Algebraic Formalisations of Fuzzy Relations and Their Representation Theorems

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https://doi.org/10.11501/3135054

出版情報：九州大学, 1997, 博士（理学）, 課程博士
バージョン：

Chapter 6

A Representation Theorem for Relation Algebras

In this chapter we consider relation algebras, which may not be Boolean, and provide their representation theorem. Relation algebras in the sense of this chapter are equivalent to Dedekind categories [Gog67] (or allegories [FS90]) with just one object. Section 5.1 proved a representation theorem for Dedekind categories, showing that a Dedekind category with a unit object satisfying the strict point axiom U1 is equivalent to a subcategory of the category of L-relations (where L is the lattice of all endomorphisms on the unit object). A unit object is an abstraction of singleton (or one-point) sets, and, following [Gog67], L-relations in section 2.2 are set-functions with values on a fixed complete distributive lattice L, that is, functions $R : X \times Y \to L$. The discussion in this chapter does not assume the existence of a unit object, and L-relations in this chapter are homogeneous relations on a set X, that is, functions $R : X \times X \to L$. This study is the first step to prove a representation theorem for Dedekind categories without unit objects.

To prove a representation theorem for relation algebras, we use concepts of scalar relations and point relations. The concept of scalar relations is an original one, which is defined in section 6.1 as a relation included in the identity relation and which commutes with the greatest relation with respect to composition. In the case of L-relations, scalar relations can be represented as scalar matrices. We use the concept of scalar relations to define a new concept of crisp relations which is called s-crisp different from that in [KF95, KFM96, Fur97a]. Also the set of all scalar relations is a complete distributive lattice, which is a sublattice of the relation algebra, and scalar relations represent membership values. The concept of point relations was introduced by Schmidt and Ströhlein in [SS85, SS93] in the context of applications of Boolean relation algebras to theories of graphs and programs, and it played an important rôle in proofs of representation theorems in [SS85, KF95, KFM96].

Section 6.1 provides definitions and some properties if scalar relations and s-crisp relations. In section 6.2 we define a "strict" point axiom by using our concepts of scalar relations and point relations. In section 6.3 we prove our representation theorem for relation algebras.

This chapter is based on [Fur97b].

6.1 Scalar relations

In this section we study a concept of scalar relations in a relation algebra $\mathcal{R}$. Note that relation algebras in this thesis which are defined in section 2.3 are not Boolean.

Throughout the chapter all discussions will assume a fixed relation algebra $\mathcal{R}$ with $\nabla \neq O$. All elements of the relation algebra $\mathcal{R}$ are called "relations" for short. A relation α is nonzero if $\alpha \neq O$.

First we provide some properties of relation algebras.

Proposition 6.1 *Let α, β, β' be relations. Then the following hold:*

(a) *If $\alpha^\sharp\alpha \sqsubseteq \mathrm{id}$, then $\alpha(\beta \sqcap \beta') = \alpha\beta \sqcap \alpha\beta'$.*

(b) *If $\alpha \sqsubseteq \mathrm{id}$ and $\beta \sqsubseteq \mathrm{id}$, then $\alpha^\sharp = \alpha\alpha = \alpha$ and $\alpha\beta = \alpha \sqcap \beta$.*

(c) *If $\beta \sqsubseteq \mathrm{id}$ and $\beta' \sqsubseteq \mathrm{id}$, then $\alpha(\beta \sqcap \beta') = \alpha\beta \sqcap \alpha\beta'$.*

Proof. (a) If $\alpha^\sharp\alpha \sqsubseteq \mathrm{id}$, then $\alpha\beta \sqcap \alpha\beta' \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\alpha\beta') \sqsubseteq \alpha(\beta \sqcap \mathrm{id}\beta') = \alpha(\beta \sqcap \beta')$ by the axiom R3.

(b) Assume that $\alpha \sqsubseteq \mathrm{id}$ and $\beta \sqsubseteq \mathrm{id}$. Then we have

$$\alpha = \alpha \sqcap \mathrm{id} \sqsubseteq \alpha(\mathrm{id} \sqcap \alpha^\sharp \mathrm{id}) \sqsubseteq \mathrm{id} \sqcap \alpha^\sharp \mathrm{id} \sqsubseteq \alpha^\sharp$$

by the axiom R3. Similarly it can be shown that $\alpha^\sharp \sqsubseteq \alpha$ holds. Also $\alpha\alpha \sqsubseteq \alpha$ is trivial, and it holds that

$$\alpha = \alpha \sqcap \nabla \sqsubseteq \alpha(\alpha \sqcap \alpha^\sharp\nabla) \sqsubseteq \alpha\alpha$$

by the axiom R3. Moreover, since $\alpha\beta \sqsubseteq \beta$, it holds that

$$\alpha\beta = \alpha\beta \sqcap \beta \sqsubseteq \alpha \sqcap \beta \quad \text{and} \quad \alpha \sqcap \beta \sqsubseteq \alpha(\mathrm{id} \sqcap \alpha^\sharp\beta) \sqsubseteq \alpha\beta$$

by the axiom R3.

(c) If $\beta \sqsubseteq \mathrm{id}$ and $\beta' \sqsubseteq \mathrm{id}$, then we have

$$\alpha\beta \sqcap \alpha\beta' \sqsubseteq (\alpha \sqcap \alpha\beta'\beta^\sharp)\beta \sqsubseteq \alpha\beta'\beta = \alpha(\beta \sqcap \beta')$$

by the axiom R3 and (b). ∎

Note that $\alpha(\sqcap_\lambda \beta_\lambda) \sqsubseteq \sqcap_\lambda(\alpha\beta_\lambda)$ and $\nabla\nabla = \nabla$ hold immediately by proposition 2.4(c).

The concepts of scalar relations and s-crisp relations in relation algebras are defined by the following:

Definition 6.1 Let $\mathcal{R}$ be a relation algebra.

(a) A relation k is called **scalar** if and only if $k \sqsubseteq \mathrm{id}$ and $k\nabla = \nabla k$.

(b) A relation α is called **s-crisp** (scalar crisp) if for all nonzero scalar relations k and all relations β, $k\beta \sqsubseteq \alpha$ implies $\beta \sqsubseteq \alpha$.

It is trivial that O and id are scalar relations, and that ∇ is s-crisp (but O and id are not necessarily s-crisp).

The concept of I-crisp relations has been defined in section 5.1 on the assumption of the existence of a unit object. The concept of crispness can also be found in section 3.1, where it is defined via semi-scalar multiplication. In this chapter we need neither a unit object, nor semi-scalar multiplication. Instead we used the concept of scalar relations to define s-crisp relations.

Next we provide some basic properties of scalar relations and s-crisp relations.

Proposition 6.2 *Let k be a scalar relation and α, β relations. Then the following holds:*

(a) $k\alpha = \alpha \sqcap k\nabla$ *and* $\alpha k = \alpha \sqcap \nabla k$*. In particular,* $k = \mathrm{id} \sqcap k\nabla$.

(b) $k\alpha = \alpha k$.

(c) $(k \sqcap k')\alpha = \alpha(k \sqcap k')$, $(k \sqcup k')\alpha = \alpha(k \sqcup k')$.

(d) *If* $k\nabla \sqsubseteq k'\nabla$*, then* $k \sqsubseteq k'$.

(e) *If α and β are s-crisp, then so is $\alpha \sqcap \beta$.*

Proof. (a) Since $k \sqsubseteq \mathrm{id}$ and $\alpha \sqsubseteq \nabla$, it holds that

$$k\alpha \sqsubseteq \alpha \sqcap k\nabla = k(k^\sharp\alpha \sqcap \nabla) = kk^\sharp\alpha = k\alpha$$

by the axion R3 and proposition 6.1(b). Similarly it can be shown that $\alpha k = \alpha \sqcap \nabla k$.
(b) From (a) it holds that $k\alpha = \alpha \sqcap k\nabla = \alpha \sqcap \nabla k = \alpha k$.
(c) It follows from

$$(k \sqcap k')\alpha = (kk')\alpha = \alpha(kk') = \alpha(k \sqcap k')$$

by proposition 6.1(b), and (b). Also it follows from

$$(k \sqcup k')\alpha = k\alpha \sqcup k'\alpha = \alpha k \sqcup \alpha k' = \alpha(k \sqcup k')$$

by proposition 2.4(b), and (b).

(d) Assume that $k\nabla \sqsubseteq k'\nabla$. Then $k = \mathrm{id} \sqcap k\nabla \sqsubseteq \mathrm{id} \sqcap k'\nabla = k'$ by (a).

(e) If $k\gamma \sqsubseteq \alpha \sqcap \beta$, then $k\gamma \sqsubseteq \alpha$ and $k\gamma \sqsubseteq \beta$ by the axiom R1. Since α and β are s-crisp, $\gamma \sqsubseteq \alpha$ and $\gamma \sqsubseteq \beta$. Thus $\gamma \sqsubseteq \alpha \sqcap \beta$ by the axiom R1. ∎

In addition to be used in the definition of s-crisp relations, scalar relations also play an important rôle in other respects. Let us denote the set of all scalar relations by L. Then L is closed under the operations supremum $\sqcup$ and infimum $\sqcap$ by proposition 6.2(c) and axiom R1. So the tuple $(L, \sqsubseteq, \sqcap, \sqcup, O, \mathrm{id})$ is a complete distributive lattice, and it is a sublattice of the relation algebra $\mathcal{R}$ with the least element O and the greatest element id.

6.2 Strict Point Axiom

This section introduces a new concept of point relations and a strict point axiom. A concept of point relations was introduced by Schmidt and Ströhlein in [SS85, SS93] to give a simple proof of a representation theorem for Boolean relation algebras and apply such algebras to computer science. We made the concept more strict in section 3.3 to prove a representation theorem for fuzzy relation algebras. The concept of point relations is defined in this chapter in the spirit of section 3.3, but we have to attend to the difference between the notions of crispness in section 3.3 and in this section.

Before define the concept of point relations, we describe properties of relations which correspond to vectors in [SS85, SS93].

Proposition 6.3 *Let α be a s-crisp relation such that $\nabla\alpha = \alpha$. Then the following three conditions are equivalent:*

(a) $\mathrm{id} \sqsubseteq \alpha\alpha^\sharp$.

(b) $\nabla = \alpha\alpha^\sharp$.

(c) $\nabla = \alpha\nabla$.

Proof. (a)$\Longrightarrow$(b) If $\mathrm{id} \sqsubseteq \alpha\alpha^\sharp$, then $\nabla = \nabla\mathrm{id} \sqsubseteq \nabla\alpha\alpha^\sharp = \alpha\alpha^\sharp$.

(b)$\Longrightarrow$(c) If $\nabla = \alpha\alpha^\sharp$, then $\nabla = \alpha\alpha^\sharp \sqsubseteq \alpha\nabla$.

