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# Arithmetic topology on braid and absolute Galois groups


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Abstract

Y. Ihara initiated the arithmetic study of a certain Galois representation that may be seen as an arithmetic analogue of the Artin representation of a pure braid group. In this thesis, we study arithmetic analogies in Ihara theory further, following after some topics in the theory of braids, and try to develop arithmetic topology in a new direction toward quantum topology. More concrete contents are as follows.

This thesis consists of a topological part (Chapters 1, 2) and an arithmetic part (Chapters 3, 4). The topological part is concerned with topics such as Milnor invariants, Johnson homomorphisms, and Gassner representations for the pure braid group, as well as their inter-relations. We give a group-theoretic exposition that serves as a useful guide for the study of the arithmetic counterpart. In the arithmetic part, we pursue the analogues of the topological part in the context of Ihara theory. We introduce $l$-adic Milnor invariants, pro-$l$ Johnson homomorphisms, and pro-$l$ Gassner representations for the absolute Galois group of a number field, and study their properties and inter-relations. We give arithmetic-topological interpretations of Jacobi sums and the Ihara power series in terms of $l$-adic Milnor numbers.

This thesis is based on [**Ko1**], [**Ko2**], and [**KMT**].

# Contents

# Notation

We denote by $\mathbb{Z}$, $\mathbb{Q}$ and, $\mathbb{C}$ the ring of rational integers, the field of rational numbers, and the field of complex numbers, respectively.

Throughout this paper, $l$ denotes a fixed prime number. We denote by $\mathbb{Z}_l$ and $\mathbb{Q}_l$ the ring of $l$-adic integers and the field of $l$-adic numbers, respectively.

For $a, b$ in a group $G$, $a \sim b$ means that $a$ is conjugate to $b$ in $G$. For subgroups $A, B$ of a (topological) group $G$, $[A, B]$ stands for the (closed) subgroup of $G$ generated by all the commutators $[a, b] := aba^{-1}b^{-1}$ with $a \in A, b \in B$.

For a group $G$, we define its lower central series by

$$\Gamma_1 G := G, \quad \Gamma_k G := [\Gamma_{k-1} G, G] \ (k \geqslant 2).$$

For each $k \geqslant 1$, we set

$$\mathrm{gr}_k(G) := \Gamma_k G / \Gamma_{k+1} G.$$

For a positive integer $n$ and a ring $R$ with an identity element, $\mathrm{M}(n; R)$ denotes the ring of $n \times n$ matrices with entries in $R$, and $\mathrm{GL}(n; R)$ denotes the group of invertible elements of $\mathrm{M}(n; R)$. We denote the group of invertible elements of $R$ by $R^\times$.

Throughout this paper, we will write the composition in a fundamental group from the left, i.e., $\gamma\gamma'$ means to go along $\gamma$ first and $\gamma'$ next.

# Intoroduction

In the early part of the 20th century, E. Artin began a mathematical study of braids and, among other things, found a representation of braid groups, called the Artin representation today ([**Ar**]). Since then, braid theory has developed as a research area in low dimensional topology, and it has provided rich soil for the growth of *quantum topology* that started with the discovery of the Jones polynomials in the 1980's.

In 1986, Y. Ihara initiated a study of a certain representation of the absolute Galois group of a number field, which may be seen as an arithmetic analogue of the Artin representation, and revealed its rich structure in connection with deep arithmetic such as Iwasawa theory on cyclotomy and complex multiplications of Fermat Jacobians ([**Ih1**]). Ihara's work has been developed extensively in the field of arithmetic algebraic geometry, including Grothendieck-Teichmüller theory, anabelian geometry, and multiple zeta values, etc.

In recent years, *arithmetic topology* has developed into a guiding principle for obtaining parallel results and analogies between three-dimensional topology and number theory ([**Ms2**]). In particular, it is known that there are intimate analogies between knot theory and Iwasawa theory. These analogies are mainly based on analogies between Galois groups (resp. ideal class groups of number fields) and 3-manifold groups (resp. homology groups of 3-manifolds).

This thesis is motivated by the general view that the position of Ihara theory relative to Iwasawa theory in number theory may be similar to that of braid theory relative to knot theory in low dimensional topology:

$$\begin{array}{ccc} & \text{Arithmetic topology} & \\ \text{Knot theory} & \longleftrightarrow & \text{Iwasawa theory} \\ \Downarrow & & \Downarrow \\ \text{Braid theory} & \longleftrightarrow & \text{Ihara theory} \end{array}$$

On the basis of this viewpoint, in this thesis, we go back to Ihara's original idea on the analogy between braid groups and absolute Galois groups and study the analogy systematically. We hope to extend arithmetic topology by drawing analogies between quantum topology and Ihara theory in the future.

Now let us introduce a basic dictionary of analogies that we will use in this thesis. We recall the analogy between the Ihara representation of the absolute Galois group of a number field and the Artin representation of a pure braid group.

Let $l$ be a prime number. Let $S := \{P_0, \ldots, P_r\}$ be a set of ordered $r+1$ ($r \geqslant 2$) distinct $\overline{\mathbb{Q}}$-rational points $P_i$ ($0 \leqslant i \leqslant r$) on the projective line $\mathbb{P}^1$ over the rational number field $\mathbb{Q}$, where $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$. Let $k := \mathbb{Q}(S \setminus \{\infty\})$, the finite algebraic number field generated by coordinates of points in $S \setminus \{\infty\}$. Note that the absolute Galois group $\mathrm{Gal}_k := \mathrm{Gal}(\overline{\mathbb{Q}}/k)$ is the étale fundamental group of $\mathrm{Spec}\, k$. Thus, it acts on the geometric fiber $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{P_0, \ldots, P_r\}$ of the fibration $\mathbb{P}^1_k \setminus \{P_0, \ldots, P_r\} \to \mathrm{Spec}\, k$ and hence on the pro-$l$ étale fundamental group $\pi_1^{\text{pro-}l}(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{P_0, \ldots, P_r\}) \simeq \mathfrak{F}_r$, where $\mathfrak{F}_r$ denotes the free pro-$l$ group on the $r$ generators $x_1, \ldots, x_r$. In [**Ih1**], Ihara initiated the study of this monodromy Galois representation

$$\mathrm{Ih}_S : \mathrm{Gal}_k \longrightarrow \mathrm{Aut}(\mathfrak{F}_r), \tag{0.0.1}$$

particularly for the case $S = \{0, 1, \infty\}$ and $k = \mathbb{Q}$, in connection with deep arithmetic such as Iwasawa theory on cyclotomy and complex multiplications of Fermat Jacobians. We note that the image of $\mathrm{Ih}_S$ is contained in the subgroup consisting of elements $\varphi \in \mathrm{Aut}(\mathfrak{F}_r)$ such that $\varphi(x_i) \sim x_i^{\alpha}$ (conjugate) for $1 \leqslant i \leqslant r$ and $\varphi(x_1 \cdots x_r) = (x_1 \cdots x_r)^{\alpha}$ for some $\alpha \in \mathbb{Z}_l^{\times}$.

As explained in [**Ih3**], the Ihara representation (0.0.1) may be regarded as an arithmetic analogue of the Artin representation of a pure braid group ([**Ar**]). Let $PB_r$ be the pure braid group with $r$ strings ($r \geqslant 2$). Note that $PB_r$ is the topological fundamental group of the configuration space $\mathrm{Config}_r(D^2)$ of ordered $r$ points on a 2-dimensional disk $D^2$. For $1 \leqslant i \leqslant r$, let $p_i$ be mutually distinct interior points of $D^2$. They define the point $(p_1, \ldots, p_r) \in \mathrm{Config}_r(D^2)$. Then $PB_r$ acts, as the monodromy, on the fiber $D^2 \setminus \{p_1, \ldots, p_r\}$ of the universal bundle over the point $(p_1, \ldots, p_r) \in \mathrm{Config}_r(D^2)$ and hence on the topological fundamental group $\pi_1(D^2 \setminus \{p_1, \ldots, p_r\}) \simeq F_r$, where $F_r$ denotes the free group on $r$ generators $x_1, \ldots, x_r$ and each $x_i$ is identified with the isotopy class of a loop encircling $p_i$ clockwise with a base point on the boundary $\partial D^2$. Thus we have the Artin representation

$$\mathrm{Ar}_r : PB_r \longrightarrow \mathrm{Aut}(F_r). \tag{0.0.2}$$

This map is an injection and its image is generated by elements $\varphi \in \mathrm{Aut}(F_r)$ such that $\varphi(x_i) \sim x_i$ for $1 \leqslant i \leqslant r$ and $\varphi(x_1 \cdots x_r) = x_1 \cdots x_r$.

We can see the following analogy between the Ihara representation (0.0.1) and the Artin representation (0.0.2):

(0.0.3)

| absolute Galois group $\mathrm{Gal}_k$ | pure braid group $PB_r$ |
|---|---|
| $\mathbb{P}^1_k \setminus \{P_0, \ldots, P_r\} \to \mathrm{Spec}\, k$ with geometric fiber $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{P_0, \ldots, P_r\}$ | universal bundle over $\mathrm{Config}_r(D^2)$ with fibers $D^2 \setminus \{p_1, \ldots, p_r\}$ |
| Ihara representation of $\mathrm{Gal}_k$ on $\pi_1^{\text{pro-}l}(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{P_0, \ldots, P_r\}) = \mathfrak{F}_r$ | Artin representation of $PB_r$ on $\pi_1(D^2 \setminus \{p_1, \ldots, p_r\}) = F_r$ |

In this thesis, with the help of the dictionaries (0.0.3), we shall investigate the arithmetic analogues in Ihara theory of the following issues and their inter-relations:

(I) Milnor invariants of links,
(II) Johnson homomorphisms for the pure braid group $PB_r$,
(III) Magnus-Gassner representations of $PB_r$,
(IV) Alexander invariants of links.

Chapters 1 and 2 deal with the topological side of these issues: Chapter 1 covers mainly (I) and (II): the Milnor invariants of a link are the higher order linking numbers. They are defined by the coefficients of the Magnus expansion of a longitude by meridians ([**Mi**]) and are interpreted in terms of Massey products in the cohomology of the link group ([**Ki**], [**T**]). Johnson homomorphisms are useful means of studying the structure of the mapping class group of a surface ([**J1**], [**J2**], [**Mt1**], [**Mt2**]). The main tools are algebraic and applicable to the study of the automorphism group of a free group ([**Ka**], [**Sa**]). Since the pure braid group $PB_r$ is a subgroup of the mapping class group of an $r$ punctured disk, the theory of Johnson homomorphisms is also applicable to $PB_r$. In Chapter 1, we show that the Johnson homomorphisms are described by Milnor invariants of pure braid links.

Chapter 2 covers mainly (III): Magnus cocycles are crossed homomorphisms of $PB_r$ defined by using the Fox free derivative ([**Bi1**, 3.1, 3.2], [**F**]). The Gassner representation is a particular case of Magnus cocycles, and it provides multivariable link invariants called the Alexander invariants ([**Bi1**, 3.3]). We show the relations of the Gassner representations with Johnson homomorphisms and Milnor invariants.

Chapters 3 and 4 deal with the arithmetic side of the above materials: In Chapter 3, we define $l$-adic Milnor invariants and the pro-$l$ Johnson homomorphism for absolute Galois groups. Among other things, we prove the following theorem that is suggested by the Alexander–Markov theorem of braid theory. This "translation" supports the idea of an analogy between braid groups and absolute Galois groups.

Let $\mathrm{Ih}_S : \mathrm{Gal}_k \to \mathrm{Aut}(\mathfrak{F}_r)$ be the Ihara action and $\chi_l : \mathrm{Gal}_k \to \mathbb{Z}_l^\times$ denote the $l$-cyclotomic character. For $g \in \mathrm{Gal}_k$, it turns out that there exists a unique word $y_i(g) \in \mathfrak{F}_r$ $(1 \leqslant i \leqslant r)$ such that $\mathrm{Ih}_S(g)(x_i) = y_i(g)x_i^{\chi_l(g)}y_i(g)^{-1}$ and the coefficient of the class of $x_i$ is 0 in the abelianization of $\mathfrak{F}_r$. We call the word $y_i(g) \in \mathfrak{F}_r$ the $i$-th longitude of $g \in \mathrm{Gal}_k$. We denote by $\mathbb{Z}\langle\langle X_1, \dots, X_r\rangle\rangle$ the ring of non-commutative formal power series over $\mathbb{Z}_l$ with variables $X_1, \dots, X_r$ and let $\Theta : \mathfrak{F}_r \to \mathbb{Z}_l\langle\langle X_1, \dots, X_r\rangle\rangle$ be the pro-$l$ Magnus embedding. Let us consider the pro-$l$ Magnus embedding of $y_i(g)$:

$$\Theta(y_i(g)) = 1 + \sum_{n\geqslant 1} \sum_{\substack{I=(i_1\cdots i_n)\\ 1\leqslant i_1,\dots,i_n\leqslant r}} \mu(g; i_1 \cdots i_r i)X_{i_1} \cdots X_{i_n}$$

For a muti-index $I$, we call the coefficient $\mu(g; I)$ the Milnor number of $g$ with respect to $I$ and we define the $l$-adic Milnor invariant $\bar{\mu}(g; I)$ of $g$ for $I$ to be the $l$-adic Milnor number $\mu(g; I)$ modulo a certain ideal $\Delta(g; I)$ of $\mathbb{Z}_l$:

$$\bar{\mu}(g; I) := \mu(g; I) \bmod \Delta(g; I).$$

Then, we have the following proposition.

Theorem 3.2.20. *For a multi-index $I$, the $l$-adic Milnor invariant $\overline{\mu}(g; I)$ of $g \in \mathrm{Gal}_k$ is preserved under the conjugate action of $\mathrm{Gal}_{k(\zeta_{l^\infty})} \subset \mathrm{Gal}_k$. More precisely, let $I$ be a multi-index with $|I| \geqslant 1$. Let $g \in \mathrm{Gal}_k$ and $h \in \mathrm{Gal}_{k(\zeta_{l^\infty})}$. Then we have $\Delta(hgh^{-1}; I) = \Delta(g; I)$ and the following equality holds:*

$$\overline{\mu}(hgh^{-1}; I) = \overline{\mu}(g; I).$$

In Chapter 4, we introduce the notion of pro-$l$ reduced Gassner representations and study the Ihara power series from the arithmetic topological viewpoints. Among other things, we give an arithmetic topological interpretation of Jacobi sums: Let $p$ be a rational prime that satisfies certain conditions on ramifications and let $\overline{p}$ be a prime of $\overline{\mathbb{Q}}$ lying over $p$. Then, $\overline{p}$ is unramified in $\overline{\mathbb{Q}}/\mathbb{Q}$ so that the Frobenius automorphism $\sigma_{\overline{p}} \in \mathrm{Gal}_{\mathbb{Q}}$ is defined. Let $n$ be a fixed positive integer and let $\mathfrak{p}_n$ be the prime of $\mathbb{Q}(\zeta_{l^n})$ lying below $\overline{p}$, where $\zeta_{l^n} \in \overline{\mathbb{Q}}$ denotes a primitive $l^n$-th root of unity. Let $\left(\frac{x}{\mathfrak{p}_n}\right)_{l^n}$ be the $l^n$-th power residue symbol at $\mathfrak{p}_n$ for a unit $x \in (\mathbb{Z}[\zeta_{l^n}]/\mathfrak{p}_n)^\times$. For $0 \neq a, b \in \mathbb{Z}/l^n\mathbb{Z}$ with $(a, b, l) = 1$, we define the Jacobi sum by

$$J_{l^n}(\mathfrak{p}_n)^{(a,b)} := \sum_{\substack{x,y \in (\mathbb{Z}[\zeta_{l^n}]/\mathfrak{p}_n)^\times \\ x+y=-1}} \left(\frac{x}{\mathfrak{p}_n}\right)_{l^n}^a \left(\frac{y}{\mathfrak{p}_n}\right)_{l^n}^b.$$

For a multi-index $I = (i_1 \cdots i_n)$ and $j \in \{1, 2\}$, let $|I|_j$ denote the number of entries $i_k$ satisfying $i_k = j$.

