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出版情報：MSJ memoirs. 21, pp.73-85, 2010-02. 日本数学会

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Masanobu Kaneko

1 Introduction

In this expository article[1], we review some aspects of poly-Bernoulli numbers and related zeta functions.

The poly-Bernoulli number is a generalization of the classical Bernoulli number using the polylogarithm series. Although its definition looks rather artificial at first glance, it has turned out recently that the poly-Bernoulli numbers of negative index have very nice combinatorial interpretations, and also they appear in special values of certain zeta functions. It may therefore be reasonable to seek arithmetic properties that may be involved with poly-Bernoulli numbers. The author made one such attempt with late Arakawa in the hope of finding a nice zeta function which connects poly-Bernoulli numbers with the so-called multiple zeta values, the subject of wide interest not only in number theory but also in numerous other branches such as topology, quantum groups, arithmetic geometry, mathematical physics etc. This work with Arakawa will be reviewed in §3, after recalling definitions and properties of poly-Bernoulli numbers in §2. In §4 we give some results and speculations concerning the "multiple harmonic sums $\bmod p$" and "multiple zeta-star values." In the final section, §5, we discuss a different type of zeta function which also has some relation to poly-Bernoulli numbers as well as to certain generalized multiple zeta values.

The author would like to take this opportunity to express his deep gratitude to late Professor Tsuneo Arakawa on the occasion of his sixtieth birthday, whose encouragement and interest at the early stage of the research on this topic greatly helped in developing the work further.

2 Poly-Bernoulli numbers

In relation to the well-known formula for the sum of consecutive powers of integers, Takakazu Seki[2] and Jacob Bernoulli[3] independently introduced a sequence of rational numbers, nowadays known as the "Bernoulli numbers," B_n $(n =$

[1] This is an extended version of [20].
[2] "Katsu-you-san-pou", published posthumously in 1712.
[3] "Ars conjectandi", published also posthumously in 1713.

$0, 1, 2, \ldots$). Their definition is by the recursion

$$\sum_{i=0}^{n} \binom{n+1}{i} B_i = n+1 \quad (n = 0, 1, 2, \ldots),$$

and this can be expressed by means of a generating series as

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

We note that the left-hand side of this differs by x from the more commonly used defining series;

$$\frac{xe^x}{e^x - 1} = \frac{x}{e^x - 1} + x,$$

and as a result, with our definition we have $B_1 = 1/2$ (instead of $-1/2$). The other values of B_n are the same in both definitions and, because $B_n = 0$ for odd $n \geq 3$, to convert any formula with one definition into the other we only need to change B_n into $(-1)^n B_n$.

According to [18] and [4], we define the poly-Bernoulli number $\mathbb{B}_n^{(k)}$ and its relative $C_n^{(k)}$, for any integers $k \in \mathbf{Z}$ and $n \geq 0$, by the generating series

$$\frac{Li_k(1 - e^{-x})}{1 - e^{-x}} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(k)} \frac{x^n}{n!} \quad \text{and} \quad \frac{Li_k(1 - e^{-x})}{e^x - 1} = \sum_{n=0}^{\infty} C_n^{(k)} \frac{x^n}{n!}$$

respectively[4]. Here, $Li_k(z)$ denotes the formal power series $\sum_{m=1}^{\infty} z^m/m^k$ (the kth polylogarithm when $k > 0$, and the rational function $(z\, d/dz)^{-k} (z/(1-z))$ when $k \leq 0$). When $k = 1$, we have $Li_1(z) = -\log(1-z)$ and these generating series become

$$\frac{xe^x}{e^x - 1} \quad \text{and} \quad \frac{x}{e^x - 1}$$

respectively, and hence each of $\mathbb{B}_n^{(k)}$ and $C_n^{(k)}$ generalizes the classical Bernoulli numbers B_n, by choosing one of the above generating series for B_n. Since the two generating series for $\mathbb{B}_n^{(k)}$ and $C_n^{(k)}$ differ by a factor e^x, the two numbers

$$\mathbb{B}_n^{(k)} = \sum_{m=0}^{n} \binom{n}{m} C_m^{(k)}, \qquad C_n^{(k)} = (-1)^n \sum_{m=0}^{n} (-1)^m \binom{n}{m} \mathbb{B}_m^{(k)}.$$

Also, using

$$\begin{aligned} \frac{Li_k(1 - e^{-x})}{1 - e^{-x}} &= e^x \frac{Li_k(1 - e^{-x})}{e^x - 1} = \frac{Li_k(1 - e^{-x})}{e^x - 1} + Li_k(1 - e^{-x}) \\ &= \frac{Li_k(1 - e^{-x})}{e^x - 1} + \int_0^x \frac{Li_{k-1}(1 - e^{-x})}{e^x - 1}\, dx, \end{aligned}$$

[4]We use the notation $\mathbb{B}_n^{(k)}$ instead of $B_n^{(k)}$, to avoid possible confusion with Carlitz's Bernoulli number of higher order.

we have the relation

$$\mathbb{B}_n^{(k)} = C_n^{(k)} + C_{n-1}^{(k-1)}.$$

In particular, specializing $k = 2$ and using the fact that $C_n^{(1)} = 0$ for odd $n \geq 3$, we have

$$\mathbb{B}_n^{(2)} = C_n^{(2)} \qquad \text{for even } n \geq 4. \tag{1}$$

We review here some of the known properties of poly-Bernoulli numbers. The first is the closed formulas in terms of the Stirling number of the second kind. The Stirling number of the second kind, denoted by $\left\{ {n \atop i} \right\}$, is the number of ways to partition a set of n elements into i nonempty subsets.