(c)$\Longrightarrow$(a) If $\nabla = \alpha\nabla$, then $\mathrm{id} = \mathrm{id} \sqcap \nabla = \mathrm{id} \sqcap \alpha\nabla \sqsubseteq \alpha(\alpha^\sharp\mathrm{id} \sqcap \nabla) = \alpha\alpha^\sharp$. ∎

The concept of point relations in relation algebras is defined as follows:

Definition 6.2 **A point relation** x is a s-crisp relation such that $x^\sharp x \sqsubseteq \mathrm{id}$, $\mathrm{id} \sqsubseteq xx^\sharp$ and $\nabla x = x$. (Point relations will be denoted by lower case Roman letters such as $x, y, z, \cdots$.) The set of all point relations is denoted by X.

Note that a point relation x is nonzero from its totality $\mathrm{id} \sqsubseteq xx^\sharp$. For point relations x and y, the relation $x^\sharp y$ is nonzero since $y \sqsubseteq x(x^\sharp y)$ by the totality $\mathrm{id} \sqsubseteq xx^\sharp$ of x.

Proposition 6.4 *Let x, x_0, y, y_0 be point relations and k a nonzero scalar. Then the following holds:*

(a) *If $kx \sqsubseteq y$, then $x = y$.*

(b) *If $kx^\sharp y \sqsubseteq x_0^\sharp y_0$, then $x = x_0$ and $y = y_0$.*

Proof. (a) Since y is s-crisp, it holds that $x \sqsubseteq y$. Using $\mathrm{id} \sqsubseteq xx^\sharp$, $x^\sharp \sqsubseteq y^\sharp$ and $y^\sharp y \sqsubseteq \mathrm{id}$ we have $y \sqsubseteq xx^\sharp y \sqsubseteq xy^\sharp y \sqsubseteq x$.

(b) Assume that $kx^\sharp y \sqsubseteq x_0^\sharp y_0$. Then we have

$$ky = k\nabla y = k\nabla x^\sharp y = \nabla kx^\sharp y \sqsubseteq \nabla x_0^\sharp y_0 = y_0$$

by proposition 6.3 and so $y = y_0$ by (a). Similarly $x = x_0$ holds. ∎

By making use of our last definition of point relations in relation algebras, we add the following axiom:

Definition 6.3 A relation algebra $\mathcal{R}$ satisfies the **strict point axiom** iff:

R5. (a) For each nonzero relation α there are a nonzero scalar relation k and two point relations x and y such that $x\alpha y^\sharp = k\nabla$.

(b) $\sqcup_{x\in X} x^\sharp x = \mathrm{id}$.

Note that the condition (b) of the strict point axiom R5 is equivalent to $\sqcup_{x\in X} x = \nabla$. In what follows we assume that the fixed relation algebra $\mathcal{R}$ satisfies the strict point axiom R5.

Proposition 6.5 *Let α be a relation, x and y point relations. Then the following holds:*

(a) *If α is a nonzero relation, then there exist a nonzero scalar relation k and point relations x and y such that $kx^\sharp y \sqsubseteq \alpha$.*

(b) *If $x \neq y$, then $x \sqcap y = O$ and $xy^\sharp = O$.*

(c) *$x\alpha y^\sharp = k\nabla$ if and only if $\alpha \sqcap x^\sharp y = k(x^\sharp y)$.*

(d) *If $\alpha \sqsubseteq x^\sharp y$, then there exists a scalar relation k such that $\alpha = kx^\sharp y$.*

Proof. (a) If $\alpha \neq O$, then then there exist a nonzero scalar relation k and point relations x and y such that $x\alpha y^\sharp = k\nabla$ by the strict point axiom R5. Since x and y are point relations, it holds that

$$kx^\sharp y = kx^\sharp \nabla y = x^\sharp k \nabla y = x^\sharp x \alpha y^\sharp y \sqsubseteq \alpha$$

by proposition 6.2(b).

(b) Assume that $x \neq y$ and $x \sqcap y \neq O$. Then there exist a nonzero scalar relation k and point relations x_0 and y_0 such that $kx_0^\sharp y_0 \sqsubseteq x \sqcap y$ by (a). From proposition 6.4(e) $x \sqcap y$ is s-crisp, so it holds that $x_0^\sharp y_0 \sqsubseteq x \sqcap y$. Thus we have

$$y_0 = \nabla x_0^\sharp y_0 \sqsubseteq \nabla(x \sqcap y) \sqsubseteq \nabla x \sqcap \nabla y = x \sqcap y$$

by proposition 6.3. Therefore $x = y_0 = y$ by the axiom R1 and proposition 6.4(a). Finally, if $x \sqcap y = O$, then we have

$$xy^\sharp = xy^\sharp \sqcap \nabla \sqsubseteq (x \sqcap \nabla y)y^\sharp = (x \sqcap y)y^\sharp = O \ .$$

(c) Assume that $\alpha \sqcap x^\sharp y = k(x^\sharp y)$. Then it holds that

$$\begin{aligned} x\alpha y^\sharp &= x\alpha y^\sharp \sqcap \nabla \\ &= x\alpha y^\sharp \sqcap (xx^\sharp)(yy^\sharp) \\ &= x(\alpha \sqcap x^\sharp y)y^\sharp \\ &= x[k(x^\sharp y)]y^\sharp \\ &= k(xx^\sharp)(yy^\sharp) \\ &= k\nabla \end{aligned}$$

by propositions 6.3, 6.1(a) and 6.2(b). Next assume that $x\alpha y^\sharp = k\nabla$. Then we have

$$\begin{aligned} \alpha \sqcap x^\sharp y &\sqsubseteq x^\sharp(x\alpha y^\sharp \sqcap \mathrm{id})y \\ &\sqsubseteq x^\sharp x\alpha y^\sharp y \\ &= x^\sharp(k\nabla)y \\ &= k(x^\sharp y) \end{aligned}$$

by the axiom R3, propositions 6.3 and 6.2(b). Conversely, it holds that

$$k(x^\sharp y) = k(x^\sharp \nabla y) = x^\sharp k \nabla y = x^\sharp(x\alpha y^\sharp)y \sqsubseteq \alpha$$

by proposition 6.2(b). Thus we have $k(x^\sharp y) \sqsubseteq \alpha \sqcap x^\sharp y$.

(d) It is trivial that if $\alpha = O$ then $\alpha = O(x^\sharp y)$. Next assume that $\alpha \neq O$. Then, by the strict point axiom R5 and (c), there are a nonzero scalar relation k and point relations x_0, y_0 such that $\alpha \sqcap x_0^\sharp y_0 = k(x_0^\sharp y_0)$. Hence we have $k(x_0^\sharp y_0) \sqsubseteq \alpha \sqsubseteq x^\sharp y$, and so $x = x_0$ and $y = y_0$ by proposition 6.4(b), which implies $\alpha = k(x^\sharp y)$. ∎

By (d) of the last proposition, for every relation α and for every two point relations x, y there exists a scalar relation k such that

$$\alpha \sqcap x^\sharp y = k(x^\sharp y) \ ,$$

and so

$$x\alpha y^\sharp = k\nabla$$

by (c) of the last proposition. Also, by proposition 3.4(d), such a scalar relation k is unique. For a relation α and point relations x, y, we define $\psi(\alpha)(x, y)$ to be the unique scalar relation k with $x\alpha y^\sharp = k\nabla$. Thus, by proposition 3.4(d), $\psi(\alpha)(x, y)$ is the unique scalar relation such that

$$x\alpha y^\sharp = \psi(\alpha)(x, y)\nabla \ .$$

Therefore $\psi(\alpha)$ defines an L-relation on the set X of all point relations in $\mathcal{R}$ since the set L of all scalar relations is a complete distributive lattice.

6.3 Representation Theorem

First we prove a representation theorem for relation algebras satisfying the strict point axiom R5. The representation problem of Boolean relation algebras was proposed by Tarski in [Tar41] and investigated for a long time, see [SS85, SS93, Mad91a] for more details on the history of the investigation of the representation theorem for Boolean relation algebras. Also we proved an algebraic representation theorem of fuzzy relations in section 3.4, and proved such theorems for Dedekind categories (or allegories) and Zadeh categories in chapter 5. The following theorem also is a representation theorem for Dedekind categories with just one object.

Theorem 6.1 (Representation Theorem) Let $\mathcal{R}$ be a relation algebra satisfying the strict point axiom. Then every relation α has a unique representation

$$\alpha = \sqcup_{x,y\in X} x^\sharp \psi(\alpha)(x, y) y \ .$$

Proof. Since $\mathrm{id} = \sqcup_{x\in X} x^\sharp x$ and $\mathrm{id} = \sqcup_{y\in X} y^\sharp y$ by the strict point axiom R5, we have

$$\begin{aligned}
\alpha &= \mathrm{id}\alpha\mathrm{id} \\
&= (\sqcup_{x\in X} x^\sharp x)\alpha(\sqcup_{y\in X} y^\sharp y) \\
&= \sqcup_{x,y\in X} x^\sharp x\alpha y^\sharp y \\
&= \sqcup_{x,y\in X} x^\sharp \psi(\alpha)(x, y)\nabla y \\
&= \sqcup_{x,y\in X} x^\sharp \psi(\alpha)(x, y) y \ .
\end{aligned}$$

Finally we show the uniqueness of the representation. Assume that

$$\alpha = \sqcup_{x,y \in X} x^\sharp k_{x,y} y \ .$$

Then for all $x_0, y_0 \in X$ we have

$$\psi(\alpha)(x_0, y_0)\nabla = x_0 \alpha y_0^\sharp = \sqcup_{x,y \in X} x_0 x^\sharp k_{x,y} y y_0^\sharp = k_{x,y}$$

by proposition 6.5(b). ∎

From the last theorem we can deduce the next property of the function $\psi : \mathcal{R} \to L\text{-}\mathbf{Rel}(X)$.

Corollary 6.1 *For every relation algebra $\mathcal{R}$ satisfying the strict point axiom, the function $\psi : \mathcal{R} \to L\text{-}\mathbf{Rel}(X)$ is bijective.*

Proof. If $\psi(\alpha) = \psi(\beta)$, then by the last theorem we have

$$\begin{aligned} \alpha &= \sqcup_{x,y \in X} x^\sharp \psi(\alpha)(x,y) \nabla y \\ &= \sqcup_{x,y \in X} x^\sharp \psi(\beta)(x,y) \nabla y \\ &= \beta \ , \end{aligned}$$

which shows that ψ is injective. Given an L-relation $R \in L\text{-}\mathbf{Rel}(X)$, we set

$$\alpha_R = \sqcup_{x,y \in X} x^\sharp R(x,y) \nabla y \ .$$

Then by the uniqueness of the representation in the last theorem we have

$$R(x,y) = \psi(\alpha_R)(x,y) \ ,$$

which shows that ψ is surjective. ∎

The following proposition shows that $\psi : \mathcal{R} \to L\text{-}\mathbf{Rel}(X)$ preserves all operations of L-relations, that is, ψ is a homomorphism of relation algebras from $\mathcal{R}$ to $L\text{-}\mathbf{Rel}(X)$.

Proposition 6.6 *Let α, β be relations. Then the following holds:*

(a) $\psi(O) = 0_X$, $\psi(\nabla) = 1_X$ *and* $\psi(\mathrm{id}) = E_X$.

(b) *If* $\alpha \sqsubseteq \beta$, *then* $\psi(\alpha) \sqsubseteq \psi(\beta)$.