For integers $n_1, n_2 \geqslant 0$ with $n_1 + n_2 \geqslant 1$ and $g \in \mathrm{Gal}_{\mathbb{Q}}$, we set

$$\mu(g; n_1, n_2) := \sum_{|I|_1 = n_1 - 1, |I|_2 = n_2} \mu(g; I12) + \sum_{|I|_1 = n_1, |I|_2 = n_2 - 1} \mu(g; I21)$$

where $\mu(g; J)$ denotes the Milnor number of $g$ with respect to the multi-index $J$.

Let $f$ denote the order of $p$ in $(\mathbb{Z}/l^n\mathbb{Z})^\times$. Then, we have the following theorem.

Theorem 4.3.5. Given the above notation, the Jacobi sum and the $l$-adic Milnor invariants satisfy

$$J_{l^n}(\mathfrak{p}_n)^{(a,b)} = 1 + \sum_{\substack{n_1, n_2 \geqslant 0 \\ n_1 + n_2 \geqslant 1}} \mu(\sigma_{\overline{p}}^f; n_1, n_2)(\zeta_{l^n}^a - 1)^{n_1}(\zeta_{l^n}^b - 1)^{n_2}.$$

We also show a formula that relates $l$-adic Milnor invariants to Soulé characters: For $a \in \mathbb{Z}/l^n\mathbb{Z}$, let $\langle a \rangle_{l^n}$ denote the integer satisfying $a = \langle a \rangle_{l^n} \bmod l^n$ with $0 \leqslant \langle a \rangle_{l^n} < l^n$. For a positive integer $m$, we set

$$\epsilon_{l^n}^{(m)} := \prod_{a \in (\mathbb{Z}/l^n\mathbb{Z})^\times} (\zeta_{l^n} - 1)^{\langle a^{m-1} \rangle_{l^n}},$$

which is an $l$-unit in $\mathbb{Q}(\zeta_{l^n})$, called a cyclotomic $l$-unit. Then, we define the $m$-th $l$-adic Soulé character $\chi^{(m)} : \mathrm{Gal}_{\mathbb{Q}} \to \mathbb{Z}_l$ as the Kummer cocycle attached to the system of cyclotomic $l$-units $\{\epsilon_{l^n}^{(m)}\}_{n \geqslant 1}$ as

$$\zeta_{l^n}^{\chi^{(m)}(g)} = \{(\epsilon_{l^n}^{(m)})^{1/l^n}\}^{g-1} (n \geqslant 1, g \in \mathrm{Gal}_{\mathbb{Q}}).$$

In addition, we set

$$\kappa_m(g) := \frac{\chi^{(m)}(g)}{1 - l^{m-1}} \quad (g \in \mathrm{Gal}_{\mathbb{Q}}).$$

Then, we have the following theorem.

Theorem 4.3.8. *Let $g \in \mathrm{Gal}_{\mathbb{Q}(\zeta_{l^\infty})}$ and let $N_1$, $N_2$ be integers with $N_1, N_2 \geqslant 0$ and $N_1 + N_2 \geqslant 1$. Then, the following equality holds:*

$$\sum_{\substack{n_1 + n_2 \geqslant 1 \\ 0 \leqslant n_1 \leqslant N_1, 0 \leqslant n_2 \leqslant N_2}} \mu(g; n_1, n_2) a_{n_1}(N_1) a_{n_2}(N_2)$$
$$= \sum_{1 \leqslant n \leqslant N_1 + N_2} \left( \frac{(-1)^n}{n!} \sum \frac{\kappa_{m_1^{(1)} + m_2^{(1)}}(g)}{m_1^{(1)}! \, m_2^{(1)}!} \cdots \frac{\kappa_{m_1^{(n)} + m_2^{(n)}}(g)}{m_1^{(n)}! \, m_2^{(n)}!} \right).$$

*Here, the last sum ranges over the integers* $m_1^{(1)}, \dots, m_1^{(n)}, m_2^{(1)}, \dots, m_2^{(n)} \geqslant 0$ *such that* $m_1^{(i)} + m_2^{(i)} \geqslant 3$ *and* $m_1^{(i)} + m_2^{(i)}$ *is odd* $(1 \leqslant i \leqslant n)$, $m_1^{(1)} + \cdots + m_1^{(n)} = N_1$ *and* $m_2^{(1)} + \cdots + m_2^{(n)} = N_2$. *For each* $j = 1, 2$, *we put*

$$a_{n_j}(N_j) := \begin{cases} 1 & (n_j = 0) \\ \displaystyle\sum_{\substack{e_1, \dots, e_{n_j} \geqslant 1 \\ e_1 + \cdots + e_{n_j} = N_j}} \frac{1}{e_1! \cdots e_{n_j}!} & (n_j \geqslant 1). \end{cases}$$

CHAPTER 1

# Pure braid groups, Milnor invariants, and Johnson homomorphisms

In this chapter, we recall the definitions of pure braid groups, Milnor invariants and Johnson homomorphisms and show their relations. More precisely, by regarding a pure braid as a mapping class of the punctured disk, we show that the Johnson homomorphism of a pure braid can be viewed as being essentially the same as the first-non-vanishing Milnor invariants of the link obtained by closing a pure braid. Moreover, we give a description of the Johnson homomorphism of a pure braid in terms of the Massey product of the associated mapping torus. This chapter is based on [**Ko1**].

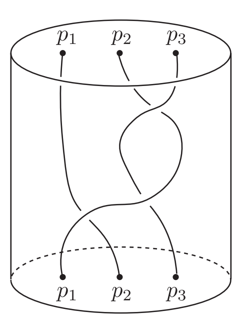
## 1.1. Pure braid groups and Artin representations

Here, we recall the definition of the pure braid group and its interpretation as the mapping class group of the punctured unit disk in the complex plane. Then, we recall the action of the pure braid group on the free group, called the Artin representation, as the induced action of the pure braid group on the fundamental group of the punctured disk.

**1.1.1. Pure braid groups.** Let $r$ be a integer with $r \geqslant 2$. Let $D^2$ be the unit disc in the complex plane $\mathbb{C}$ with center $(\frac{1}{2}, 0)$ and $p_i = (\frac{i}{r+1}, 0)$ $(1 \leqslant i \leqslant r)$ be a point in $D^2$. Let $I$ denote the unit interval $[0,1]$ and $I_i$ $(1 \leqslant i \leqslant r)$ denote its copy. We consider an embedding $b : \bigsqcup_{i=1}^{r} I_i$ (disjoint union) $\to D^2 \times I$ satisfying the following conditions:

(1) $b_i(0) = p_i$, $b_i(1) = p_{k_i}$ for some $k_i$ $(1 \leqslant k_i \leqslant r)$ with $k_i \neq k_j$ $(i \neq j)$
(2) $b_i(t) \in D^2 \times \{t\}$

where we denote the restriction of $b$ to $I_i$ by $b_i$. Note that from the definition $b$ induces the permutation $\overline{b}$ of $\{1, \ldots, r\}$. Hence, condition (1) is written as $b_i(0) = p_i, b_i(1) = p_{\overline{b}(i)}$. Such a $b$ is called a *braid*, and the $b_i$ $(1 \leqslant i \leqslant r)$ are called *strings*. We often identify a braid $b$ and its image $b(\bigsqcup_{i=1}^{r} I_i)$ in $D^2 \times I$.



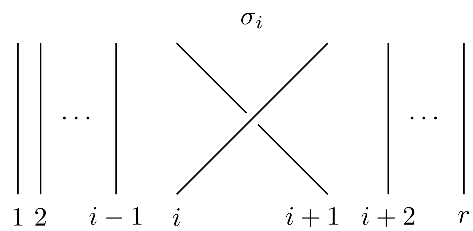
By projecting the image of $b$ to the plane $\mathbb{R} \times I$, we get its *braid diagram*, which possesses information on the crossings of the strands. In a braid diagram, we draw $D^2 \times \{0\}$ as the bottom plane and $D^2 \times \{1\}$ as the top plane.

For two braids $b, b'$, we say that $b$ and $b'$ are isotopic if there is a level preserving ambient isotopy $H_t : D^2 \times I \to D^2 \times I$ $(t \in [0,1])$ such that $H_t$ fixes the boundary of $D^2 \times I$ and $H_0 = \mathrm{id}, H_1(b) = b'$.

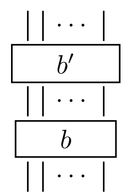
Now, let $B_r$ be the group of isotopy classes of braids. It is called the *braid group with $r$ strings* and is generated by $\sigma_1, \ldots, \sigma_{r-1}$ satisfying the following relations:

$$\begin{array}{rcll} \sigma_i\sigma_{i+1}\sigma_{i+1} & = & \sigma_{i+1}\sigma_i\sigma_{i+1} & (1 \leqslant i \leqslant r-2) \\ \sigma_i\sigma_j & = & \sigma_j\sigma_i & (\mid i-j \mid > 1). \end{array}$$

The following braid diagram depicts each generator $\sigma_i$.




For $b, b' \in B_r$, its product $bb'$ is defined by stacking $b'$ on $b$, like in the following picture:



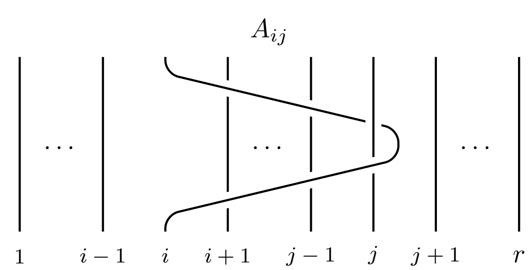

Now there exists a natural surjective homomorphism,

$$\chi : B_r \longrightarrow S_r; \quad b \mapsto \overline{b}$$

where $S_r$ denotes the $r$-th symmetric group. We set $PB_r := \mathrm{Ker}(\chi)$ and call it the *pure braid group of $n$ strings*. Each generator $A_{ij}$ $(1 \leqslant i < j \leqslant r)$ of $PB_r$ is presented in terms of a generator $\sigma_i$ $(1 \leqslant i \leqslant r-1)$ of $B_r$:

$$A_{ij} = \sigma_i\sigma_{i+1}\cdots\sigma_{j-2}\sigma_{j-1}^2\sigma_{j-2}^{-1}\cdots\sigma_{i+1}^{-1}\sigma_i^{-1}$$

which is depicted as the following braid diagram.



The generators $A_{ij}$ $(1 \leqslant i < j \leqslant r)$ of $PB_r$ are subject to the following relations:

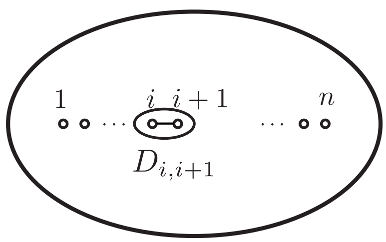
$$A_{rs}A_{ij}A_{rs}^{-1} = \begin{cases} A_{ij} & (\text{if } s < i \text{ or } i < r < s < j), \\ A_{rj}^{-1}A_{ij}A_{rj} & (\text{if } s = i), \\ A_{rj}^{-1}A_{sj}^{-1}A_{ij}A_{sj}A_{rj} & (\text{if } i = r < s < j), \\ A_{rj}^{-1}A_{sj}^{-1}A_{rj}A_{sj}A_{ij}A_{sj}^{-1}A_{rj}^{-1}A_{sj}A_{rj} & (\text{if } r < i < s < j). \end{cases}$$

Noting that in the case of a pure braid $b$ each strand $b_i$ connects $p_i \times \{0\}$ and $p_i \times \{1\}$, we call $b_i$ the *i-th string* of $b$.

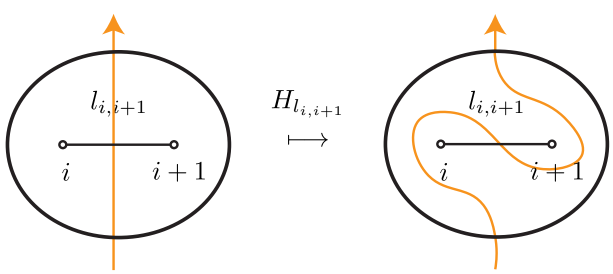
**1.1.2. The Artin representation of the pure braid group.** Next, we recall the interpretation of the braid group as the mapping class group of a surface. Let $D_r = D^2 \backslash \{p_1, \dots, p_r\}$ be the 2-dimensional disc in the complex plane with $r$ punctured points. By tracing the punctured points permuted by a mapping class of $D_r$, we have the natural homomorphism,

$$\chi' : \mathcal{M}(D_r) \longrightarrow S_r$$

and we set $\mathcal{PM}(D_r) := \mathrm{Ker}(\chi')$. Now, the braid group $B_r$ induces homeomorphisms of $D_r$ as follows: Let us consider a simple proper arc $l_{i,i+1}$ connecting the $i$-th and $i+1$-th punctures and a disk $D_{i,i+1}$, which contains only the $i$-th and $i+1$-th punctures corresponding to the following picture.




Each generator $\sigma_i$ can be viewed as the isotopy of $D^2$ between the identity map and the roation map which rotates the arc $l_{i,i+1}$ in $D^2$ clockwise about its midpoint by an angle $\pi$. As a result of this isotopy, we have a homeomorphism $H_{l_{i,i+1}}$ of $D_n$ with support on $D_{i,i+1}$, called the *half twist* along the arc $l_{i,i+1}$, described as follows.




It turns out that this correspondence gives a homomorphism $B_r \to \mathcal{M}(D_r)$. The following proposition is known.

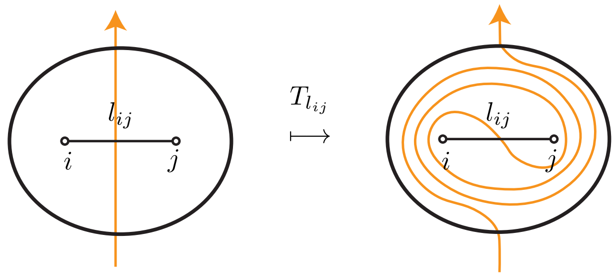
Proposition 1.1.1 (see [**Bi1**, Theorem 1.10], [**KT**, Theorem 1.33]). *The above correspondence induces an isomorphism*

$$B_r \cong \mathcal{M}(D_r)$$

*and so*

$$PB_r \cong \mathcal{PM}(D_r).$$

Similarly, for each generator $A_{ij}$ of $PB_r$, we have the mapping class represented by the full twist $T_{l_{ij}}$ along $l_{ij}$, which has support on $D_{ij}$, and can be described with the following picture:




We take a base point $p_0$ on the boundary $\partial D_r$. The fundamental group of $\pi_1(D_n, p_0)$ is the free group $F_r$ generated by $x_1, \ldots, x_r$, where $x_i$ is a small loop encircling the $i$-th puncture clockwise. So the mapping class group $\mathcal{PM}(D_r) \cong PB_r$ acts naturally on $F_r$ from the left. Therefore, we have a homomorphism,

$$\mathrm{Ar}_r : PB_r = \mathcal{PM}(D_r) \longrightarrow \mathrm{Aut}(\pi_1(D_r, p_0)) = \mathrm{Aut}(F_r); \quad \sigma \mapsto \sigma_*.$$

Accordingly, we can prove the following proposition.

