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バージョン：

Discrete Differential Geometry of Curves

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outline of my talk

- what is DDG?
- plane curve
  - continuous
  - discrete
- deformation of plane curve
  - continuous
  - semi-discrete
  - discrete
- explicit formula
- deformation of space curve

what is DDG?

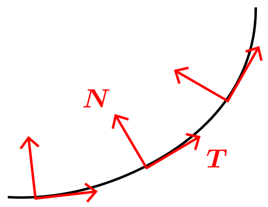
soliton equation ↔	differential geometry
↕	↕
discrete soliton equation ↔	DDG

an example

modified KdV equation ↔	deformation of curve
↕	↕
discrete mKdV equation ↔	?

plane curve

continuous



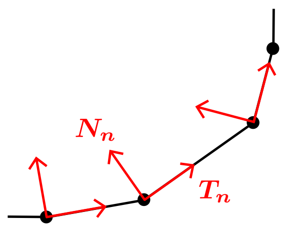
$$|\gamma'| = 1$$

$$T := \gamma' = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\phi := (T, N) \in SO(2)$$

$$\phi' = \phi \begin{pmatrix} 0 & -\theta' \\ \theta' & 0 \end{pmatrix}$$

discrete



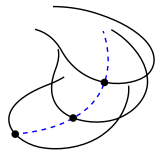
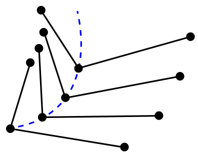
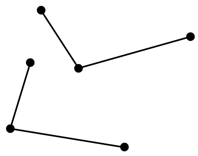
$$a_n := |\gamma_{n+1} - \gamma_n|$$

$$T_n := \frac{\gamma_{n+1} - \gamma_n}{a_n} = \begin{pmatrix} \cos\sigma_n \\ \sin\sigma_n \end{pmatrix}$$

$$\phi_n := (T_n, N_n) \in SO(2)$$

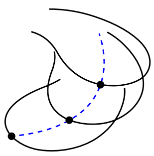
$$\phi_{n+1} = \phi_n \begin{pmatrix} \cos(\sigma_{n+1} - \sigma_n) & -\sin(\sigma_{n+1} - \sigma_n) \\ \sin(\sigma_{n+1} - \sigma_n) & \cos(\sigma_{n+1} - \sigma_n) \end{pmatrix}$$

deformation of plane curve

system	curve	deformation	visual image
continuous	continuous	continuous	
semi-discrete	discrete	continuous	
discrete	discrete	discrete	

we are interested in isoperimetric deformation
isoperimetric = arc length (segment length) preserving

deformation of plane curve (continuous)



$\gamma = \gamma(x, t)$ an isoperimetric deformation of a unit speed curve $\gamma(x, 0)$

that is, $\left|\frac{\partial}{\partial x}\gamma\right| = 1$ for all t

- we write $' = \frac{\partial}{\partial x}$ and $\dot{} = \frac{\partial}{\partial t}$
- decompose $\dot{\gamma}$ in the form $\dot{\gamma} = fT + gN$

  then, we have $\begin{cases} f' = \theta' g & \text{(isoperimetric condition)} \\ \dot{\theta} = g' + \theta' f & \text{(compatibility condition)} \end{cases}$

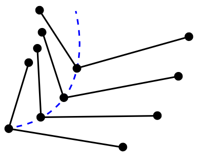
  where $\theta = \arg T$
- when we determine the deformation by $(f, g) = \left(\frac{1}{2}(\theta')^2, \theta''\right)$,

  the angle θ obeys the potential modified KdV equation

  $\dot{\theta} = \theta''' + \frac{1}{2}(\theta')^3$ and hence the curvature $\kappa := \theta'$ obeys

  the modified KdV equation $\dot{\kappa} = \kappa''' + \frac{3}{2}\kappa^2\kappa'$

deformation of plane curve (semi-discrete)



