Spectrum of non-commutative harmonic oscillators and residual modular forms

Kimoto, Kazufumi
Department of Mathematical Sciences, University of the Ryukyus

Wakayama, Masato
Institute of Mathematics for Industry, Kyushu University

出版情報：Noncommutative Geometry and Physics. 3, 2013-03. World Scientific
バージョン：

MI Preprint Series

Kyushu University
The Global COE Program
Math-for-Industry Education & Research Hub

Spectrum of non-commutative harmonic oscillators and residual modular forms

Kazufumi Kimoto
& Masato Wakayama

MI 2012-1

(Received January 30, 2012)

Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

Spectrum of non-commutative harmonic oscillators and residual modular forms

Kazufumi Kimoto[1] and Masato Wakayama[2]

[1] Department of Mathematical Sciences, University of the Ryukyus
kimoto@math.u-ryukyu.ac.jp
[2] Institute of Mathematics for Industry, Kyushu University
wakayama@imi.kyushu-u.ac.jp

Summary. Special values $\zeta_Q(k)$ $(k = 2, 3, 4, ...)$ of the spectral zeta function $\zeta_Q(s)$ of the non-commutative harmonic oscillator Q are discussed. Particular emphasis is put on basic modular properties of the generating function $w_k(t)$ of *Apéry-like numbers* which is appeared in analysis on the *first anomaly* of each special value. Here the first anomaly is defined to be the "1st order" difference of $\zeta_Q(k)$ from $\zeta(k)$, $\zeta(s)$ being the Riemann zeta function. In order to describe such modular properties for $k \geq 4$, we introduce a notion of *residual modular forms* for congruence subgroups of $SL_2(\mathbb{Z})$ which contains the classical notion of Eichler integrals as a particular case. Further, we define *differential Eisenstein series*, which are residual modular forms. Using such differential Eisenstein series, for example, one obtains an explicit description of $w_4(t)$. A certain Eichler cohomology group associated to such residual modular forms plays also an important role in the discussion.

1 Introduction

Let Q be an ordinary differential operator having two real parameters α, β defined by

$$Q = Q_{\alpha,\beta} = \left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\right)\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} + \left(x\frac{d}{dx} + \frac{1}{2}\right)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

The system defined by Q is called the *non-commutative harmonic oscillator*, which was introduced in [22, 23] (see [21] for a detailed study of the spectral problem of Q and [19] for a particular interpretation of the problem in terms of Fuchsian ordinary differential equations with four regular singular points in a complex domain). Throughout the paper, we assume that $\alpha, \beta > 0$ and $\alpha\beta > 1$. Under this assumption, Q becomes a positive self-adjoint unbounded operator on $L^2(\mathbb{R}; \mathbb{C}^2)$, the space of $\mathbb{C}^2$-valued square-integrable functions on $\mathbb{R}$, and hence Q has only a discrete spectrum. Denote the eigenvalues of Q by $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots (\to \infty)$. One knows in [23] that the multiplicity of each eigenvalue is at most 3 (see also [10], [21] for certain stronger but conditional estimates of the multiplicities). However, nothing is known explicitly about a real shape of eigenvalues/eigenfunctions of Q if $\alpha \neq \beta$. Let us then consider a series defined by $\zeta_Q(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}$. This series is absolutely convergent and defines a holomorphic function in s in the region $\Re s > 1$. We call $\zeta_Q(s)$ the *spectral zeta function* [8] for the non-commutative harmonic oscillator Q. The spectral zeta function $\zeta_Q(s)$ is analytically continued to the whole complex plane $\mathbb{C}$ as a single-valued meromorphic function that is holomorphic, except a simple pole at $s = 1$. It is notable that $\zeta_Q(s)$ has 'trivial zeros' at $s = 0, -2, -4, \ldots$. When $\alpha = \beta (> 1)$, $\zeta_Q(s)$ is essentially identified with the Riemann zeta function $\zeta(s)$ (see Remark 2).

The aim of the present paper is to investigate modular properties of special values of the spectral zeta function $\zeta_Q(s)$ at $s = 2, 3, 4, \ldots$. Similarly to the Apéry numbers which were introduced in 1978 by R. Apéry for proving the irrationality of $\zeta(2)$ and $\zeta(3)$ (see, e.g. [3]), *Apéry-like numbers* have been introduced in [9] for the description of the special values $\zeta_Q(2)$ and $\zeta_Q(3)$. These Apéry-like numbers $J_2(n)$ and $J_3(n)$ share with many of the properties of the original Apéry numbers, e.g. recurrence equations, congruence properties, etc (see [13, 11]). Actually, the Apéry-like numbers $J_2(n)$ for $\zeta_Q(2)$ obtain a remarkable modular form interpretation as the Apéry numbers possess shown by F. Beukers [3]. We have shown in [14] that the differential equation satisfied by the generating function $w_2(t)$ of $J_2(n)$ is the Picard-Fuchs equation for the universal family of elliptic curves equipped with rational 4-torsion. The parameter t of this family is regarded as a modular function for the congruence subgroup $\Gamma_0(4) (\cong \Gamma(2)) \subset SL_2(\mathbb{Z})$. Moreover, one observes ([14]) that $w_2(t)$ is considered as a $\Gamma_0(4)$ meromorphic modular form of weight 1 in the variable τ as the classical Legendre modular function $t(\tau) = -\frac{\theta_4(\tau)^2}{\theta_4(\tau)^4}$. We also remark that the modular form $w_2(t)$ can be found at #19 in the list of [29].

At the beginning of the paper, we describe the special values $\zeta_Q(k)$ in terms of certain integrals. The formulas for the general cases $k \geq 4$ are much complicated than those of $k = 2, 3$. Thus, we will focus only on the *first anomaly* $R_{k,1}(x)$ (see §3) which expresses the 1st order difference (in a suitable sense) of $\zeta_Q(k)$ from $\zeta(k)$ with respect to the parameters α, β. The first anomaly $R_{k,1}(x)$ for $x = 1/\sqrt{\alpha\beta - 1}$ describes the special value $\zeta_Q(k)$ partly. Notice that when $k = 2, 3$, $R_{k,1}(x)$ possesses full information of each special value. The Taylor expansion of $R_{k,1}(x)$ in x yields numbers $J_k(n)$ what we call k-th Apéry-like numbers. Then, remarkably, one can show that the generating function $w_k(t)$ of $J_k(n)$ satisfies an inhomogeneous differential equation whose homogeneous part is given by the same Fuchsian differential operator which annihilates $w_2(t)$.

In order to solve this differential equation for $w_4(t)$, it is necessary to integrate a certain explicitly given modular form. Employing a simple lemma which is essentially given in [28], we arrive a consequence which claims the generating function $w_4(t)$ can be expressed as a differential of an Eichler integral (or automorphic integral) multiplied by a modular form (a product and quotient of theta functions) for $\Gamma(2)$. Note that Eichler integrals are known as a generalization of the Abelian integrals [5]. At this point, we will introduce a notion of *residual modular forms* which contains Eichler integrals and the Eisenstein series $E_2(\tau)$ of weight 2 for $SL_2(\mathbb{Z})$. The name “residual” comes from the following two facts. 1) Eichler's integral possesses a “integral constant” given by a polynomial in τ which is known as a period function and computed as residues of the integral when one performs the inverse Mellin transform of L-function of the corresponding modular form. 2) To obtain another meaningful expression of such Eichler's integral, we will define *differential Eisenstein series* by a derivative of the analytic continuation of generalized Eisenstein series [2, 18] at negative integer points like in, e.g. [26, 24]. In particular, one can give an explicit expression of $w_4(t)$ by a sum of two such differential Eisenstein series. We remark that the residual part of a differential Eisenstein series is in general given by a rational function in τ, whence it can not be handled in a framework of the Eichler integrals.

Furthermore, to understand the structure, especially the dimension of a space of residual modular forms, it is important to consider the Eichler cohomology groups [5, 6, 16] associated with several $\Gamma(2)$-modules made by a set of certain functions on the Poincaré upper half plane, such as the space (field) of rational functions $\mathbb{C}(\tau)$, the space of holomorphic/meromorphic functions with some decay condition at the infinity (cusps), etc. In the very end of the paper, we focus on a particular subgroup of the Eichler cohomology group which we call a periodic cohomology for the explicit determination of the space of residual modular forms which contains $w_4(t)$.

2 Special values of the spectral zeta function

The first two special values $\zeta_Q(2)$ and $\zeta_Q(3)$ have been calculated in [9]. For instance, the value $\zeta_Q(2)$ is represented essentially by a contour integral of a holomorphic solution of some Fuchsian differential equation. Actually, these values are represented by the contour integral expressions of solutions of certain special type of Heun differential equations. Later, Ochiai [20] gave an expression of $\zeta_Q(2)$ using the complete elliptic integral or the hypergeometric function, and the authors [13] gave a formula for $\zeta_Q(3)$ similar to the Ochiai's one.

Now we give a general formula for the spectral zeta values $\zeta_Q(k)$ ($k = 2, 3, 4, \dots$). We refer to [12] for its proof. For $\boldsymbol{u} = (u_1, u_2, \dots, u_k)$, we define the k by k matrix $\Delta_k(\boldsymbol{u})$ ([9]) by

$$\Delta_k(\boldsymbol{u}) :=$$
$$\begin{pmatrix}
\frac{1-u_k^4u_1^4}{(1-u_k^4)(1-u_1^4)} & \frac{-u_1^2}{1-u_1^4} & 0 & 0 & \cdots & \frac{-u_k^2}{1-u_k^4} \\
\frac{-u_1^2}{1-u_1^4} & \frac{1-u_1^4u_2^4}{(1-u_1^4)(1-u_2^4)} & \frac{-u_2^2}{1-u_2^4} & 0 & \cdots & 0 \\
0 & \frac{-u_2^2}{1-u_2^4} & \frac{1-u_2^4u_3^4}{(1-u_2^4)(1-u_3^4)} & \frac{-u_3^2}{1-u_3^4} & \cdots & 0 \\
0 & 0 & \frac{-u_3^2}{1-u_3^4} & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \frac{-u_{k-1}^2}{1-u_{k-1}^4} \\
\frac{-u_k^2}{1-u_k^4} & 0 & 0 & \cdots & \frac{-u_{k-1}^2}{1-u_{k-1}^4} & \frac{1-u_{k-1}^4u_k^4}{(1-u_{k-1}^4)(1-u_k^4)}
\end{pmatrix}$$
$$= \sum_{i=1}^{k} \left\{ \left(E_{ii}^{(k)} + E_{i+1,i+1}^{(k)}\right)\left(\frac{1}{1-u_i^4} - \frac{1}{2}\right) + \left(E_{i,i+1}^{(k)} + E_{i+1,i}^{(k)}\right)\frac{-u_i^2}{1-u_i^4} \right\}.$$

Here $E_{ij}^{(k)}$ denotes the (i,j)-matrix unit of size k. We also assume that the indices of $E_{ij}^{(k)}$ are understood modulo k, i.e. $E_{0,j}^{(k)} = E_{k,j}^{(k)}$, $E_{k+1,j}^{(k)} = E_{1,j}^{(k)}$, etc. Notice that $\Delta_k(\boldsymbol{u})$ is real symmetric and positive definite for any $\boldsymbol{u} \in (0,1)^k$. For $\{i_1, i_2, \ldots, i_{2j}\} \subset [k] = \{1, 2, \ldots, k\}$, we also put

$$\Xi_k(i_1, \ldots, i_{2j}) := \sqrt{-1}\sum_{r=1}^{2j}(-1)^r E_{i_r,i_r}^{(k)}.$$

Theorem 1. *For each positive integer $n \geq 2$, one has*

$$\begin{aligned}\zeta_Q(k) = 2 &\left(\frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\right)^k \\ &\times \left(\zeta\Big(k, \frac{1}{2}\Big) + \sum_{0<2j\leq k}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^{2j} R_{k,j}\left(\frac{1}{\sqrt{\alpha\beta-1}}\right)\right).\end{aligned} \tag{1}$$

Here $R_{k,j}(x)$ is given by a sum of integrals

$$R_{k,j}(x) = \sum_{1\leq i_1<i_2<\cdots<i_{2j}\leq k} \int_{[0,1]^k} \frac{2^k du_1 \ldots du_k}{\sqrt{\mathcal{W}_k(\boldsymbol{u}; x; i_1, \ldots, i_{2j})}},$$

where the function $\mathcal{W}_k(\boldsymbol{u}; x; i_1, \ldots, i_{2j})$ is given by

$$\mathcal{W}_k(\boldsymbol{u}; x; i_1, \ldots, i_{2j}) = \det\left(\Delta_k(\boldsymbol{u}) + x\Xi_k(i_1, \ldots, i_{2j})\right)\prod_{i=1}^{k}(1-u_i^4).$$

□

Remark 1. It follows from the fact $\zeta_Q(k) \in \mathbb{R}$ that $\mathcal{W}_k(\boldsymbol{u}; x; i_1, \ldots, i_{2j})$ is even as a polynomial in x.

Remark 2. If $\alpha = \beta$, then we have $\zeta_Q(k) = 2(\alpha^2-1)^{-k/2}(2^k-1)\zeta(k)$. This follows from the fact $\zeta_Q(s) = 2(\alpha^2-1)^{-s/2}\zeta(s,1/2)$ when $\alpha=\beta$. In fact, when $\alpha=\beta$, it is known in [23] that Q is unitarily equivalent to a couple of the harmonic oscillators $\sqrt{\alpha^2-1}\left(-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2\right)I$. If $\alpha\neq\beta$, however, it seems hard to expect a symmetry of $\mathfrak{sl}_2(\mathbb{C})$ (the oscillator representation of $\mathfrak{sl}_2(\mathbb{C})$ (see, e.g. [7]). Hence the eigenvalue problem of Q is being highly non-trivial in general.