Theorem 1 *We have the following formulas:*
1) For any $k \in \mathbf{Z}$ and $n \geq 0$,

$$\mathbb{B}_n^{(k)} = (-1)^n \sum_{i=0}^{n} \frac{(-1)^i i! \left\{ {n \atop i} \right\}}{(i+1)^k}, \quad C_n^{(k)} = (-1)^n \sum_{i=0}^{n} \frac{(-1)^i i! \left\{ {n+1 \atop i+1} \right\}}{(i+1)^k}.$$

2) For $k, n \geq 0$,

$$\begin{aligned} \mathbb{B}_n^{(-k)} &= \sum_{j=0}^{\min(n,k)} (j!)^2 \left\{ {n+1 \atop j+1} \right\} \left\{ {k+1 \atop j+1} \right\}, \\ C_n^{(-k-1)} &= \sum_{j=0}^{\min(n,k)} j!(j+1)! \left\{ {n+1 \atop j+1} \right\} \left\{ {k+1 \atop j+1} \right\}. \end{aligned}$$

Proof. To prove 1), we expand the defining generating series by using the formula (see e.g. [17])

$$(e^x - 1)^i = i! \sum_{n=i}^{\infty} \left\{ {n \atop i} \right\} \frac{x^n}{n!} \tag{2}$$

for the Stirling numbers and compare the coefficients.

For 2), we calculate the two variable generating series

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathbb{B}_n^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} \quad \text{and} \quad \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} C_n^{(-k-1)} \frac{x^n}{n!} \frac{y^k}{k!}$$

by using formulas in 1), and as a result we have

$$\frac{e^{x+y}}{e^x + e^y - e^{x+y}} \quad \text{and} \quad \frac{e^{x+y}}{(e^x + e^y - e^{x+y})^2}$$

respectively. Writing the first expression as

$$\begin{aligned}
\frac{e^{x+y}}{e^x+e^y-e^{x+y}} &= \frac{e^{x+y}}{1-(e^x-1)(e^y-1)} \\
&= e^{x+y}\sum_{j=0}^{\infty}(e^x-1)^j(e^y-1)^j \\
&= \sum_{j=0}^{\infty}\frac{1}{(j+1)^2}\frac{d}{dx}(e^x-1)^{j+1}\frac{d}{dy}(e^y-1)^{j+1}
\end{aligned}$$

and using (2), we obtain the first formula of 2). The second can be proved similarly by using

$$\begin{aligned}
\frac{e^{x+y}}{(e^x+e^y-e^{x+y})^2} &= \frac{e^{x+y}}{\big(1-(e^x-1)(e^y-1)\big)^2} \\
&= e^{x+y}\sum_{j=0}^{\infty}(j+1)(e^x-1)^j(e^y-1)^j \\
&= \sum_{j=0}^{\infty}\frac{1}{(j+1)}\frac{d}{dx}(e^x-1)^{j+1}\frac{d}{dy}(e^y-1)^{j+1}.
\end{aligned}$$

The $\min(n,k)$ in the upper limits in the formulas is because the Stirling number $\left\{ {n \atop k} \right\}$ is 0 when $n<k$. $\square$

Corollary For $k,n\geq 0$, we have the symmetries

$$\mathbb{B}_n^{(-k)}=\mathbb{B}_k^{(-n)} \quad \text{and} \quad C_n^{(-k-1)}=C_k^{(-n-1)}.$$

With the aid of the above explicit formulas, C. Brewbaker [8, 9] and S. Launois [22] found beautiful combinatorial interpretations of $\mathbb{B}_n^{(-k)}$, which we now describe briefly.

A *lonesum matrix* is a matrix with entries 0 and 1 whose row-sums and column-sums determine the matrix uniquely. For instance, the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$ gives ${}^t(1,2,1)$ and $(3,1)$ as row- and column-sums respectively, and from these two vectors, the original matrix is recovered uniquely. The theorem of Brewbaker states that the number of lonesum matrices of a given size is equal to the poly-Bernoulli number.

Theorem (Brewbaker [8, 9]) *For $k,n\geq 1$, the number of $k\times n$ lonesum matrices is equal to $\mathbb{B}_n^{(-k)}$.*

The key fact in order to count the total number of lonesum matrices is the characterization (using an old result of Ryser [27]) to the effect that a $(0,1)$-matrix is lonesum if and only if it has no 2×2 minor of the form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ nor

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For a complete proof, we refer the reader to the original papers [8, 9].

The second combinatorial interpretation of $\mathbb{B}_n^{(-k)}$ is related to the number of special type of permutations. Let $\mathfrak{S}_n$ denote the symmetric group of order n, identified with the set of all permutations on the set $\{1, 2, \ldots, n\}$. Launois proved the following.