$\gamma_n = \gamma_n(t)$ an isoperimetric deformation of a discrete curve $\gamma_n(0)$

that is, $\dfrac{\partial}{\partial t}|\gamma_{n+1} - \gamma_n| = 0$

- decompose $\dot{\gamma}_n$ in the form $\dot{\gamma}_n = f_n T_n + g_n N_n$

  where $T_n := \dfrac{\gamma_{n+1} - \gamma_n}{|\gamma_{n+1} - \gamma_n|} = \dfrac{\gamma_{n+1} - \gamma_n}{a_n}$ and $N_n := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T_n$

- then, we have

$$\begin{cases} f_{n+1}\cos\kappa_{n+1} - f_n = g_{n+1}\sin\kappa_{n+1} \\ \dot{\kappa}_n = \dfrac{f_{n+1}\sin\kappa_{n+1} + g_{n+1}\cos\kappa_{n+1} - g_n}{a_n} - \dfrac{f_n\sin\kappa_n + g_n\cos\kappa_n - g_{n-1}}{a_{n-1}} \end{cases}$$

  where $\kappa_n := \angle(T_n, T_{n-1})$

- when a_n is constant, we determine the deformation by $(f_n, g_n) = \left(1, -\tan\dfrac{\kappa_n}{2}\right)$ so that κ_n obeys the semi-discrete modified KdV equation $\dot{\kappa}_n = \dfrac{1}{a}\left(\tan\dfrac{\kappa_{n+1}}{2} - \tan\dfrac{\kappa_{n-1}}{2}\right)$

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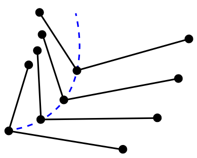
deformation of plane curve (semi-discrete)



$\gamma_n = \gamma_n(t)$ the isoperimetric deformation of a discrete curve $\gamma_n(0)$

defined by $\dot{\gamma}_n = T_n - \tan\dfrac{\kappa_n}{2} N_n$

- $\kappa_n = \angle(T_n, T_{n-1})$ varies according to the semi-discrete mKdV equation $\dot{\kappa}_n = \dfrac{1}{a}\left(\tan\dfrac{\kappa_{n+1}}{2} - \tan\dfrac{\kappa_{n-1}}{2}\right)$

- the semi-discrete potential mKdV equation

$$\dot{\theta}_n = \frac{2}{a}\tan\frac{\theta_{n+1} - \theta_{n-1}}{4}$$

  where $\kappa_n = \dfrac{\theta_{n+1} - \theta_{n-1}}{2}$

- therefore $\arg T_n = \dfrac{\theta_{n+1} + \theta_n}{2}$

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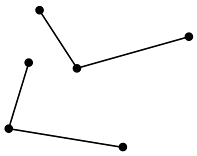
deformation of plane curve (discrete)



γ_n^m an isoperimetric deformation of a discrete curve γ_n^0

that is, $\left|\gamma_{n+1}^{m+1} - \gamma_n^{m+1}\right| = \left|\gamma_{n+1}^m - \gamma_n^m\right| =: a_n$

- we also impose the equidistant condition on γ_n^m, which means that $\left|\gamma_{n+1}^{m+1} - \gamma_{n+1}^m\right| = \left|\gamma_n^{m+1} - \gamma_n^m\right| =: b_m$
- decompose $\dfrac{\gamma_n^{m+1} - \gamma_n^m}{b_m}$ in the form

$$\begin{aligned}\frac{\gamma_n^{m+1} - \gamma_n^m}{b_m} &= f_n^m T_n^m + g_n^m N_n^m \\ &= \cos w_n^m \, T_n^m + \sin w_n^m \, N_n^m = \begin{pmatrix} \cos w_n^m & -\sin w_n^m \\ \sin w_n^m & \cos w_n^m \end{pmatrix} T_n^m\end{aligned}$$

then, we have

$$\begin{cases} a_n \sin \dfrac{\kappa_{n+1}^m + w_{n+1}^m + w_n^m}{2} = b_m \sin \dfrac{\kappa_{n+1}^m + w_{n+1}^m - w_n^m}{2} \\ \kappa_n^{m+1} - \kappa_{n+1}^m = w_{n+1}^m - w_{n-1}^m \end{cases}$$

where $\kappa_n^m := \angle(T_n^m, T_{n-1}^m)$

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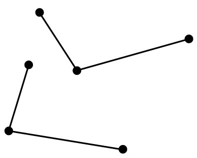
deformation of plane curve (discrete)



