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SOME REMARKS ON ESTIMATION OF DIFFUSION COEFFICIENTS FOR JUMP-DIFFUSIONS FROM FINITE SAMPLES

By

Yasutaka SHIMIZU*

Abstract

Diffusion coefficients of jump-diffusion processes with finite Lévy measure are estimated from discrete observations at points $t_i^n = i/n$ ($i = 0, 1, \dots, n$) using filtered conditional moments of increments. The estimation is based on the local time for jump-diffusions and the consistency result is obtained. This is an extension of the result for pure diffusion cases by Florens-Zmirou (1993).

Key Words and Phrases: diffusion coefficients, jump-diffusions, discrete observations, asymptotic filter, local times, infinitesimal moments.

1. Introduction

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq 1}, P)$, we consider an one-dimensional jump-diffusion process X satisfying the following stochastic differential equation:

$$X_t = X_0 + \int_0^t a(X_{s-}) ds + \int_0^t b(X_{s-}) dW_s + \int_0^t \int_{\mathbb{R}} c(X_{s-}, z) \mu(ds, dz), \quad (1)$$

where X_0 is a random variable, coefficients a, b and c are measurable functions such that $a : \mathbb{R} \rightarrow \mathbb{R}$, $b : \mathbb{R} \rightarrow \mathbb{R}$ and $c : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, respectively, W is a Wiener process, $\mu(dt, dz)$ is a Poisson random measure with the Lévy density π ; $E[\mu(dt, dz)] = \pi(z) dz dt$. We assume throughout this paper that $\int_{\mathbb{R}} \pi(z) dz < \infty$. Therefore (1) is well defined.

This paper is devoted to the nonparametric estimation of the diffusion function $\beta(\cdot) := b^2(\cdot)$ from discrete observations obtained in $[0, 1]$ -time interval, that is, $\{X_{t_i^n}\}_{i=0}^n$, where $t_i^n = i/n$; $i = 0, 1, \dots, n$.

Recently, several authors have studied the asymptotic inference for discretely observed jump-diffusions; see e.g. Bandi and Nguyen (2003), Mancini (2004), Shimizu and Yoshida (2006) and Shimizu (2006a,b) etc. All their works are discussed under the sampling scheme such that $t_i^n = i \times h_n$; $i = 0, 1, \dots, n$ for a sequence $h_n > 0$ satisfying $h_n \rightarrow 0$ and $nh_n \rightarrow \infty$ as $n \rightarrow \infty$. This is because they tried to estimate not only the diffusion but also the drift and the Lévy characteristics such as $a(x)$, $c(x, z)$ and π . We know well that it is impossible to estimate these characteristics without the asymptotics $nh_n \rightarrow \infty$. However it is nothing unusual in applications that the terminal nh_n is fixed. In this paper we suppose that the observations are obtained in $[0, 1]$ -interval,

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and try to estimate the diffusion function with the estimation of the drift and the Lévy characteristics abandoned.

When X is a diffusion process *without* jumps; $\mu \equiv 0$, Florens-Zmirou (1993) proposed a nonparametric estimator of the diffusion coefficient in view of the fact that $t^{-1}E[(X_{s+t} - X_s)^2 | X_s = x] \rightarrow \beta(x)$ as $t \rightarrow 0$ from the same type of observations as ours. However when X is a jump-diffusion, the limit is not only $\beta(\cdot)$. Bandi and Nguyen (2003) extended their work to the case of jump-diffusions although the sampling scheme is different; $h_n \rightarrow 0$ and $nh_n \rightarrow \infty$. They also focused on the fact that

$$\lim_{t \rightarrow 0} \frac{1}{t} E[(X_{s+t} - X_s)^2 | X_s = x] = \beta(x) + \int_{\mathbb{R}} c^2(x, z) \pi(z) dz, \quad (2)$$

and discretized the above conditional expectation when t is small by using the local time estimators for X . Such a procedure is essentially the same as in Florens-Zmirou (1993), but the assumption that $nh_n \rightarrow \infty$ is necessary since the right-hand side of (2) includes the Lévy characteristics. Moreover the fact (2) clearly does not indicate the possibility of separate estimation of $\beta(\cdot)$ and $\int c^2(\cdot, z) \pi(z) dz$.

