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Soliton Equations generated by the Bäcklund transformation for the generalizd discrete KP equation

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Soliton Equations generated by the Bäcklund transformation for the generalizd discrete KP equation

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Abstract

We explain how to transform the bilinear discrete equations into the nonlinear discrete equation of ordinary forms using the generalized discrete KP equation.

1 Generalized discrete KP equation

We have the generalized discrete KP(g-dKP) equation in a bilinear form.

$$[c_1\exp(D_1)+c_2\exp(D_2)+c_3\exp(D_3)]f\cdot f=0, \tag{1}$$

where D_1, D_2, D_3 are the bilinear operators which are a linear combination of the operators $D_l, D_m, D_n, etc.$, and c_1, c_2, c_3 are arbitrary constants satisfying a relation $c_1+c_2+c_3=0$ for soliton solutions.
The Bäcklund transformation for the g-KP equation is known,

$$[\alpha_1 e^{-(D_1-D_3)/2}-e^{(D_1-D_3)/2}+\beta_1 e^{-(D_2-D_4)/2}]f\cdot g=0, \tag{2}$$
$$[\alpha_2 e^{-(D_2-D_3)/2}-e^{(D_2-D_3)/2}+\beta_2 e^{-(D_1-D_4)/2}]f\cdot g=0, \tag{3}$$

where $D_4=-D_1-D_2-D_3$ and $\alpha_1=1-\beta_1, \alpha_2=1-\beta_2$ for soliton soloutions,and β_1, β_2 are constants satisfying a relation $c_1\beta_1+c_2\beta_2=0$.

2 Transformation of the discrete bilinear forms into the nonlinear discrete wave equations.

We show that Eq.(1) and its Bäcklund transformation,Eqs.(2) and (3) are transformed into nonlinear soliton equations of ordinary form.

2.1 3-D Soliton Equation

We shall transform Eq.(1) into a 3-dimensional discrete nonlinear wave equation, which exhibits solitons. We write Eq.(1) as

$$[(c_1+c_2)e^{D_t} - c_1 e^{D_m} - c_2 e^{D_n}]f \cdot f = 0, \quad c_1, c_2 > 0. \tag{4}$$

First we introduce dependent variables $w^t_{m,n}$ and $z^t_{m,n}$,

$$w^t_{m,n} = \frac{e^{D_m} f \cdot f}{e^{D_n} f \cdot f} = \frac{f^t_{m+1,n} f^t_{m-1,n}}{f^t_{m,n+1} f^t_{m,n-1}}, \tag{5}$$

$$z^t_{m,n} = \frac{e^{D_t} f \cdot f}{e^{D_n} f \cdot f} = \frac{f^{t+1}_{m,n} f^{t-1}_{m,n}}{f^t_{m,n+1} f^t_{m,n-1}}. \tag{6}$$

Then Eq.(4) is reduced to

$$(c_1+c_2) z^t_{m,n} = c_1 w^t_{m,n} + c_2. \tag{7}$$

Next we look for another relation than Eq.(7) between $w^t_{m,n}$ and $z^t_{m,n}$.
For this purpose let us introduce shift operators p, q, s operating on an aritrary function $h^t_{m.n}$,

$$p^\alpha h^t_{m,n} = h^t_{m+\alpha,n}, \tag{8}$$

$$q^\beta h^t_{m,n} = h^t_{m,n+\beta}, \tag{9}$$

$$s^\gamma h^t_{m,n} = h^{t+\gamma}_{m,n}. \tag{10}$$

Then the logarithm of Eqs.(6) and (5) are expressed by the shift operators

$$\log z = (s + s^{-1} - q - q^{-1}) \log f, \tag{11}$$

$$\log w = (p + p^{-1} - q - q^{-1}) \log f. \tag{12}$$

Accordingly we find a relation,

$$(s + s^{-1} - q - q^{-1}) \log w = (p + p^{-1} - q - q^{-1}) \log z, \tag{13}$$

which is equivalent to

$$\frac{w^{t+1}_{m,n} w^{t-1}_{m,n}}{w^t_{m,n+1} w^t_{m,n-1}} = \frac{z^t_{m+1,n} z^t_{m-1,n}}{z^t_{m,n+1} z^t_{m,n-1}}, \tag{14}$$

which is transformed, using Eq.(7), to a 3-dimentional discrete nonlinear wave equation of $w^t_{m,n}$,

$$w^{t+1}_{m,n} w^{t-1}_{m,n} = \frac{[w^t_{m+1,n} + (c_2/c_1)][w^t_{m-1,n} + (c_2/c_1)]}{[1 + (c_2/c_1)/w^t_{m,n+1}][1 + (c_2/c_1)/w^t_{m,n-1}]}. \tag{15}$$