Example 1. The values $\zeta_Q(2)$ and $\zeta_Q(3)$ are given by

$$\zeta_Q(2) = 2\left(\frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\right)^2\left(\zeta(2,1/2)+\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 R_{2,1}\Big(\frac{1}{\sqrt{\alpha\beta-1}}\Big)\right),$$
$$\zeta_Q(3) = 2\left(\frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\right)^3\left(\zeta(3,1/2)+\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 R_{3,1}\Big(\frac{1}{\sqrt{\alpha\beta-1}}\Big)\right)$$

with

$$R_{2,1}(x) = \int_{[0,1]^2}\frac{4du_1du_2}{\sqrt{(1-u_1^2u_2^2)^2+x^2(1-u_1^4)(1-u_2^4)}},$$
$$R_{3,1}(x) = 3\int_{[0,1]^3}\frac{8du_1du_2du_3}{\sqrt{(1-u_1^2u_2^2u_3^2)^2+x^2(1-u_1^4)(1-u_2^4u_3^4)}}.$$

This recovers the result in [9].

If we define the numbers $J_2(n)$ $(n\geq 0)$ by the expansion

$$R_{2,1}(x) = \sum_{n=0}^{\infty}\binom{-1/2}{n}J_2(n)x^{2n},$$

then they satisfy the three-term recurrence relation

$$4n^2J_2(n)-(8n^2-8n+3)J_2(n-1)+4(n-1)^2J_2(n-2)=0 \quad (n\geq 1). \tag{2}$$

This implies that the generating function $w_2(z)=\sum_{n=0}^{\infty}J_2(n)z^n$ satisfies

$$\left\{z(1-z)^2\frac{d^2}{dz^2}+(1-3z)(1-z)\frac{d}{dz}+z-\frac{3}{4}\right\}w_2(z)=0, \tag{3}$$

which looks a confluent Heun differential equation [9]. This equation, however, can be reduced to the Gaussian hypergeometric differential equation by a suitable change of variable and solved as follows [20]:

$$w_2(z)=\frac{3\zeta(2)}{1-z}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{z}{z-1}\right), \tag{4}$$

from which, using the Clausen identity, one obtains

$$R_{2,1}(x) = 3\zeta(2)_2F_1\left(\frac{1}{4},\frac{3}{4};1;-x^2\right)^2.$$

Thus we have the following expression of $\zeta_Q(2)$ [20]:

$$\zeta_Q(2) = \left(\frac{\pi(\alpha+\beta)}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\right)^2\left(1+\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{1}{1-\alpha\beta}\right)^2\right).$$

We also have similar expression for $\zeta_Q(3)$ in [13].

3 Apéry-like numbers

In what follows, we restrict our attention to the quantities $R_{k,1}(x)$ appearing in the special value formula for $\zeta_Q(s)$. We sometimes refer to $R_{k,1}(x)$ as the *first anomaly* in $\zeta_Q(k)$ for short.

3.1 Apéry-like numbers associated to the first anomalies

Let us define the numbers $J_k(n)$ for $k = 2,3,4,\dots$ as coefficients in the Taylor expansion of the first anomaly $R_{k,1}(x)$

$$\begin{aligned} R_{k,1}(x) &= \frac{k}{2}\sum_{r=1}^{k-1}\int_{[0,1]^k}\frac{2^k du_1\cdots du_k}{\sqrt{(1-u_1^2\cdots u_k^2)^2+x^2(1-u_1^4\cdots u_r^4)(1-u_{r+1}^4\cdots u_k^4)}} \\ &= \frac{k}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}J_k(n)x^{2n}, \end{aligned}$$

and call the numbers $J_k(n)$ the *Apéry-like numbers* associated to the first anomaly $R_{k,1}(x)$ of $\zeta_Q(k)$, or *k-th Apéry-like numbers* for short. For convenience, we define numbers $J_0(n)$ and $J_1(n)$ by

$$J_0(n) = 0, \qquad J_1(n) = \frac{2^n n!}{(2n+1)!!} = \frac{(1)_n(1)_n}{(\frac{3}{2})_n(1)_n} \qquad (n = 0,1,2,\dots),$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. An elementary manipulation shows that

$$\begin{aligned} J_k(n) &= \frac{1}{2^{2n+1}}\int_0^{\infty}\frac{u^{k-2}}{(k-2)!}B_n(u)du, \\ B_n(u) &= \frac{e^{nu}}{(\sinh\frac{u}{2})^{2n+1}}\int_0^u(1-e^{-2t})^n(1-e^{-2(u-t)})^n dt \end{aligned}$$

for $k = 2,3,4,\dots$ and $n = 0,1,2,\dots$. We notice that the function $B_n(u)$ is continuous at $u = 0$ and is of exponential decay as $u \to +\infty$ (see Proposition 4.10 in [9]).

Example 2 (Initial values). We see that

$$\begin{aligned} B_0(u) &= \frac{1}{\sinh\frac{u}{2}}\int_0^u dt = \frac{u}{\sinh\frac{u}{2}}, \\ B_1(u) &= \frac{e^u}{(\sinh\frac{u}{2})^3}\int_0^u (1-e^{-2t})(1-e^{-2(u-t)})dt \\ &= 4\frac{u}{\sinh\frac{u}{2}} + 2\frac{u}{(\sinh\frac{u}{2})^3} - 4\frac{\cosh\frac{u}{2}}{(\sinh\frac{u}{2})^2}. \end{aligned}$$

Thus we have

$$\begin{aligned} J_k(0) &= \frac{1}{2\cdot(k-2)!}\int_0^\infty \frac{u^{k-1}}{\sinh\frac{u}{2}}du, \\ J_k(1) &= \frac{1}{2\cdot(k-2)!}\int_0^\infty \frac{u^{k-1}}{\sinh\frac{u}{2}}du + \frac{1}{4\cdot(k-2)!}\int_0^\infty \frac{u^{k-1}}{(\sinh\frac{u}{2})^3}du \\ &\quad - \frac{1}{2\cdot(k-2)!}\int_0^\infty \frac{\cosh\frac{u}{2}}{(\sinh\frac{u}{2})^2}u^{k-2}du. \end{aligned}$$

Using the formulas

$$\begin{aligned} \zeta\Big(s,\frac{1}{2}\Big) &= \frac{1}{2\Gamma(s)}\int_0^\infty \frac{u^{s-1}}{\sinh\frac{u}{2}}du \\ &= \frac{1}{4\Gamma(s+1)}\int_0^\infty \frac{\cosh\frac{u}{2}}{(\sinh\frac{u}{2})^2}u^s du \quad (\Re(s)>1), \\ \int_0^\infty \frac{u^{s-1}}{(\sinh\frac{u}{2})^3}du &= (s-1)\int_0^\infty \frac{\cosh\frac{u}{2}}{(\sinh\frac{u}{2})^2}u^{s-2}du \\ &\quad - \frac{1}{2}\int_0^\infty \frac{u^{s-1}}{\sinh\frac{u}{2}}du \quad (\Re(s)>3), \end{aligned}$$

we get

$$\begin{aligned} J_k(0) &= (k-1)\zeta\Big(k,\frac{1}{2}\Big), \\ J_k(1) &= (k-3)\zeta\Big(k-2,\frac{1}{2}\Big) + \frac{3(k-1)}{4}\zeta\Big(k,\frac{1}{2}\Big)\Big(= J_{k-2}(0) + \frac{3}{4}J_k(0)\Big) \end{aligned}$$

for $k \geq 4$. It is worth noting that these formulas are also valid for $k=2$ and $k=3$:

$$\begin{aligned} J_2(0) &= \zeta\Big(2,\frac{1}{2}\Big), \quad J_2(1) = \frac{3}{4}\zeta\Big(2,\frac{1}{2}\Big), \\ J_3(0) &= 2\zeta\Big(3,\frac{1}{2}\Big), \quad J_3(1) = 1 + \frac{3}{2}\zeta\Big(3,\frac{1}{2}\Big). \end{aligned}$$

Here we use the fact that

$$\zeta\Big(0,\frac{1}{2}\Big) = 0, \qquad \lim_{s\to 1}(s-1)\zeta\Big(s,\frac{1}{2}\Big) = 1.$$

Remark 3 (Remarks on conventions for Apéry-like numbers). $J_2(n)$ in this article is equal to J_n in [9] (and $J_2(n)$ in [13]). $J_3(n)$ in this article is equal to $2J_n^1$ in [9] (and $2J_3(n)$ in [13]), since our $J_3(n)$ is defined to be the sum $J_{1,3-1}(n) + J_{2,3-2}(n)$, each summand in which is equal to J_n^1 in [9].

As we have mentioned above, the second Apéry-like numbers $J_2(n)$ satisfy the three-term recurrence formula (2), which also implies the second order differential equation (3) for their generating function $w_2(z)$. By developing the discussion in [9], we can prove that the Apéry-like numbers $J_k(n)$ also satisfy similar three-term recurrence formula for each $k = 2, 3, 4, \dots$ in general as follows.

Theorem 2.

$$4n^2 J_k(n) - (8n^2 - 8n + 3)J_k(n-1) + 4(n-1)^2 J_k(n-2) = 4J_{k-2}(n-1) \tag{5}$$

for $k \geq 2$ and $n \geq 2$. □

For $k \geq 0$, we define

$$w_k(z) = \sum_{n=0}^{\infty} J_k(n) z^n.$$

It is immediate to see that $w_0(z) = 0$ and

$$w_1(z) = {}_2F_1\left(1, 1; \frac{3}{2}; z\right)$$

The formula (5) into the differential equations for the generating functions $w_k(z)$ as follows.

Theorem 3. *One has*

$$\left\{ z(1-z)^2 \frac{d^2}{dz^2} + (1-z)(1-3z)\frac{d}{dz} + z - \frac{3}{4} \right\} w_k(z) = w_{k-2}(z) \tag{6}$$

for $k \geq 2$. □

Remark 4. We have

$$\left\{ z(1-z)^2 \frac{d^2}{dz^2} + (1-z)(1-3z)\frac{d}{dz} + z - \frac{3}{4} \right\}^k w_{2k}(z) = 0$$

and

$$\left\{ z(1-z) \frac{d^2}{dz^2} + \frac{3}{2}(1-2z)\frac{d}{dz} - 1 \right\} \left\{ z(1-z)^2 \frac{d^2}{dz^2} + (1-z)(1-3z)\frac{d}{dz} + z - \frac{3}{4} \right\}^k w_{2k+1}(z) = 0$$

for each $k \geq 0$. This shows that each $w_k(z)$ is a formal power series solution of a linear differential equation.

To find an explicit formula for $J_k(n)$, it is useful to introduce the function

$$v_k(t) = (1-z)w_k(z), \quad t = \frac{z}{z-1} \qquad \left(\Longleftrightarrow \; w_k(z) = (1-t)v_k(t), \quad z = \frac{t}{t-1} \right).$$

Note that

$$\begin{aligned} v_2(t) &= J_2(0) \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; t\right) = J_2(0) \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n}^2 t^n, \\ v_1(t) &= \frac{1}{1-t} {}_2F_1\left(1, 1; \frac{3}{2}; \frac{t}{t-1}\right) = \sum_{n=0}^{\infty} \frac{t^n}{2n+1}. \end{aligned}$$

The formula (6) is translated equivalently as

$$\left\{ t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{1}{4} \right\} v_k(t) = -v_{k-2}(t).$$

Let us look at the (hypergeometric differential) operator

$$D = t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{1}{4}.$$

It is straightforward to check that the polynomial

$$p_n(t) = -\frac{1}{(n+\frac{1}{2})^2} \binom{-\frac{1}{2}}{n}^{-2} \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k}^2 t^k \quad (n = 0, 1, 2, \dots)$$

satisfy $Dp_n(t) = t^n$ (see §4 of [13]). Thus, if we put

$$\xi_l(t) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n}^2 A_{l,n} t^n \quad (l \geq 0),$$

then

$$D\left\{ -\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n}^2 A_{l,n} p_n(t) \right\} = -\xi_l(t).$$

On the other hand, we see that

$$\begin{aligned} -\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n}^2 A_{l,n} p_n(t) &= \sum_{n=0}^{\infty} A_{l,n} \frac{1}{(n+\frac{1}{2})^2} \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k}^2 t^k \\ &= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k}^2 \left\{ \sum_{n=k}^{\infty} \frac{A_{l,n}}{(n+\frac{1}{2})^2} \right\} t^k. \end{aligned}$$

Hence, if we *assume* that the numbers $A_{l,k}$ satisfy the condition

$$A_{l+2,k} = \sum_{n=k}^{\infty} \frac{A_{l,n}}{(n+\frac{1}{2})^2}, \tag{7}$$

then the functions $\xi_l(t)$ satisfy the relation

$$D\xi_{l+2}(t) = -\xi_l(t) \quad (l \geq 0).$$

Notice that we have

$$A_{l+2m,n} = \sum_{n \leq k_1 \leq k_2 \leq \cdots \leq k_m} \frac{A_{l,k_m}}{(k_1+\frac{1}{2})^2(k_2+\frac{1}{2})^2 \cdots (k_m+\frac{1}{2})^2} \tag{8}$$

under the assumption (7).

Now we determine the numbers $A_{l,n}$ so that they satisfy (7). If we set

$$A_{l,n} = \begin{cases} \dfrac{1}{2}\dfrac{1}{n+\frac{1}{2}}\dbinom{-\frac{1}{2}}{n}^{-2} & l=1, \\ J_2(0) & l=2 \end{cases}$$

and extend by the relation (8), then the relation (7) is surely satisfied. We remark that the series (8) indeed converges since $A_{1,n}$ and $A_{2,n}$ are bounded so that the positive series $A_{l+2m,n}$ is majorated by a constant multiple of the series (*multiple zeta-star value*) $\sum_{0<k_1 \leq k_2 \leq \cdots \leq k_m} (k_1 k_2 \ldots k_m)^{-2}$. Notice that

$$\begin{aligned} \xi_1(t) &= \frac{1}{1-t}\,{}_2F_1\left(1,1;\frac{3}{2};\frac{t}{t-1}\right) = v_1(t), \\ \xi_2(t) &= J_2(0) \cdot {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;t\right) = v_2(t). \end{aligned}$$

As a result, we see that $v_l(t)$ is of the form

$$v_l(t) = \xi_l(t) + \sum_{0<j \leq l/2} C_{l-2j} v_{2j}(t),$$

and the coefficients $C_{l-2}, C_{l-4}, \ldots$ are determined inductively. Indeed, if $v_l(t)$ is given as above, then

$$\begin{aligned} D\xi_{l+2}(t) = -\xi_l(t) &= -v_l(t) + \sum_{0<j \leq l/2} C_{l-2j} v_{2j}(t) \\ &= D\big(v_{l+2}(t) - \sum_{0<j \leq l/2} C_{l-2j} v_{2j+2}(t)\big), \end{aligned}$$

which implies that there exists certain constant C_l such that

$$v_{l+2}(t) - \xi_{l+2}(t) - \sum_{0<j\le l/2} C_{l-2j} v_{2j+2}(t) = C_l v_2(t).$$

We also have

$$w_l(z) = \frac{1}{1-z}\xi_l\left(\frac{z}{z-1}\right) + \sum_{0<j\le 2l} C_{l-2j} w_{2j}(z),$$

and we can obtain explicit formulas of $J_k(n)$ for each k.

