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Addition in Jacobians of hyperelliptic curves and the periodic discrete Toda lattice

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Abstract

We give a geometric realization of the $(g+1)$-periodic discrete Toda lattice. The realization is given as the addition of g-tuples of points on the hyperelliptic curve of genus g, which is the spectral curve of the $(g+1)$-periodic discrete Toda lattice. We show that the addition on the hyperelliptic curve is induced from its Jacobian through a surjection and is realized by using the intersection of the hyperelliptic curve and a curve of genus 0.

1 Addition in Jacobians of hyperelliptic curves

Let $h(X)$ be the monic polynomial of degree $2g+2 \geq 4$:

$$h(X) = X^{2g+2} + a_{2g+1}X^{2g+1} + a_{2g}X^{2g} + \cdots + a_1 X + a_0.$$

Consider the hyperelliptic curve H defined by $h(X)$:

$$H = \left\{ P = (x,y) \in \mathbb{C}^2 \mid y^2 - h(x) = 0 \right\} \cup \left\{ P_\infty, P'_\infty \right\},$$

where P_∞ and P'_∞ are the points at infinity and we assume that the equation $h(x) = 0$ has no multiple root. There exist exactly two points (x,y) and $(x,-y)$ on H for any $x \in \mathbb{C}$ such that $h(x) = y^2 \neq 0$. We denote these points P and P', and call P' the conjugate of P.

Let $\mathcal{D}_0(H)$ be the group of divisors of degree 0 on H. Also let $\mathcal{D}_l(H)$ be the group of principal divisors of rational functions on H. We define the Picard group $\mathrm{Pic}^0(H)$ to be the residue class group $\mathrm{Pic}^0(H) = \mathcal{D}_0(H)/\mathcal{D}_l(H)$. Note that $\mathrm{Pic}^0(H)$ is isomorphic to the Jacobian $\mathrm{Jac}(H)$ of H.

Let $\mathcal{D}_g^+(H)$ be the group of effective divisors of degree g on H:

$$\mathcal{D}_g^+(H) = \{ D \in \mathcal{D}(H) \mid D > 0, \deg D = g \},$$

where $\mathcal{D}(H)$ is the divisor group of H. Fix an element D^* of $\mathcal{D}_g^+(H)$. Define the map $\Phi_{D^*} : \mathcal{D}_g^+(H) \to \mathrm{Pic}^0(H)$ to be

$$\Phi_{D^*}(A) :\equiv A - D^* \pmod{\mathcal{D}_l(H)} \qquad \text{for } A \in \mathcal{D}_g^+(H).$$

We then have the following theorem [5]

Theorem 1 *The map Φ_{D^*} is surjective. In particular, Φ_{D^*} is bijective if and only if $g = 1$.*

For simplicity, we denote the element $P_1 + P_2 + \cdots + P_g$ of $\mathcal{D}_g^+(H)$ as $\boldsymbol{P} := P_1 + P_2 + \cdots + P_g$. We have $\mathrm{Pic}^0(H) = \left\{ \Phi_{D^*}(\boldsymbol{P}) \mid \boldsymbol{P} \in \mathcal{D}_g^+(H) \right\}$ because Φ_{D^*} is surjective. Thus we can define the addition $\{P_1, P_2, \cdots, P_g\} \oplus \{Q_1, Q_2, \cdots, Q_g\}$ of g-tuples of points on H as the adition $\Phi_{D^*}(\boldsymbol{P}) + \Phi_{D^*}(\boldsymbol{Q})$ in $\mathrm{Pic}^0(H)$:

$$\{P_1, P_2, \cdots, P_g\} \oplus \{Q_1, Q_2, \cdots, Q_g\} := \Phi_{D^*}(\boldsymbol{P}) + \Phi_{D^*}(\boldsymbol{Q}).$$

Since $\Phi_{D^*}(D^*) = 0$, we choose the g-tuple of points consisting of D^* as the unit $\mathcal{O}$ of addition on H.

