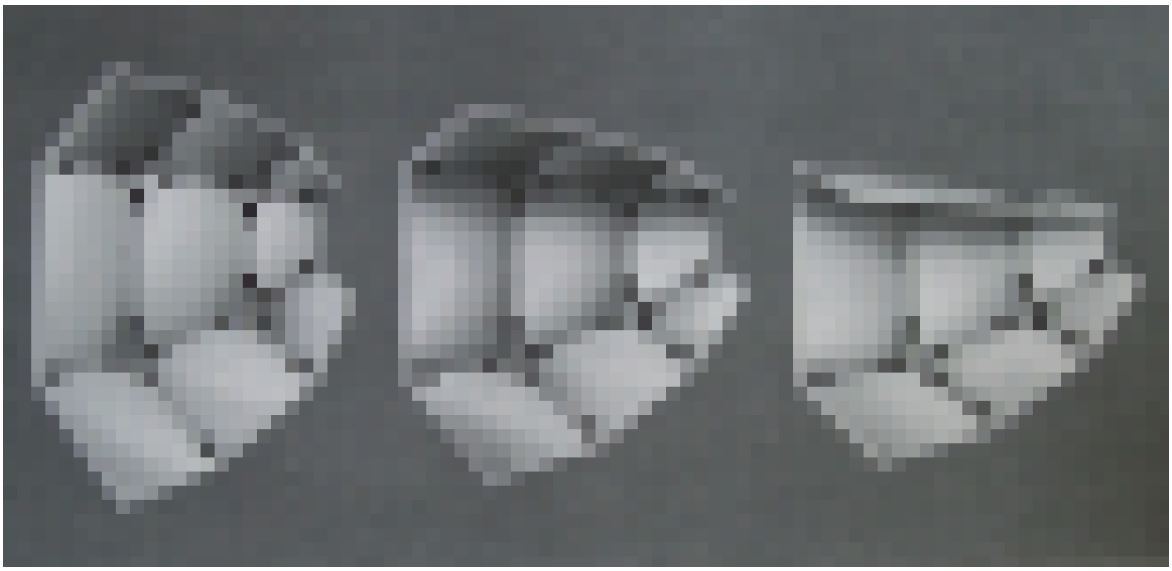
0. The problem



Sauer & Graf, Mathematische Annalen (1930)

Note: The classical analogue was shown to be 'integrable' by Bianchi in 1892!

1. Infinitesimal deformations

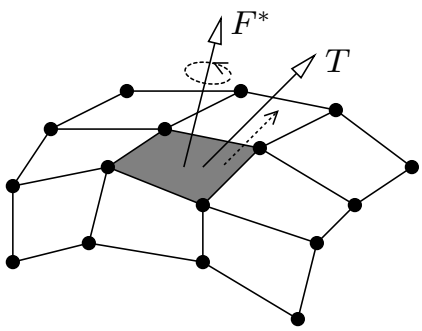
Consider a generic discrete surface (net)

$$F : \mathbb{Z}^2 \to \mathbb{R}^3$$

and a 'small' deformation

$$F^\epsilon = F + \epsilon \bar{F}.$$

Displacement of the vertices: $\bar{F} : \mathbb{Z}^2 \to \mathbb{R}^3$



Infinitesimal isometric deformation F^ϵ: A deformation which does not change the quadrilaterals in the order $O(\epsilon)$.

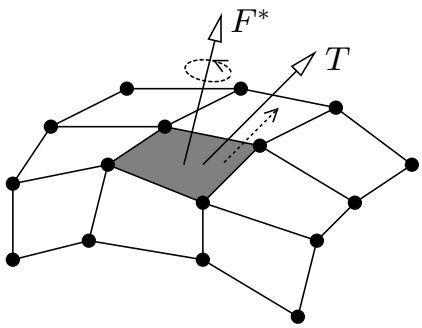
Kinematic interpretation. Any quadrilateral undergoes an infinitesimal rigid motion, that is

$$\boxed{\text{infinitesimal translation } T \quad \text{"}+\text{"} \quad \text{infinitesimal rotation } F^*.}$$