Our goal in this paper is to estimate $\beta(\cdot)$ separately under the sampling scheme that nh_n is fixed. In order to execute the separate estimation as above, we need to discriminate whether the increment of neighboring data $\Delta_i X^n = X_{t_i^n} - X_{t_{i-1}^n}$ includes a jump or does not. An asymptotic filter proposed by Shimizu and Yoshida (2006) or Shimizu (2006a) enables us to judge whether a jump occurred or not in $(t_{i-1}^n, t_i^n]$. Although it is a stochastic judgment, the error rate asymptotically decreases as $n \rightarrow \infty$; see Section 2. Using increments that are expected not to include any jump, we take the similar argument to the one in Florens-Zmirou (1993); see Section 3, where some estimators of the local time of X are presented. The estimator of $\beta(x)$; say $\hat{\beta}_n(x)$ is constructed in combination with their local time estimators. Our main result is for the consistency: $\lim_{n \rightarrow \infty} \hat{\beta}_n(x) = \beta(x)$ in probability for appropriate points $x \in \mathbb{R}$, which is presented in Theorem 3.7.

2. The basic idea of estimation

Let Y be a one-dimensional diffusion process that follows $dY_t = \tilde{a}(Y_t) dt + \tilde{b}(Y_t) dW_t$. As already described, Florens-Zmirou (1993) focused attention to the following *infinitesimal conditional moment* to construct an estimator of \tilde{b}^2 :

$$\lim_{\tau \rightarrow 0} \frac{1}{\tau} E[(Y_{t+\tau} - Y_t)^2 | Y_t = x] = \tilde{b}^2(x). \quad (3)$$

This fact also indicates the possibility that we can extract only the diffusion part if we can identify the increment from the time t to the first jump time. For that purpose, we introduce the asymptotic filter proposed in Shimizu (2006a), which is originally due to Shimizu and Yoshida (2006). The filter can separate the jumps of the process and the increment by diffusions asymptotically, and it helps us to discretize the expectation in (3). This is the basic idea in this paper. Indeed, the estimator of $\beta(\cdot)$ we propose in Theorem 3.7 later would look like a discrete approximation of $\tau^{-1} E[(Y_{t+\tau} - Y_t)^2 | Y_t = x]$ for small τ .

Before introducing the filter, we make the following assumptions, which is the similar setting as in Florens-Zmirou (1993) and Shimizu (2006a).

A 1 For a constant L and a function $\zeta(z)$ of polynomial growth in z , $|a(x) - a(y)| + |b(x) - b(y)| \leq L|x - y|$, $|c(x, z) - c(y, z)| \leq \zeta(z)|x - y|$, and $|c(x, z)| \leq \zeta(z)(1 + |x|)$.

A 2 The Lévy density π is bounded and satisfies that $\int_{\mathbb{R}} |z|^p \pi(z) dz < \infty$ for any $p > 0$. Moreover $\sup_{t \in [0, 1]} E|X_t|^p < \infty$ for any $p > 0$.

A 3 There exists $c_0 > 0$ such that $\inf_{x \in \mathbb{R}} |c(x, z)| \geq c_0|z|$ in a neighborhood of $z = 0$.

A 4 The coefficient a is bounded and twice continuously differentiable with bounded derivatives. The coefficient b is three times continuously differentiable with bounded derivatives such that there exists constants k and K with $0 < k \leq b(x) \leq K$.

In the sequel, we use the following notation. Denote by $\Delta_i X^n := X_{t_i^n} - X_{t_{i-1}^n}$; the increments of X in the interval $(t_{i-1}^n, t_i^n]$, and denote by J_i^n the number of jumps in that interval. The symbol P_{i-1}^n stands for the conditional probability on σ -field $\mathcal{F}_{t_{i-1}^n}$, and E_{i-1}^n means that the expectation with respect to P_{i-1}^n , that is, $P_{i-1}^n\{\cdot\} = P\{\cdot | \mathcal{F}_{t_{i-1}^n}\}$ and $E_{i-1}^n[\cdot] = E[\cdot | \mathcal{F}_{t_{i-1}^n}]$. Moreover $X \lesssim Y$ means there exists a constant $c > 0$ such that $X \leq cY$. Let u_n be a real valued sequence, and $R : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a function for which there exists a constant C such that $R(u_n, x) \lesssim u_n(1 + |x|)^C$ for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$.