Let P_i, Q_i, R_i $(i = 1, 2, \ldots, g)$ be the points on H satisfying the addition formula

$$\{P_1, P_2, \cdots, P_g\} \oplus \{Q_1, Q_2, \cdots, Q_g\} \oplus \{R_1, R_2, \cdots, R_g\} = \mathcal{O}. \tag{1}$$

This can be written by the divisors

$$\sum_{i=1}^{g} (P_i + Q_i + R_i) - 3D^* \sim 0, \tag{2}$$

where $\sim$ stands for the equivalence of divisors. The formula (2) is equivalent to the existence of the rational function $k \in L(3D^*)$ whose zeros are the $3g$ points P_i, Q_i, R_i $(i = 1, 2, \ldots, g)$ on H, where $L(3D^*) := \{k \in \mathbb{C}(H) \mid (k) + 3D^* > 0\}$ is the linear system of rational functions on H and $\mathbb{C}(H)$ is the field of rational functions on H. Let C be the curve defined by $k \in L(3D^*)$. Then the zeros P_i, Q_i, R_i $(i = 1, 2, \ldots, g)$ of k are the points on C. Since these points are on H by definition, these points are the intersection points of H and C. Thus the addition (1) is realized by using the intersection of H and C.

Now let us fix D^* as follows

$$D^* = \begin{cases} \dfrac{g}{2}(P_\infty + P'_\infty) & \text{If } g \text{ is even,} \\ \dfrac{g-1}{2}(P_\infty + P'_\infty) + P_\infty & \text{If } g \text{ is odd.} \end{cases}$$

Put $I_n := \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. We then have the following lemma [5].

Lemma 1 *The kernel of Φ_{D^*} is given by*

$$\ker \Phi_{D^*} = \left\{ \sum_{i=1}^{g} P_i \in \mathcal{D}_g^+ \;\middle|\; {}^\forall i \in I_g \; {}^\exists j \in I_g \; s.t. \; P_j = P'_i, \, j \neq i \right\}$$

if g is even and by

$$\ker \Phi_{D^*} = \left\{ \sum_{i=1}^{g-1} P_i + P_\infty \in \mathcal{D}_g^+ \;\middle|\; {}^\forall i \in I_{g-1} \; {}^\exists j \in I_{g-1} \; s.t. \; P_j = P'_i, \, j \neq i. \right\}$$

if g is odd. Moreover, if $\boldsymbol{P} \notin \ker \Phi_{D^*}$ then we have

$$\boldsymbol{P} = \boldsymbol{Q} \quad \Longleftrightarrow \quad \{P_1, P_2, \ldots, P_g\} = \{Q_1, Q_2, \ldots, Q_g\}.$$

Put $\mathcal{G}(H) := \left(\mathcal{D}_g^+ \setminus \ker \Phi_{D^*}\right) \cup \{D^*\}$. Then we have $\mathcal{G}(H) = \mathcal{D}_g^+ / \ker \Phi_{D^*} \simeq \mathrm{Pic}^0(H)$ by lemma 1. Thus $\mathcal{G}(H)$ has the additive group structure equipped with the unit of addition D^*.

Theorem 2 *The reduced map $\Phi_{D^*} : \mathcal{G}(H) \to \mathrm{Pic}^0(H)$ is the group isomorphism.*

2 A geometric realization of the periodic discrete Toda lattice

Let us consider the $(g+1)$-periodic Toda lattice in discrete time [2, 4, 3]

$$I_i^{t+1} = I_i^t + V_i^t - V_{i-1}^{t+1}, \qquad V_i^{t+1} = \frac{I_{i+1}^t V_i^t}{I_i^{t+1}}, \qquad (i = 1, 2, \ldots, g+1, \; t \in \mathbb{Z}). \tag{3}$$

The Lax form of (3) is given as follows

$$L^{t+1}(y)M^t(y) = M^t(y)L^t(y), \tag{4}$$

where $y \in \mathbb{C}$ is the spectral parameter and the matrices $L^t(y)$ and $M^t(y)$ are defined to be

$$L^t(y) := \begin{pmatrix} a_1^t & 1 & & & (-1)^g b_1^t/y \\ b_2^t & a_2^t & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & b_g^t & a_g^t & 1 \\ (-1)^g y & & & b_{g+1}^t & a_{g+1}^t \end{pmatrix}, \qquad M^t(y) := \begin{pmatrix} I_2^t & 1 & & & \\ & I_3^t & 1 & & \\ & & \ddots & \ddots & \\ & & & I_{g+1}^t & 1 \\ y & & & & I_1^t \end{pmatrix}$$

for $a_i^t = I_{i+1}^t + V_i^t$ and $b_i^t = I_i^t V_i^t$ $(i = 1, 2, \dots, g+1,\ t \in \mathbb{Z})$.