Example 3. For $k = 2, 3, 4$ one has

$$\begin{aligned}
J_2(n) &= \zeta\Big(2, \frac{1}{2}\Big) \sum_{k=0}^{n} (-1)^k \binom{-\frac{1}{2}}{k}^2 \binom{n}{k}, \\
J_3(n) &= -\frac{1}{2} \sum_{k=0}^{n} (-1)^k \binom{-\frac{1}{2}}{k}^2 \binom{n}{k} \sum_{0\le j<k} \frac{1}{(j+\frac{1}{2})^3} \binom{-\frac{1}{2}}{j}^{-2} \\
&\quad + 2\zeta\Big(3, \frac{1}{2}\Big) \sum_{k=0}^{n} (-1)^k \binom{-\frac{1}{2}}{k}^2 \binom{n}{k}, \\
J_4(n) &= -\zeta\Big(2, \frac{1}{2}\Big) \sum_{k=0}^{n} (-1)^k \binom{-\frac{1}{2}}{k}^2 \binom{n}{k} \sum_{0\le j<k} \frac{1}{(j+\frac{1}{2})^2} \\
&\quad + 3\zeta\Big(4, \frac{1}{2}\Big) \sum_{k=0}^{n} (-1)^k \binom{-\frac{1}{2}}{k}^2 \binom{n}{k}.
\end{aligned}$$

4 Modular forms and Apéry-like numbers

In this section, we focus on the study of modular properties of the generating functions $w_k(t)$ of Apéry-like numbers. In particular, we obtain an explicit expression of $w_4(t)$ in terms of a newly introduced functions which we call *differential Eisenstein series.*

4.1 Modular interpretation of w_2 — a motivating example

Let $\Gamma(2) := \{\gamma \in SL_2(\mathbb{Z}) \,|\, \gamma \equiv I_2 \mod 2\}$, the principal congruence subgroup of level 2. Let $\tau \in \mathfrak{h}$, $\mathfrak{h}$ being the complex upper half plane. We recall the following standard functions (the elliptic theta functions $\theta_2(\tau)$, $\theta_3(\tau)$, $\theta_4(\tau)$ and normalized Eisenstein series $E_k(\tau)$ for $k = 2, 4, 6, \dots$):

$$\theta_2(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i (n+1/2)^2 \tau}, \quad \theta_3(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau},$$

$$\theta_4(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i n^2 \tau}, \quad E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n \tau}.$$

Put

$$t = t(\tau) = -\frac{\theta_2(\tau)^4}{\theta_4(\tau)^4}, \tag{9}$$

which is a $\Gamma(2)$-modular function such that $t(i\infty) = 0$. Notice that

$$1 - t(\tau) = \frac{\theta_3(\tau)^4}{\theta_4(\tau)^4}, \qquad \frac{t(\tau)}{t(\tau)-1} = \frac{\theta_2(\tau)^4}{\theta_3(\tau)^4}$$

by the identity $\theta_2(\tau)^4 + \theta_4(\tau)^4 = \theta_3(\tau)^4$. By the formula (§22.3 in [27])

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\theta_2(\tau)^4}{\theta_3(\tau)^4}\right) = \theta_3(\tau)^2,$$

it follows from (4) that

$$w_2(t) = \frac{J_2(0)}{1-t(\tau)}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{t(\tau)}{t(\tau)-1}\right) = \frac{\theta_4(\tau)^4}{\theta_3(\tau)^2},$$

which is a $\Gamma(2)$-modular form of weight 1.

Remark 5. The differential equation (3) satisfied by the generating function $w_2(z)$ of Apéry-like numbers $J_2(n)$ is the Picard-Fuchs equation for the universal family of elliptic curves equipped with rational 4-torsion. In fact, each elliptic curve in the family is birationally equivalent to one of the curves $(1-u^2v^2)^2 + x^2(1-u^4)(1-v^4) = 0$ in the (u,v)-plane, which are appeared in the denominator of the integrand of $R_{2,1}(x)$.

This fact naturally leads us to a question what the nature of $w_k(t)$ is in general from a geometric viewpoint. In order to answer this question for the special case $w_4(t)$ at the beginning, we recall a lemma [28] (Lemma 1). We slightly generalize the statement of this lemma for our purpose. The proof is essentially the same.

Lemma 1. *Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$ commensurable with the modular group. Let $A(\tau)$ be a modular form of weight k and $t(\tau)$ be a non-constant modular function on Γ such that $t(i\infty) = 0$. Let $\vartheta = t\frac{d}{dt}$. Let $L := \vartheta^{k+1} + r_k(t)\vartheta^k + \cdots + r_0(t)$ be the differential operator with rational coefficients $r_j(t)$. Assume that $LA(t) = 0$. Let $g(t) = g(t(\tau))$ be a modular form. Then a solution of the inhomogeneous differential equation $LB(t) = g(t)$ is given by*

$$B(t) = A(t) \underbrace{\int^q \cdots \int^q}_{k+1} \Big(\frac{q\,dt/dq}{t}\Big)^{k+1} \frac{g(t)}{A(t)} \frac{dq}{q} \cdots \frac{dq}{q},$$

where the integration is iterated $k+1$ times and $q := e^{2\pi i\tau}$. □

From Theorem 3, it follows that

$$\Big(z(1-z)^2\frac{d^2}{dz^2}+(1-z)(1-3z)\frac{d}{dz}+z-\frac{3}{4}\Big)^k w_{2k+r}(z)=w_r(z) \tag{10}$$

for $k\geq 1$ and $r\geq 0$, which can be also written in terms of the Euler operator $\vartheta=t\frac{d}{dt}$ as

$$\Big(\vartheta^{2k}+\cdots\Big)w_{2k+r}(z)=\frac{z^k}{(1-z)^{2k}}w_r(z) \qquad (k\geq 1, r\geq 0). \tag{11}$$

Let us consider the function

$$\begin{aligned}W_{k,r}(t) &:= w_2(t)\underbrace{\int_0^q\cdots\int_0^q}_{2k}\Big(\frac{qdt/dq}{t}\Big)^{2k}\frac{t^k}{(1-t)^{2k}}\frac{w_r(t)}{w_2(t)}\frac{dq}{q}\cdots\frac{dq}{q}\\ &=\Big(-\frac{1}{2}\Big)^k J_2(0)\frac{\theta_4(\tau)^4}{\theta_3(\tau)^2}\underbrace{\int_0^q\cdots\int_0^q}_{2k}\Big(\theta_2(\tau)^4\theta_4(\tau)^4\Big)^k\frac{\theta_3(\tau)^2}{\theta_4(\tau)^4}w_r(t)\frac{dq}{q}\cdots\frac{dq}{q}.\end{aligned}$$

Here we use the fact

$$\frac{q}{t}\frac{dt}{dq}=\frac{1}{2}\theta_3(\tau)^4.$$

Obviously, Lemma 1 is applicable to (11) if $k=1$ and $r=2$, and then a solution to (11) is given by the integral $W_{1,2}(t)$. Thus we have the following.

Lemma 2.

$$w_4(t)=W_{1,2}(t)+\pi^2 w_2(t).$$

Proof. It is clear that $w_4(t)$ is of the form

$$w_4(t)=W_{1,2}(t)+Cw_2(t)$$

with some constant C. Notice that $w_2(0)=\zeta(2,\frac{1}{2})=3\zeta(2)=3\cdot\frac{\pi^2}{6}$ and $w_4(0)=J_4(0)=3\zeta(4,\frac{1}{2})=3\cdot 15\cdot\zeta(4)=3\cdot 15\cdot\frac{\pi^4}{90}$. Hence the result follows immediately from the fact $W_{1,2}(0)=W_{1,2}(t(\tau))|_{\tau=i\infty}=0$.

In what follows, we consider $W_{k,2}(t)$ for $k\in\mathbb{N}$ in general. For convenience, let us put

$$f(\tau)=\theta_2(\tau)^4\theta_4(\tau)^4=\frac{1}{15}\big(E_4(\tau)-17E_4(2\tau)+16E_4(4\tau)\big), \tag{12}$$

$$\mathit{\Lambda}_k(s)=\int_0^\infty t^s f(it)^k\frac{dt}{t}, \tag{13}$$

$$\boldsymbol{E}_k(\tau)=\underbrace{\int_0^q\cdots\int_0^q}_{2k} f(\tau)^k\frac{dq}{q}\cdots\frac{dq}{q}. \tag{14}$$

Notice that

$$W_{k,2}(t) = \Big(-\frac{1}{2}\Big)^k J_2(0)\frac{\theta_4(\tau)^4}{\theta_3(\tau)^2}\boldsymbol{E}_k(\tau). \tag{15}$$

Since $f(\tau)$ is a modular form of weight 4 with respect to $\Gamma(2)$, the corresponding L-function $\Lambda_k(s)$ of $f(\tau)^k$ satisfies the functional equation $\Lambda_k(4k-s) = \Lambda_k(s)$. By the inversion formula of Mellin's transform, one notices that

$$f(iy)^k = \frac{1}{2\pi i}\int_{\Re s=\alpha} y^{-s}\Lambda_k(s)ds \qquad (y>0, \alpha \gg 0).$$

4.2 Modular interpretation of w_4

Let us consider the case where $k=1$. $\Lambda_1(s)$ satisfies the functional equation $\Lambda_1(4-s) = \Lambda_1(s)$. If we put

$$\Xi_1(s) = \frac{\Lambda_1(s+2)}{(s+1)s(s-1)},$$

then the functional equation for $\Lambda_1(s)$ implies the oddness $\Xi_1(-s) = -\Xi_1(s)$. For later use, we denote by $\rho_{1,j}$ the residue of $\Xi_1(s)$ at $s=j$ for $j=-1,0,1$. Explicitly, we have

$$\Lambda_1(s) = 16\zeta(s)\zeta(s-3)(1-2^{-s})(1-2^{4-s}),$$
$$\rho_{1,-1} = \rho_{1,1} = \frac{7\zeta(3)}{\pi^3}, \qquad \rho_{1,0} = -\frac{1}{2}.$$

Let us introduce

$$\boldsymbol{G}_1(\tau) = \int_0^q\int_0^q\int_0^q f(\tau)\frac{dq}{q}\frac{dq}{q}\frac{dq}{q} = \int_0^q \boldsymbol{E}_1(\tau)\frac{dq}{q}.$$

Clearly, $\boldsymbol{G}_1(\tau)$ is a periodic function with period 2 and $\boldsymbol{G}_1(i\infty) = 0$.

Lemma 3. *For $\beta \gg 0$, one has*

$$\boldsymbol{E}_1(\tau) = \frac{(2\pi)^2}{2\pi i}\int_{\Re s=\beta}(s-1)\Big(\frac{\tau}{i}\Big)^{-s}\Xi_1(s)ds,$$
$$\boldsymbol{G}_1(\tau) = \frac{(2\pi)^3}{2\pi i}\int_{\Re s=\beta}\Big(\frac{\tau}{i}\Big)^{1-s}\Xi_1(s)ds$$

and

$$\frac{d}{d\tau}\boldsymbol{G}_1(\tau) = 2\pi i\boldsymbol{E}_1(\tau).$$

Proof. For simplicity, we restrict our discussion on the upper imaginary axis, that is, we assume that $\tau = iy$ $(y > 0)$. We see that $q = e^{-2\pi y}$ and

$$\frac{dq}{q} = -2\pi dy.$$

It follows that

$$\begin{aligned}
&\boldsymbol{E}_1(iy) \\
&= (-2\pi)^2 \int_\infty^y \int_\infty^y f(iy) dy dy = \frac{(2\pi)^2}{2\pi i} \int_{\Re s = \alpha} \left\{ \int_\infty^y \int_\infty^y y^{-s} dy dy \right\} \Lambda_1(s) ds \\
&= \frac{(2\pi)^2}{2\pi i} \int_{\Re s = \alpha} \frac{y^{2-s} \Lambda_1(s)}{(s-1)(s-2)} ds = \frac{(2\pi)^2}{2\pi i} \int_{\Re s = \alpha - 2} \frac{y^{-s} \Lambda_1(s+2)}{s(s+1)} ds \\
&= \frac{(2\pi)^2}{2\pi i} \int_{\Re s = \alpha - 2} (s-1) y^{-s} \varXi_1(s) ds,
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{G}_1(iy) &= (-2\pi)^3 \int_\infty^y \int_\infty^y \int_\infty^y f(iy) dy dy dy \\
&= \frac{-(2\pi)^3}{2\pi i} \int_{\Re s = \alpha - 2} (s-1) \left\{ \int_\infty^y y^{-s} dy \right\} \varXi_1(s) ds \\
&= \frac{(2\pi)^3}{2\pi i} \int_{\Re s = \alpha - 2} y^{1-s} \varXi_1(s) ds.
\end{aligned}$$

Lemma 4. *One has*

$$\boldsymbol{G}_1\Big(-\frac{1}{\tau}\Big) - \tilde{\rho}_{1,1} = \tau^{-2} \Big(\boldsymbol{G}_1(\tau) - \tilde{\rho}_{1,1} - \frac{\tilde{\rho}_{1,0}}{i} \tau \Big).$$

Here $\tilde{\rho}_{1,j} = (2\pi)^3 \rho_{1,j}$.