Accordingly, the displacement $P^\epsilon - P = \epsilon \bar{P}$ of a point P on a quadrilateral is given by

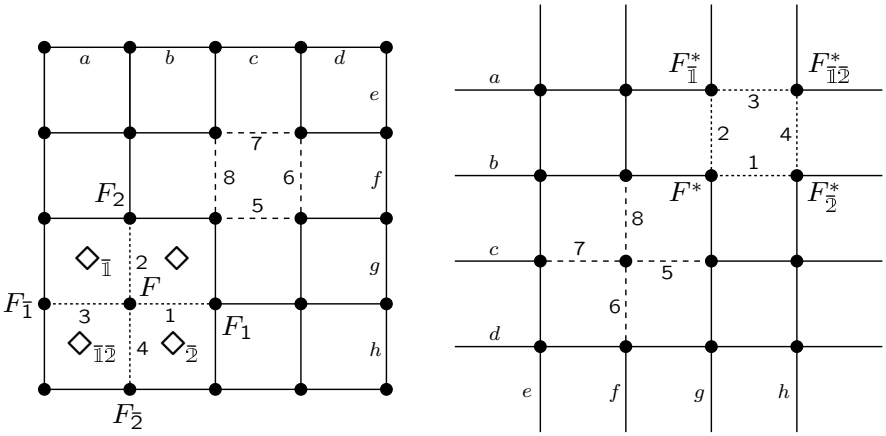
$$\bar{P} = T + F^* \times P,$$

where the translation T and the oriented axis of rotation F^* are independent of P.



Since each quadrilateral is associated with a vector of rotation F^*, we may regard F^* as another discrete surface which is defined on the lattice dual to $\mathbb{Z}^2$, that is

$$F^* : (\mathbb{Z}^2)^* \simeq \mathbb{Z}^2 \to \mathbb{R}^3 .$$



Consider the edge $[F, F_2]$: On the one hand, we have

$$\bar{F} = T + F^* \times F, \quad \bar{F}_2 = T + F^* \times F_2 .$$

On the other hand:

$$\bar{F} = T_{\bar{1}} + F^*_{\bar{1}} \times F, \quad \bar{F}_2 = T_{\bar{1}} + F^*_{\bar{1}} \times F_2$$

Conclusion. The dual edges $[F^*, F^*_{\bar{1}}]$ and $[F, F_2]$ are parallel, that is

$$\boxed{(F^*_{\bar{1}} - F^*) \times (F_2 - F) = 0 .}$$

Kinematic interpretation. The relative motion of two adjacent quadrilaterals represents an infinitesimal rotation about the common edge.

Thus, we have been led to (half of) the following result (Sauer 1970):

Definition. Two combinatorially dual discrete surfaces are termed reciprocal-parallel if dual edges are parallel.

Theorem. A discrete surface F admits an infinitesimal isometric deformation if and only if there exists a reciprocal-parallel discrete surface F^*.

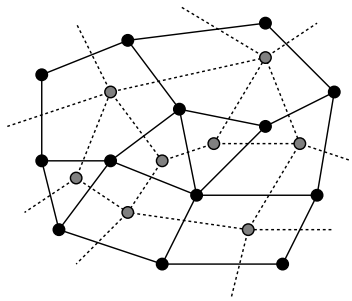
Interlude. It is evident that the above theorem may be extended to discrete surfaces of the type

$$F : V(\mathcal{G}) \to \mathbb{R}^3,$$

where $V(\mathcal{G})$ denotes the set of vertices of a cellular decomposition of the plane and

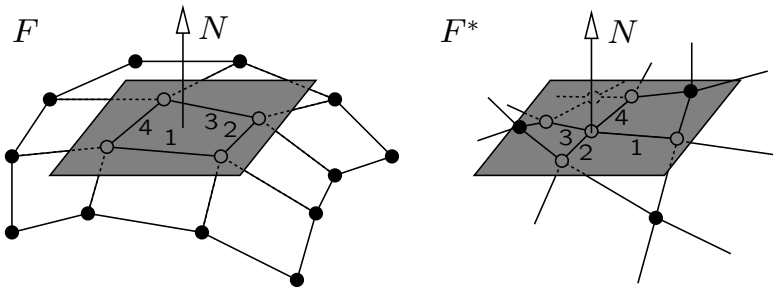
$$F^* : V(\mathcal{G}^*) \to \mathbb{R}^3$$

is defined on the vertices of the dual cellular decomposition.