The following lemma is due to Shimizu (2006a), which is the key result in this paper.

LEMMA 2.1. *Suppose Assumptions A 1–A 3. Fix an arbitrary constant $L > 0$ and $\rho \in [0, 1/2)$. Then the following inequalities are valid for any $p \geq 1$ and sufficiently large $n \in \mathbb{N}$:*

$$\begin{aligned} P_{i-1}^n\{|\Delta_i X^n| \leq Ln^{-\rho}, J_i^n = 0\} &= R(1, X_{t_{i-1}^n}), \\ P_{i-1}^n\{|\Delta_i X^n| \leq Ln^{-\rho}, J_i^n = 1\} &= R(n^{-(1+\rho)}, X_{t_{i-1}^n}), \\ P_{i-1}^n\{|\Delta_i X^n| \leq Ln^{-\rho}, J_i^n \geq 2\} &= R(n^{-2}, X_{t_{i-1}^n}), \\ P_{i-1}^n\{|\Delta_i X^n| > Ln^{-\rho}, J_i^n = 0\} &= R(n^{-p}, X_{t_{i-1}^n}), \\ P_{i-1}^n\{|\Delta_i X^n| > Ln^{-\rho}, J_i^n = 1\} &= R(n^{-1}, X_{t_{i-1}^n}), \\ P_{i-1}^n\{|\Delta_i X^n| > Ln^{-\rho}, J_i^n \geq 2\} &= R(n^{-2}, X_{t_{i-1}^n}). \end{aligned}$$

This lemma implies that, under $n \rightarrow \infty$, we can judge that the interval $(t_{i-1}^n, t_i^n]$ has no jump if $|\Delta_i X^n| \leq Ln^{-\rho}$ and the interval has a single jump if $|\Delta_i X^n| > Ln^{-\rho}$, and that we can ignore the event which includes more than two jumps in that interval. Although this lemma holds for any $L > 0$ and $\rho \in [0, 1/2)$ if we take sufficiently large $n \in \mathbb{N}$, it is important in applications to choose L and ρ suitably for fixed n . On some methods to choose a suitable (L, ρ) , see Shimizu (2007b).

Let us define the further notation: $C_i^n = \cup_{j=1}^2 C_{i,j}^n$; $i = 1, \dots, n$, where

$$\begin{aligned} C_{i,j}^n &:= \{|\Delta_i X^n| \leq Ln^{-\rho}, J_i^n = j\} \quad (j = 0, 1), \\ C_{i,2}^n &:= \{|\Delta_i X^n| \leq Ln^{-\rho}, J_i^n \geq 2\}. \end{aligned}$$

Thanks to the previous lemma, we obtain the following estimates for moments of the increments $\Delta_i X^n$, which are provided in Shimizu and Yoshida (2006), Proposition 3.2. Therefore their proofs are omitted here.

LEMMA 2.2. *Suppose Assumptions A 1–A 3. Then*

$$E_{i-1}^n [\Delta_i X^n \mathbf{1}(\mathcal{C}_{i,0}^n)] = a(X_{t_{i-1}^n})n^{-1} + R(n^{-2}, X_{t_{i-1}^n}), \quad (4)$$

$$E_{i-1}^n [(\Delta_i X^n)^2 \mathbf{1}(\mathcal{C}_{i,0}^n)] = \beta(X_{t_{i-1}^n})n^{-1} + R(n^{-2}, X_{t_{i-1}^n}), \quad (5)$$

$$E_{i-1}^n [(\Delta_i X^n)^3 \mathbf{1}(\mathcal{C}_{i,0}^n)] = R(n^{-2}, X_{t_{i-1}^n}), \quad (6)$$

$$E_{i-1}^n [(\Delta_i X^n)^4 \mathbf{1}(\mathcal{C}_{i,0}^n)] = 3\beta^2(X_{t_{i-1}^n})n^{-2} + R(n^{-3}, X_{t_{i-1}^n}), \quad (7)$$

where $\mathbf{1}(A)$ is the indicator function of a set A .

REMARK. Under the same assumptions as in Lemma 2.2, we see that

$$E_{i-1}^n [|\Delta_i X^n|^N \mathbf{1}(\mathcal{C}_{i,0}^n)] = R(n^{-N/2}, X_{t_{i-1}^n}) \quad (8)$$

for any $N \geq 1$. See Remark 3.1 in Shimizu and Yoshida (2006).