Proposition 1.1.2. *The homomorphism $\psi$ gives an isomorphism*

$$\mathrm{Ar}_r : PB_r \xrightarrow{\sim} \mathrm{Aut}_0(F_r)$$

*where we set*

$$\mathrm{Aut}_0(F_r) = \{\varphi \in \mathrm{Aut}(F_r) \mid \varphi(x_i) = y_i x_i y_i^{-1} \ (1 \leqslant i \leqslant r),\ \varphi(x_1 \cdots x_r) = x_1 \cdots x_r\}$$

*and each $y_i$ $(1 \leqslant i \leqslant r)$ is some element of $F_r$. Furthermore, $y_i$ is uniquely determined under the condition that the exponent sum of $x_i$ in $y_i$ $(1 \leqslant j \leqslant r)$ is 0.*

Proof. The first part is a special case of [**Bi1**, Corollary 1.8.3] and [**Bi1**, Theorem 1.9]. For uniqueness, we assume that there are two elements $y_i$ and $z_i$ in $F_r$ that satisfy $\varphi(x_i) = y_i x_i y_i^{-1} = z_i x_i z_i^{-1}$ and the condition on the exponent sum. Then we have $x_i(y_i^{-1} z_i) = (y_i^{-1} z_i) x_i$. Since an element of the centralizer of $x_i$ is given by $x_i^l$ with some $l \in \mathbb{Z}$, we have $y_i z_i^{-1} = x_i^l$. From the condition on the exponent sum, we have $l = 0$, and so $y_i = z_i$. □

Remark 1.1.3. In [**Bi1**], the action of the braid group on $F_r$ is given by the right action. Here, we think that the braid group acts on $F_r$ from the left in the following manner: For $b \in B_r$ and $x \in F_r$, we set $b_* b'_*(x) := x(b_* b'_*)^{\mathrm{op}}$. Here, op denotes the reverse order of the product, i.e., $(b_* b'_*)^{\mathrm{op}} = b'_* b_*$. Hence, in our notation, the action of the braid group is given by

$$(\sigma_k)_*(x_i) = \begin{cases} x_{i-1} & (\text{if } k = i-1), \\ x_i x_{i+1} x_i^{-1} & (\text{if } k = i), \\ x_i & (\text{otherwise}). \end{cases} \tag{1.1.4}$$

The action of the inverse $\sigma_i^{-1}$ is given by

$$(1.1.5) \qquad (\sigma_k^{-1})_*(x_i) = \begin{cases} x_i^{-1}x_{i-1}x_i & (\text{if } k = i-1), \\ x_{i+1} & (\text{if } k = i), \\ x_i & (\text{otherwise}). \end{cases}$$

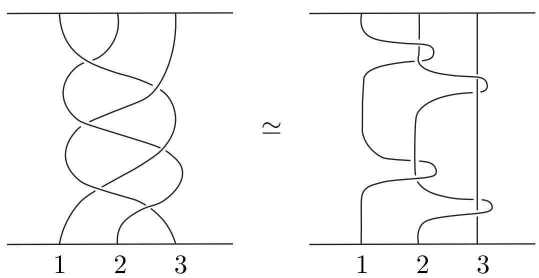
In what follows, we often simply denote by $b(x)$ the action of $b \in B_r$ on $x \in F_r$.

The action of $PB_r$ on the free group $F_r$ is expressed as follows:

$$(1.1.6) \qquad A_{kl}(x_i) = \begin{cases} x_k x_l x_i x_l^{-1} x_k^{-1} & (\text{if } k = i), \\ x_k x_i x_k^{-1} & (\text{if } l = i), \\ x_k x_l x_k^{-1} x_l^{-1} x_i x_l x_k x_l^{-1} x_k^{-1} & (\text{if } k < i < l), \\ x_i & (\text{if } i < k \text{ or } l < i). \end{cases}$$

$$(1.1.7) \qquad A_{kl}^{-1}(x_i) = \begin{cases} x_l^{-1} x_i x_l & (\text{if } k = i), \\ x_i^{-1} x_k^{-1} x_i x_k x_i & (\text{if } l = i), \\ x_l^{-1} x_k^{-1} x_l x_k x_i x_k^{-1} x_l^{-1} x_k x_l & (\text{if } k < i < l), \\ x_i & (\text{if } i < k \text{ or } l < i). \end{cases}$$

Example 1.1.8. Let $b_{\text{Borr}}$ be the following pure braid.




Moreover, we have $b_{\text{Borr}} = \sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1} = A_{23}A_{12}A_{23}^{-1}A_{12}^{-1}$. The mapping class corresponding to $b_{\text{Borr}}$ is represented by $T_{\alpha_{23}} \circ T_{\alpha_{12}} \circ T_{\alpha_{23}}^{-1} \circ T_{\alpha_{12}}^{-1}$. The action of $b_{\text{Borr}}$ on the fundamental group is given by

$$\begin{aligned} \text{Ar}_3(b_{\text{Borr}})(x_1) &= [x_1x_2x_1^{-1}, x_3]x_1[x_1x_2x_1^{-1}]^{-1}, \\ \text{Ar}_3(b_{\text{Borr}})(x_2) &= [x_3^{-1}, x_1^{-1}]x_2[x_3^{-1}, x_1^{-1}], \\ \text{Ar}_3(b_{\text{Borr}})(x_3) &= [x_3^{-1}x_1^{-1}x_3, x_1x_2^{-1}x_1^{-1}]x_3[x_3^{-1}x_1^{-1}x_3, x_1x_2^{-1}x_1^{-1}]^{-1} \end{aligned}$$

Hence, we have

$$\begin{aligned} y_1 &= [x_1x_2x_1^{-1}, x_3] \\ y_2 &= [x_3^{-1}, x_1^{-1}], \\ y_3 &= [x_3^{-1}x_1^{-1}x_3, x_1x_2^{-1}x_1^{-1}]. \end{aligned}$$

Remark 1.1.9. The pure braid group $PB_r$ is also considered to be the fundamental group of the configuration space $\text{Config}_r(D^2) = \{(p_1, ..., p_r) \in (D^2)^r \mid p_i \neq p_j \text{ (if } i \neq j)\}$, which is the moduli space of $r$ ordered distinct points on the 2-dimensional disc $D^2$. Let $h : \mathcal{E} \to \text{Config}_r(D^2)$ be the universal bundle such that the fibre of $(p_1, \ldots, p_r) \in \text{Config}_r(D^2)$ is $h^{-1}((p_1, ..., p_r)) = D^2 \backslash \{p_1, ..., p_r\}$.

Then the representation $\mathrm{Ar}_r : PB_r \to \mathrm{Aut}(F_r)$ can be interpreted as the monodromy representation of $\pi_1(\mathrm{Config}_r(D^2))$ on the fundamental group of the fibre $h^{-1}((p_1, ..., p_r))$.

## 1.2. Milnor invariants

Here, we recall the Milnor invariants of a pure braid link and introduce the Milnor filtration of the pure braid group.

**1.2.1. The Magnus expansion and Fox free derivatives.** Let $\mathbb{Z}\langle\langle X_1, \cdots X_r\rangle\rangle$ be the algebra of non-commutative formal power series of $r$ variables $X_1, \cdots, X_r$ over $\mathbb{Z}$. Let $F_r$ be the free group generated by $x_1, \ldots, x_r$. Let $\mathbb{Z}[F_r]$ be the group algebra of $F_r$ over $\mathbb{Z}$ and let $\epsilon : \mathbb{Z}[F_r] \to \mathbb{Z}$ be the augmentation map. We define the $\mathbb{Z}$-algebra homomorphism, called the *Magnus homomorphism*,

$$\theta : \mathbb{Z}[F_r] \to \mathbb{Z}\langle\langle X_1, \cdots, X_r\rangle\rangle$$

by

$$\theta(x_i) := 1 + X_i, \quad \theta(x_i^{-1}) := 1 - X_i + X_i^{-2} - \cdots \quad (1 \leqslant i \leqslant r).$$

Remark 1.2.1. It is known that the Magnus homomorphism $\theta$ is injective (cf.[**MKS**, 5.5]).

For $\alpha \in \mathbb{Z}[F_r]$, we have

$$\theta(\alpha) = \epsilon(\alpha) + \sum_{n\geqslant 1} \sum_{\substack{I=(i_1,\cdots,i_n) \\ 1\leqslant i_1,\ldots,i_n\leqslant r}} \mu(I;\alpha)X_I, \quad X_I := X_{i_1} \cdots X_{i_n}. \tag{1.2.2}$$

We call it the *Magnus expansion* of $\alpha$ and call the integer $\mu(I;\alpha)$ the *Magnus coefficient* of $\alpha$ with respect to $I$.

For $1 \leqslant j \leqslant r$, let

$$\frac{\partial}{\partial x_j} : \mathbb{Z}[F_r] \to \mathbb{Z}[F_r]$$

be the Fox free derivative given by the following properties (cf.[**MKS**, 5.15]):

$$\frac{\partial x_i}{\partial x_k} = \delta_{i,k}, \quad \frac{\partial(\alpha\beta)}{\partial x_k} = \frac{\partial\alpha}{\partial x_k}\epsilon(\beta) + \alpha\frac{\partial\beta}{\partial x_k} \quad (\alpha, \beta \in \mathbb{Z}[F_r])$$

Higher order derivatives are defined inductively by

$$\frac{\partial^n\alpha}{\partial x_{i_1}\cdots\partial x_{i_n}} := \frac{\partial}{\partial x_{i_1}}\left(\frac{\partial^{n-1}\alpha}{\partial x_{i_2}\cdots\partial x_{i_n}}\right)(\alpha \in \mathbb{Z}[F_r]).$$

The Magnus coefficients can be written in terms of the Fox free derivatives:

$$\mu(i_1\cdots i_n;\alpha) = \epsilon\left(\frac{\partial^n\alpha}{\partial x_{i_1}\cdots\partial x_{i_n}}\right). \tag{1.2.3}$$

Remark 1.2.4. Note that for $m \geq 2$

$$f \in \Gamma_m F_r \iff \text{for any } I \text{ with } 1 \leqslant |I| < m, \text{we have } \mu(I;f) = 0$$

where $|I|$ means the length of multi index $I$.

**1.2.2. Milnor invariants.** Next, we recall Milnor's theorem on the presentation of a link group in our context. Let $x_1, \cdots, x_r$ be a free generator of $F_r = \pi_1(D_r, p_0)$ and $y_i$ $(1 \leqslant i \leqslant r)$ be the word of $x_1, \cdots, x_r$ that is uniquely determined by a pure braid $b$, as in Proposition 1.1.2. For a braid $b$, we denote by $\widehat{b}$ the link obtained by closing $b$. In particular, for a pure braid $b$, we denote by $\widehat{b}_i$ the $i$-th component of $\widehat{b}$ which is obtained by closing the $i$-th strings $b_i$. Here, one can easily prove the following proposition, for example, by using the Wirtinger presentation.

PROPOSITION 1.2.5. *Let $b$ a pure braid in $PB_r$ with $b(x_i) = y_i x_i y_i^{-1}$. The link group $G_{\widehat{b}} := \pi_1(S^3 \backslash \widehat{b})$ of the pure braid link $\widehat{b}$ has the following presentation*

$$G_{\widehat{b}} = \langle x_1, \cdots, x_r \mid [y_1, x_1] = \cdots = [y_r, x_r] = 1 \rangle,$$

*where $x_i$ and $y_i$ may also be regarded as the words representing a meridian and a longitude of $\widehat{b}_i$, respectively.*

Now, let us recall the Milnor invariants of $\widehat{b}$. Following [**Mi**], we consider the Magnus expansion of the $i$-th longitude $y_i$ in $\mathbb{Z}\langle\langle X_1, ..., X_r \rangle\rangle$:

$$\theta(y_i) = 1 + \sum_{n \geqslant 1} \sum_{\substack{I=(i_1 \cdots i_n) \\ 1 \leqslant i_1, \cdots, i_n \leqslant r}} \mu(b; i_1 \cdots i_n i) X_{i_1} \cdots X_{i_n}. \tag{1.2.6}$$

and the coefficient $\mu(b; i_1 \cdots i_n i)$ is called the *Milnor number* or *Milnor $\mu$ invariant* of $b$ with respect to the multi-index $I = (i_1 \cdots i_n i)$. From (1.2.3), we have the following description of Milnor numbers in the light of Fox free derivatives:

$$\mu(b; i_1 \cdots i_n i) = \mu(i_1 \cdots i_n; y_i) = \epsilon \left( \frac{\partial^n y_i}{\partial x_{i_1} \cdots \partial x_{i_n}} \right).$$

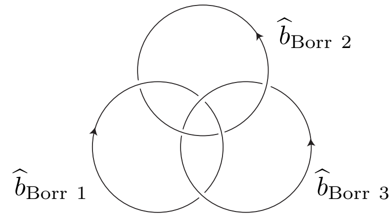
To get the isotopy invariants of links, we need to consider the residue class of $\mu(b; I)$ in order to get rid of the indeterminacies of the choices of meridians and longitudes and of the group presentation of $\widehat{b}$. Here, we set

$$\overline{\mu}(\widehat{b}; I) := \mu(b; I) \bmod \Delta(I)$$

where $\Delta(I)$ denotes that the ideal of $\mathbb{Z}$ generated by $\mu(\widehat{b}; J)$ ($J$ runs over all cyclic permutations of proper subsequences of $I$). Then $\overline{\mu}(\widehat{b}; I)$ is known to be an isotopy invariant of $\widehat{b}$ and is called the *Milnor $\overline{\mu}$ invariant* of $\widehat{b}$ with respect to the multi-index $I$.

REMARK 1.2.7. (1) In [**MK**, Definition 4.3], the Milnor number $\mu(b \mid I)$ of a pure braid $b$ is defined for a multi-index $I$ consisting of distinct integers. It coincides with our $\mu(b; I)$.
(2) Let $m$ be a integer greater than 1. If $\overline{\mu}(\widehat{b}; I) = 0$ for $|I| \leqslant m$, then $\overline{\mu}(\widehat{b}; I) = \mu(b; I)$ for $|I| = m + 1$.

EXAMPLE 1.2.8. Let $b_{\text{Borr}}$ be the pure braid in Example 1.1.8. Then $\widehat{b}_{\text{Borr}}$ is the following link, called the Borromean rings. Here, we give $b_{\text{Borr}}$ an orientation downward.




From (1.1.9), we have

$$\begin{aligned}\theta(y_1) &= 1 + X_2X_3 - X_3X_2 + (\text{higher degree terms}),\\ \theta(y_2) &= 1 + X_3X_1 - X_3X_1 + (\text{higher degree terms}),\\ \theta(y_3) &= 1 + X_1X_2 - X_2X_1 + (\text{higher degree terms}).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\overline{\mu}(\widehat{b}_{\mathrm{Borr}}; 123) = \overline{\mu}(\widehat{b}_{\mathrm{Borr}}; 231) = \overline{\mu}(\widehat{b}_{\mathrm{Borr}}; 312) = 1,\\ &\overline{\mu}(\widehat{b}_{\mathrm{Borr}}; 132) = \overline{\mu}(\widehat{b}_{\mathrm{Borr}}; 321) = \overline{\mu}(\widehat{b}_{\mathrm{Borr}}; 213) = -1,\\ &\overline{\mu}(\widehat{b}_{\mathrm{Borr}}; ijk) = 0 \quad (\text{otherwise}).\end{aligned}$$

The Milnor invariants of pure braid links induce a filtration of $PB_r$ as follows. We denote by $PB_r^{\mathrm{Mil}}(m)$ the normal subgroup of $PB_r$ consisting of elements whose Milnor invariants of length $\leqslant m$ vanish, i.e.,

$$PB_r^{\mathrm{Mil}}(m) := \{b \in PB_r \mid \overline{\mu}(\widehat{b}; I) = 0 \quad (|I| \leqslant m)\}.$$

We then have the descending series

$$PB_r = PB_r^{\mathrm{Mil}}(1) \supset PB_r^{\mathrm{Mil}}(2) \supset \cdots \supset PB_r^{\mathrm{Mil}}(m) \supset \cdots .$$

and $\{P_r^{\mathrm{Mil}}(m)\}_{m\geqslant 1}$ is called the *Milnor filtration* of $PB_r$ ([**Oh**]).