Definition. A discrete net is termed discrete conjugate if all quadrilaterals are planar.

Definition. A discrete net is termed discrete asymptotic if all stars are planar.



The following theorem is evident:

Theorem. A 'non-degenerate' discrete conjugate net is infinitesimally and isometrically deformable if and only if there exists a reciprocal-parallel discrete asymptotic net. The latter is uniquely determined up to a scaling and hence the deformation is unique.

2. Finite deformations

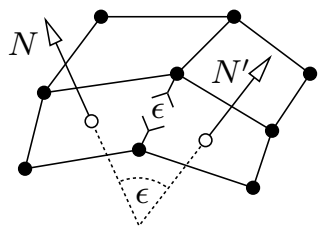
A (finite) isometric deformation of a discrete conjugate net $F : \mathbb{Z}^2 \to \mathbb{R}^3$ is a one-parameter family of discrete surfaces

$$F^\epsilon : \mathbb{Z}^2 \to \mathbb{R}^3, \qquad F^0 = F, \qquad \diamond^\epsilon \cong \diamond^0.$$

The preceding analysis implies the following:

Observation. Isometric deformations of (non-degenerate) conjugate nets are unique if they exist.

In fact, isometric deformations of 2×2 complexes always exist. If we hold constant the dihedral angle ϵ between two adjacent quadrilaterals then the complex is rigid and if the complex forms part of a discrete conjugate net then the entire discrete surface is rigid.

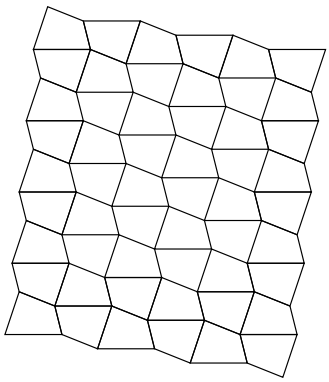


Remark. The deformability of a discrete conjugate net is a property of its normals only and may therefore be dealt within the realm of spherical geometry.

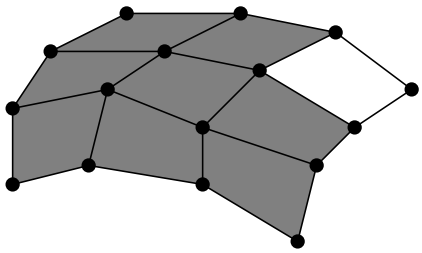
Examples of isometrically deformable discrete surfaces:

Discrete Voss surfaces (Sauer & Graf 1930).
(Classical Voss surfaces are defined as surfaces which admit two families of geodesic conjugate lines.)

A 'tessellation' of the plane (Kokotsakis 1932)



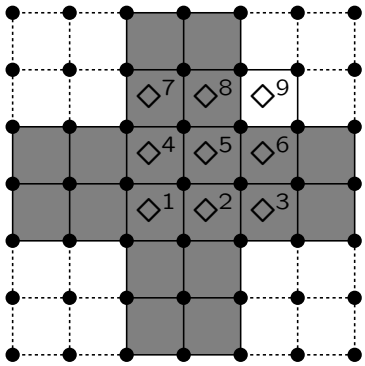
The problem of isometric deformations becomes non-trivial as soon as the discrete conjugate net consists of at least 3×3 quadrilaterals.