(c) $\psi(\alpha \sqcup \beta) = \psi(\alpha) \cup \psi(\beta)$,

(d) $\psi(\alpha \sqcap \beta) = \psi(\alpha) \cap \psi(\beta)$.

(e) $\psi(\alpha^\sharp) = \psi(\alpha)^\cup$.

(f) $\psi(\alpha\beta) = \psi(\alpha)\psi(\beta)$.

Proof. (a) The first follows from $\psi(O)(x,y)\nabla = xOy^\sharp = O\nabla$, the second follows from $\psi(\nabla)(x,y)\nabla = x\nabla y^\sharp = \mathrm{id}\nabla$ by proposition 6.3. Remarking $\psi(\mathrm{id})(x,y)\nabla = x\mathrm{id}y^\sharp = xy^\sharp$, the last follows from $\psi(\mathrm{id})(x,y)\nabla = \mathrm{id}\nabla$ if $x = y$ and $\psi(\mathrm{id})(x,y)\nabla = O\nabla$, otherwise by propositions 6.3 and 6.5(b).

(b) If $\alpha \sqsubseteq \beta$, then $\psi(\alpha)(x,y)\nabla = x\alpha y^\sharp \sqsubseteq x\beta y^\sharp = \psi(\beta)(x,y)\nabla$.

(c) It follows from

$$\begin{aligned}
\psi(\alpha \sqcup \beta)(x,y)\nabla &= x(\alpha \sqcup \beta)y^\sharp \\
&= x\alpha y^\sharp \sqcup x\beta y^\sharp \\
&= \psi(\alpha)(x,y)\nabla \sqcup \psi(\beta)(x,y)\nabla \\
&= [\psi(\alpha)(x,y) \sqcup \psi(\beta)(x,y)]\nabla \\
&= [\psi(\alpha) \cup \psi(\beta)](x,y)\nabla\ .
\end{aligned}$$

(d) It follows from

$$\begin{aligned}
\psi(\alpha \sqcap \beta)(x,y)\nabla &= x(\alpha \sqcap \beta)y^\sharp \\
&= x\alpha y^\sharp \sqcap x\beta y^\sharp \\
&= \psi(\alpha)(x,y)\nabla \sqcap \psi(\beta)(x,y)\nabla \\
&= [\psi(\alpha)(x,y) \sqcap \psi(\beta)(x,y)]\nabla \\
&= [\psi(\alpha) \cap \psi(\beta)](x,y)\nabla\ ,
\end{aligned}$$

by propositions 6.1(a) and 6.1(c) since x and y are point relations and

$$\psi(\alpha)(x,y), \psi(\beta)(x,y) \sqsubseteq \mathrm{id}\ .$$

(e) It follows from

$$\begin{aligned}
\psi(\alpha^\sharp)(x,y)\nabla &= x\alpha^\sharp y^\sharp \\
&= (y\alpha x^\sharp)^\sharp \\
&= (\psi(\alpha)(y,x)\nabla)^\sharp \\
&= \psi(\alpha)(y,x)\nabla \\
&= \psi(\alpha)^\cup(x,y)\nabla
\end{aligned}$$

since $\psi(\alpha)(y, x)$ is a scalar relation.

(f) It follows from

$$\begin{array}{rcl}\psi(\alpha\beta)(x,y)\nabla & = & x(\alpha\beta)y^\sharp \\ & = & x\alpha\mathrm{id}\beta y^\sharp \\ & = & x\alpha(\sqcup_{z\in X} z^\sharp z)\beta y^\sharp \\ & = & \sqcup_{z\in X} x\alpha z^\sharp z\beta y^\sharp \\ & = & \sqcup_{z\in X}\psi(\alpha)(x,z)\nabla\psi(\beta)(z,y)\nabla \\ & = & \sqcup_{z\in X}\psi(\alpha)(x,z)\psi(\beta)(z,y)\nabla \\ & = & \sqcup_{z\in X}[\psi(\alpha)(x,z)\sqcap\psi(\beta)(z,y)]\nabla \\ & = & (\psi(\alpha)\psi(\beta))(x,y)\nabla\ ,\end{array}$$

since $\psi(\alpha)(x, z)$ and $\psi(\beta)(z, y)$ are scalar relations. ∎

It is now obvious that ψ^{-1} is a function and is a homomorphism of algebras of L-relations from L-**Rel**(X) to $\mathcal{R}$. Consequently the following corollary is deduced:

Corollary 6.2 (Isomorphism Theorem) Every relation algebra $\mathcal{R}$ satisfying the strict point axiom is isomorphic to the algebra L-**Rel**(X) of L-relations on the set X of all point relations of $\mathcal{R}$, where L is the distributive lattice of scalar relations in $\mathcal{R}$.

In this chapter we proved a representation theorem for homogeneous relation algebras $\mathcal{R}$ satisfying the strict point axiom R5, which can be considered as Dedekind categories with just one object, using concepts of scalar relations and point relations. In section 5.1 such a theorem for Dedekind category was proved without using the concept of scalar relations. But in that section, the existence of the unit object was assumed to prove the theorem. The contribution of this chapter is so show that such a representation theorem can be proved without assuming the existence of a unit object, using instead our new algebraically defined concept of scalar relations.

Chapter 7

Crispness and Representation Theorem in Dedekind Categories

In this chapter we consider Dedekind categories named by Olivier and Serrato [OS95]. One of the aim of this chapter is to study notions of crispness and scalar relations in Dedekind categories. A notion of crispness was introduced in section 5.1 under the assumption that Dedekind categories have unit objects which are an abstraction of singleton (or one-point) sets. To capture the notion of crispness without such assumption, we use a notion of scalar relations. The notion of scalar relations in homogeneous relation algebras was introduced in section 6.1. The other aim of this chapter is to prove a representation theorem for Dedekind categories. Such a theorem for Dedekind categories with a unit object satisfying strict point axiom was also proved in section 5.1.

This chapter is organized as follows:

In section 7.1 we define a preoder among objects of Dedekind categories which compares the lattice structures on objects in a sense. Section 7.2 studies notions of scalars and crispness for Dedekind categories. The scalars on an object form a distributive lattice, which would be seen as the underlying lattice structure. In section 7.3 we recall the definition of L-relations, due to Goguen [Gog67], and illustrate a few relationships between crispness and lattice structures of scalars. In section 7.4 we show a representation theorem for uniform Dedekind categories satisfying the strict point axiom without the assumption of existence of unit objects, and it is proved that the representation function is a bijection preserving all operations of Dedekind categories.

7.1 Preorder among Objects of Dedekind Categories

In this section we provide a preorder among objects of Dedekind categories which compares the lattice structures on objects in a sense.

First, we define a function $\phi_W : \mathcal{D}(X,Y) \to \mathcal{D}(W,W)$ by

$$\phi_W(\xi) = \nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W : W \to W$$

for a morphism $\xi : X \to Y$ and an object W of a Dedekind category $\mathcal{D}$. This function is related to scalars; the relationship will be described in the next section, and the following lemma holds:

Lemma 7.1 (a) *$\phi_W(\xi)\nabla_{WZ} = \nabla_{WX}\xi\nabla_{YZ}$ and $\nabla_{ZW}\phi_W(\xi) = \nabla_{ZX}\xi\nabla_{YW}$ for each object Z.*

(b) $\phi_W(\phi_X(\xi)) = \phi_W(\phi_Y(\xi)) = \phi_W(\xi)$.

(c) $\phi_W(\xi) = \phi_W(\xi^\sharp)$.

(d) *If $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, then $\xi \sqsubseteq \nabla_{XW}\phi_W(\xi)\nabla_{WY}$.*

(e) *If $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, then $\phi_W(\xi) = O_{WW}$ is equivalent to $\xi = O_{XY}$.*

Proof. (a) The former follows from

$$\begin{aligned}
\phi_W(\xi)\nabla_{WZ} &= (\nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W)\nabla_{WZ} \\
&\sqsubseteq \nabla_{WX}\xi\nabla_{YW}\nabla_{WZ} \\
&\sqsubseteq \nabla_{WX}\xi\nabla_{YZ} \\
&= \nabla_{WX}\xi\nabla_{YZ} \sqcap \nabla_{WZ} \\
&\sqsubseteq (\nabla_{WX}\xi\nabla_{YZ}\nabla_{WZ}^\sharp \sqcap \mathrm{id}_W)\nabla_{WZ} \\
&\sqsubseteq (\nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W)\nabla_{WZ} \\
&= \phi_W(\xi)\nabla_{WZ} \ .
\end{aligned}$$

The latter is similar.

(b) follows from

$$
\begin{aligned}
\phi_W(\phi_X(\xi)) &= \nabla_{WX}\phi_X(\xi)\nabla_{XW} \sqcap \mathrm{id}_W && (\text{ Definition of } \phi_W \text{ }) \\
&= \nabla_{WX}\nabla_{XX}\xi\nabla_{YW} \sqcap \mathrm{id}_W && (\ \phi_X(\xi)\nabla_{XW} = \nabla_{XX}\xi\nabla_{YW}\ ) \\
&= \nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W && (\ \nabla_{WX}\nabla_{XX} = \nabla_{WX}\ ) \\
&= \phi_W(\xi) && (\text{ Definition of } \phi_W \text{ })
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_W(\phi_Y(\xi)) &= \nabla_{WY}\phi_Y(\xi)\nabla_{YW} \sqcap \mathrm{id}_W && (\text{ Definition of } \phi_W \text{ }) \\
&= \nabla_{WX}\xi\nabla_{YY}\nabla_{YW} \sqcap \mathrm{id}_W && (\ \nabla_{WX}\phi_Y(\xi) = \nabla_{WX}\xi\nabla_{YY}\ ) \\
&= \nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W && (\ \nabla_{YY}\nabla_{YW} = \nabla_{YW}\ ) \\
&= \phi_W(\xi) && (\text{ Definition of } \phi_W \text{ })\ .
\end{aligned}
$$

$$
\begin{tikzcd}
\mathcal{D}(X,Y) \arrow[r, "\phi_X"] \arrow[d, "\phi_Y"'] & \mathcal{D}(X,X) \arrow[d, "\phi_W"] \\
\mathcal{D}(Y,Y) \arrow[r, "\phi_W"'] & \mathcal{D}(W,W)
\end{tikzcd}
$$

(c) follows from

$$
\begin{aligned}
\phi_W(\xi^\sharp) &= (\phi_W(\xi^\sharp))^\sharp \\
&= (\nabla_{WY}\xi^\sharp\nabla_{XW} \sqcap \mathrm{id}_W)^\sharp \\
&= \nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W \\
&= \phi_W(\xi)\ .
\end{aligned}
$$

(d) If $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, then

$$
\begin{aligned}
\xi &= \xi \sqcap \nabla_{XY} \\
&= \xi \sqcap \nabla_{XW}\nabla_{WY} \\
&\sqsubseteq \nabla_{XW}(\nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W)\nabla_{WY} \\
&= \nabla_{XW}\phi_{XYW}(\xi)\nabla_{WY}\ .
\end{aligned}
$$

(e) is immediate from (d). ∎

A binary relation $\prec$ among objects of $\mathcal{D}$ is defined as follows: For two objects X and Y, the relation $X \prec Y$ holds if and only if $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$. (Note that the three conditions $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$, $\mathrm{id}_X \sqsubseteq \nabla_{XY}\nabla_{YX}$ and $\phi_X(\mathrm{id}_Y) = \mathrm{id}_X$ are mutually equivalent.) It is easy to see that $\prec$ is a preorder, that is, reflexive and transitive. For $\nabla_{XX} = \nabla_{XX}\nabla_{XX}$, and if $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$ and $\nabla_{YY} = \nabla_{YZ}\nabla_{ZY}$, then

$$\nabla_{XX} = \nabla_{XY}\nabla_{YY}\nabla_{YX} = \nabla_{XY}\nabla_{YZ}\nabla_{ZY}\nabla_{YX} \sqsubseteq \nabla_{XZ}\nabla_{ZX}\ .$$

Hence its symmetric kernel with $X \sim Y$ if and only if $X \prec Y$ and $Y \prec X$, is an equivalence relation. Remark that in the category Rel_0 of example 2.1, two distinct objects are never equivalent.