In the next section, we propose some estimators of the local time of X , which will play the role of τ in (3), and together with Lemma 2.2 enables us to discretize the infinitesimal conditional moments in (3).

3. Local time estimates

Before the discussion for estimators of the local time, we shall state some auxiliary results for the local time of semimartingales.

Notice that the assumption $\int_{\mathbb{R}} \pi(z) dz < \infty$ implies that $\sum_{0 \leq t \leq 1} |\Delta X_t| < \infty$ a.s., which is an important hypothesis for the following useful lemma; see e.g. Protter (2004).

LEMMA 3.1. *We denote by $L_t(x)$ the local time of the jump-diffusion (1) in x during $[0, t]$, then $L_t(x)$ has a version with càdlàg in x and continuous in t , and the following equalities hold true:*

$$\begin{aligned} L_t(x) &= L_t(x+) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}(\{x \leq X_s \leq x + \epsilon\}) d[X, X]_s^c \quad a.s., \\ L_t(x-) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}(\{x - \epsilon \leq X_s \leq x\}) d[X, X]_s^c \quad a.s., \end{aligned}$$

where $[X, X]^c$ is the path-by-path continuous part of the quadratic variation of X .

Moreover we denote by $\bar{L}_t(x)$ the symmetric version of the local time, that is,

$$\bar{L}_t(x) := \frac{L_t(x+) + L_t(x-)}{2} = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}(\{|X_s - x| \leq \epsilon\}) d[X, X]_s^c \quad a.s.$$

The next lemma is known as the *occupation time formula*.

LEMMA 3.2. *Let X be a semimartingale with the local time $L_t(x)$ in x during $[0, t]$, and let g be a bounded Borel function. Then*

$$\int_{-\infty}^{\infty} L_t(x) g(x) dx = \int_0^t g(X_{s-}) d[X, X]_s^c \quad a.s.$$

Now, let us construct estimators of the local time $\bar{L}_t(\cdot)$ from the data $(X_{t_i^n})_{0 \leq i \leq n}$ using the asymptotic filter $\mathcal{C}^n := \{\mathcal{C}_i^n\}_{i=1}^n$, and show their asymptotic behaviors.

First we prepare a lemma. Hereafter we use a convention $\sum_{i=1}^0 \equiv 0$ and use

$$I^+(y) = \mathbf{1}(\{0 \leq y \leq 1\}), \quad I^-(y) = \mathbf{1}(\{-1 \leq y \leq 0\}), \quad I(y) = \frac{1}{2} \mathbf{1}(\{|y| \leq 1\}).$$

LEMMA 3.3. *Suppose Assumption A 4. Let $\{\delta_n\}_{n \in \mathbb{N}}$ be a sequence such that $\delta_n \rightarrow 0$ and $n\delta_n^2 \rightarrow \infty$ as $n \rightarrow \infty$, and let $\Lambda_n = \sqrt{n}e^{-Cn\delta_n^2}$ for a constant $C > 0$. Then it follows for each $i = 1, \dots, n$ and any $s \in (t_{i-1}^n, t_i^n]$ that*

$$\left\| \left\{ I \left(\frac{X_{t_{i-1}^n}}{\delta_n} \right) - I \left(\frac{X_{s-}}{\delta_n} \right) \right\} \mathbf{1}(\{J_i^n = 0\}) \right\|_{L^1} = O(\Lambda_n)$$

as $n \rightarrow \infty$.

PROOF. Let Y be a strong solution process to the SDE $dY_t = a(Y_t) dt + b(Y_t) dW_t$ and $Y_{t_{i-1}^n} = X_{t_{i-1}^n}$. Then $(X_t)_{t \in (t_{i-1}^n, t_i^n]}$ is indistinguishable from $(Y_t)_{t \in (t_{i-1}^n, t_i^n]}$ on the set $\{J_i^n = 0\}$. Hence it follows for any $s \in (t_{i-1}^n, t_i^n]$; $i = 1, \dots, n$, that