Theorem (Launois [22]) *Let k and n be positive integers. The cardinality of the set*

$$\{\sigma \in \mathfrak{S}_{k+n} | -k \leq \sigma(i) - i \leq n, \ 1 \leq \forall i \leq k+n\}$$

is equal to $\mathbb{B}_n^{(-k)}$.

We omit the proof and only refer to [22]. It may be an interesting problem to establish a natural bijection between the sets of $n \times k$ lonesum matrices and the above permutations. We note that either of these interpretations of $\mathbb{B}_n^{(-k)}$ makes the above duality formula $\mathbb{B}_n^{(-k)} = \mathbb{B}_k^{(-n)}$ apparent.

Further results obtained in [18, 5, 19] include Clausen-von Staudt type theorems for $\mathbb{B}_n^{(k)}$, and an analogue for $C_n^{(k)}$ of the Akiyama-Tanigawa algorithm for computing Bernoulli numbers (similar to Pascal's triangle for binomial coefficients). As for the Clausen-von Staudt type result, a complete description of denominators of $\mathbb{B}_n^{(2)}$ (di-Bernoulli numbers) is given in [18] and partial results are obtained in [5] for general k.

An important open problem is to find a Kummer type congruence for poly-Bernoulli numbers. This and its generalization may be of importance also in the theory of p-adic multiple zeta values, as surmised by Furusho [13].

As another topic of further investigation, we point out that the extra symmetries or other nice properties of dilogarithm function (see [32]) may force di-Bernoulli numbers (the case of the upper index $k = 2$) to have the more rich properties than the other ones ($k \neq 2$). We also point out that in the di-Bernoulli case, both numbers $\mathbb{B}_n^{(2)}$ and $C_n^{(2)}$ coincide when n is even, as noted before (1).

3 Multiple zeta values and a zeta function

The *multiple zeta value* (MZV) is a real number associated to each index set $(k_1, k_2, \ldots, k_n)$ of positive integers with $k_1 \geq 2$, defined by the convergent series

$$\zeta(k_1, k_2, \ldots, k_n) := \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}.$$

This is a rather naive generalization of values of the Riemann zeta function $\zeta(s)$ at positive integer arguments, whose study was initiated by Euler [12] in the case of "depth" $n = 2$. Since 1990's when connections to quantum field theory, knot theory, mixed Tate motive, or quantum groups were found ([10], [23], [14], [11]), the MZV has become a topic of intensive study. We refer the interested reader to Mike Hoffman's web page [16] for extensive references on MZV's.

In [4], we studied the function $\xi_k(s)$ $(k \geq 1)$ defined by

$$\xi_k(s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} Li_k(1 - e^{-t}) dt, \tag{3}$$

with the intention that we might be able to find a generalization of Euler's celebrated formulas

$$\zeta(2k) = (-1)^{k-1} \frac{B_{2k}}{2(2k)!} (2\pi)^{2k}, \quad \zeta(1-k) = -\frac{B_k}{k} \quad (k \geq 1),$$

with multiple zeta values on one hand, and poly-Bernoulli numbers on the other. What we obtained in [4] is the following theorem.

Theorem 2 ([4]) *1) The integral* (3) *converges for* $Re(s) > 0$ *and the function* $\xi_k(s)$ *analytically continues to an entire function of* s.

2) We have the relation

$$\begin{aligned} \xi_k(s) \;&=\; (-1)^{k-1} \big[\zeta(s, \underbrace{2, 1, \ldots, 1}_{k-1}) + \zeta(s, \underbrace{1, 2, 1, \ldots, 1}_{k-1}) + \cdots + \zeta(s, \underbrace{1, \ldots, 1, 2}_{k-1}) \\ &\quad + s \cdot \zeta(s+1, \underbrace{1, \ldots, 1}_{k-1}) \big] + \sum_{j=0}^{k-2} (-1)^j \zeta(k-j) \cdot \zeta(s, \underbrace{1, \ldots, 1}_{j}). \end{aligned} \tag{4}$$

Here, $\zeta(s, k_2, \ldots, k_n)$ *is a one variable function in* s *with fixed* $k_2, \ldots, k_n$, *continued meromorphically to the whole* s*-plane.*

In particular, the values of $\xi_k(s)$ *at positive integers are written in terms of MZV's.*

3) At non-positive integer arguments, we have

$$\xi_k(-n) = (-1)^n C_n^{(k)} \quad (n = 0, 1, 2, \ldots). \tag{5}$$

We only briefly mention to the proof. Once the holomorphic continuation is established in a standard manner such as used in [33, §4 of Part 1], the relation (4) is obtained by computing the integral

$$\int_0^\infty \cdots \int_0^\infty x_k^{s-1} \cdot \frac{x_1 + \cdots + x_k}{e^{x_1 + \cdots + x_k} - 1} \cdot \frac{1}{e^{x_2 + \cdots + x_k} - 1} \cdots \frac{1}{e^{x_k} - 1} dx_1 \cdots dx_k$$

in two ways, one is by the repeated use of the Mellin transform and the other is the integration by parts using

$$\frac{\partial}{\partial x_1} Li_2(1 - e^{-x_1 - \cdots - x_k}) = \frac{x_1 + \cdots + x_k}{e^{x_1 + \cdots + x_k} - 1}$$

and similar formulas for higher Li_k. Formula (5) in 3) is deduced also in a standard way. Or rather, we have *so defined* the function $\xi_k(s)$ *that* we have (5). (Note, however, we face with a convergence problem if we want $\mathbb{B}_n^{(k)}$ instead of $C_n^{(k)}$.) Interesting point is that the function $\xi_k(s)$ has the expression (4) in terms of the multiple zeta function.

