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Solutions to a q-analog of Painlevé III equation of type $D_7^{(1)}$

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Abstract. This paper deals with a q-analog of Painlevé III equation of type $D_7^{(1)}$. We study its algebraic function solutions and transcendental function solutions. We construct algebraic function solutions expressed by Laurent polynomials and prove irreducibility in the sense of decomposable extensions.

Keywords: q-Painlevé equation; q-difference equation; algebraic function solution; τ function; irreducibility
MSC 2010: 33E17; 34M55; 39A13

1 Introduction

In the 1900s, Painlevé equations, $\mathrm{P_I}$, $\mathrm{P_{II}}$, $\ldots$ and $\mathrm{P_{VI}}$, are defined by P. Painlevé and B. Gambier. In the recent research, $\mathrm{P_{III}}$ was divided into three equations, $\mathrm{P_{III}^{D_6^{(1)}}}$, $\mathrm{P_{III}^{D_7^{(1)}}}$ and $\mathrm{P_{III}^{D_8^{(1)}}}$. Since the 1990s, discrete Painlevé equations have been studied actively from various points of view. Painlevé and discrete Painlevé equations (Painlevé systems) are now regarded as one of the most important classes of equations in the theory of integrable systems (see, for example, [3]). In 2001, Sakai stated that Painlevé systems are classified by theory of rational surfaces and the system of discrete Painlevé equations is constructed in a unified manner as the birational action of a translation of the corresponding affine Weyl group on a certain family of rational surfaces[15].

There are many kind of discrete Painlevé equations and some of them are regarded as the discrete analog of Painlevé equations. We note here that a discrete analog of $\mathrm{P_X}$ is a discrete Painlevé equation which leads to $\mathrm{P_X}$ by a continuous limit without loss of parameters. Most of discrete analogs of Painlevé equations have been already found, but discrete analogs of $\mathrm{P_{III}^{D_7^{(1)}}}$ and $\mathrm{P_{III}^{D_8^{(1)}}}$ were not known.

We show that the q-difference equation,

$$f(p^2t)f(t) = \frac{(f(pt)+p^{-1}t^{-1})(f(pt)+p^{-1}t^{-1}\alpha)}{1+f(pt)}, \tag{1.1}$$

where $\alpha, t, p \in \mathbb{C}^\times$, is the first model of a discrete analog of $\mathrm{P_{III}^{D_7^{(1)}}}$. In fact, by setting

$$(1+\alpha)t^{-1} = (1-p)^4 As, \quad \alpha t^{-2} = -(1-p)^6 s^2, \quad f(t) = (1-p)^2 X(s), \tag{1.2}$$

and letting $p \to 1$, we obtain $\mathrm{P_{III}^{D_7^{(1)}}}$,

$$X'' = \frac{(X')^2}{X} - \frac{X'}{s} - \frac{X^2}{s^2} + \frac{A}{s} - \frac{1}{X}\,, \tag{1.3}$$

where $X'=\frac{\mathrm{d}X}{\mathrm{d}s}$. Therefore we call (1.1) a q-Painlevé III equation of type $D_7^{(1)}$ (q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$). We note here that (1.1) is obtained by substituting

$$\beta=\alpha^{-1},\quad \gamma=1, \tag{1.4}$$

and putting

$$q=p^2,\quad g(t)=f(q^{-1/2}t), \tag{1.5}$$

in a q-analog of $\mathrm{P_V}$ (q-$\mathrm{P_V}$)[15],

$$g(qt)g(t)=\frac{(f(t)+t^{-1})(f(t)+\alpha t^{-1})}{1+\gamma f(t)},\quad f(q^{-1}t)f(t)=\frac{(g(t)+q^{1/2}\alpha\beta t^{-1})(g(t)+q^{1/2}\beta^{-1}t^{-1})}{1+\gamma^{-1}g(t)}, \tag{1.6}$$

where $\alpha,\beta,\gamma,t,q\in\mathbb{C}^\times$ are parameters. This specialization is called a projective reduction[4].

It is well known that $\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$ has the following properties:

(i) existence of algebraic function solutions which are rational functions of $s^{1/3}$;

(ii) irreducibility in the sense of P. Painlevé and H. Umemura.

The aim of this paper is to show that q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$ has quite similar properties to the above.

This paper is organized as follows: in Section 2, we introduce a representation of the affine Weyl group of type $A_4^{(1)}$, and then derive q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$ from the affine Weyl group. In Section 3, we construct algebraic function solutions to q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$ and show that each of them is expressed as a ratio of Laurent polynomials in $t^{1/3}$. In Section 4, we prove irreducibility of q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$ in the sense of decomposable extensions, which implies that any transcendental solution cannot be algebraically expressed by solutions of linear difference equations and solutions of first order algebraic difference equations. Concluding remarks are given in Section 5.

Throughout this paper, we use the following conventions of q-analysis with $|q|<1$ (cf. the books [2, 5]).
q-Shifted factorials:

$$(a;q)_k=\prod_{i=1}^{k}(1-aq^{i-1}),\quad (a_1,\cdots,a_s;q)_n=\prod_{j=1}^{s}(a_j;q)_n \tag{1.7}$$

$$(a;p,q)_k=\prod_{i,j=0}^{k-1}(1-p^iq^ja),\quad (a_1,\cdots,a_s;p,q)_n=\prod_{j=1}^{s}(a_j;p,q)_n. \tag{1.8}$$

Jacobi theta function:

$$\Theta(a;q)=(a;q)_\infty(qa^{-1};q)_\infty. \tag{1.9}$$

It holds that

$$(qa;q)_\infty=\frac{(a;q)_\infty}{1-a}, \tag{1.10}$$

$$(pa;p,q)_\infty=\frac{(a;p,q)_\infty}{(a;q)_\infty}, \tag{1.11}$$

$$\Theta(qa;q)=-\frac{\Theta(a;q)}{a}. \tag{1.12}$$

2 Affine Weyl group of type $A_4^{(1)}$

2.1 Projective reduction to q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$

We formulate the family of Bäcklund transformations of q-P_{V}, (1.6), as a birational representation of the affine Weyl group of type $A_4^{(1)}$. We refer to [12] for basic ideas of this formulation.

We define the transformations s_i $(i = 0, 1, 2, 3, 4)$, σ, and ι on variables f_j $(j = 0, 1, 2, 3, 4)$ and parameters a_k $(k = 0, 1, 2, 3, 4)$ by

$$s_i(a_j) = a_j {a_i}^{-a_{ij}}, \quad s_i(f_{i+2}) = \frac{a_{i+3}a_{i+4}(a_i a_{i+1} + a_{i+3} f_i)}{{a_{i+1}}^2 f_{i+3}}, \quad s_i(f_{i+4}) = \frac{a_{i+4}(a_{i+2} + a_{i+4} a_i f_{i+1})}{a_i a_{i+1} {a_{i+2}}^2 f_{i+3}},$$
$$s_i(f_j) = f_j \quad (j \neq i+2, i+4), \quad \sigma(a_i) = a_{i+1}, \quad \sigma(f_i) = f_{i+1}, \quad i \in \mathbb{Z}/5\mathbb{Z},$$
$$\iota : (a_0, a_1, a_2, a_3, a_4, f_0, f_1, f_2, f_3, f_4) \mapsto ({a_0}^{-1}, {a_4}^{-1}, {a_3}^{-1}, {a_2}^{-1}, {a_1}^{-1}, f_1, f_0, f_4, f_3, f_2).$$

Here the symmetric 5×5 matrix

$$A = (a_{ij})_{i,j=0}^{4} = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}, \tag{2.1}$$

is the Cartan matrix of type $A_4^{(1)}$. Note that f_i's have the conditions

$${a_{i+3}}^2 a_{i+4} f_i = a_{i+1}(a_i a_{i+1} f_{i+2} f_{i+3} - a_{i+3} a_{i+4}) \quad (i \in \mathbb{Z}/5\mathbb{Z}). \tag{2.2}$$

Proposition 2.1. *The group of birational transformations $\langle s_0, s_1, s_2, s_3, s_4, \sigma, \iota \rangle$ forms the affine Weyl group of type $A_4^{(1)}$, denoted by $\widetilde{W}(A_4^{(1)})$. Namely, the transformations satisfy the fundamental relations*

$${s_i}^2 = 1, \quad (s_i s_{i\pm 1})^3 = 1, \quad (s_i s_j)^2 = 1 \quad (j \neq i \pm 1), \quad \sigma^5 = 1, \quad \sigma s_i = s_{i+1}\sigma,$$
$$\iota^2 = 1, \quad \iota s_0 = s_0 \iota, \quad \iota s_1 = s_4 \iota, \quad \iota s_2 = s_3 \iota, \quad i \in \mathbb{Z}/5\mathbb{Z}.$$

In general, for a function $F = F(a_i, f_j)$, we let an element $w \in \widetilde{W}(A_4^{(1)})$ act as $w.F(a_i, f_j) = F(w.a_i.w, f_j)$, that is, w is an injective homomorphism. Note that $q = a_0 a_1 a_2 a_3 a_4$ is invariant under any action of $\langle s_0, s_1, s_2, s_3, s_4, \sigma \rangle$. We define the translations T_i $(i = 0, 1, 2, 3, 4)$ by

$$T_0 = \sigma s_4 s_3 s_2 s_1, \quad T_1 = \sigma s_0 s_4 s_3 s_2, \quad T_2 = \sigma s_1 s_0 s_4 s_3, \quad T_3 = \sigma s_2 s_1 s_0 s_4, \quad T_4 = \sigma s_3 s_2 s_1 s_0. \tag{2.3}$$