γ_n^0 a discrete curve

b_m a sequence of positive numbers

γ_n^m the isoperimetric & equidistant deformation of γ_n^0 defined by $\dfrac{\gamma_n^{m+1} - \gamma_n^m}{b_m} = \cos w_n^m \, T_n^m + \sin w_n^m \, N_n^m$ and $a_n \sin \dfrac{\kappa_{n+1}^m + w_{n+1}^m + w_n^m}{2} = b_m \sin \dfrac{\kappa_{n+1}^m + w_{n+1}^m - w_n^m}{2}$

- the angle $w_n^m = \angle\left(\gamma_n^{m+1} - \gamma_n^m, \ \gamma_{n+1}^m - \gamma_n^m\right)$ obeys the discrete mKdV equation

$$w_{n+1}^{m+1} - w_n^m = 2\arctan\left(\frac{b_{m+1} + a_n}{b_{m+1} - a_n} \tan \frac{w_n^{m+1}}{2}\right) - 2\arctan\left(\frac{b_m + a_{n+1}}{b_m - a_{n+1}} \tan \frac{w_{n+1}^m}{2}\right)$$

- the compatibility condition $\kappa_n^{m+1} - \kappa_{n+1}^m = w_{n+1}^m - w_{n-1}^m$ implies $\kappa_n^m = \dfrac{\theta_{n+1}^m - \theta_{n-1}^m}{2}, \quad w_n^m = \dfrac{\theta_n^{m+1} - \theta_{n+1}^m}{2}$ for some θ_n^m

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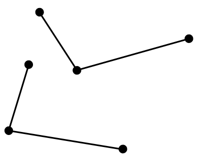
deformation of plane curve (discrete)



γ_n^0 a discrete curve

b_m a sequence of positive numbers

γ_n^m the isoperimetric & equidistant deformation of γ_n^0

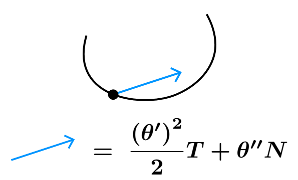
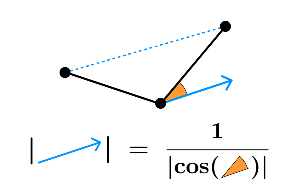
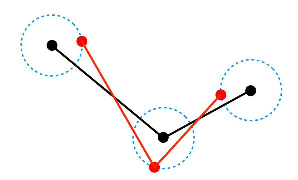
defined by $\dfrac{\gamma_n^{m+1}-\gamma_n^m}{b_m} = \cos w_n^m\, T_n^m + \sin w_n^m\, N_n^m$

and $a_n \sin \dfrac{\kappa_{n+1}^m + w_{n+1}^m + w_n^m}{2} = b_m \sin \dfrac{\kappa_{n+1}^m + w_{n+1}^m - w_n^m}{2}$

- $\kappa_n^m = \dfrac{\theta_{n+1}^m - \theta_{n-1}^m}{2}, \quad w_n^m = \dfrac{\theta_n^{m+1} - \theta_{n+1}^m}{2}$ for some θ_n^m
- the isoperimetric & equidistant condition becomes
  the discrete potential modified KdV equation
  $$\tan \frac{\theta_{n+1}^{m+1} - \theta_n^m}{4} = \frac{b_m + a_n}{b_m - a_n} \tan \frac{\theta_n^{m+1} - \theta_{n+1}^m}{4}$$
- $\arg T_n^m = \dfrac{\theta_{n+1}^m + \theta_n^m}{2}$

deformation of plane curve

summary: we derive isoperimetric deformations which are governed by the potential modified KdV equations

continuous	semi-discrete	discrete
		

potential modified KdV equations

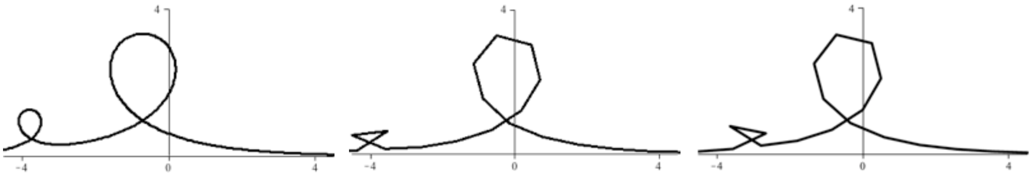
$$\dot{\theta} = \theta''' + \frac{1}{2}(\theta')^3, \quad \arg T = \theta$$