$$\begin{aligned} & \left\| \left\{ I \left(\frac{X_{t_{i-1}^n}}{\delta_n} \right) - I \left(\frac{X_{s-}}{\delta_n} \right) \right\} \mathbf{1}(\{J_i^n = 0\}) \right\|_{L^1} \\ & \leq \left\| \mathbf{1}(\{|X_{t_{i-1}^n}| \leq \delta_n\}) - \mathbf{1}(\{|Y_{s-}| \leq \delta_n\}) \right\|_{L^1} \\ & = \left\| \mathbf{1}(\{|X_{t_{i-1}^n}| \leq \delta_n, |Y_{s-}| > \delta_n\}) \right\|_{L^1} + \left\| \mathbf{1}(\{|X_{t_{i-1}^n}| > \delta_n, |Y_{s-}| \leq \delta_n\}) \right\|_{L^1} \\ & = P \left\{ |X_{t_{i-1}^n}| \leq \delta_n, |Y_{s-}| > \delta_n \right\} + P \left\{ |X_{t_{i-1}^n}| > \delta_n, |Y_{s-}| \leq \delta_n \right\}. \end{aligned}$$

Here we denote by $p_t(x, y)$ the transition kernel of the process Y from x to y in time t . Then it follows under A 4 that

$$p_t(x, y) \lesssim \frac{1}{\sqrt{t}} \exp \left(-\frac{c(x-y)^2}{2t} \right) \quad (9)$$

for a positive constant c ; see Friedman (1964) p257. Therefore we easily see that $\sup_{|x| < \delta_n} |p_t(x, y)| \lesssim t^{-1/2} \exp(-cy^2/2t)$; see also Lemma 1 in Florens-Zmirou (1993). Therefore, denoting by $D_i^n(dx)$ the distribution of $X_{t_{i-1}^n}$, we have

$$\begin{aligned} P \left\{ |X_{t_{i-1}^n}| \leq \delta_n, |Y_{s-}| > \delta_n \right\} & = \int_{-\delta_n}^{\delta_n} \int_{|y| > \delta_n} p_{s-t_{i-1}^n}(x, y) dy D_i^n(dx) \\ & \lesssim \int_{-\delta_n}^{\delta_n} \int_{|y| > \delta_n} \frac{1}{\sqrt{s-t_{i-1}^n}} e^{-\frac{cy^2}{2(s-t_{i-1}^n)}} dy D_i^n(dx) \\ & \lesssim \sqrt{n} e^{-Cn\delta_n^2} \quad (C > 0). \end{aligned}$$

The same argument as above is possible on the term $P\{|X_{t_{i-1}^n}| > \delta_n, |Y_{s-}| \leq \delta_n\}$ since we also see by (9) that $\sup_{|y| < \delta_n} |p_t(x, y)| \lesssim t^{-1/2} \exp(-c'y^2/2t)$ ($c' > 0$). \square

LEMMA 3.4. *Suppose Assumptions A 1–A 4, and that $\delta_n \downarrow 0$, $n\delta_n^2 \rightarrow \infty$ and $\delta_n^{-1}\Lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Let g be a function that is bounded and Lipschitz continuous, and let*

$$G_t^{(n)}(x) = \frac{1}{n\delta_n} \sum_{i=1}^{[nt]} g(X_{t_{i-1}^n}) I\left(\frac{X_{t_{i-1}^n} - x}{\delta_n}\right) \mathbf{1}(\mathcal{C}_i^n)$$

for an arbitrary $t \in (0, 1]$. Then the following convergence holds true for any $x \in \mathbb{R}$:

$$\lim_{n \rightarrow \infty} \|G_t^{(n)}(x) - g(x)\beta^{-1}(x)\bar{L}_t(x)\|_{L^1} = 0.$$