$c_1 = c_2$ のとき、方程式 (15) は可解格子模型における Y-systems と同じであり、双線形方程式 (4) は T-systems に対応している。

2.1.1 Special solutions to the 3-D ultradiscrete equation of $w^t_{m,n}$

Let

$$w^t_{m,n} = e^{w(t,m,n)/\epsilon}, \quad c_2/c_1 = e^{d/\epsilon}. \tag{16}$$

Then Eq.(15) is transformed into a ultradiscrete form,

$$\begin{aligned} & w(t+1,m,n) + w(t-1,m,n) \\ & \qquad = \max(w(t,m+1,n),d) + \max(w(t,m-1,n),d) \\ & \qquad - \max(0,-w(t,m,n+1)+d) - \max(0,-w(t,m,n-1)+d). \end{aligned} \tag{17}$$

We consider Eq.(17) under the condition $c_1 = c_2$. Then Eq.(17) is reduced to

$$\begin{aligned} & w(t+1,m,n) + w(t-1,m,n) \\ & \qquad = \max(0,w(t,m+1,n)) + \max(0,w(t,m-1,n)) \\ & \qquad - \max(0,-w(t,m,n+1)) - \max(0,-w(t,m,n-1)). \end{aligned} \tag{18}$$

We have special solutions to Eq.(18). There are two types of solutions.
One is "positive wave" travelling along the m-axis with a speed $c_0(=\pm 1)$ and another is "negative wave" travelling along the n-axis with a speed $c_0(=\pm 1)$:

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1. Positive wave. Let $w(t,m,n) > 0$. Then Eq.(18) is reduced to

$$\begin{aligned} & w(t+1,m,n) + w(t-1,m,n) \\ & \qquad = \max(0,w(t,m+1,n)) + \max(0,w(t,m-1,n)) \\ & \qquad - \max(0,-w(t,m,n+1)) - \max(0,-w(t,m,n-1)). \\ & \qquad = w(t,m+1,n) + w(t,m-1,n). \end{aligned} \tag{19}$$

   which is solved if

$$w(t,m,n) = h(t - c_0 m), \quad c_0 = \pm 1, \tag{20}$$

   where $h(t - c_0 m) > 0$ denotes a positive wave travelling along the m-axis with a speed $c_0(=\pm 1)$.

2. Negative wave. Let $w(t,m,n) < 0$. Then Eq.(18) is reduced to

$$\begin{aligned} & w(t+1,m,n) + w(t-1,m,n) \\ & \qquad = -\max(0,-w(t,m,n+1)) - \max(0,-w(t,m,n-1)), \\ & \qquad = w(t,m,n+1) + w(t,m,n-1). \end{aligned} \tag{21}$$

which is solved if

$$w(t,m,n) = h'(t - c_0 n), \quad c_0 = \pm 1, \tag{22}$$

where $h'(t - c_0 n) < 0$ denotes a negative wave travelling along the n-axis with a speed $c_0(= \pm 1)$.

We note that these positive and negative waves have no corresponding solutions to the discrete wave equation of $w^t_{m,n}$,Eq.(15) with $c_1 = c_2$.