Their actions on a_i's are given by

$$\begin{aligned} T_0 &: (a_0, a_1, a_2, a_3, a_4) \mapsto (q a_0, q^{-1} a_1, a_2, a_3, a_4), \\ T_1 &: (a_0, a_1, a_2, a_3, a_4) \mapsto (a_0, q a_1, q^{-1} a_2, a_3, a_4), \\ T_2 &: (a_0, a_1, a_2, a_3, a_4) \mapsto (a_0, a_1, q a_2, q^{-1} a_3, a_4), \\ T_3 &: (a_0, a_1, a_2, a_3, a_4) \mapsto (a_0, a_1, a_2, q a_3, q^{-1} a_4), \\ T_4 &: (a_0, a_1, a_2, a_3, a_4) \mapsto (q^{-1} a_0, a_1, a_2, a_3, q a_4). \end{aligned}$$

Note that T_i's commute with each other and $T_0 T_1 T_2 T_3 T_4 = 1$. We introduce α, β, γ, t, f, and g by

$$\alpha = {a_4}^{-1}, \quad \beta = {a_2}^{1/2} {a_4}^{1/2}, \quad \gamma = q^{-1/4} {a_2}^{1/4} {a_3}^{1/2} {a_4}^{1/4}, \quad t = q^{-1/4} a_0 {a_2}^{1/4} {a_3}^{1/2} {a_4}^{1/4}, \tag{2.4}$$

$$f = q^{-3/4} {a_2}^{3/4} {a_3}^{3/2} {a_4}^{3/4} f_0, \quad g = q^{3/4} {a_2}^{-3/4} {a_3}^{-3/2} {a_4}^{-3/4} f_2. \tag{2.5}$$

Then the action of T_0 on f and g are expressed as

$$T_0(g)g = \frac{(f+t^{-1})(f+\alpha t^{-1})}{1+\gamma f}, \quad {T_0}^{-1}(f)f = \frac{(g+q^{1/2}\alpha\beta t^{-1})(g+q^{1/2}\beta^{-1}t^{-1})}{1+\gamma^{-1}g}, \tag{2.6}$$

which is equivalent to q-$\mathrm{P_V}$, (1.6). We regard T_0 and T_i $(i=1,2,3,4)$ as the time evolution and Bäcklund transformations of q-$\mathrm{P_V}$, respectively.

In order to derive q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$, we introduce the transformations R_0 and R_{13} defined by

$$R_0 = \sigma^3 s_2 s_1, \quad R_{13} = \sigma s_0 s_4 s_2. \tag{2.7}$$

Note that R_0 and R_{13} commute with each other. The transformations are not translations but their squares are translations,

$$ {R_0}^2 = T_0, \quad {R_{13}}^2 = T_1 T_3. \tag{2.8}$$

Considering the projection of the action of R_0 and R_{13} on the subspace of the parameter space $\beta = \alpha^{-1}$ and $\gamma = 1$ (or, $a_0 a_1 = a_3$ and $a_2 = a_4$), we have

$$\begin{aligned} &R_0 : \ (\alpha, t) \mapsto (\alpha, q^{1/2}t), \\ &R_{13} : \ (\alpha, t) \mapsto (q^{1/2}\alpha, t). \end{aligned}$$

Then the action of R_0 can be expressed as

$$R_0(f) = \frac{(f+t^{-1})(f+\alpha t^{-1})}{g(1+f)}, \qquad {R_0}^{-1}(f) = g, \tag{2.9}$$

which is equivalent to q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$, (1.1). We regard R_0 as the time evolution of q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$. The action of R_{13} can be expressed as

$$R_{13}(f) = \frac{q^{1/2}\alpha + q^{1/2}\alpha t f + tg}{t^2 fg}, \qquad {R_{13}}^{-1}(f) = \frac{q^{1/2}\alpha + q^{1/2}tf + \alpha t g + \alpha t f g}{q^{1/2} t f(tf+\alpha)}, \tag{2.10}$$

which is a Bäcklund transformation of (2.9) because of commutative property between R_0 and R_{13}. Therefore we obtain the following proposition:

Proposition 2.2. *Transformations T and T^{-1},*

$$T : (\alpha, t, f(t)) \mapsto \left(p\alpha, t, \frac{p\alpha + p\alpha t f(t) + tf(p^{-1}t)}{t^2 f(t) f(p^{-1}t)}\right), \tag{2.11}$$

$$T^{-1} : (\alpha, t, f(t)) \mapsto \left(p\alpha^{-1}, t, \frac{p\alpha + ptf(t) + \alpha t f(p^{-1}t) + \alpha t f(t) f(p^{-1}t)}{ptf(t)(tf(t)+\alpha)}\right), \tag{2.12}$$

are Bäcklund transformations of q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$, (1.1).

In general, we can derive various discrete Painlevé systems from elements of infinite order of affine Weyl groups that are not necessarily translations by taking a projection on a certain subspace of the parameter space. We call such a procedure a projective reduction[4].

2.2 τ function

We introduce the new variables τ_i $(i = 1, 2, \cdots, 7)$ with

$$f_2 = \frac{\tau_4\tau_5}{\tau_6\tau_7}, \quad f_4 = \frac{\tau_1\tau_2}{\tau_3\tau_7}, \tag{2.13}$$

and lift the representation of the affine Weyl group to τ_i's-level[16]:

Proposition 2.3. *The actions of s_i $(i = 0, 1, 2, 3, 4)$, ι, and σ are expressed on τ_k by the following*:

$$\begin{aligned}
&s_0(\tau_1) = \frac{a_4\left(a_0\tau_3\tau_4\tau_5 + a_2a_3\tau_1\tau_2\tau_6 + a_0a_3\tau_3\tau_6\tau_7\right)}{{a_0}^2a_1a_2\tau_4\tau_7}, \quad s_0(\tau_i) = \tau_i \ (i = 2, 3, 5, 6),\\
&s_0(\tau_4) = \frac{a_0a_4\left(a_0\tau_3\tau_4\tau_5 + a_2a_3\tau_1\tau_2\tau_6 + a_3\tau_3\tau_6\tau_7\right)}{a_1a_2\tau_1\tau_7},\\
&s_0(\tau_7) = \frac{a_4\left(a_0^2\tau_3\tau_4\tau_5 + a_3a_0\tau_3\tau_6\tau_7 + a_2a_3\tau_1\tau_2\tau_6\right)}{a_0a_1a_2\tau_1\tau_4},\\
&s_1(\tau_1) = \tau_2, \quad s_1(\tau_2) = \tau_1, \quad s_1(\tau_i) = \tau_i \ (i = 3, \cdots, 7),\\
&s_2(\tau_1) = \frac{a_0a_1\left(a_0\tau_4\tau_5 + a_2a_3\tau_6\tau_7\right)}{{a_3}^2\tau_3}, \quad s_2(\tau_3) = \frac{a_0a_1\left(a_0\tau_4\tau_5 + a_3\tau_6\tau_7\right)}{a_2{a_3}^2\tau_1},\\
&s_2(\tau_i) = \tau_i \ (i = 2, 4, 5, 6, 7),\\
&s_3(\tau_i) = \tau_i \ (i = 1, 2, 3, 5, 7),\\
&s_3(\tau_4) = \frac{a_2\left(a_2a_3\tau_1\tau_2 + a_0\tau_3\tau_7\right)}{{a_0}^2a_3a_4\tau_6}, \quad s_3(\tau_6) = \frac{a_2a_3\left(a_2\tau_1\tau_2 + a_0\tau_3\tau_7\right)}{{a_0}^2a_4\tau_4},\\
&s_4(\tau_4) = s_4(\tau_5), \quad s_4(\tau_5) = s_4(\tau_4), \quad s_4(\tau_i) = \tau_i \ (i = 1, 2, 3, 6, 7),\\
&\iota : (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7) = (\tau_4, \tau_5, \tau_6, \tau_1, \tau_2, \tau_3, \tau_7),\\
&\sigma(\tau_1) = \frac{a_0a_1\left(a_0\tau_4\tau_5 + a_3\tau_6\tau_7\right)}{a_2{a_3}^2\tau_1}, \quad \sigma(\tau_2) = \tau_3, \quad \sigma(\tau_3) = \tau_6,\\
&\sigma(\tau_4) = \frac{a_4\left({a_0}^2\tau_3\tau_4\tau_5 + a_3a_0\tau_3\tau_6\tau_7 + a_2a_3\tau_1\tau_2\tau_6\right)}{a_0a_1a_2\tau_1\tau_4}, \quad \sigma(\tau_5) = \tau_7,\\
&\sigma(\tau_6) = \tau_5, \quad \sigma(\tau_7) = \tau_2.
\end{aligned}$$

Since we are studying the property of $q\text{-P}_{\rm III}^{D_7^{(1)}}$, we consider the τ functions under the conditions $\beta = \alpha^{-1}$ and $\gamma = 1$ (or, $a_0a_1 = a_3$ and $a_2 = a_4$). We define the τ functions τ_N^n $(n, N \in \mathbb{Z})$ by

$$\tau_N^n = {R_0}^n {R_{13}}^N(\tau_4). \tag{2.14}$$

We note that τ_i's are expressed by τ_N^n as follows (Figure 1):

$$\tau_1 = \tau_2^0, \quad \tau_2 = \tau_1^2, \quad \tau_3 = \tau_2^2, \quad \tau_4 = \tau_0^0, \quad \tau_5 = \tau_2^1, \quad \tau_6 = \tau_1^1, \quad \tau_7 = \tau_1^0, \tag{2.15}$$

and (2.4) and (2.5) are rewritten as

$$f = f_0 = \frac{\tau_0^1\tau_2^2}{\tau_1^1\tau_1^2}, \quad t = a_0, \quad \alpha = {a_2}^{-1}. \tag{2.16}$$

3 Algebraic function solutions to $q\text{-}\mathbf{P}_{\rm III}^{D_7^{(1)}}$

In this section, we use the notation $\overline{F} = F(pt)$ for arbitrary function $F = F(t)$.