PROOF. Notice the following decomposition:

$$\begin{aligned} & \|G_t^{(n)}(x) - g(x)\beta^{-1}(x)\bar{L}_t(x)\|_{L^1} \\ & \leq \left\| \frac{1}{n\delta_n} \sum_{i=1}^{[nt]} g(X_{t_{i-1}^n}) I\left(\frac{X_{t_{i-1}^n} - x}{\delta_n}\right) \mathbf{1}(\{J_i^n = 0\}) - g(x)\beta^{-1}(x)\bar{L}_t(x) \right\|_{L^1} \\ & \quad + \left\| \frac{1}{n\delta_n} \sum_{i=1}^{[nt]} g(X_{t_{i-1}^n}) I\left(\frac{X_{t_{i-1}^n} - x}{\delta_n}\right) \mathbf{1}(\mathcal{C}_{i,1}^n \cup \mathcal{C}_{i,2}^n) \right\|_{L^1} \\ & \quad + \left\| \frac{1}{n\delta_n} \sum_{i=1}^{[nt]} g(X_{t_{i-1}^n}) I\left(\frac{X_{t_{i-1}^n} - x}{\delta_n}\right) \mathbf{1}(\{|\Delta_i X^n| > Ln^{-\rho}, J_i^n = 0\}) \right\|_{L^1} \\ & =: \alpha_1 + \alpha_2 + \alpha_3. \end{aligned}$$

On the second term α_2 , it is easily seen that $\alpha_2 \leq \delta_n^{-1}P\{\mathcal{C}_{i,1}^n \cup \mathcal{C}_{i,2}^n\}$ since g and I are bounded. Thus $\alpha_2 = O(\delta_n^{-1}n^{-(1+\rho)}) \rightarrow 0$ as $n \rightarrow \infty$. Similarly, we see that $\alpha_3 \rightarrow 0$.

Let us estimate the first term α_1 . In the sequel, we assume $t = 1$ and $x = 0$ without loss of generality; see Florens-Zmirou (1993), and denote by $G_1^{(n)}(0) = G^{(n)}$ and $\bar{L}_1(0) = \bar{L}$. We show $G^{(n)}$ converges to $g(0)\beta^{-1}(0)\bar{L}$ in L^1 -sense.

Notice that

$$\begin{aligned} \alpha_1 & \leq \left\| G^{(n)}\mathbf{1}(\{J_i^n = 0\}) - \int_0^1 \frac{g(X_{s-})}{\delta_n\beta(X_{s-})} I\left(\frac{X_{s-}}{\delta_n}\right) d[X, X]_s^c \right\|_{L^1} \\ & \quad + \left\| \int_0^1 \frac{g(X_{s-})}{\delta_n\beta(X_{s-})} I\left(\frac{X_{s-}}{\delta_n}\right) d[X, X]_s^c - g(0)\beta^{-1}(0)\bar{L} \right\|_{L^1} \\ & =: \gamma_1 + \gamma_2. \end{aligned}$$

On the term γ_1 , noticing that $d[X, X]_t^c = \beta(X_t) dt$, the Lipschitz condition for g , and

that g and I are bounded function, it follows by Lemma 3.3 that

$$\begin{aligned}
 \gamma_1 &\leq \frac{1}{\delta_n} \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \left\| g(X_{t_{i-1}^n}) \left\{ I\left(\frac{X_{t_{i-1}^n}}{\delta_n}\right) - I\left(\frac{X_{s-}}{\delta_n}\right) \right\} \mathbf{1}(\{J_i^n = 0\}) \right\|_{L^1} ds \\
 &\quad + \frac{1}{\delta_n} \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \left\| I\left(\frac{X_{s-}}{\delta_n}\right) \left\{ g(X_{t_{i-1}^n}) - g(X_{s-}) \right\} \right\|_{L^1} ds \\
 &\lesssim \delta_n^{-1} \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \left\| I\left(\frac{X_{t_{i-1}^n}}{\delta_n}\right) - I\left(\frac{X_{s-}}{\delta_n}\right) \right\|_{L^1} ds + \delta_n^{-1} \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \|X_{s-} - X_{t_{i-1}^n}\|_{L^1} ds \\
 &= O\left(\delta_n^{-1}(\Lambda_n + n^{-1/2})\right) \rightarrow 0 \quad (n \rightarrow \infty).
 \end{aligned}$$