**1.2.3. Massey products for a link complement.** In this section, we recall the definition of Massey products for cohomology and their relation with Magnus coefficients. Then, we recall the result of Turaev and Porter that relates Massey products for a link complement to the Milnor invariant of a link in our context.

Let $X$ be a topological space. In the following, the cohomology group of $X$ stands for the singular cohomology with integral coefficients. Let $\alpha_1, \ldots, \alpha_m \in H^1(X, \mathbb{Z})$ be cohomology classes. A *Massey product* $\langle \alpha_1, \ldots, \alpha_m \rangle$ is said to be *defined* if there is an array $A$

$$A = \{a_{ij} \in C^1(X, \mathbb{Z}) \mid 1 \leqslant i < j \leqslant m+1, (i, j) \neq (1, m+1)\}$$

such that

$$\begin{cases} [a_{i,i+1}] = a_i \quad (1 \leqslant i \leqslant m)\\ da_{ij} = \sum_{k=i+1}^{j-1} a_{ik} \cup a_{kj} \quad (j \neq i+1).\end{cases}$$

Such an array $A$ is called a *defining system* for $\langle \alpha_1, \ldots, \alpha_m \rangle$. Then, for a defining system $A$, we define the cohomology class $\langle \alpha_1, \ldots, \alpha_m \rangle_A$ of $H^2(X, \mathbb{Z})$ represented by the 2-cocycle

$$\sum_{k=2}^{m} a_{1k} \cup a_{k,m+1}.$$

We then define a Massey product of $\alpha_1, \ldots, \alpha_m$ as the subset of $H^2(X, \mathbb{Z})$ by

$$\langle \alpha_1, \ldots, \alpha_m \rangle := \{\langle \alpha_1, \ldots, \alpha_m \rangle_A \in H^2(X, \mathbb{Z}) \mid A \text{ ranges over defining systems}\}.$$

REMARK 1.2.9. (1) *The Massey product $\langle \alpha_1 \rangle$ is $\alpha_1$ and its defining system $A$ consists of any 1-cocycle representing $\alpha_1$. The Massey product $\langle \alpha_1, \alpha_2 \rangle$ is the cup product $\alpha_1 \cup \alpha_2$. For $m \geqslant 3$, the Massey product $\langle \alpha_1, \ldots, \alpha_m \rangle$ is defined and consists of a single element if $\langle \alpha_{i_1}, \ldots, \alpha_{i_r} \rangle = 0$ for any proper subset $\{i_1, \ldots, i_r\}$ $(r \geqslant 2)$ of $\{1, \ldots, m\}$.*
(2)(*The naturality of the Massey products*) *Let $X$ and $X'$ be topological spaces and $f : X \to X'$ be a continuous map. We assume that $\langle \alpha_1, \ldots, \alpha_m \rangle$ is defined for $\alpha_i \in H^1(X', \mathbb{Z})$ $(1 \leqslant i \leqslant m)$ with the defining system $A = (a_{ij})$. Then, $\langle f^*(\alpha_1), \ldots, f^*(\alpha_m) \rangle$ is defined for $f^*(\alpha_i) \in H^1(X, \mathbb{Z})$ $(1 \leqslant i \leqslant m)$ with the defining system $A^* = (f^*(a_{ij}))$ and we have $f^*(\langle \alpha_1, \ldots, \alpha_m \rangle) \subset \langle f^*(\alpha_1), \cdots, f^*(\alpha_m) \rangle$.*

Next, let us recall the relation between Massey products and Magnus coefficients. Let $G$ be a finitiely generated group with minimal generators $g_1, \ldots, g_r$. The group cohomology $H^*(G, \mathbb{Z})$ is given by the singular cohomology $H^*(K(G,1), \mathbb{Z})$ of the Eilenberg-Maclane space $K(G, 1)$. Let

$$1 \longrightarrow R \longrightarrow F_r \xrightarrow{\pi} G \longrightarrow 1 \tag{1.2.10}$$

be a presentation of $G$ such that $\pi$ sends each generator $x_i$ $(1 \leqslant i \leqslant r)$ of $F_r$ to $g_i$ and $\pi$ induces the isomorphism $F_r^{\mathrm{ab}} \cong G^{\mathrm{ab}}$. The subgroup $R$ is generated normally by the relators of $G$. Now we have an isomorphism $H^1(G, \mathbb{Z}) \cong H^1(F_r, \mathbb{Z})$ induced by $\pi$. Moreover, we have an isomorphism, called the Hopf isomorphism ([**Br**, Theorem 5.3])

$$h : H_2(G, \mathbb{Z}) \longrightarrow H_1(R, \mathbb{Z})_G = R/[R, F_r]. \tag{1.2.11}$$

The following proposition yields the relation between Massey products and Magnus coefficients.

PROPOSITION 1.2.12. *With the notation as above, let $\alpha_1, \ldots, \alpha_m \in H^1(G, \mathbb{Z})$ and let $A = (a_{ij})$ be a defining system for the Massey product $\langle \alpha_1, \ldots, \alpha_m \rangle$. Let $f \in R$ and set $\eta := h^{-1}(f \bmod [R, F_r])$. Then we have*

$$\begin{aligned} &\langle \alpha_1, \ldots, \alpha_m \rangle_A(\eta) \\ =&\sum_{j=1}^{m} (-1)^{j+1} \sum_{c_1 + \cdots c_j = m} \sum_{1 \leqslant i_1, \cdots, i_j \leqslant s} a_{1,1+c_1}(g_{i_1}) \cdots a_{m+1-c_j, m+1}(g_{i_j}) \mu(i_1, \ldots, i_j; f), \end{aligned}$$

*where $c_i$ $(1 \leqslant i \leqslant j)$ runs over positive integers satisfying $c_1 + \cdots + c_j = m$ and $g_i := \pi(x_i)$ $(1 \leqslant i \leqslant r)$ and $\mu(i_1 \cdots i_j; f)$ is the Magnus coefficient of $f$ withe respect to $I = (i_1 \cdots i_j)$.*

Next, let us recall the result of Turaev ([**T**]) and Porter ([**P**]) on the interpretation of Milnor invariants as Massey products for the cohomology of a link complement in our context.

For $b \in PB_r^{\mathrm{Mil}}(m)$ $(m \geqslant 1)$, we have its closed pure link $\widehat{b}$. By Proposition 1.2.5, a free basis of $H_1(S^3 \backslash \widehat{b}, \mathbb{Z})$ is given by $[x_1], \ldots, [x_r]$ and its dual basis is given by $x_1^*, \ldots, x_r^*$ in $H^1(S^3 \backslash \widehat{b}, \mathbb{Z})$. For a tubular neighborhood $V_i$ around the $i$-th component $\widehat{b}_i$, we consider the homology class $\eta_i$ in $H_2(S^3 \backslash \widehat{b}, \mathbb{Z})$ realizing the boundary $\partial V_i$. Then Turaev-Porter's result can be expressed as follows.

THEOREM 1.2.13 ([**T**],[**P**]). *For* $1 \leqslant i_1, \cdots, i_{m+1} \leqslant r$*, there is a uniquely defined Massey product* $\langle x^*_{i_1}, \cdots, x^*_{i_{m+1}} \rangle \in H^2(S^3 \setminus \widehat{b}, \mathbb{Z})$ *such that we have*

$$\langle x^*_{i_1}, \cdots, x^*_{i_{m+1}} \rangle (\eta_i) = \begin{cases} (-1)^{m+1} \left( \overline{\mu}(\widehat{b}; i_2 \cdots i_{m+1} i_1) - \delta_{i_1, i_{m+1}} \overline{\mu}(\widehat{b}; i_1 \cdots i_m i_{m+1}) \right) & (\textit{if } i = i_1) \\ (-1)^{m+1} \left( \overline{\mu}(\widehat{b}; i_2 \cdots i_{m+1} i_1) \delta_{i_1, i_{m+1}} - \overline{\mu}(\widehat{b}; i_1 \cdots i_m i_{m+1}) \right) & (\textit{if } i = i_{m+1}) \\ 0 & (\textit{otherwise}). \end{cases}$$

## 1.3. Johnson homomorphisms

**1.3.1. Johnson homomorphisms.** Let $\Sigma = \Sigma_g^{1,r}$ be an oriented surface of genus $g \geqslant 0$ with $r \geqslant 0$ punctured points and one boundary component $\partial\Sigma$. Let $\mathcal{M}(\Sigma)$ denote the mapping class group of $\Sigma$, i.e., the group of isotopy classes of orientation-preserving self-homeomorphisms of $\Sigma$ which fix the boundary pointwisely. Taking a base point $p_0 \in \partial\Sigma$, we have a group isomorphism $\pi_1(\Sigma, p_0) \cong F_{2g+r}$. Since $\mathcal{M}(\Sigma)$ acts naturally on the fundamental group $\pi_1(\Sigma, p_0)$, we have a homomorphism

$$\psi : \mathcal{M}(\Sigma) \longrightarrow \mathrm{Aut}(\pi_1(\Sigma, p)) \cong \mathrm{Aut}(F_{2g+r}); \quad \phi \mapsto \phi_*.$$

Since $\Gamma_m F_{2g+r}$ is a characteristic subgroup of $F_{2g+r}$, any mapping class $\phi \in \mathcal{M}(\Sigma)$ induces the automorphism $[\phi_*]_m$ of $F_{2g+r}/\Gamma_{m+1}F_{2g+r}$. Thus, we have the homomorphism

$$\psi_m : \mathcal{M}(\Sigma) \longrightarrow \mathrm{Aut}(F_{2g+r}/\Gamma_{m+1}F_{2g+r}); \quad \phi \mapsto [\phi_*]_m.$$

We denote the kernel of $\psi_m$ by $\mathcal{M}(\Sigma)^{\mathrm{Joh}}(m)$, i.e.,

$$\begin{aligned} \mathcal{M}(\Sigma)^{\mathrm{Joh}}(m) &:= \mathrm{Ker}(\psi_m) \\ &= \{\phi \in \mathcal{M}(\Sigma) \mid \phi_*(g)g^{-1} \in \Gamma_{m+1}F_{2g+r}\} \ (m \geqslant 0). \end{aligned}$$

We then have the descending series

$$\mathcal{M}(\Sigma) = \mathcal{M}(\Sigma)^{\mathrm{Joh}}(0) \supset \mathcal{M}(\Sigma)^{\mathrm{Joh}}(1) \supset \cdots \supset \mathcal{M}(\Sigma)^{\mathrm{Joh}}(m) \supset \cdots \tag{1.3.1}$$

and $\{\mathcal{M}(\Sigma)^{\mathrm{Joh}}(m)\}_{m \geqslant 0}$ is called the *Johnson filtration* of $\mathcal{M}(\Sigma)$. Let $H$ denote the first homology group of $\Sigma$ with integer coefficients:

$$H_{\mathbb{Z}} := H_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{\oplus(2g+r)}.$$

Then we define the map

$$\tau_m : \mathcal{M}(\Sigma)^{\mathrm{Joh}}(m) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathrm{gr}_{m+1}(F_{2g+r}))$$

as follows. First, we define a map $\tau_m(\phi) : H_{\mathbb{Z}} \to \mathrm{gr}_{m+1}(F_{2g+r})$ for any $\phi \in \mathcal{M}(\Sigma)^{\mathrm{Joh}}(m)$ in the following way: For $[\gamma] \in H_{\mathbb{Z}}$ with $\gamma \in F_{2g+r}$, we have $\phi_*(\gamma)\gamma^{-1} \in \Gamma_{m+1}$. Net, we set $\tau_m(\phi)([\gamma]) := \phi_*(\gamma)\gamma^{-1} \bmod \Gamma_{m+2}F_{2g+r} \in \mathrm{gr}_{m+1}(F_{2g+r})$. We can easily see that the map $\tau_m(\phi)$ is well defined homomorphism.

This leads us to the following proposition (for the proof, see [**Sa**]).

PROPOSITION 1.3.2. *For* $m \geqslant 1$*, the map* $\tau_m$ *is a group homomorphism.*

For $m \geqslant 1$, the homomorphism

$$\tau_m : \mathcal{M}(\Sigma)^{\mathrm{Joh}}(m) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathrm{gr}_{m+1}(F_{2g+r})) \tag{1.3.3}$$

is called the $m$-th *Johnson homomorphism.*

**1.3.2. Magnus coefficients of Johnson homomorphisms.** Let us go back to the setting of Section 1.2.2. Let us consider the Magnus coefficients of the image of the Johnson homomorphism of a mapping class.

The ring of non-commutative formal power series $\mathbb{Z}\langle\langle X_1, \ldots, X_{2g+r}\rangle\rangle$ can be identified with the completed tensor algebra of $H$ over $\mathbb{Z}$:

$$\mathbb{Z}\langle\langle X_1, \ldots, X_{2g+r}\rangle\rangle = \prod_{m\geqslant 0} H_{\mathbb{Z}}^{\otimes m},$$

where $H_{\mathbb{Z}}^{\otimes m}$ is the submodule generated by monomials of $X_1, \ldots, X_{2g+r}$ of degree $m$. Noting Remark 1.2.4, the restriction of the Magnus homomorphism $\theta$ to $\Gamma_{m+1}F_{2g+r}$ induces the homomorphism, for $m \geqslant 0$,

$$\theta_m : \mathrm{gr}_{m+1}(F_{2g+r}) \to H_{\mathbb{Z}}^{\otimes(m+1)},$$

which is written as

$$\theta_m([\gamma]) = \sum_{1\leqslant i_1,\cdots,i_{m+1}\leqslant 2g+r} \mu(i_1\cdots i_{m+1};\gamma)X_{i_1}\cdots X_{i_{m+1}}$$

for $\gamma \in \Gamma_{m+1}F_{2g+r}$. Composing the $m$-th Johnson homomorphism $\tau_m$ in (1.3.3) with $\theta_m$, we have the homomorphism, for $m \geqslant 1$,

$$\tau_m^\theta := \theta_m \circ \tau_m : \mathcal{M}(\Sigma)^{\mathrm{Joh}}(m) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, H_{\mathbb{Z}}^{\otimes(m+1)}); \quad \phi \mapsto \theta_m \circ \tau_m(\phi).$$

For each $\phi \in \mathcal{M}(\Sigma)^{\mathrm{Joh}}(m)$ and the basis $[x_i]$ $(1 \leqslant i \leqslant 2g+r)$ of $H_{\mathbb{Z}}$, we have

$$\begin{aligned}\tau_m^\theta(\phi)([x_i]) &= \theta_m(\tau_m(\phi)([x_i])) \\ &= \sum_{1\leqslant i_1,\ldots,i_{m+1}\leqslant 2g+r} \mu(i_1\cdots i_{m+1};\phi(x_i)x_i^{-1})X_{i_1}\cdots X_{i_{m+1}}.\end{aligned} \tag{1.3.4}$$

One can see that the Magnus coefficients of $\tau_m^\theta(\phi)([x_i])$ contain all the information of $\tau_m(\phi)([x_i])$ as integers.

**1.3.3. Johnson homomorphisms for pure braid groups.** In this section, we prove our first theorem, which gives the explicit relation between Johnson coefficients and Milnor invariants of a pure braid link.

From Proposition 1.1.1, we may regard $PB_r$ as $\mathcal{PM}(D_r)$. We define the *Johnson filtration* $\{PB_r^{\mathrm{Joh}}(m)\}_{m\geqslant 0}$ of $PB_r$ by

$$PB_r^{\mathrm{Joh}}(m) := PB_r \ \cap\ \mathcal{M}^{\mathrm{Joh}}(D_r)(m).$$

LEMMA 1.3.5. $PB_r = PB_r^{\mathrm{Joh}}(1)$.