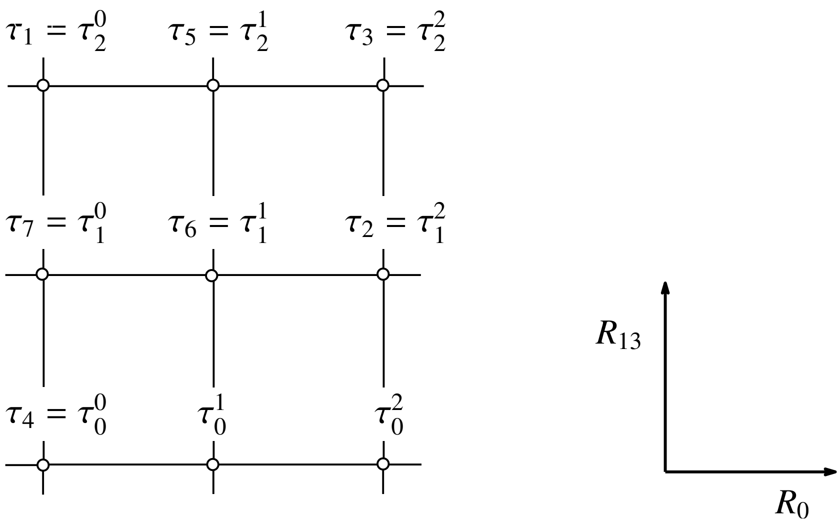

Figure 1. Configuration of the τ functions on the lattice.

3.1 Puiseux series representation of algebraic function solutions

Proposition 3.1. *Let f be an algebraic function solution to* (1.1). *Then*

$$f = \sum_{i=-2}^{\infty} c_i t^{i/3} \in \mathbb{C}(t^{1/3}), \tag{3.1}$$

where $c_i \in \mathbb{C}$ and $c_{-2} \neq 0$.

Proof. Let $L = \mathbb{C}(t, f, \overline{f})$. By Lemma 12 in [11], $L = \mathbb{C}(x)$ where $x^n = t$ ($n \in \mathbb{Z}_{>0}$) and $\overline{x} = p^{1/n}x$. Express f as

$$f = \frac{P}{Q}, \tag{3.2}$$

where $P, Q \in \mathbb{C}[x] \setminus \{0\}$ and P and Q are relatively prime. From (1.1), we obtain

$$x^{2n}\overline{P}\underline{P}(P + Q)Q = \overline{Q}\underline{Q}(x^n P + Q)(x^n P + \alpha Q). \tag{3.3}$$

Let $v_0(F)$ denote the maximum number k such that $x^k \mid F$ for $F \in \mathbb{C}[x] \setminus \{0\}$. Assume $x \nmid Q$. Then, from (3.3), it follows that

$$2n + 2v_0(P) + v_0(P + Q) = 0, \tag{3.4}$$

which implies $n = 0$. Therefore we obtain $x \mid Q$ and so $x \nmid P$. Put $m = v_0(Q) \in \mathbb{Z}_{>0}$. From (3.3), it follows that

$$2n + m \geq 2m + 2\min(n, m), \tag{3.5}$$

which implies $2n = 3m$.

We can express f as

$$f = \sum_{i=-m}^{\infty} c_i x^i, \tag{3.6}$$

where $c_i \in \mathbb{C}$, $c_{-m} \neq 0$ To show $m/2 \nmid i \Rightarrow c_i = 0$, we assume that there exists i such that $m/2 \nmid i$ and $c_i \neq 0$. Let $k \cdot \frac{m}{2} + l = \min\{i \mid m/2 \nmid i \text{ and } c_i \neq 0\}$, $1 \leq l < m/2$. Then f can be expressed as

$$f = c_{-m}x^{-m} + c_{-m/2}x^{-m/2} + \cdots + c_{km/2}x^{km/2} + c_{km/2+l}x^{km/2+l} + \cdots .$$

From (1.1), we obtain

$$\overline{f}\underline{f}(1+f) = (f + x^{-3m/2})(f + \alpha x^{-3m/2}). \tag{3.7}$$

The coefficient of $x^{km/2+l-2m} = x^{(k-4)m/2+l}$ of the left side is

$$\begin{aligned} &c_{km/2+l}p^{km/2n+l/n}c_{-m}p^{m/n}c_{-m} + c_{-m}p^{-m/n}c_{km/2+l}p^{-km/2n-l/n}c_{-m} \\ &+ c_{-m}p^{-m/n}c_{-m}p^{m/n}c_{km/2+l} \\ =&(p^{(k+2)m/2n+l/n} + p^{-(k+2)m/2n-l/n} + 1)c_{-m}^2 c_{km/2+l} \\ \neq& 0, \end{aligned} \tag{3.8}$$

and one of the right side is 0, a contradiction.

Therefore we find $f \in \mathbb{C}((x^{m/2})) \cap \mathbb{C}(x) = \mathbb{C}(x^{m/2})$. Then we have $L \subset \mathbb{C}(x^{m/2}) \subset \mathbb{C}(x) = L$, which yields $L = \mathbb{C}(x^{m/2}) = \mathbb{C}(x)$, and so $m = 2$ and $n = 3$ are obtained. Since $x^3 = t$, we can express f as

$$f = \sum_{i=-2}^{\infty} c_i t^{i/3} \in \mathbb{C}(t^{1/3}), \quad c_i \in \mathbb{C},\ c_{-2} \neq 0. \tag{3.9}$$

This expression is what we want. ■

In fact, when $\alpha = -1$, q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$ has the algebraic function solution,

$$f = -t^{-2/3}. \tag{3.10}$$

Moreover, using the Bäcklund transformation (2.11), we obtain the algebraic function solutions:

$$f = -t^{-2/3}(p^{1/3} + (1-p^{1/3})t^{1/3}) \quad (\alpha = -p), \tag{3.11}$$

$$f = -t^{-2/3}\frac{p^{4/3} + p^{1/3}(1-p^{4/3})t^{1/3} + p^{-1/3}(1-p^{1/3})(1-p^{5/3})t^{2/3}}{(p^{1/3} + (1-p^{1/3})t^{1/3})(p^{1/3} + p^{-1/3}(1-p^{1/3})t^{1/3})} \quad (\alpha = -p^2). \tag{3.12}$$

By using the continuous limit (1.2) and $p \to 1$, we obtain the algebraic function solution to $\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$ as follows:

$$X = -s^{2/3} \quad (A = 0), \tag{3.13}$$

$$X = \frac{2}{9}s^{1/3}\left(-\frac{3+9s^{1/3}}{2}\right) \quad (A = 1), \tag{3.14}$$

$$X = \frac{2}{3}s^{1/3}\frac{\begin{vmatrix} \dfrac{35+90s^{1/3}+81s^{2/3}}{8} & -\dfrac{105+315s^{1/3}+405s^{2/3}+243s}{16} \\ 1 & -\dfrac{3+9s^{1/3}}{2} \end{vmatrix}}{\left(-\dfrac{3+9s^{1/3}}{2}\right)^2} \quad (A = 2). \tag{3.15}$$

These solutions to $\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$ will be seen in [13] and K. Kajiwara, T. Masuda, and Y. Ohta anticipate determinant formula of them as follows:

$$X = \frac{2(2N-1)}{9} s^{1/3} \frac{\psi_N \psi_{N-2}}{{\psi_{N-1}}^2}, \quad (A = N \in \mathbb{Z}_{\geq 0}), \tag{3.16}$$

where

$$\psi_{-2} = \frac{9}{2} s^{1/3}, \quad \psi_{-1} = \psi_0 = 1, \tag{3.17}$$

$$\psi_N = \begin{vmatrix} P_N^{(N-1)}(s) & P_{N+1}^{(N-1)}(s) & \cdots & P_{2N-1}^{(N-1)}(s) \\ P_{N-2}^{(N-2)}(s) & P_{N-1}^{(N-2)}(s) & \cdots & P_{2N-3}^{(N-2)}(s) \\ \vdots & \vdots & \ddots & \vdots \\ P_{-N+2}^{(0)}(s) & P_{-N+3}^{(0)}(s) & \cdots & P_1^{(0)}(s) \end{vmatrix} \quad (N > 0), \tag{3.18}$$

$$P_n^{(k)}(s) = 0 \quad (n < 0), \quad P_n^{(k)}(s) = L_n^{(-k-n-3/2)}\left(\frac{9}{2} s^{1/3}\right) \quad (n \geq 0). \tag{3.19}$$

Here $L_n^{(\alpha)}(x)$ is Laguerre Polynomial,

$$L_n^{(\alpha)}(x) = \sum_{r=0}^{n} \frac{(-1)^r \Pi_{k=1}^{n-r}(n+\alpha-k+1)}{(n-r)!r!} x^r. \tag{3.20}$$

3.2 Algebraic τ function

In [7, 8, 9], τ functions expressed by gauge functions and basic hypergeometric function are called hypergeometric τ functions. In this section, we construct τ functions expressed by gauge functions and algebraic functions. We call them algebraic τ functions.