We remark that the multi-variable function

$$\zeta(s_1, s_2, \ldots, s_n) := \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_n^{s_n}}$$

is also meromorphically continued to $\mathbf{C}^n$, thanks to the works of Akiyama-Egami-Tanigawa [2] and Zhao [34]. To seek for a connection between poly-Bernoulli numbers (or its generalization) and the values of $\zeta(s_1, s_2, \ldots, s_n)$ at non-positive integers may be an interesting problem, but, as described in [2], those points are "points of indeterminacy" and we have no canonical values there. Still, it is possible to find a connection with any fixed way of limiting process.

As for values at positive integers of $\xi_k(s)$, formulas

$$\xi_k(n) = \sum_{\substack{a_1+\cdots+a_k=n-1 \\ \forall a_j \geq 0}} (a_1+1)\zeta(a_1+2, a_2+1, \ldots, a_k+1) \quad (k, n \geq 1),$$

and

$$\xi_k(2) = \frac{1}{2}\sum_{i=0}^{k-2}(-1)^i \zeta(i+2)\zeta(k-i) \quad (k\text{: even } \geq 2)$$

are obtained in [4]. Y. Ohno discovered, as an application of his renowned formula [25], that the first expression can be transformed into the following simple formula:

Theorem (Ohno [25]) *For $k, n \geq 1$, we have*

$$\xi_k(n) = \zeta^\star(k+1, \underbrace{1, \ldots, 1}_{n-1}), \tag{6}$$

where

$$\zeta^\star(k_1, k_2, \ldots, k_n) := \sum_{m_1 \geq m_2 \geq \cdots \geq m_n \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}$$

is the multiple "zeta-star" value.

This formula plays an interesting role in finding a "duality" phenomenon of multiple zeta-star values, which we discuss in the next section.

4 Finite multiple zeta sums $\bmod p$ and multiple zeta-star values

Let p be an odd prime number. Consider the finite sums obtained by truncating the series for $\zeta(k_1, k_2, \ldots, k_n)$ and $\zeta^\star(k_1, k_2, \ldots, k_n)$ right before the prime p appears in denominators;

$$\begin{aligned} H_p(k_1, k_2, \ldots, k_n) &:= \sum_{p-1 \geq m_1 > m_2 > \cdots > m_n \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}, \\ H_p^\star(k_1, k_2, \ldots, k_n) &:= \sum_{p-1 \geq m_1 \geq m_2 \geq \cdots \geq m_n \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}. \end{aligned}$$

Hoffman [15] and Zhao [35] studied these sums ("multiple harmonic sums" in Hoffman's terminology) mod p. For a particular type of index ("height 1" case), Hoffman showed that there is essentially no difference in modulo p between "zetas" and "zeta-stars":

Theorem (Hoffman [15, Th. 5.1]) *For prime $p > n$, it holds the congruence*

$$H_p^\star(k, \underbrace{1, \ldots, 1}_{n-1}) \equiv (-1)^k H_p(k, \underbrace{1, \ldots, 1}_{n-1}) \bmod p.$$

And he conjectures that all sums $H_p(k_1, k_2, \ldots, k_n)$ and $H_p^\star(k_1, k_2, \ldots, k_n)$ can be written $\bmod p$ as sums of products of the height one sums $H_p(k, 1, \ldots, 1)$.

For these conjectural "building blocks" $H_p(k, 1, \ldots, 1) \bmod p$, he gave a closed formula as a sum involving the Stirling numbers of the second kind. A simple manipulation using Fermat's little theorem and the closed formula of $C_n^{(k)}$ in Theorem 1-2) shows that his formula can be stated simply in terms of poly-Bernoulli numbers as follows.

Theorem (Hoffman [15, Th. 5.4]) For $k, n \geq 1$ and any prime $p > n$, we have

$$H_p^\star(k, \underbrace{1, \ldots, 1}_{n-1}) \equiv (-1)^n C_{p-1-n}^{(k-1)} \bmod p. \tag{7}$$

Proof. The formula of Hoffman [15, Th. 5.4] reads

$$H_p(k, \underbrace{1, \ldots, 1}_{n-1}) \equiv \sum_{j=1}^{p-n} (-1)^j (-j)^{p-k} (j-1)! \left\{ {p-n \atop j} \right\} \bmod p.$$