On the term γ_2 , thanks to Lemma 3.2, we obtain that

$$\begin{aligned}
 \gamma_2 &= \left\| \int_{-\infty}^{\infty} \bar{L}_1(r) \frac{g(r)}{\delta_n \beta(r)} I\left(\frac{r}{\delta_n}\right) dr - g(0) \beta^{-1}(0) \bar{L} \right\|_{L^1} \\
 &= \int_{\mathbb{R}} \left\| \frac{g(\delta_n q)}{\beta(\delta_n q)} \bar{L}_1(\delta_n q) - \frac{g(0)}{\beta(0)} \bar{L} \right\|_{L^1} I(q) dq \\
 &\lesssim \frac{1}{2} \int_0^{\infty} \|\bar{L}(\delta_n q) - \bar{L}_1(0+)\|_{L^1} I(q) dq \\
 &\quad + \frac{1}{2} \int_{-\infty}^0 \|\bar{L}(\delta_n q) - \bar{L}_1(0-)\|_{L^1} I(q) dq \\
 &\quad + \|\bar{L}\|_{L^1} \int_{\mathbb{R}} \left| \frac{g(\delta_n q)}{\beta(\delta_n q)} - \frac{g(0)}{\beta(0)} \right| I(q) dq.
 \end{aligned}$$

Here we see that $\bar{L}_1(\delta_n q)$ is uniformly integrable since $\sup_{n, q: |q| \leq 1} E|\bar{L}(\delta_n q)|^2 < \infty$, which can be shown by Assumption A 2 and Tanaka's formula (e.g. Protter (2004), p.213, Theorem 68). Moreover $\bar{L}(\pm \delta_n q) \rightarrow \bar{L}(0\pm)$ a.s. by Lemma 3.1. Hence $\bar{L}(\pm \delta_n q) \rightarrow \bar{L}(0\pm)$ in L^1 . Furthermore the bounded term $g(\delta_n q) \beta^{-1}(\delta_n q) \rightarrow g(0) \beta^{-1}(0)$ by the continuity. Therefore the last three terms converge to zero as $n \rightarrow \infty$ by the Lebesgue convergence theorem since $I(q) dq$ is a probability measure. This completes the proof. \square

REMARK. Suppose the same assumptions as in Lemma 3.4, and let

$$\tilde{G}_t^{(n)}(x) = \frac{1}{n\delta_n} \sum_{i=1}^{[nt]} g(X_{t_{i-1}^n}) I\left(\frac{X_{t_{i-1}^n} - x}{\delta_n}\right),$$

Then we also see that $\lim_{n \rightarrow \infty} \|\tilde{G}_t^{(n)}(x) - g(x) \beta^{-1}(x) \bar{L}_t(x)\|_{L^1} = 0$ for all $x \in \mathbb{R}$ since

$$\begin{aligned}
 \|G_t^{(n)}(x) - \tilde{G}_t^{(n)}(x)\|_{L^1} &= \left\| \frac{1}{n\delta_n} \sum_{i=1}^{[nt]} g(X_{t_{i-1}^n}) I\left(\frac{X_{t_{i-1}^n} - x}{\delta_n}\right) \mathbf{1}(\{|\Delta_i X^n| > L n^{-\rho}\}) \right\|_{L^1} \\
 &= O(n^{-1} \delta_n^{-1}),
 \end{aligned}$$

which tends to zero as $n \rightarrow \infty$.

By Remark 3., we have the following.

COROLLARY 3.5. *Suppose the same assumptions as in Lemma 3.4. Then*

$$\begin{aligned} L_t^{(n)}(x) &:= \frac{1}{n\delta_n} \sum_{i=1}^{[nt]} I\left(\frac{X_{t_{i-1}^n} - x}{\delta_n}\right) \rightarrow \beta^{-1}(x)\bar{L}_t(x), \\ B_t^{(n)}(x) &:= \frac{1}{n\delta_n} \sum_{i=1}^{[nt]} \beta(X_{t_{i-1}^n}) I\left(\frac{X_{t_{i-1}^n} - x}{\delta_n}\right) \rightarrow \bar{L}_t(x) \end{aligned}$$

in L^1 -sense as $n \rightarrow \infty$.

The above estimator $B_t^{(n)}(\cdot)$ still includes the unknown function β . Therefore we provide its empirical version as follows.