We assume that τ_N^n are functions of t such that

$$\tau_N^n = \tau_N^0(p^n t), \tag{3.21}$$

and

$$\alpha = -p^2. \tag{3.22}$$

By the action of the affine Weyl group, τ_N^n is determined as a rational function in τ_0^n, τ_1^n, and τ_2^n (or $\tau_1, \tau_2, \ldots, \tau_7$). Thus, we only have to determine τ_0^n, τ_1^n, and τ_2^n. From (2.7), (2.16), and Proposition 2.3, we see that the action of R_0 on τ_i is given by

$$R_0(\tau_1) = \tau_5, \quad R_0(\tau_5) = \tau_3, \quad R_0(\tau_6) = \tau_2, \quad R_0(\tau_7) = \tau_6, \tag{3.23}$$

$$R_0(\tau_2) = \frac{t^2\tau_3(t\tau_4\tau_5 + p\alpha\tau_6\tau_7) + pt^2\tau_1\tau_2\tau_6}{p\alpha\tau_1\tau_7}, \tag{3.24}$$

$$R_0(\tau_3) = \frac{t\tau_3(t\tau_4\tau_5 + p\alpha\tau_6\tau_7) + p\tau_1\tau_2\tau_6}{p\alpha\tau_1\tau_4}, \tag{3.25}$$

$$R_0(\tau_4) = \frac{t\tau_4\tau_5 + p\alpha\tau_6\tau_7}{p\tau_1}, \tag{3.26}$$

$$R_0^{-1}(\tau_1) = \frac{p\alpha t^2\tau_3\tau_7(\tau_4\tau_5 + \tau_6\tau_7) + t\tau_1\tau_2(t\tau_4\tau_5 + p\tau_6\tau_7)}{p^3\alpha\tau_2\tau_3\tau_4}, \tag{3.27}$$

$$R_0^{-1}(\tau_4) = \frac{t\tau_4\tau_5 + p\tau_6\tau_7}{p\alpha\tau_3}, \tag{3.28}$$

$$R_0^{-1}(\tau_7) = \frac{t^2(\tau_4\tau_5 + \tau_6\tau_7)}{p^2\alpha\tau_2}. \tag{3.29}$$

Lemma 3.1. *Equations* (3.24), (3.25), *and* (3.27) *can be eliminated.*

Proof. Erasing the term "$t\tau_4\tau_5 + p\alpha\tau_6\tau_7$" from (3.24) by using (3.26), we obtain

$$\tau_7 = \frac{t^2(R_0(\tau_4)\tau_3 + \tau_2\tau_6)}{\alpha R_0(\tau_2)}, \tag{3.30}$$

which is equivalent to (3.29). Erasing the term "$t\tau_4\tau_5 + p\alpha\tau_6\tau_7$" from (3.25) by using (3.26), we obtain

$$\tau_4 = \frac{t\tau_3 R_0(\tau_4) + \tau_2\tau_6}{\alpha R_0(\tau_3)}, \tag{3.31}$$

which is equivalent to (3.28). Finally, erasing the terms "$\tau_4\tau_5 + \tau_6\tau_7$" and "$t\tau_4\tau_5 + p\tau_6\tau_7$" from (3.27) by using (3.28) and (3.29), we obtain

$$\tau_4 = \frac{t{R_0}^{-1}(\tau_4)\tau_1 + p^2\alpha\tau_7{R_0}^{-1}(\tau_7)}{p^2{R_0}^{-1}(\tau_1)}, \tag{3.32}$$

which is equivalent to (3.26). ■

By (2.15) and (3.22), we rewrite (3.26), (3.28), and (3.29) as follows:

$$\tau_2^1\tau_0^2 = t\tau_0^1\tau_2^2 - p^2\tau_1^1\tau_1^2, \tag{3.33}$$
$$\tau_2^3\tau_0^0 = -p^{-2}t\tau_0^1\tau_2^2 - p^{-2}\tau_1^1\tau_1^2, \tag{3.34}$$
$$\tau_1^3\tau_1^0 = -p^{-2}t^2\tau_0^1\tau_2^2 - p^{-2}t^2\tau_1^1\tau_1^2, \tag{3.35}$$

respectively. Thus, the action of R_0 on τ_i is equivalent to the bilinear equations (3.33)–(3.35). By elementary calculations, we can verify that

$$\begin{aligned}\tau_0^n =& -p^{(13+24n)/12}t^2\\ &\times\frac{(p^{n/3}t^{1/3}, -p^{(-1+n)/3}t^{1/3}; p^{1/3}, p^{2/3})_\infty (p^{(-5+4n)/12}t^{1/3}; p^{1/3}, p^{1/6})_\infty}{(p^{(11-4n)/12}t^{-1/3}; p^{1/3}, p^{1/6})_\infty}\\ &\times\left(\frac{\Theta(p^{(1+4n)/12}t^{1/3}; p^{1/3})}{\Theta(p^{(-5+4n)/12}t^{1/3}; p^{1/3})}\right)^{1/4},\end{aligned} \tag{3.36}$$

$$\begin{aligned}\tau_1^n =& p^n t(p^{1/3} + p^{(n-2)/3}(1-p^{1/3})t^{1/3})\\ &\times\frac{(p^{n/3}t^{1/3}, -p^{(-2+n)/3}t^{1/3}; p^{1/3}, p^{2/3})_\infty (p^{(-7+4n)/12}t^{1/3}; p^{1/3}, p^{1/6})_\infty}{(p^{(13-4n)/12}t^{-1/3}; p^{1/3}, p^{1/6})_\infty}\\ &\times\left(\frac{\Theta(p^{(-5+4n)/12}t^{1/3}; p^{1/3})}{\Theta(p^{(1+4n)/12}t^{1/3}; p^{1/3})}\right)^{1/4},\end{aligned} \tag{3.37}$$

$$\begin{aligned}\tau_2^n =& \big(p^{4/3} + p^{(n-1)/3}(1-p^{4/3})t^{1/3} + p^{(2n-5)/3}(1-p^{1/3})(1-p^{5/3})t^{2/3}\big)\\ &\times\frac{(p^{n/3}t^{1/3}, -p^{(-3+n)/3}t^{1/3}; p^{1/3}, p^{2/3})_\infty (p^{(-9+4n)/12}t^{1/3}; p^{1/3}, p^{1/6})_\infty}{(p^{(15-4n)/12}t^{-1/3}; p^{1/3}, p^{1/6})_\infty}\\ &\times\left(\frac{\Theta(p^{(1+4n)/12}t^{1/3}; p^{1/3})}{\Theta(p^{(-5+4n)/12}t^{1/3}; p^{1/3})}\right)^{1/4},\end{aligned} \tag{3.38}$$

is a solution to (3.33)–(3.35). Incidentally, the algebraic function solution (3.12),

$$f = \frac{\tau_0^1\tau_2^2}{\tau_1^1\tau_1^2} = -t^{-2/3}\frac{p^{4/3} + p^{1/3}(1-p^{4/3})t^{1/3} + p^{-1/3}(1-p^{1/3})(1-p^{5/3})t^{2/3}}{(p^{1/3} + (1-p^{1/3})t^{1/3})(p^{1/3} + p^{-1/3}(1-p^{1/3})t^{1/3})}, \tag{3.39}$$

is useful to find out this solution.