PROOF. For any $b \in PB_r$, we have

$$\begin{aligned} b_*(x_i)x_i^{-1} &= y_i x_i y_i^{-1} x_i^{-1} \\ &= [y_i, x_i] \in \Gamma_2 F_r. \end{aligned}$$

This implies $b \in PB_r^{\mathrm{Joh}}(1)$. Hence $PB_r = PB_r^{\mathrm{Joh}}(1)$. □

Now we can prove the following proposition that shows the equivalence of the Johnson filtration and the Milnor filtration of the pure braid group.

PROPOSITION 1.3.6. *For $m \geqslant 1$, the Johnson filtration and the Milnor filtration of the pure braid group coincide, i.e., we have*

$$PB_r^{\mathrm{Joh}}(m) = PB_r^{\mathrm{Mil}}(m).$$

PROOF. For any $b \in PB_r$, from Remark 1.2.4 and (1.2.6), we have

$$b \in PB_r^{\mathrm{Mil}}(m) \iff y_i \in \Gamma_m F_r \quad (1 \leqslant i \leqslant r).$$

On the other hand,

$$\begin{aligned} b \in PB_r^{\mathrm{Joh}}(m) &\iff b_*(x_i)x_i^{-1} \in \Gamma_{m+1}F_r \quad (1 \leqslant i \leqslant r) \\ &\iff [y_i, x_i] \in \Gamma_{m+1}F_r \quad (1 \leqslant i \leqslant r) \\ &\iff y_i \in \Gamma_m F_r \quad (1 \leqslant i \leqslant r). \end{aligned}$$

Therefore, $PB_r^{\mathrm{Joh}}(m) = PB_r^{\mathrm{Mil}}(m)$. □

In the following, we simply denote by $PB_r(m)$ the $m$-th term of the Johnson (or Milnor) filtration for $m \geqslant 0$.

The following theorem states that the Johnson homomorphisms of a pure braid are essentially same as the first non-vanishing Milnor invariants of the pure braid link.

THEOREM 1.3.7. *For each $b \in PB_r(m)$, each basis $[x_i] \in H_{\mathbb{Z}}$ and multi-index $I = (i_1 \cdots i_{m+1})$ of length $m+1$, we have*

$$\begin{aligned} &\mu(i_1 \cdots i_{m+1}; \tau_m(b)(x_i)) \\ &= \begin{cases} -\overline{\mu}(\widehat{b}; i_2 \cdots i_{m+1} i_1) + \delta_{i_1, i_{m+1}} \overline{\mu}(\widehat{b}; i_1 \cdots i_m i_{m+1}) & (\textit{if } i = i_1) \\ -\overline{\mu}(\widehat{b}; i_2 \cdots i_{m+1} i_1)\delta_{i_1, i_{m+1}} + \overline{\mu}(\widehat{b}; i_1 \cdots i_m i_{m+1}) & (\textit{if } i_1 = i_{m+1}) \\ 0 & (\textit{otherwise}). \end{cases} \end{aligned}$$

PROOF. From (1.3.4), we have

$$\tau_m^\theta(b)([x_i]) = \sum_{|I|=m+1} \mu(I; y_i x_i y_i^{-1} x_i^{-1}) X_I.$$

From proposition 1.3.6, $b \in PB_r(m) \iff y_i \in \Gamma_m F_r$. Therefore, from Remark 1.2.4 and (1.2.6), we have $\theta(y_i) = 1 + Y$ where $Y = \sum_{|I'| \geqslant m} \mu(\widehat{b}; I'i) X_{I'}$. Thus, we have

$$\begin{aligned} \theta(y_i x_i y_i^{-1} x_i^{-1}) &= \theta(y_i)\theta(x_i)\theta(y_i^{-1})\theta(x_i)^{-1} \\ &= (1+Y)(1+X_i)(1 - Y + Y^2 - \cdots)(1 - X_i + X_i^2 - \cdots) \\ &= 1 + YX_i - X_i Y + (\text{higher degree terms}). \end{aligned}$$

Then, the homogeneous degree $m+1$ part of $\theta(y_i x_i y_i^{-1} x_i^{-1})$ is given by

$$\sum_{|I'|=m} \mu(b; I'i)(X_{I'} X_i - X_i X_{I'}).$$

By carefully comparing the coefficients, the assertion follows. □

EXAMPLE 1.3.8. Let $b_{\mathrm{Borr}}$ be the pure braid as in Example 1.1.8. As is shown in Example 1.2.8, we can see that $b_{\mathrm{Borr}} \in P_3(2)$. Then, the image of the Johnson homomorphism of $b_{\mathrm{Borr}}$ is given by

$$\tau_2^\theta(b_{\mathrm{Borr}}) = [x_1]^* \otimes [[X_2, X_3], X_1] + [x_2]^* \otimes [[X_3, X_1], X_2] + [x_3]^* \otimes [[X_1, X_2], X_3]$$

where $[x_i]^*$ denotes the Kronecker dual of $[x_i]$ with $1 \leqslant i \leqslant 3$.

REMARK 1.3.9. Note that the relation between Milnor invariants and Johnson homomorphisms is also shown by Habegger in [**Ha**].

**1.3.4. Massey products for a mapping torus.** First, we describe the punctured disc $\Sigma_0^{1,r} = D_r$ counterpart of the Kitano's result on Massey products for a mapping torus of a surface $\Sigma_g^{1,0}$ ([**Ki**]).

Let $b \in PB_r(m)$ $(m \geqslant 1)$: we consider the mapping torus of $b$,

$$X_b := D_r \times [0,1]/\sim,$$

where we define the equivalence relation $\sim$ by identifying $x \times \{0\}$ with $b_*(x) \times \{1\}$. The Seifert-van Kampen theorem gives

$$\begin{aligned}\pi_1(X_b) &= \langle x_1, \ldots, x_r, t \mid [x_1, t] b_*(x_1) x_1^{-1}, \cdots, [x_r, t] b_*(x_r) x_r^{-1} \rangle \\ &= \langle x_1, \ldots, x_r, t \mid [x_1, t][y_1, x_1], \cdots, [x_r, t][y_r, x_r] \rangle\end{aligned}$$

where $x_i$ and $y_i$ are the words in Proposition 1.2.5. Now, $[x_1], \cdots, [x_r], [t]$ forms a free basis of $H_1(X_b, \mathbb{Z})$. Let $x_i^* \in H^1(X_b, \mathbb{Z})$ $(1 \leqslant i \leqslant r+1)$ denote the dual basis of $H^1(X_b, \mathbb{Z})$ given by

$$\begin{cases} x_i^*([x_j]) = \delta_{ij} \ (1 \leqslant i, j \leqslant r) \\ x_{r+1}^*([t]) = 1 \\ x_{r+1}^*([x_i]) = 0 \ (1 \leqslant i \leqslant r). \end{cases}$$

Since $X_b$ is an Eilenberg-Maclane space $K(\pi_1(X_b), 1)$, we have $H_*(X_b, \mathbb{Z}) \cong H_*(\pi_1(X_b), \mathbb{Z})$. Let $\xi_j$ be the homology class in $H_2(X_b, \mathbb{Z})$, which corresponds to the homology class of $H_2(X_b, \mathbb{Z})$ representing the relator $[x_j, t][y_i, x_j]$ via the Hopf isomorphism $h$ in (1.2.11). Then, the punctured disc analogue of Kitano's result is as follows.

THEOREM 1.3.10. *Let $b \in PB_r(m)$. For $1 \leqslant i_1, \ldots, i_{m+1} \leqslant r$, the Massey product $\langle x_{i_1}^*, \ldots, x_{i_{m+1}}^* \rangle \in H^2(X_b, \mathbb{Z})$ is uniquely defined and its evaluation on $\xi_i \in H_2(X_b, \mathbb{Z})$ is given by*

$$\langle x_{i_1}^*, \ldots, x_{i_{m+1}}^* \rangle (\xi_i) = (-1)^m \mu(i_1 \cdots i_{m+1}; \tau_m(b)(x_i)).$$

By using Theorems 1.2.13, 1.3.7, and 1.3.10, we can easily show the following.

COROLLARY 1.3.11. *The punctured disk analogue of Kitano's result and Turaev–Porter's result are equivalent.*

CHAPTER 2

# Reduced Gassner representations of pure braid groups

In this chapter, we recall the notions of Gassner representations and the reduced version. Each representation has two definitions. One is defined as the induced action of the pure braid group on the first homology group of the universal abelian covering space of a punctured disc. The other is defined as a special case of the Magnus representations of the mapping class group of a surface. First, we show that these definitions are equivalent. Second, we give explicit formulas for the (reduced) Gassner representation in terms of the Milnor invariants of a pure braid link. Finally, we give the relation between (reduced) Gassner representations and Johnson homomorphisms. This chapter is based on [**Ko2**].

## 2.1. Gassner representations

In this section, we recall two different definitions of reduced Gassner representations of pure braid groups: One is a special case of homological representations and the other is a special case of Magnus representations. Then, we prove the equivalence of these two different definitions.

**2.1.1. Homological Gassner representations.** Here, we recall the definition of the reduced Gassner representation derived from the induced action of the pure braid group on the abelian covering space of the $r$-punctured disk $D_r$. (For details on the homological representations of the mapping class group, for example, see [**KT**].)

Take a base point $p_0$ on the boundary $\partial D_r$ and consider the fundamental group $\pi_1(D_r, p_0) = F_r$. From Hurewicz theorem, we have a natural homomorphism $\mathrm{ab} : \pi_1(D_r, p_0) = F_r \to H_1(D_r) = F_r^{\mathrm{ab}} := F_r/\Gamma_2 F_r = \mathbb{Z}^{\oplus r}$. Let $D_r^{\mathrm{ab}}$ be the universal abelian covering space of $D_r$ corresponding to $\mathrm{Ker}(\mathrm{ab}) = \Gamma_2 F_r$. Then, the group of covering transformations $\mathrm{Aut}(D_r^{\mathrm{ab}}/D_r)$ of $h : D_r^{\mathrm{ab}} \to D_r$ is identified with $F_r^{\mathrm{ab}}$.

Let us consider the relative homology group $H_1(D_r^{\mathrm{ab}}, h^{-1}(p_0))$. Since the action of $F_r^{\mathrm{ab}}$ on $D_r^{\mathrm{ab}}$ induces the action on $H_1(D_r^{\mathrm{ab}}, h^{-1}(p_0))$, the relative homology group $H_1(D_r^{\mathrm{ab}}, h^{-1}(p_0))$ is endowed with the structure of a $\mathbb{Z}[F_r^{\mathrm{ab}}]$-module. In the following, we shall identify $\mathbb{Z}[F_r^{\mathrm{ab}}]$ with the ring of Laurent polynomials $\Lambda_r := \mathbb{Z}[t_1^{\pm}, \ldots, t_r^{\pm}]$ over $\mathbb{Z}$ with variables $t_1, \ldots, t_r$. Since $D_r$ is homotopy equivalent to a bouquet of $r$ circles, one may see that $H_1(D_r^{\mathrm{ab}}, h^{-1}(p_0))$ is a free $\Lambda_r$-module of rank $r$, i.e., $H_1(D_r^{\mathrm{ab}}, h^{-1}(p_0)) \cong {\Lambda_r}^{\oplus r}$.

As explained in Section 1.1.2, by viewing the pure braid group $PB_r$ as a subgroup of the mapping class group of $D_r$, $PB_r$ acts on $\pi_1(D_r, p_0)$. We can see that $b_*$ commutes with ab for each $b \in PB_r$ by Proposition 1.1.2, i.e., $\mathrm{ab} \circ b_* = \mathrm{ab}$. Take a point $\widetilde{p_0}$ on the fiber $h^{-1}(p_0)$ of $p$. An automorphism of $D_r$ representing $b \in PB_r$

has a unique lift $\widetilde{b} : D_r^{\mathrm{ab}} \to D_r^{\mathrm{ab}}$ fixing $\widetilde{p_0}$. Since $\widetilde{b}$ commutes with the action of $F_r^{\mathrm{ab}}$ on $D_r^{\mathrm{ab}}$, we have $\widetilde{b}(g\widetilde{p_0}) = g\widetilde{b}(\widetilde{p_0}) = g\widetilde{p_0}$ $(g \in F_r^{\mathrm{ab}})$. Therefore, the lift $\widetilde{b}$ fixes the fiber $h^{-1}(p_0)$ pointwise.

The lift $\widetilde{b} : D_r^{\mathrm{ab}} \to D_r^{\mathrm{ab}}$ induces the automorphism

$$\widetilde{b}_* : H_1(D_r^{\mathrm{ab}}, h^{-1}(p_0)) \to H_1(D_r^{\mathrm{ab}}, h^{-1}(p_0))$$

and $\widetilde{b}_*$ is a $\Lambda_r$-linear map since $\widetilde{b}_*$ commutes with $F_r^{\mathrm{ab}}$. Hence, we have the homomorphism

$$\mathrm{Gass}_r^{\mathrm{H}} : PB_r \longrightarrow \mathrm{Aut}(H_1(D_r^{\mathrm{ab}}, h^{-1}(p_0))) = \mathrm{GL}(r; \Lambda_r); \quad b \mapsto \widetilde{b}_*.$$

We call this representation *the homological Gassner representation* of $PB_r$.

By extending this representation to the full braid group $B_r$, we obtain a 1-cocycle

$$\mathrm{Gass}_r^{\mathrm{H}} : B_r \longrightarrow \mathrm{GL}(r; \Lambda_r); \quad b \mapsto \widetilde{b}_*$$

since $\widetilde{b}$ acts on $\Lambda_n$ by a permutation of $\{t_1, \ldots, t_n\}$. We call this 1-cocycle the *homological Gassner cocycle* of $B_r$.

In the following, for each $b \in PB_r$ as long as there is no risk of confusion, we will denote the lift $\widetilde{b}_*$ by using the same $b$.

Noting that $H_1(D_r, h^{-1}(p_0))$ is freely generated by the classes $[x_1], \ldots, [x_r]$, by (1.1.6), the representation matrix of the homological Gassner representation is explicitly given as follows.

Proposition 2.1.1. *For each generator $A_{ij} \in PB_r$, we have*

$$A_{kl}([x_i]) = \begin{cases} (1 + t_k(t_l - 1))[x_k] + t_k(1 - t_k)[x_l] & (\textit{if } k = i), \\ (1 - t_l)[x_k] + t_k[x_l] & (\textit{if } l = i), \\ (1 - t_l)(1 - t_i)[x_k] + (1 - t_k)(t_i - 1)[x_l] + [x_i] & (\textit{if } k < i < l), \\ [x_i] & (\textit{if } i < k \textit{ or } l < i). \end{cases}$$

Similarly, the homological Gassner cocycle is explicitly given as follows by (1.1.4).

Proposition 2.1.2. *For each generator $\sigma_k \in B_r$, we have*

$$\sigma_k([x_i]) = \begin{cases} [x_{i-1}] & (\textit{if } k = i - 1), \\ (1 - t_{i+1})[x_i] + t_i[x_{i+1}] & (\textit{if } k = i), \\ [x_i] & (\textit{if otherwise}). \end{cases}$$

**2.1.2. Homological reduced Gassner representations.** Next, let us define a free submodule $L_r^{\mathrm{prim}} \subset H_1(D_r, h^{-1}(p_0))$ with rank $r - 1$ and define the reduced homological reduced Gassner representation of $PB_r$ as its induced action on $L_r^{\mathrm{prim}}$.