By Fermat's little theorem, $j^{p-k} \equiv j^{1-k} \bmod p$, the right-hand side is congruent to

$$
\begin{aligned}
& (-1)^{p-k}\sum_{j=1}^{p-n}\frac{(-1)^j(j-1)!\left\{{p-n \atop j}\right\}}{j^{k-1}} \bmod p \\
\equiv\ & (-1)^{k+n}(-1)^{p-n-1}\sum_{j=0}^{p-n-1}\frac{(-1)^j j!\left\{{p-n \atop j+1}\right\}}{(j+1)^{k-1}} \bmod p \qquad (j\to j+1) \\
\equiv\ & (-1)^{k+n}C_{p-1-n}^{(k-1)} \bmod p \qquad \text{(by Theorem 1).}
\end{aligned}
$$

This congruence, combined with the previous theorem of Hoffman, establishes the theorem. □

With (7) and (5) together, we obtain

$$
H_p^\star(k,\underbrace{1,\ldots,1}_{n-1}) \equiv \xi_{k-1}(-p+1+n) \bmod p. \tag{8}
$$

In view of the formula (6) of Ohno, this congruence looks very suggestive, although for the moment it is only a superficial curiosity. The curious point is this: Start with the value of the function $\xi_{k-1}(s)$ at a positive integer n. This is, by Ohno's Theorem (6), the multiple zeta-star value $\zeta^\star(k,\underbrace{1,\ldots,1}_{n-1})$. Take an odd prime p and truncate this series to get $H_p^\star(k,\underbrace{1,\ldots,1}_{n-1})$, and reduce it modulo p. Then the resulting value is congruent $\bmod\, p$ to the value of $\xi_{k-1}(s)$ at $n-(p-1)$, the shift of the initial n by $p-1$!

$$
\xi_{k-1}(n)=\zeta^\star(k,\underbrace{1,\ldots,1}_{n-1}) \overset{\text{truncate}}{\Longrightarrow} H_p^\star(k,\underbrace{1,\ldots,1}_{n-1}) \overset{\text{mod } p}{\Longrightarrow} \xi_{k-1}(n-(p-1)).
$$

Now, we trace our original thinking to get the idea of a kind of "duality" for multiple zeta-stars.

In the same paper, Hoffman also proved the duality congruence ([15, Th.5.2])

$$
(-1)^n H_p^\star(n,\underbrace{1,\ldots,1}_{k-1}) \equiv (-1)^k H_p^\star(k,\underbrace{1,\ldots,1}_{n-1}) \bmod p.
$$

Given the relation to poly-Bernoulli numbers (7), this is just a consequence of the duality of poly-Bernoulli numbers in Corollary to Theorem 1. As a possible different approach however, first note the congruence of truncated Riemann zeta values

$$
1+\frac{1}{2^n}+\frac{1}{3^n}+\cdots+\frac{1}{(p-1)^n} \equiv 0 \bmod p
$$

which is valid for all $p>n+1$. This is because the left-hand side is equal to $1+2+\cdots+(p-1)=p(p-1)/2\equiv 0 \bmod p$. The above mentioned duality of Hoffman would follow from this if the difference

$$
(-1)^n H_p^\star(n,\underbrace{1,\ldots,1}_{k-1}) - (-1)^k H_p^\star(k,\underbrace{1,\ldots,1}_{n-1})
$$

could be expressed as a polynomial in the truncated Riemann zeta values. Actually, Hoffman proved in [15] many of the congruences in this way. However, the duality in question is proved in another way and we do not know if there is such an expression.

Anyway, inspired by this and the above mentioned curious analogy, we surmised that the difference of the two multiple zeta-star values

$$(-1)^n\zeta^\star(n,\underbrace{1,\ldots,1}_{k-1})-(-1)^k\zeta^\star(k,\underbrace{1,\ldots,1}_{k-1}) \tag{9}$$

may be written as a polynomial over $\mathbf{Q}$ in the Riemann zeta values, and did numerical experiments. The result was in favor of the speculation, and soon after the author had informed him of this speculation, Yasuo Ohno proved that this was indeed true. He obtained, using (6) and (4) together with his main result in [25], the formula

$$\begin{aligned}
&(-1)^n\zeta^\star(n,\underbrace{1,\ldots,1}_{k-1})-(-1)^k\zeta^\star(k,\underbrace{1,\ldots,1}_{n-1})\\
= \;&(k-1)\zeta(k+1,\underbrace{1,\ldots,1}_{n-2})-(n-1)\zeta(n+1,\underbrace{1,\ldots,1}_{k-2})\\
&+(-1)^k\sum_{j=1}^{k-2}(-1)^j\zeta(k-j)\zeta(n,\underbrace{1,\ldots,1}_{j-1})\\
&-(-1)^n\sum_{j=1}^{n-2}(-1)^j\zeta(n-j)\zeta(k,\underbrace{1,\ldots,1}_{j-1}).
\end{aligned}$$

Since we know that the multiple zeta values of height 1 (= of type $\zeta(m,1,\ldots,1)$) can be expressed as polynomials over $\mathbf{Q}$ in the Riemann zeta values ([3], [11], see also [26]), we conclude that the quantity (9) is a polynomial in the Riemann zeta values. Using this formula, we can compute the two variable generating series of (9):

$$\begin{aligned}
&\sum_{k,n\geq 2}\big((-1)^n\zeta^\star(n,\underbrace{1,\ldots,1}_{k-1})-(-1)^k\zeta^\star(k,\underbrace{1,\ldots,1}_{k-1})\big)x^{k-1}y^{n-1}\\
&\qquad=\psi(x)-\psi(y)+\pi\left(\cot(\pi x)-\cot(\pi y)\right)\frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}.
\end{aligned}$$