We next express τ_N^n for a general $N\in\mathbb{Z}$ by gauge functions and algebraic functions. From (3.26), (3.28), and (3.29), we obtain the following bilinear equations:

$$p^{N+2}\tau_{N+1}^{n+1}\tau_{N+1}^{n}=-\tau_{N+2}^{n}\tau_{N}^{n+1}+p^{n-1}t\tau_{N+2}^{n+1}\tau_{N}^{n},\tag{3.40}$$
$$\tau_{N+1}^{n+1}\tau_{N+1}^{n}=-p^{N+2}\tau_{N+2}^{n+2}\tau_{N}^{n-1}-p^{n-1}t\tau_{N+2}^{n+1}\tau_{N}^{n},\tag{3.41}$$
$$\tau_{N+2}^{n}\tau_{N}^{n-1}=-\tau_{N+1}^{n-1}\tau_{N+1}^{n}-p^{6-2n+N}t^{-2}\tau_{N+1}^{n-2}\tau_{N+1}^{n+1}.\tag{3.42}$$

We define ψ_N^n by

$$\begin{aligned}\tau_N^n=&\frac{(p^{n/3}t^{1/3},-p^{(-1+n-N)/3}t^{1/3};p^{1/3},p^{2/3})_\infty(p^{(-5+4n-2N)/12}t^{1/3};p^{1/3},p^{1/6})_\infty}{(p^{(11-4n+2N)/12}t^{-1/3};p^{1/3},p^{1/6})_\infty}\\&\times\left(\frac{\Theta(p^{(1+4n)/12}t^{1/3};p^{1/3})}{\Theta(p^{(-5+4n)/12}t^{1/3};p^{1/3})}\right)^{(-1)^N/4}\psi_N^n.\end{aligned}\tag{3.43}$$

Then it holds that

$$\psi_0^n=-p^{(13+24n)/12}t^2,\tag{3.44}$$
$$\psi_1^n=p^nt(p^{1/3}+p^{(n-2)/3}(1-p^{1/3})t^{1/3}),\tag{3.45}$$
$$\psi_2^n=p^{4/3}+p^{(n-1)/3}(1-p^{4/3})t^{1/3}+p^{(2n-5)/3}(1-p^{1/3})(1-p^{5/3})t^{2/3},\tag{3.46}$$

and (3.40)–(3.42) are rewritten as

$$\begin{aligned}&(-1)^{N+1}p^{(8+4N-(1-2n)(-1)^N)/4}t^{((-1)^N-1)/2}\psi_{N+1}^{n+1}\psi_{N+1}^{n}\\&\quad=p^{(-4-N+2n)/4}(1+p^{(-3+n-N)/3}t^{1/3})\psi_{N+2}^{n}\psi_{N}^{n+1}-p^{(-8+N+10n)/12}t^{1/3}\psi_{N+2}^{n+1}\psi_{N}^{n},\end{aligned}\tag{3.47}$$
$$\begin{aligned}&(-1)^{N+1}p^{-(1-2n)(-1)^N/4}t^{((-1)^N-1)/2}\psi_{N+1}^{n+1}\psi_{N+1}^{n}\\&\quad=p^{(4+3N+2n)/4}(1-p^{(n-1)/3}t^{1/3})\psi_{N+2}^{n+2}\psi_{N}^{n-1}+p^{(-8+N+10n)/12}t^{1/3}\psi_{N+2}^{n+1}\psi_{N}^{n},\end{aligned}\tag{3.48}$$
$$\begin{aligned}&(-1)^{N+1}p^{(6+9(-1)^N-2(1+3(-1)^N)n+N)/12}t^{-(1+3(-1)^N)/6}\psi_{N+2}^{n}\psi_{N}^{n-1}\\&\quad=\psi_{N+1}^{n-1}\psi_{N+1}^{n}+t^{-2/3}(p^{(2-n)/3}-t^{1/3})(p^{(4-n+N)/3}+t^{1/3})\psi_{N+1}^{n-2}\psi_{N+1}^{n+1},\end{aligned}\tag{3.49}$$

respectively. For convenience, we introduce

$$u=t^{1/3}.\tag{3.50}$$

It is obvious that ψ_N^n is a rational function for u from (3.49) and initial conditions (3.44)–(3.46). Set

$$\psi_N^0=\psi_N(t)=u^{e_N}\frac{P_N(u)}{Q_N(u)},\tag{3.51}$$

where P_N is a polynomial, Q_N is a monic polynomial, $e_N\in\mathbb{Z}$, P_N and Q_N are relatively prime, and $P_N(0),Q_N(0)\neq 0$. We shall show that ψ_N^n is a Laurent polynomial by the following lemma:

Lemma 3.2. *It holds that*

(i) $Q_N=1$*;*

(ii) $P_N,\overline{P}_N$ *:relatively prime;*

(iii) $P_N,\overline{\overline{P}}_N$ *:relatively prime.*

Proof. We shall prove this lemma by induction for $N \in \mathbb{Z}_{\geq 0}$. It is obvious for or $N = 0, 1$. We assume that (i)–(iii) are hold for $N = 0, 1, \ldots, M-1$ ($M \geq 2$). From (3.49),

$$\psi_M \underline{\psi}_{M-2} = p^{-e_{M-2}/3} u^{e_M + e_{M-2}} \frac{P_M \underline{P}_{M-2}}{Q_M}, \tag{3.52}$$

is a Laurent polynomial, which implies

$$Q_M \mid \underline{P}_{M-2}. \tag{3.53}$$

From (3.47) and the fact that $\overline{\psi}_M \psi_{M-2}$ is a Laurent polynomial,

$$(1 + p^{(-1-M)/3} u) \psi_M \overline{\psi}_{M-2} = p^{e_{M-2}/3} (1 + p^{(-1-M)/3} u) u^{e_M + e_{M-2}} \frac{P_M \overline{P}_{M-2}}{Q_M}, \tag{3.54}$$

is a Laurent polynomial, which implies

$$Q_M \mid (1 + p^{(-1-M)/3} u) \overline{P}_{M-2}. \tag{3.55}$$

From (3.48) and the fact that $\overline{\psi}_M \psi_{M-2}$ is a Laurent polynomial,

$$(1 - p^{-1/3} u) \overline{\overline{\psi}}_M \underline{\psi}_{M-2} = p^{(2e_M - e_{M-2})/3} u^{e_M + e_{M-2}} (1 - p^{-1/3} u) \frac{\overline{\overline{P}}_M \underline{P}_{M-2}}{\overline{\overline{Q}}_M}, \tag{3.56}$$

is a Laurent polynomial, which implies

$$Q_M \mid (1 - p^{-1} u) \underline{\underline{\underline{P}}}_{M-2}. \tag{3.57}$$

From (ii), (iii), (3.53), (3.55), and (3.57), it holds that

$$Q_M \mid (1 + p^{(-1-M)/3} u), \tag{3.58}$$

$$Q_M \mid (1 - p^{-1} u). \tag{3.59}$$

Therefore $Q_M = 1$, and so (i) holds for $N = M$.

We next prove (ii). Set $S_1 = gcd(P_M, \overline{P}_M)$ where S_1 is a monic polynomial. From (3.47)–(3.49), we have

$$\begin{aligned}
&(-1)^{M+1} p^{(4M-(-1)^M)/4 + e_{M-1}/3} u^{3((-1)^M - 1)/2 + 2e_{M-1}} \overline{P}_{M-1} P_{M-1} \\
&\quad = p^{(-2-M)/4 + e_{M-2}/3} u^{e_M + e_{M-2}} (1 + p^{(-1-M)/3} u) P_M \overline{P}_{M-2} \\
&\quad - p^{(-10+M)/12 + e_M/3} u^{1 + e_M + e_{M-2}} \overline{P}_M P_{M-2},
\end{aligned} \tag{3.60}$$

$$\begin{aligned}
&(-1)^{M-1} p^{(-1)^{M+1}/4 + e_{M-1}/3} u^{3((-1)^M - 1)/2 + 2e_{M-1}} \overline{P}_{M-1} P_{M-1} \\
&\quad = p^{(-2+3M)/4 + (2e_M - e_{M-2})/3} u^{e_M + e_{M-2}} (1 - p^{-1/3} u) \overline{\overline{P}}_M \underline{P}_{M-2} \\
&\quad + p^{(-10+M)/12 + e_M/3} u^{1 + e_M + e_{M-2}} \overline{P}_M P_{M-2},
\end{aligned} \tag{3.61}$$

$$\begin{aligned}
&(p^{2/3} - u)(p^{(2+M)/3} + u) p^{-e_{M-1}/3} u^{-2 + 2e_{M-1}} \underline{\underline{P}}_{M-1} \overline{P}_{M-1} \\
&\quad = (-1)^{M+1} p^{(4 + 9(-1)^M + M)/12 - e_{M-2}/3} u^{-(1 + 3(-1)^M)/2 + e_M + e_{M-2}} P_M \underline{P}_{M-2} \\
&\quad - p^{-e_{M-1}/3} u^{2e_{M-1}} \underline{P}_{M-1} P_{M-1}.
\end{aligned} \tag{3.62}$$

From (3.60) and (3.61), we obtain

$$S_1 \mid \overline{P}_{M-1} P_{M-1}, \tag{3.63}$$

$$S_1 \mid P_{M-1} \underline{P}_{M-1}, \tag{3.64}$$

respectively. From (3.62) and (3.64), we obtain

$$S_1 \mid (p^{2/3} - u)(p^{(2+M)/3} + u)\underline{\underline{P}}_{M-1}\overline{P}_{M-1}. \tag{3.65}$$

From (3.62) and (3.63), we obtain

$$S_1 \mid (p^{2/3} - p^{1/3}u)(p^{(2+M)/3} + p^{1/3}u)\underline{P}_{M-1}\overline{\overline{P}}_{M-1}. \tag{3.66}$$

Since $\overline{P}_{M-1}P_{M-1}$ and $\underline{P}_{M-1}\overline{\overline{P}}_{M-1}$ are relatively prime, it follows by (3.63) and (3.66) that

$$S_1 \mid (p^{2/3} - p^{1/3}u)(p^{(2+M)/3} + p^{1/3}u). \tag{3.67}$$

Since $P_{M-1}\underline{P}_{M-1}$ and $\underline{\underline{P}}_{M-1}\overline{P}_{M-1}$ are relatively prime, it follows by (3.64) and (3.65) that

$$S_1 \mid (p^{2/3} - u)(p^{(2+M)/3} + u). \tag{3.68}$$

Therefore $S_1 = 1$, and so (ii) holds for $N = M$.