LEMMA 3.6. *Let*

$$\hat{B}_t^{(n)}(x) = \frac{1}{\delta_n} \sum_{i=1}^{[nt]} I\left(\frac{X_{t_{i-1}^n} - x}{\delta_n}\right) (\Delta_i X^n)^2 \mathbf{1}(\mathcal{C}_i^n).$$

Suppose the same assumptions as in Lemma 3.4. Then, for all $x \in \mathbb{R}$ and $t \in (0, 1]$

$$\hat{B}_t^{(n)}(x) \xrightarrow{p} \bar{L}_t(x). \quad (10)$$

PROOF. We shall show that $\hat{B}_t^{(n)}(x) - B_t^{(n)}(x) \xrightarrow{p} 0$, then Corollary 3.5 yields the consequence. For that purpose, it suffices to show the following two conditions; see Genon-Catalot and Jacod (1993), Lemma 9: as $n \rightarrow \infty$,

$$\sum_{i=1}^{[nt]} E_{i-1}^n \left[\frac{1}{\delta_n} I\left(\frac{X_{t_{i-1}^n} - x}{\delta_n}\right) (\Delta_i X^n)^2 \mathbf{1}(\mathcal{C}_i^n) \right] - B_t^{(n)}(x) \xrightarrow{p} 0, \quad (11)$$

$$\sum_{i=1}^{[nt]} E_{i-1}^n \left[\left| \frac{1}{\delta_n} I\left(\frac{X_{t_{i-1}^n} - x}{\delta_n}\right) (\Delta_i X^n)^2 \mathbf{1}(\mathcal{C}_i^n) \right|^2 \right] \xrightarrow{p} 0. \quad (12)$$

On (11), we see by Lemma 2.2, (5) that

$$\begin{aligned} & \sum_{i=1}^{[nt]} E_{i-1}^n \left[\frac{1}{\delta_n} I\left(\frac{X_{t_{i-1}^n} - x}{\delta_n}\right) (\Delta_i X^n)^2 \mathbf{1}(\mathcal{C}_i^n) \right] - B_t^{(n)}(x) \\ &= \frac{1}{n\delta_n} \sum_{i=1}^{[nt]} I\left(\frac{X_{t_{i-1}^n} - x}{\delta_n}\right) E_{i-1}^n \left[n(\Delta_i X^n)^2 \mathbf{1}(\mathcal{C}_i^n) - \beta(X_{t_{i-1}^n}) \right] \\ &= O_p\left(\frac{1}{n\delta_n}\right) \xrightarrow{p} 0. \end{aligned}$$

Moreover, on (12), we see by Lemma 2.2, (7) that

$$\begin{aligned} & \sum_{i=1}^{[nt]} E_{i-1}^n \left[\left| \frac{1}{\delta_n} I \left(\frac{X_{t_{i-1}^n} - x}{\delta_n} \right) (\Delta_i X^n)^2 \mathbf{1}(C_i^n) \right|^2 \right] \\ &= \frac{1}{\delta_n^2} \sum_{i=1}^{[nt]} I \left(\frac{X_{t_{i-1}^n} - x}{\delta_n} \right) E_{i-1}^n [(\Delta_i X^n)^4 \mathbf{1}(C_i^n)] \\ &= O_p \left(\frac{1}{n\delta_n^2} \right) \xrightarrow{p} 0. \end{aligned}$$

This ends the proof. \square

The next theorem is our main result, which is the direct consequence from Corollary 3.5 and Lemma 3.6.

THEOREM 3.7. *Suppose Assumptions A 1–A 4, and that $\delta_n \downarrow 0$, $n\delta_n^2 \rightarrow \infty$ and $\delta_n^{-1}\Lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Then the estimator $\hat{\beta}_n(x) := \hat{B}_1^{(n)}(x)/L_1^{(n)}(x)$ is consistent to $\beta(x)$ for all $x \in \mathbb{R}$ with $\bar{L}_1(x) > 0$:*

$$\hat{\beta}_n(x) \xrightarrow{p} \beta(x)$$

as $n \rightarrow \infty$.

REMARK. Mancini (2004) also proposes the same type of the asymptotic filter, which can also be applied to our purpose. However, in that case, the order condition for δ_n would be changed.

REMARK. Under the asymptotics $h_n \rightarrow 0$ and $nh_n \rightarrow \infty$, it would also be possible to estimate $\beta(\cdot)$ and $\int c^2(\cdot, z)\pi(z) dz$ separately, and also $a(\cdot)$ by operating the asymptotic filter to kernel estimators by Bandi and Nguyen (2003). In particular, when $c(x, z)$ is known, Shimizu (2007a) proposes a nonparametric estimator of $\int G(\cdot, z)\pi(z) dz$ for any known function G .

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