Proposition 7.1 *Assume that $X \prec Y$. If $u \sqsubseteq \mathrm{id}_X$, $u \sqsubseteq \mathrm{id}_X$ and $u\nabla_{XY} \sqsubseteq v\nabla_{XY}$ for $u, v : X \rightharpoonup X$, then $u \sqsubseteq v$.*

Proof. It follows from $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$ that $u = \mathrm{id}_X \sqcap u\nabla_{XX} = \mathrm{id}_X \sqcap u\nabla_{XY}\nabla_{YX}$. ∎

Definition 7.1 A Dedekind category $\mathcal{D}$ is **uniform** if all pairs of objects of $\mathcal{D}$ are equivalent, that is, if $X \sim Y$ for all objects X and Y of $\mathcal{D}$.

A morphism $f : X \rightharpoonup Y$ such that $f^\sharp f \sqsubseteq \mathrm{id}_Y$ (*univalent*) and $\mathrm{id}_X \sqsubseteq ff^\sharp$ (*total*) is called a *function* and may be introduced as $f : X \to Y$.

Proposition 7.2 (a) *If there exists at least one total morphism $\alpha : X \rightharpoonup Y$, then $X \prec Y$.*

(b) *If there exists at least one function $f : X \to Y$, then $X \prec Y$.*

(c) *If $X \prec W$ or $Y \prec W$, then $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$.*

(d) *If $X \prec Y$ and $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, then $X \prec W$.*

(e) *If $\nabla_{XY} = p^\sharp q$ for some functions $p : W \to X$ and $q : W \to Y$ and if $X \prec Y$, then $X \sim W$.*

Proof. (a) Assume that α is total, then we have $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp \sqsubseteq \nabla_{XY}\nabla_{YX}$.

(b) It is a just corollary of (a).

(c) If $\nabla_{XX} = \nabla_{XW}\nabla_{WX}$, then $\nabla_{XY} = \nabla_{XX}\nabla_{XY} = \nabla_{XW}\nabla_{WX}\nabla_{XY} \sqsubseteq \nabla_{XW}\nabla_{WY}$.

(d) $\nabla_{XX} = \nabla_{XY}\nabla_{YX} = \nabla_{XW}\nabla_{WY}\nabla_{YX} \sqsubseteq \nabla_{XW}\nabla_{WX}$.

(e) First note that $W \prec X$ by (a). Next $\nabla_{XY} = p^\sharp q \sqsubseteq \nabla_{XW}\nabla_{WX}$ and so it follows from (d) that $X \prec W$. ∎

7.2 Scalars and Crispness

We now introduce the two notions of scalars and of s-crisp relations as a preparation for defining a concept of points with a separation property, that is, different points never meet.

Definition 7.2 A **scalar** k on X is a morphism $k : X \rightharpoonup X$ of $\mathcal{D}$ such that $k \sqsubseteq \mathrm{id}_X$ and $k\nabla_{XX} = \nabla_{XX}k$.

A scalar k on X commutes with all endomorphisms $\alpha : X \rightharpoonup X$, that is, $k\alpha = \alpha k$, because

$$k\alpha = \alpha \sqcap k\nabla_{XX} = \alpha \sqcap \nabla_{XX}k = \alpha k \ .$$

It is trivial that the zero morphism $0_{XX} : X \rightharpoonup X$ and the identity morphism $\mathrm{id}_X : X \rightharpoonup X$ are scalars on X. The set of all scalars on X is denoted by $\mathcal{F}(X)$. It is clear that $\mathcal{F}(X)$ is a complete distributive lattice for all objects X. A morphism $\xi : X \rightharpoonup Y$ is called an *ideal* if $\nabla_{XX}\xi\nabla_{YY} = \xi$. The notion of ideals in relation algebras was initially introduced by Jónsson and Tarski [JT52]. The following lemma shows that scalars bijectively correspond to ideals.

Lemma 7.2 (a) *If $\iota : X \rightharpoonup X$ is an ideal, then $k = \iota \sqcap \mathrm{id}_X$ is a scalar on X such that $\iota = k\nabla_{XX}$.*

(b) *If k is a scalar on X, then $\iota = k\nabla_{XX}$ is an ideal such that $k = \iota \sqcap \mathrm{id}_X$.*

Proof. (a) Assume that ι is an ideal on an object X, then we have

$$(\iota \sqcap \mathrm{id}_X)\nabla_{XX} \sqsubseteq \iota\nabla_{XX} = \iota = \iota \sqcap \mathrm{id}_X\nabla_{XX} \sqsubseteq (\iota\nabla_{XX}^\sharp \sqcap \mathrm{id}_X)\nabla_{XX} = (\iota \sqcap \mathrm{id}_X)\nabla_{XX} \ ,$$

and so $(\iota \sqcap \mathrm{id}_X)\nabla_{XX} = \iota = \nabla_{XX}(\iota \sqcap \mathrm{id}_X)$.
(b) Assume that k is a scalar on an object X, then we have

$$\nabla_{XX}(k\nabla_{XX})\nabla_{XX} = k\nabla_{XX}\nabla_{XX}\nabla_{XX} = k\nabla_{XX}$$

and

$$k = k\mathrm{id}_X = k = k\nabla_{XX} \sqcap \mathrm{id}_X \ .$$

∎

Proposition 7.3 *Let $\xi : X \rightharpoondown Y$ be a morphism. Then the following holds:*

(a) *$\phi_W(\xi)$ is a scalar on W.*

(b) *If $X \prec Y$, then $\phi_X(\phi_Y(k)) = k$ for all scalars $k \in \mathcal{F}(X)$.*

(c) *If $X \sim Y$, then $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ are isomorphic as lattices.*

(d) *$\phi_X(k)\xi = \xi\phi_Y(k)$ for all scalars k on W.*

(e) *If $\xi \neq O_{XY}$, then there is a nonzero scalar $k \in \mathcal{F}(X)$ such that $\nabla_{XX}\xi\nabla_{YY} = k\nabla_{XY}$.*

Proof. (a) Set $W = Z$ in lemma 7.1(a). Then $\phi_W(\xi)\nabla_{WW} = \nabla_{WX}\xi\nabla_{YW} = \nabla_{WW}\phi_W(\xi)$.

(b) First note that $\phi_Y(k)\nabla_{YX} = \nabla_{YX}k\nabla_{XX}$ by lemma 7.1(a) and so

$$\begin{aligned}
\nabla_{XY}\phi_Y(k)\nabla_{YX} &= \nabla_{XY}\nabla_{YX}k\nabla_{XX} \\
&= \nabla_{XX}k\nabla_{XX} && (\text{by } \nabla_{XX} = \nabla_{XY}\nabla_{YX}\ ) \\
&= k\nabla_{XX} && (\text{since } k \text{ is a scalar }) \ .
\end{aligned}$$

Hence we have

$$\begin{aligned}
\phi_X(\phi_Y(k)) &= \nabla_{XY}\phi_Y(k)\nabla_{YX} \sqcap \mathrm{id}_X \\
&= k\nabla_{XX} \sqcap \mathrm{id}_X \\
&= k \ .
\end{aligned}$$

(c) It is obvious from (b).

(d) By lemma 7.1(a) we have $\phi_X(k)\nabla_{XY} = \nabla_{XW}k\nabla_{WY} = \nabla_{XY}\phi_Y(k)$ and consequently $\phi_X(k)\alpha = \alpha \sqcap \phi_X(k)\nabla_{XY} = \alpha \sqcap \nabla_{XY}\phi_Y(k) = \alpha\phi_Y(k)$.

(e) Set $k = \phi_X(\xi)$. Then it is clear that k is a scalar on X by (a) and $\nabla_{XX}\xi\nabla_{YY} = k\nabla_{XY}$ by lemma 7.1(a). And k is nonzero by lemma 7.1(d), since ξ is nonzero. (Cf. [KFM96, Theorem 5.4])

∎

From the above lemma 7.1(a) we have ϕ_W as a mapping $\phi_W : \mathcal{D}(X, Y) \to \mathcal{F}(W)$.

Fact 7.1

$$\begin{aligned}\phi_W(\phi_X(\xi)) &= \nabla_{WX}\phi_X(\xi)\nabla_{XW}\sqcap \mathrm{id}_W && (\text{Definition of } \phi_W)\\ &= \nabla_{WX}\xi\nabla_{YW}\sqcap \mathrm{id}_W && (\phi_X(\xi)\nabla_{XW} = \nabla_{XX}\xi\nabla_{YW})\\ &= \nabla_{WX}\xi\nabla_{YX}\nabla_{XW}\sqcap \mathrm{id}_W && (\nabla_{WX}\phi_X(\xi) = \nabla_{WX}\xi\nabla_{YX})\end{aligned}$$

and

$$\begin{aligned}\phi_W(\phi_Y(\xi)) &= \nabla_{WY}\phi_Y(\xi)\nabla_{YW}\sqcap \mathrm{id}_W && (\text{Definition of } \phi_W)\\ &= \nabla_{WX}\xi\nabla_{YW}\sqcap \mathrm{id}_W && (\nabla_{WY}\phi_Y(\xi) = \nabla_{WX}\xi\nabla_{YY})\\ &= \nabla_{WY}\nabla_{YX}\xi\nabla_{YW}\sqcap \mathrm{id}_W && (\phi_Y(\xi)\nabla_{YX} = \nabla_{YX}\xi\nabla_{YX})\ .\end{aligned}$$

In particular, the following holds for $\xi = \nabla_{XY}$:

$$\begin{aligned}\nabla_{WX}\nabla_{XY}\nabla_{YW}\sqcap \mathrm{id}_W &= \nabla_{WX}\nabla_{XY}\nabla_{YX}\nabla_{XW}\sqcap \mathrm{id}_W\\ &= \nabla_{WY}\nabla_{YX}\nabla_{XY}\nabla_{YW}\sqcap \mathrm{id}_W\ .\end{aligned}$$

The Tarski rule for Boolean relation algebras are introduced by Tarski [JT52, SS85, SS93, Tar41]. A Boolean relation algebra which satisfies Tarski rule has no ideal except for the zero relation and the universal relation. The next proposition corresponds to the suggestion.