We finally prove (iii). Set $S_2 = gcd(P_M, \overline{\overline{P}}_M)$ where S_2 is a monic polynomial. From (3.47) and (3.48), we obtain

$$\begin{aligned}(p^M + 1)(-1)^{M+1}p^{-(-1)^M/4+e_{M-1}/3}u^{3(-1)^M/2+2e_{M-1}}\overline{P}_{M-1}P_{M-1}\\ = p^{(-2-M)/4+e_{M-2}/3}u^{3/2+e_M+e_{M-2}}(1 + p^{(-1-M)/3}u)P_M\overline{P}_{M-2}\\ + p^{(-2+3M)/4+(2e_M-e_{M-2})/3}u^{3/2+e_M+e_{M-2}}(1 - p^{-1/3}u)\overline{\overline{P}}_M\underline{P}_{M-2},\end{aligned} \tag{3.69}$$

which implies

$$S_2 \mid \overline{P}_{M-1}P_{M-1}. \tag{3.70}$$

From (3.60), (3.70), and (ii), we obtain

$$S_2 \mid P_{M-2}. \tag{3.71}$$

From $\overline{S}_2 \mid \overline{P}_{M-2}$ and (3.60), we obtain

$$S_2 \mid P_{M-1}\underline{P}_{M-1}. \tag{3.72}$$

From $\underline{S}_2 \mid \underline{P}_{M-2}$ and (3.61), we obtain

$$S_2 \mid \overline{\overline{P}}_{M-1}\overline{P}_{M-1}. \tag{3.73}$$

From (3.70), (3.72), and (3.73), $S_2 = 1$, and so (iii) holds for $N = M$. We can prove this lemma for $N \in \mathbb{Z}_{<0}$ in a similar way. ■

Therefore we obtain the following theorem:

Theorem 3.1. *The functions,*

$$f = (-1)^N p^{(2-3(-1)^N+N)/12} t^{-(1+3(-1)^N)/6} \frac{\psi_N(pt)\psi_{N+2}(p^2t)}{\psi_{N+1}(pt)\psi_{N+1}(p^2t)}, \tag{3.74}$$

are algebraic function solutions to (1.1) *with* $\alpha = -p^{2+N}$. *Here* $\psi_N(t)$ *is a Laurent polynomial for* $t^{1/3}$ *constructed by*

$$\begin{aligned}(-1)^{N+1}p^{(6+9(-1)^N+N)/12}t^{-(1+3(-1)^N)/6}\psi_{N+2}(t)\psi_N(p^{-1}t)\\ = \psi_{N+1}(p^{-1}t)\psi_{N+1}(t) + (p^{2/3} - t^{1/3})(p^{(4+N)/3} + t^{1/3})t^{-2/3}\psi_{N+1}(p^{-2}t)\psi_{N+1}(pt),\end{aligned} \tag{3.75}$$

under the initial conditions

$$\psi_0(t) = -p^{13/12}t^2, \tag{3.76}$$

$$\psi_1(t) = t(p^{1/3} + p^{-2/3}(1 - p^{1/3})t^{1/3}). \tag{3.77}$$

4 Irreducibility of q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$

In this section, we prove irreducibility of q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$. We use the following terms of difference algebra.

Throughout this section every field is of characteristic zero. When K is a field and τ is an isomorphism of K into itself, namely an injective endomorphism, the pair $\mathcal{K} = (K, \tau)$ is called a difference field. We call τ the (transforming) operator and K the underlying field. For $a \in K$, an element $\tau^n a \in K$, $n \in \mathbb{Z}$, is called the n-th transform of a and is frequently denoted by a_n if it exists. If $\tau K = K$, we say that $\mathcal{K}$ is inversive. If $K/\tau K$ is algebraic, we say that $\mathcal{K}$ is almost inversive. For difference fields $\mathcal{K} = (K, \tau)$ and $\mathcal{K}' = (K', \tau')$, $\mathcal{K}'/\mathcal{K}$ is called a difference field extension if K'/K is a field extension and $\tau'|_K = \tau$. In this case we say that $\mathcal{K}'$ is a difference overfield of $\mathcal{K}$ or $\mathcal{K}$ is a difference subfield of $\mathcal{K}'$. For brevity we sometimes use (K, τ') instead of $(K, \tau'|_K)$. Let $\mathcal{K}$ be a difference field, $\mathcal{L} = (L, \tau)$ a difference overfield of $\mathcal{K}$ and B a subset of L. The difference subfield $\mathcal{K}\langle B\rangle_{\mathcal{L}}$ of $\mathcal{L}$ is defined to be the difference field $(K(B, \tau B, \tau^2 B, \dots), \tau)$ and is denoted by $\mathcal{K}\langle B\rangle$ for brevity. A solution of a difference equation over $\mathcal{K}$ is defined to be an element of some difference overfield of $\mathcal{K}$ which satisfies the equation (cf. the books [1, 6]).

We say that q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$ is irreducible if there is no transcendental function solution in any decomposable extension of $(\mathbb{C}(t), t \mapsto pt)$ (cf. [10, 11]). The irreducibility implies that any transcendental function solution cannot be algebraically expressed by solutions of linear difference equations and solutions of first order algebraic difference equations. To obtain the irreducibility of q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$, we only have to prove the following theorem because of Lemma 4.10 in the paper[10] (cf. Lemma 9 in [11]):

Lemma 4.1 (Lemma 4.10 in [10])**.** *Let $\mathcal{K}$ be an almost inversive difference field, $\mathcal{D}$ a decomposable extension of $\mathcal{K}$ and $B \subset D$. Suppose that for any inversive difference overfield $\mathcal{L}$ of $\mathcal{K}$ and for any difference overfield $\mathcal{U}$ of $\mathcal{L}$ with $\mathcal{K}\langle B\rangle_{\mathcal{D}} \subset \mathcal{U}$, the following holds,*

$$\text{tr. deg } \mathcal{L}\langle B\rangle_{\mathcal{U}}/\mathcal{L} \leq 1 \Rightarrow \text{any } f \in B \text{ is algebraic over } L.$$

Then any $f \in B$ is algebraic over K.

Theorem 4.1. *Let $p \in \mathbb{C}^\times$ be not a root of unity, $\mathcal{L}$ an inversive difference overfield of $(\mathbb{C}(t), t \mapsto pt)$, $\mathcal{U} = (U, \tau)$ a difference overfield of $\mathcal{L}$ and $f \in U$ satisfy*

$$\overline{\overline{f}} f = \frac{(\overline{f} + p^{-1}t^{-1})(\overline{f} + p^{-1}t^{-1}\alpha^{-1})}{1 + \overline{f}}, \tag{4.1}$$

where $\overline{f} = \tau f$ and $\overline{\overline{f}} = \tau^2 f$. Then we obtain

$$\text{tr. deg } \mathcal{L}\langle f\rangle/\mathcal{L} \leq 1 \Rightarrow f \text{ is algebraic over } L. \tag{4.2}$$

The proof of Theorem 4.1 is given later. We now prove the following lemma:

Lemma 4.2. *Let $p \in \mathbb{C}^\times$ be not a root of unity, $\mathcal{L}$ an inversive difference overfield of $(\mathbb{C}(t), t \mapsto pt)$, $\mathcal{U} = (U, \tau)$ a difference overfield of $\mathcal{L}$ and $f \in \mathcal{U}$ satisfy* (4.1)*. If tr. deg $\mathcal{L}\langle f\rangle/\mathcal{L} = 1$, then there are $n, l \in \mathbb{Z}_{\geq 0}$ such that*

$$\alpha^{n-2l} = p^n, \tag{4.3}$$

$0 \leq l \leq n$, $n \neq 0$ and $2l \neq n$.

Proof. There is an irreducible polynomial $F \in L[X,Y] \setminus \{0\}$ such that $F(f,\overline{f}) = 0$. We set

$$F = \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} a_{i,j} X^i Y^j, \tag{4.4}$$

where $a_{i,j} \in L$ and define $F^*, F_0, F_1 \in L[X,Y] \setminus \{0\}$ as

$$F^* = \sum_{i=0}^{n_0} \sum_{j=0}^{n_1} \tau(a_{i,j}) X^i Y^j, \tag{4.5}$$

$$F_0(X,Y) = Y^{n_0}(1+X)^{n_0} F\left(\frac{(X+p^{-1}t^{-1})(X+p^{-1}t^{-1}\alpha^{-1})}{Y(1+X)}, X\right), \tag{4.6}$$

$$F_1(X,Y) = X^{n_1}(1+Y)^{n_1} F^*\left(Y, \frac{(Y+p^{-1}t^{-1})(Y+p^{-1}t^{-1}\alpha^{-1})}{X(1+Y)}\right), \tag{4.7}$$

respectively. Now we prove this lemma by dividing the proof into the following eight steps:

Step 1. *We prove that $F|F_1$ and $F^*|F_0$ in $L[X,Y]$.*

We find that $(f,\overline{f})$ is a zero of F and F_1, and $(\overline{f},\overline{\overline{f}})$ is a zero of F^* and F_0. By the assumption $\mathrm{tr.deg}\,\mathcal{L}\langle f\rangle/\mathcal{L} = 1$, we see that $\overline{f}$ is transcendental over L, which implies $F|F_1$ and $F^*|F_0$, the required (cf. the book [17], Ch. II, §13, Lemma 2).

