On Multiple Zeta Values of Maximal Height

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On Multiple Zeta Values of Maximal Height

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1 Introduction

The multiple zeta values are real numbers first studied by Leonhard Euler. These numbers have been appeared in various contexts in number theory, geometry, knot theory, mathematical physics and arithmetical algebraic geometry. In 1994, Don Zagier made a conjecture about the dimensions of the vector spaces spanned by the multiple zeta values. This conjecture was partially solved by Tomohide Terasoma, Alexander Goncharov and Pierre Deligne in 2000's. According to this result, there are many relations over $\mathbb{Q}$ among the multiple zeta values. One of the main problems in the theory of multiple zeta values is to clarify all relations among multiple zeta values. In Section 2, we present some basics of the multiple zeta values and introduce several well-known relations among them.

Tsuneo Arakawa and Masanobu Kaneko introduced a function as a generalization of the Riemann zeta function. This function is related to multiple zeta functions and to poly-Bernoulli numbers. In particular the special values at positive integers of the Arakawa-Kaneko zeta function are expressed by multiple zeta values. We discuss them in Section 3.

In Section 4, we give the following identities, all of them involving multiple zeta values of maximal height:

Theorem 4.1. *For any integers $r, k \geq 1$, we have*

$$\zeta(\underbrace{1,\dots,1}_{r-1},k+1) = \sum_{j=1}^{\min(r,k)} (-1)^{j-1} \sum_{\substack{a_1+\cdots+a_j=r \\ \forall a_i \geq 1}} \sum_{\substack{b_1+\cdots+b_j=k \\ \forall b_i \geq 1}} \zeta(a_1+b_1,\dots,a_j+b_j).$$

The right-hand side of this formula is symmetric in r and k, and thus the formula makes the duality for the multiple zeta values of height one visible. The proof uses results on the special values of the Arakawa-Kaneko zeta function. The next result is a generalization of Theorem 4.1.

Theorem 4.2. *For $\mathbf{k} = (k_1,\dots,k_d) \in \mathbb{N}^d$ and $r \in \mathbb{N}$ with $r \geq d$, we have*

$$\sum_{\substack{r_1+\cdots+r_d=r \\ \forall r_i \geq 1}} \zeta(\underbrace{1,\dots,1}_{r_1-1},k_1+1,\dots,\underbrace{1,\dots,1}_{r_d-1},k_d+1) = \sum_{\substack{\mathbf{k}' \succeq \mathbf{k} \\ \mathrm{dep}(\mathbf{k}') \leq r}} \sum_{\substack{\mathrm{wt}(\mathbf{r})=r \\ \mathrm{dep}(\mathbf{r})=\mathrm{dep}(\mathbf{k}')}} (-1)^{\mathrm{dep}(\mathbf{k}')-d} \zeta(\mathbf{k}'+\mathbf{r}),$$

where $\mathbf{k}' \succeq \mathbf{k}$ means $\mathbf{k}'$ is a refinement of $\mathbf{k}$.

This theorem has two proofs. The first is combinatorial by using Ohno's relation and the second is algebraic by using the derivation relation. We explain them in Section 5. Also we apply this method to the finite multiple zeta values in Section 6.

In the last section 7, we give a purely algebraic version of the Taylor series for the reciprocal gamma function (Theorem 7.1). This series as well as the Taylor series of the gamma function play an important role in the theory of regularization of MZVs. We review this theory in subsection 2.3. The coefficients of the Taylor series of $\Gamma(x)^{-1}$ are described in terms of the multiple zeta values of maximal height:

Corollary 7.2.

$$\frac{1}{\Gamma(x)} = x + \gamma x^2$$
$$+\sum_{n=2}^{\infty}\left(\frac{\gamma^n}{n!}+\sum_{k=0}^{n-2}\frac{(-1)^{n-k}\gamma^k}{k!}\sum_{\substack{k_1+\cdots+k_r=n-k\\ r\geq 1,\forall k_i\geq 2}}(-1)^r\frac{(k_1-1)\cdots(k_r-1)}{k_1!\cdots k_r!}\zeta(k_1,\ldots,k_r)\right)x^{n+1}.$$

2 Multiple zeta values

In this section, we recall basic facts on the multiple zeta values. The multiple zeta value (MZV) and the multiple zeta star value (MZSV) are real numbers given by the nested series

$$\zeta(k_1,\ldots,k_r) := \sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}},$$
$$\zeta^\star(k_1,\ldots,k_r) := \sum_{0<m_1\leq\cdots\leq m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$$

for each index set $\mathbf{k}=(k_1,\ldots,k_r)$ of positive integers k_i, with the last entry $k_r\geq 2$ for convergence. The quantities $\mathrm{wt}(\mathbf{k}) := k_1+\cdots+k_r$, $\mathrm{dep}(\mathbf{k}) := r$, and $\mathrm{ht}(\mathbf{k}) := \#\{i\,|\,k_i\geq 2, 1\leq i\leq r\}$ are called respectively the weight, the depth, and the height of the index set $\mathbf{k}$ (or of the multiple zeta value $\zeta(\mathbf{k})=\zeta(k_1,\ldots,k_r)$). A list of MZVs in low weights is given in the table below.

Table 1: MZVs in low weights

	wt = 2	wt = 3	wt = 4	wt = 5
dep = 1	$\zeta(2)$	$\zeta(3)$	$\zeta(4)$	$\zeta(5)$
dep = 2		$\zeta(1,2)$	$\zeta(1,3),\zeta(2,2)$	$\zeta(1,4),\zeta(2,3),\zeta(3,2)$
dep = 3			$\zeta(1,1,2)$	$\zeta(1,1,3),\zeta(1,2,2),\zeta(2,1,2)$
dep = 4				$\zeta(1,1,1,2)$

	wt = 6
dep = 1	$\zeta(6)$
dep = 2	$\zeta(1,5),\zeta(2,4),\zeta(3,3),\zeta(4,2)$
dep = 3	$\zeta(1,1,4),\zeta(1,2,3),\zeta(1,3,2),\zeta(2,1,3),\zeta(2,2,2),\zeta(3,1,2)$
dep = 4	$\zeta(1,1,1,3),\zeta(1,1,2,2),\zeta(1,2,1,2),\zeta(2,1,1,2)$
dep = 5	$\zeta(1,1,1,1,2)$

We note that the number of MZVs of weight k and depth r is $\binom{k-2}{r-1}$, and of weight k is 2^{k-2}.

We shall also need the following multiple zeta function of one variable:

$$\zeta(k_1, \dots, k_{r-1}, k_r + s) := \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} m_r^{k_r+s}}.$$

It is shown in [2] that the function $\zeta(k_1, \dots, k_{r-1}, k_r+s)$ can be meromorphically continued to the whole s-plane and has a pole at $s = 0$ when $k_r = 1$. We need the description of the principal part at $s = 0$ in terms of regularized polynomials, which we explain in subsection 2.3.

Further we mention a conjecture for MZVs by Zagier [33]. For each $k \geq 0$, we define the $\mathbb{Q}$-vector space $\mathcal{Z}_k$ by

$$\mathcal{Z}_0 := \mathbb{Q}, \quad \mathcal{Z}_1 := \{0\},$$
$$\mathcal{Z}_k := \sum_{\substack{k_1+\cdots+k_r=k \\ r\geq 1, k_i\in\mathbb{N}, k_r\geq 2}} \mathbb{Q} \cdot \zeta(k_1, \dots, k_r) \quad (k \geq 2).$$

In 1994, Zagier gave a conjecture for the dimension of $\mathcal{Z}_k$. Let a sequence $\{d_k\}_{k\geq 0}$ be defined recursively by

$$d_0 = 1, d_1 = 0, d_2 = 1, d_k = d_{k-2} + d_{k-3} \ (k \geq 3).$$

Conjecture 2.1 (Zagier [33]). *For any integer $k \geq 0$, the equality*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k$$

holds.

Table 2: d_k

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
d_k	1	0	1	1	1	2	2	3	4	5	7	9	12	16	21	28
2^{k-2}	–	–	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192

That the number d_k gives an upper bound of the dimension of $\mathcal{Z}_k$ is proved by Goncharov, Terasoma and Deligne–Goncharov.

Theorem 2.2 (Goncharov [8], Terasoma [31], Deligne–Goncharov [6]). *For any integer k, the inequality*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$$

holds.

Also we define the $\mathbb{Q}$-vector space $\mathcal{Z}$ generated by all MZVs:

$$\mathcal{Z} := \sum_{k=0}^{\infty} \mathcal{Z}_k.$$

The vector space $\mathcal{Z}$ becomes a $\mathbb{Q}$-algebra. Moreover, $\mathcal{Z}$ has two product rules, which are called the harmonic (stuffle) product and the shuffle product. The product of MZVs with weights k and k' is a linear combination of MZVs with weight $k+k'$, i.e., $\mathcal{Z}_k \cdot \mathcal{Z}_{k'} \subseteq \mathcal{Z}_{k+k'}$. In 1997, Michael Hoffman made the following conjecture [11]. This conjecture was proved by Francis Brown [5].

Theorem 2.3 (Brown [5]). *The space $\mathcal{Z}$ is spanned by $\zeta(k_1, \ldots, k_r)$'s with $k_i \in \{2, 3\}$.*

2.1 Algebraic setup

We recall the algebraic setup of MZVs that was introduced by Hoffman [11] with a slightly different convention. Let $\mathscr{R}$ be the $\mathbb{Q}$-vector space

$$\mathscr{R} = \bigoplus_{r=0}^{\infty} \mathbb{Q}[\mathbb{N}^r]$$

spanned by a finite formal $\mathbb{Q}$-linear combination of symbols $[\mathbf{k}] = [k_1, \ldots, k_r]$ with $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$ for some r. We understand $\mathbb{Q}[\mathbb{N}^0] = \mathbb{Q}[\emptyset]$ for $r = 0$. Further let $\mathscr{R}^0$ denote the subspace of $\mathscr{R}$ spanned by the *admissible* symbols, i.e., by $[\emptyset]$ and the symbols $[k_1, \ldots, k_r]$ with $k_r \geq 2$. First, we introduce the *harmonic* (stuffle) product on $\mathscr{R}$. We consider the $\mathbb{Q}$-bilinear product $*$ which is defined inductively as:

1. for any index $\mathbf{k}$, $[\emptyset] * [\mathbf{k}] = [\mathbf{k}] * [\emptyset] = [\mathbf{k}]$;
2. for any indices $\mathbf{k} = (k_1, \ldots, k_r)$ and $\mathbf{l} = (l_1, \ldots, l_s)$ with $r, s \geq 1$,

$$[\mathbf{k}] * [\mathbf{l}] = [[\mathbf{k}_-] * [\mathbf{l}], k_r] + [[\mathbf{k}] * [\mathbf{l}_-], l_s] + [[\mathbf{k}_-] * [\mathbf{l}_-], k_r + l_s],$$

where $\mathbf{k}_- = (k_1, \ldots, k_{r-1}), \mathbf{l}_- = (l_1, \ldots, l_{s-1})$. Hoffman proved that $\mathscr{R}_* := (\mathscr{R}, *)$ is a commutative and associative $\mathbb{Q}$-algebra and that $\mathscr{R}^0_*$ is a subalgebra of $\mathscr{R}_*$ [11]. Moreover, he proved that the evaluation map $\zeta : \mathscr{R}^0_* \ni [k_1, \ldots, k_r] \mapsto \zeta(k_1, \ldots, k_r) \in \mathbb{R}$, being extended $\mathbb{Q}$-linearly, is an algebra homomorphism from $\mathscr{R}^0_*$ to $\mathbb{R}$, i.e., we have

$$\zeta([\mathbf{k}] * [\mathbf{l}]) = \zeta(\mathbf{k})\zeta(\mathbf{l}) \tag{1}$$

for any $\mathbf{k}, \mathbf{l} \in \mathscr{R}^0$.

Secondly, we introduce the *shuffle* product. Let $\mathfrak{H} := \mathbb{Q}\langle e_0, e_1 \rangle$ be the noncommutative polynomial ring in two indeterminates e_0, e_1, and $\mathfrak{H}^1$ (resp. $\mathfrak{H}^0$) its subring $\mathbb{Q} + e_1\mathfrak{H}$ (resp. $\mathbb{Q}+e_1\mathfrak{H}e_0$). On $\mathfrak{H}$, we consider the $\mathbb{Q}$-bilinear shuffle product $\sqcup\!\sqcup$ which is defined inductively as:

1. for any words w, $1 ш w = w ш 1 = w$;
2. for any words $u_1, u_2 \in \{e_0, e_1\}$ and $w_1, w_2 \in \mathfrak{H}$,

$$u_1w_1 ш u_2w_2 = u_1(w_1 ш u_2w_2) + u_2(u_1w_1 ш w_2).$$

We set $e_k := e_1e_0^{k-1}$ ($k \in \mathbb{N}$). Then $\mathfrak{H}^1$ is freely generated by $\{e_k\}_{k\geq 1}$. As a $\mathbb{Q}$-vector space, $\mathfrak{H}^1$ is identified with $\mathscr{R}$ under the correspondence $[k_1,\ldots,k_r] \leftrightarrow e_{k_1}e_{k_2}\cdots e_{k_r}$. We define ш on $\mathscr{R}$ by using the identification $[k_1,\ldots,k_r] \leftrightarrow e_{k_1}e_{k_2}\cdots e_{k_r}$. For instance, $[2] ш [2] = 2[2,2] + 4[1,3]$ because $[2] \leftrightarrow e_1e_0$ and $e_1e_0 ш e_1e_0 = 2e_1e_0e_1e_0 + 4e_1e_1e_0e_0$. Then $\mathscr{R}_ш := (\mathscr{R}, ш)$ is also a commutative and associative $\mathbb{Q}$-algebra and $\mathscr{R}^0_ш$ becomes a subalgebra of $\mathscr{R}_ш$. Further the evaluation map $\zeta : \mathscr{R}^0_ш \ni [k_1,\ldots,k_r] \mapsto \zeta(k_1,\ldots,k_r) \in \mathbb{R}$, being extended $\mathbb{Q}$-linearly, is an algebra homomorphism from $\mathscr{R}^0_ш$ to $\mathbb{R}$, i.e. we have

$$\zeta([\mathbf{k}] ш [\mathbf{l}]) = \zeta(\mathbf{k})\zeta(\mathbf{l}) \tag{2}$$

for any $\mathbf{k}, \mathbf{l} \in \mathscr{R}^0$. By (1) and (2), we have

$$\zeta([\mathbf{k}] * [\mathbf{l}]) = \zeta(\mathbf{k})\zeta(\mathbf{l}) = \zeta([\mathbf{k}] ш [\mathbf{l}])$$

for any $\mathbf{k}, \mathbf{l} \in \mathscr{R}^0$. This is called the *finite double shuffle relation.*

Example 2.4. When $\mathbf{k} = (2), \mathbf{l} = (2)$, we calculate $\zeta(2)\cdot\zeta(2)$ by harmonic product $*$ and shuffle product ш respectively:

$$\zeta([2] * [2]) = 2\zeta(2,2) + \zeta(4),$$
$$\zeta([2] ш [2]) = 2\zeta(2,2) + 4\zeta(1,3).$$

Then we get

$$\zeta(1,3) = \frac{1}{4}\zeta(4).$$

2.2 Some relations for the MZVs

We introduce some relations among MZVs. The MZV is expressed by an iterated integral.