Proposition 7.4 *If the Tarski rule holds in $\mathcal{D}$, that is, all nonzero morphisms $\alpha : X \rightarrow X$ satisfy $\nabla_{XX}\alpha\nabla_{XX} = \nabla_{XX}$, then there is no scalar on X except for the zero morphism O_{XX} and the identity id_X.*

Proof. Let k be a nonzero scalar on X. Then, by the Tarski rule, we have

$$k\nabla_{XX} = k\nabla_{XX}\nabla_{XX} = \nabla_{XX}k\nabla_{XX} = \nabla_{XX}\ ,$$

which means that k is total, and so $\mathrm{id}_X \sqsubseteq kk^\sharp = k$ by $k \sqsubseteq \mathrm{id}_X$. ∎

By using the notion of scalar, we define a crispness which called s-crispness (scalar crispness).

Definition 7.3 A morphism $\alpha : X \rightarrow Y$ is **s-crisp** if $k\tau \sqsubseteq \alpha$ implies $\tau \sqsubseteq \alpha$ for all nonzero scalars $k : X \rightarrow X$ and all morphisms $\tau : X \rightarrow Y$.

It is trivial from the above definition that every universal morphism ∇_{XY} is s-crisp.

Proposition 7.5 (a) *A morphism is s-crisp if and only if its converse is s-crisp.*

(b) *The infimum of two s-crisp morphisms is s-crisp.*

(c) *If $f : X \to Y$ is a function and a morphism $\beta : Y \rightharpoonup Z$ is s-crisp, then the composite $f\beta : X \rightharpoonup Z$ is s-crisp.*

(d) *If the identity id_Y is s-crisp, then so are all functions $f : X \to Y$.*

(e) *A morphism $\alpha : X \rightharpoonup Y$ is s-crisp if and only if its relative pseudo-complement $\alpha' \Rightarrow \alpha$ is s-crisp for every morphism $\alpha' : X \rightharpoonup Y$.*

Proof. (a) Assume that $\alpha : X \rightharpoonup Y$ is s-crisp and $k\tau \sqsubseteq \alpha^\sharp$ for a nonzero scalar k on Y and a morphism $\tau : Y \rightharpoonup X$. Then $\phi_X(k)\tau^\sharp = \tau^\sharp k = (k\tau)^\sharp \sqsubseteq (\alpha^\sharp)^\sharp = \alpha$ and so $\tau^\sharp \sqsubseteq \alpha$, since $\phi_X(k)$ is a nonzero scalar on X by lemma 7.1(e). Hence $\tau \sqsubseteq \alpha^\sharp$.

(b) Assume that $\alpha_i : X \rightharpoonup Y$ is s-crisp for $i = 0$ or 1 and $k\tau \sqsubseteq \alpha_0 \sqcap \alpha_1$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoonup Y$. Then we have $k\tau \sqsubseteq \alpha_0$ and $k\tau \sqsubseteq \alpha_1$, and so $\tau \sqsubseteq \alpha_0$ and $\tau \sqsubseteq \alpha_1$ by s-crispness. Hence $\tau \sqsubseteq \alpha_0 \sqcap \alpha_1$.

(c) Assume that $k\tau \sqsubseteq f\beta$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoonup Z$. First note that $\phi_Y(k)$ is a nonzero scalar by lemma 7.1(e) and $\phi_Y(k)f^\sharp = f^\sharp k$ by proposition 7.3(d). Then we have

$$\phi_Y(k)f^\sharp\tau = f^\sharp k\tau \sqsubseteq f^\sharp f\beta \sqsubseteq \beta$$

and so $f^\sharp\tau \sqsubseteq \beta$ by the s-crispness of β. Therefore $\tau \sqsubseteq ff^\sharp\tau \sqsubseteq f\beta$, which completes the proof.

(d) is a special case of (b).

(e) First assume that $\alpha : X \rightharpoonup Y$ is s-crisp and $k\tau \sqsubseteq \alpha' \Rightarrow \alpha$ for a nonzero scalar k and morphisms $\tau, \alpha' : X \rightharpoonup Y$. Then we have

$$k(\tau \sqcap \alpha') = k\tau \sqcap \alpha' \sqsubseteq \alpha$$

and so $\tau \sqcap \alpha' \sqsubseteq \alpha$, since $\alpha : X \rightharpoonup Y$ is s-crisp. Therefore $\tau \sqsubseteq \alpha' \Rightarrow \alpha$. Conversely, if $\alpha' \Rightarrow \alpha$ is s-crisp for all morphisms $\alpha' : X \rightharpoonup Y$, then $\alpha = \nabla_{XY} \Rightarrow \alpha$ is s-crisp. This completes the proof. ∎

It immediately follows from the last proposition 7.5(c) that every composite of s-crisp functions is also an s-crisp function.

A morphism $\alpha : X \rightharpoonup Y$ is *complemented* if it has a complement morphism $\overline{\alpha} : X \rightharpoonup Y$ such that $\alpha \sqcup \overline{\alpha} = \nabla_{XY}$ and $\alpha \sqcap \overline{\alpha} = O_{XY}$.

Theorem 7.1 *The following four statements are equivalent:*

(a) *If $k \neq O_{XX}$ and $k \sqcap k' = O_{XX}$ for scalars $k, k' \in \mathcal{F}(X)$, then $k' = O_{XX}$.*

(b) *The zero morphism O_{XY} is s-crisp for every object Y (that is, if $k\tau = O_{XY}$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoonup Y$, then $\tau = O_{XY}$).*

(c) *For every morphism $\alpha : X \rightharpoonup Y$, its pseudo-complement $\neg\alpha : X \rightharpoonup Y$ is s-crisp.*

(d) *Every complemented morphism $\alpha : X \rightharpoonup Y$ is s-crisp.*

Proof. (a)$\Longrightarrow$(b) Assume that $k\tau = O_{XY}$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoonup Y$. Recall that $\phi_X(\tau)$ is a scalar on X. Hence we have

$$\begin{aligned} k \sqcap \phi_X(\tau) = k\phi_X(\tau) &= k(\nabla_{XX}\tau\nabla_{YX} \sqcap \mathrm{id}_X) \\ &\sqsubseteq k\nabla_{XX}\tau\nabla_{XY} \\ &= \nabla_{XX}k\tau\nabla_{YX} \\ &= O_{XX} \ . \end{aligned}$$

It follows from (a) that $\phi_X(\tau) = O_{XX}$ and so $\tau = O_{XY}$ by lemma 7.1(e). Hence O_{XY} is s-crisp.

(b)$\Longrightarrow$(a) is trivial.

(b) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d) is a corollary of the last lemma. ∎

Definition 7.4 A scalar k on X is called **linear** if and only if for every scalar k' on X an equation $k \sqcap k' = O_{XX}$ implies $k' = O_{XX}$.

Let $\mathcal{W}(X)$ denote the set of all linear scalars on X. Every identity id_X is obviously linear. Note that a scalar k on X is linear if and only if its pseudo-complement $\neg k$ $(= \mathrm{id}_X \sqcap (k \Rightarrow O_{XX}))$ in $\mathcal{F}(X)$ is equal to O_{XX}.

Lemma 7.3 *If X is a nonempty object, then $\mathcal{W}(X)$ is a filter of $\mathcal{F}(X)$.*

Proof. 0) It is trivial that O_{XX} is not a linear scalar, whenever X is nonempty.

i) If $k_0, k_1 \in \mathcal{W}(X)$, then $k_0 \sqcap k_1 \in \mathcal{W}(X)$: Assume $(k_0 \sqcap k_1) \sqcap k' = O_{XX}$. Then $k_0 \sqcap (k_1 \sqcap k') = O_{XX}$ and so $k_1 \sqcap k' = O_{XX}$, which shows $k' = O_{XX}$.

ii) If $k_0 \in \mathcal{W}(X)$ and $k_1 \in \mathcal{F}(X)$ with $k_0 \sqsubseteq k_1$, then $k_1 \in \mathcal{W}(X)$: Assume $k_1 \sqcap k' = O_{XX}$. Then $k_0 \sqcap k' = O_{XX}$ and so $k' = O_{XX}$. ∎

So the set of linear scalars on X is a sublattice of the lattice $\mathcal{F}(X)$ of all scalars on X, and as such it is distributive.

Definition 7.5 A morphism $\alpha : X \rightharpoonup Y$ is **l-crisp** if $k\tau \sqsubseteq \alpha$ implies $\tau \sqsubseteq \alpha$ for all linear scalars $k : X \rightharpoonup X$ and all morphisms $\tau : X \rightharpoonup Y$.

Proposition 7.6 *Every zero morphism O_{XY} is l-crisp.*

Proof. Assume that $k\tau = O_{XY}$ for a linear scalar on X and a morphism $\tau : X \rightharpoonup Y$. Then we have

$$\begin{array}{rcl} k \sqcap \phi_X(\tau) & = & k\phi_X(\tau) \\ & = & k(\nabla_{XX}\tau\nabla_{YX} \sqcap \mathrm{id}_X) \\ & \sqsubseteq & k\nabla_{XX}\tau\nabla_{YX} \\ & \sqsubseteq & \nabla_{XX}k\tau\nabla_{YX} \\ & = & O_{XY} \end{array}$$

and so $\phi_X(\tau) = O_{XX}$. Hence $\tau = O_{XY}$ by lemma 7.1(e). ∎

7.3 Crispness in L-Relations

Obviously an L-relation $k : X \rightharpoonup X$ is a scalar on X if and only if

$$\forall x, x' \in X : k(x, x) = k(x', x') \text{ and } x \neq x' \Longrightarrow k(x, x') = 0 \ .$$

An L-relation $R : X \rightharpoonup Y$ is called *0-1* crisp [Gog67] if $R(x, y) = \mathbf{0}$ or $R(x, y) = \mathbf{1}$ for all $(x, y) \in X \times Y$. Of course the zero relation 0_{XY}, the universal relation 1_{XY} and the identity relation E_X are 0-1 crisp. For a 0-1 crisp L-relation $R : X \rightharpoonup Y$ define an L-relation $\overline{R} : X \rightharpoonup Y$ by $\overline{R}(x, y) = \mathbf{0}$ if $R(x, y) = \mathbf{1}$ and $\overline{R}(x, y) = \mathbf{1}$ otherwise. Then $R \cup \overline{R} = 1_{XY}$ and $R \cap \overline{R} = 0_{XY}$. This fact means that all 0-1 crisp L-relations are complemented.