To begin with, let us review the Crowell exact sequence (cf.[**Cr**]). Let $G$ be a group with finite presentation

$$G = \langle x_1, \ldots, x_r \mid r_1 = \cdots = r_m = 1 \rangle.$$

Let $\pi : F_r \to G$ be a natural projection and we denote the induced $\mathbb{Z}$-linear map of the group algebra of the same $\pi$, i.e., $\pi : \mathbb{Z}[F_r] \to \mathbb{Z}[G]$. Take any group $H$. Let $\Psi : G \to L$ be a surjective homomorphism and, we will use the same $\Psi$ to denote the induced $\mathbb{Z}$-linear map $\Psi : \mathbb{Z}[G] \to \mathbb{Z}[L]$. By setting $N := \mathrm{Ker}(\Psi)$, we have the following exact sequence:

$$1 \longrightarrow N \to G \xrightarrow{\Psi} L \longrightarrow 1.$$

Moreover, by setting

$$\begin{aligned} Q_\Psi &:= \left(\Psi \circ \pi \left(\frac{\partial r_i}{\partial x_j}\right)\right) \in \mathrm{M}_{m,r}(\mathbb{Z}[L]), \\ A_\Psi &:= \mathrm{Coker}(\mathbb{Z}[L]^m \overset{Q_\Psi}{\to} \mathbb{Z}[L]^r) \\ &= \mathbb{Z}[L]^r/\mathrm{Im}(Q_\Psi), \end{aligned}$$

we have the following exact sequence of $\mathbb{Z}[L]$-modules

$$\mathbb{Z}[L]^m \xrightarrow{Q_\Psi} \mathbb{Z}[L]^r \longrightarrow A_\Psi \to 0.$$

Since the group $L$ acts on the abelianization $N^{\mathrm{ab}} := N/[N,N]$ through the conjugate, $N^{\mathrm{ab}}$ is endowed with the structure of a $\mathbb{Z}[L]$-module. In fact, one can easily see that the $L$ action defined by $l \cdot [n] := [\widetilde{l} n \widetilde{l}^{-1}]$ $([n] \in N^{\mathrm{ab}}, l \in L, \Psi(\widetilde{l}) = l)$ is well-defined. Then, we have the following theorem.

Theorem 2.1.3. (*Crowell exact sequence* [**Cr**]) *The following exact sequence of $\mathbb{Z}[L]$-modules exists:*

$$0 \longrightarrow N^{\mathrm{ab}} \xrightarrow{\theta_1} A_\Psi \xrightarrow{\theta_2} \mathbb{Z}[L] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

*where*

$$\theta_1(n \bmod [N,N]) := \left(\Psi \circ \pi \left(\frac{\partial f}{\partial x_i}\right)\right) \bmod \mathrm{Im}(Q_\Psi) \quad (\pi(f) = n) \tag{2.1.4}$$

*and*

$$\theta_2((\alpha_1, \dots, \alpha_r) \bmod \mathrm{Im}(Q_\Psi) := \sum_{j=1}^{n} \alpha_j(\Psi \circ \pi(x_j) - 1).$$

*Here $\epsilon : \mathbb{Z}[L] \to \mathbb{Z}$ is the augmentation map.*

We set $G = F_r$, $\pi = \mathrm{id}_{F_r}$, $L = F_r^{\mathrm{ab}} = F_r/[F_r, F_r]$ and $\Psi = \mathrm{ab} : F_r \to F_r^{\mathrm{ab}}$. Then, the kernel $N$ of ab is $N = \mathrm{Ker}(\mathrm{ab}) = [F_r, F_r]$ and its abelianization $N^{\mathrm{ab}}$ is the meta-abelian quotient $N^{\mathrm{ab}} = [F_r, F_r]/[[F_r, F_r], [F_r, F_r]]$ of $F_r$. Since the free group $F_r$ has no relation, we have $Q_\Psi = 0$ and therefore $A_\Psi = \mathbb{Z}[L]^r = \mathbb{Z}[F_r^{\mathrm{ab}}]^r = {\Lambda_r}^{\oplus r}$. Then, by Theorem2.1.3, we have the following exact sequence of $\mathbb{Z}[H]$-modules:

$$0 \longrightarrow [F_r, F_r]/[[F_r, F_r], [F_r, F_r]] \xrightarrow{\theta_1} {\Lambda_r}^{\oplus r} \xrightarrow{\theta_2} \Lambda_r \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where

$$\theta_1(f \bmod [[F_r, F_r], [F_r, F_r]]) = \Psi\left(\frac{\partial f}{\partial x_j}\right), \tag{2.1.5}$$

and

$$\theta_2((\alpha_1, \dots, \alpha_r)) = \sum_{j=1}^{r} \alpha_j(t_j - 1).$$

This exact sequence is called the *Blanchfield-Lyndon exact sequence*. It can be identified with the exact sequence of the relative homology group for the pair of topological spaces $(D_r^{\mathrm{ab}}, h^{-1}(p_0))$;
(2.1.6)

$$\begin{array}{ccccccccc} 0 \longrightarrow & H_1(D_r^{\mathrm{ab}}) & \longrightarrow & H_1(D_r^{\mathrm{ab}}, h^{-1}(p_0)) & \longrightarrow & H_0(h^{-1}(p_0)) & \longrightarrow & H_0(D_r^{\mathrm{ab}}) & \longrightarrow 0. \\ & \| & & \| & & \| & & \| & \\ & [F_r, F_r]/[[F_r, F_r], [F_r, F_r]] & & \Lambda_r^{\oplus r} & & \Lambda_r & & \mathbb{Z} & \end{array}$$

Setting the augmentation ideal of $\Lambda_r$ by $I_{\Lambda_r}$ and $L_r := [F_r, F_r]/[[F_r, F_r], [F_r, F_r]]$, (2.1.6) leads to the following exact sequence

$$0 \longrightarrow L_r \xrightarrow{\theta_1} \Lambda_r^{\oplus r} \xrightarrow{\theta_2} I_{\Lambda_r} \longrightarrow 0. \tag{2.1.7}$$

Moreover, we have the isomorphism of $\Lambda_r$-modules induced by $\theta_1$, called the *Blanchfield-Lyndon isomorphism*,

$$\theta_1 : L_r \xrightarrow{\sim} \{(\alpha_i) \in \Lambda_r^{\oplus r} \mid \sum_{j=1}^{r} \alpha_j (t_j - 1) = 1\}. \tag{2.1.8}$$

In particular, we can see that $H_1(D_r^{\mathrm{ab}}; \mathbb{Z}) = [F_r, F_r]/[[F_r, F_r], [F_r, F_r]]$ has a basis $[x_i, x_{i+1}]$ with $1 \leqslant i \leqslant r-1$.

To define the reduced Gassner representation, we define the submodule $L_r^{\mathrm{prim}}$ of $H_1(D_r^{\mathrm{ab}}; \mathbb{Z})$ algebraically. This construction is inspired by the one by Oda ([**O2**]) of its pro-$l$ analogue. For $1 \leqslant i \leqslant r$, let $R_i$ be the normal closure of $x_i$ in $F_r$, i.e. $R_i := \langle\langle x_i \rangle\rangle$. We set $F_r^{(i)} := F_r/R_i$ and set $\Lambda_r^{(i)} := \mathbb{Z}[t_1^{\pm}, \ldots, \widehat{t_i^{\pm}}, \ldots, t_r^{\pm}]$ with the augmentation ideal $I_{\Lambda_r^{(i)}}$. Here, $\widehat{t_i^{\pm}}$ means deleting $t_i^{\pm}$. We define a $\mathbb{Z}$-algebra homomorphism $\delta_i : \Lambda_r \longrightarrow \Lambda_r^{(i)}$ by

$$\delta_i(t_j) = \begin{cases} t_j & (j \neq i), \\ 0 & (j = i). \end{cases}$$

Note that $\Lambda_r^{(i)}$-module can be regarded as a $\Lambda_r$-module through $\delta_i$. We set $L_r^{(i)} := [F_r^{(i)}, F_r^{(i)}]/[[F_r^{(i)}, F_r^{(i)}], [F_r^{(i)}, F_r^{(i)}]]$. Let $\xi_i : L_r \to L_r^{(i)}$ be the $\Lambda_r$-algebra homomorphism induced by the natural homomorphism $F_r \to F_r^{(i)}$. We define the *primitive part* $L_r^{\mathrm{prim}}$ of $L_r$ by

$$L_r^{\mathrm{prim}} := \bigcap_{i=1}^{r} \mathrm{Ker}\ (\xi_i) \subset H_1(D_r^{\mathrm{ab}}; \mathbb{Z}). \tag{2.1.9}$$

We set $U_i := (1-t_1) \cdots \widehat{(1-t_i)} \cdots (1-t_r)$. Here $\widehat{(1-t_i)}$ means that we remove the term $(1-t_i)$.

THEOREM 2.1.10. (1) *The Blanchfiled-Lyndon isomorphism* $\theta_1$ *in* (2.1.8) *is restricted to the isomorphism of the* $\Lambda_r$*-module*

$$L_r^{\mathrm{prim}} \xrightarrow{\sim} \{(\alpha_j U_j) \in \Lambda_r^{\oplus r} \mid \alpha_j \in \Lambda_r,\ \sum_{j=1}^{r} \alpha_j = 0\}.$$

(2) *The submodule* $L_r^{\mathrm{prim}}$ *is stable under the induced action of* $B_r$.
(3) *As a* $\Lambda_r$*-module,* $L_2^{\mathrm{prim}} = H_1(D_2^{\mathrm{ab}}; \mathbb{Z})$, $L_r^{\mathrm{prim}} \neq H_1(D_r^{\mathrm{ab}}; \mathbb{Z})$ $(r \geqslant 3)$.
(4) $L_r^{\mathrm{prim}} \otimes_{\Lambda_n} \mathbb{Q}(t_1, \ldots, t_r) = H_1(D_r^{\mathrm{ab}}; \mathbb{Z}) \otimes_{\Lambda_r} \mathbb{Q}(t_1, \ldots, t_r)$.

PROOF. We extend $\xi$ to the $\Lambda_r$-homomorphism $\widetilde{\xi} : \Lambda_r^{\oplus r} \to (\Lambda_r^{(i)})^{\oplus(r-1)}$ by

$$\widetilde{\xi}_i(\alpha_1, \dots, \alpha_r) := (\delta_i(\alpha_1), \dots, \delta_i(\alpha_{i-1}), \delta_i(\alpha_{i+1}), \dots, \delta_i(\alpha_r)).$$

Thus, we have the commutative diagram of $\Lambda_r$-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_r & \longrightarrow & \Lambda_r^{\oplus r} & \longrightarrow & I_{\Lambda_r} & \longrightarrow & 0 \\ & & \downarrow \xi_i & & \downarrow \tilde{\xi}_i & & \downarrow \delta_i & & \\ 0 & \longrightarrow & L_r^{(i)} & \longrightarrow & (\Lambda_r^{(i)})^{\oplus(r-1)} & \longrightarrow & I_{\Lambda_r^{(i)}} & \longrightarrow & 0 \end{array}$$

where the two rows are the Crowel exact sequences. Thus, we can see that $\mathrm{Ker}(\widetilde{\xi}_i)$ is given by

$$\mathrm{Ker}\ (\widetilde{\xi}_i) = \{(\alpha_1(t_i - 1), \dots, \alpha_{i-1}(t_i - 1), \alpha_i, \alpha_{i+1}(t_i - 1), \dots, \alpha_r(t_i - 1))\}$$

where $\alpha_j \in \Lambda_r$ $(1 \leqslant i \leqslant r)$. Thus, from (2.1.8) and (2.1.9), we have

$$L_r^{\mathrm{prim}} = \{(\alpha_j) \in \Lambda_r^{\oplus r} \mid \sum_{j=1}^{r} \alpha_j(t_j - 1) = 0,\ \alpha_j \equiv 0 \bmod (t_i - 1) \text{ if } i \neq j\}$$

The assertion (1) follows, since $\Lambda_r$ is U.F.D.
(2) Since the Artin representation $\mathrm{Ar}_r(\sigma)$ sends $x_i$ to the conjugate of $x_{\chi(\sigma)(i)}$, the definition (2.1.9) implies that $L_r^{\mathrm{prim}}$ is stable under the induced action of $B_r$.
(3) and (4) are immediate consequences of (1) and (2). □

Since, for $b \in B_r$, we have the induced action,

$$\widetilde{b}_* : H_1(D_r^{\mathrm{ab}}) \to H_1(D_r^{\mathrm{ab}})$$

from the lift $\widetilde{b} : D_r^{\mathrm{ab}} \to D_r^{\mathrm{ab}}$ of a homeomorphism representing $b$ and the submodule $L_r^{\mathrm{prim}}$ is invariant under the action of $\widetilde{b}_*$ (Theorem 2.1.10(1)), we obtain a 1-cocycle

$$\mathrm{Gass}_r^{\mathrm{H,red}} : B_r \longrightarrow \mathrm{Aut}(L_r^{\mathrm{prim}}) = \mathrm{GL}(r-1; \Lambda_r) \subset \mathrm{Aut}(H_1(D_r^{\mathrm{ab}})); \quad b \mapsto \widetilde{b}_*|_{L_r^{\mathrm{prim}}}.$$

We call this 1-cocycle the *homological reduced Gassner 1-cocycle* of $B_r$.

By restricting $\mathrm{Gass}_r^{\mathrm{H,red}}$ to $PB_r$, we have the homomorphism

$$\mathrm{Gass}_r^{\mathrm{H,red}} : PB_r \longrightarrow \mathrm{Aut}(L_r^{\mathrm{prim}}) = \mathrm{GL}(r-1; \Lambda_r) \subset \mathrm{Aut}(H_1(D_r^{\mathrm{ab}})).$$

We call it the *homological reduced Gassner representation* of $PB_r$.

As explained in the Appendix, this definition of the reduced Gassner representation is equivalent to original definition (cf.[**Bi1**, 3.3]).

**2.1.3. A representation matrix of the homological reduced Gassener representation.** Here, we calculate a representation matrix of the homological reduced Gassner representation of $PB_r$.

For this, we need some lemmas. In what follows, we simply denote by $[x_i, x_j]$ the class $[x_i, x_j] \bmod [[F_r, F_r], [F_r, F_r]]$ for $x_i, x_j \in F_r$ $(1 \leqslant i, j \leqslant r)$.

LEMMA 2.1.11. *We have*

$$A_{kl}[x_i, x_{i+1}] = \begin{cases} t_i t_{i+1}[x_i, x_{i+1}] & (\textit{if } k = i < l = i+1), \\ [x_i, x_{i+1}] + t_{i+1}(1 - t_i)[x_{i+1}, x_l] & (\textit{if } k = i+1 < l), \\ [x_i, x_{i+1}] + (1 - t_{i+1})[x_k, x_i] & (\textit{if } k < l = i), \\ [x_i, x_{i+1}] - (1 - t_{i+1})[x_i, x_l] & (\textit{if } k = i < i+1 < l), \\ [x_i, x_{i+1}] - t_{i+1}(1 - t_i)[x_k, x_{i+1}] & (\textit{if } k < i < i+1 = l), \\ [x_i, x_{i+1}] & (\textit{otherwise}), \end{cases}$$

*and*

$$A_{kl}^{-1}[x_i, x_{i+1}] = \begin{cases} t_i^{-1}t_{i+1}^{-1}[x_i, x_{i+1}] & (\textit{if } k = i < l = i+1), \\ [x_i, x_{i+1}] - t_l^{-1}(1-t_i)[x_{i+1}, x_l] & (\textit{if } k = i+1 < l), \\ [x_i, x_{i+1}] - t_k^{-1}t_i^{-1}(1-t_{i+1})[x_k, x_i] & (\textit{if } k < l = i), \\ [x_i, x_{i+1}] - t_i^{-1}t_l^{-1}(1-t_{i+1})[x_l, x_i] & (\textit{if } k = i < i+1 < l), \\ [x_i, x_{i+1}] - t_k^{-1}(1-t_i)[x_{i+1}, x_k] & (\textit{if } k < i < i+1 = l). \\ [x_i, x_{i+1}] & (\textit{otherwise}). \end{cases}$$

We set $U_{ij} := (1-t_1)\cdots\widehat{(1-t_i)}\cdots\widehat{(1-t_j)}\cdots(1-t_r)$, where $\widehat{(1-t_i)}$ means that we remove the term $(1-t_i)$. We also put $U_i := (1-t_1)\cdots\widehat{(1-t_i)}\cdots(1-t_r)$. Moreover, we set

$$E_{ij} := U_{ij}[x_i, x_j] \quad (1 \leqslant i < j \leqslant r).$$

Then, we have

LEMMA 2.1.12. *Using the above notation, for $1 \leqslant i < j \leqslant r$, we have*

$$E_{ij} = \sum_{k=i}^{j-1} E_{i,i+1}$$

PROOF. From the Blanchfield Lyndon exact sequence, it is enough to prove that

$$\theta_1(E_{ij}) = \sum_{k=i}^{j-1} \theta_1(E_{i,i+1}). \tag{2.1.13}$$

By the definition of $\theta_1$, we have

$$\theta_1(E_{ij}) = (0, \ldots, 0, \overbrace{U_i}^{i\text{-th}}, 0, \ldots, 0, \overbrace{-U_j}^{j\text{-th}}, 0, \ldots, 0)$$

and one may easily obtain equation (2.1.13). □

Theorem 2.1.10 enable us to see that $L_r^{\mathrm{prim}}$ is spanned by $E_i := E_{i,i+1}$ $(1 \leqslant i \leqslant r-1)$ as a $\Lambda_r$-module. Then, by direct computation using Lemma 2.1.11 and Lemma 2.1.12, we arrive at the following.