2.2 Coupled 3-D soliton equation

The Bäcklund transformation with $\beta_1 c_1 + \beta_2 c_2 = 0$,

$$[\alpha_1 e^{-(D_1-D_3)/2} - e^{(D_1-D_3)/2} + \beta_1 e^{-(D_2-D_4)/2}] f \cdot g = 0, \tag{23}$$
$$[\alpha_2 e^{-(D_2-D_3)/2} - e^{(D_2-D_3)/2} + \beta_2 e^{-(D_1-D_4)/2}] f \cdot g = 0, \tag{24}$$

are transformed into a coupled soliton equations.
We choose the parameters α_1, α_2 as

$$\alpha_1 = 1 - \beta_1, \quad \alpha_2 = 1 - \beta_2, \tag{25}$$

so that Eqs.(23) and (24) have soliton solutions.
First we arrange Eqs.(23) and (24) as

$$e^{(D_1-D_3)/2} f \cdot g = (1-\beta_1) e^{-(D_1-D_3)/2} f \cdot g + \beta_1 e^{-(D_2-D_4)/2} f \cdot g, \tag{26}$$
$$e^{-(D_1-D_4)/2} f \cdot g = (1-\beta_2^{-1}) e^{-(D_2-D_3)/2} f \cdot g + \beta_2^{-1} e^{(D_2-D_3)/2} f \cdot g. \tag{27}$$

Let us introduce new dependent variables $\hat{v}_1, \hat{v}_2, \hat{z}_1$ and $\hat{z}_2$ by

$$\begin{aligned}
\hat{v}_1(x_1,x_2,x_3) &= \frac{e^{-(D_2-D_4)/2} f \cdot g}{e^{-(D_1-D_3)/2} f \cdot g} \\
&= \frac{f(x_1-1/2, x_2-1, x_3-1/2) g(x_1+1/2, x_2+1, x_3+1/2)}{f(x_1-1/2, x_2, x_3+1/2) g(x_1+1/2, x_2, x_3-1/2)}, \\
\hat{z}_1(x_1,x_2,x_3) &= \frac{e^{(D_1-D_3)/2} f \cdot g}{e^{-(D_1-D_3)/2} f \cdot g} \\
&= \frac{f(x_1+1/2, x_2, x_3-1/2) g(x_1-1/2, x_2, x_3+1/2)}{f(x_1-1/2, x_2, x_3+1/2) g(x_1+1/2, x_2, x_3-1/2)},
\end{aligned}$$

$$\begin{aligned}
\hat{v}_2(x_1,x_2,x_3) &= \frac{e^{(D_2-D_3)/2} f \cdot g}{e^{-(D_2-D_3)/2} f \cdot g} \\
&= \frac{f(x_1, x_2+1/2, x_3-1/2) g(x_1, x_2-1/2, x_3+1/2)}{f(x_1, x_2-1/2, x_3+1/2) g(x_1, x_2+1/2, x_3-1/2)},
\end{aligned}$$

$$\hat{z}_2(x_1,x_2,x_3) = \frac{e^{-(D_1-D_4)/2} f\cdot g}{e^{-(D_2-D_3)/2} f\cdot g}$$
$$= \frac{f(x_1-1,x_2-1/2,x_3-1/2)g(x_1+1,x_2+1/2,x_3+1/2)}{f(x_1,x_2-1/2,x_3+1/2)g(x_1,x_2+1/2,x_3-1/2)}.$$

Accordingly Eqs.(26) and (27) are reduced to relations among $\hat{v}_1, \hat{v}_2, \hat{z}_1$ and $\hat{z}_2$

$$\hat{z}_1(x_1,x_2,x_3) = 1-\beta_1+\beta_1\hat{v}_1(x_1,x_2,x_3), \tag{28}$$
$$\hat{z}_2(x_1,x_2,x_3) = 1-\beta_2^{-1}+\beta_2^{-1}\hat{v}_2(x_1,x_2,x_3). \tag{29}$$

We define new variables v_1, v_2, z_1 and z_2, for notational convenience, shifting independent variables x_1, x_2 and x_3 by $1/2$