Proposition 2.5 (Iterated integral expression)**.** *For any index $\mathbf{k} = (k_1,\ldots,k_r)$ with $k_r \geq 2$,*

$$\zeta(k_1,\ldots,k_r) = \int\limits_{0<t_1<\cdots<t_k<1} \frac{dt_1}{1-t_1}\underbrace{\frac{dt_2}{t_2}\cdots\frac{dt_{k_1}}{t_{k_1}}}_{k_1-1}\frac{dt_{k_1+1}}{1-t_{k_1+1}}\underbrace{\frac{dt_{k_1+2}}{t_{k_1+2}}\cdots\frac{dt_{k_1+k_2}}{t_{k_1+k_2}}}_{k_2-1}$$
$$\cdots\frac{dt_{k-k_r+1}}{1-t_{k-k_r+1}}\underbrace{\frac{dt_{k-k_r+2}}{t_{k-k_r+2}}\cdots\frac{dt_k}{t_k}}_{k_r-1},$$

where $k = k_1 + \cdots + k_r$.

Example 2.6. When $\mathbf{k}=(3,2)$,

$$\begin{aligned}
\int_{0<t_1<\cdots<t_5<1} \frac{dt_1}{1-t_1}\underbrace{\frac{dt_2}{t_2}\frac{dt_3}{t_3}}_{3-1}\frac{dt_4}{1-t_4}\underbrace{\frac{dt_5}{t_5}}_{2-1} &= \int_{0<t_2<\cdots<t_5<1}\left(\int_0^{t_2}\frac{dt_1}{1-t_1}\right)\frac{dt_2}{t_2}\frac{dt_3}{t_3}\frac{dt_4}{1-t_4}\frac{dt_5}{t_5}\\
&= \int_{0<t_2<\cdots<t_5<1}\left(\sum_{m=1}^{\infty}\frac{t_2^m}{m}\right)\frac{dt_2}{t_2}\frac{dt_3}{t_3}\frac{dt_4}{1-t_4}\frac{dt_5}{t_5}\\
&= \int_{0<t_4<t_5<1}\left(\sum_{m=1}^{\infty}\frac{t_4^m}{m^3}\right)\frac{dt_4}{1-t_4}\frac{dt_5}{t_5}\\
&= \int_{0<t_4<t_5<1}\left(\sum_{m=1}^{\infty}\frac{t_4^m}{m^3}\right)\sum_{n=1}^{\infty}t_4^{n-1}dt_4\frac{dt_5}{t_5}\\
&= \int_{0<t_4<t_5<1}\left(\sum_{m,n=1}^{\infty}\frac{t_4^{m+n-1}}{m^3}dt_4\right)\frac{dt_5}{t_5}\\
&= \int_{0<t_5<1}\left(\sum_{0<m<n}^{\infty}\frac{t_5^n}{m^3n}\right)\frac{dt_5}{t_5}\\
&= \zeta(3,2).
\end{aligned}$$

Let $\mathbf{k}=(k_1,\dots,k_r)$ be an index set with $k_r\geq 2$. We write

$$\mathbf{k}=(\underbrace{1,\dots,1}_{a_1-1},b_1+1,\dots,\underbrace{1,\dots,1}_{a_s-1},b_s+1)$$

with $a_p,b_q\geq 1$. Then, we define the dual index set of $\mathbf{k}$ as

$$\mathbf{k}^*=(\underbrace{1,\dots,1}_{b_s-1},a_s+1,\dots,\underbrace{1,\dots,1}_{b_1-1},a_1+1).$$

The following result, which is a direct consequence of the iterated integral expression, provides the so-called *duality theorem* for the MZVs.

Theorem 2.7 (Duality theorem). *For $\mathbf{k}=(k_1,\dots,k_r)$ with $k_r\geq 2$, we have*

$$\zeta(\mathbf{k})=\zeta(\mathbf{k}^*). \tag{3}$$

The sum of all MZVs with fixed weight and depth is equal to the Riemann zeta value of that weight. This identity is called the *sum formula* conjectured by Courtney Moen and Hoffman, and proved by Andrew Granville, Zagier and others.

Theorem 2.8 (Sum formula). *For $0<r<k$, we have*

$$\sum_{\substack{k_1+\cdots+k_r=k\\ {}^\forall k_i\geq 1,\, k_r\geq 2}}\zeta(k_1,\dots,k_r)=\zeta(k). \tag{4}$$

In 1992, Hoffman proved the following identity [10].

Theorem 2.9 (Hoffman's relation)**.** *For any index* $\mathbf{k} = (k_1, \ldots, k_r)$ *with* $k_r \geq 2$*, we have*

$$\sum_{i=1}^{r} \zeta(k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_r) = \sum_{\substack{1 \leq l \leq r \\ k_l \geq 2}} \sum_{j=1}^{k_l - 1} \zeta(k_1, \ldots, k_{l-1}, k_l - j, j + 1, k_{l+1}, \ldots, k_r). \tag{5}$$

Yasuo Ohno introduced a generalization of all of the duality theorem and the sum formula and Hoffman's relation in [24].

Theorem 2.10 (Ohno's relation)**.** *For* $\mathbf{k} = (k_1, \ldots, k_r)$ *with* $k_r \geq 2$ *and* $m \in \mathbb{Z}_{\geq 0}$*, we have*

$$\sum_{\substack{\varepsilon_1 + \cdots + \varepsilon_r = m \\ {}^\forall \varepsilon_i \geq 0}} \zeta(k_1 + \varepsilon_1, k_2 + \varepsilon_2, \ldots, k_r + \varepsilon_r) = \sum_{\substack{\varepsilon'_1 + \cdots + \varepsilon'_{r'} = m \\ {}^\forall \varepsilon'_i \geq 0}} \zeta(k'_1 + \varepsilon'_1, k'_2 + \varepsilon'_2, \cdots, k'_{r'} + \varepsilon'_{r'}), \tag{6}$$

where $(k'_1, \ldots, k'_{r'})$ *is the dual of* $\mathbf{k}$*.*

When $m = 0$ in (6), we get the duality theorem (3). When $m = 1$, (6) and the duality theorem give Hoffman's relation (5). Specializing (6) to $\mathbf{k} = (\underbrace{1, \ldots, 1}_{r-1}, 2)$ and $m = k - r - 1$, we obtain the sum formula (4).

In 1997, Tu Quoc Thang Le and Jun Murakami proved the following theorem by using the theory of knot invariants [19].

Theorem 2.11 (Le-Murakami's relation)**.** *For any integers* k *and* s *with* $1 \leq s \leq k$*, we have*

$$\sum_{\substack{\mathrm{wt}(\mathbf{k}) = 2k \\ \mathrm{ht}(\mathbf{k}) = s}} (-1)^{\mathrm{dep}(\mathbf{k})} \zeta(\mathbf{k}) = \frac{(-1)^k}{(2k+1)!} \sum_{i=0}^{k-s} \binom{2k+1}{2i} (2 - 2^{2i}) B_{2i} \pi^{2k}, \tag{7}$$

where B_n *is the* n*-th Bernoulli number.*

Note that the right-hand side is expressed as the product of the MZVs and the MZSVs:

$$\sum_{\substack{\mathrm{wt}(\mathbf{k}) = 2k \\ \mathrm{ht}(\mathbf{k}) = s}} (-1)^{\mathrm{dep}(\mathbf{k})} \zeta(\mathbf{k}) = (-1)^k \sum_{i=0}^{k-s} (-1)^i \zeta^\star(\underbrace{2, \ldots, 2}_{i}) \zeta(\underbrace{2, \ldots, 2}_{k-i}),$$

because

$$\zeta(\underbrace{2, \ldots, 2}_{r}) = \frac{\pi^{2r}}{(2r+1)!} \quad \text{and} \quad \zeta^\star(\underbrace{2, \ldots, 2}_{r}) = (-1)^r \frac{(2 - 2^{2r}) B_{2r}}{(2r)!} \pi^{2r}.$$

(cf. [20], [34])

The generating function of the MZVs with fixed weight, depth and height is given by Ohno and Zagier [25]. Let $I(k,r,s)$ be the set of indices $\mathbf{k}$ of weight k, depth r, height s, and $I_0(k,r,s)$ be the subset of *admissible* indices, i.e., indices with the extra requirement that $k_r \geq 2$. The set $I_0(k,r,s)$ is non-empty only if the indices k, n and s satisfy the inequality $s \geq 1$, $r \geq s$, and $k \geq r+s$.

Theorem 2.12 (Ohno–Zagier [25]). *The generating function of MZVs of fixed weight, depth and height is given by*

$$\begin{aligned}\sum_{k,r,s\geq 0}\left(\sum_{\mathbf{k}\in I_0(k,r,s)}\zeta(\mathbf{k})\right)x^{k-r-s}y^{r-s}z^{s-1}\\ =\frac{1}{xy-z}\left(1-\exp\left(\sum_{n=2}^{\infty}\frac{\zeta(n)}{n}(x^n+y^n-\alpha^n-\beta^n)\right)\right),\end{aligned} \tag{8}$$

where α and β are $\alpha+\beta=x+y$, $\alpha\beta=z$.

Specializing (8) to $z=xy$, we have the sum formula (4). If $y=-x$ in (8), then we obtain Le-Murakami's relation (7). Specializing (8) to $z=0$, we get a formula given by Kazuhiko Aomoto and Vladimir Gershonovich Drinfel'd [1, 7]:

$$1-\sum_{r,k\geq 1}\zeta(\underbrace{1,\ldots,1}_{r-1},k+1)x^ry^k=\frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}=\exp\left(\sum_{n=2}^{\infty}\frac{\zeta(n)}{n}\left(x^n+y^n-(x+y)^n\right)\right).$$

Next we explain the derivation relation for the MZVs. A derivation ∂ on $\mathfrak{H}$ is a $\mathbb{Q}$-linear endomorphism of $\mathfrak{H}$ satisfying Leibniz's rule $\partial(ww')=\partial(w)w'+w\partial(w')$. Such a derivation is uniquely determined by its images of generators e_0 and e_1. For each $l\in\mathbb{N}$, the derivation ∂_l on $\mathfrak{H}$ is defined by $\partial_l(e_0):=e_1(e_0+e_1)^{l-1}e_0$ and $\partial_l(e_1):=-e_1(e_0+e_1)^{l-1}e_0$. We note that $\partial_l(1)=0$ and $\partial_l(e_0+e_1)=0$. We define the $\mathbb{Q}$-linear map $Z:\mathfrak{H}^0\to\mathbb{R}$ by $Z(1):=1$ and $Z(e_{k_1}\cdots e_{k_r}):=\zeta(k_1,\ldots,k_r)$. Kentaro Ihara, Kaneko and Zagier proved the derivation relations for the MZVs [13].

Theorem 2.13 (Derivation relation). *For $l\in\mathbb{N}$, we have*

$$Z(\partial_l(w))=0\quad(w\in\mathfrak{H}^0).$$

This relation is a generalization of Hoffman's relation (5) which is equivalent to the case $l=1$. Also it is known that Ohno's relation is equivalent to the union of the duality formula and the derivation relation.

2.3 Regularization of divergent MZVs

We review the theory of regularization of MZVs. For any index $\mathbf{k}$, we may write

$$[\mathbf{k}] = \sum_{i=0}^{p} a_i * [1]^{*i} \ \in \mathscr{R}_*^0[[1]],$$
$$[\mathbf{k}] = \sum_{j=0}^{q} b_j ш [1]^{шj} \ \in \mathscr{R}_ш^0[[1]]$$

with $a_i \in \mathscr{R}_*^0$ and $b_j \in \mathscr{R}_ш^0$ ($[1]^{*i} = \underbrace{[1] * \cdots * [1]}_{i}$ and $[1]^{шj} = \underbrace{[1] ш \cdots ш [1]}_{j}$), because of the isomorphisms $\mathscr{R}_* \cong \mathscr{R}_*^0[[1]]$ and $\mathscr{R}_ш \cong \mathscr{R}_ш^0[[1]]$. And define $\zeta_*(\mathbf{k}; T)$ and $\zeta_ш(\mathbf{k}; T)$ in $\mathbb{R}[T]$ by

$$\zeta_*(\mathbf{k}; T) = \sum_{i=0}^{p} \zeta(a_i) T^i \quad \text{and} \quad \zeta_ш(\mathbf{k}; T) = \sum_{j=0}^{q} \zeta(b_j) T^j.$$

These are $\mathbb{Q}$-algebra homomorphisms $\zeta_* : \mathscr{R}_* \to \mathbb{R}[T]$ and $\zeta_ш : \mathscr{R}_ш \to \mathbb{R}[T]$ from $\mathscr{R}$ to the polynomial algebra $\mathbb{R}[T]$ extending ζ and satisfying $\zeta_*([1]; T) = \zeta_ш([1]; T) = T$. Also these are the polynomials in $\mathbb{R}[T]$ uniquely characterized by the asymptotics

$$\sum_{0<m_1<\cdots<m_r<M} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} = \zeta_*(\mathbf{k}; \log M + \gamma) + O(M^{-\varepsilon}) \quad \text{as } M \to \infty \text{ for some } \varepsilon > 0$$

(γ is Euler's constant) and

$$\mathrm{Li}_{k_1,\ldots,k_r}(x) = \zeta_ш(\mathbf{k}; -\log(1-x)) + O((1-x)^{\varepsilon}) \quad \text{as } x \to 1 \text{ for some } \varepsilon > 0,$$

where $\mathrm{Li}_{k_1,\ldots,k_r}(x)$ is the multiple polylogarithm function defined by

$$\mathrm{Li}_{k_1,\ldots,k_r}(x) = \sum_{0<m_1<\cdots<m_r} \frac{x^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}}.$$

The fundamental theorem of regularizations of MZVs claims that the two polynomials $\zeta_*(\mathbf{k}; T)$ and $\zeta_ш(\mathbf{k}; T)$ are related with each other by an $\mathbb{R}$-linear map coming from the Taylor series of the gamma function $\Gamma(x)$. Define an $\mathbb{R}$-linear endomorphism ρ on $\mathbb{R}[T]$ by the equality

$$\rho(e^{Tx}) = A(x) e^{Tx}$$

in the formal power series algebra $\mathbb{R}[T][[x]]$, on which ρ acts coefficientwise, where

$$A(x) = \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) x^n \right) \in \mathbb{R}[[x]].$$

Note that $A(x) = e^{\gamma x}\Gamma(1+x)$. We will discuss the Taylor series of $A(x)^{-1}$ in Section 7. Ihara, Kaneko and Zagier proved the following theorem in [13].