Here, $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the digamma function. To compute this, we use the formula of Aomoto [1] and Drinfeld [11]

$$\sum_{k,n\geq 1}\zeta(k+1,\underbrace{1,\ldots,1}_{n-1})x^ky^n=1-\frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}$$

and the well-known Taylor expansion of the (logarithm of) gamma function

$$\Gamma(1+x)=\exp\left(-\gamma x+\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}x^n\right)\quad(|x|<1,\gamma:\text{Euler's constant}).$$

From this we have

$$
\begin{aligned}
\frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} &= \exp\left(\sum_{n=2}^{\infty}\frac{\zeta(n)}{n}\left(x^n+y^n-(x+y)^n\right)\right), \\
\psi(x) &= -\frac{1}{x}-\gamma+\sum_{n=2}^{\infty}(-1)^n\zeta(n)x^{n-1}, \\
\pi\cot(\pi x) &= \frac{1}{x}+\psi(1-x)-\psi(1+x)=\frac{1}{x}-2\sum_{n=1}^{\infty}\zeta(2n)x^{2n-1},
\end{aligned}
$$

and expanding these out we obtain a rather complicated (in fact too complicated to neatly describe, because we have to expand the exponential) expression of

$$
(-1)^n\zeta^\star(n,\underbrace{1,\dots,1}_{k-1})-(-1)^k\zeta^\star(k,\underbrace{1,\dots,1}_{k-1})
$$

as a polynomial in Riemann zeta values. All the details and possible generalizations will be discussed in a joint paper [21].

Recall the duality (in height 1 case) for the usual multiple zeta values;

$$
\zeta(n+1,\underbrace{1,\dots,1}_{k-1})=\zeta(k+1,\underbrace{1,\dots,1}_{n-1}).
$$

This does not hold for $\zeta^\star$-values when we just replace ζ by $\zeta^\star$, and, to the best of our knowledge, no duality-like formula for $\zeta^\star$ is known so far. The established assertion

$$
(-1)^n\zeta^\star(n,\underbrace{1,\dots,1}_{k-1})-(-1)^k\zeta^\star(k,\underbrace{1,\dots,1}_{n-1})\in\mathbf{Q}[\zeta(2),\zeta(3),\zeta(5),\dots]
$$

may be regarded as a kind of duality (modulo the ring of Riemann zeta values $\mathbf{Q}[\zeta(2),\zeta(3),\zeta(5),\dots]$). We do not know the reason why the correspondence of indices

$$
(n,\underbrace{1,\dots,1}_{k-1})\longleftrightarrow(k,\underbrace{1,\dots,1}_{n-1})
$$

for this $\zeta^\star$ case is different from that of the duality of usual multiple zeta values,

$$
(n+1,\underbrace{1,\dots,1}_{k-1})\longleftrightarrow(k+1,\underbrace{1,\dots,1}_{n-1}).
$$

Finally, we point out the potential importance of studying further the function $\xi(k_1,\dots,k_r;s)$, a multiple generalization of $\xi_k(s)$ introduced in [4], in order to understand and generalize properties and phenomena of the multiple zeta-star values discussed in this section.

5 Values of the central binomial series

In this final section, we review some facts on the values of the "central binomial series" $\zeta_{CB}(s)$, defined by the following absolutely convergent Dirichlet series;

$$\zeta_{CB}(s) := \sum_{m=1}^{\infty} \frac{1}{m^s \binom{2m}{m}} \quad (\forall s \in \mathbf{C}).$$

In [6], Borwein, Broadhurst and Kamnitzer show that the value $\zeta_{CB}(k)$ for each positive integer $k \geq 2$ is written as a $\mathbf{Q}$-linear combination of multiple zeta values (of height 1) and multiple Clausen and Glaisher values. The latter two are real or imaginary parts (according to the parity of weights) of values at a 6th root of unity of the multiple polylogarithm function

$$Li_{k_1,\ldots,k_n}(z) := \sum_{m_1 > \cdots > m_n > 0} \frac{z^{m_1}}{m_1^{k_1} \cdots m_n^{k_n}}.$$

In analogy with "Zagier's conjecture" for multiple zeta values[5], they conjecture the following

Conjecture ([6]) Consider the following dimensions of the $\mathbf{Q}$-vector spaces

$$a_k := \dim_{\mathbf{Q}} \sum_{\substack{k_1+\cdots+k_n=k \\ k_i \geq 1,\, n \geq 1}} \mathbf{Q} \cdot \mathrm{Re}\left(i^{k_1+\cdots+k_n} Li_{k_1,\ldots,k_n}(e^{\pi i/3})\right),$$

$$b_k := \dim_{\mathbf{Q}} \sum_{\substack{k_1+\cdots+k_n=k \\ k_i \geq 1,\, n \geq 1}} \mathbf{Q} \cdot \mathrm{Im}\left(i^{k_1+\cdots+k_n} Li_{k_1,\ldots,k_n}(e^{\pi i/3})\right).$$

Then, these numbers a_k and b_k are determined recursively by

$$a_0 = a_1 = 1,\ b_0 = b_1 = 0,$$
$$a_n = a_{n-1} + b_{n-2}, \quad b_n = b_{n-1} + a_{n-2}.$$

(In particular, the number $a_n + b_n$ is the Fibonacci number.)