$$\dot{\theta}_n = \frac{2}{a} \tan \frac{\theta_{n+1} - \theta_{n-1}}{4}, \quad \arg T_n = \frac{\theta_{n+1} + \theta_n}{2}$$

$$\tan \frac{\theta_{n+1}^{m+1} - \theta_n^m}{4} = \frac{b_m + a_n}{b_m - a_n} \tan \frac{\theta_n^{m+1} - \theta_{n+1}^m}{4}, \quad \arg T_n^m = \frac{\theta_{n+1}^m + \theta_n^m}{2}$$

deformation of plane curve

animation



procedure for making an animation

① construct a solution θ to the potential mKdV equation

② integrate $\gamma' = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ to obtain the deformation γ explicitly

③ plot γ

explicit formula (continuous)

① construct a solution θ to the potential mKdV equation

- write $\theta = \frac{2}{\sqrt{-1}} \log \frac{\tau}{\tau^*}$, where τ^* is the complex conjugate of τ
- potential mKdV equation

$$(D_x{}^3 + D_t)\, \tau \cdot \tau^* = 0, \quad D_x{}^2\, \tau \cdot \tau^* = 0$$

- N-soliton solution

$$\tau = \det\left(f_{j-1}^i\right)_{i,\,j=1,\ldots,N}, \quad f_j^i = \alpha_i\, p_i{}^j \exp\left(p_i x - 4 p_i{}^3 t\right) + \begin{bmatrix} p_i \leftrightarrow -p_i \\ \alpha_i \leftrightarrow \beta_i \end{bmatrix}$$

② integrate $\gamma' = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ to obtain the curve γ explicitly

- explicit formula

$$\gamma = \frac{\partial}{\partial y}\bigg|_{y=0} \begin{pmatrix} -\Re \log\tau \\ \Im \log \tau \end{pmatrix}$$

- y-dependence

$$D_x D_y\, \tau \cdot \tau = -2\, (\tau^*)^2$$

- revisit the N-soliton solution

$$\tau = \exp(-yx) \det\left(f_{j-1}^i\right), \quad f_j^i = \alpha_i\, p_i{}^j \exp\left(\frac{y}{p_i} + p_i x - 4 p_i{}^3 t\right) + \begin{bmatrix} p_i \leftrightarrow -p_i \\ \alpha_i \leftrightarrow \beta_i \end{bmatrix}$$

explicit formula

- explicit formula of the isoperimetric deformation

$$\gamma = \frac{\partial}{\partial y}\bigg|_{y=0} \begin{pmatrix} -\Re \log \tau \\ \Im \log \tau \end{pmatrix}$$

- N-soliton solution

continuous

$$\tau = \tau(x,t) = \exp\left(-yx\right) \det\left(f^i_{j-1}\right)_{i,\,j=1,\ldots,N}$$

$$f^i_j = \alpha_i \, {p_i}^j \exp\left(\frac{y}{p_i} + p_i \, x - 4\, {p_i}^3 \, t\right) + \begin{bmatrix} p_i \leftrightarrow -p_i \\ \alpha_i \leftrightarrow \beta_i \end{bmatrix}$$

semi-discrete

$$\tau = \tau_n(t) = \exp\left(-yt - yan\,\right) \det\left(f^i_{j-1}\right)_{i,\,j=1,\ldots,N}$$

$$f^i_j = \frac{\alpha_i \, {p_i}^j}{\left(1 - ap_i\right)^n} \exp\left(\frac{y}{p_i} + \frac{p_i}{1 - a^2 {p_i}^2}\, t\right) + \begin{bmatrix} p_i \leftrightarrow -p_i \\ \alpha_i \leftrightarrow \beta_i \end{bmatrix}$$