Proof. Using the functional equation $\varXi_1(-s) = -\varXi_1(s)$, one observes

$$\begin{aligned}
\frac{2\pi i}{(2\pi)^3} \boldsymbol{G}_1\Big(-\frac{1}{iy}\Big) &= -\int_{\Re s = \beta} \Big(\frac{1}{y}\Big)^{1-s} \varXi_1(-s) ds = -\int_{\Re s = -\beta} \Big(\frac{1}{y}\Big)^{1+s} \varXi_1(s) ds \\
&= -y^{-2} \int_{\Re s = -\beta} y^{1-s} \varXi_1(s) ds \\
&= (iy)^{-2} \left\{ \int_{\Re s = \beta} y^{1-s} \varXi_1(s) ds - 2\pi i (y^2 \rho_{1,-1} + y \rho_{1,0} + \rho_{1,1}) \right\} \\
&= (iy)^{-2} \frac{2\pi i}{(2\pi)^3} \left\{ \boldsymbol{G}_1(iy) - (y^2 \tilde{\rho}_{1,1} + y \tilde{\rho}_{1,0} + \tilde{\rho}_{1,1}) \right\}.
\end{aligned}$$

It then follows that

$$\boldsymbol{G}_1\Big(-\frac{1}{iy}\Big) - \tilde{\rho}_{1,1} = (iy)^{-2} \Big(\boldsymbol{G}_1(iy) - \tilde{\rho}_{1,1} - y \tilde{\rho}_{1,0} \Big).$$

This is the desired conclusion.

Define $\psi_1(\tau)$ by

$$\psi_1(\tau) = \boldsymbol{G}_1(\tau) - \tilde{\rho}_{1,1}. \tag{16}$$

Lemma 5.

$$\psi_1\Big(-\frac{1}{\tau}\Big) = \tau^{-2}\Big\{\psi_1(\tau) - \tilde{\rho}_{1,0}\frac{\tau}{i}\Big\}.$$

Since $\psi_1(\tau)$ is a constant shift of the 2-periodic function $\boldsymbol{G}_1(\tau)$, it is also invariant under the translation $\tau \mapsto \tau + 2$ but $\psi_1(i\infty) \neq 0$.

4.3 General case for $k > 1$

Put

$$\Xi_k(s) = \frac{\Lambda_k(s+2k)}{\prod_{j=1-2k}^{2k-1}(s-j)}.$$

Then the functional equation $\Lambda_k(4k-s) = \Lambda_k(s)$ implies the oddness $\Xi_k(-s) = -\Xi_k(s)$. For later use, we denote by $\rho_{k,j}$ the residue of $\Xi_k(s)$ of a (possible) pole at $s = j$ for $j = 1-2k, \dots, 2k-1$. Notice that $\rho_{k,-j} = \rho_{k,j}$. Put $\tilde{\rho}_{k,j} = (2\pi)^{4k-1}\rho_{k,j}$.

Let us introduce

$$\boldsymbol{G}_k(\tau) = \underbrace{\int_0^q \cdots \int_0^q}_{4k-1} f(\tau)^k \frac{dq}{q} \cdots \frac{dq}{q} = \underbrace{\int_0^q \cdots \int_0^q}_{2k-1} \boldsymbol{E}_k(\tau) \frac{dq}{q} \cdots \frac{dq}{q}.$$

Clearly, $\boldsymbol{G}_k(\tau)$ is a periodic function with period 2 and $\boldsymbol{G}_k(i\infty) = 0$.

Lemma 6. *For $\beta \gg 0$, one has*

$$\boldsymbol{G}_k(\tau) = \frac{(2\pi)^{4k-1}}{2\pi i} \int_{\Re s = \beta} \Big(\frac{\tau}{i}\Big)^{-s+2k-1} \Xi_k(s) ds$$

and

$$\frac{d^{2k-1}}{d\tau^{2k-1}} \boldsymbol{G}_k(\tau) = (2\pi i)^{2k-1} \boldsymbol{E}_k(\tau).$$

Proof.

$$
\begin{aligned}
\boldsymbol{G}_k(\tau) &= (-2\pi)^{4k-1}\int_{\infty}^{y}\cdots\int_{\infty}^{y} f(iy)^k dy\cdots dy \\
&= -\frac{(2\pi)^{4k-1}}{2\pi i}\int_{\Re s=\alpha} \mathit{\Lambda}_k(s)\left\{\int_{\infty}^{y}\cdots\int_{\infty}^{y} y^{-s}dy\cdots dy\right\}ds \\
&= \frac{(2\pi)^{4k-1}}{2\pi i}\int_{\Re s=\alpha}\frac{\mathit{\Lambda}_k(s)y^{-s+4k-1}}{(s-1)(s-2)\cdots(s-4k+1)}ds \\
&= \frac{(2\pi)^{4k-1}}{2\pi i}\int_{\Re s=\alpha-2k}\frac{\mathit{\Lambda}_k(s+2k)y^{-s+2k-1}}{(s+2k-1)(s+2k-2)\cdots(s-2k+1)}ds \\
&= \frac{(2\pi)^{4k-1}}{2\pi i}\int_{\Re s=\alpha-2k} y^{-s+2k-1}\mathit{\Xi}_k(s)ds.
\end{aligned}
$$

Lemma 7.

$$
\boldsymbol{G}_k\Big(-\frac{1}{\tau}\Big) = \tau^{2-4k}\Big\{\boldsymbol{G}_k(\tau) - \sum_{j=1-2k}^{2k-1}\tilde{\rho}_{k,j}\Big(\frac{\tau}{i}\Big)^{2k-1-j}\Big\}.
$$

Proof. Since $\mathit{\Xi}_k(-s) = \mathit{\Xi}_k(s)$, we have

$$
\begin{aligned}
&\frac{2\pi i}{(2\pi)^{4k-1}}\boldsymbol{G}_k\Big(-\frac{1}{\tau}\Big) \\
&= -\int_{\Re s=\beta}\Big(\frac{1}{y}\Big)^{-s+2k-1}\mathit{\Xi}_k(-s)ds = -\int_{\Re s=-\beta}\Big(\frac{1}{y}\Big)^{s+2k-1}\mathit{\Xi}_k(s)ds \\
&= -y^{2-4k}\int_{\Re s=-\beta} y^{-s+2k-1}\mathit{\Xi}_k(s)ds \\
&= (iy)^{2-4k}\left\{\int_{\Re s=\beta} y^{-s+2k-1}\mathit{\Xi}_k(s)ds - 2\pi i\sum_{j=1-2k}^{2k-1}\rho_{k,j}y^{2k-1-j}\right\} \\
&= (iy)^{2-4k}\frac{2\pi i}{(2\pi)^{4k-1}}\left\{\boldsymbol{G}_k(iy) - \sum_{j=1-2k}^{2k-1}\rho_{k,j}y^{2k-1-j}\right\}.
\end{aligned}
$$

Define $R_S^k(\tau)$ by

$$
R_S^k(\tau) = -\sum_{j=1-2k}^{2k-1}\tilde{\rho}_{k,j}\Big(\frac{\tau}{i}\Big)^{2k-1-j}.
$$

Notice that $R_S^k(\tau)$ is a polynomial in τ of degree $4k-2$. Summarizing the discussion above, we obtain the

Theorem 4. *One has*

$$
\boldsymbol{G}_k(\tau+2) = \boldsymbol{G}_k(\tau), \qquad \boldsymbol{G}_k\Big(-\frac{1}{\tau}\Big) = \tau^{2-4k}\Big\{\boldsymbol{G}_k(\tau) + R_S^k(\tau)\Big\}.
$$

□

5 Residual modular forms

We introduce the notion of *residual modular forms*, which is a generalization of the classical holomorphic modular forms and Eichler integrals (or automorphic integrals) [5, 16].

5.1 Definition

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, we put $j(\gamma, \tau) := c\tau + d$. Let m be an integer. Define a slash operator $f|_m\gamma$ for a function f on $\mathfrak{h}$ by

$$(f|_m\gamma)(\tau) := j(\gamma, \tau)^{-m} f(\gamma\tau). \tag{17}$$

Let $G(\subset SL_2(\mathbb{Z}))$ be a congruence subgroup of level N. Let X be a G-invariant subspace of the vector space $F(\mathfrak{h})$ of all complex-valued functions on $\mathfrak{h}$ under the action $f \mapsto f|_m\gamma,\ (\gamma \in G)$. The vector spaces $C^\infty(\mathfrak{h})$ of C^∞-functions, $H(\mathfrak{h})$ of holomorphic functions, $M(\mathfrak{h})$ of holomorphic functions on $\mathfrak{h}$ with certain decay conditions at cusps, and the space of rational functions $\mathbb{C}(\tau)$ are typical examples of such X. If $m < 0$, the space $\mathbb{C}[\tau]_{-m}$ of all polynomials of degree at most $-m$ is also an example of X for the action $f|_m\gamma$.

Definition 1 (Residual modular forms). *Let $m \in \mathbb{Z}$. We define*

$$M_m(G, X) := \left\{ f \colon \mathfrak{h} \xrightarrow{hol.} \mathbb{C} \;\middle|\; \begin{array}{l} f(\tau + N) = f(\tau), \\ (f|_m\gamma)(\tau) - f(\tau) \in X \ (\forall \gamma \in G) \\ f \textit{ is holomorphic at each cusp of } G \end{array} \right\}.$$

We call an element in $M_m(G, X)$ a residual modular form for G of weight m. The second condition can be replaced by the one for only generators of G. One may also define the notion of residual cusp forms in an obvious way:

$$C_m(G, X) := \{ f \in M_m(G, X) \,|\, f \textit{ vanishes at each cusp of} G \}.$$

Remark 6. When m is positive, an element of $M_m(G) := M_m(G, 0)$ (reps. $C_m(G) := C_m(G, 0)$) is the classical modular (reps. cusp) form of weight m. Since the Eisenstein series $E_2(\tau)$ of weight 2 satisfies $\tau^{-2}E_2(-\frac{1}{\tau}) = E_2(\tau) + \frac{12}{2\pi i\tau}$, it is an element of $M_2(SL_2(\mathbb{Z}), \mathbb{C}(\tau))$.

Remark 7. Suppose that the space X contains constant functions. Then, if $f(\tau) \in M_m(G, X)$, it is clear that any constant shift $f(\tau) + c$ $(c \in \mathbb{C})$ belongs to $M_m(G, X)$. In this case, it is natural to study the quotient $M_m(G, X)/(\text{constant shift})$.

Remark 8. When $m < 0$, an element of $M_m(G, \mathbb{C}[\tau]_{-m})$ is also known as an *Eichler integral* or *automorphic integral* [17]. The notion of Eichler integrals is a generalization of the classical Abelian integrals, which occur as the case $m = 0$.

Define $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let Γ be a subgroup of $SL_2(\mathbb{Z})$ (or $PSL_2(\mathbb{Z})$ practically) defined by $\Gamma := \langle T^2, S \rangle$. Notice that

$$\Gamma \supset \Gamma(2) = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle = \langle T^2, ST^{-2}S^{-1} \rangle .$$

For convenience, we give the definitions of the space of residual modular forms (with characters) in terms of the generators for the specific groups Γ and $\Gamma(2)$.

Definition 2 (Residual modular forms).

$$M_m^{\pm}(\Gamma, X) := \left\{ f\colon \mathfrak{h} \xrightarrow{hol.} \mathbb{C} \;\middle|\; \begin{array}{l} f(\tau+2) = f(\tau), \\ \tau^{-m} f\Big(-\dfrac{1}{\tau}\Big) \mp f(\tau) \in X \\ f \textit{ is holomorphic at each cusp of } \Gamma \end{array} \right\},$$

$$M_m(\Gamma(2), X) := \left\{ f\colon \mathfrak{h} \xrightarrow{hol.} \mathbb{C} \;\middle|\; \begin{array}{l} f(\tau+2) = f(\tau), \\ (2\tau+1)^{-m} f\Big(\dfrac{\tau}{2\tau+1}\Big) - f(\tau) \in X \\ f \textit{ is holomorphic at each cusp of } \Gamma(2) \end{array} \right\}.$$

Notice that $M_m^{+}(\Gamma, X)$ is identified with $M_m(\Gamma, X)$ in Definition 1.

Now Theorem 4 in the previous section may be simply restated as follows.

Theorem 5. *One has $\boldsymbol{G}_k(\tau) \in M_{2-4k}(\Gamma, \mathbb{C}[\tau]_{4k-2})$ for each positive integer k. Moreover, $\boldsymbol{G}_k(\tau)$ is a residual cusp form.* $\square$

Remark 9. Let $f(\tau) \in M_m^{\pm}(\Gamma, X)$. Put

$$R(\tau) = \tau^{-m} f\Big(-\frac{1}{\tau}\Big) \mp f(\tau) \in X.$$

Then we have

$$f\Big(\frac{\tau}{2\tau+1}\Big) = (-1)^m (2\tau+1)^m \Big\{ f(\tau) \pm R(\tau) + \tau^{-m} R\Big(-\frac{2\tau+1}{\tau}\Big)\Big\}$$

Hence, in particular, one has $M_{2m}^{\pm}(\Gamma, \mathbb{C}(\tau)) \subset M_{2m}(\Gamma(2), \mathbb{C}(\tau))$. Notice also that, when $-m = k \in \mathbb{N}$ one has $\tau^{2k} R\Big(-\frac{2\tau+1}{\tau}\Big) \in \mathbb{C}[\tau]_{2k}$ for $R(\tau) \in \mathbb{C}[\tau]_{2k}$. Thus, in particular, $M_{-2k}^{\pm}(\Gamma, \mathbb{C}[\tau]_{2k}) \subset M_{-2k}(\Gamma(2), \mathbb{C}[\tau]_{2k})$.

5.2 Differential Eisenstein series

We always assume that $-\pi \leq \arg z < \pi$ for $z \in \mathbb{C}$ to determine the branch of complex powers.