$$v_1(x_1,x_2,x_3) = \hat{v}_1(x_1+1/2,x_2,x_3+1/2), \\ = \frac{f(x_1,x_2-1,x_3)g(x_1+1,x_2+1,x_3+1)}{f(x_1,x_2,x_3+1)g(x_1+1,x_2,x_3)}, \tag{30}$$
$$z_1(x_1,x_2,x_3) = \hat{z}_1(x_1+1/2,x_2,x_3+1/2), \\ = \frac{f(x_1+1,x_2,x_3)g(x_1,x_2,x_3+1)}{f(x_1,x_2,x_3+1)g(x_1+1,x_2,x_3)}, \tag{31}$$
$$v_2(x_1,x_2,x_3) = \hat{v}_2(x_1,x_2+1/2,x_3+1/2), \\ = \frac{f(x_1,x_2+1,x_3)g(x_1,x_2,x_3+1)}{f(x_1,x_2,x_3+1)g(x_1,x_2+1,x_3)}, \tag{32}$$
$$z_2(x_1,x_2,x_3) = \hat{z}_2(x_1,x_2+1/2,x_3+1/2), \\ = \frac{f(x_1-1,x_2,x_3)g(x_1+1,x_2+1,x_3+1)}{f(x_1,x_2,x_3+1)g(x_1,x_2+1,x_3)}. \tag{33}$$

Accordingly Eqs.(28) and (29) read as

$$z_1(x_1,x_2,x_3) = 1-\beta_1+\beta_1 v_1(x_1,x_2,x_3), \tag{34}$$
$$z_2(x_1,x_2,x_3) = 1-\beta_2^{-1}+\beta_2^{-1} v_2(x_1,x_2,x_3). \tag{35}$$

Secondly we look for other relations among v_1, v_2, z_1 and z_2 than Eqs.(34) and (35).
Let us introduce shift operators p, q, s operating on an aritrary function $h(x_1,x_2,x_3)$ by

$$p^\alpha h(x_1,x_2,x_3) = h(x_1+\alpha,x_2,x_3), \tag{36}$$
$$q^\beta h(x_1,x_2,x_3) = h(x_1,x_2+\beta,x_3), \tag{37}$$
$$s^\gamma h(x_1,x_2,x_3) = h(x_1,x_2,x_3+\gamma). \tag{38}$$

Then logarithm of Eqs.(30),(32),(31),(33) are expressed by

$$\log v_1 = (q^{-1}-s)\log f + (pqs-p)\log g$$

$$
\begin{aligned}
&= (q^{-1} - s)[\log f - pq \log g], && (39)\\
\log v_2 &= (q - s)\log f + (s - q)\log g\\
&= (q - s)[\log f - \log g], && (40)\\
\log z_1 &= (p - s)\log f + (s - p)\log g\\
&= (p - s)[\log f - \log g], && (41)\\
\log z_2 &= (p^{-1} - s)\log f + (pqs - q)\log g\\
&= (p^{-1} - s)[\log f - pq\log g]. && (42)
\end{aligned}
$$

These expressions give us new relations among v_1, v_2, z_1 and z_2. From Eqs.(39),(42) we obtain

$$
(p^{-1} - s)\log v_1 = (q^{-1} - s)\log z_2, \tag{43}
$$

and from Eqs.(40),(41) we obtain

$$
(p - s)\log v_2 = (q - s)\log z_1. \tag{44}
$$

They are equivalent to the relations

$$
\frac{v_1(x_1 - 1, x_2, x_3)}{v_1(x_1, x_2, x_3 + 1)} = \frac{z_2(x_1, x_2 - 1, x_3)}{z_2(x_1, x_2, x_3 + 1)}, \tag{45}
$$

$$
\frac{v_2(x_1 + 1, x_2, x_3)}{v_2(x_1, x_2, x_3 + 1)} = \frac{z_1(x_1, x_2 + 1, x_3)}{z_1(x_1, x_2, x_3 + 1)}. \tag{46}
$$

Substituting Eqs.(34) and (35) into Eqs.(45) and (46) respectively we obtain a coupled 3-D soliton equation of v_1 and v_2,

$$
\frac{v_1(x_1 - 1, x_2, x_3)}{v_1(x_1, x_2, x_3 + 1)} = \frac{1 - \beta_2^{-1} + \beta_2^{-1} v_2(x_1, x_2 - 1, x_3)}{1 - \beta_2^{-1} + \beta_2^{-1} v_2(x_1, x_2, x_3 + 1)}, \tag{47}
$$

$$
\frac{v_2(x_1 + 1, x_2, x_3)}{v_2(x_1, x_2, x_3 + 1)} = \frac{1 - \beta_1 + \beta_1 v_1(x_1, x_2 + 1, x_3)}{1 - \beta_1 + \beta_1 v_1(x_1, x_2, x_3 + 1)}. \tag{48}
$$