discrete

$$\tau = \tau^m_n = \exp\left(-y \textstyle\sum_k^{n-1} a_k - y \sum_k^{m-1} b_k\right) \det\left(f^i_{j-1}\right)_{i,\,j=1,\ldots,N}$$

$$f^i_j = \alpha_i \, {p_i}^j \exp \frac{y}{p_i} \prod_k^{n-1} \frac{1}{1 - a_k p_i} \prod_{k'}^{m-1} \frac{1}{1 - b_{k'} p_i} + \begin{bmatrix} p_i \leftrightarrow -p_i \\ \alpha_i \leftrightarrow \beta_i \end{bmatrix}$$

explicit formula

- bilinear structure

continuous

$$\left(D_x{}^3 + D_t\right) \tau \cdot \tau^* = 0$$
$$D_x{}^2 \, \tau \cdot \tau^* = 0$$
$$D_x D_y \, \tau \cdot \tau = -2 \left(\tau^*\right)^2$$

semi-discrete

$$D_t \, \tau^*{}_n \cdot \tau_n = \frac{1}{2a} \left(\tau^*{}_{n+1}\tau_{n-1} - \tau^*{}_{n-1}\tau_{n+1}\right)$$
$$\tau^*{}_n \, \tau_n = \frac{1}{2} \left(\tau^*{}_{n+1}\tau_{n-1} + \tau^*{}_{n-1}\tau_{n+1}\right)$$
$$D_t D_y \, \tau_n \cdot \tau_n = -2\tau^*{}_{n+1}\tau^*{}_{n-1}$$
$$D_y \, \tau_{n+1} \cdot \tau_n = -a\tau^*{}_{n+1}\tau^*{}_n$$

discrete

$$b_m \tau^{*m+1}_{\ n} \tau^m_{n+1} - a_n \tau^{*m}_{\ n+1} \tau^{m+1}_n + \left(a_n - b_m\right) \tau^{*m+1}_{\ n+1} \tau^m_n = 0$$ ☜
$$D_y \, \tau^m_{n+1} \cdot \tau^m_n = -a_n \tau^{*m}_{\ n+1} \tau^{*m}_{\ n}$$
$$D_y \, \tau^{m+1}_n \cdot \tau^m_n = -b_m \tau^{*m+1}_{\ n} \tau^{*m}_{\ n}$$

discrete pot. mKdV eq. (reduction of the Hirowa-Miwa eq.)

$$b_m\tau_n^{m+1}(s+1)\tau_{n+1}^m(s)-a_n\tau_{n+1}^m(s+1)\tau_n^{m+1}(s)=-\left(a_n-b_m\right)\tau_{n+1}^{m+1}(s+1)\tau_n^m(s)$$

impose $\tau_n^m(s+1)=\text{const.}\times\tau^{*m}_n(s)$

$$\therefore\quad\begin{cases} b_m\tau^{*m+1}_n\tau_{n+1}^m-a_n\tau^{*m}_{n+1}\tau_n^{m+1}=-\left(a_n-b_m\right)\tau^{*m+1}_{n+1}\tau_n^m\\ b_m\tau_n^{m+1}\tau^{*m}_{n+1}-a_n\tau_{n+1}^m\tau^{*m+1}_n=-\left(a_n-b_m\right)\tau_{n+1}^{m+1}\tau^{*m}_n\end{cases}$$

$$\therefore\quad\begin{cases} b_m\exp\left(\sqrt{-1}\dfrac{\theta_{n+1}^m}{2}\right)-a_n\exp\left(\sqrt{-1}\dfrac{\theta_n^{m+1}}{2}\right)=-\left(a_n-b_m\right)\dfrac{\tau^{*m+1}_{n+1}\tau_n^m}{\tau^{*m+1}_n\tau^{*m}_{n+1}}\\ b_m\exp\left(\sqrt{-1}\dfrac{\theta_n^{m+1}}{2}\right)-a_n\exp\left(\sqrt{-1}\dfrac{\theta_{n+1}^m}{2}\right)=-\left(a_n-b_m\right)\dfrac{\tau_{n+1}^{m+1}\tau^{*m}_n}{\tau^{*m+1}_n\tau^{*m}_{n+1}}\end{cases}$$