Proposition 7.7 *All s-crisp L-relations are 0-1 crisp.*

Proof. Let an L-relation $R : X \rightharpoonup Y$ be s-crisp. Assume that $a = R(x_0, y_0)$ is not equal to $\mathbf{0} \in L$ for some point $(x_0, y_0) \in X \times Y$. Consider a scalar k on X such that $k(x, x') = a$ if $x = x'$ and $k(x, x') = \mathbf{0}$ otherwise, and an L-relation $T : X \rightharpoonup Y$ such that $T(x, y) = a \Rightarrow R(x, y)$ for all $(x, y) \in X \times Y$. Then we have $kT \sqsubseteq R$, since

$$(kT)(x, y) = a \wedge (a \Rightarrow R(x, y)) \leq R(x, y)$$

for all $(x, y) \in X \times Y$. Hence $T \sqsubseteq R$ follows from the fact that $R : X \rightharpoonup Y$ is s-crisp. Finally we have $\mathbf{1} = (a \Rightarrow a) = T(x_0, y_0) \leq R(x_0, y_0)$, which shows R is 0-1 crisp. ∎

The converse of the last proposition does not hold in general. Its necessary and sufficient condition is given by the following:

Proposition 7.8 *For L-relations the following statements are equivalent:*

C0. $\forall a, b \in L : a \wedge b = \mathbf{0} \Longrightarrow a = \mathbf{0}$ *or* $b = \mathbf{0}$.

K0. All 0-1 crisp L-relations are s-crisp.

Proof. First assume that C0 and $kT \sqsubseteq R$ for a scalar k on X, an L-relation $T : X \rightharpoonup Y$, and a 0-1 crisp L-relation $R : X \rightharpoonup Y$. To prove that R is s-crisp we have to show that $T(x, y) \leq R(x, y)$ for all $(x, y) \in X \times Y$. Since $R(x, y) = \mathbf{0}$ or $\mathbf{1}$ by the 0-1 crispness of R it is enough to show that if $R(x, y) = \mathbf{0}$ then $T(x, y) = \mathbf{0}$. But

$(kT)(x,y) = k(x,x) \wedge T(x,y) \leq R(x,y)$. Hence when $R(x,y) = \mathbf{0}$, we have $T(x,y) = \mathbf{0}$ from C0 and $k(x,x) \neq \mathbf{0}$. Conversely assume that K0 and $a \wedge b = \mathbf{0}$ for $a, b \in L$. Define a scalar k on a singleton set $I = \{*\}$ and an L-relation $R : I \rightharpoonup I$ by $k(*,*) = a$ and $T(*,*) = b$, respectively. Then $kT = 0_{II}$ and so $k = 0_{II}$ or $T = 0_{II}$ since 0_{II} is s-crisp by the assumption K0. ∎

Proposition 7.9 *For L-relations the following statements are equivalent:*

C1. $\forall a, b \in L : a \wedge b = \mathbf{0}$ *and* $a \vee b = \mathbf{1} \Longrightarrow a = \mathbf{0}$ *or* $b = \mathbf{0}$.

K1. All complemented L-relations are 0-1 crisp.

K2. All L-relations which are functions are 0-1 crisp.

Proof. Trivial. ∎

Definition 7.6 An element x of a lattice L is called **linear** if $x \wedge y = \mathbf{0}$ implies $y = \mathbf{0}$ for $y \in L$.

Let $k : X \rightharpoonup X$ be an L-relation on a nonempty set X. If k is a linear scalar, then $k(x,x)$ is linear in L for all $x \in X$.

Assume that $k(x,x) \wedge a = \mathbf{0}$ for $a \in L$. Now consider a scalar $k' : X \rightharpoonup X$ such that $k'(x,x') = a$ if $x = y$, and $k'(x,x') = \mathbf{0}$ otherwise. Then $k \sqcap k' = 0_{XX}$ and so $k' = 0_{XX}$ by the linearity of k. Hence $a = \mathbf{0}$, which proves that $k(x,x)$ is linear.

Proposition 7.10 *All 0-1 crisp L-relations are l-crisp.*

Proof. Let an L-relation $R : X \rightharpoonup Y$ be 0-1 crisp and assume that $kT \sqsubseteq R$ for a linear scalar k on X and an L-relation $T : X \rightharpoonup Y$. We have to show that $T(x,y) \leq R(x,y)$ for all $(x,y) \in X \times Y$. Now $k(x,x) \wedge T(x,y) \leq R(x,y) = (kT)(x,y) \subseteq R(x,y)$, and since $k(x,x)$ is linear, it follows that $R(x,y) = \mathbf{0}$ implies $T(x,y) = \mathbf{0}$, which is sufficient since $R(x,y)$ can only be $\mathbf{0}$ or $\mathbf{1}$ by 0-1 crispness. ∎

The converse of the above proposition does not hold: Consider a Boolean lattice L having a nontrivial element s such that $s \neq \mathbf{0}$ and $s \neq \mathbf{1}$, and define an L-relation $R_s : X \rightharpoonup X$ by $R(x, x') = s$ if $x = x'$ and $R(x, x') = \mathbf{0}$ otherwise. Then it is clear that R_s is l-crisp, but not 0-1 crisp. Generally for a Boolean lattice L every L-relation is l-crisp since the identity E_X is a unique linear scalar on X.

7.4 Representation Theorem

In this section we first introduce the concept of points in Dedekind categories. Then some useful properties on points, due to Schmidt and Ströhlein [SS85], and a point axiom will be stated to show a representation theorem in uniform Dedekind categories. In particular, the point axiom induces a function assigning a concrete L-relation between the sets of point relations to an abstract relation in Dedekind categories. In view of [Fur97b, KF95, SS85] the concept of points in Dedekind categories is defined as follows:

Definition 7.7 Let $\mathcal{D}$ be a Dedekind category. A **point** x of X is an s-crisp function $x : X \rightarrow X$ such that $\nabla_{XX} x = x$.

We will denote the set of all points of X by $\chi(X)$.

Lemma 7.4 *Let x and x' be points of X. Then the following holds:*

(a) *If $\nabla_{XX}\rho = \rho$ and $\rho \sqsubseteq x$ for a morphism $\rho : X \rightharpoonup X$, then $\rho = kx$ for a unique scalar k on X.*

(b) *If $x \neq x'$, then $x \sqcap x' = 0_{XX}$ and $xx'^{\sharp} = 0_{XX}$.*

Proof. (a) First set $k = \phi_X(\rho x^{\sharp})$. Then by proposition 7.3(a) k is a scalar on X, and $k = \rho x^{\sharp} \sqcap \mathrm{id}_X$ from $\nabla_{XX} x = x$ and $\nabla_{XX}\rho = \rho$. Moreover we have

$$\rho = \rho \sqcap x \sqsubseteq (\rho x^{\sharp} \sqcap \mathrm{id}_X) x \sqsubseteq \rho x^{\sharp} x \sqsubseteq \rho \ .$$

Finally the uniqueness of k follows from $k = k\nabla_{XX} \sqcap \mathrm{id}_X = kx\nabla_{XX} \sqcap \mathrm{id}_X = \rho\nabla_{XX} \sqcap \mathrm{id}_X$.
(b) It is enough to show that if $x \sqcap x' \neq O_{XX}$ then $x = x'$. As $x \sqcap x' \sqsubseteq x$ and $\nabla_{XX}(x \sqcap x') = x \sqcap x'$, by (a) there is a unique scalar $k : X \rightharpoonup X$ such that $x \sqcap x' = kx$. If $x \sqcap x' \neq O_{XX}$, then $k \neq O_{XX}$ and so $x \sqsubseteq x'$, because $kx \sqsubseteq x'$ and x' is s-crisp. If $x \sqcap x' = O_{XX}$, then $xx'^\sharp = xx'^\sharp \sqcap \nabla_{XX} \sqsubseteq (x \sqcap \nabla_{XX}x')x'^\sharp = (x \sqcap x')x'^\sharp = O_{XX}$. This completes the proof. ∎

Set $L = \mathcal{F}(W)$ for a fixed object W. Then L is a complete distributive lattice. A function $\chi(\alpha) : \chi(X) \times \chi(Y) \to L$ assigning $\chi(\alpha)(x, y) = \phi_W(x\alpha y^\sharp) \in L$ to a pair (x, y) of points x of X and y of Y, gives an L-relation of $\chi(X)$ into $\chi(Y)$. Thus we have a function $\chi : \mathcal{D}(X, Y) \to L\text{-}\mathbf{Rel}(\chi(X), \chi(Y))$.

Proposition 7.11 *If $\mathcal{D}$ is a uniform Dedekind category, then the function $\chi : \mathcal{D}(X, Y) \to L\text{-}\mathbf{Rel}(\chi(X), \chi(Y))$ satisfies the following properties:*

(a) $\chi(O_{XY}) = 0_{\chi(X)\chi(Y)}$, $\chi(\nabla_{XY}) = 1_{\chi(X)\chi(Y)}$ *and* $\chi(\mathrm{id}_X) = E_{\chi(X)}$.

(b) $\chi(\alpha \sqcup \alpha') = \chi(\alpha) \cup \chi(\alpha')$ *and* $\chi(\alpha \sqcap \alpha') = \chi(\alpha) \cap \chi(\alpha')$.

(c) $\chi(\alpha^\sharp) = \chi(\alpha)^\cup$.

(d) $\chi(\alpha)\chi(\beta) = \chi(\alpha[\sqcup_{y \in \chi(Y)} y^\sharp y]\beta)$.

(e) *The function $\chi : \mathcal{D}(X, Y) \to L\text{-}\mathbf{Rel}(\chi(X), \chi(Y))$ is surjective.*

Proof. Recall that $\chi(\alpha)(x, y) = \phi_W\phi_X(x\alpha y^\sharp) = \phi_W\phi_Y(x\alpha y^\sharp)$ by lemma 7.1(b).
(a) It is immediate that $\chi(O_{XY})(x, y) = O_{WW}$. Note that $x\nabla_{XY}y^\sharp = \nabla_{XY}$ from $x\nabla_{XX} = \nabla_{XX}$ and $y\nabla_{YY} = \nabla_{YY}$. The second equality follows from

$$\begin{array}{rcll} \phi_W(x\nabla_{XY}y^\sharp) & = & \phi_W(\nabla_{XY}) & (\text{by } x\nabla_{XX} = \nabla_{XX} \text{ and } y\nabla_{YY} = \nabla_{YY}) \\ & = & \phi_W\phi_X(\nabla_{XY}) & (\text{by lemma 7.1(b)}) \\ & = & \phi_W(\mathrm{id}_X) & (\text{by } X \sim Y) \\ & = & \mathrm{id}_W & (\text{by } X \sim W) \end{array}$$

and the third holds from $\phi_X(x\mathrm{id}_X x'^\sharp) = \nabla_{XX} x x'^\sharp \nabla_{XX} \sqcap \mathrm{id}_X = x x'^\sharp \sqcap \mathrm{id}_X$ and lemma 7.4(b).