Let us consider the eigenpolynomial of $L^t(y)$

$$\tilde{f}(x, y) := y \det\left(x\mathbb{I} + L^t(y)\right) = y^2 + y\left(x^{g+1} + c_g x^g + \cdots + c_1 x + c_0\right) + c_{-1},$$

where $\mathbb{I}$ is the identity matrix of degree $g+1$ and the coefficients $c_g, c_{g-1}, \dots, c_{-1}, c_0$ are given by

$$c_g = \sum_{1 \le i \le g+1} (I_i + V_i), \quad c_{g-1} = \sum_{1 \le i < j \le g+1} (I_i I_j + V_i V_j) + \sum_{1 \le i, j \le g+1, j \ne i, i-1} I_i V_j,$$
$$\cdots, \quad c_0 = \prod_{i=1}^{g+1} I_i + \prod_{i=1}^{g+1} V_i, \quad c_{-1} = \prod_{i=1}^{g+1} I_i V_i. \tag{5}$$

Since the eigenpolynomial of $L^t(y)$ does not depend on t, the coefficients $c_g, c_{g-1}, \dots, c_{-1}, c_0$ of $\tilde{f}(x, y)$ are the conserved quantities of the time evolution (3). The spectral curve $\tilde{\gamma}_c$ of the $(g+1)$-periodic discrete Toda lattice is defined to be

$$\tilde{\gamma}_c = \left\{P = (x, y) \in \mathbb{C}^2 \mid \tilde{f}(x, y) = 0\right\} \cup \left\{P_\infty, P'_\infty\right\}.$$

For generic c_i the spectral curve $\tilde{\gamma}_c$ is the hyperelliptic curve of genus g. Applying the birational transformation on $\mathbb{P}^2(\mathbb{C})$

$$(x, y) \to (u, v) = \left(x, 2y + x^{g+1} + c_g x^g + \cdots + c_1 x + c_0\right), \tag{6}$$

we obtain the canonical form γ_c of $\tilde{\gamma}_c$ defined by

$$f(u, v) := v^2 - \left(u^{g+1} + c_g u^g + \cdots + c_1 u + c_0\right)^2 + 4c_{-1}.$$

The hyperelliptic curve γ_c is of degree $2g+2$ and of genus g. Hereafter, we consider γ_c and f instead of $\tilde{\gamma}_c$ and $\tilde{f}$.

Let the phase space of (3) be $\mathcal{U} := \left\{U^t := (I_1^t, \dots, I_{g+1}^t, V_1^t, \dots, V_{g+1}^t) \mid t \in \mathbb{Z}\right\} \simeq \mathbb{C}^{2(g+1)}$. Also let the moduli space of γ_c be $\mathcal{C} := \{c := (c_{-1}, \dots, c_g)\} \simeq \mathbb{C}^{g+2}$. Consider the map $\psi : \mathcal{U} \to \mathcal{C}$, $U^t \mapsto c$ defined by (5). We define the isolevel set $\mathcal{U}_c$ of the $(g+1)$-periodic discrete Toda lattice to be $\mathcal{U}_c := \left\{U^t \in \mathcal{U} \mid U^t = \psi^{-1}(c)\right\}$. The isolevel set $\mathcal{U}_c$ is isomorphic to the affine part of the Jacobian $\mathrm{Jac}(\gamma_c)$ of γ_c, and the time evolution (3) is linearized on it [1, 4].