Theorem 2.14 (Fundamental theorem of regularization)**.** *For any index* $\mathbf{k}$*, we have*

$$\zeta_{ш}(\mathbf{k};T) = \rho(\zeta_*(\mathbf{k};T)).$$

We denote the constant term $\zeta_{ш}(\mathbf{k};0)$ of the shuffle-regularized polynomial $\zeta_{ш}(\mathbf{k};T)$ by $\zeta_{ш}(\mathbf{k})$ and call it the shuffle-regularized value of (possibly divergent) $\zeta(\mathbf{k})$. If $\mathbf{k}$ is of the form $\mathbf{k} = (k_1,\dots,k_n,\underbrace{1,\dots,1}_{m})$ with $k_n \geq 2, m \geq 0$, then both $\zeta_{ш}(\mathbf{k};T)$ and $\zeta_*(\mathbf{k};T)$ are of degree m and each coefficient of T^i is a linear combination of multiple zeta values of weight $k_1+\cdots+k_n+m-i$. If $m=0$ (and so $n=r$), then $\zeta_{ш}(\mathbf{k};T) = \zeta_*(\mathbf{k};T) = \zeta_{ш}(\mathbf{k};0) = \zeta_*(\mathbf{k};0) = \zeta(k_1,\dots,k_n)$.

Now write

$$\zeta_*(\mathbf{k};T) = \sum_{i=0}^{m} c_i(\mathbf{k})\frac{(T-\gamma)^i}{i!} \quad \text{and} \quad \zeta_{ш}(\mathbf{k};T) = \sum_{i=0}^{m} c_i'(\mathbf{k})\frac{T^i}{i!}.$$

Then, as was shown in [3], the principal parts at $s=0$ of $\zeta(k_1,\dots,k_{r-1},k_r+s)$ and $\Gamma(s+1)\zeta(k_1,\dots,k_{r-1},k_r+s)$ are given respectively by

$$\zeta(k_1,\dots,k_{r-1},k_r+s) = \sum_{i=0}^{m}\frac{c_i(\mathbf{k})}{s^i} + O(s) \quad (s\to 0) \tag{9}$$

and

$$\Gamma(s+1)\zeta(k_1,\dots,k_{r-1},k_r+s) = \sum_{i=0}^{m}\frac{c_i'(\mathbf{k})}{s^i} + O(s) \quad (s\to 0). \tag{10}$$

As a corollary, we obtain the following result which will be needed later.

Proposition 2.15. *For any index* $\mathbf{k} = (k_1,\dots,k_r)$*, the constant terms of Laurent series of* $\zeta(k_1,\dots,k_{r-1},k_r+s)$ *and* $\Gamma(s+1)\zeta(k_1,\dots,k_{r-1},k_r+s)$ *are given respectively by* $\zeta_*(\mathbf{k};\gamma)$ *and* $\zeta_{ш}(\mathbf{k})$*.*

3 The Arakawa-Kaneko zeta function

In this section, we review the Arakawa-Kaneko zeta function which will be used in the proof of Theorem 4.1. In 1999, Arakawa and Kaneko introduced a generalization of the Riemann zeta function by using the polylogarithm. The Arakawa-Kaneko zeta function $\xi_k(s)$ is defined by

$$\xi_k(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}dt$$

for $k \geq 1$, where $\mathrm{Li}_k(z)$ is the polylogarithm function $\sum_{n=1}^\infty z^n/n^k$. The integral converges for $\Re(s) > 0$. When $k=1$, $\xi_1(s)$ is equal to $s\zeta(s+1)$. The Arakawa-Kaneko zeta function is related to poly-Bernoulli numbers and to multiple zeta functions. For any integers k, poly-Bernoulli numbers $B_n^{(k)}$ and $C_n^{(k)}$ are defined by the generating functions

$$\sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \quad \text{and} \quad \sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}.$$

Theorem 3.1 (Arakawa–Kaneko [2]). *(i) The function $\xi_k(s)$ continues to an entire function of s, and the special values at non-positive integers are given by*

$$\xi_k(-m) = (-1)^m C_m^{(k)} = \sum_{l=0}^{m} (-1)^l \binom{m}{l} B_l^{(k)} \quad (m \in \mathbb{Z}_{\geq 0}).$$

(ii) The function $\xi_k(s)$ can be written in terms of the zeta functions $\zeta(k_1, \dots, k_l, s)$ as

$$\xi_k(s) = (-1)^{k-1} \left\{ \sum_{i=0}^{k-2} \zeta(\underbrace{1, \dots, 1}_{i}, 2, \underbrace{1, \dots, 1}_{k-2-i}, s) + s \cdot \zeta(\underbrace{1, \dots, 1}_{k-1}, s+1) \right\}$$
$$+ \sum_{j=0}^{k-2} (-1)^j \zeta(k-j) \cdot \zeta(\underbrace{1, \dots, 1}_{j}, s).$$

The special values at positive integers of the function $\xi_k(s)$ are expressed in terms of MZVs.

Theorem 3.2 (Arakawa–Kaneko [2]). *For $k \geq 1$ and $m \geq 0$,*

$$\xi_k(m+1) = \sum_{\substack{a_1+\cdots+a_k=m \\ \forall a_j \geq 0}} (a_k+1) \zeta(a_1+1, \dots, a_{k-1}+1, a_k+2). \tag{11}$$

Ohno proved that (11) is equal to the MZSV of height one.

Theorem 3.3 (Ohno [24]). *For $k \geq 1$ and $m \geq 0$,*

$$\xi_k(m+1) = \zeta^\star(\underbrace{1, \dots, 1}_{m-1}, k+1).$$

Also Arakawa and Kaneko studied a generalization of the function $\xi_k(s)$. The generalized Arakawa-Kaneko zeta function $\xi(k_1, \dots, k_r; s)$ is defined by

$$\xi(k_1, \dots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{Li}_{k_1, \dots, k_r}(1-e^{-t})}{e^t-1} dt \tag{12}$$

for any integers $r \geq 1$ and $k_1, \dots, k_r \geq 1$. This function is absolutely convergent for $\Re(s) > 0$. When $r = 1$, $\xi(k; s)$ equals $\xi_k(s)$. The function $\xi(k_1, \dots, k_r; s)$ also relates with multi-poly-Bernoulli numbers and multiple zeta values [17].

Theorem 3.4 (Arakawa–Kaneko [2]). *For $r, k \geq 1$, we have*

$$\xi(\underbrace{1,\ldots,1}_{k-1}, r; s+1) = (-1)^{r-1} \sum_{\substack{a_1+\cdots+a_r=k \\ \forall a_p \geq 0}} \binom{s+a_r}{a_r} \zeta(a_1+1,\ldots,a_{r-1}+1, a_r+1+s)$$
$$+ \sum_{i=0}^{r-2} (-1)^i \zeta(\underbrace{1,\ldots,1}_{k-1}, r-i) \zeta(\underbrace{1,\ldots,1}_{i}, 1+s). \tag{13}$$

The special values at positive arguments of $\xi(\underbrace{1,\ldots,1}_{k-1}, r; s+1)$ are expressed by multiple zeta values.

Theorem 3.5 (Arakawa–Kaneko [2]). *Let $m \geq 0, r \geq 1$, and $k \geq 1$ be integers. Then*

$$\xi(\underbrace{1,\ldots,1}_{k-1}, r; m+1)$$
$$= \sum_{\substack{a_1+\cdots+a_r=m \\ \forall a_j \geq 0}} \binom{a_r+k}{k} \zeta(a_1+1,\ldots,a_{r-1}+1, a_r+k+1). \tag{14}$$

4 Main results

We now present our main results, all of which are identities involving multiple zeta values of maximal height (no component of the index is one).

Theorem 4.1 (Kaneko–Sakata [16]). *For any integers $r, k \geq 1$, we have*

$$\zeta(\underbrace{1,\ldots,1}_{r-1}, k+1) = \sum_{j=1}^{\min(r,k)} (-1)^{j-1} \sum_{\substack{\mathrm{wt}(\mathbf{a})=k,\, \mathrm{wt}(\mathbf{b})=r \\ \mathrm{dep}(\mathbf{a})=\mathrm{dep}(\mathbf{b})=j}} \zeta(\mathbf{a}+\mathbf{b}), \tag{15}$$

where, for two indices $\mathbf{a} = (a_1, \ldots, a_j)$ and $\mathbf{b} = (b_1, \ldots, b_j)$ of the same depth, we write $\zeta(\mathbf{a}+\mathbf{b})$ for $\zeta(a_1+b_1, \ldots, a_j+b_j)$.

Note that the right-hand side of this formula is symmetric in r and k, and thus the formula makes the duality $\zeta(\underbrace{1,\ldots,1}_{r-1}, k+1) = \zeta(\underbrace{1,\ldots,1}_{k-1}, r+1)$ visible. (N.B. We use the duality in our proof, so that we are not giving an alternative proof of the duality.) To our knowledge, no such symmetric explicit formula for the height-one MZV has been known, except for the well-known symmetric generating function [1, 7]:

$$1 - \sum_{r,k \geq 1} \zeta(\underbrace{1,\ldots,1}_{r-1}, k+1) x^r y^k = \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} = \exp\left(\sum_{n=2}^{\infty} \frac{\zeta(n)}{n} (x^n + y^n - (x+y)^n) \right).$$

Also, we should remark that the right-hand side of the theorem is symmetric with respect to any permutations of the arguments, so that the theorem of Hoffman [10, Theorem 2.2] ensures the right-hand side is a polynomial in the Riemann zeta values $\zeta(n)$, the fact also can be seen from the generating function above. Moreover, we note that all the MZVs appearing on the right-hand side is of maximal height.

As a final remark, the case of $r = 2$ gives nothing but the sum formula for depth 2 ($r = 1$ gives the trivial identity $\zeta(k+2) = \zeta(k+2)$). It was Hirofumi Tsumura who first remarked that we could obtain the depth 2 case of the sum formula (4) if we looked at the behavior at $s = 0$ of the identity (13) for $r = 2$. The proof of Theorem 4.1 is given in Section 5

Further we give a generalization of Theorem 4.1. For two indices $\mathbf{k}, \mathbf{k}'$, we say $\mathbf{k}'$ refines $\mathbf{k}$ (denoted $\mathbf{k}' \succeq \mathbf{k}$) if $\mathbf{k}$ can be obtained from $\mathbf{k}'$ by combining some of its adjacent parts. For example, $(1,2,3,4) \succeq (1+2, 3+4) = (3,7)$.

Theorem 4.2 (Murahara–Sakata [23])**.** *For $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$ and $r \in \mathbb{N}$ with $r \geq d$, we have*

$$\sum_{\substack{r_1+\cdots+r_d=r \\ \forall r_i \geq 1}} \zeta(\underbrace{1,\dots,1}_{r_1-1}, k_1+1, \dots, \underbrace{1,\dots,1}_{r_d-1}, k_d+1) = \sum_{\substack{\mathbf{k}' \succeq \mathbf{k} \\ \mathrm{dep}(\mathbf{k}') \leq r}} \sum_{\substack{\mathrm{wt}(\mathbf{r})=r \\ \mathrm{dep}(\mathbf{r})=\mathrm{dep}(\mathbf{k}')}} (-1)^{\mathrm{dep}(\mathbf{k}')-d} \zeta(\mathbf{k}'+\mathbf{r}).$$

The case $d = 1$ of Theorem 4.2 gives Theorem 4.1. In Section 5, we will give two different proofs of Theorem 4.2. The first proof is based on the duality theorem and Ohno's relations, and the second is on the derivation relations. In Section 6, we will also prove a counterpart of this theorem for the finite multiple zeta values.

Example 4.3. When $k = 3, r = 4$ in Theorem 4.1, or $\mathbf{k} = (3), r = 4$ in Theorem 4.2, we have

$$\begin{aligned}\zeta(1,1,1,4) = \zeta(7) &- (\zeta(5,2) + 2\zeta(4,3) + 2\zeta(3,4) + \zeta(2,5)) \\ &+ \zeta(3,2,2) + \zeta(2,3,2) + \zeta(2,2,3).\end{aligned}$$

Example 4.4. When $\mathbf{k} = (3,2), r = 4$ in Theorem 4.2, we have

$$\begin{aligned}&\zeta(4,1,1,3) + \zeta(1,4,1,3) + \zeta(1,1,4,3) \\ &= \zeta(6,3) + \zeta(5,4) + \zeta(4,5) \\ &\quad - (\zeta(5,2,2) + \zeta(4,3,2) + 2\zeta(4,2,3) + 2\zeta(3,3,3) + \zeta(3,2,4) + \zeta(2,4,3) + \zeta(2,3,4)) \\ &\quad + \zeta(3,2,2,2) + \zeta(2,3,2,2) + \zeta(2,2,2,3).\end{aligned}$$

Recall the classical sum formula states that the sum of all MZVs of fixed weight and depth is equal to the Riemann zeta value of that weight (Theorem 2.8). If we extend the sum to include non-convergent MZVs with the shuffle regularization, the result will be the height-one MZV (up to sign).