(b) The former equality is trivial from $\phi_W(x(\alpha \sqcup \alpha')y^\sharp) = \phi_W(x\alpha y^\sharp) \sqcup \phi_W(x\alpha' y^\sharp)$, and the latter follows from

$$\begin{array}{rcl}
\phi_W(x(\alpha \sqcap \alpha')y^\sharp) & = & \nabla_{WX}(x\alpha y^\sharp \sqcap x\alpha' y^\sharp)\nabla_{YW} \sqcap \mathrm{id}_W \\
& \sqsubseteq & \nabla_{WX} x\alpha y^\sharp \nabla_{YW} \sqcap \nabla_{WX} x\alpha' y^\sharp \nabla_{YW} \sqcap \mathrm{id}_W \\
& (= & \phi_W(x\alpha y^\sharp) \sqcap \phi_W(x\alpha' y^\sharp)\ ) \\
& \sqsubseteq & \nabla_{WX}(x\alpha y^\sharp \sqcap \nabla_{XW}\nabla_{WX} x\alpha' y^\sharp \nabla_{YW}\nabla_{WY})\nabla_{YW} \sqcap \mathrm{id}_W \\
& \sqsubseteq & \nabla_{WX}(x\alpha y^\sharp \sqcap \nabla_{XX} x\alpha' y^\sharp \nabla_{YY})\nabla_{YW} \sqcap \mathrm{id}_W \\
& \sqsubseteq & \nabla_{WX}(x\alpha y^\sharp \sqcap x\alpha' y^\sharp)\nabla_{YW} \sqcap \mathrm{id}_W \\
& = & \phi_W(x(\alpha \sqcap \alpha')y^\sharp)\ .
\end{array}$$

(c) It directly follows from lemma 7.1(c).

(d) First note that $\chi(\alpha)(x,y) \sqcap \chi(\beta)(y,z) = \chi(\alpha y^\sharp y \beta)(x,z)$ for $(x,y,z) \in \chi(X) \times \chi(Y) \times \chi(Z)$, since

$$\begin{array}{rcl}
\phi_W(x\alpha y^\sharp) \sqcap \phi_W(y\beta z^\sharp) & = & \nabla_{WX} x\alpha y^\sharp \nabla_{YW} \sqcap \nabla_{WY} y\beta z^\sharp \nabla_{ZW} \sqcap \mathrm{id}_W \\
& \sqsubseteq & \nabla_{WX} x\alpha y^\sharp(\nabla_{YW} \sqcap y\alpha^\sharp x^\sharp \nabla_{XW}\nabla_{WY} y\beta z^\sharp \nabla_{ZW}) \sqcap \mathrm{id}_W \\
& \sqsubseteq & \nabla_{WX} x\alpha y^\sharp \nabla_{YY} y\beta z^\sharp \nabla_{ZW} \sqcap \mathrm{id}_W \\
& = & \nabla_{WX} x\alpha y^\sharp y\beta z^\sharp \nabla_{ZW} \sqcap \mathrm{id}_W \\
& (= & \phi_W(x\alpha y^\sharp y\beta z^\sharp)\ ) \\
& = & (\nabla_{WX} x\alpha y^\sharp \sqcap \nabla_{WZ} z\beta^\sharp y^\sharp)(y\alpha^\sharp x^\sharp \nabla_{XW} \sqcap y\beta z^\sharp \nabla_{ZW}) \sqcap \mathrm{id}_W \\
& \sqsubseteq & \nabla_{WX} x\alpha y^\sharp \nabla_{YW} \sqcap \nabla_{WY} y\beta z^\sharp \nabla_{ZW} \sqcap \mathrm{id}_W \\
& = & \phi_W(x\alpha y^\sharp) \sqcap \phi_W(y\beta z^\sharp)\ .
\end{array}$$

Therefore we have

$$\begin{array}{rcl}
\chi(\alpha)\chi(\beta)(x,z) & = & \sqcup_{y\in\chi(Y)}[\chi(\alpha)(x,y) \sqcap \chi(\beta)(y,z)] \\
& = & \sqcup_{y\in\chi(Y)}\chi(\alpha y^\sharp y\beta)(x,z) \\
& = & \chi(\alpha[\sqcup_{y\in\chi(Y)} y^\sharp y]\beta)(x,z)\ .
\end{array}$$

(e) Let $R : \chi(X) \rightharpoonup \chi(Y)$ be an L-relation. Noticing $L = \mathcal{F}(W)$ we define a morphism $\alpha_R : X \rightharpoonup Y$ by

$$\alpha_R = \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \phi_X(R(x,y))x^\sharp \nabla_{XY} y\ .$$

Then we have $\phi_X(x_0\alpha_R y_0^\sharp) = \phi_X(R(x_0,y_0))$ from

$$\phi_X(x_0\alpha_R y_0^\sharp)\nabla_{XY} = x_0\alpha_R y_0{}^\sharp = \phi_X(R(x_0,y_0))\nabla_{XY}\ .$$

Hence we have

$$\chi(\alpha_R)(x_0, y_0) = \phi_W(x_0 \alpha_R y_0^\sharp) = \phi_W \phi_X(x_0 \alpha_R y_0^\sharp) = \phi_W \phi_X(R(x_0, y_0)) = R(x_0, y_0) \ ,$$

which completes the proof. ∎

Definition 7.8 A Dedekind category $\mathcal{D}$ satisfies the **strict point axiom** iff:

$$\sqcup_{x \in \chi(X)} x = \nabla_{XX}$$

for all objects X.

Assume that $\sqcup_{x \in \chi(X)} x = \nabla_{XX}$. Then it follows from $\mathrm{id}_X \sqcap x \sqsubseteq (\mathrm{id}_X x^\sharp \sqcap \mathrm{id}_X) x \sqsubseteq x^\sharp x$ that $\mathrm{id}_X = \mathrm{id}_X \sqcap \nabla_{XX} = \mathrm{id}_X \sqcap (\sqcup_{x \in \chi(X)} x) = \sqcup_{x \in \chi(X)} (\mathrm{id}_X \sqcap x) \sqsubseteq \sqcup_{x \in \chi(X)} x^\sharp x$. Hence $\sqcup_{x \in \chi(X)} x^\sharp x = \mathrm{id}_X$. Conversely assume that $\sqcup_{x \in \chi(X)} x^\sharp x = \mathrm{id}_X$. Then $\nabla_{XX} = \nabla_{XX} \mathrm{id}_X = \nabla_{XX} (\sqcup_{x \in \chi(X)} x^\sharp x) = \sqcup_{x \in \chi(X)} \nabla_{XX} x^\sharp x = \sqcup_{x \in \chi(X)} \nabla_{XX} x = \sqcup_{x \in \chi(X)} x$. Therefore the condition $\sqcup_{x \in \chi(X)} x = \nabla_{XX}$ is equivalent to $\sqcup_{x \in \chi(X)} x^\sharp x = \mathrm{id}_X$.

Proposition 7.12 *If a Dedekind category $\mathcal{D}$ satisfies the strict point axiom, then for all objects X the identity morphism id_X is complemented. Moreover, if the statement (a) of theorem 7.1 is valid in $\mathcal{D}$, then id_X is s-crisp.*

Proof. Assume that $\nabla_{XX} = \sqcup_{x \in \chi(X)} x$. Then it is obvious that

$$\nabla_{XX} = \nabla_{XX} \nabla_{XX} = (\sqcup_{x \in \chi(X)} x^\sharp)(\sqcup_{y \in \chi(X)} y) = \mathrm{id}_X \sqcup (\sqcup_{x \neq y \in \chi(X)} x^\sharp y) \ .$$

Here note that for $x \neq y \in \chi(X)$ we have $\mathrm{id}_X \sqcap x^\sharp y \sqsubseteq x^\sharp (x \mathrm{id}_X \sqcap y) = O_{XX}$. Hence this shows that $\sqcup_{x \neq y \in \chi(X)} x^\sharp y$ is the complement of id_X. ∎

Theorem 7.2 (Representation Theorem) *Assume that $\mathcal{D}$ is a uniform Dedekind and satisfies the strict point axiom. Then every morphism $\alpha : X \rightarrow Y$ has a unique representation*

$$\alpha = \sqcup_{x \in \chi(X)} \sqcup_{y \in \chi(Y)} k_{x,y} x^\sharp \nabla_{XY} y \ ,$$

where $k_{x,y}$ is a scalar on X for all $(x, y) \in \chi(X) \times \chi(Y)$.

Proof. Note that $x\alpha y^\sharp = \phi(x\alpha y^\sharp)\nabla_{XY}$ for $x \in \chi(X)$ and $y \in \chi(Y)$, because $x\alpha y^\sharp = \nabla_{XX} x\alpha y^\sharp \nabla_{YY} = \phi_X(x\alpha y^\sharp)\nabla_{XY}$ by lemma 7.1(a). We now show the uniqueness of the representation. Assume $\alpha = \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} k_{x,y} x^\sharp \nabla_{XY} y$. Then for all $(x, y) \in \chi(X) \times \chi(Y)$ we have $k_{x,y}\nabla_{XY} = x\alpha y^\sharp = \phi_X(x\alpha y^\sharp)\nabla_{XY}$ and so $k_{x,y} = \phi_X(x\alpha y^\sharp)$ by proposition 7.1. Hence it suffices to see that $\alpha = \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \phi_X(x\alpha y^\sharp) x^\sharp \nabla_{XY} y$. Since $\mathrm{id}_X = \sqcup_{x\in\chi(X)} x^\sharp x$ and $\mathrm{id}_Y = \sqcup_{y\in\chi(Y)} y^\sharp y$ by the strict point axiom, we have

$$\begin{aligned}
\alpha &= \mathrm{id}_X \alpha \mathrm{id}_Y \\
&= (\sqcup_{x\in\chi(X)} x^\sharp x)\alpha(\sqcup_{y\in\chi(Y)} y^\sharp y) \\
&= \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} x^\sharp x\alpha y^\sharp y \\
&= \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} x^\sharp \phi_X(x\alpha y^\sharp)\nabla_{XY} y \\
&= \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \phi_X(x\alpha y^\sharp) x^\sharp \nabla_{XY} y \ .
\end{aligned}$$

This completes the proof. ∎

Corollary 7.1 *A uniform Dedekind category $\mathcal{D}$ satisfies the strict point axiom if and only if the function $\chi : \mathcal{D}(X, X) \to L\text{-}\mathbf{Rel}(\chi(X), \chi(X))$ is injective for all objects X.*

Proof. First assume that the function χ is injective. Then it follows from proposition 7.11(a) and (d) that $\mathrm{id}_X = \sqcup_{x\in\chi(X)} x^\sharp x$, which is equivalent to $\nabla_X = \sqcup_{x\in\chi(X)} x$. Secondly assume that the point axiom and consequently the representation theorem 7.2 hold. Let $\chi(\alpha) = \chi(\alpha')$ for $\alpha, \alpha' : X \to Y$. Then $\phi_W(x\alpha y^\sharp) = \phi_W(x\alpha' y^\sharp)$ for all $(x, y) \in \chi(X) \times \chi(Y)$. Since $\mathcal{D}$ is uniform, $\phi_X(x\alpha y^\sharp) = \phi_Y(x\alpha' y^\sharp)$ for all $(x, y) \in \chi(X) \times \chi(Y)$and so $\alpha = \alpha'$ by the virtue of the representation theorem. ∎

From the proof of proposition 2.4(d) it is easy to see that $\nabla_{XY} \neq O_{XY}$ for all nonempty objects X and Y if $\mathcal{D}$ has a unit object I and satisfies the strict point axiom.

As a result we have proved that a Dedekind category which has a unit object satisfying the strict point axiom is equivalent to a subcategory of a category of L-relations.

Let I and X be objects in $\mathcal{D}$. An I-point of X is an s-crisp function $p : I \to X$ such that $p = \nabla_{II} p$. Thus, when I is a unit object in $\mathcal{D}$, an I-point of X is just an s-crisp function from I to X. The set of all I-points of X will be denoted by $Q(X)$.