$$\therefore\quad\frac{b_m\exp\left(\sqrt{-1}\dfrac{\theta_{n+1}^m}{2}\right)-a_n\exp\left(\sqrt{-1}\dfrac{\theta_n^{m+1}}{2}\right)}{b_m\exp\left(\sqrt{-1}\dfrac{\theta_n^{m+1}}{2}\right)-a_n\exp\left(\sqrt{-1}\dfrac{\theta_{n+1}^m}{2}\right)}=\exp\left(\sqrt{-1}\frac{\theta_n^m}{2}-\sqrt{-1}\frac{\theta_{n+1}^{m+1}}{2}\right)$$

$$\therefore\quad a_n\sin\frac{\theta_{n+1}^{m+1}-\theta_n^m+\theta_n^{m+1}-\theta_{n+1}^m}{4}=b_m\sin\frac{\theta_{n+1}^{m+1}-\theta_n^m-\theta_n^{m+1}+\theta_{n+1}^m}{4}$$

$$\therefore\quad\tan\frac{\theta_{n+1}^{m+1}-\theta_n^m}{4}=\frac{b_m+a_n}{b_m-a_n}\tan\frac{\theta_n^{m+1}-\theta_{n+1}^m}{4}$$

proof of the explicit formula (discrete)

- $\gamma_n^m=\dfrac{\partial}{\partial y}\Big|_{y=0}\begin{pmatrix}-\Re\log\tau_n^m\\ \Im\log\tau_n^m\end{pmatrix}=-\dfrac{1}{2}\dfrac{\partial}{\partial y}\Big|_{y=0}\begin{pmatrix}\log\left(\tau_n^m\tau^{*m}_n\right)\\ \sqrt{-1}\log\left(\tau_n^m/\tau^{*m}_n\right)\end{pmatrix}=:\begin{pmatrix}X_n^m\\ Y_n^m\end{pmatrix}$

- $$\begin{aligned} X_{n+1}^m-X_n^m&=-\frac{1}{2}\frac{\partial}{\partial y}\left(\log\left(\tau_{n+1}^m\tau^{*m}_{n+1}\right)-\log\left(\tau_n^m\tau^{*m}_n\right)\right)\\ &=-\frac{1}{2}\frac{\partial}{\partial y}\left(\log\frac{\tau_{n+1}^m}{\tau_n^m}+\log\frac{\tau^{*m}_{n+1}}{\tau^{*m}_n}\right)\\ &=\frac{a_n}{2}\left(\frac{\tau^{*m}_{n+1}\tau^{*m}_n}{\tau_{n+1}^m\tau_n^m}+\frac{\tau_{n+1}^m\tau_n^m}{\tau^{*m}_{n+1}\tau^{*m}_n}\right)\qquad\because\ D_y\,\tau_{n+1}^m\cdot\tau_n^m=-a_n\tau^{*m}_{n+1}\tau^{*m}_n\\ &=\frac{a_n}{2}\left(\exp\left(-\sqrt{-1}\frac{\theta_{n+1}^m+\theta_n^m}{2}\right)+\exp\left(\sqrt{-1}\frac{\theta_{n+1}^m+\theta_n^m}{2}\right)\right)\\ &=a_n\cos\frac{\theta_{n+1}^m+\theta_n^m}{2}\end{aligned}$$

  $$Y_{n+1}^m-Y_n^m=a_n\sin\frac{\theta_{n+1}^m+\theta_n^m}{2}$$

  thus, $T_n^m:=\dfrac{\gamma_{n+1}^m-\gamma_n^m}{a_n}$ satisfies $|T_n^m|=1,\quad\arg T_n^m=\dfrac{\theta_{n+1}^m+\theta_n^m}{2}$

  similarly, $S_n^m:=\dfrac{\gamma_n^{m+1}-\gamma_n^m}{b_m}$ satisfies $|S_n^m|=1,\quad\arg S_n^m=\dfrac{\theta_n^{m+1}+\theta_n^m}{2}$

  consequently, $w_n^m:=\angle\left(\gamma_n^{m+1}-\gamma_n^m,\ \gamma_{n+1}^m-\gamma_n^m\right)=\dfrac{\theta_n^{m+1}-\theta_{n+1}^m}{2}$

other equations

plane curve

- modified KdV equation (centroaffine geometry)
- KdV equation (equicentroaffine geometry)
- Burgers equation (similarity geometry)
- Sawada-Kotera equation (equiaffine geometry)
- ……