Theorem 4.5 (Shuffle-regularized sum formula, [16])**.** *For any integers $r, k \geq 1$, we have*

$$\sum_{\substack{\mathrm{wt}(\mathbf{k})=r+k \\ \mathrm{dep}(\mathbf{k})=r}} \zeta_{ш}(\mathbf{k}) = (-1)^{r-1} \zeta(\underbrace{1,\dots,1}_{r-1}, k+1),$$

where $\zeta_{ш}(\mathbf{k})$ is the shuffle regularized value which was recalled in subsection 2.3.

Although this theorem may be well known, our proof seems to be new and hopefully be of some interest. We do not know if there exists any nice stuffle-regularized sum formula.

Remark 4.6. Shuji Yamamoto also obtained a generalization of Theorem 4.1 and Theorem 4.5 by using generating function [32]. For any integers $k_1, \ldots, k_r, j \in \mathbb{Z}_{\geq 0}$, set

$$U_r(k_1, \ldots, k_r, j) = \sum_{\substack{a_{1,1}+\cdots+a_{1,j}=k_1 \\ \forall a_{1,i} \geq 1}} \cdots \sum_{\substack{a_{r,1}+\cdots+a_{r,j}=k_r \\ \forall a_{r,i} \geq 1}} \zeta(a_{1,1}+\cdots+a_{r,1}, \cdots, a_{1,j}+\cdots+a_{r,j}).$$

Here $U_r(k_1, \ldots, k_r, j) = 0$ except $k_1, \ldots, k_r \geq j$. When $j = 0$, we define

$$U_r(k_1, \ldots, k_r, 0) = \begin{cases} 1 & \text{if} \quad k_1 = \cdots = k_r = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Then, the generating function of $U_r(k_1 \ldots, k_r, j)$ is given by

$$\sum_{k_1,\ldots,k_r \geq j \geq 0} U_r(k_1, \ldots, k_r, j) x_1^{k_1-j} \cdots x_r^{k_r-j} w^j = \prod_{m=1}^{\infty} \left(1 + \frac{w}{(m-x_1)\cdots(m-x_r)}\right) \tag{16}$$

$$= \frac{\Gamma(1-x_1)\cdots\Gamma(1-x_r)}{\Gamma(1-\alpha_1)\cdots\Gamma(1-\alpha_r)}, \tag{17}$$

where $\alpha_1, \ldots, \alpha_r$ are the roots of $(X - x_1)\cdots(X - x_r) + w = 0$. When $r = 2$ and $w = z - xy$, (17) and Theorem 2.12 give the following:

$$\sum_{k_1,k_2 \geq j \geq 0} U_2(k_1, k_2, j) x^{k_1-j} y^{k_2-j} (z - xy)^j$$

$$= 1 + (z - xy) \sum_{k,r,s \geq 0} \left(\sum_{\mathbf{k} \in I_0(k,r,s)} \zeta(\mathbf{k}) \right) x^{k-r-s} y^{r-s} z^{s-1}.$$

Therefore we obtain

$$\sum_{\mathbf{k} \in I_0(k,r,s)} \zeta(\mathbf{k}) = \sum_{j=s}^{\min(k-r,r)} (-1)^{j-s} \binom{j-1}{s-1} U_2(k-r, r, j), \tag{18}$$

for $r \geq s \geq 1, k \geq r + s$. From this we have, for $k_1, k_2 \geq j \geq 1$,

$$U_2(k_1, k_2, j) = \sum_{s=j}^{\min(k_1,k_2)} \binom{s-1}{j-1} \sum_{\mathbf{k} \in I_0(k_1+k_2,k_2,s)} \zeta(\mathbf{k}). \tag{19}$$

Specializing (18) to $s = 1$, we have Theorem 4.1. Equation (18) can be obtained from Theorem 4.2.

Further Yamamoto gave alternative proof of Theorem 4.5. Set

$$U_1^{ш}(k, r)(T) = \sum_{\substack{a_1+\cdots+a_r=k \\ \forall a_i \geq 1}} \zeta_{ш}(a_1, \ldots, a_r; T) \quad \in \mathbb{R}[T].$$

Then, the generating function of $U_1^{ш}(k,r)(T)$ is given by

$$\sum_{k\geq r\geq 0} U_1^{ш}(k,r)(T)x^{k-r}w^r = e^{Tw}\frac{\Gamma(1-x)\Gamma(1+w)}{\Gamma(1-x+w)}.$$

By Theorem 2.12 and this generating function, we get

$$U_1^{ш}(k,r)(T) = \sum_{m=0}^{r-1}(-1)^{r-m-1} \sum_{\mathbf{k}\in I_0(k-r+m,r-m,1)} \zeta(\mathbf{k})\frac{T^m}{m!}.$$

When $T=0$, this equation gives Theorem 4.5.

Finally, we give a kind of sum formula for the maximal-height MZVs in the form of generating function. This is also essentially known (can be deduced from Theorem 2.12), but may be new in this form of presentation. Let $T(k)$ be the sum of all multiple zeta values of weight k and of maximal height:

$$T(k) := \sum_{\substack{k_1+\cdots+k_r=k \\ r\geq 1,\ {}^\forall k_i\geq 2}} \zeta(k_1,\ldots,k_r).$$

Theorem 4.7 (Kaneko–Sakata [16]). *We have the generating series identity*

$$1+\sum_{k=2}^{\infty} T(k)x^k = \left(1+\sum_{n=1}^{\infty}\zeta^\star(\underbrace{2,\ldots,2}_{n})x^{2n}\right)\left(1+\sum_{n=1}^{\infty}\zeta(\underbrace{3,\ldots,3}_{n})x^{3n}\right).$$

We prove these results in the next section.

5 Proofs

First we prove Theorem 4.1 by studying relations between the Arakawa-Kaneko zeta function and the MZVs in detail. Secondly we prove Theorem 4.2 in two ways by using Ohno's relation and the derivation relation.

5.1 Proof of Theorem 4.1

Since we have the duality $\zeta(\underbrace{1,\ldots,1}_{r-1},k+1) = \zeta(\underbrace{1,\ldots,1}_{k-1},r+1)$ and the right-hand side of (15) is symmetric in r and k, it is enough to prove the theorem under the assumption $k\geq r$. We proceed by induction on r. When $r=1$, both sides become $\zeta(k+1)$ and the assertion is true for all $k\geq 1$. Suppose $r\geq 2$ and the theorem is true when the depth on the left is less than r (and k is greater than or equal to the depth).

We look at the values at $s=0$ of both sides of (13). The value $\xi(\underbrace{1,\ldots,1}_{k-1},r;1)$ on the left is evaluated in (14) and is equal to $\zeta(\underbrace{1,\ldots,1}_{r-1},k+1)$. Since the functions

$\zeta(a_1+1,\ldots,a_{r-1}+1,a_r+1+s)$ with $a_r=0$ as well as $\zeta(\underbrace{1,\ldots,1}_{i},1+s)$ on the right have poles at $s=0$, we need to look at the constant term of the Laurent expansion of the right-hand side. (Because $\xi(\underbrace{1,\ldots,1}_{k-1},r;s+1)$ is entire, all the poles on the right actually cancel out.) In what follows within the proof of Theorem 4.1, we simply write the constant term of the Laurent expansion at $s=0$ of $\zeta(k_1,\ldots,k_{r-1},k_r+s)$ as $\zeta(k_1,\ldots,k_{r-1},k_r)$ even when $k_r=1$, which is equal to $\zeta_*(k_1,\ldots,k_r;\gamma)$ by Proposition 2.15 in subsection 2.3. Note that these values satisfy the harmonic product rule. With this convention, we have

$$\zeta(\underbrace{1,\ldots,1}_{r-1},k+1)=(-1)^{r-1}\sum_{\substack{a_1+\cdots+a_r=k\\ \forall a_p\geq 0}}\zeta(a_1+1,\ldots,a_r+1)$$
$$+\sum_{i=0}^{r-2}(-1)^i\zeta(\underbrace{1,\ldots,1}_{k-1},r-i)\cdot\zeta(\underbrace{1,\ldots,1}_{i+1}).$$

We apply the duality $\zeta(\underbrace{1,\ldots 1}_{k-1},r-i)=\zeta(\underbrace{1,\ldots,1}_{r-i-2},k+1)$ in the second sum on the right and use the induction hypothesis (since $r-i-1<r$) to obtain

$$\begin{aligned}\zeta(\underbrace{1,\ldots,1}_{r-1},k+1)&=(-1)^{r-1}\sum_{\substack{a_1+\cdots+a_r=k\\ \forall a_p\geq 0}}\zeta(a_1+1,\ldots,a_r+1)\\
&\quad+\sum_{i=0}^{r-2}(-1)^i\sum_{j=1}^{r-i-1}(-1)^{j-1}\sum_{\substack{\mathrm{wt}(\mathbf{a})=k,\,\mathrm{wt}(\mathbf{b})=r-i-1\\ \mathrm{dep}(\mathbf{a})=\mathrm{dep}(\mathbf{b})=j}}\zeta(\mathbf{a}+\mathbf{b})\cdot\zeta(\underbrace{1,\ldots,1}_{i+1})\\
&=(-1)^{r-1}\sum_{\substack{a_1+\cdots+a_r=k\\ \forall a_p\geq 0}}\zeta(a_1+1,\ldots,a_r+1)\\
&\quad+\sum_{j=1}^{r-1}(-1)^{j-1}\sum_{\substack{\mathrm{wt}(\mathbf{a})=k\\ \mathrm{dep}(\mathbf{a})=j}}\sum_{i=0}^{r-j-1}(-1)^i\sum_{\substack{\mathrm{wt}(\mathbf{b})=r-i-1\\ \mathrm{dep}(\mathbf{b})=j}}\zeta(\mathbf{a}+\mathbf{b})\cdot\zeta(\underbrace{1,\ldots,1}_{i+1}).\end{aligned}$$

Now we expand the product $\zeta(\mathbf{a}+\mathbf{b})\cdot\zeta(\underbrace{1,\ldots,1}_{i+1})$ by using the harmonic product and re-arrange the terms according to the number of 1's to compute the inner sum

$$\sum_{i=0}^{r-j-1}(-1)^i\sum_{\substack{\mathrm{wt}(\mathbf{b})=r-i-1\\ \mathrm{dep}(\mathbf{b})=j}}\zeta(\mathbf{a}+\mathbf{b})\cdot\zeta(\underbrace{1,\ldots,1}_{i+1}).$$

For that purpose, we introduce another notation. For a fixed index $\mathbf{a}=(a_1,\ldots,a_j)$ of depth j and integers $l,n\geq 0$, we set

$$S(\mathbf{a},l,n):=\sum_{\substack{\mathrm{wt}(\mathbf{b})=r-l\\ \mathrm{dep}(\mathbf{b})=j,\,\mathrm{ht}(\mathbf{b})=n}}\zeta(a_1+b_1,\ldots,1,\ldots,a_s+b_s,\ldots,1,\ldots,a_j+b_j),$$

where the sum runs over all $\mathbf{b} = (b_1, \ldots, b_j)$ of weight $r-l$, depth j, and height n, and over all possible positions of exactly l 1's in the arguments. Then, by the harmonic product rule, we have

$$\sum_{\substack{\mathrm{wt}(\mathbf{b})=r-i-1\\ \mathrm{dep}(\mathbf{b})=j}} \zeta(\mathbf{a}+\mathbf{b}) \cdot \zeta(\underbrace{1, \ldots, 1}_{i+1}) = \sum_{l=\max(0,i+1-j)}^{i+1} \sum_{n=i+1-l}^{j} \binom{n}{i+1-l} S(\mathbf{a}, l, n).$$

We note that, when we expand $\zeta(\mathbf{a}+\mathbf{b})\zeta(\underbrace{1, \ldots, 1}_{i+1})$ by the harmonic product, the number of 1's in each term should at least $i+1-j$ when $j < i+1$. And if the number of 1's is l, then the height n on the right varies from $i+1-l$ to j. A particular term in the sum $S(\mathbf{a}, l, n)$ on the right comes in exactly $\binom{n}{i+1-l}$ ways from the product $\zeta(\mathbf{a}+\mathbf{b})\zeta(\underbrace{1, \ldots, 1}_{i+1})$ on the left, because there are $i+1-l$ out of n positions of the index $\mathbf{a}+\mathbf{b}$ on the left which produces that particular term on the right by colliding $i+1-l$ 1's at those positions.