It would be a very interesting problem to find an arithmetic/geometric interpretation of the conjecture and to prove, as in Goncharov [14] and Terasoma [30], that these numbers actually give upper bounds of the spaces.

On the other hand, all the values $\zeta_{CB}(k)$ for $k \leq 1$ are $\mathbf{Q}$-linear combinations of 1 and $\pi/\sqrt{3}$. This fact follows from a result due to D. H. Lehmer [24], who adopted the formula

$$\frac{2x \arcsin(x)}{\sqrt{1-x^2}} = \sum_{m=1}^{\infty} \frac{(2x)^{2m}}{m\binom{2m}{m}} \quad |x| < 1$$

[5] This is a conjecture posed in [31] concerning the dimension of the $\mathbf{Q}$-vector space spanned by MZV's of fixed weight. The conjecture predicts that the dimensions in question satisfy a simple Fibonacci-like recursion. Decisive result to the effect that the conjectural dimension does give an upper bound was given by Goncharov [14] and Terasoma [30].

and its successive differentiations to derive the following explicit formula.

Define two sequences of polynomials $\{p_k(t)\}$ and $\{q_k(t)\}$ $(k=-1,0,1,2,\ldots)$ over $\mathbf{Z}$ by $p_{-1}(t)=0,\ q_{-1}(t)=1$ and the recursion

$$\begin{aligned} p_{k+1}(t) &= 2(kt+1)p_k(t)+2t(1-t)p_k'(t)+q_k(t) \quad (k\geq -1),\\ q_{k+1}(t) &= (2(k+1)t+1)\,q_k(t)+2t(1-t)q_k'(t) \quad (k\geq -1). \end{aligned}$$

The first few examples are

$$\begin{aligned} &p_0(t)=1,\ p_1(t)=3,\ p_2(t)=8t+7,\ p_3(t)=20t^2+70t+15,\ldots,\\ &q_0(t)=1,\ q_1(t)=2t+1,\ q_2(t)=4t^2+10t+1,\ldots. \end{aligned}$$

Then we have for $k\geq -1$

$$\sum_{m=1}^{\infty}\frac{(2m)^k(2x)^{2m}}{\binom{2m}{m}}=\frac{x}{(1-x^2)^{k+3/2}}\left(x\sqrt{1-x^2}p_k(x^2)+\arcsin(x)q_k(x^2)\right)$$

and consequently

$$\zeta_{CB}(-k)=\frac{1}{3}\left(\frac{2}{3}\right)^k p_k\left(\frac{1}{4}\right)+\frac{1}{3}\left(\frac{2}{3}\right)^{k+1} q_k\left(\frac{1}{4}\right)\frac{\pi}{\sqrt{3}} \quad (k\geq -1). \tag{10}$$

This shows that the values $\zeta_{CB}(k)$ $(k\leq 1)$ all lie in the two dimensional $\mathbf{Q}$-vector space spanned by 1 and $\pi/\sqrt{3}$, the fact which is parallel to the result of Euler for $\zeta(s)$: Namely, the values of $\zeta(s)$ at positive integers give variety of (conjecturally independent) transcendental numbers including powers of π (at even arguments) and almost unknown $\zeta(\text{odd})$, whereas the values at negative integers all lie in the one dimensional $\mathbf{Q}$-vector space, $\mathbf{Q}$ itself, and these values are explicitly described by the Bernoulli numbers.

It is therefore interesting to note that, for the value (10), R. Stephan [29] observed the (still conjectural) formula

$$\left(\frac{2}{3}\right)^k p_k\left(\frac{1}{4}\right)=\sum_{j=0}^{k}\mathbb{B}_{k-j}^{(-j)}.$$

An explicit formula given in [7] may be of help to establish this identity. It would also be interesting if we could find any connection of the coefficient of $\pi/\sqrt{3}$ in (10) to poly-Bernoulli or allied numbers, but so far no such connection seems to have been found.

We may consider various analogues of the function $\zeta_{CB}(s)$ and its values at integer arguments. It is possible that among them there are similar descriptions as in the case of $\zeta_{CB}(s)$ described above.

References

[1] A. Adelberg, *Kummer congruences for universal Bernoulli numbers and related congruences for poly-Bernoulli numbers*, Int. Math. J, **1**, 53–63 (2002).

[2] S, Akiyama, S. Egami and Y. Tanigawa, *Analytic continuation of multiple zeta-functions and their values at non-positive integers*, Acta Arith., **98-2**, 107–116 (2001).

[3] K. Aomoto, *Special values of hyperlogarithms and linear difference schemes*, Illinois J. of Math., **34-2**, 191–216 (1990).

[4] T. Arakawa and M. Kaneko, *Multiple zeta values, poly-Bernoulli numbers, and related zeta functions*, Nagoya Math. J. **153**, 1–21 (1999).