Step 2. *We prove $n_0 = n_1$.*

By using $F|F_1$ and $F^*|F_0$ in $L[X,Y]$, we get the following:

$$n_0 = \deg_X F \le \deg_X F_1 \le n_1, \tag{4.8}$$

$$n_0 \ge \deg_Y F_0 \ge \deg_Y F^* = \deg_Y F = n_1. \tag{4.9}$$

We set

$$n = n_0 = n_1, \tag{4.10}$$

and define $P \in L[X,Y]$ as

$$F_1 = PF. \tag{4.11}$$

Step 3. *We prove that the following equations hold:*

$$(Y+p^{-1}t^{-1})^n (Y+p^{-1}t^{-1}\alpha^{-1})^n \sum_{i=0}^{n} \tau(a_{i,n}) Y^i = P \sum_{i=0}^{n} a_{0,i} Y^i, \tag{4.12}$$

$$(1+Y)^j (Y+p^{-1}t^{-1})^{n-j} (Y+p^{-1}t^{-1}\alpha^{-1})^{n-j} \sum_{i=0}^{n} \tau(a_{i,n-j}) Y^i = P \sum_{i=0}^{n} a_{j,i} Y^i, \tag{4.13}$$

$$(1+Y)^n \sum_{i=0}^{n} \tau(a_{i,0}) Y^i = P \sum_{i=0}^{n} a_{n,i} Y^i, \tag{4.14}$$

where $j = 1, 2, \cdots, n-1$.

We find $P \in L[Y]$ because

$$\deg_X P = \deg_X F_1 - \deg_X F = 0. \tag{4.15}$$

We obtain

$$\begin{aligned} F_1(X,Y) &= \sum_{j=0}^{n} \left\{ (1+Y)^{n-j}(Y+p^{-1}t^{-1})^j (Y+p^{-1}t^{-1}\alpha^{-1})^j \sum_{i=0}^{n} \tau(a_{i,j}) Y^i \right\} X^{n-j} \\ &= \sum_{j=0}^{n} \left\{ (1+Y)^{j}(Y+p^{-1}t^{-1})^{n-j} (Y+p^{-1}t^{-1}\alpha^{-1})^{n-j} \sum_{i=0}^{n} \tau(a_{i,n-j}) Y^i \right\} X^{j}, \end{aligned} \tag{4.16}$$

$$PF(X,Y) = P \sum_{i,j=0}^{n} a_{i,j} X^i Y^j = \sum_{j,i=0}^{n} P a_{j,i} X^j Y^i = \sum_{j=0}^{n} \left(P \sum_{i=0}^{n} a_{j,i} Y^i \right) X^j, \tag{4.17}$$

By comparing coefficients of X of (4.11), we get (4.12)–(4.14).

Step 4. *We prove* $\sum_{i=0}^{n} a_{0,i} Y^i \neq 0$ *and* $\sum_{i=0}^{n} a_{n,i} Y^i \neq 0$.

Supposing $\sum_{i=0}^{n} a_{0,i} Y^i = 0$, we get

$$a_{0,n} = a_{1,n} = \cdots = a_{n,n} = 0, \tag{4.18}$$

from (4.12). This contradicts $\deg_Y F = n$.

Supposing $\sum_{i=0}^{n} a_{n,i} Y^i = 0$, we get

$$a_{n,0} = a_{n,1} = \cdots = a_{n,n} = 0, \tag{4.19}$$

This contradicts $\deg_X F = n$.

Step 5. *We prove that there are* $A_0 \in L$ *and* $l, m \in \mathbb{Z}_{\geq 0}$ *which satisfy*

$$P = A_0 (1+Y)^m (Y+p^{-1}t^{-1})^l (Y+p^{-1}t^{-1}\alpha^{-1})^{n-l}, \tag{4.20}$$

where $0 \leq l, m \leq n$.

We define l_1 and l_2 by $(Y+p^{-1}t^{-1})^{l_1} || P$ and $(Y+p^{-1}t^{-1}\alpha^{-1})^{l_2} || P$, respectively. We get

$$l_1 + l_2 = n, \tag{4.21}$$

because

$$l_1 + l_2 \geq n, \tag{4.22}$$
$$l_1 + l_2 \leq n, \tag{4.23}$$

hold from (4.12) and (4.14), respectively. We set

$$l = l_1. \tag{4.24}$$

We get

$$(Y+p^{-1}t^{-1})^l (Y+p^{-1}t^{-1}\alpha^{-1})^{n-l} || \sum_{i=0}^{n} \tau(a_{i,0}) Y^i, \tag{4.25}$$

from (4.14). Considering the coefficient of Y^n of the above, we get

$$\sum_{i=0}^{n} \tau(a_{i,0})Y^i = \tau(a_{n,0})(Y + p^{-1}t^{-1})^l (Y + p^{-1}t^{-1}\alpha^{-1})^{n-l}. \tag{4.26}$$

Substituting this into (4.14), we get

$$\tau(a_{n,0})(1+Y)^n (Y + p^{-1}t^{-1})^l (Y + p^{-1}t^{-1}\alpha^{-1})^{n-l} = P \sum_{i=0}^{n} a_{n,i}Y^i. \tag{4.27}$$

Therefore we can set

$$P = A_0(1+Y)^m (Y + p^{-1}t^{-1})^l (Y + p^{-1}t^{-1}\alpha^{-1})^{n-l}. \tag{4.28}$$

Step 6. *We prove $a_{n,n-m} \neq 0$ and $a_{m,n} \neq 0$.*

By using (4.20) and (4.26) in (4.14), we get

$$\tau(a_{n,0})(1+Y)^{n-m} = A_0 \sum_{i=0}^{n} a_{n,i}Y^i. \tag{4.29}$$

Comparing the coefficients of Y^{n-m} of (4.29), we get

$$a_{n,n-m} \neq 0. \tag{4.30}$$

By using (4.20) in (4.12), we get

$$(Y + p^{-1}t^{-1})^{n-l} (Y + p^{-1}t^{-1}\alpha^{-1})^l \sum_{i=0}^{n} \tau(a_{i,n})Y^i = A_0(1+Y)^m \sum_{i=0}^{n} a_{0,i}Y^i. \tag{4.31}$$

From (4.31), we get

$$(Y + p^{-1}t^{-1})^{n-l} (Y + p^{-1}t^{-1}\alpha^1)^l || \sum_{i=0}^{n} a_{0,i}Y^i, \tag{4.32}$$

$$\deg_Y \left(\sum_{i=0}^{n} \tau(a_{i,n})Y^i \right) = m. \tag{4.33}$$

Therefore we obtain $a_{m,n} \neq 0$.

Step 7. *We prove $2m = n$.*

By using (4.20) and (4.26) in (4.13), we get

$$\begin{aligned}(Y + p^{-1}t^{-1})^{n-m} (Y + p^{-1}t^{-1}\alpha^{-1})^{n-m} \sum_{i=0}^{n} \tau(a_{i,n-m})Y^i \\ = A_0(Y + p^{-1}t^{-1})^l (Y + p^{-1}t^{-1}\alpha^{-1})^{n-l} \sum_{i=0}^{n} a_{m,i}Y^i.\end{aligned} \tag{4.34}$$

By comparing degrees of Y of (4.34)

$$2m = n. \tag{4.35}$$

Step 8. *We prove that the following equations hold*:

$$\tau(a_{0,0}) = p^{-n}\alpha^{l-n}t^{-n}\tau(a_{n,0}), \tag{4.36}$$
$$\tau(a_{n,0}) = A_0 a_{n,0}, \tag{4.37}$$
$$\tau(a_{n,0}) = A_0 a_{n,m}, \tag{4.38}$$
$$\tau(a_{0,n}) = A_0 p^n \alpha^l t^n a_{0,0}, \tag{4.39}$$
$$\tau(a_{m,n}) = A_0 a_{0,n}, \tag{4.40}$$
$$\tau(a_{n,m}) = A_0 a_{m,n}. \tag{4.41}$$

From the coefficient of Y^0 of (4.26), we get (4.36). From the coefficients of Y^0 and Y^m of (4.29), we get (4.37) and (4.38), respectively. From the coefficients of Y^0 and Y^{n+m} of (4.31), we get (4.39) and (4.40), respectively. From the coefficient of Y^{2n} of (4.34), we get (4.41).