When we sum this up alternatingly for $i = 0, \ldots, r-j-1$ with signs, all coefficients of $S(\mathbf{a}, l, n)$ with $n, l \geq 1$ vanish, because of the binomial identity

$$\sum_{i=l-1}^{n+l-1} (-1)^i \binom{n}{i+1-l} = 0$$

if $n, l \geq 1$. Hence, also by the identity

$$\sum_{i=0}^{n-1} (-1)^i \binom{n}{i+1} = 1$$

if $n \geq 1$ (the case $l = 0$), we obtain

$$\sum_{i=0}^{r-j-1} (-1)^i \sum_{\substack{\mathrm{wt}(\mathbf{b})=r-i-1\\ \mathrm{dep}(\mathbf{b})=j}} \zeta(\mathbf{a}+\mathbf{b}) \cdot \zeta(\underbrace{1, \ldots, 1}_{i+1}) = \sum_{n=1}^{j} S(\mathbf{a}, 0, n) + (-1)^{r-j-1} S(\mathbf{a}, r-j, 0).$$

When $j \leq r-1$, we have

$$\sum_{n=1}^{j} S(\mathbf{a}, 0, n) = \sum_{\mathrm{wt}(\mathbf{b})=r,\, \mathrm{dep}(\mathbf{b})=j} \zeta(\mathbf{a}+\mathbf{b})$$

and this gives

$$\sum_{j=1}^{r-1} (-1)^{j-1} \sum_{\substack{\mathrm{wt}(\mathbf{a})=k,\, \mathrm{wt}(\mathbf{b})=r\\ \mathrm{dep}(\mathbf{a})=\mathrm{dep}(\mathbf{b})=j}} \zeta(\mathbf{a}+\mathbf{b}). \tag{20}$$

Finally, we have

$$\begin{aligned}
&\sum_{j=1}^{r-1}(-1)^{j-1}\sum_{\substack{\mathrm{wt}(\mathbf{a})=k\\ \mathrm{dep}(\mathbf{a})=j}}(-1)^{r-j-1}S(\mathbf{a},r-j,0)\\
&=(-1)^{r}\sum_{j=1}^{r-1}\sum_{\substack{\mathrm{wt}(\mathbf{a})=k\\ \mathrm{dep}(\mathbf{a})=j}}S(\mathbf{a},r-j,0)\\
&=(-1)^{r}\sum_{\substack{a_1+\cdots+a_r=k\\ a_p\geq 0,\,\text{at least one } a_p=0}}\zeta(a_1+1,\dots,a_r+1).
\end{aligned}$$

Hence, this and the terms in

$$(-1)^{r-1}\sum_{\substack{a_1+\cdots+a_r=k\\ {}^{\forall}a_p\geq 0}}\zeta(a_1+1,\dots,a_r+1)$$

with at least one $a_p=0$ cancel out, thereby remains the term

$$(-1)^{r-1}\sum_{\substack{\mathrm{wt}(\mathbf{a})=k,\,\mathrm{wt}(\mathbf{b})=r\\ \mathrm{dep}(\mathbf{a})=\mathrm{dep}(\mathbf{b})=r}}\zeta(\mathbf{a}+\mathbf{b}). \tag{21}$$

The sum of (20) and (21) gives the right-hand side of the theorem, and our proof is done.
□

5.2 Proof of Theorem 4.2 by using Ohno's relation

We set $k:=\mathrm{wt}(\mathbf{k})$.

$$\begin{aligned}
R.H.S.&=\sum_{\substack{\mathbf{k}'\succeq\mathbf{k}\\ \mathrm{dep}(\mathbf{k}')\leq r}}\sum_{\substack{\mathrm{wt}(\mathbf{r})=r\\ \mathrm{dep}(\mathbf{r})=\mathrm{dep}(\mathbf{k}')}}(-1)^{\mathrm{dep}(\mathbf{k}')-d}\zeta(\mathbf{k}'+\mathbf{r})\\
&=\sum_{j=d}^{\min(k,r)}(-1)^{j-d}\sum_{(k'_1,\dots,k'_j)\succeq\mathbf{k}}\sum_{\substack{r_1+\cdots+r_j=r\\ {}^{\forall}r_i\geq 1}}\zeta(k'_1+r_1,\dots,k'_j+r_j).
\end{aligned}$$

From Ohno's relations, we have

$$\begin{aligned}
&\sum_{\substack{r_1+\cdots+r_j=r\\ {}^{\forall}r_i\geq 1}}\zeta(k'_1+r_1,\dots,k'_j+r_j)\\
&=\sum_{\substack{r_1+\cdots+r_j=r-j\\ {}^{\forall}r_i\geq 0}}\zeta(k'_1+r_1+1,\dots,k'_j+r_j+1)\\
&=\sum_{\substack{r_1+\cdots+r_k=r-j\\ {}^{\forall}r_i\geq 0}}\zeta(\underbrace{r_1+1,\dots,r_{k'_j-1}+1}_{k'_j-1},r_{k'_j}+2,\dots,\underbrace{r_{k-k'_1+1}+1,\dots,r_{k-1}+1}_{k'_1-1},r_k+2).
\end{aligned}$$

Then,

$$R.H.S. = \sum_{j=d}^{\min(k,r)} (-1)^{j-d} \sum_{(k'_1,\dots,k'_j)\succeq \mathbf{k}} \sum_{\substack{r_1+\cdots+r_k=r-j \\ \forall r_i \geq 0}} \zeta(\underbrace{r_1+1,\dots r_{k'_j-1}+1}_{k'_j-1}, r_{k'_j}+2,\dots$$
$$\dots, \underbrace{r_{k-k'_1+1}+1,\dots,r_{k-1}+1}_{k'_1-1}, r_k+2). \tag{22}$$

Now, for a fixed $\mathbf{k}=(k_1,\dots,k_d)\in\mathbb{N}^d$ of weight k and $r\in\mathbb{N}$ with $r\geq d$, and a variable $s\in\mathbb{N}$, we set

$$\begin{aligned} F(s) &:= F_{\mathbf{k},r}(s) \\ &:= \sum_{\substack{r_1+\cdots+r_k=r-d \\ \forall r_i\geq 0,\, \mathrm{ht}=s}} \zeta(\underbrace{r_1+1,\dots,r_{k_d-1}+1}_{k_d-1}, r_{k_d}+2,\dots,\underbrace{r_{k-k_1+1}+1,\dots,r_{k-1}+1}_{k_1-1}, r_k+2), \end{aligned}$$

where the sum runs over those r_is such that the argument of ζ is of height s. For each j in equation (22), the number of each height s MZV is $\binom{s-d}{j-d}$ because it is determined by the partitions of $\mathbf{k}$ and the allocation of r. (Here, we note that there are $s-d$ places with a component greater than 1 that are determined by the partitions of $\mathbf{k}$ and the allocation of r, and there are $j-d$ places which depend on the various partitions of $\mathbf{k}$.) Thus, by focusing on the height and retaking the sums, we have

$$R.H.S. = \sum_{j=d}^{\min(k,r)} (-1)^{j-d} \sum_{s=j}^{\min(k,r)} \binom{s-d}{j-d} F(s).$$

From the binomial theorem and the duality theorem, we have

$$\begin{aligned} R.H.S. =& \binom{0}{0} F(d) + \left(\binom{1}{0} - \binom{1}{1}\right) F(d+1) \\ & + \cdots + \left(\sum_{m=0}^{\min(k,r)-d} (-1)^m \binom{\min(k,r)-d}{m}\right) F(\min(k,r)) \\ =& F(d) \\ =& \sum_{\substack{r_1+\cdots+r_d=r-d \\ \forall r_i\geq 0}} \zeta(\underbrace{1,\dots,1}_{k_d-1}, r_1+2,\dots,\underbrace{1,\dots,1}_{k_1-1}, r_d+2) \\ =& \sum_{\substack{r_1+\cdots+r_d=r \\ \forall r_i\geq 1}} \zeta(\underbrace{1,\dots,1}_{k_d-1}, r_1+1,\dots,\underbrace{1,\dots,1}_{k_1-1}, r_d+1) \\ =& \sum_{\substack{r_1+\cdots+r_d=r \\ \forall r_i\geq 1}} \zeta(\underbrace{1,\dots,1}_{r_1-1}, k_1+1,\dots,\underbrace{1,\dots,1}_{r_d-1}, k_d+1) \\ =& L.H.S. \end{aligned}$$

□

5.3 Proof of Theorem 4.2 by using the derivation relation

In this section, we give an alternative proof of Theorem 4.2. Let α be the endomorphism on $\mathfrak{H}$ such that $\alpha(e_0) := e_0 - e_1e_0$ and $\alpha(e_1) := -e_1e_0$, and τ be the anti-automorphism on $\mathfrak{H}$ such that $\tau(e_0) := e_1$ and $\tau(e_1) := e_0$. For each $l \in \mathbb{N}$, we define the derivation D_l on $\mathfrak{H}$ by $D_l(e_0) := 0$ and $D_l(e_1) := e_1e_0^l$. Set $\sigma := \exp\left(\sum_{l=1}^{\infty} \frac{D_l}{l}\right)$. Then, we find the map σ is an automorphism on $\widehat{\mathfrak{H}} = \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$, and we see that $\sigma(e_0) = e_0$ and $\sigma(e_1) = e_1\frac{1}{1-e_0}$. (See also [13, Section 6] and [30, Appendix].)

According to Ihara, Kaneko and Zagier in [13, Proof of Theorem 3], we first note that

$$\sigma - \tau\sigma\tau = \left(1 - \exp\left(\sum_{l=1}^{\infty} \frac{\partial_l}{l}\right)\right)\sigma. \tag{23}$$

By Theorem 2.13, we have

$$Z\left(\left(1 - \exp\left(\sum_{l=1}^{\infty} \frac{\partial_l}{l}\right)\right)\sigma(w)\right) = 0 \quad (w \in \mathfrak{H}^0).$$

From the equality (23) and by putting $\alpha(e_1e_0^{k_1-1}\cdots e_1e_0^{k_d-1})$ into w,

$$Z((\sigma - \tau\sigma\tau)\alpha(e_1e_0^{k_1-1}\cdots e_1e_0^{k_d-1})) = 0.$$

Here,

$$\begin{aligned}
&\sigma\alpha(e_1e_0^{k_1-1}\cdots e_1e_0^{k_d-1})\\
&= \sigma((-e_1e_0)(e_0 - e_1e_0)^{k_1-1}\cdots(-e_1e_0)(e_0 - e_1e_0)^{k_d-1})\\
&= (-1)^d\sigma(e_1e_0(e_0 - e_1e_0)^{k_1-1}\cdots e_1e_0(e_0 - e_1e_0)^{k_d-1})\\
&= (-1)^d e_1\frac{e_0}{1-e_0}\left(e_0 - e_1\frac{e_0}{1-e_0}\right)^{k_1-1}\cdots e_1\frac{e_0}{1-e_0}\left(e_0 - e_1\frac{e_0}{1-e_0}\right)^{k_d-1}\\
&= (-1)^d\sum_{r=d}^{\infty}\sum_{\substack{\varepsilon_{1,1}+\cdots+\varepsilon_{d,k_d}=r\\ \varepsilon_{i,j}\ge 0\,(2\le j\le k_i)\\ \varepsilon_{i,1}\ge 1}} e_1e_0^{\varepsilon_{1,1}}(-e_1e_0^{\varepsilon_{1,2}})\cdots(-e_1e_0^{\varepsilon_{1,k_1}})\\
&\qquad\qquad\cdots\cdots e_1e_0^{\varepsilon_{d,1}}(-e_1e_0^{\varepsilon_{d,2}})\cdots(-e_1e_0^{\varepsilon_{d,k_d}}).
\end{aligned}$$

When $\varepsilon_{i,j} = 0$, we understand $e_1e_0^{\varepsilon_{i,j}} = -e_0$. On the other hand,

$$\begin{aligned}
\tau\sigma\tau\alpha(e_1e_0^{k_1-1}\cdots e_1e_0^{k_d-1}) &= \tau\sigma\tau((-e_1e_0)(e_0 - e_1e_0)^{k_1-1}\cdots(-e_1e_0)(e_0 - e_1e_0)^{k_d-1})\\
&= (-1)^d\tau\sigma\tau(e_1e_0(e_0 - e_1e_0)^{k_1-1}\cdots e_1e_0(e_0 - e_1e_0)^{k_d-1})\\
&= (-1)^d\tau\sigma((e_1 - e_1e_0)^{k_d-1}e_1e_0\cdots(e_1 - e_1e_0)^{k_1-1}e_1e_0)\\
&= (-1)^d\tau\left(e_1^{k_d}\frac{e_0}{1-e_0}\cdots e_1^{k_1}\frac{e_0}{1-e_0}\right)\\
&= (-1)^d\frac{e_1}{1-e_1}e_0^{k_1}\cdots\frac{e_1}{1-e_1}e_0^{k_d}
\end{aligned}$$

$$= (-1)^d \sum_{r=d}^{\infty} \sum_{\substack{r_1+\cdots+r_d=r \\ \forall r_i \geq 1}} e_1^{r_1} e_0^{k_1} \cdots e_1^{r_d} e_0^{k_d}.$$

Then, we have

$$\sum_{r=d}^{\infty} \sum_{\substack{\varepsilon_{1,1}+\cdots+\varepsilon_{d,k_d}=r \\ \varepsilon_{i,j}\geq 0\,(2\leq j\leq k_i) \\ \varepsilon_{i,1}\geq 1}} e_1 e_0^{\varepsilon_{1,1}} (-e_1 e_0^{\varepsilon_{1,2}}) \cdots (-e_1 e_0^{\varepsilon_{1,k_1}})$$

$$\cdots\cdots\cdots e_1 e_0^{\varepsilon_{d,1}} (-e_1 e_0^{\varepsilon_{d,2}}) \cdots (-e_1 e_0^{\varepsilon_{d,k_d}})$$

$$- \sum_{r=d}^{\infty} \sum_{\substack{r_1+\cdots+r_d=r \\ \forall r_i \geq 1}} e_1^{r_1} e_0^{k_1} \cdots e_1^{r_d} e_0^{k_d} \ \in \operatorname{Ker} Z.$$

This finishes the proof of the theorem. $\square$

5.4 Proof of Theorem 4.5

We multiply $\Gamma(s+1)$ on both sides of the identity (13) and look at the constant terms of the Laurent expansions at $s = 0$. The left-hand side is holomorphic at $s = 0$ and gives the value $\zeta(\underbrace{1,\ldots,1}_{r-1}, k+1)$ as we already saw in subsection 5.1. The function $\binom{s+a_r}{a_r}\Gamma(s+1)\zeta(a_1+1,\ldots,a_{r-1}+1, a_r+1+s)$ on the right is holomorphic at $s = 0$ if $a_r > 1$, and in that case gives the value $\zeta(a_1+1,\ldots,a_{r-1}+1, a_r+1)$. If $a_r = 0$, then

$$\binom{s+a_r}{a_r}\Gamma(s+1)\zeta(a_1+1,\ldots,a_{r-1}+1, a_r+1+s) = \Gamma(s+1)\zeta(a_1+1,\ldots,a_{r-1}+1, 1+s)$$

has a pole at $s = 0$ and its constant term of the Laurent expansion is $\zeta_{ш}(a_1+1,\ldots,a_r+1)$ by Proposition 2.15. On the other hand, the function $\Gamma(s+1)\zeta(\underbrace{1,\ldots,1}_{i}, 1+s)$ has no constant term at $s = 0$ because $\zeta_{ш}(\underbrace{1,\ldots,1}_{i+1}; T) = T^{i+1}/(i+1)!$, and hence we conclude the proof of the theorem. $\square$

We remark that we can also prove the theorem by computing directly the left-hand side using the regularization formula [13, (5.2)].

5.5 Proof of Theorem 4.7

This is almost obvious if we write k_i (≥ 2) as $k_i = 2 + \cdots + 2$ (k_i: even) or $k_i = 3 + 2 + \cdots + 2$ (k_i: odd), and consider the harmonic product of $\zeta^\star(2,\ldots,2)\zeta(3,\ldots,3)$ after writing $\zeta^\star(2,\ldots,2)$ as sums of ordinary multiple zeta values.