Let $\varphi^t(x, y) = {}^t\left(\varphi_1^t, \varphi_2^t, \cdots, \varphi_g^t, -\varphi_{g+1}^t\right)$ be the eigenvector of $L^t(y)$. The elements φ_i^t are

$$\varphi_i^t(x, y) := \det \begin{array}{c} \begin{matrix} 1 & 2 & \cdots & i & \cdots & g \end{matrix} \\ \begin{pmatrix} l_{11} + x & l_{12} & \cdots & l_{1,g+1} & \cdots & l_{1g} \\ l_{21} & l_{22} + x & \cdots & l_{2,g+1} & \cdots & l_{2g} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ l_{g1} & l_{g2} & \cdots & l_{g,g+1} & \cdots & l_{gg} + x \end{pmatrix} \end{array} \quad \text{for } i = 1, 2, \dots, g$$

and

$$\varphi_{g+1}^t(x) := \det \begin{pmatrix} l_{11}+x & l_{12} & \cdots & l_{1g} \\ l_{21} & l_{22}+x & \cdots & l_{2g} \\ \vdots & \vdots & \cdots & \vdots \\ l_{g1} & l_{g2} & \cdots & l_{gg}+x \end{pmatrix},$$

where l_{ij} is the (i,j)-element of $L^t(y)$. Note that φ_{g+1}^t is independent of y.

Let the curve defined by $\varphi_i^t(x,y)$ be $\tilde{v}_i$ $(i=1,2,\ldots,g)$. Also let the curve obtained from $\tilde{v}_i$ by applying the birational transformation (6) be v_i $(i=1,2,\ldots,g)$. We choose the zero $P_i^t := (u_i^t, v_i^t)$ $(i=1,2,\ldots,g)$ of $\varphi^t(x,y)$, *i.e.*, the intersection point of v_1, v_1, $\ldots$, and v_g, as a representative of $\mathrm{Pic}^g(\gamma_c)$. Then the eigenvector map $\phi : \mathcal{U}_c \hookrightarrow \mathrm{Pic}^g(\gamma_c) \simeq \mathrm{Jac}(\gamma_c)$ is defined to be $\phi(U^t) = \sum_{i=1}^g P_i^t =: \boldsymbol{P}^t$.

By using the eigenvector map, we obtain the following theorem from lemma 1.

Theorem 3 *We have $\phi(U^t) \in \mathcal{G}(\gamma_c)$ if and only if there exists no polynomial $\xi(x)$ in x such that $\varphi_{g+1}^t(x) = \xi(x)^2$.*

Since the time evolution (3) of the $(g+1)$-periodic discrete Toda lattice is linearized on $\mathrm{Jac}(\gamma_c) \simeq \mathrm{Pic}^0(\gamma_c)$, it can be realized as the addition of g-tuples of points on the spectral curve γ_c:

$$\left\{P_1^{t+1}, P_2^{t+1}, \cdots, P_g^{t+1}\right\} = \left\{P_1^t, P_2^t, \cdots, P_g^t\right\} \oplus \{T_1, T_2, \cdots, T_g\},$$

where P_i^t and P_i^{t+1} are the points on γ_c given by the eigenvector map: $\boldsymbol{P}^{t+1} = \sum_{i=1}^g P_i^{t+1} = \phi(U^{t+1})$ and $\boldsymbol{P}^t = \sum_{i=1}^g P_i^t = \phi(U^t)$. For generic $2g$ points P_i^t and P_i^{t+1} $(i=1,2,\ldots,g)$, there exists a unique curve C passing through these points and defined by a rational function $k \in L(3D^*)$. If both $\boldsymbol{P}^{t+1}$ and $\boldsymbol{P}^t$ are in $\mathcal{G}(\gamma_c)$ then $\boldsymbol{T} = \sum_{i=1}^g T_i$ is uniquely determined as an element of $\mathcal{G}(\gamma_c)$ via the intersection of γ_c and C.

In the following subsections, we give several examples of geometric realizations of the $(g+1)$-periodic discrete Toda lattice in terms of the additions of g-tuples of points on the spectral curves.

2.1 The case of $g = 1$

Suppose $g=1$. Then the matrices in (4) reduce to[1]

$$L^t(y) = \begin{pmatrix} -a_2^t & 1 - b_1^t/y \\ b_2^t - y & -a_1^t \end{pmatrix}, \qquad M^t(y) = \begin{pmatrix} I_2^t & 1 \\ -y & I_1^t \end{pmatrix}.$$