[5] T. Arakawa and M. Kaneko, *On Poly-Bernoulli numbers*, Comment. Math. Univ. Sanct. Pauli, **48-2**, 159–167 (1999).

[6] J. Borwein, D. Broadhurst and J. Kamnitzer, *Central binomial sums, multiple Clausen values, and zeta values*, Experiment. Math., **10**, 25–34 (2001).

[7] J. Borwein and R. Girgensohn, *Evaluations of binomial series*, Aequationes Math., **70**, 25–36 (2005).

[8] C. Brewbaker, *Lonesum (0,1)-matrices and poly-Bernoulli numbers of negative index*, Master's thesis, Iowa State University, 2005.

[9] C. Brewbaker, *A combinatorial interpretation of the poly-Bernoulli numbers and two Fermat analogues*, INTEGERS: Electr. J. Comb. Num. Th. , **8** (2008), #A02.

[10] D. J. Broadhurst and D. Kreimer, Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops, Physics Lett. B **393**, 403–412 (1997).

[11] V. G. Drinfel'd, *On quasitriangular quasi-Hopf algebras and a group closely connected with Gal(*$\bar{\mathbf{Q}}/\mathbf{Q}$*)*, Leningrad Math. J. **2**, 829–860 (1991).

[12] L. Euler, Meditationes circa singulare serierum genus, Novi Comm. Acad. Sci. Petropol **20** (1775), 140–186, reprinted in Opera Omnia ser. I, vol. 15, B. G. Teubner, Berlin (1927), 217–267.

[13] H. Furusho, *p-adic multiple zeta values I, p-adic multiple polylogarithms and the p-adic KZ equation*, Invent. Math., **155** no. 2, 253–286, (2004).

[14] A. B. Goncharov, Multiple polylogarithms, cyclotomy and modular complexes, Math. Res. Letters, **5**, 497–516 (1998).

[15] M. Hoffman, *Quasi-symmetric functions and mod p multiple harmonic sums*, preprint, `arXiv:math/0401319v2` [math.NT] 17 Aug. 2007.

[16] M. Hoffman, References on multiple zeta values and Euler sums, `http://www.usna.edu/Users/math/meh/biblio.html`

[17] Charles Jordan, Calculus of Finite Differences, Chelsea Publ. Co., New York, (1950).

[18] M. Kaneko, *Poly-Bernoulli numbers*, J. de Théorie des Nombres **9**, 199–206 (1997).

[19] M. Kaneko, *The Akiyama-Tanigawa algorithm for Bernoulli numbers*, Electronic J. of Integer Sequences, **3**, article 00.2.9, (2000). `http://www.research.att.com/ njas/sequences/JIS/`

[20] M. Kaneko, *A note on poly-Bernoulli numbers and multiple zeta values*, Diophantine analysis and related fields (DARF 2007/2008), AIP Conf. Proc. **976**, 118–124, Amer. Inst. Phys., Melville, NY, (2008).

[21] M. Kaneko and Y. Ohno, *On a kind of duality of multiple zeta-star values*, in preparation.

[22] S. Launois, *Rank t $\mathcal{H}$-primes in quantum matrices*, Comm. Alg., **33-3**, 837–854 (2005).

[23] T. Q. T. Le and J. Murakami, Kontsevich's integral for the Homfly polynomial and relations between values of multiple zeta functions, Topology and its Applications, **62**, 193–206 (1995).

[24] D. H. Lehmer, *Interesting series involving the central binomial coefficient*, Amer. Math. Monthly **92-7**, 449–457 (1985).

[25] Y. Ohno, *A generalization of the duality and sum formulas on the multiple zeta values*, J. Number Th. **74**, 39–43 (1999).

[26] Y. Ohno and D. Zagier, *Multiple zeta values of fixed weight, depth, and height*, Indag. Math., **12** (4), 483–487 (2001).

[27] H. J. Ryser, *Combinatorial Properties of Matrices of Zeros and Ones*, Canadian Journal of Mathematics, **9**, 371–377 (1957).

[28] R. Sanchez-Peregrino, *A note on a closed formula for poly-Bernoulli numbers*, Amer. Math. Month., **109-8** 755–756 (2002).

[29] R. Stephan, Sequence number A098830 in *On-line Encyclopedia of Integer Sequences*,

`http://www.research.att.com/ njas/sequences/Seis.html`

[30] T. Terasoma, *Mixed Tate motives and multiple zeta values*, Invent. Math., **149**, no. 2, 339–369 (2002).

[31] D. Zagier, *Values of zeta functions and their applications*, in ECM volume, Progress in Math., **120** 497–512 (1994).

[32] D. Zagier, *The dilogarithm function*, Frontiers in number theory, physics, and geometry. II, 3–65, Springer, Berlin, (2007).

[33] Zagier, D. : Zetafunktionen und quadratische Körper, Springer, (1981).

[34] J. Zhao, *Analytic continuation of multiple zeta functions*, Proc. Amer. Math. Soc., **128** 1275–1283, (2000).

[35] J. Zhao, *Wolstenholme type theorem for multiple harmonic sums*, Int. J. Number Theory, **4** no. 1, 73–106, (2008).

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