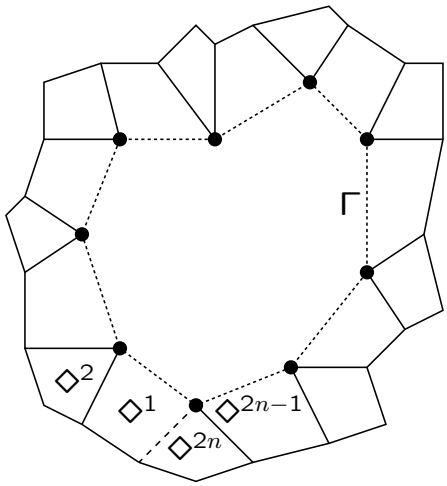
In fact, it is sufficient to focus on 3×3 complexes in the following sense:

Theorem. A non-degenerate discrete conjugate net is isometrically deformable if and only if its 3×3 complexes are isometrically deformable.

Proof →



Kokotsakis (1932) investigated infinitesimal and finite deformations of a special class of (open) polyhedral surfaces.



Results:

Analytic description of infinitesimal deformations (generalization of Sauer's result).

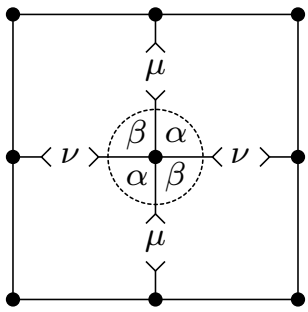
Cauchy's theorem in the case of convex octahedra and the infinitesimal deformability of the octahedra of Bricard type.

Finite deformations of discrete Voss surfaces.

The delimitation of finite deformations is an outstanding problem. Various algebraic/analytic formulations are at hand but unwieldy – to say the least.

3. Discrete Voss surfaces

Definition. A non-degenerate conjugate net which is such that opposite angles made by the edges of any star are equal is termed a discrete Voss surface.



Application of spherical geometry yields:

Lemma. The angles α, β and μ, ν of a 2×2 complex of Voss type are related by (Sauer & Graf 1930)

$$\boxed{\tan\frac{\mu}{2}\tan\frac{\nu}{2} = \frac{\sin(\alpha+\beta)}{\sin\alpha+\sin\beta}.} \tag{1}$$

Its one-parameter (λ) family of isometric deformations is given by

$$\boxed{\tan\frac{\mu}{2} \to \lambda\tan\frac{\mu}{2}, \quad \tan\frac{\nu}{2} \to \frac{1}{\lambda}\tan\frac{\nu}{2}.}$$

The deformability of discrete Voss surfaces is an immediate consequence of the above lemma:

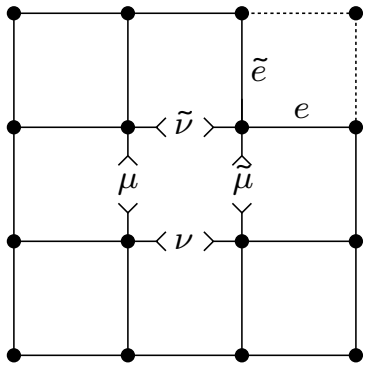
Theorem. Discrete Voss surfaces admit a one-parameter family of isometric deformations.

Proof.

$$\tan\frac{\mu}{2} \to \lambda \tan\frac{\mu}{2}, \quad \tan\frac{\nu}{2} \to \frac{1}{\lambda}\tan\frac{\nu}{2}$$

$$\tan\frac{\tilde{\mu}}{2} \to \lambda \tan\frac{\tilde{\mu}}{2}, \quad \tan\frac{\tilde{\nu}}{2} \to \frac{1}{\lambda}\tan\frac{\tilde{\nu}}{2}$$

$$\Rightarrow \tan\frac{\tilde{\mu}}{2}\tan\frac{\tilde{\nu}}{2} \text{ is preserved!}$$



Relation (1) regarded as a lattice equation is another avatar of Hirota's (1977) integrable discrete sine-Gordon equation and the deformation parameter λ plays the role of the spectral parameter!