The canonical form γ_c of the spectral curve is given by $f(u,v) = v^2 - \left(u^2 + c_1 u + c_0\right)^2 + 4c_{-1}$, where $c_1 = -a_1^t - a_2^t$, $c_0 = a_1^t a_2^t - b_1^t - b_2^t$, and $c_{-1} = b_1^t b_2^t$. The eigenvector φ^t of $L^t(y)$ is

$$\varphi^t = \begin{pmatrix} \varphi_1^t \\ -\varphi_2^t \end{pmatrix} = \begin{pmatrix} l_{12} \\ -l_{11} - x \end{pmatrix} = \begin{pmatrix} 1 - b_1^t/y \\ a_2^t - x \end{pmatrix}.$$

The curve v_1 defined by φ_1^t is $v - \left(u^2 + c_1 u + c_0 + 2b_1^t\right) = 0$. Let one of the intersection point of γ_c and v be $P_1^t = \left(u_1^t, v_1^t\right) = \left(a_2^t, b_1^t - b_2^t\right)$. The eigenvector map is $\phi(I_1^t, I_2^t, V_1^t, V_2^t) = P_1^t$.

Now we fix D^* on γ_c as $D^* = P_\infty$. Then, by lemma 1, we have $\ker \Phi_{D^*} = \{D^*\}$, and hence $\mathcal{G}(\gamma_c) = \mathcal{D}_1^+(\gamma_c) = \gamma_c \simeq \mathrm{Pic}^0(\gamma_c)$. The time evolution of the 2-periodic discrete Toda lattice is realized as the addition on γ_c:

$$P_1^{t+1} = P_1^t \oplus T_1 \qquad \Longleftrightarrow \qquad -P_1^{t+1} \oplus P_1^t \oplus T_1 = \mathcal{O}, \tag{7}$$

where T_1 is a point on γ_c and $\mathcal{O} = D^* = P_\infty$.

[1] We can assume $L^t(y)$ has the form in order the geometric realization of the 2-periodic discrete Toda lattice to be simple.

In order to obtain the point T_1 in (7), we consider the curve C defined by a rational function $k \in L(3D^*) = \langle 1, Y + X^2 + c_1 X, X\left(Y + X^2 + c_1 X + c_0\right)\rangle$, where X and Y are the rational function on γ_c such that $X(P) = u$ and $Y(P) = v$ for $P = (u,v) \in \gamma_c$ and $L(3D^*)$ is the vector space over $\mathbb{C}$ spanned by the rational functions 1, $Y + X^2 + c_1 X$, and $X\left(Y + X^2 + c_1 X + c_0\right)$ on γ_c. A rational function in $L(3D^*)$ can be considered to be defined on $\mathbb{C}^2$. Suppose the points $-P_1^{t+1}$ and P_1^t on γ_c to be in generic position. Then there exists a unique curve C passing through both $-P_1^{t+1}$ and P_1^t. Let the third intersection point of γ_c and C be T_1 then we have $(k) = -P_1^{t+1} + P_1^t + T_1 - 3D^*$, which is equivalent to (7). Noting the fact $-P_1^{t+1} \oplus P_1^{t+1} = \mathcal{O}$ and the time evolution (3), we have[2] $-P_1^{t+1} = (I_2^t + V_2^t, I_2^t V_1^t - I_1^t V_2^t)$. Therefore the curve C passing through both $-P_1^{t+1}$ and P_1^t is uniquely determined by the rational function

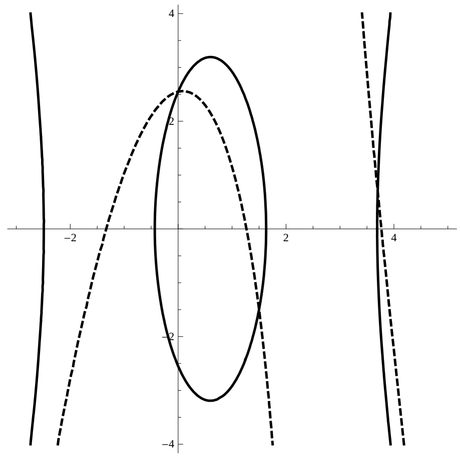

Figure 1: A geometric realization of 2-periodic discrete Toda lattice.

$$k(u,v) = V_2^t\left(I_1^t I_2^t - V_1^t V_2^t\right) + c_0 u + (u - V_2^t)(c_1 u + u^2 + v).$$