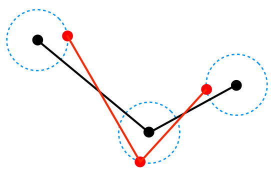
c	s-d	d
○		
○	○	○
○		○
○		

space curve

- modified KdV equation (euclidean geometry)
- ……

○	○	△

other equations (discrete KdV equation)

discrete mKdV	discrete KdV
$\frac{w^{m+1}_{n+1}}{2} - \frac{w^m_n}{2} = \arctan\left(\frac{b_{m+1}+a_n}{b_{m+1}-a_n}\tan\frac{w^{m+1}_n}{2}\right) - \arctan\left(\frac{b_m+a_{n+1}}{b_m-a_{n+1}}\tan\frac{w^m_{n+1}}{2}\right)$	$\left(\frac{1}{b_{m+1}}-\frac{1}{a_{n+1}}\right)\frac{1}{v^{m+1}_{n+1}} - \left(\frac{1}{b_m}-\frac{1}{a_n}\right)\frac{1}{v^m_n} = \left(\frac{1}{b_{m+1}}+\frac{1}{a_n}\right)v^{m+1}_n - \left(\frac{1}{b_m}+\frac{1}{a_{n+1}}\right)v^m_{n+1}$
euclidean geometry	equicentroaffine geometry
deformation preserves lengths $a_n = \lvert\gamma^m_{n+1}-\gamma^m_n\rvert,\ b_m = \lvert\gamma^{m+1}_n-\gamma^m_n\rvert$	deformation preserves areas $a_n = \det(\gamma^m_{n+1},\gamma^m_n),\ b_m = \det(\gamma^{m+1}_n,\gamma^m_n)$
	
$w^m_n = \angle(\gamma^{m+1}_n-\gamma^m_n,\ \gamma^m_{n+1}-\gamma^m_n)$	$v^m_n = \frac{\det(\gamma^{m+1}_n-\gamma^m_{n+1},\ \gamma^m_n)}{\det(\gamma^{m+1}_n-\gamma^m_{n+1},\ \gamma^m_{n+1})}$

space curve

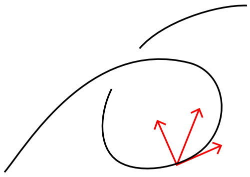
continuous

$$|\gamma'| = 1$$

$$T := \gamma', \quad N := \frac{T'}{|T'|}, \quad B := T \times N$$

$$\phi := (T, N, B) \in \mathrm{SO}(3)$$

$$\phi' = \phi \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\lambda \\ 0 & \lambda & 0 \end{pmatrix}, \quad \kappa := |T'|, \quad \lambda := -\langle N, B' \rangle$$



discrete

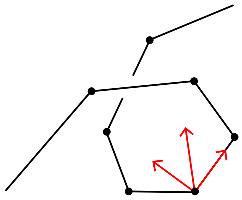
$$a_n := |\gamma_{n+1} - \gamma_n|$$

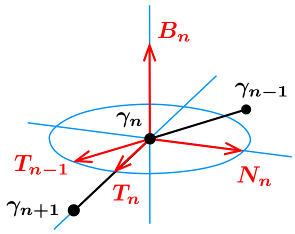
$$T_n := \frac{\gamma_{n+1} - \gamma_n}{a_n}$$

$$N_n := \frac{\Delta T_n - \langle \Delta T_n, T_n \rangle T_n}{|\Delta T_n - \langle \Delta T_n, T_n \rangle T_n|}, \quad \Delta T_n := \frac{T_n - T_{n-1}}{a_n + a_{n-1}}$$

$$B_n := T_n \times N_n$$

$$\phi_n := (T_n, N_n, B_n) \in \mathrm{SO}(3)$$





remark $\quad B_n = \dfrac{T_{n-1} \times T_n}{|T_{n-1} \times T_n|}, \quad N_n = B_n \times T_n$

point $\quad N_n \in \mathrm{span}\,\{T_n, T_{n-1}\}$

space curve

continuous

$$\phi' = \phi \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\lambda \\ 0 & \lambda & 0 \end{pmatrix}, \quad \kappa = |T'|, \quad \lambda = -\langle N, B' \rangle$$