4. 'Integrability' of finite deformations

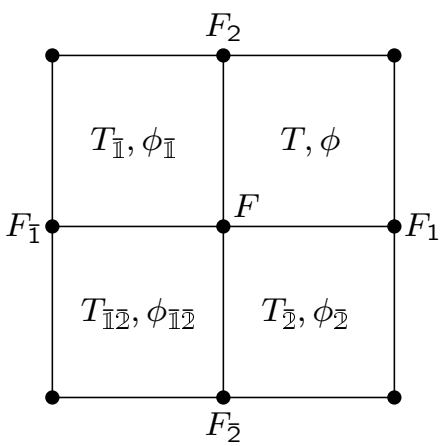
Finite deformations.

isometric deformation $\to$ rigid motion of quadrilaterals

Each quadrilateral is associated with a translation $T \in su(2)$ and a rotation $\phi \in SU(2)$ such that

$$P^{\epsilon} = T(\epsilon) + \phi^{-1}(\epsilon)P\phi(\epsilon),$$

where we have identified the ambient space $\mathbb{R}^3$ with the Lie algebra $su(2)$.



Given an isometrically deformable surface, the relations

$$\begin{aligned} F^\epsilon &= T + \phi^{-1} F \phi, & F_2^\epsilon &= T + \phi^{-1} F_2 \phi \\ F^\epsilon &= T_{\bar{\mathbb{1}}} + \phi_{\bar{\mathbb{1}}}^{-1} F \phi_{\bar{\mathbb{1}}}, & F_2^\epsilon &= T_{\bar{\mathbb{1}}} + \phi_{\bar{\mathbb{1}}}^{-1} F_2 \phi_{\bar{\mathbb{1}}} \end{aligned}$$

imply that

$$[F_{\mathbb{1}}^* - F^*, \phi_1 \phi^{-1}] = 0, \quad [F_2^* - F^*, \phi_2 \phi^{-1}] = 0$$

Accordingly, there exist real lattice functions $a(\epsilon), b(\epsilon)$ and $c(\epsilon), d(\epsilon)$ such that

$$\boxed{\begin{aligned} \phi_1 &= \mathcal{L}(\epsilon)\phi, & \mathcal{L}(\epsilon) &= a(\epsilon)(F_{\mathbb{1}}^* - F^*) + b(\epsilon)\mathbb{1} \\ \phi_2 &= \mathcal{M}(\epsilon)\phi, & \mathcal{M}(\epsilon) &= c(\epsilon)(F_2^* - F^*) + d(\epsilon)\mathbb{1}. \end{aligned}}$$

'Lax pair'

The compatibility condition $\phi_{12} = \phi_{21}$ yields

$$\boxed{\mathcal{L}_2(\epsilon)\mathcal{M}(\epsilon) = \mathcal{M}_1(\epsilon)\mathcal{L}(\epsilon).} \qquad \rightarrow \qquad \text{'soliton equation'}$$

The converse of the preceding may also be formulated in a precise manner. Thus:

matrix of rotation	= 'eigenfunction'
deformation parameter	= 'spectral parameter'

Example:

$$\phi_1 = (\epsilon \mathcal{N}_1 \mathcal{N} + \mathbb{1})\phi, \quad \phi_2 = (-\epsilon \mathcal{N}_2 \mathcal{N} + \mathbb{1})\phi$$

with

$$F_1^* - F^* = \mathcal{N}_1 \times \mathcal{N}, \quad F_2^* - F^* = \mathcal{N} \times \mathcal{N}_2, \quad |\mathcal{N}| = 1$$

Compatibility condition:

$$\boxed{\mathcal{N}_{12} + \mathcal{N} = \frac{\langle \mathcal{N}, \mathcal{N}_1 + \mathcal{N}_2 \rangle}{1 + \langle \mathcal{N}_1, \mathcal{N}_2 \rangle}(\mathcal{N}_1 + \mathcal{N}_2)}$$

This is the 'Gauß map' of both

F : discrete Voss surfaces

F^* : discrete surfaces of constant negative Gaußian curvature