An alternative proof is given by using the main identity in [25]. As is already remarked there, if we specialize $y = 0$ and $z = x^2$ in equation (8), we obtain

$$1 + \sum_{k=2}^{\infty} T(k)x^k = \exp\left(\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n} x^{2n}\right) \cdot \exp\left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\zeta(3n)}{n} x^{3n}\right).$$

It is standard that

$$\exp\left(\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n} x^{2n}\right) = \Gamma(1+x)\Gamma(1-x) = \prod_{m=1}^{\infty}\left(1 - \frac{x^2}{m^2}\right)^{-1} = 1 + \sum_{n=1}^{\infty} \zeta^{\star}(\underbrace{2, \ldots, 2}_{n}) x^{2n},$$

whereas the identity

$$\exp\left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\zeta(3n)}{n} x^{3n}\right) = 1 + \sum_{n=1}^{\infty} \zeta(\underbrace{3, \ldots, 3}_{n}) x^{3n}$$

is a special case of [13, Corollary 2 of Proposition 4]. □

6 Finite multiple zeta values

In this section, we prove the counterpart of Theorem 4.2 for what we call *finite multiple zeta values* (FMZVs), a generic term for the $\mathcal{A}$-finite multiple zeta values and the symmetrized multiple zeta values.

We consider the collection of truncated sums

$$\zeta_p(k_1, \ldots, k_r) := \sum_{0 < n_1 < \cdots < n_r < p} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

modulo all primes p in the quotient ring $\mathcal{A} = \left(\prod_p \mathbb{Z}/p\mathbb{Z}\right) \Big/ \left(\bigoplus_p \mathbb{Z}/p\mathbb{Z}\right)$, which is a $\mathbb{Q}$-algebra. Elements of $\mathcal{A}$ are represented by $(a_p)_p$, where $a_p \in \mathbb{Z}/p\mathbb{Z}$, and two elements $(a_p)_p$ and $(b_p)_p$ are identified if and only if $a_p = b_p$ for all but finitely many primes p. For integers $k_1, \ldots, k_r \in \mathbb{N}$, the $\mathcal{A}$-finite multiple zeta value ($\mathcal{A}$-FMZV) and the $\mathcal{A}$-finite multiple zeta star value ($\mathcal{A}$-FMZSV) are defined by

$$\zeta^{\mathcal{A}}(k_1, \ldots, k_r) := \left(\sum_{0 < n_1 < \cdots < n_r < p} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \bmod p\right)_p \in \mathcal{A},$$

$$\zeta^{\mathcal{A},\star}(k_1, \ldots, k_r) := \left(\sum_{0 < n_1 \le \cdots \le n_r \le p-1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \bmod p\right)_p \in \mathcal{A}.$$

Also we introduce a conjecture for $\mathcal{A}$-FMZVs. For each $k \geq 0$, we define a $\mathbb{Q}$-vector subspace $\mathcal{Z}_{\mathcal{A}}$ in $\mathcal{A}$ by

$$\begin{aligned} \mathcal{Z}_{\mathcal{A},0} &:= \mathbb{Q}, \\ \mathcal{Z}_{\mathcal{A},k} &:= \sum_{\substack{k_1+\cdots+k_r=k \\ r\geq 1, k_i \in \mathbb{N}}} \mathbb{Q} \cdot \zeta^{\mathcal{A}}(k_1, \ldots, k_r), \\ \mathcal{Z}_{\mathcal{A}} &:= \sum_{k=0}^{\infty} \mathcal{Z}_{\mathcal{A},k}. \end{aligned}$$

The dimension of the $\mathbb{Q}$-vector space generated by $\mathcal{A}$-FMZVs of weight k was conjectured by Zagier:

Conjecture 6.1 (Zagier)**.** *For any integer $k \geq 0$, the equality*

$$\dim_{\mathbb{Q}} \mathcal{Z}_{\mathcal{A},k} = d_{k-3}$$

holds.

Here d_k is the non-negative integer satisfying the following recursion (the same sequence as in Conjecture 2.1):

$$d_k = d_{k-2} + d_{k-3} \quad (k \geq 3), \; d_{<0} = 0, \; d_0 = 1, \; d_1 = 0, \; d_2 = 1.$$

The symmetrized multiple zeta values were first introduced by Kaneko and Zagier in [15, 18]. For integers $k_1, \ldots, k_r \in \mathbb{N}$, we let

$$\begin{aligned} \zeta_*^{\mathcal{S}}(k_1, \ldots, k_r) &:= \sum_{i=0}^{r} (-1)^{k_{i+1}+\cdots+k_r} \zeta_*(k_1, \ldots, k_i; T) \zeta_*(k_r, \ldots, k_{i+1}; T), \\ \zeta_{ш}^{\mathcal{S}}(k_1, \ldots, k_r) &:= \sum_{i=0}^{r} (-1)^{k_{i+1}+\cdots+k_r} \zeta_{ш}(k_1, \ldots, k_i; T) \zeta_{ш}(k_r, \ldots, k_{i+1}; T). \end{aligned}$$

Here, the symbols ζ_* and $\zeta_{ш}$ on the right-hand sides stand for the regularized polynomials coming from harmonic and shuffle regularizations as explained in subsection 2.3. In the sum, we understand $\zeta_*(\emptyset; T) = \zeta_{ш}(\emptyset; T) = 1$. Kaneko and Zagier proved that both $\zeta_*^{\mathcal{S}}(k_1, \ldots, k_r)$ and $\zeta_{ш}^{\mathcal{S}}(k_1, \ldots, k_r)$ are elements in $\mathcal{Z}$ independent of T and the congruence

$$\zeta_*^{\mathcal{S}}(k_1 \ldots, k_r) \equiv \zeta_{ш}^{\mathcal{S}}(k_1 \ldots, k_r) \mod \zeta(2)$$

holds in the $\mathbb{Q}$-algebra $\mathcal{Z}$. The symmetrized multiple zeta value (SMZV) is defined as an element in the quotient ring $\mathcal{Z}/\zeta(2)$ by

$$\zeta^{\mathcal{S}}(k_1, \ldots, k_r) := \zeta_*^{\mathcal{S}}(k_1, \ldots, k_r) \mod \zeta(2).$$

Kaneko and Zagier conjectured that $\mathcal{A}$-FMZVs connect with SMZVs.

Conjecture 6.2 (Kaneko–Zagier [18])**.** *There exists an algebra isomorphism ϕ between $\mathcal{Z}_{\mathcal{A}}$ and $\mathcal{Z}/\zeta(2)$ such that $\phi : \mathcal{Z}_{\mathcal{A}} \ni \zeta^{\mathcal{A}}(k_1, \ldots, k_r) \mapsto \zeta^{\mathcal{S}}(k_1 \ldots, k_r) \in \mathcal{Z}/\zeta(2)$.*

Next we introduce the harmonic product and a shuffle identity for FMZVs. The harmonic product is immediate from the definition, and the shuffle identities for $\mathcal{A}$-FMZVs and SMZVs are proved by Kaneko and Zagier.

Theorem 6.3 (Kaneko–Zagier [18])**.** *For any indices* $\mathbf{k} = (k_1, \dots, k_r)$ *and* $\mathbf{l} = (l_1, \dots, l_s)$*, we have*

$$\begin{aligned}
\zeta^{\mathcal{F}}([\mathbf{k}] * [\mathbf{l}]) &= \zeta^{\mathcal{F}}(\mathbf{k})\zeta^{\mathcal{F}}(\mathbf{l}),\\
\zeta^{\mathcal{F}}([\mathbf{k}] \mathbin{ш} [\mathbf{l}]) &= (-1)^{\mathrm{wt}(\mathbf{l})}\zeta^{\mathcal{F}}(\mathbf{k}, \overleftarrow{\mathbf{l}}),
\end{aligned}$$

where $\overleftarrow{\mathbf{l}}$ *is the reversal* $(l_s, \dots, l_1)$ *and* $\mathcal{F}$ *is either* $\mathcal{A}$ *or* $\mathcal{S}$*.*

6.1 Some relations for the FMZVs

We introduce several relations among FMZVs. For $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$, we define Hoffman's dual of $\mathbf{k}$ by

$$\mathbf{k}^\vee := (\underbrace{1, \dots, 1}_{k_1} + \underbrace{1, \dots, 1}_{k_2} + 1, \dots, 1 + \underbrace{1, \dots, 1}_{k_r}).$$

Note that $(\mathbf{k}^\vee)^\vee = \mathbf{k}$, $\mathrm{wt}(\mathbf{k}^\vee) = \mathrm{wt}(\mathbf{k})$, $\mathrm{dep}(\mathbf{k}) + \mathrm{dep}(\mathbf{k}^\vee) = \mathrm{wt}(\mathbf{k}) + 1$ by definition. Hoffman proved the following property for $\mathcal{A}$-FMZSVs:

Theorem 6.4 (Hoffman [12])**.** *For any indices* $\mathbf{k}$*, we have*

$$\zeta^{\mathcal{A},\star}(\mathbf{k}^\vee) = -\zeta^{\mathcal{A},\star}(\mathbf{k}). \tag{24}$$

The sum formula for $\mathcal{A}$-FMZVs is conjectured by Kaneko in [14], proved by Shingo Saito and Noriko Wakabayashi in [28]. Recently the sum formula for SMZV was proved by Hideki Murahara [21]. For $k, r, i \in \mathbb{Z}$ with $1 \le i \le n \le k-1$, set

$$I_{k,r,i} = \{(k_1, \dots, k_r) \in \mathbb{N}^r \mid k_1 + \cdots + k_r = k, k_i \ge 2\}.$$

Theorem 6.5 (Saito–Wakabayashi [28], Murahara [21])**.** *For* $k, r, i \in \mathbb{Z}$ *with* $1 \le i \le n \le k-1$*, we have*

$$\sum_{(k_1,\dots,k_r)\in I_{k,r,i}} \zeta^{\mathcal{F}}(k_1, \dots, k_r) = (-1)^{i-1}\left(\binom{k-1}{i-1} + (-1)^n\binom{k-1}{n-i}\right)\mathfrak{Z}(k), \tag{25}$$

where

$$\mathfrak{Z}(k) = \begin{cases} \left(\frac{B_{p-k}}{k}\right)_p & (\mathcal{F} = \mathcal{A}),\\ \zeta(k) \mod \zeta(2) & (\mathcal{F} = \mathcal{S}). \end{cases}$$

Kojiro Oyama proved Ohno type relations for the FMZVs, which were first conjectured by Kaneko in [15].

Theorem 6.6 (Oyama [26])**.** *For* $(k_1, \ldots, k_r) \in \mathbb{N}^r$ *and* $m \in \mathbb{Z}_{\geq 0}$*, we have*

$$\sum_{\substack{\varepsilon_1+\cdots+\varepsilon_r=m \\ \forall \varepsilon_i \geq 0}} \zeta^{\mathcal{F}}(k_1+\varepsilon_1, \ldots, k_r+\varepsilon_r) = \sum_{\substack{\varepsilon'_1+\cdots+\varepsilon'_{r'}=m \\ \forall \varepsilon'_i \geq 0}} \zeta^{\mathcal{F}}((k'_1+\varepsilon'_1, \ldots, k'_{r'}+\varepsilon'_{r'})^{\vee}), \tag{26}$$

where $(k'_1, \ldots, k'_{r'}) = (k_1, \ldots, k_r)^{\vee}$ *is Hoffman's dual of* $(k_1, \ldots, k_r)$ *and* $\mathcal{F} = \mathcal{A}$ *or* $\mathcal{S}$*.*

The derivation relations for the FMZVs were conjectured by Oyama and proved by Murahara in [22]. We define two $\mathbb{Q}$-linear maps $Z_{\mathcal{A}} \colon \mathfrak{H}^1 \to \mathcal{A}$ and $Z_{\mathcal{S}} \colon \mathfrak{H}^1 \to \mathcal{Z}_{\mathbb{R}}/\zeta(2)$ respectively by $Z_{\mathcal{A}}(1) := 1$ and $Z_{\mathcal{A}}(e_{k_1} \cdots e_{k_r}) := \zeta^{\mathcal{A}}(k_1, \ldots, k_r)$, and $Z_{\mathcal{S}}(1) := 1$ and $Z_{\mathcal{S}}(e_{k_1} \cdots e_{k_r}) := \zeta^{\mathcal{S}}(k_1, \ldots, k_r)$. We also define the $\mathbb{Q}$-linear operator R_{e_0} on $\mathfrak{H}$ by $R_{e_0} := we_0$ $(w \in \mathfrak{H})$.