Definition 3 (Generalized Eisenstein series). *Define*

$$G(s,x,\tau) := \sum_{m,n\in\mathbb{Z}}{}' (m\tau+n+x)^{-s},$$
$$G(s,\tau) := G(s,0,\tau),$$
$$G^{a,b}(s,\tau) := \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv a \,(\mathrm{mod}\, 2)\\ n\equiv b\,(\mathrm{mod}\,2)}}{}' (m\tau+n)^{-s} \qquad (a,b\in\{0,1\})$$

for $s\in\mathbb{C}$ such that $\Re(s)>2$. Here $\sum'_{m,n\in\mathbb{Z}}$ means the sum over all pairs (m,n) of integers such that the summand is defined.

Remark that $G^{0,0}(s,\tau)=2^{-s}G(s,\tau)$,

$$G^{a,b}(s,\tau)=2^{-s}G\Big(s,\frac{a\tau+b}{2},\tau\Big).$$

It is known that $G(s,x,\tau)$ is analytically continued to the whole s-plane, and $G(s,x,\tau)$ can be written in the form

$$G(s,x,\tau)=\sum_{n>-x}\frac{1}{(n+x)^s}+\frac{1}{\Gamma(s)}A(s,x,\tau),$$

when $x\in\mathbb{R}$, where $A(s,x,\tau)$ is holomorphic in s and τ. In particular, we see that

$$G(-2k,\tau)=G^{1,1}(-2k,\tau)=0$$

for any positive integer k (see [18, Theorem 1]; see also [2]). We now introduce the *differential Eisenstein series.*

Definition 4 (Differential Eisenstein series). *For $m\in\mathbb{Z}$, define*

$$dE_m(\tau) := \frac{\partial}{\partial s}G(s,\tau)\Big|_{s=m},$$
$$dE_m^{a,b}(\tau) := \frac{\partial}{\partial s}G^{a,b}(s,\tau)\Big|_{s=m} \qquad (a,b\in\{0,1\})$$

It is immediate to see that $dE_m(\tau+1)=dE_m(\tau)$ and $dE_m^{a,b}(\tau+2)=dE_m^{a,b}(\tau)$. For later use, we recall the definitions and several results on the double zeta functions and double Bernoulli numbers [1].

Definition 5 (Barnes' double zeta function).

$$\zeta_2(s,z\,|\,\underline{\omega}) := \sum_{m,n\geq 0}(m\omega_1+n\omega_2+z)^{-s}$$

for $\underline{\omega}=(\omega_1,\omega_2)$.

Definition 6 (Double Bernoulli polynomials). *The* double Bernoulli polynomials $B_{2,k}(z\,|\,\underline{\omega})$ *are defined by the expansion*

$$\frac{t^2 e^{zt}}{(e^{\omega_1 t}-1)(e^{\omega_2 t}-1)} = \sum_{k=0}^{\infty} B_{2,k}(z\,|\,\underline{\omega})\frac{t^k}{k!}.$$

The following is well-known (see, e.g. [1]).

Lemma 8. *For each $m \in \mathbb{N}$, one has*

$$\zeta_2(1-m, z\,|\,\underline{\omega}) = \frac{B_{2,m+1}(z\,|\,\underline{\omega})}{m(m+1)}.$$

Example 4.

$$\zeta_2\Big(-2k, \frac{\tau-1}{2}\,\Big|\,(-1,\tau)\Big) = \frac{B_{2,2k+2}(\frac{\tau-1}{2}\,|\,(-1,\tau))}{(2k+1)(2k+2)} \in \frac{1}{\tau}\mathbb{C}[\tau],$$
$$\zeta_2(-2k, \tau\,|\,(-1,\tau)) = \frac{B_{2,2k+2}(\tau\,|\,(-1,\tau))}{(2k+1)(2k+2)} \in \frac{1}{\tau}\mathbb{C}[\tau].$$

5.3 Residual-modularity of dE_{-2k}

We notice the following elementary fact.

Lemma 9. *If $\tau \in \mathfrak{h}$ and $(a,b) \in \mathbb{R}^2 - \{(0,0)\}$, then*

$$\arg\Big(-\frac{1}{\tau}\Big) + \arg(a\tau+b) \geq \pi \iff a>0,\, b\leq 0.$$

□

Lemma 10. *For each $k \in \mathbb{N}$, one has*

$$dE_{-2k}\Big(-\frac{1}{\tau}\Big) = \Big(-\frac{1}{\tau}\Big)^{2k}\Big\{dE_{-2k}(\tau) - 4k\pi i\zeta_2\big(-2k, \tau\,|\,(-1,\tau)\big)\Big\}.$$

Proof. It follows from Lemma 9 that

$$\begin{aligned}
G\Big(s, -\frac{1}{\tau}\Big) &= \sum_{m,n\in\mathbb{Z}}{}' \Big(-m\frac{1}{\tau}+n\Big)^{-s} = \sum_{m,n\in\mathbb{Z}}{}' \Big(\Big(-\frac{1}{\tau}\Big)(m\tau+n)\Big)^{-s} \\
&= \Big(-\frac{1}{\tau}\Big)^{-s}\Big\{\sum_{m,n\in\mathbb{Z}}{}'(m\tau+n)^{-s} + (e^{2\pi i s}-1)\sum_{\substack{m>0,\\ n\leq 0}}(m\tau+n)^{-s}\Big\} \\
&= \Big(-\frac{1}{\tau}\Big)^{-s}\Big\{G(s,\tau) + (e^{2\pi i s}-1)\zeta_2\big(s,\tau\,|\,(-1,\tau)\big)\Big\}.
\end{aligned}$$

This yields that

$$
\begin{aligned}
&\frac{\partial}{\partial s} G\Big(s, -\frac{1}{\tau}\Big)\Big|_{s=-2k} \\
&= \frac{\partial}{\partial s}\Big(-\frac{1}{\tau}\Big)^{-s}\Big|_{s=-2k} \Big\{G(-2k,\tau) + (e^{-4k\pi} - 1)\zeta_2\big(-2k, \tau \,|\, (-1,\tau)\big)\Big\} \\
&\quad + \Big(-\frac{1}{\tau}\Big)^{2k} \frac{\partial}{\partial s}\Big\{G(s,\tau) + (e^{2\pi i s} - 1)\zeta_2\big(s, \tau \,|\, (-1,\tau)\big)\Big\}\Big|_{s=-2k} \\
&= \Big(-\frac{1}{\tau}\Big)^{2k} \Big\{\frac{\partial}{\partial s} G(s,\tau)\Big|_{s=-2k} - 4k\pi i \zeta_2\big(-2k, \tau \,|\, (-1,\tau)\big)\Big\}.
\end{aligned}
$$

Thus we have

$$
dE_{-2k}\Big(-\frac{1}{\tau}\Big) = \Big(-\frac{1}{\tau}\Big)^{2k}\Big\{dE_{-2k}(\tau) - 4k\pi i \zeta_2\big(-2k, \tau \,|\, (-1,\tau)\big)\Big\}.
$$

Lemma 11. *For each $k \in \mathbb{N}$, one has*

$$
dE^{1,1}_{-2k}\Big(-\frac{1}{\tau}\Big) = \tau^{-2k}\Big(dE^{1,1}_{-2k}(\tau) - 4k\pi i \zeta_2(-2k, \tau - 1 \,\big|\, (-2, 2\tau))\Big).
$$

Proof. It follows from Lemma 9 that

$$
\begin{aligned}
&G^{1,1}\Big(s, -\frac{1}{\tau}\Big) \\
&= \sum_{m,n\in\mathbb{Z}} \Big(-(2m+1)\frac{1}{\tau} + (2n+1)\Big)^{-s} \\
&= \sum_{m,n\in\mathbb{Z}} \Big(\Big(-\frac{1}{\tau}\Big)\big((2m+1)\tau + 2n + 1\big)\Big)^{-s} \\
&= \Big(-\frac{1}{\tau}\Big)^{-s}\Big\{G^{1,1}(s,\tau) + (e^{2\pi i s} - 1)\sum_{\substack{m\geq 0\\ n<0}}\big((2m+1)\tau + 2n + 1\big)^{-s}\Big\} \\
&= \Big(-\frac{1}{\tau}\Big)^{-s}\Big\{G^{1,1}(s,\tau) + (e^{2\pi i s} - 1)\zeta_2\Big(s, \tau - 1 \,\big|\, (-2, 2\tau)\Big)\Big\}
\end{aligned}
$$

Hence, by the same discussion as in the proof of the previous lemma, we get

$$
dE^{1,1}_{-2k}\Big(-\frac{1}{\tau}\Big) = \tau^{-2k}\Big(dE^{1,1}_{-2k}(\tau) - 4k\pi i \zeta_2(-2k, \tau - 1 \,\big|\, (-2, 2\tau))\Big)
$$

as desired.

Corollary 1. *Suppose $k \in \mathbb{N}$. Then, one has $dE_{-2k}(\tau) \in M_{-2k}(SL_2(\mathbb{Z}), \mathbb{C}(\tau))$ and $dE^{0,0}_{-2k}(\tau), dE^{1,1}_{-2k}(\tau) \in M^{+}_{-2k}(\Gamma, \mathbb{C}(\tau))$.* □

Remark 10. Notice that

$$
dE_{-2k}(\tau) \notin M_{-2k}(SL_2(\mathbb{Z}), \mathbb{C}[\tau]), \quad dE^{0,0}_{-2k}(\tau), dE^{1,1}_{-2k}(\tau) \notin M^{+}_{-2k}(\Gamma, \mathbb{C}[\tau])
$$

for $k > 0$. In other words, neither $dE^{1,1}_{-2k}(\tau)$ nor $dE^{0,0}_{-2k}(\tau)$ is Eichler's integral.

Remark 11. A recent calculation due to G. Shibukawa on the same analysis of the lemmas above shows that $dE_{2k+1}(\tau) \in M_{-2k}(SL_2(\mathbb{Z}), M(\mathfrak{h}))$ but $\notin M_{-2k}(SL_2(\mathbb{Z}), \mathbb{C}(\tau))$ for $k > 0$.

Let us look at the case where $k = 1$. Using the special value formula of $\zeta_2(s, z \,|\, \underline{\omega})$ for negative integers s, we have

$$dE_{-2}^{1,1}\Big(-\frac{1}{\tau}\Big) = \tau^{-2}\Big(dE_{-2}(\tau) - \frac{\pi i}{3}B_{2,4}(\tau - 1 \,|\, (-2, 2\tau))\Big),$$
$$dE_{-2}\Big(-\frac{1}{\tau}\Big) = \tau^{-2}\Big(dE_{-2}(\tau) - \frac{\pi i}{3}B_{2,4}(\tau \,|\, (-1, \tau))\Big).$$

Lemma 12.

$$7B_{2,4}\big(\tau \,\big|\, (1, \tau)\big) - 2B_{2,4}\big(\tau - 1 \,\big|\, (-2, 2\tau)\big) = \frac{3}{2}\tau.$$

Proof. Straightforward calculation.

Recall the function $\psi_1(\tau) = \boldsymbol{G}_1(\tau) - \tilde{\rho}_{1,1} \in M_{-2}(\Gamma, \mathbb{C}[\tau]_2)$ in (16).

Corollary 2. *The function $\psi_1(\tau)$ is given by*

$$\psi_1(\tau) = -\frac{2\tilde{\rho}_{1,0}}{\pi}\Big\{7dE_{-2}(\tau) - 2dE_{-2}^{1,1}(\tau)\Big\}.$$

Proof. Denote the right hand side by $\phi_1(\tau)$. Then, obviously $\phi(\tau)$ satisfies

$$\phi_1(\tau + 2) = \phi_1(\tau), \quad \phi_1\Big(-\frac{1}{\tau}\Big) = \tau^{-2}\Big(\phi_1(\tau) - \tilde{\rho}_{1,0}\frac{\tau}{i}\Big).$$

Hence $\psi_1(\tau) - \phi_1(\tau) \in M_{-2}(\Gamma, 0) \subset M_{-2}(\Gamma(2), 0) = M_{-2}(\Gamma(2))$, the space of classical holomorphic modular forms of weight -2. Since $M_{-2}(\Gamma(2)) = \{0\}$, the result follows.

Corollary 3. *One has*

$$w_4(t) = w_4(t(\tau)) = \pi^2\frac{\theta_4(\tau)^4}{\theta_3(\tau)^2}\left[1 + i\pi\frac{d}{d\tau}\Big\{7dE_{-2}(\tau) - 2dE_{-2}^{1,1}(\tau)\Big\}\right]. \quad (18)$$

Proof. Since $2\pi i\boldsymbol{E}_k(\tau) = \frac{d}{d\tau}\psi_1(\tau)$, the expression follows immediately from Lemma 2 and (15).

6 Eichler cohomology for residual modular forms

We construct a cochain complex arising from residual modular forms. We then focus on a particular cohomology which we call a periodic Eichler cohomology. We start by the first cohomology in §6.1 and discuss later a general cochain cohomology in §6.2.

6.1 First cohomology group

Let m be an integer. Suppose that X is a G-invariant subspace of the space of complex-valued functions on $\mathfrak{h}$ under the action $f|_m\gamma\,(\gamma \in G)$ (see §5.1). Namely, we assume that X is a G-module.

Suppose that f be a function on $\mathfrak{h}$ which obeys the following equation for some $R_f^m(\gamma)(\tau) \in X$:

$$f|_m\gamma - f = R_f^m(\gamma) \quad (\forall\gamma \in G),$$

that is,

$$f(\gamma\tau) = j(\gamma,\tau)^m(f(\tau) + R_f^m(\gamma)(\tau)) \quad (\forall\gamma \in G).$$

Obviously $R_f^m(e)(\tau) \equiv 0$. Notice also that $R_f^m(T^N) = 0$ if $f \in M_m(G,X)$ whenever G is a congruence subgroup of level N. In order to recall the Eichler cohomology group (see, e.g. [5, 6]) in this setting, one notices first the following equation for $R_f^m(\gamma)$.