Eliminating v from $f(u,v) = 0$ and $k(u,v) = 0$, we obtain $u(u - a_1^{t+1})\varphi_3^t(u) = 0$. Solving this equation, we obtain the third intersection point T_1 of γ_c and C as $T_1 = \left(0, I_1^t I_2^t - V_1^t V_2^t\right)$. It should be noted that the point T_1 is independent of t because we have $\left(I_1^t I_2^t - V_1^t V_2^t\right)^2 = c_0^2 - 4c_{-1}$. Thus the 2-periodic discrete Toda lattice is realized as the addition of points on the elliptic curve γ_c by using the intersection of γ_c and the curve C of degree 3 (see figure 1). Thus we see that the 2-periodic discrete Toda lattice is nothing but a member of the celebrated QRT family.

2.2 The case of $g = 2$

Next suppose $g = 2$. Then the Lax matrices in (4) reduce to

$$L^t(y) = \begin{pmatrix} a_1^t & 1 & b_1^t/y \\ b_2^t & a_2^t & 1 \\ y & b_3^t & a_3^t \end{pmatrix} \qquad M^t(y) = \begin{pmatrix} I_2^t & 1 & 0 \\ 0 & I_3^t & 1 \\ y & 0 & I_1^t \end{pmatrix}.$$

The canonical form γ_c of the spectral curve is given by $f(u,v) = v^2 - \left(u^3 + c_2 u^2 + c_1 u + c_0\right)^2 + 4c_{-1}$, where $c_2 = a_1^t + a_2^t + a_3^t$, $c_1 = a_1^t a_2^t + a_2^t a_3^t + a_3^t a_1^t - b_1^t - b_2^t - b_3^t$, $c_0 = a_1^t a_2^t a_3^t - a_2^t b_1^t - a_3^t b_2^t - a_1^t b_3^t$, and $c_{-1} = b_1^t b_2^t b_3^t$. The eigenvector φ^t of $L^t(y)$ is

$$\varphi^t = \begin{pmatrix} \varphi_1^t \\ \varphi_2^t \\ -\varphi_3^t \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} l_{13} & l_{12} \\ l_{23} & l_{22} + x \end{vmatrix} \\ \begin{vmatrix} l_{11} + x & l_{13} \\ l_{21} & l_{23} \end{vmatrix} \\ -\begin{vmatrix} l_{11} + x & l_{12} \\ l_{21} & l_{22} + x \end{vmatrix} \end{pmatrix} = \begin{pmatrix} (a_2^t + x)\dfrac{b_1^t}{y} - 1 \\ a_1^t + x - \dfrac{b_1^t b_2^t}{y} \\ -(a_1^t + x)(a_2^t + x) + b_2^t \end{pmatrix}.$$

The curves υ_1 and υ_2 respectively defined by φ_1^t and φ_2^t are given as follows

$$\begin{aligned} \upsilon_1 : &\quad v - (b_1^t - b_3^t)u - a_2^t b_1^t + a_1^t b_3^t = 0, \\ \upsilon_2 : &\quad (u + a_1^t)\left\{v + b_3^t(a_1^t - a_2^t)\right\} + b_2^t(b_3^t - b_1^t) = 0. \end{aligned}$$

[2] The intersection point of γ_c and the curve passing through P_1^{t+1} and defined by $k \in L(2D^*)$ is $-P_1^{t+1}$ [5].

By eliminating v form v_1 and v_2, we obtain $\varphi_3^t(u) = (a_1^t + u)(a_2^t + u) - b_2^t = 0$.

Let the intersection points of v_1 and v_2 be $P_i^t = (u_i^t, v_i^t)$ $(i = 1, 2)$. Then we have

$$u_1^t + u_2^t = -(a_1^t + a_2^t), \quad u_1^t u_2^t = a_1^t a_2^t - b_2^t, \quad v_i^t = (b_1^t - b_3^t)u_i^t + a_2^t b_1^t - a_1^t b_3^t \quad (i = 1, 2).$$

The eigenvector map is $\phi(I_1^t, I_2^t, I_3^t, V_1^t, V_2^t, V_3^t) = P_1^t + P_2^t = \boldsymbol{P}^t$.