Theorem 6.7 (Murahara [22])**.** *For* $l \in \mathbb{N}$*, we have*

$$Z_{\mathcal{F}}(R_{e_0}^{-1} \partial_l R_{e_0}(w)) = 0 \quad (w \in \mathfrak{H}^1, \mathcal{F} = \mathcal{A} \text{ or } \mathcal{S}).$$

Our result is the following:

Theorem 6.8 (Murahara–Sakata [23])**.** *For* $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ *and* $r \in \mathbb{N}$ *with* $r \geq d$*, we have*

$$\begin{aligned} &\sum_{\substack{r_0+\cdots+r_d=r+1 \\ \forall r_i \geq 1}} \zeta^{\mathcal{F}}(\underbrace{1, \ldots, 1}_{r_0-1}, k_1+1, \underbrace{1, \ldots, 1}_{r_1-1}, \ldots, \underbrace{1, \ldots, 1}_{r_{d-1}-1}, k_d+1, \underbrace{1, \ldots, 1}_{r_d-1}) \\ &= \sum_{\substack{\mathbf{k}' \succeq \mathbf{k} \\ \mathrm{dep}(\mathbf{k}') \leq r}} \sum_{\substack{\mathrm{wt}(\mathbf{r})=r \\ \mathrm{dep}(\mathbf{r})=\mathrm{dep}(\mathbf{k}')}} (-1)^{\mathrm{dep}(\mathbf{k}')-d} \zeta^{\mathcal{F}}(\mathbf{k}'+\mathbf{r}) \qquad (\mathcal{F} = \mathcal{A} \text{ or } \mathcal{S}). \end{aligned}$$

6.2 Proof of Theorem 6.8 by using Ohno type relation

We can prove Theorem 6.8 in the same manner as in the proof of Theorem 4.2. From Ohno type relations (26), we have

$$\begin{aligned} &R.H.S. \\ &= \sum_{\substack{\mathbf{k}' \succeq \mathbf{k} \\ \mathrm{dep}(\mathbf{k}') \leq r}} \sum_{\substack{\mathrm{wt}(\mathbf{r})=r \\ \mathrm{dep}(\mathbf{r})=\mathrm{dep}(\mathbf{k}')}} (-1)^{\mathrm{dep}(\mathbf{k}')-d} \zeta^{\mathcal{F}}(\mathbf{k}'+\mathbf{r}) \\ &= \sum_{j=d}^{\min(k,r)} (-1)^{j-d} \sum_{(k'_1,\ldots,k'_j) \succeq \mathbf{k}} \sum_{\substack{r_1+\cdots+r_j=r-j \\ \forall r_i \geq 0}} \zeta^{\mathcal{F}}(k'_1+r_1+1, \ldots, k'_j+r_j+1) \end{aligned}$$

$$= \sum_{j=d}^{\min(k,r)} (-1)^{j-d} \sum_{(k'_1,\dots,k'_j)\succeq \mathbf{k}} \sum_{\substack{r_0+\cdots+r_k=r-j \\ \forall r_i \geq 0}} \zeta^{\mathcal{F}}((\underbrace{r_0+1,\dots,r_{k'_1-1}+1}_{k'_1}, r_{k'_1}+2,$$

$$\underbrace{r_{k'_1+1}+1,\dots,r_{k'_1+k'_2-1}+1}_{k'_2-1},\dots,\underbrace{r_{k-k'_{j-1}-k'_j+1}+1,\dots,r_{k-k'_j-1}+1}_{k'_{j-1}-1},$$

$$r_{k-k'_j}+2,\underbrace{r_{k-k'_j+1}+1,\dots,r_k+1}_{k'_j})^{\vee}).$$

For a fixed $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$ and $r \in \mathbb{N}$ with $r \geq d$, and a variable $s \in \mathbb{N}$, we set

$$G(s) := G_{\mathbf{k},r}(s) := \sum_{\substack{r_0+\cdots+r_k=r-d-2 \\ \forall r_i \geq 0,\, \mathrm{ht}=s}} \zeta^{\mathcal{F}}((r_0+2,\underbrace{r_1+1,\dots,r_{k_1-1}+1}_{k_1-1}, r_{k_1}+2,$$

$$\underbrace{r_{k_1+1}+1,\dots,r_{k_1+k_2-1}+1}_{k_2-1},\dots,\underbrace{r_{k-k_{d-1}-k_d+1}+1,\dots,r_{k-k_d-1}+1}_{k_{d-1}-1},$$

$$r_{k-k_d}+2,\underbrace{r_{k-k_d+1}+1,\dots,r_{k-1}+1}_{k_d-1},r_k+2)^{\vee}),$$

$$G'(s) := G'_{\mathbf{k},r}(s) := \sum_{\substack{r_1+\cdots+r_k=r-d-1 \\ \forall r_i \geq 0,\, \mathrm{ht}=s}} \zeta^{\mathcal{F}}((1,\underbrace{r_1+1,\dots,r_{k_1-1}+1}_{k_1-1}, r_{k_1}+2,$$

$$\underbrace{r_{k_1+1}+1,\dots,r_{k_1+k_2-1}+1}_{k_2-1},\dots,\underbrace{r_{k-k_{d-1}-k_d+1}+1,\dots,r_{k-k_d-1}+1}_{k_{d-1}-1},$$

$$r_{k-k_d}+2,\underbrace{r_{k-k_d+1}+1,\dots,r_{k-1}+1}_{k_d-1},r_k+2)^{\vee})$$

$$+ \sum_{\substack{r_0+\cdots+r_{k-1}=r-d-1 \\ \forall r_i \geq 0,\, \mathrm{ht}=s}} \zeta^{\mathcal{F}}((r_0+2,\underbrace{r_1+1,\dots,r_{k_1-1}+1}_{k_1-1}, r_{k_1}+2,$$

$$\underbrace{r_{k_1+1}+1,\dots,r_{k_1+k_2-1}+1}_{k_2-1},\dots,\underbrace{r_{k-k_{d-1}-k_d+1}+1,\dots,r_{k-k_d-1}+1}_{k_{d-1}-1},$$

$$r_{k-k_d}+2,\underbrace{r_{k-k_d+1}+1,\dots,r_{k-1}+1}_{k_d-1},1)^{\vee}),$$

$$G''(s) := G''_{\mathbf{k},r}(s) := \sum_{\substack{r_1+\cdots+r_{k-1}=r-d \\ \forall r_i \geq 0,\, \mathrm{ht}=s}} \zeta^{\mathcal{F}}((1,\underbrace{r_1+1,\dots,r_{k_1-1}+1}_{k_1-1}, r_{k_1}+2,$$

$$\underbrace{r_{k_1+1}+1,\dots,r_{k_1+k_2-1}+1}_{k_2-1},\dots,\underbrace{r_{k-k_{d-1}-k_d+1}+1,\dots,r_{k-k_d-1}+1}_{k_{d-1}-1},$$

$$r_{k-k_d}+2,\underbrace{r_{k-k_d+1}+1,\dots,r_{k-1}+1}_{k_d-1},1)^{\vee}),$$

where the sum runs over those r_is such that the argument of $\zeta^{\mathcal{F}}$ is of height s. By concentrating on the height, and adding up all the terms, then re-arranging the sums; we have

$$\begin{aligned}
R.H.S. = &\sum_{j=d+1}^{\min(k+1,r-1)} (-1)^{j-d-1} \sum_{s=j}^{\min(k+1,r-1)} \binom{s-d-1}{j-d-1} G(s) \\
&+ \sum_{j=d}^{\min(k,r-1)} (-1)^{j-d} \sum_{s=j}^{\min(k,r-1)} \binom{s-d}{j-d} G'(s) \\
&+ \sum_{j=d-1}^{\min(k-1,r-1)} (-1)^{j-d+1} \sum_{s=j}^{\min(k-1,r-1)} \binom{s-d+1}{j-d+1} G''(s) \\
&= G(d+1) + G'(d) + G''(d-1) \\
&= \sum_{\substack{r_0+\cdots+r_d=r-d \\ \forall r_i \geq 0}} \zeta^{\mathcal{F}}((r_0+1, \underbrace{1,\ldots,1}_{k_1-1}, r_1+2, \ldots, r_{d-1}+2, \underbrace{1,\ldots,1}_{k_d-1}, r_d+1)^{\vee}) \\
&= \sum_{\substack{r_0+\cdots+r_d=r+1 \\ \forall r_i \geq 1}} \zeta^{\mathcal{F}}(\underbrace{1,\ldots,1}_{r_0-1}, k_1+1, \underbrace{1,\ldots,1}_{r_1-1}, \ldots, \underbrace{1,\ldots,1}_{r_{d-1}-1}, k_d+1, \underbrace{1,\ldots,1}_{r_d-1}) \\
&= L.H.S.
\end{aligned}$$

□

6.3 Proof of Theorem 6.8 by using the derivation relation

By Theorem 6.7, we have

$$Z_{\mathcal{F}}\left(R_{e_0}^{-1}\left(1-\exp\left(\sum_{l=1}^{\infty}\frac{\partial_l}{l}\right)\right)R_{e_0}\sigma(w)\right)=0 \quad (w\in\mathfrak{H}^1).$$

Since $R_{e_0}\sigma = \sigma R_{e_0}$,

$$Z_{\mathcal{F}}\left(R_{e_0}^{-1}\left(1-\exp\left(\sum_{l=1}^{\infty}\frac{\partial_l}{l}\right)\right)\sigma R_{e_0}(w)\right)=0 \quad (w\in\mathfrak{H}^1).$$

From the equality (23) and by putting $\alpha(e_1e_0^{k_1-1}\cdots e_1e_0x^{k_d-1})$ into w, we have

$$Z_{\mathcal{F}}(R_{e_0}^{-1}(\sigma-\tau\sigma\tau)R_{e_0}\alpha(e_1e_0^{k_1-1}\cdots e_1e_0^{k_d-1}))=0.$$

Since $R_{e_0}^{-1}\sigma R_{e_0}=\sigma$ and by the same calculation in subsection 5.3, we have

$$
\begin{aligned}
&R_{e_0}^{-1}\sigma R_{e_0}\alpha(e_1e_0^{k_1-1}\cdots e_1e_0^{k_d-1})\\
&=\sigma\alpha(e_1e_0^{k_1-1}\cdots e_1e_0^{k_d-1})\\
&=(-1)^d\sum_{r=d}^{\infty}\sum_{\substack{\varepsilon_{1,1}+\cdots+\varepsilon_{d,k_d}=r\\ \varepsilon_{i,j}\geq 0\,(2\leq j\leq k_i)\\ \varepsilon_{i,j}\geq 1\,(j=1)}} e_1e_0^{\varepsilon_{1,1}}(-e_1e_0^{\varepsilon_{1,2}})\cdots(-e_1e_0^{\varepsilon_{1,k_1}})\\
&\qquad\qquad\cdots\cdots\cdots e_1e_0^{\varepsilon_{d,1}}(-e_1e_0^{\varepsilon_{d,2}})\cdots(-e_1e_0^{\varepsilon_{d,k_d}}).
\end{aligned}
$$

When $\varepsilon_{i,j}=0$, we understand $e_1e_0^{\varepsilon_{i,j}}=-e_0$. On the other hand,

$$
\begin{aligned}
&R_{e_0}^{-1}\tau\sigma\tau R_{e_0}\alpha(e_1e_0^{k_1-1}\cdots e_1e_0^{k_d-1})\\
&=R_{e_0}^{-1}\tau\sigma\tau R_{e_0}((-e_1e_0)(e_0-e_1e_0)^{k_1-1}\cdots(-e_1e_0)(e_0-e_1e_0)^{k_d-1})\\
&=(-1)^dR_{e_0}^{-1}\tau\sigma\tau R_{e_0}(e_1e_0(e_0-e_1e_0)^{k_1-1}\cdots e_1e_0(e_0-e_1e_0)^{k_d-1})\\
&=(-1)^dR_{e_0}^{-1}\tau\sigma\tau(e_1e_0(e_0-e_1e_0)^{k_1-1}\cdots e_1e_0(e_0-e_1e_0)^{k_d-1}e_0)\\
&=(-1)^dR_{e_0}^{-1}\tau\sigma(e_1(e_1-e_1e_0)^{k_d-1}e_1e_0\cdots(e_1-e_1e_0)^{k_1-1}e_1e_0)\\
&=(-1)^dR_{e_0}^{-1}\tau\left(e_1\frac{1}{1-e_0}e_1^{k_d}\frac{e_0}{1-e_0}\cdots e_1^{k_1}\frac{e_0}{1-e_0}\right)\\
&=(-1)^d\frac{e_1}{1-e_1}e_0^{k_1}\cdots\frac{e_1}{1-e_1}e_0^{k_d}\frac{1}{1-e_1}\\
&=(-1)^d\sum_{r=d}^{\infty}\sum_{\substack{r_0+\cdots+r_d=r+1\\ {}^{\forall}r_i\geq 1}} e_1^{r_0}e_0^{k_1}e_1^{r_1}\cdots e_1^{r_d}e_0^{k_d}e_1^{r_d-1}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&\sum_{r=d}^{\infty}\sum_{\substack{\varepsilon_{1,1}+\cdots+\varepsilon_{d,k_d}=r\\ \varepsilon_{i,j}\geq 0\,(2\leq j\leq k_i)\\ \varepsilon_{i,j}\geq 1\,(j=1)}} e_1e_0^{\varepsilon_{1,1}}(-e_1e_0^{\varepsilon_{1,2}})\cdots(-e_1e_0^{\varepsilon_{1,k_1}})\\
&\qquad\qquad\cdots\cdots\cdots e_1e_0^{\varepsilon_{d,1}}(-e_1e_0^{\varepsilon_{d,2}})\cdots(-e_1e_0^{\varepsilon_{d,k_d}})\\
&=\sum_{r=d}^{\infty}\sum_{\substack{r_0+\cdots+r_d=r+1\\ {}^{\forall}r_i\geq 1}} e_1^{r_0}e_0^{k_1}e_1^{r_1}\cdots e_1^{r_d}e_0^{k_d-1}.
\end{aligned}
$$

This completes the proof of the theorem. □

7 Taylor series for the reciprocal gamma function

In this section, we consider the Taylor series of $A^{-1}(x)$ in subsection 2.3. In [4], the following formula for the Taylor series of the function $A^{-1}(x) = \exp(\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n}\zeta(n)x^n)$ is given:

$$\exp\left(\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n}\zeta(n)x^n\right)$$
$$= 1 + \sum_{k=2}^{\infty}(-1)^k \sum_{\substack{k_1+\cdots+k_r=k \\ r\geq 1, \forall k_i \geq 2}} (-1)^r \frac{(k_1-1)\cdots(k_r-1)}{k_1!\cdots k_r!}\zeta(k_1,\ldots,k_r)x^k. \tag{27}$$

We note that the function on the left-hand side of (27) is equal to $e^{-\gamma x}/\Gamma(1+x)$, where γ is Euler's constant. In [13] and [27], this function plays an important role in the theory of regularization of MZVs. Arakawa and Kaneko proved (27) by using the Weierstrass infinite product of the gamma function:

$$\begin{aligned}
\frac{e^{-\gamma x}}{\Gamma(1+x)} &= \prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/n} \\
&= \prod_{n=1}^{\infty}\left(1+\sum_{k=2}^{\infty}(-1)^{k-1}\frac{(k-1)x^k}{k!n^k}\right) \\
&= 1 + \sum_{k=2}^{\infty}(-1)^k \sum_{\substack{k_1+\cdots+k_r=k \\ r\geq 1, \forall k_i \geq 2}} (-1)^r \frac{(k_1-1)\cdots(k_r-1)}{k_1!\cdots k_r!}\zeta(k_1,\ldots,k_r)x^k.
\end{aligned}$$

The aim of this section is to give an alternative, purely algebraic proof of the formula in the setting of abstract algebra of MZVs.

Our result is the following:

Theorem 7.1 (Sakata [29]). *Let $\mathscr{R}_*[[x]]$ be the ring of formal power series over $\mathscr{R}_*$. The equality*

$$\exp_*\left(\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n}[n]x^n\right)$$
$$= 1 + \sum_{k=2}^{\infty}(-1)^k \sum_{\substack{k_1+\cdots+k_r=k \\ r\geq 1, \forall k_i \geq 2}} (-1)^r \frac{(k_1-1)\cdots(k_r-1)}{k_1!\cdots k_r!}[k_1,\ldots,k_r]\,x^k \tag{28}$$

holds in $\mathscr{R}_*[[x]]$. Where $\exp_*$ is the exponential $\exp_*(f) = \sum_{n=0}^{\infty}\frac{f^n}{n!}$ with f^n being the power in the ring $\mathscr{R}_*[[x]]$.