PROPOSITION 2.1.14. *For $1 \leqslant k < l \leqslant n$ and for $1 \leqslant i \leqslant r-1$, we have*

$$A_{kl}(E_i) = \begin{cases} t_i t_{i+1} E_i & (\textit{if } k = i < l = i+1), \\ E_i + t_{i+1}(1-t_l)\displaystyle\sum_{m=i+1}^{l-1} E_m & (\textit{if } k = i+1 < l), \\ E_i + (1-t_k)\displaystyle\sum_{m=k}^{i-1} E_m & (\textit{if } k < l = i), \\ t_l E_i - (1-t_l)\displaystyle\sum_{m=i+1}^{l-1} E_m & (\textit{if } k = i < i+1 < l), \\ (1 - t_{i+1}(1-t_k))E_i - t_{i+1}(1-t_k)\displaystyle\sum_{m=k}^{i-1} E_m & (\textit{if } k < i < i+1 = l), \\ E_i & (\textit{otherwise}), \end{cases}$$

*and*

$$A_{kl}^{-1}(E_i) = \begin{cases} t_i^{-1}t_{i+1}^{-1}E_i & (\textit{if } k = i < l = i+1), \\ E_i + (1-t_l^{-1})\displaystyle\sum_{m=i+1}^{l-1} E_m & (\textit{if } k = i+1 < l), \\ E_i + t_i^{-1}(1-t_k^{-1})\displaystyle\sum_{m=k}^{i-1} E_m & (\textit{if } k < l = i), \\ (1-t_i^{-1}(1-t_l^{-1}))E_i - t_i^{-1}(1-t_l^{-1})\displaystyle\sum_{m=i+1}^{l-1} E_m & (\textit{if } k = i < i+1 < l), \\ t_k^{-1}E_i - (1-t_k^{-1})\displaystyle\sum_{m=k}^{i-1} E_m & (\textit{if } k < i < i+1 = l), \\ E_i & (\textit{otherwise}). \end{cases}$$

Example 2.1.15. Let $b_{\mathrm{Borr}}$ be a pure braid as in Example 1.1.8. Then, the homological reduced Gassner representation of $b_{\mathrm{Borr}}$ is given by the following matrix:

$$\mathrm{Gass}_3^{\mathrm{H,red}}(b_{\mathrm{Borr}}) = \begin{pmatrix} 1 & 0 \\ -t_1^{-1}t_2^{-1}(t_1-1)(t_2t_3-1) & 1 \end{pmatrix}.$$

Remark 2.1.16. (1) It is known that Gassner representation may be defined (co)homologically and extended to representations of string links by le Dimet, Kirk, Livingston, and Wang ([**lD**], [**KLW**]). However, our construction of the reduced Gassner representation is different from their one.
(2)It is known that the monodromy representation of the KZ equation with values in the null vector space of the tensor product of the Verma module of $\mathfrak{sl}_2(\mathbb{C})$ gives the reduced Gassner representation with generic parameter (cf. [**Koh**]). We can prove this fact by direct computation in terms of the above proposition.

**2.1.4. (Reduced) Gassner representations via Magnus representations.** Here, we define the (reduced) Gassner representation as a special case of the Magnus representations of the pure braid groups. (For more details on the Magnus representation, we refer the reader to [**Bi1**], [**Mt1**], and [**Sa**].)

To begin with, let us recall the Magnus 1-cocycle of the automorphism group of free groups. Let $\mathrm{Aut}(F_r)$ be the group of automorphisms of $F_r$ and $\mathbb{Z}[F_r]$ be the group ring of $F_r$ over the ring of rational integers $\mathbb{Z}$. For a given free basis $\mathbf{x} = \{x_1, \ldots, x_r\}$, let $\frac{\partial}{\partial x_i} : \mathbb{Z}[F_r] \to \mathbb{Z}[F_r]$ be the Fox free derivative with respect to $\mathbf{x}$. Let $G$ be the quotient group of $F_r$ by a characteristic subgroup $R \subset F_r$, i.e., $G = F_r/R$, and let $f : F_r \to G$ be the canonical projection.

In the following, we restrict ourselves to the case that $G$ is an abelian group.

Then, for any $\alpha \in \mathrm{Aut}(F_r)$ and for any free basis $\mathbf{x} = \{x_1, \ldots, x_r\}$ of $F_r$, we define the map $\mathrm{M}_f : \mathrm{Aut}(F_r) \to \mathrm{GL}(r; \mathbb{Z}[G])$ by

$$\mathrm{M}_f := {}^t\left(f\left(\frac{\partial \alpha(x_i)}{\partial x_j}\right)\right)$$

where ${}^t : \mathrm{GL}(r;\mathbb{Z}[G]) \to \mathrm{GL}(r;\mathbb{Z}[G])$ denotes an anti-automorohism that sends each matrix to its transposed matrix. It is known that the map $\mathrm{M}_f$ is a 1-cocycle, i.e., $\mathrm{M}_f(\alpha\beta) = \mathrm{M}_f(\alpha)^\alpha(\mathrm{M}_f(\beta))$ for $\alpha, \beta \in \mathrm{Aut}(F_r)$. Here, $^\alpha(\mathrm{M}_f(\beta))$ is the matrix obtained from $\mathrm{M}_f(\beta)$ by the induced action of $\alpha$ on each matrix element.

Remark 2.1.17. Our definition of the Magnus representation is slightly different from that of [**Mt1**]. If we restrict ourselves to the case that $G$ is an abelian group, the Magnus representation becomes a 1-cocycle before taking $^- : \mathrm{GL}(r;\mathbb{Z}[G]) \to \mathrm{GL}(r;\mathbb{Z}[G])$. Here, $^- : \mathrm{GL}(r;\mathbb{Z}[G]) \to \mathrm{GL}(r;\mathbb{Z}[G])$ is the automorphism induced by the involution $g \mapsto g^{-1}$.[1]

We take $f = \mathrm{ab}$, $G = F_r^{\mathrm{ab}}$ and $R = \Gamma_2 F_r$. By composing the Artin representation $\mathrm{Ar}_r$ of $B_r$ and $\mathrm{M}_{\mathrm{ab}}$, we obtain the 1-cocycle,

$$\mathrm{Gass}_r^{\mathrm{M}} := \mathrm{M}_{\mathrm{ab}}(\mathrm{Ar}_r(b)) : B_r \longrightarrow \mathrm{GL}(r;\Lambda_r); \quad b \mapsto {}^t\left(\mathrm{ab}\left(\frac{\partial b(x_i)}{\partial x_j}\right)\right).$$

We call this 1-cocycle $\mathrm{Gass}_r^{\mathrm{M}}$ the *Magnus-Gassner 1-cocycle* of $B_r$. By restricting $\mathrm{Gass}_r^{\mathrm{M}}$ to $PB_r$, we obtain the homomorpshism

$$\mathrm{Gass}_r^{\mathrm{M}} : PB_r \longrightarrow \mathrm{GL}(r;\Lambda_r),$$

and we call it the *Magnus-Gassner representation* of $PB_r$.

Proposition 2.1.18. *For each generator $A_{ij} \in PB_r$, we have*

$$\left(\mathrm{ab}\left(\frac{\partial A_{kl}(x_i)}{\partial x_j}\right)\right) = \begin{cases} 1 + t_k(t_l - 1) & (k = i = j) \\ t_k(t_k - 1) & (l = j, k = i) \\ 1 - t_l & (l = i, k = j) \\ t_k & (l = i = j) \\ (1 - t_l)(1 - t_i) & (k < i < l, j = k) \\ (1 - t_k)(t_i - 1) & (k < i < l, l = j) \\ \delta_{ij} & (\mathit{otherwise}) \end{cases}$$

Proof. We can easily obtain the above formulas by direct computation. □

Remark 2.1.19. Note that the formula of Magnus-Gassner representation in [**Bi1**, p119] contains some errors. For more details, see also its errata [**Bi2**].

From Proposition 2.1.1 and Proposition 2.1.18, we can deduce the following.

Corollary 2.1.20. *The homological Gassner representation and Magnus-Gassner representation are equivalent.*

For $b \in PB_r$, let us consider the matrix

$$D^{-1}\mathrm{Gass}_n^{\mathrm{M}}(b)D$$

where the matrix $D$ is given by

$$D = \begin{pmatrix} U_1 & 0 & 0 & \cdots & 0 & 0 \\ -U_2 & U_2 & 0 & \cdots & 0 & 0 \\ 0 & -U_3 & U_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_{r-1} & 0 \\ 0 & 0 & 0 & \cdots & -U_r & 1 \end{pmatrix}.$$

[1] Note that , in the case that $G$ is *not* an abelian group, $^- : \mathbb{Z}[G] \to \mathbb{Z}[G]$ is an *anti*-automorphism.

Here, one can see that the $r$-th row of the above matrix is $(0, \ldots, 0, *)$, where $*$ denotes some element of $\Lambda_r$. Hence, the Magnus-Gassner representation can be reduced to an $r-1$ dimensional representation. We denote by $\mathrm{Gass}_r^{\mathrm{M,red}}(b)$ the matrix obtained from $D^{-1}\mathrm{Gass}_r^{\mathrm{M}}(b)D$ by eliminating the $r$-th row and column.

We call this representation $\mathrm{Gass}_r^{\mathrm{M,red}} : PB_r \to \mathrm{GL}(r-1; \Lambda_r)$ the *reduced Magnus-Gassner representation* of $PB_r$. Then, we have

Theorem 2.1.21. *The reduced homological-Gassner representation is equivalent to the reduced Magnus-Gassner representation:, i.e., we have the following commutative diagram:*

$$\begin{array}{ccc} PB_r & \xrightarrow{\quad = \quad} & PB_r \\ \Big\downarrow{\scriptstyle \mathrm{Gass}_r^{\mathrm{M,red}}} & & \Big\downarrow{\scriptstyle \mathrm{Gass}_r^{\mathrm{H,red}}} \\ \mathrm{GL}(r-1; \Lambda_r) & \xrightarrow{\ \mu\ } & \mathrm{Aut}_{\Lambda_r}(L_r^{\mathrm{prim}}) \end{array}$$

Proof. For the proof, we need the following lemma, which can be proved by direct computation:

Lemma 2.1.22. *For any $A = (a_{ij}) \in \mathrm{GL}(r; \Lambda_r)$, the entries $a'_{ij}$ $(1 \leqslant i, j \leqslant r-1)$ of the matrix*

$$D^{-1}AD = (a'_{ij})_{1 \leqslant i < j \leqslant r}$$

*are given by*

$$\begin{aligned} a'_{1j} &= \frac{1-t_1}{1-t_j} a_{1j} - \frac{1-t_1}{1-t_{j+1}} a_{1,j+1} \quad (1 \leqslant j \leqslant r-1) \\ a'_{ij} &= a'_{i-1,j} + \frac{1-t_i}{1-t_j} a_{ij} - \frac{1-t_i}{1-t_{j+1}} a_{i,j+1} \quad (2 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant r-1). \end{aligned}$$

By Lemma 2.1.22, we can see that $\mathrm{Gass}_r^{\mathrm{M,red}}(A_{ij}) = \mathrm{Gass}_r^{\mathrm{H,red}}(A_{ij})$ where the righthand side denotes the representation matrix given in Proposition 2.1.14. This completes the proof. □

Remark 2.1.23. The above equivalence of the representations also follows from Corollary 2.1.20 and the Crowell exact sequence. Since each basis $E_i$ $(1 \leqslant i \leqslant r)$ of $\mathrm{L}_r^{\mathrm{prim}}$ corresponds to $U_i[x_i] - U_{i+1}[x_{i+1}]$ in $H_1(D_n^{\mathrm{ab}}; h^{-1}(p_0))$, the above matrix $D$ is nothing but the basis transformation matrix from $[x_1], \ldots, [x_r]$ to $U_1[x_1] - U_2[x_2], \ldots, U_{r-1}[x_{r-1}] - U_r[x_r], [x_r]$.

Since the homological and Magnus Gassner representations are equivalent, in the following, the (reduced) Gassner representation always means the (reduced) Magnus-Gassner representation, and we will denote them by $\mathrm{Gass}_r$ and $\mathrm{Gass}_r^{\mathrm{red}}$, respectively.

Remark 2.1.24. By setting $t_1 = \cdots = t_r = t$, we obtain the *Burau representation* of $B_r$

$$\mathrm{Bur}_r := \mathrm{Gass}_r|_{t_1 = \cdots = t_r = t} : B_r \longrightarrow \mathrm{GL}(r; \Lambda)$$

and the *reduced Burau representation* of $B_r$

$$\mathrm{Bur}_r^{\mathrm{red}} := \mathrm{Gass}_r^{\mathrm{red}}\Big|_{t_1 = \cdots = t_r = t} : B_r \longrightarrow \mathrm{GL}(r-1; \Lambda)$$

form the Gassner representation and the reduced Gassner representation. Here we put $\Lambda := \mathbb{Z}[t^{\pm}]$.

### 2.2. Gassner representations and Milnor numbers

In this section, we give some formulas that relate the Gassner representation and the Milnor number of a pure braid.