Now we fix D^* as $D^* = P_\infty + P'_\infty$. Then, by lemma 1, we have $\ker \Phi_{D^*} = \{P + P' \mid P \in \gamma_c\}$, and hence $\mathcal{G}(\gamma_c) = \left(\mathcal{D}_2^+(\gamma_c) \setminus \{P + P' \mid P \in \gamma_c\}\right) \cup \{D^*\} \simeq \mathrm{Pic}^0(\gamma_c)$. Noting theorem 3, we assume that $\varphi_3^t(u)$ has no double root. Then we have $\boldsymbol{P}^t, \boldsymbol{P}^{t+1} \in \mathcal{G}_2(\gamma_c)$ and the time evolution of the 3-periodic discrete Toda lattice is realized as the addition of couples of points on γ_c:

$$\left\{P_1^{t+1}, P_2^{t+1}\right\} = \left\{P_1^t, P_2^t\right\} \oplus \{T_1, T_2\} \iff \left\{P_1^{t+1\prime}, P_2^{t+1\prime}\right\} \oplus \left\{P_1^t, P_2^t\right\} \oplus \{T_1, T_2\} = \mathcal{O}, \qquad (8)$$

where $\boldsymbol{T} = T_1 + T_2 \in \mathcal{G}_2(\gamma_c)$, $\mathcal{O} = \{P_\infty, P'_\infty\}$ and we use the fact[3] $\left\{P_1^{t+1\prime}, P_2^{t+1\prime}\right\} = -\left\{P_1^{t+1}, P_2^{t+1}\right\}$.

Consider the curve C defined by a rational function $k \in L(3D^*) = \langle 1, X, X^2, X^3, Y\rangle$. Assume that the points $P_1^{t+1\prime}, P_2^{t+1\prime}, P_1^t, P_2^t$ are in generic position and C passes through these points. Then C is uniquely determined and is given by the rational function

$$\begin{aligned} k(u,v) = (I_1^t I_2^t I_3^t - V_1^t V_2^t V_3^t) + (I_1^t I_2^t - I_2^t I_3^t + I_3^t I_1^t - V_1^t V_2^t - V_2^t V_3^t - V_3^t V_1^t + I_1^t V_2^t - I_2^t V_3^t - I_3^t V_1^t)u \\ +(2I_1^t - c_2)u^2 - u^3 + v. \end{aligned}$$

Let the remaining two intersection points of γ_c and C be T_1 and T_2. Then we have $(k) = P_1^{t+1\prime} + P_2^{t+1\prime} + P_1^t + P_2^t + T_1 + T_2 - 3D^*$, which is equivalent to (8). Eliminating v from $f(u,v) = 0$ and $k(u,v) = 0$, we obtain $u\varphi_3^t(u)\varphi_3^{t+1}(u) = 0$. By solving $u\varphi_3^t(u)\varphi_3^{t+1}(u) = 0$ we conclude that $\{T_1, T_2\} = \{(0, V_1^t V_2^t V_3^t - I_1^t I_2^t I_3^t), P_\infty\}$. Since we have $\left(V_1^t V_2^t V_3^t - I_1^t I_2^t I_3^t\right)^2 = c_0^2 - 4c_{-1}$, the points T_1 and T_2 are independent of t. Thus the 3-periodic discrete Toda lattice is realized as the addition of points on the hyperelliptic curve γ_c by using the intersection of γ_c and the curve C of degree 3 (see figure 2).

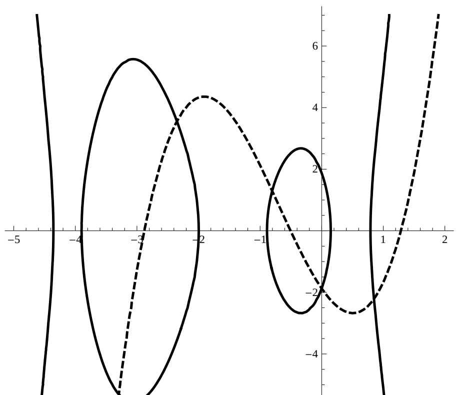

Figure 2: A geometric realization of 3-periodic discrete Toda lattice.

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[3]If g is even then we have $\left\{P_1', P_2', \cdots, P_g'\right\} = -\{P_1, P_2, \cdots, P_g\}$ [5].