Lemma 13.

$$R_f^m(\gamma_1\gamma_2)(\tau) = R_f^m(\gamma_2)(\tau) + j(\gamma_2,\tau)^{-m}R_f^m(\gamma_1)(\gamma_2\tau).$$

Proof. Since $j(\gamma_1\gamma_2,\tau) = j(\gamma_1,\gamma_2\tau)j(\gamma_2,\tau)$, one has

$$f(\gamma_1\gamma_2\tau) = j(\gamma_1,\gamma_2\tau)^m j(\gamma_2,\tau)^m\Big(f(\tau) + R_f^m(\gamma_1\gamma_2)(\tau)\Big).$$

On the other hand,

$$\begin{aligned} f(\gamma_1\gamma_2\tau) &= j(\gamma_1,\gamma_2\tau)^m(f(\gamma_2\tau) + R_f^m(\gamma_1)(\gamma_2\tau)) \\ &= j(\gamma_1,\gamma_2\tau)^m\Big\{j(\gamma_2,\tau)^m(f(\tau) + R_f^m(\gamma_2)(\tau)) + R_f^m(\gamma_1)(\gamma_2\tau)\Big\}. \end{aligned}$$

Hence the claim follows.

Let $C^1(G,X)$ be a space of all maps from G to X. We call $R \in C^1(G,X)$ a (*twisted*) 1-*cocycle* of weight m if it satisfies

$$R(\gamma_1\gamma_2) = R(\gamma_2) - R(\gamma_1)|_m\gamma_2.$$

We denote by $Z^1_{[m]}(G,X)$ the set of all (twisted) 1-cocycles of weight m. Obviously $Z^1_{[m]}(G,X)$ is a subspace of $C^1(G,X)$.

Define the element $\partial R \in C^1(G,X)$ for $R \in X$ by

$$\partial R\colon \mathit{\Gamma} \ni \gamma \mapsto R - R|_m\gamma,$$

that is,

$$(\partial R)(\gamma)(\tau) = R(\tau) - j(\gamma,\tau)^{-m}R(\gamma\tau).$$

Lemma 14. $\partial R \in Z^1_{[m]}(G,X)$.

Proof. The lemma follows from

$$\begin{aligned}(\partial R)(\gamma_1\gamma_2)(\tau) &= R(\tau) - j(\gamma_1\gamma_2,\tau)^{-m}R(\gamma_1\gamma_2\tau)\\ &= R(\tau) - j(\gamma_1,\gamma_2\tau)^{-m}j(\gamma_2,\tau)^{-m}R(\gamma_1\cdot\gamma_2\tau)\\ &= R(\tau) + j(\gamma_2,\tau)^{-m}\Big((\partial R)(\gamma_1)(\gamma_2\tau) - R(\gamma_2\tau)\Big)\\ &= (\partial R)(\gamma_2)(\tau) + j(\gamma_2,\tau)^{-m}(\partial R)(\gamma_1)(\gamma_2\tau).\end{aligned}$$

Define a subgroup $B^1_{[m]}(G,X)$ of $Z^1_{[m]}(G,X)$ by

$$B^1_{[m]}(G,X) = \big\{\partial R \,\big|\, R \in X\big\}.$$

We call an element of $B^1_{[m]}(G,X)$ by a (*twisted*) 1-*coboundary* of weight m. The quotient group defined by

$$H^1_{[m]}(G,X) := Z^1_{[m]}(G,X)/B^1_{[m]}(G,X)$$

is called the 1st *Eichler cohomology group* of weight m for the G-module X. (Notice that $\partial^1\circ\partial = 0$ in the notation of §6.2.)

Define subspaces $\tilde{Z}^1_{[m]}(G,X)$ and $\tilde{B}^1_{[m]}(G,X)$ of $Z^1_{[m]}(G,X)$ and $B^1_{[m]}(G,X)$ respectively by

$$\begin{aligned}\tilde{Z}^1_{[m]}(G,X) :=& \{R \in Z^1_{[m]}(G,X) \mid R(T^N) = 0\},\\ \tilde{B}^1_{[m]}(G,X) :=& \{\partial R \in B^1_{[m]}(G,X) \mid \partial R(T^N) = 0\}\\ =& \partial\{R \in X \mid R(\tau+N) = R(\tau),\ \forall\tau\in\mathfrak{h}\,\}.\end{aligned}$$

Then we may define the 1st *periodic* Eichler cohomology group by the quotient:

$$\tilde{H}^1_{[m]}(G,X) := \tilde{Z}^1_{[m]}(G,X)/\tilde{B}^1_{[m]}(G,X).$$

The following lemma is obvious.

Lemma 15. *Assume that a congruence subgroup G of level N contains S. If $f \in M_{-k}(G,X)$ we have*

$$R_f^{-k}(T^N)(\tau) = 0, \quad R_f^{-k}(S)(S\tau) = -\tau^{-k}R_f^{-k}(S)(\tau).$$

In particular, $R_f^{-k}(\gamma) \in \tilde{Z}^1_{[m]}(G,X)$. From the cocycle condition, one knows that $R \in \tilde{Z}^1_{[m]}(G,X)$ is determined by the double coset of $\varGamma_\infty = \langle T^N\rangle$:

$$R(T^N\gamma)(\tau) = R(\gamma)(\tau), \quad R(\gamma T^N)(\tau) = R(\gamma)(T^N\tau).$$

Definition 7. *Suppose X contains a constant function. Define*

$$M^*_m(G,X) := M_m(G,X)/\textit{(constant shift)}.$$

Notice that we may always take a cusp form as a representative of $M^*_m(G,X)$.

Lemma 16. *Let $k \in \mathbb{N}$. Let X be either $\mathbb{C}(\tau)$ or $\mathbb{C}[\tau]_k$. For $f \in M^*_{-k}(G,X)$, define*

$$R_f(\gamma) := f|_{-k}\gamma - f \in X.$$

Then the map $f \mapsto R_f$ induces an injective map from $M^*_{-k}(G,X)$ to the 1st periodic cohomology group $\tilde{H}^1_{[-k]}(G,X)$.

Proof. Since $f(T^N\tau) = f(\tau)$, we have the map

$$M^*_{-k}(G,X) \ni f \mapsto R_f \in \tilde{Z}^1_{[-k]}(G,X).$$

Suppose $R_f \in \tilde{B}^1_{[-k]}(G,X)$. Since X is either $\mathbb{C}(\tau)$ or $\mathbb{C}[\tau]_k$, it is clear that $\tilde{B}^1_{[-k]}(G,X) = \partial\{\text{constant functions}\}$. Hence, for some $c \in \mathbb{C}$, one has

$$R_f(\gamma)(\tau) = j(\gamma,\tau)^k f(\gamma\tau) - f(\tau) = cj(\gamma,\tau)^k - c.$$

It follows that $f(\tau) - c \in M_{-k}(G)(= M_{-k}(G,0)) = \{0\}$. This shows that the map $f \mapsto R_f$ induces a well-defined map from $M^*_{-k}(G,X)$ to $\tilde{H}^1_{[-k]}(G,X)$, which is injective.

Lemma 17. *Retain the assumption of Lemma 16. Then*

$$\dim_{\mathbb{C}} M^*_{-k}(G,X) \leq \dim_{\mathbb{C}} \tilde{H}^1_{[-k]}(G,X) \leq \dim_{\mathbb{C}} H^1_{[-k]}(G,X) - 1.$$

Proof. The first inequality follows immediately from Lemma 16.

In order to prove the second inequality, let us consider the natural inclusion $\tilde{Z}^1_{[-k]}(G,X) \hookrightarrow Z^1_{[-k]}(G,X)$. Suppose the image R (denoting the same letter) of $R \in \tilde{Z}^1_{[-k]}(G,X)$ belongs to $B^1_{[-k]}(G,X)$. Then one sees that $R(\gamma)(\tau) = \partial P(\gamma)(\tau) = P(\tau) - (P|_{-k}\gamma)(\tau)$ for some $P \in X$. It follows in particular that $0 = R(T^N)(\tau) = P(\tau) - P(T^N\tau)$. This shows that P is a constant, whence $R \in \tilde{B}^1_{[-k]}(G,X)$. One can therefore naturally define the "inclusion" map $\tilde{H}^1_{[-k]}(G,X) \ni R \mapsto R \in H^1_{[-k]}(G,X)$.

We now construct an element of $H^1_{[-k]}(G,X)\backslash\tilde{H}^1_{[-k]}(G,X)$. Let P be a non-constant element in X. Then, since $\tilde{B}^1_{[-k]}(G,X) = \partial\{\text{constant functions}\}$, $\partial P(\in B^1_{[-k]}(G,X))$ satisfies $\partial P(T^N) \neq 0$. By Lemma 16, there exists an element $R \in \tilde{Z}^1_{[-k]}(G,X)(\subset Z^1_{[-k]}(G,X))$ but $R \notin \tilde{B}^1_{[-k]}(G,X)$. Put $L := R + \partial P \in Z^1_{[-k]}(G,X)$. Then, obviously $L(T^N) = \partial P(T^N) \neq 0$. Further, one sees that $L \notin B^1_{[-k]}(G,X)$. In fact, suppose otherwise. Then $R \in B^1_{[-k]}(G,X)$, that is, $R \in B^1_{[-k]}(G,X) \cap \tilde{Z}^1_{[-k]}(G,X) = \tilde{B}^1_{[-k]}(G,X)$, whence the contradiction. This shows that L defines the element of $H^1_{[-k]}(G,X)\backslash\tilde{H}^1_{[-k]}(G,X)$. Hence the second inequality follows. This proves the lemma.

Corollary 4.

$$M_{-2}(\mathit{\Gamma},\mathbb{C}[\tau]_2) = M_{-2}(\mathit{\Gamma}(2),\mathbb{C}[\tau]_2) = \mathbb{C}\cdot\psi_1 \oplus \mathbb{C}\cdot 1.$$

Proof. Notice that $1 \in M_{-2}(\mathit{\Gamma},\mathbb{C}[\tau]_2)$ because $1 = j(\gamma,\tau)^{-2}\{1-(1-j(\gamma,\tau)^2)\}$. Since $\psi_1 \in M_{-2}(\mathit{\Gamma},\mathbb{C}[\tau]_2)$, by the lemma above we observe that

$$\begin{aligned}1 &\leq \dim_{\mathbb{C}} M^*_{-2}(\mathit{\Gamma},\mathbb{C}[\tau]_2) \leq \dim_{\mathbb{C}} M^*_{-2}(\mathit{\Gamma}(2),\mathbb{C}[\tau]_2)\\ &\leq \dim_{\mathbb{C}} \tilde{H}^1_{[-2]}(\mathit{\Gamma}(2),\mathbb{C}[\tau]_2) \leq \dim_{\mathbb{C}} H^1_{[-2]}(\mathit{\Gamma}(2),\mathbb{C}[\tau]_2) - 1.\end{aligned}$$

It is known in [6] that $H^1_{[-2k]}(\mathit{\Gamma}(2),\mathbb{C}[\tau]_{2k})$ is isomorphic to the direct sum $M_{2k+2}(\mathit{\Gamma}(2)) \oplus C_{2k+2}(\mathit{\Gamma}(2))$, $C_{2k+2}(\mathit{\Gamma}(2))$ being the space of cusp forms of weight $2k+2$ for $\mathit{\Gamma}(2)$. Since $\dim_{\mathbb{C}} M_4(\mathit{\Gamma}(2)) = 2$ and $\dim_{\mathbb{C}} C_4(\mathit{\Gamma}(2)) = 0$ (see, e.g. [25]), one concludes that $\dim_{\mathbb{C}} M^*_{-2}(\mathit{\Gamma}(2),\mathbb{C}[\tau]_2) = 1$. This proves the lemma.

Remark 12. Let $m \geq 0$. It is worth noting the following classical result due to G. Bol [4]:

$$\frac{d^{m+1}}{d\tau^{m+1}}\{j(\gamma,\tau)^m F(\gamma\tau)\} = j(\gamma,\tau)^{-m-2}F^{(m+1)}(\gamma\tau)$$

for any γ with $\det(\gamma) = 1$ and any function F with sufficiently many derivatives. Actually, this identity bridges between Eichler integrals of weight $-m-2$ and classical modular forms of weight m.

Remark 13. It is obvious that $R(\gamma) \in \tilde{Z}^1_{[-2d]}(\mathit{\Gamma},\mathbb{C}[\tau]_{2d})$ is completely determined by $R(S)(\tau)$. Lemma 15 asserts that $R(S)(\tau) \in \mathbb{C}[\tau]_{2d}$ is anti-self-reciprocal. Namely, one has $\dim_{\mathbb{C}} \tilde{Z}^1_{[-2d]}(\mathit{\Gamma},\mathbb{C}[\tau]_{2d}) = d+1$. Since the space $\tilde{B}^1_{[-2d]}(\mathit{\Gamma},\mathbb{C}[\tau]_{2d})$ is one-dimensional, one finds that $\dim_{\mathbb{C}} \tilde{H}^1_{[-2d]}(\mathit{\Gamma},\mathbb{C}[\tau]_{2d}) = d$. This gives another proof of Corollary 4.

Remark 14. Let $f \in M^-_{-2d}(\mathit{\Gamma},\mathbb{C}[\tau]_{2d})$. Then

$$R_f(S)(\tau) = \tau^{2d} f\Big(-\frac{1}{\tau}\Big) + f(\tau) \in \mathbb{C}[\tau]_{2d}$$

and

$$f\Big(\frac{\tau}{2\tau+1}\Big) = (2\tau+1)^{-2d}\Big\{f(\tau) - R_f(S)(\tau) + \tau^{2d}R_f(S)\Big(-\frac{2\tau+1}{\tau}\Big)\Big\}$$

Notice that $R_f(S)(\tau)$ is self-reciprocal, that is, $\tau^{2d}R_f(S)(-\frac{1}{\tau}) = R_f(S)(\tau)$. Theorefore, if $d=1$ there exists $c \in \mathbb{C}$ such that $R_f(S)(\tau) = c(1+\tau^2)$. Hence

$$\begin{aligned}R_f\Big(\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\Big)(\tau) &= -R_f(S)(\tau) + \tau^2 R_f(S)\Big(-\frac{2\tau+1}{\tau}\Big)\\ &= -c(1-(2\tau+1)^2) \in \mathbb{C}\cdot\Big(1 - j\Big(\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix},\tau\Big)^2\Big).\end{aligned}$$

This implies that $R_f(\gamma) \in \tilde{B}^1_{[-2]}(\mathit{\Gamma},\mathbb{C}[\tau]_2)$ for $f \in M^-_{-2}(\mathit{\Gamma},\mathbb{C}[\tau]_2)$, which meets the result in Corollary 4.