Now we prove Lemma 4.2. From (4.37) and (4.38), we get

$$a_{n,0} = a_{n,m}. \tag{4.42}$$

By using (4.36), (4.37), (4.39), (4.40), (4.41), and (4.42), we obtain

$$\tau(a_{0,0}) = p^{-n}\alpha^{l-n}t^{-n}\tau(a_{n,0}) \tag{4.43}$$
$$\Leftrightarrow a_{0,0} = \alpha^{l-n}t^{-n}a_{n,0} \tag{4.44}$$
$$\Leftrightarrow \tau(a_{0,n}) = A_0 p^n \alpha^{2l-n} a_{n,0} \tag{4.45}$$
$$\Leftrightarrow \tau(a_{0,n}) = p^n \alpha^{2l-n}\tau(a_{n,0}) \tag{4.46}$$
$$\Leftrightarrow a_{0,n} = p^n \alpha^{2l-n} a_{n,0} \tag{4.47}$$
$$\Leftrightarrow \tau(a_{m,n}) = A_0 p^n \alpha^{2l-n} a_{n,0} \tag{4.48}$$
$$\Leftrightarrow \tau(a_{m,n}) = p^n \alpha^{2l-n}\tau(a_{n,0}) \tag{4.49}$$
$$\Leftrightarrow a_{m,n} = p^n \alpha^{2l-n} a_{n,0} \tag{4.50}$$
$$\Leftrightarrow \tau(a_{n,m}) = A_0 p^n \alpha^{2l-n} a_{n,0} \tag{4.51}$$
$$\Leftrightarrow \tau(a_{n,m}) = p^n \alpha^{2l-n}\tau(a_{n,0}) \tag{4.52}$$
$$\Leftrightarrow a_{n,m} = p^n \alpha^{2l-n} a_{n,0} \tag{4.53}$$
$$\Leftrightarrow \alpha^{n-2l} = p^n. \tag{4.54}$$

We also get

$$2l \neq n, \tag{4.55}$$

because p is not a root of unity. ■

In order to prove Theorem 4.1, we use the following lemma:

Lemma 4.3. Let $\mathcal{L}$ be an inversive difference overfield of $(\mathbb{C}(t), t \mapsto pt)$ and $\mathcal{U}$ a difference overfield of $\mathcal{L}$. Using the Bäcklund transformations, (2.11) and (2.12), we find that if there is a solution $f \in \mathcal{U}$ to (4.1) with $\alpha = A$ satisfying $\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} = 1$, then for all $j \in \mathbb{Z}$, there is a solution $g \in \mathcal{U}$ to (4.1) with $\alpha = p^j A$ satisfying $\operatorname{tr.deg} \mathcal{L}\langle g\rangle/\mathcal{L} = 1$.

Now we prove Theorem 4.1 by using Lemma 4.2 and Lemma 4.3.

Proof of theorem 4.1. We assume $\operatorname{tr.deg} \mathcal{L}\langle f\rangle/\mathcal{L} = 1$. We will derive a contradiction. We consider the set Φ of all $(A, j) \in \mathbb{C}^\times \times \mathbb{Z}_{\geq 0}$ satisfying the following conditions: (i) there is a solution g to (4.1) with $\alpha = A$ such that $\operatorname{tr.deg} \mathcal{L}\langle g\rangle/\mathcal{L} = 1$; (ii) there is $i \in \mathbb{Z} \setminus \{0\}$ which satisfies $A^i = p^j$. We find $\Phi \neq \{\}$ by Lemma 4.2. We choose $(A_1, k) \in \Phi$ whose k is minimum. There is $i \in \mathbb{Z} \setminus \{0\}$ which satisfies

$$A_1{}^i = p^k, \tag{4.56}$$

and from Lemma 4.2, there are $n, l \in \mathbb{Z}_{\geq 0}$ which satisfy

$$A_1{}^{n-2l} = p^n, \tag{4.57}$$

where $0 \leq l \leq n$, $n \neq 0$ and $2l \neq n$. Supposing $k = 0$, we get

$$p^{in} = A_1{}^{(n-2l)i} = 1, \tag{4.58}$$

from (4.57) because

$$A_1{}^i = 1, \tag{4.59}$$

from (4.56). This contradicts the assumption that p is not a root of unity. Therefore

$$k \neq 0. \tag{4.60}$$

We obtain

$$in = k(n-2l), \tag{4.61}$$

because

$$p^n = A_1{}^{n-2l} \Rightarrow p^{in} = A_1{}^{i(n-2l)} \Leftrightarrow p^{in} = p^{k(n-2l)} \Leftrightarrow in = k(n-2l), \tag{4.62}$$

from (4.56) and (4.57).

Now we consider by dividing the range of l into the following four types:

(i) $0 < 2l < n$**.** We find $(p^{-1}A_1, |k-i|) \in \Phi$ because

$$(p^{-1}A_1)^i = p^{-i}A_1{}^i = p^{k-i}. \tag{4.63}$$

This contradicts the minimality of k because we get

$$\begin{aligned}
&0 < 2l < n &(4.64)\\
&\Leftrightarrow 0 < n-2l < n &(4.65)\\
&\Leftrightarrow 0 < \frac{k(n-2l)}{n} < k &(4.66)\\
&\Leftrightarrow 0 < i < k &(4.67)\\
&\Leftrightarrow -k < -i < 0 &(4.68)\\
&\Leftrightarrow 0 < k-i < k &(4.69)\\
&\Leftrightarrow |k-i| < k, &(4.70)
\end{aligned}$$

from (4.61).

(ii) $n < 2l < 2n$**.** In the same way as (i), we can prove the contradiction.

(iii) $l = 0$**.** We get

$$A_1{}^n = p^n, \tag{4.71}$$

from (4.57). From Lemma 4.3, we find that the assumption holds for $\alpha = p^{-1}A_1$. From Lemma 4.2, there is $n' \in \mathbb{Z}_{>0}$ and $l' \in \mathbb{Z}_{\geq 0}$ which satisfy

$$\left(p^{-1}A_1\right)^{n'-2l'} = p^{n'} \tag{4.72}$$

and $n' \geq l'$. This contradicts that p is not a root of unity from

$$p^{nn'} = \left(\left(p^{-1}A_1\right)^n\right)^{n'-2l'} = 1. \tag{4.73}$$

(iv) $l = n$**.** In the same way as (iii), we can prove the incoherence.

Therefore we find a contradiction in any case, and so we conclude $\mathrm{tr.deg}\, \mathcal{L}\langle f\rangle/\mathcal{L} \neq 1$, which implies

$$\mathrm{tr.deg}\, \mathcal{L}\langle f\rangle/\mathcal{L} \leq 1 \Rightarrow f \text{ is algebraic over } L, \tag{4.74}$$

the required. ■

5 Concluding remarks

In this paper, we have studied algebraic function solutions to q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$ and proved its irreducibility.

Before closing, we mention a determinant formula of algebraic function solutions to q-$\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$. As mentioned in Section 3.1, K. Kajiwara et al. anticipate the determinant formula of algebraic function solutions to $\mathrm{P}_{\mathrm{III}}^{D_7^{(1)}}$. By the continuous limit, the Laurent polynomials are reduced to the determinants of which the determinant formula above are composed. This gives us expectation that each of the Laurent polynomials has an analogous determinant expression.

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References

[1] R.M. Cohn, Difference Algebra, Interscience Publishers, (New York · London · Sydney, 1965).

[2] G. Gasper and M. Rahman, Basic Hypergeometric Series (Second edition), Encyclopedia of Mathematics and Its Applications **96** (Cambridge University Press, Cambridge, 2004).

[3] B. Grammaticos and A. Ramani, Discrete Painlevé equations: a review, Lect. Notes Phys. **644** (2004) 245–321.

[4] K. Kajiwara, N. Nakazono and T. Tsuda, Projective reduction of the discrete Painlevé system of type $(A_2 + A_1)^{(1)}$, Int. Math. Res. Not. **2011** (2011) 930–966.

[5] R. Koekoek, P.A. Lesky and R.F. Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010.

[6] A. Levin, Difference Algebra, (Springer Science+Business Media B.V., 2008).

[7] T. Masuda, Hypergeometric τ-functions of the q-Painlevé system of type $E_7^{(1)}$, *SIGMA Symmetry Integrability Geom. Methods Appl.* **5** (2009).

[8] T. Masuda, Hypergeometric τ-functions of the q-Painlevé system of type $E_8^{(1)}$, Ramanujan J. **24** (2011).

[9] N. Nakazono, Hypergeometric τ Functions of the q-Painlevé Systems of Type $(A_2{+}A_1)^{(1)}$, *SIGMA Symmetry Integrability Geom. Methods Appl.* **6** (2010).

[10] S. Nishioka, Solvability and irreducibility of difference equations, Dr. thesis (2010).

[11] S. Nishioka, Decomposable extensions of difference fields, Funkcial. Ekvac. **53** (2010) 489–501.

[12] M. Noumi, Painlevé equations through symmetry (American Mathematical Society, Providence, 2004).

[13] Y. Ohyama, H. Kawamuko, H. Sakai and K. Okamoto, Studies on the Painlevé equations. V. Third Painlevé equations of special type $P_{\rm III}(D_7)$ and $P_{\rm III}(D_8)$, J. Math. Sci. Univ. Tokyo **13** (2006) 145–204.

[14] A. Ramani, B. Grammaticos and J. Hietarinta, Discrete versions of the Painlevé equations, Phys. Rev. Lett. **67** (1991) 1829–1832.

[15] H. Sakai, Rational surfaces associated with affine root systems and geometry of the Painlevé equations, Comm. Math. Phys. **220** (2001) 165–229.

[16] T. Tsuda, Tau functions of q-Painlevé III and IV equations, Lett. Math. Phys. **75** (2006) 39–47.

[17] O. Zariski and P. Samuel, Commutative Algebra Volume I, (Springer-Verlag, New York, NY, 1958).

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