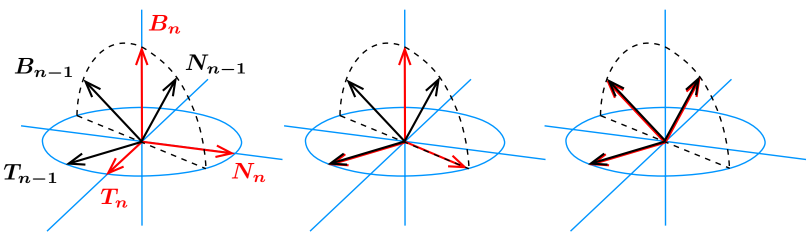
discrete

$$\phi_{n-1} = \phi_n \begin{pmatrix} \cos\kappa_n & \sin\kappa_n & 0 \\ -\sin\kappa_n & \cos\kappa_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda_n & -\sin\lambda_n \\ 0 & \sin\lambda_n & \cos\lambda_n \end{pmatrix}$$

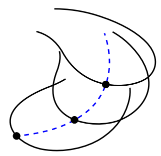
where κ_n $(0 < \kappa_n < \pi)$ and λ_n $(0 \leq \lambda_n < 2\pi)$ are defined by

$$\langle T_n, T_{n-1} \rangle = \cos\kappa_n, \quad \langle B_n, B_{n-1} \rangle = \cos\lambda_n, \quad \langle B_n, N_{n-1} \rangle = \sin\lambda_n$$

proof



deformation of space curve



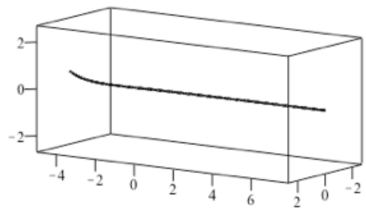
$\gamma = \gamma(x,t)$ space curves of constant torsion λ

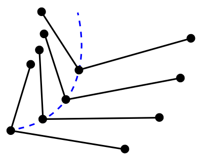
$$\dot{\gamma} = \left(\frac{\kappa^2}{2} - 3\lambda^2\right) T + \kappa' N - 2\lambda\kappa B$$

then, the curvature κ obeys the mKdV equation

$$\dot{\kappa} = \kappa''' + \frac{3}{2}\kappa^2\kappa'$$

we can construct explicit formulas of the deformation γ





$\gamma = \gamma_n(t)$ discrete space curves

assume that both $\lambda = \angle(B_n, B_{n-1})$ and $a = |\gamma_{n+1} - \gamma_n|$ are constant

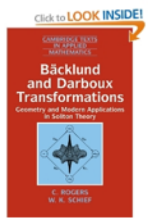
$$\dot{\gamma}_n = \cos\lambda\, T_n - \cos\lambda \tan\frac{\kappa_n}{2} N_n + \sin\lambda \tan\frac{\kappa_n}{2} B_n$$

then, the angle $\kappa_n = \angle(T_n, T_{n-1})$ obeys the semi-discrete mKdV equation $\dot{\kappa}_n = \frac{1}{a}\left(\tan\frac{\kappa_{n+1}}{2} - \tan\frac{\kappa_{n-1}}{2}\right)$

references (DDG of curves)

continuous

- Inoguchi (in Japanese)
- Rogers-Schief



semi-discrete

- Inoguchi-Kajiwara-M-Ohta, Explicit solutions to the semi-discrete modified KdV equation and motion of discrete plane curves, J. Phys. A: Math. Theor. 45 (2012), 045206 (16pp)

discrete

- Hoffmann, COE Lecture Note 18 (2009)
- Inoguchi-Kajiwara-M-Ohta, Motion and Bäcklund transformations of discrete plane curves, to appear in Kyushu J. Math.