Remark 15. Similarly to the classical automorphic forms, it is expected that negatively weighted Poincaré series defined below may give the basis of the space $M^*_{-2k}(\Gamma, \mathbb{C}(\tau))$. Let N be a non-negative integer. Define a generalized Poincaré series by

$$\begin{aligned} P^N(s, z, \tau) &:= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, \tau)^{-s} \exp\left(2\pi i N \gamma\tau\right), \\ P^N(s, \tau) &:= P^N(s, 0, \tau), \end{aligned}$$

where $\Gamma_\infty = \left\langle T^2 \right\rangle$ is the stabilizer of ∞. Then one defines the negatively weighted Poincaré series as

$$P^N_{-2k}(\tau) := \frac{\partial}{\partial s} P^N(s, \tau)\Big|_{s=-2k}.$$

6.2 Cochain complex

Retain the assumption on G and X. Fix an integer m. Let us put

$$C^n = C^n(G, X) := \mathrm{Map}(G^n, X),$$

for $n = 1, 2, 3, \ldots$ and $C^0 = C^0(G, X) := X$. Define the linear operator $\partial^n \colon C^n \to C^{n+1}$ by

$$\begin{aligned} (\partial^n f)(\gamma_1, \ldots, \gamma_{n+1})(\tau) := f(\gamma_1, \ldots, \gamma_n)(\tau) & \\ + (-1)^{n+1} j(\gamma_1, \tau)^{-m} f(\gamma_2, \ldots, \gamma_n)(\gamma_1\tau) & \\ + \sum_{j=1}^{n} (-1)^{n+1-j} f(\gamma_1, \ldots, \underset{j\text{-th}}{\gamma_{j+1}\gamma_j}, \ldots, \gamma_{n+1})(\tau) & \end{aligned} \tag{19}$$

Lemma 18. $\partial^{n+1} \circ \partial^n = 0$.

Proof. Take arbitrary $f(\gamma_1, \ldots, \gamma_n)(\tau) \in C^n$. One has

$$\begin{aligned} &((\partial^{n+1} \circ \partial^n) f)(\gamma_1, \ldots, \gamma_{n+2})(\tau) \\ &= (\partial^n f)(\gamma_1, \ldots, \gamma_{n+1})(\tau) + (-1)^{n+2} j(\gamma_1, \tau)^{-m} (\partial^n f)(\gamma_2, \ldots, \gamma_{n+2})(\gamma_1\tau) \\ &\quad + \sum_{k=1}^{n+1} (-1)^{n+2-k} (\partial^n f)(\gamma_1, \ldots, \underset{k\text{-th}}{\gamma_{k+1}\gamma_k}, \ldots, \gamma_{n+2})(\tau) \end{aligned}$$

$$
\begin{aligned}
&= f(\gamma_1,\dots,\gamma_n)(\tau) + (-1)^{n+1} j(\gamma_1,\tau)^{-m} f(\gamma_2,\dots,\gamma_{n+1})(\gamma_1\tau)\\
&\quad + \sum_{j=1}^{n} (-1)^{n+1-j} f(\gamma_1,\dots,\underset{j\text{-th}}{\gamma_{j+1}\gamma_j},\dots,\gamma_{n+1})(\tau)\\
&\quad + (-1)^n j(\gamma_1,\tau)^{-m}\Bigg[f(\gamma_2,\dots,\gamma_{n+1})(\gamma_1\tau)\\
&\qquad + (-1)^{n+1} j(\gamma_2,\gamma_1\tau)^{-m} f(\gamma_3,\dots,\gamma_{n+2})(\gamma_2\gamma_1\tau)\\
&\qquad\quad + \sum_{j=1}^{n} (-1)^{n+1-j} f(\gamma_2,\dots,\underset{j\text{-th}}{\gamma_{j+2}\gamma_{j+1}},\dots,\gamma_{n+2})(\gamma_1\tau)\Bigg]\\
&\quad + \sum_{k=1}^{n+1} (-1)^{n-k} \Bigg[\begin{Bmatrix} f(\gamma_1,\dots,\underset{k\text{-th}}{\gamma_{k+1}\gamma_k},\dots,\gamma_{n+1})(z) & (1 \le k \le n)\\ f(\gamma_1,\dots,\gamma_n)(z) & (k=n+1) \end{Bmatrix}\\
&\quad + (-1)^{n+1} \begin{Bmatrix} j(\gamma_2\gamma_1,\tau)^{-m} f(\gamma_3,\dots,\gamma_{n+2})(\gamma_2\gamma_1\tau) & (k=1)\\ j(\gamma_1,\tau)^{-m} f(\gamma_2,\dots,\underset{(k-1)\text{-th}}{\gamma_{k+1}\gamma_k},\dots,\gamma_{n+2})(\gamma_1\tau) & (2 \le k \le n+1) \end{Bmatrix}\\
&\qquad + \sum_{j=1}^{n} (-1)^{n+1-j} \begin{Bmatrix} f(\gamma_1,\dots,\underset{j\text{-th}}{\gamma_{j+2}\gamma_{j+1}\gamma_j},\dots,\gamma_{n+2})(\tau) & (k=j,j+1)\\ f(\dots,\underset{j\text{-th}}{\gamma_{j+1}\gamma_j},\dots,\underset{(k-1)\text{-th}}{\gamma_{k+1}\gamma_k},\dots)(\tau) & (j \le k-2)\\ f(\dots,\underset{k\text{-th}}{\gamma_{k+1}\gamma_k},\dots,\underset{j\text{-th}}{\gamma_{j+2}\gamma_{j+1}},\dots)(\tau) & (j \ge k+1) \end{Bmatrix} \Bigg],
\end{aligned}
\qquad \blacksquare
$$

which is verified to vanish.

Thus, for a fixed $m \in \mathbb{Z}$, we define cocycles and coboundaries by

$$Z^n_{[m]}(G,X) := \ker \partial^n, \quad B^n_{[m]}(G,X) := \operatorname{im} \partial^{n-1}$$

in $C^n(G,X)$, and the cohomology group

$$H^n_{[m]}(G,X) := Z^n_{[m]}(G,X)/B^n_{[m]}(G,X)$$

for each $n = 0,1,2,\dots$. One may obviously define a periodic cohomology group $\tilde{H}^n_{[m]}(G,X)$.

Example 5. Recall the congruence subgroup $\Gamma = \langle T^2, S \rangle\, (\supset \Gamma(2))$. Let us look at $H^0_{[-k]}(\Gamma, \mathbb{C}(\tau))$ for $k \in \mathbb{N}$. Noticing that

$$
\begin{aligned}
&R \in Z^0_{[-k]}(\Gamma, \mathbb{C}(\tau)) \subset \mathbb{C}(\tau)\\
\Longrightarrow\ & 0 = (\partial^0 R)(\gamma)(\tau) = R(\tau) - j(\gamma,\tau)^k R(\gamma\tau) \quad (\forall \gamma \in \Gamma)\\
\Longrightarrow\ & R(\tau+N) = R(\tau), \quad R(-1/\tau) = \tau^{-k} f(\tau) \implies R(\tau) = 0,
\end{aligned}
$$

we conclude that $H^0_{[-k]}(G, \mathbb{C}(\tau)) = \{0\}$.

We will make much systematic study on the residual modular forms and related Eichler cohomology groups arising from the spectrum of the non-commutative harmonic oscillators in [15].

Acknowledgement. The authors thank to the referee for his/her careful reading and useful comments. The first author is partially supported by Grand-in-Aid for Scientific Research (C) No. 22540028. The second author is partially supported by Grand-in-Aid for Scientific Research (B) No. 21340011.

References

1. E. W. Barnes, *On the theory of the multiple gamma function*, Trans. Cambridge Philos. Soc. **19** (1904), 374–425.
2. B. C. Berndt, *Generalized Eisenstein series and modified Dedekind sums*, J. Reine Angew. Math. **272** (1974), 182–193.
3. F. Beukers, *Irrationality of π^2, periods of an elliptic curve and $\Gamma_1(5)$*, Diophantine approximations and transcendental numbers (Luminy, 1982), 47–66, Progr. Math. **31**, Birkhäuser, Boston, 1983.
4. G. Bol, *Invarianten linearer Differentialgleichungen*, Abh. Math. Sem. Univ. Hamburg **16** (1949), 1–28.
5. M. Eichler, *Eine Verallgemeinerung der Abelschen Integrale*, Math. Z. **67** (1957), 267–298.
6. R. C. Gunning, *The Eichler cohomology groups and automorphic forms*, Trans. Amer. Math. Soc. **100** (1961), 44–62.
7. R. Howe and E. C. Tan, Non-Abelian Harmonic Analysis. Applications of $SL(2,\mathbb{R})$. Springer, 1992.
8. T. Ichinose and M. Wakayama, *Zeta functions for the spectrum of the non-commutative harmonic oscillators*, Commun. Math. Phys. **258** (2005), 697–739.
9. T. Ichinose and M. Wakayama, *Special values of the spectral zeta function of the non-commutative harmonic oscillator and confluent Heun equations*, Kyushu J. Math. **59** (2005), 39–100.
10. T. Ichinose and M. Wakayama, *On the spectral zeta function for the noncommutative harmonic oscillator*, Rep. Math. Phys. **59** (2007), 421–432.
11. K. Kimoto, *Higher Apéry-like numbers arising from special values of the spectral zeta function for the non-commutative harmonic oscillator*, `arXiv:0901.0658` (2009).
12. K. Kimoto, *Special value formula for the spectral zeta function of the non-commutative harmonic oscillator*. `arXiv:0903.5165` (2009).
13. K. Kimoto and M. Wakayama, *Apéry-like numbers arising from special values of spectral zeta functions for non-commutative harmonic oscillators*, Kyushu J. Math. **60** (2006), 383–404.
14. K. Kimoto and M. Wakayama, *Elliptic curves arising from the spectral zeta function for non-commutative harmonic oscillators and $\Gamma_0(4)$-modular forms*, Proceedings Conf. L-functions (eds. L. Weng, M. Kaneko), 201–218, World Scientific, 2007.
15. K. Kimoto and M. Wakayama, *Residual modular forms and Eichler cohomology groups arising from non-commutative harmonic oscillators*, in preparation.

16. M. Knopp and H. Mawi, *Eichler cohomology theorem for automorphic forms of small weights*, Proc. Amer. Math. Soc. **138** (2010), 395–404.
17. J. Lehner, *Automorphic integrals with preassigned periods*, J. Res. Nat. Bur. Standards Sect. B **73B** (1969), 153–161.
18. J. Lewittes, *Analytic continuation of Eisenstein series*, Trans. Amer. Math. Soc. **171** (1972), 469–490.
19. H. Ochiai, *Non-commutative harmonic oscillators and Fuchsian ordinary differential operators*, Commun. Math. Phys. **217** (2001), 357–373.
20. H. Ochiai, *A special value of the spectral zeta function of the non-commutative harmonic oscillators*, Ramanujan J. **15** (2008), 31–36.
21. A. Parmeggiani, Spectral Theory of Non-commutative Harmonic Oscillators: An Introduction. Lecture Notes in Math. **1992**, Springer, 2010.
22. A. Parmeggiani and M. Wakayama, *Oscillator representations and systems of ordinary differential equations*, Proc. Natl. Acad. Sci. USA **98** (2001), 26–30.
23. A. Parmeggiani and M. Wakayama, *Non-commutative harmonic oscillators-I, II, Corrigenda and remarks to I*, Forum. Math. **14** (2002), 539–604, 669–690, ibid **15** (2003), 955–963.
24. G. Shibukawa, *Bilateral zeta functions and their applications*, `arXiv:1106.1754` (2011).
25. G. Shimura, Introduction to the Arithmetic Theory of Automorphic Functions. Princeton Univ. Press, 1971.
26. M. Wakayama and Y. Yamasaki, *Higher regularizations for zeros of cuspidal automorphic L-functions of GL_d*, J. Theor. Nombres Bordeaux, **23** (2011), 751–767.
27. E. T. Whittaker and G. N. Watson, A course of modern analysis, Fourth edition. Cambridge Univ. Press, 1927.
28. Y. Yang, *Apéry limits and special values of L-functions*, J. Math. Anal. Appl. **343** (2008), 492–513.
29. D. Zagier, *Integral solutions of Apéry-like recurrence equations*, Groups and symmetries, 349–366, CRM Proc. Lecture Notes **47**, Amer. Math. Soc., Providence, RI, 2009.