Proposition 7.13 *Let I and X be objects in $\mathcal{D}$. Then the following holds:*

(a) *If $X \prec I$, then a morphism $x = \nabla_{XI}p : X \rightharpoonup X$ is a point of X for an I-point $p : I \rightharpoonup X$ of X.*

(b) *If $I \prec X$, then a morphism $p = \nabla_{IX}x : I \rightharpoonup X$ is an I-point of X for a point $x : X \rightharpoonup X$ of X.*

(c) *If $X \sim I$, then $\nabla_{IX} = \sqcup_{p\in Q(X)}p$ is equivalent to $\nabla_{XX} = \sqcup_{x\in X(X)}x$.*

Proof. (a) First note that

$$\nabla_{XX}x = \nabla_{XX}\nabla_{XI}p = \nabla_{XI}p = x \ ,$$

$$x^\sharp x = (\nabla_{XI}p)^\sharp(\nabla_{XI}p) = p^\sharp\nabla_{IX}\nabla_{XI}p \sqsubseteq p^\sharp\nabla_{II}p = p^\sharp p \sqsubseteq \mathrm{id}_X \ ,$$

and

$$xx^\sharp = (\nabla_{XI}p)(\nabla_{XI}p)^\sharp = \nabla_{XI}pp^\sharp\nabla_{IX} \sqsupseteq \nabla_{XI}\nabla_{IX} = \nabla_{XX}$$

by $X \prec I$. Next assume that $k\tau \sqsubseteq \nabla_{XI}p(= x)$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoonup X$. Then $\phi_I(k)\nabla_{IX}\tau = \nabla_{IX}k\tau \sqsubseteq \nabla_{IX}\nabla_{XI}p \sqsubseteq \nabla_{II}p = p$ and so $\nabla_{IX}\tau \sqsubseteq p$, since $\phi_I(k) \neq O_{II}$ by lemma 7.1(e) and p is s-crisp. Hence $\tau \sqsubseteq \nabla_{XX}\tau = \nabla_{XI}\nabla_{IX}\tau \sqsubseteq \nabla_{XI}p = x$ by $X \prec I$.
(b) First note that

$$\nabla_{II}p = \nabla_{II}\nabla_{IX}x = \nabla_{IX}x = p \ ,$$

$$p^\sharp p = (\nabla_{IX}x)^\sharp(\nabla_{IX}x) = x^\sharp\nabla_{XI}\nabla_{IX}x \sqsubseteq x^\sharp\nabla_{XX}x = x^\sharp x \sqsubseteq \mathrm{id}_X \ ,$$

and

$$pp^\sharp = (\nabla_{IX}x)(\nabla_{IX}x)^\sharp = \nabla_{IX}xx^\sharp\nabla_{XI} = \nabla_{IX}\nabla_{XX}\nabla_{XI} = \nabla_{II} \sqsupseteq \mathrm{id}_I$$

by $I \prec X$. Next assume that $k\tau \sqsubseteq \nabla_{IX}x(= p)$ for a nonzero scalar k on I and a morphism $\tau : I \rightharpoonup X$. Then $\phi_X(k)\nabla_{XI}\tau = \nabla_{XI}k\tau \sqsubseteq \nabla_{XI}\nabla_{IX}x \sqsubseteq \nabla_{XX}x = x$

and so $\nabla_{XI}\tau \sqsubseteq x$, since $\phi_X(k) \neq O_{XX}$ by lemma 7.1(e) and x is s-crisp. Hence $\tau \sqsubseteq \nabla_{II}\tau = \nabla_{IX}\nabla_{XI}\tau \sqsubseteq \nabla_{IX}x = p$ by $I \prec X$.

(c) First assume that $\nabla_{IX} = \sqcup_{p \in Q(X)} p$. Then

$$\sqcup_{x \in \chi(X)} x = \sqcup_{p \in Q(X)} \nabla_{XI} p = \nabla_{XI} \sqcup_{p \in Q(X)} p = \nabla_{XI}\nabla_{IX} = \nabla_{XX}$$

by $X \prec I$. Conversely assume that $\nabla_{XX} = \sqcup_{x \in \chi(X)} x$. Then we have

$$\sqcup_{p \in Q(X)} p = \sqcup_{x \in \chi(X)} \nabla_{IX} x = \nabla_{IX} \sqcup_{x \in \chi(X)} x = \nabla_{IX}\nabla_{XX} = \nabla_{IX} \ .$$

∎

In this chapter, we defined a notion of s-crisp and points. Unfortunately s-crispness is not equivalent to 0-1 crispness in L-relations but just a sufficient condition for 0-1 crispness. So we gave a condition the two crispness to be equivalent. However the notion of s-crispness is enough to make points satisfy separate property, and we proved representation theorem for Dedekind categories without assumption of existence of unit objects.

Chapter 8

Conclusion

The contribution of this thesis is as follows:

(1) We proposed new two algebraic formalisations of fuzzy relations which are fuzzy relation algebras and Zadeh categories, and proved their representation theorems. To prove such theorems, we used a notion of point relations with a separation property, that is, different point relations never meet. In order to make point relations satisfy the property, a notion of crispness is necessary. In the two formalisations, we defined a notion of crispness via scalar multiplications, which is equivalent to an intuitive element-wise definition of crispness of fuzzy relations, namely 0-1 crispness.

(2) We proved representation theorems for relation algebras and Dedekind categories. As in the case of fuzzy relations, we used a notion of point relations. Since neither relation algebras nor Dedekind categories have scalar multiplications, we introduced a notion of scalar relations and defined the crispness by using the notion of scalar relations. Of course the crispness also provided the separation property of point relations.

The list of our future researches is below:

(a) As we described in (1), the notion of crispness in fuzzy relation algebras and Zadeh categories is well defined via scalar multiplications. But in relation algebras and Dedekind categories, the notion is not so well defined, that is, definition of crispness in the two frameworks are not equivalent to 0-1 crispness of L-relations. The notion of s-crispness is just a sufficient condition for 0-1 crispness of L-relations.

(b) We would like to investigate the aspect of new applications of our fuzzy relational calculus.

The suggestion (a) proposes the necessity to continue studying crispness in Dedekind categories. Especially the author is interested in the case that L-relations take values in a Boolean algebras; for example power set $\mathcal{P}(\{a, b\})$ of a set $\{a, b\}$. In this case, the notion of s-crispness is too strict to characterize 0-1 crispness in Dedekind categories.

In spite of (b), already, fuzzy relation algebras [KF95] which were introduced in chapter 3 gave a theoretical basis to theory of fuzzy difunctional dependency in fuzzy relational databases [OJ96], and a rewriting system of fuzzy graphs by using single pushouts [MoK97] based on a study of Zadeh categories [KFM96] which were introduced in chapter 5. Besides that, in the future, our calculus would be applied to *graded accessibility* and *fuzzy possible world semantics* introduced by Suzuki [Suz96]. In the research, accessibility relations correspond to L-relations which satisfy condition C0 provided in chapter 7. The relations may be useful tool to investigate accessibility and reliability of networks. Also the results in Boolean relation algebraic approach to theory of natural languages [Bot92a, Bot92b, Sup76, Sup79, Sup81] suggest that our calculus may enable them to treat fuzziness in element-free style. But, in order to consider applications of our calculus to relational modelling of fuzzy systems, we should study fuzzy relational equations in our frameworks.

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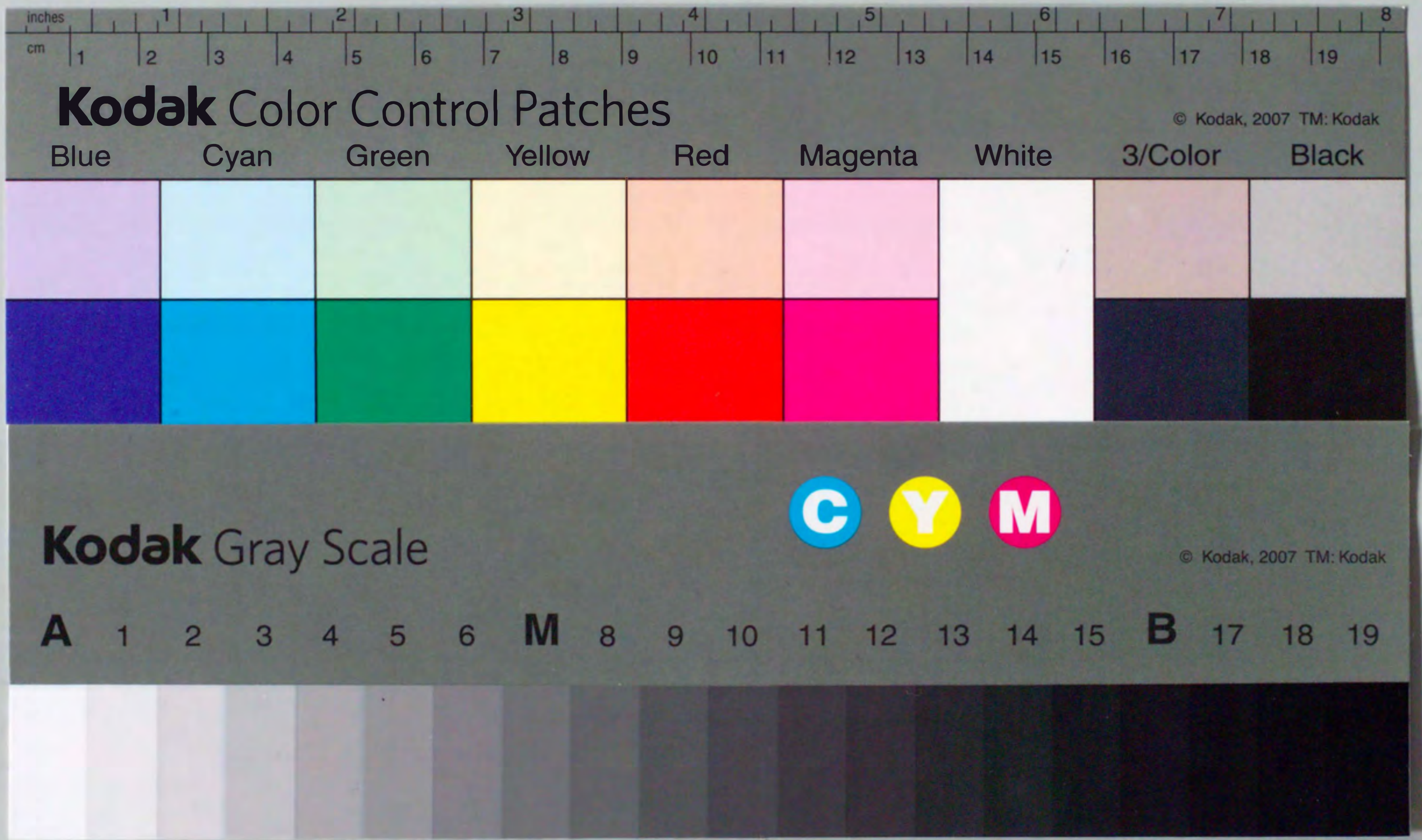

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