Applying the evaluation map ζ coefficient-wise to both sides of (28), we obtain (27). It is worth remarking that the right-hand sides of both (27) and (28) take exactly the same form. As a corollary, we obtain an explicit formula for the Taylor series for the reciprocal gamma function:

Corollary 7.2.

$$\frac{1}{\Gamma(x)} = x + \gamma x^2$$
$$+ \sum_{n=2}^{\infty} \left(\frac{\gamma^n}{n!} + \sum_{k=0}^{n-2} \frac{(-1)^{n-k}\gamma^k}{k!} \sum_{\substack{k_1+\cdots+k_r=n-k \\ r\geq 1, \forall k_i \geq 2}} (-1)^r \frac{(k_1-1)\cdots(k_r-1)}{k_1!\cdots k_r!} \zeta(k_1,\ldots,k_r) \right) x^{n+1}.$$

7.1 Proof

Set

$$S(k) := \sum_{\substack{k_1+\cdots+k_r=k \\ r\geq 1, \forall k_i \geq 2}} (-1)^r \frac{(k_1-1)\cdots(k_r-1)}{k_1!\cdots k_r!} [k_1,\ldots,k_r]$$

for $k \geq 2$, and put $S(0)=1, S(1)=0$. By taking the $\log_*$ of both sides of the equation (28) and then taking $x \cdot \partial/\partial x$, we see that (since both sides are 1 for $x=0$) equation (28) is equivalent to

$$\left(\sum_{n=2}^{\infty} (-1)^{n-1}[n]x^n\right) * \left(\sum_{m=0}^{\infty} (-1)^m S(m) x^m\right) = \sum_{n=2}^{\infty} (-1)^n n S(n) x^n,$$

which in turn is equivalent to

$$\sum_{m=2}^{n} [m] * S(n-m) = -nS(n)$$

for $n \geq 2$. We compute $[m]*[k_1,\ldots,k_r]$ in $[m]*S(n-m)$ by the harmonic product:

$$\begin{aligned} L.H.S. &= \sum_{m=2}^{n} \sum_{\substack{k_1+\cdots+k_r=n-m \\ r\geq 1, \forall k_i \geq 2}} (-1)^r \frac{(k_1-1)\cdots(k_r-1)}{k_1!\cdots k_r!} [m]*[k_1,\ldots,k_r] \\ &= \sum_{r=1}^{\left[\frac{n-2}{2}\right]} (-1)^r \sum_{m=2}^{n-2r} \sum_{\substack{k_1+\cdots+k_r=n-m \\ \forall k_i \geq 2}} \frac{(k_1-1)\cdots(k_r-1)}{k_1!\cdots k_r!} \\ &\quad \times \left(\sum_{l=1}^{r} [k_1,\ldots,k_l+m,\ldots,k_r] + \sum_{l=1}^{r+1} [k_1,\ldots,\underset{l\text{-th}}{\underset{\wedge}{m}},\ldots,k_r] \right). \end{aligned}$$

On the other hand,

$$R.H.S. = -n \sum_{r=1}^{\left[\frac{n}{2}\right]} (-1)^r \sum_{\substack{k_1+\cdots+k_r=n \\ \forall k_i \geq 2}} \frac{(k_1-1)\cdots(k_r-1)}{k_1!\cdots k_r!} [k_1,\ldots,k_r].$$

Therefore, the proof is completed if we show that the terms of length r on both sides coincide for each r, i.e.,

$$
\begin{aligned}
&\sum_{m=2}^{n-2r} \sum_{\substack{k_1+\cdots+k_r=n-m \\ \forall k_i \geq 2}} \frac{(k_1-1)\cdots(k_r-1)}{k_1!\cdots k_r!} \sum_{l=1}^{r} [k_1,\ldots,k_l+m,\ldots,k_r] \\
&- \sum_{m=2}^{n-2r+2} \sum_{\substack{k_1+\cdots+k_{r-1}=n-m \\ \forall k_i \geq 2}} \frac{(k_1-1)\cdots(k_{r-1}-1)}{k_1!\cdots k_{r-1}!} \sum_{l=1}^{r} [k_1,\ldots,\underset{l\text{-th}}{\overset{}{\underset{\wedge}{m}}},\ldots,k_{r-1}] \\
=& -n \sum_{\substack{k_1+\cdots+k_r=n \\ \forall k_i \geq 2}} \frac{(k_1-1)\cdots(k_r-1)}{k_1!\cdots k_r!} [k_1,\ldots,k_r].
\end{aligned} \tag{29}
$$

Let us compute the first sum on the left-hand side of the last equation (29) by putting $k_l+m=h$. Then, we have

$$
\begin{aligned}
&\sum_{m=2}^{n-2r} \sum_{\substack{k_1+\cdots+k_r=n-m \\ \forall k_i \geq 2}} \frac{(k_1-1)\cdots(k_r-1)}{k_1!\cdots k_r!} \sum_{l=1}^{r} [k_1,\ldots,k_l+m,\ldots,k_r] \\
=& \sum_{l=1}^{r} \sum_{m=2}^{n-2r} \sum_{h=m+2}^{n-2r+2} \sum_{\substack{k_1+\cdots+k_{r-1}=n-h \\ \forall k_i \geq 2}} \frac{(k_1-1)\cdots(k_{r-1}-1)}{k_1!\cdots k_{r-1}!} \frac{(h-m-1)}{(h-m)!} [k_1,\ldots,\underset{l\text{-th}}{\underset{\wedge}{h}},\ldots,k_{r-1}] \\
=& \sum_{l=1}^{r} \sum_{h=4}^{n-2r+2} \sum_{\substack{k_1+\cdots+k_{r-1}=n-h \\ \forall k_i \geq 2}} \frac{(k_1-1)\cdots(k_{r-1}-1)}{k_1!\cdots k_{r-1}!} \left(\sum_{m=2}^{h-2} \frac{(h-m-1)}{(h-m)!} \right) [k_1,\ldots,\underset{l\text{-th}}{\underset{\wedge}{h}},\ldots,k_{r-1}].
\end{aligned}
$$

Since

$$
\sum_{m=2}^{h-2} \frac{(h-m-1)}{(h-m)!} = \sum_{m=2}^{h-2} \left(\frac{1}{(h-m-1)!} - \frac{1}{(h-m)!} \right) = -\frac{1}{(h-2)!} + 1,
$$

the first sum of (29) is equal to

$$
-\sum_{l=1}^{r} \sum_{h=2}^{n-2r+2} \sum_{\substack{k_1+\cdots+k_{r-1}=n-h \\ \forall k_i \geq 2}} \frac{(k_1-1)\cdots(k_{r-1}-1)}{k_1!\cdots k_{r-1}!} \left(h\frac{(h-1)}{h!} \right) [k_1,\ldots,\underset{l\text{-th}}{\underset{\wedge}{h}},\ldots,k_{r-1}] \tag{30}
$$

$$
+\sum_{l=1}^{r} \sum_{h=2}^{n-2r+2} \sum_{\substack{k_1+\cdots+k_{r-1}=n-h \\ \forall k_i \geq 2}} \frac{(k_1-1)\cdots(k_{r-1}-1)}{k_1!\cdots k_{r-1}!} [k_1,\ldots,\underset{l\text{-th}}{\underset{\wedge}{h}},\ldots,k_{r-1}]. \tag{31}
$$

Because the terms of $h = 2, 3$ in (30) cancel out the terms of $h = 2, 3$ in (31) respectively, we include the terms of $h = 2, 3$ in the sums. Replacing h in the first sum by k_l, we have

$$\begin{aligned}
&-\sum_{l=1}^{r} \sum_{\substack{k_1+\cdots+k_r=n \\ {}^\forall k_i \geq 2}} k_l \frac{(k_1-1)\cdots(k_r-1)}{k_1!\cdots k_r!}[k_1,\ldots,k_l,\ldots,k_r] \\
&+\sum_{l=1}^{r} \sum_{h=2}^{n-2r+2} \sum_{\substack{k_1+\cdots+k_{r-1}=n-h \\ {}^\forall k_i \geq 2}} \frac{(k_1-1)\cdots(k_{r-1}-1)}{k_1!\cdots k_{r-1}!}[k_1,\ldots,\underset{l\text{-th}}{\overset{}{\underset{\wedge}{h}}},\ldots,k_{r-1}] \\
=&-n \sum_{\substack{k_1+\cdots+k_r=n \\ {}^\forall k_i \geq 2}} \frac{(k_1-1)\cdots(k_r-1)}{k_1!\cdots k_r!}[k_1,\ldots,k_r] \\
&+\sum_{h=2}^{n-2r+2} \sum_{\substack{k_1+\cdots+k_{r-1}=n-h \\ {}^\forall k_i \geq 2}} \frac{(k_1-1)\cdots(k_{r-1}-1)}{k_1!\cdots k_{r-1}!} \sum_{l=1}^{r}[k_1,\ldots,\underset{l\text{-th}}{\underset{\wedge}{h}},\ldots,k_{r-1}].
\end{aligned}$$

This gives equation (29) and hence completes the proof. $\square$

References

[1] K. Aomoto, Special values of hyperlogarithms and linear difference schemes, Illinois J. Math., **34-2** (1990), 191–216.

[2] T. Arakawa and M. Kaneko, Multiple zeta values, poly-Bernoulli numbers, and related zeta functions, Nagoya Math. J., **153** (1999), 189–209.

[3] T. Arakawa and M. Kaneko, On multiple L-values, J. Math. Soc. Japan, **56** (2004), 967–991.

[4] T. Arakawa and M. Kaneko, Introduction to Multiple Zeta Values (in Japanese). MI Lecture Note Series, vol. **23**. Kyushu University, Kyushu (2010).

[5] F. C. S. Brown, Mixed Tate motives over $\mathbb{Z}$, Annals of Math., **175** (2012), 949–976.

[6] P. Deligne and A. Goncharov, Groupes fondamentaux motiviques de Tate mixte, Ann. Sci. Ecole Norm. Sup. (4) **38** (2005), 1–56.

[7] V. G. Drinfel'd, On quasitriangular quasi-Hopf algebras and a group closely connected with Gal$(\bar{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J., **2** (1991), 829–860.

[8] A. B. Goncharov, Multiple ζ-values, Galois groups, and geometry of modular varieties, in European Congress of Mathematics (Barcelona, 2000), Vol. I, Progress in Math., **201**, Birkhäuser, Basel, (2001), 361–392.

[9] A. Granville, A decomposition of Riemann's zeta-function, in London Math. Soc. Lecture Note Ser. **247**, Cambridge, 1997, 95–101.

[10] M. Hoffman, Multiple harmonic series, Pacific J. Math., **152** (1992), 275–290.

[11] M. Hoffman, The algebra of multiple harmonic series, J. Algebra, **194** (1997), 477–495.

[12] M. Hoffman, Quasi-symmetric functions and mod p multiple harmonic sums, Kyushu J. Math., **69** (2015), 345–366.

[13] K. Ihara, M. Kaneko and D. Zagier, Derivation and double shuffle relations for multiple zeta values, Compositio Math., **142** (2006), 307–338.

[14] M. Kaneko, Finite multiple zeta values mod p and relations among multiple zeta values (in Japanese), Sûrikaisekikenkyûsho Kôkyûroku (2012), no. **1813**, 27–31, Various aspects of multiple zeta values (Kyoto, 2010).

[15] M. Kaneko, Finite multiple zeta values (in Japanese), to appear in RIMS Kôkyûroku Bessatsu.

[16] M. Kaneko and M. Sakata, On multiple zeta values of extremal height, Bull. Aust. Math. Soc., **93** (2016), 186–193.

[17] M. Kaneko and H. Tsumura, Multi-poly-Bernoulli numbers and related zeta functions, to appear in Nagoya Math. J., arXiv:1503.02156[NT].

[18] M. Kaneko and D. Zagier, Finite multiple zeta values, in preparation.

[19] T. Q. T. Le and J. Murakami, Kontsevich's integral for the Homfly polynomial and relations between values of the multiple zeta functions, Topology Appl., **62** (1995), 193–206.

[20] S. Muneta, On some explicit evaluations of multiple star-zeta values, J. Number Theory, **128** (2008), 2538–2548.

[21] H. Murahara, A note on finite real multiple zeta values, Kyushu J. Math., **70** (2016), 197–204.

[22] H. Murahara, Derivation relations for finite multiple zeta values, to appear in Int. J. Number Theory, arXiv:1411.1123.

[23] H. Murahara and M. Sakata, On multiple zeta values and finite multiple zeta values of maximal height, preprint, arXiv:1612.04071[NT].

[24] Y. Ohno, A generalization of the duality and sum formulas on the multiple zeta values, J. Number Theory, **74** (1999), 39–43.

[25] Y. Ohno and D. Zagier, Multiple zeta values of fixed weight, depth, and height, Indag. Math., **12** (2001), 483–487.

[26] K. Oyama, Ohno's relation for finite multiple zeta values, preprint, arXiv:1506.00833[NT].

[27] G. Racinet, Doubles mélanges des polylogarithmes multiples aux racines de l'unité, Publ. Math. Inst. Hautes Études Sci., **95** (2002), 185–231.

[28] S. Saito and N. Wakabayashi, Sum formula for finite multiple zeta values, J. Math. Soc. Japan, **67** (2015), 1069–1076.

[29] M. Sakata, Taylor series for the reciprocal gamma function and multiple zeta values, preprint.

[30] T. Tanaka, On the quasi-derivation relation for multiple zeta values, J. Number Theory, **129** (2009), 2021–2034.

[31] T. Terasoma, Mixed Tate motives and multiple zeta values, Invent. Math., **149** (2002), 339–369.

[32] S. Yamamoto, On multiple sums of multiple zeta values (in Japanese), unpublished note.

[33] D. Zagier, Values of zeta functions and their applications, in First European Congress of Mathematics, Volume II, Progress in Math., **120**, Birkhäuser-Verlag, Basel, (1994), 497–512.

[34] S. A. Zlobin, Generating functions for a multiple zeta function (in Russian), Vestnik Moskov. Univ. Ser. I Mat. Mekh. 2005, 55–59; English translation in Moscow Univ. Math. Bull., **60** (2005), 44–48.