List of MI Preprint Series, Kyushu University

The Global COE Program
Math-for-Industry Education & Research Hub

MI

MI2008-1 Takahiro ITO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Abstract collision systems simulated by cellular automata

MI2008-2 Eiji ONODERA
The intial value problem for a third-order dispersive flow into compact almost Hermitian manifolds

MI2008-3 Hiroaki KIDO
On isosceles sets in the 4-dimensional Euclidean space

MI2008-4 Hirofumi NOTSU
Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme

MI2008-5 Yoshiyasu OZEKI
Torsion points of abelian varieties with values in nfinite extensions over a p-adic field

MI2008-6 Yoshiyuki TOMIYAMA
Lifting Galois representations over arbitrary number fields

MI2008-7 Takehiro HIROTSU & Setsuo TANIGUCHI
The random walk model revisited

MI2008-8 Silvia GANDY, Masaaki KANNO, Hirokazu ANAI & Kazuhiro YOKOYAMA
Optimizing a particular real root of a polynomial by a special cylindrical algebraic decomposition

MI2008-9 Kazufumi KIMOTO, Sho MATSUMOTO & Masato WAKAYAMA
Alpha-determinant cyclic modules and Jacobi polynomials

MI2008-10 Sangyeol LEE & Hiroki MASUDA
Jarque-Bera Normality Test for the Driving Lévy Process of a Discretely Observed Univariate SDE

MI2008-11 Hiroyuki CHIHARA & Eiji ONODERA
A third order dispersive flow for closed curves into almost Hermitian manifolds

MI2008-12 Takehiko KINOSHITA, Kouji HASHIMOTO and Mitsuhiro T. NAKAO
On the L^2 a priori error estimates to the finite element solution of elliptic problems with singular adjoint operator

MI2008-13 Jacques FARAUT and Masato WAKAYAMA
Hermitian symmetric spaces of tube type and multivariate Meixner-Pollaczek polynomials

MI2008-14 Takashi NAKAMURA
Riemann zeta-values, Euler polynomials and the best constant of Sobolev inequality

MI2008-15 Takashi NAKAMURA
Some topics related to Hurwitz-Lerch zeta functions

MI2009-1 Yasuhide FUKUMOTO
Global time evolution of viscous vortex rings

MI2009-2 Hidetoshi MATSUI & Sadanori KONISHI
Regularized functional regression modeling for functional response and predictors

MI2009-3 Hidetoshi MATSUI & Sadanori KONISHI
Variable selection for functional regression model via the L_1 regularization

MI2009-4 Shuichi KAWANO & Sadanori KONISHI
Nonlinear logistic discrimination via regularized Gaussian basis expansions

MI2009-5 Toshiro HIRANOUCHI & Yuichiro TAGUCHII
Flat modules and Groebner bases over truncated discrete valuation rings

MI2009-6 Kenji KAJIWARA & Yasuhiro OHTA
Bilinearization and Casorati determinant solutions to non-autonomous 1+1 dimensional discrete soliton equations

MI2009-7 Yoshiyuki KAGEI
Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow

MI2009-8 Shohei TATEISHI, Hidetoshi MATSUI & Sadanori KONISHI
Nonlinear regression modeling via the lasso-type regularization

MI2009-9 Takeshi TAKAISHI & Masato KIMURA
Phase field model for mode III crack growth in two dimensional elasticity

MI2009-10 Shingo SAITO
Generalisation of Mack's formula for claims reserving with arbitrary exponents for the variance assumption

MI2009-11 Kenji KAJIWARA, Masanobu KANEKO, Atsushi NOBE & Teruhisa TSUDA
Ultradiscretization of a solvable two-dimensional chaotic map associated with the Hesse cubic curve

MI2009-12 Tetsu MASUDA
Hypergeometric τ-functions of the q-Painlevé system of type $E_8^{(1)}$

MI2009-13 Hidenao IWANE, Hitoshi YANAMI, Hirokazu ANAI & Kazuhiro YOKOYAMA
A Practical Implementation of a Symbolic-Numeric Cylindrical Algebraic Decomposition for Quantifier Elimination

MI2009-14 Yasunori MAEKAWA
On Gaussian decay estimates of solutions to some linear elliptic equations and its applications

MI2009-15 Yuya ISHIHARA & Yoshiyuki KAGEI
Large time behavior of the semigroup on L^p spaces associated with the linearized compressible Navier-Stokes equation in a cylindrical domain

MI2009-16 Chikashi ARITA, Atsuo KUNIBA, Kazumitsu SAKAI & Tsuyoshi SAWABE
Spectrum in multi-species asymmetric simple exclusion process on a ring

MI2009-17 Masato WAKAYAMA & Keitaro YAMAMOTO
Non-linear algebraic differential equations satisfied by certain family of elliptic functions

MI2009-18 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of an Elliptical Flow Subjected to a Coriolis Force

MI2009-19 Mitsunori KAYANO & Sadanori KONISHI
Sparse functional principal component analysis via regularized basis expansions and its application

MI2009-20 Shuichi KAWANO & Sadanori KONISHI
Semi-supervised logistic discrimination via regularized Gaussian basis expansions

MI2009-21 Hiroshi YOSHIDA, Yoshihiro MIWA & Masanobu KANEKO
Elliptic curves and Fibonacci numbers arising from Lindenmayer system with symbolic computations

MI2009-22 Eiji ONODERA
A remark on the global existence of a third order dispersive flow into locally Hermitian symmetric spaces

MI2009-23 Stjepan LUGOMER & Yasuhide FUKUMOTO
Generation of ribbons, helicoids and complex scherk surface in laser-matter Interactions

MI2009-24 Yu KAWAKAMI
Recent progress in value distribution of the hyperbolic Gauss map

MI2009-25 Takehiko KINOSHITA & Mitsuhiro T. NAKAO
On very accurate enclosure of the optimal constant in the a priori error estimates for H_0^2-projection

MI2009-26 Manabu YOSHIDA
Ramification of local fields and Fontaine's property (Pm)

MI2009-27 Yu KAWAKAMI
Value distribution of the hyperbolic Gauss maps for flat fronts in hyperbolic three-space

MI2009-28 Masahisa TABATA
Numerical simulation of fluid movement in an hourglass by an energy-stable finite element scheme

MI2009-29 Yoshiyuki KAGEI & Yasunori MAEKAWA
Asymptotic behaviors of solutions to evolution equations in the presence of translation and scaling invariance

MI2009-30 Yoshiyuki KAGEI & Yasunori MAEKAWA
On asymptotic behaviors of solutions to parabolic systems modelling chemotaxis

MI2009-31 Masato WAKAYAMA & Yoshinori YAMASAKI
Hecke's zeros and higher depth determinants

MI2009-32 Olivier PIRONNEAU & Masahisa TABATA
Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type

MI2009-33 Chikashi ARITA
Queueing process with excluded-volume effect

MI2009-34 Kenji KAJIWARA, Nobutaka NAKAZONO & Teruhisa TSUDA
Projective reduction of the discrete Painlevé system of type$(A_2 + A_1)^{(1)}$

MI2009-35 Yosuke MIZUYAMA, Takamasa SHINDE, Masahisa TABATA & Daisuke TAGAMI
Finite element computation for scattering problems of micro-hologram using DtN map

MI2009-36 Reiichiro KAWAI & Hiroki MASUDA
Exact simulation of finite variation tempered stable Ornstein-Uhlenbeck processes

MI2009-37 Hiroki MASUDA
On statistical aspects in calibrating a geometric skewed stable asset price model

MI2010-1 Hiroki MASUDA
Approximate self-weighted LAD estimation of discretely observed ergodic Ornstein-Uhlenbeck processes

MI2010-2 Reiichiro KAWAI & Hiroki MASUDA
Infinite variation tempered stable Ornstein-Uhlenbeck processes with discrete observations

MI2010-3 Kei HIROSE, Shuichi KAWANO, Daisuke MIIKE & Sadanori KONISHI
Hyper-parameter selection in Bayesian structural equation models

MI2010-4 Nobuyuki IKEDA & Setsuo TANIGUCHI
The Itô-Nisio theorem, quadratic Wiener functionals, and 1-solitons

MI2010-5 Shohei TATEISHI & Sadanori KONISHI
Nonlinear regression modeling and detecting change point via the relevance vector machine

MI2010-6 Shuichi KAWANO, Toshihiro MISUMI & Sadanori KONISHI
Semi-supervised logistic discrimination via graph-based regularization

MI2010-7 Teruhisa TSUDA
UC hierarchy and monodromy preserving deformation

MI2010-8 Takahiro ITO
Abstract collision systems on groups

MI2010-9 Hiroshi YOSHIDA, Kinji KIMURA, Naoki YOSHIDA, Junko TANAKA & Yoshihiro MIWA
An algebraic approach to underdetermined experiments

MI2010-10 Kei HIROSE & Sadanori KONISHI
Variable selection via the grouped weighted lasso for factor analysis models

MI2010-11 Katsusuke NABESHIMA & Hiroshi YOSHIDA
Derivation of specific conditions with Comprehensive Groebner Systems

MI2010-12 Yoshiyuki KAGEI, Yu NAGAFUCHI & Takeshi SUDOU
Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Poiseuille type flow

MI2010-13 Reiichiro KAWAI & Hiroki MASUDA
On simulation of tempered stable random variates

MI2010-14 Yoshiyasu OZEKI
Non-existence of certain Galois representations with a uniform tame inertia weight

MI2010-15 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of a Rotating Flow Driven by Precession of Arbitrary Frequency

MI2010-16 Yu KAWAKAMI & Daisuke NAKAJO
The value distribution of the Gauss map of improper affine spheres

MI2010-17 Kazunori YASUTAKE
On the classification of rank 2 almost Fano bundles on projective space

MI2010-18 Toshimitsu TAKAESU
Scaling limits for the system of semi-relativistic particles coupled to a scalar bose field

MI2010-19 Reiichiro KAWAI & Hiroki MASUDA
Local asymptotic normality for normal inverse Gaussian Lévy processes with high-frequency sampling

MI2010-20 Yasuhide FUKUMOTO, Makoto HIROTA & Youichi MIE
Lagrangian approach to weakly nonlinear stability of an elliptical flow

MI2010-21 Hiroki MASUDA
Approximate quadratic estimating function for discretely observed Lévy driven SDEs with application to a noise normality test

MI2010-22 Toshimitsu TAKAESU
A Generalized Scaling Limit and its Application to the Semi-Relativistic Particles System Coupled to a Bose Field with Removing Ultraviolet Cutoffs

MI2010-23 Takahiro ITO, Mitsuhiko FUJIO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Composition, union and division of cellular automata on groups

MI2010-24 Toshimitsu TAKAESU
A Hardy's Uncertainty Principle Lemma in Weak Commutation Relations of Heisenberg-Lie Algebra

MI2010-25 Toshimitsu TAKAESU
On the Essential Self-Adjointness of Anti-Commutative Operators

MI2010-26 Reiichiro KAWAI & Hiroki MASUDA
On the local asymptotic behavior of the likelihood function for Meixner Lévy processes under high-frequency sampling

MI2010-27 Chikashi ARITA & Daichi YANAGISAWA
Exclusive Queueing Process with Discrete Time

MI2010-28 Jun-ichi INOGUCHI, Kenji KAJIWARA, Nozomu MATSUURA & Yasuhiro OHTA
Motion and Bäcklund transformations of discrete plane curves

MI2010-29 Takanori YASUDA, Masaya YASUDA, Takeshi SHIMOYAMA & Jun KOGURE
On the Number of the Pairing-friendly Curves

MI2010-30 Chikashi ARITA & Kohei MOTEGI
Spin-spin correlation functions of the q-VBS state of an integer spin model

MI2010-31 Shohei TATEISHI & Sadanori KONISHI
Nonlinear regression modeling and spike detection via Gaussian basis expansions

MI2010-32 Nobutaka NAKAZONO
Hypergeometric τ functions of the q-Painlevé systems of type $(A_2 + A_1)^{(1)}$

MI2010-33 Yoshiyuki KAGEI
Global existence of solutions to the compressible Navier-Stokes equation around parallel flows

MI2010-34 Nobushige KUROKAWA, Masato WAKAYAMA & Yoshinori YAMASAKI
Milnor-Selberg zeta functions and zeta regularizations

MI2010-35 Kissani PERERA & Yoshihiro MIZOGUCHI
Laplacian energy of directed graphs and minimizing maximum outdegree algorithms

MI2010-36 Takanori YASUDA
CAP representations of inner forms of $Sp(4)$ with respect to Klingen parabolic subgroup

MI2010-37 Chikashi ARITA & Andreas SCHADSCHNEIDER
Dynamical analysis of the exclusive queueing process

MI2011-1 Yasuhide FUKUMOTO& Alexander B. SAMOKHIN
Singular electromagnetic modes in an anisotropic medium

MI2011-2 Hiroki KONDO, Shingo SAITO & Setsuo TANIGUCHI
Asymptotic tail dependence of the normal copula

MI2011-3 Takehiro HIROTSU, Hiroki KONDO, Shingo SAITO, Takuya SATO, Tatsushi TANAKA & Setsuo TANIGUCHI
Anderson-Darling test and the Malliavin calculus

MI2011-4 Hiroshi INOUE, Shohei TATEISHI & Sadanori KONISHI
Nonlinear regression modeling via Compressed Sensing

MI2011-5 Hiroshi INOUE
Implications in Compressed Sensing and the Restricted Isometry Property

MI2011-6 Daeju KIM & Sadanori KONISHI
Predictive information criterion for nonlinear regression model based on basis expansion methods

MI2011-7 Shohei TATEISHI, Chiaki KINJYO & Sadanori KONISHI
Group variable selection via relevance vector machine

MI2011-8 Jan BREZINA & Yoshiyuki KAGEI
Decay properties of solutions to the linearized compressible Navier-Stokes equation around time-periodic parallel flow
Group variable selection via relevance vector machine

MI2011-9 Chikashi ARITA, Arvind AYYER, Kirone MALLICK & Sylvain PROLHAC
Recursive structures in the multispecies TASEP

MI2011-10 Kazunori YASUTAKE
On projective space bundle with nef normalized tautological line bundle

MI2011-11 Hisashi ANDO, Mike HAY, Kenji KAJIWARA & Tetsu MASUDA
An explicit formula for the discrete power function associated with circle patterns of Schramm type

MI2011-12 Yoshiyuki KAGEI
Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a parallel flow

MI2011-13 Vladimír CHALUPECKÝ & Adrian MUNTEAN
Semi-discrete finite difference multiscale scheme for a concrete corrosion model: approximation estimates and convergence

MI2011-14 Jun-ichi INOGUCHI, Kenji KAJIWARA, Nozomu MATSUURA & Yasuhiro OHTA
Explicit solutions to the semi-discrete modified KdV equation and motion of discrete plane curves

MI2011-15 Hiroshi INOUE
A generalization of restricted isometry property and applications to compressed sensing

MI2011-16 Yu KAWAKAMI
A ramification theorem for the ratio of canonical forms of flat surfaces in hyperbolic three-space

MI2011-17 Naoyuki KAMIYAMA
Matroid intersection with priority constraints

MI2012-1 Kazufumi KIMOTO & Masato WAKAYAMA
Spectrum of non-commutative harmonic oscillators and residual modular forms