Let $\theta^{\mathrm{ab}} : \Lambda_r \to \mathbb{Z}[[T_1, \ldots, T_r]]$ be a homomorphism defined by $\theta^{\mathrm{ab}}(t_i) = 1 + T_i$ and let us consider the composition $\mathrm{Gass}_r^\theta := \theta^{\mathrm{ab}} \circ \mathrm{Gass}_r$. Here, we denote the map obtained by applying $\theta^{\mathrm{ab}}$ on each matrix element by the same $\theta^{\mathrm{ab}}$. Then, we have

Proposition 2.2.1. *For any $b \in PB_r$, we write that $\mathrm{Gass}_r^\theta(b) = (\theta^{\mathrm{ab}}(a_{ij}))_{1\leqslant i,j\leqslant r}$. Then, we have*

$$\begin{aligned}\theta(a_{ij}) &= \delta_{ij} - \delta_{ij}\sum_{k\geqslant 1}\sum_{1\leqslant i_i,\ldots,i_k\leqslant r}\mu(i_1\cdots i_k j)T_{i_1}\cdots T_{i_k}\\ &\quad+\sum_{k\geqslant 0}\sum_{1\leqslant i_1,\ldots,i_k\leqslant r}\mu(i_1\cdots i_k ij)T_{i_1}\cdots T_{i_k}T_j.\end{aligned}$$

Proof. To prove the above proposition, we begin by writing

$$\begin{aligned}&\mathrm{ab}\left(\frac{\partial[y_j,x_j]}{\partial x_i}\right)\\ =\;&\mathrm{ab}\left(\frac{\partial y_j}{\partial x_i} + y_j\frac{\partial x_j}{\partial x_i} - y_jx_jy_j^{-1}\frac{\partial y_j}{\partial x_i} - y_jx_jy_j^{-1}x_j^{-1}\frac{\partial y_j}{\partial x_i}\right)\\ =\;&(\mathrm{ab}(y_j)-1)\mathrm{ab}\left(\frac{\partial x_j}{\partial x_i}\right) + (1-\mathrm{ab}(x_j))\mathrm{ab}\left(\frac{\partial y_j}{\partial x_i}\right)\end{aligned}$$

Noting that

$$\mathrm{ab}\left(\frac{\partial[y_j,x_j]}{\partial x_i}\right) = \frac{\partial y_jx_jy_j^{-1}}{\partial x_i} - \frac{\partial x_j}{\partial x_i},$$

we have

$$a_{ij} = \mathrm{ab}\left(\frac{\partial[y_j,x_j]}{\partial x_i}\right) + \delta_{ij}$$

where we set $(b_{ij}) = \mathrm{Gass}_n(b)$. Hence, $\theta^{\mathrm{ab}}(a_{ij})$ is given by

$$\begin{aligned}\theta^{\mathrm{ab}}(a_{ij}) &= \delta_{ij} - \delta_{ij}\sum_{k\geqslant 1}\sum_{1\leqslant i_i,\ldots,i_k\leqslant r}\mu(i_1\cdots i_k j)T_{i_1}\cdots T_{i_k}\\ &\quad+\sum_{k\geqslant 0}\sum_{1\leqslant i_1,\ldots,\leqslant r}\mu(i_1\cdots i_k ij)T_{i_1}\cdots T_{i_k}T_j.\end{aligned}$$

This completes the proof. □

Theorem 2.2.2. *For any $b \in PB_r$, we write that $\mathrm{Gass}_r^{\mathrm{red}}(b) = (b_{ij})_{1\leqslant i,j\leqslant r}$. Then, we have*

$$\begin{aligned}b_{1j} &= \mathrm{ab}(y_1)\delta_{1j} + (1-t_1)\mathrm{ab}\left(\frac{\partial y_j}{\partial x_1} - \frac{\partial y_{j+1}}{\partial x_1}\right)\\ b_{ij} &= b_{i-1,j} + (\delta_{ij} - \delta_{i,j+1})\mathrm{ab}(y_i)\\ &\qquad+(1-t_i)\mathrm{ab}\left(\frac{\partial y_j}{\partial x_i} - \frac{\partial y_{j+1}}{\partial x_i}\right)\quad (1 < i \leqslant r-1, 1 \leqslant j \leqslant r-1).\end{aligned}$$

Proof. By Lemma 2.1.22, the above formulas can be obtained by direct computation. □

Then, by applying $\theta^{ab}$, we have

THEOREM 2.2.3. *For any $b \in PB_r$, we write that* $\mathrm{Gass}_r^{\mathrm{red},\theta}(b) = (\theta^{\mathrm{ab}}(b_{ij}))$. *Then, we have*

$$\begin{aligned}
\theta^{\mathrm{ab}}(b_{1j}) &= \delta_{1j}\left(\sum_{k\geqslant 0}\sum_{1\leqslant i_1,\ldots,i_k\leqslant r}\mu(i_1\cdots i_k 1)T_{i_1}\cdots T_{i_k}\right) \\
&\quad - \left(\sum_{k\geqslant 1}\sum_{1\leqslant i_1,\ldots,i_{k-1}\leqslant r}(\mu(i_1\cdots i_{k-1}1j)-\mu(i_1\cdots i_{k-1}1j+1))T_{i_1}\cdots T_{i_{k-1}}T_1\right), \\
\theta^{\mathrm{ab}}(b_{ij}) &= \theta^{\mathrm{ab}}(b_{i-1,j}) + (\delta_{ij}-\delta_{i,j+1})\left(\sum_{k\geqslant 0}\sum_{1\leqslant i_1,\ldots,i_k\leqslant r}\mu(i_1\cdots i_k i)T_{i_1}\cdots T_{i_k}\right) \\
&\quad - \left(\sum_{k\geqslant 1}\sum_{1\leqslant i_1,\ldots,i_{k-1}\leqslant r}(\mu(i_1\cdots i_{k-1}ij)-\mu(i_1\cdots i_{k-1}ij+1))T_{i_1}\cdots T_{i_{k-1}}T_1\right) \\
&\quad (1 < i \leqslant r-1, 1 \leqslant j \leqslant r-1).
\end{aligned}$$

### 2.3. Gassner representations, Milnor invariants, and Johnson homomorphisms

Here, we define the Gassner representation of the Johnson homomorphisms of the automorphism group of free groups as a special case of the Magnus representation of them ([**Mt1**],[**Sa**]). Then, we show the explicit relation between the Gassner representation of the Johnson homomorphisms and the Milnor invariants for a pure braid.

Using the notation in §1.3.1, for any $\tau \in \mathrm{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathrm{gr}_{k+1}(F_r))$, we define the map $||\cdot|| : \mathrm{Aut}(F_r) \to \mathrm{M}(r; I_{\Lambda_r}^k/I_{\Lambda_r}^{k+1})$ by

$$||\tau|| := {}^t\left(\mathrm{ab}\left(\frac{\partial \tau(x_i)}{\partial x_j}\right)\right)$$

where we consider any lift of the element $\tau(x_i)$ in $\mathrm{gr}_{k+1}(F_r) = \Gamma_{k+1}F_r/\Gamma_{k+2}F_r$ to $\Gamma_{k+1}F_r$. Note that the above definition does not depend on the choice of the lift of $\tau(x_i)$.

For any $b \in PB_r(k)$, we obtain the representation

$$||\tau_k|| : PB_r(k) \longrightarrow \mathrm{M}(r; I_{\Lambda_r}^k/I_{\Lambda_r}^{k+1}).$$

We call it the *Gassner representation of the k-th Johnson homomorphism* for $PB_r$. Then, we have the following theorem.

THEOREM 2.3.1 ([**Mt1**]). *The Gassner representation* $\mathrm{Gass}_r : PB_r \to \mathrm{GL}(r; \Lambda_r)$ *induces the homomorphism,*

$$\mathrm{Gass}_r^{[k]} : PB_r(k) \longrightarrow \mathrm{GL}(r; \Lambda_r/I_{\Lambda_r}^{k+1}).$$

*Moreover, for any $b \in PB_r(k)$, we have*

$$\mathrm{Gass}_r^{[k]}(b) = E_r + ||\tau_k(b)||$$

*where $E_r$ is the identity matrix of degree $r$.*

Then, from Theorem 2.3.1 and Theorem 2.2.2, we can deduce the following.

Corollary 2.3.2. *For any $b \in PB_r(k)$, let $||\tau_k^\theta(b)||$ be the matrix obtained from $||\tau_k(b)||$ by applying $\theta^{\mathrm{ab}}$ to each entries. Then, we have*

$$\begin{aligned} ||\tau_k^\theta(b)|| &= -\delta_{ij} \sum_{1\leqslant i_i,\ldots,i_k\leqslant r} \mu(i_1\cdots i_k i) T_{i_1}\cdots T_{i_k} \\ &\quad + \sum_{1\leqslant i_1,\ldots,i_{k-1}\leqslant r} \mu(i_1\cdots i_{k-1} j i) T_{i_1}\cdots T_{i_{k-1}} T_j. \end{aligned}$$

We can also define the reduced Gassner representation of the Johnson homomorphisms for $PB_r$ as follows: For $b \in PB_r(k)$, let $||\tau_k^{\mathrm{red}}(b)||$ be the matrix obtained from the matrix,

$$D^{-1}||\tau_k(b)||D,$$

by eliminating the $r$-th column and row where the matrix $D$ is as in §2.1.4. We call this representation the *reduced Gassner representation of the $k$-th Johnson homomorphism* for $PB_r$. Then, we have

Theorem 2.3.3. *The reduced Gassner representation* $\mathrm{Gass}_n^{\mathrm{red}} : PB_r \to \mathrm{GL}(r-1;\Lambda_r)$ *induces a homomorphism*

$$\mathrm{Gass}_r^{\mathrm{red}[k]} : PB_r(k) \longrightarrow \mathrm{GL}(r-1;\Lambda_r/I_{\Lambda_r}^{k+1}).$$

*Moreover, for any $b \in PB_r(k)$, we have*

$$\mathrm{Gass}_r^{\mathrm{red}[k]}(b) = E_{r-1} + ||\tau_k^{\mathrm{red}}(b)||$$

*where $E_{r-1}$ is the identity matrix of degree $r-1$.*

Proof. The first part is clear from the definition. The second part is proven as follows. From Lemma 2.1.22 and the definition, we can see that the matrix $||\tau_k^{\mathrm{red}}(b)|| = (d_{ij})$ is given by

$$\begin{aligned} d_{1j} &= (\mathrm{ab}(y_1)-1)\delta_{1j} + (1-t_1)\mathrm{ab}\left(\frac{\partial y_j}{\partial x_1} - \frac{\partial y_{j+1}}{\partial x_1}\right) \\ d_{ij} &= d_{i-1,j} + (\delta_{ij}-\delta_{i,j+1})(\mathrm{ab}(y_i)-1) \\ &\quad +(1-t_i)\mathrm{ab}\left(\frac{\partial y_j}{\partial x_i} - \frac{\partial y_{j+1}}{\partial x_i}\right) \quad (1 < i \leqslant r-1, 1 \leqslant j \leqslant r-1). \end{aligned}$$

By careful computation, we obtain

$$\begin{aligned} d_{1j} &= b_{1j} - \delta_{1j} \\ d_{ij} &= b_{ij} - \delta_{1j} - (\delta_{2j}-\delta_{2,j+1}) - \cdots - (\delta_{ij}-\delta_{i,j+1}) \\ &= b_{ij} - \delta_{ij} \quad (1 < i \leqslant r-1, 1 \leqslant j \leqslant r-1) \end{aligned}$$

from Theorem 2.2.2. Here, we use the same notation as in Theorem 2.2.2 for the induced representation $\mathrm{Gass}_r^{\mathrm{red}[k]}$. This completes the proof. □

From Theorem 2.3.3 and Theorem 2.2.3, we have the following reduced Gassner version of Corollary 2.3.2.

Corollary 2.3.4. *For any $b \in P_n(k)$, let $||\tau_k^{\mathrm{red},\theta}(b)|| = (\theta^{\mathrm{ab}}(d_{ij}))_{1\leqslant i,j\leqslant r-1}$ be the matrix obtained from $||\tau_k^{\mathrm{red}}(b)||$ by applying $\theta^{\mathrm{ab}}$ to each entry. Accordingly, we have*

$$
\begin{aligned}
\theta^{\mathrm{ab}}(d_{1j}) \quad &= \delta_{1j} \left( \sum_{1 \leqslant i_1, \ldots, i_k \leqslant r} \mu(i_1 \cdots i_k 1) T_{i_1} \cdots T_{i_k} \right) \\
&\qquad - \left( \sum_{1 \leqslant i_1, \ldots, i_{k-1} \leqslant r} (\mu(i_1 \cdots i_{k-1} 1 j) - \mu(i_1 \cdots i_{k-1} 1 j+1)) T_{i_1} \cdots T_{i_{k-1}} T_1 \right), \\
\theta^{\mathrm{ab}}(d_{ij}) \quad &= \theta^{\mathrm{ab}}(d_{i-1,j}) + (\delta_{ij} - \delta_{i,j+1}) \left( \sum_{1 \leqslant i_1, \ldots, i_k \leqslant r} \mu(i_1 \cdots i_k i) T_{i_1} \cdots T_{i_k} \right) \\
&\qquad - \left( \sum_{1 \leqslant i_1, \ldots, i_{k-1} \leqslant r} (\mu(i_1 \cdots i_{k-1} i j) - \mu(i_1 \cdots i_{k-1} i j+1)) T_{i_1} \cdots T_{i_{k-1}} T_1 \right) \\
&\quad (1 < i \leqslant r-1, 1 \leqslant j \leqslant r-1).
\end{aligned}
$$

CHAPTER 3

# Absolute Galois groups, $l$-adic Milnor invariants, and pro-$l$ Johnson homomorphism

In this chapter, we study the arithmetic analogue of chapter 1. Precisely speaking, we study the action of the absolute Galois group of a number field on the étale fundamental group of the projective line minus $r+1$-points by defining and studying the $l$-adic Milnor invariants and the pro-$l$ Johnson homomorphism in an analogous way to which the pure braid group was studied in Chapter 1. This chapter is based on [**KMT**, Sections 1, 2, and 3].

## 3.1. Absolute Galois groups and the Ihara action

In this section, we recall the definition of absolute Galois groups and recall the set-up and some results on the Galois representation introduced by Ihara in [**Ih1**].

**3.1.1. Absolute Galois groups.** Here, we recall basics of absolute Galois groups. For more details on this materials, see [**Sz**].

Let $k$ be a field. An extension $L/k$ is called *algebraic* if every element $\alpha$ of $L$ is a root of some polynomial with coefficients in $k$. This polynomial is called the *minimal polynomial* of $\alpha$ if it is monic and irreducible over $k$. We may easily see that finite extension $L/k$ is algebraic.

A polynomial $f \in k[x]$ is *separable* if it has no multiple roots is some algebraic closure of $k$. An algebraic extension $L/k$ is *separable* over $k$ if the minimal polynomial of any $\alpha \in L/k$ is separable. Note that, in the case of characteristic 0, separability automatically follows.

For an extension $L$ of $k$, let $\mathrm{Aut}(L/k)$ denote the group of field automorphisms of $L$ fixing $k$ elementwise.

An algebraic extension $L$ of $k$ is called a *Galois extension* of $k$ if the elements of $L$ that remain fixed under the action of $\mathrm{Aut}(L/k)$ are exactly those of $k$. When $L/k$ is a Galois extension, we denote $\mathrm{Aut}(L/k)$ by $\mathrm{Gal}(L/k)$ and call it the *Galois group* of $L$ over $k$.

Remark 3.1.1. It is known that an algebraic extension $L/k$ is Galois if and only if $L/k$ is separable and the minimal polynomial over $k$ of any $\alpha \in L$ splits into linear factors in $L$.

Let $\overline{k}$ be an algebraic closure of $k$. We define the separable closure $k_s$ over $k$ in $\overline{k}$ as the compositum of all finite separable subextensions of $\overline{k}$. Then, we can see that $k_s/k$ is Galois extension as follows: For $\alpha \in k_s \setminus k$, let $\alpha' \in k_s$ be another root of the minimal polynomial of $\alpha$. Let us consider the isomorphism of field extensions $k(\alpha) \to k(\alpha')$ given by sending $\alpha$ to $\alpha'$. This isomorphism may be extended to an isomorphism of the algebraic closure $\overline{k}$. Noting that each automorphism of

$\mathrm{Aut}(\overline{k}/k)$ sends an element of $\overline{k}$ to another root of its minimal polynomial, we may see that $k_s$ is stable under the action of $\mathrm{Aut}(\overline{k}/k)$.

The group $\mathrm{Gal}(k_s/k)$ is called the *absolute Galois group* of $k$. It is known that the absolute Galois group $\mathrm{Gal}(k_s/k)$ is a profinite group:

$$\mathrm{Gal}(k_s/k) = \varprojlim \mathrm{Gal}(L/k)$$

where $L$ runs over all finite Galois extensions of $k$.

**3.1.2. The outer Galois representation.** Let $x_1, \ldots, x_r$ be the $r$ letters ($r \geqslant 2$) and let $F_r$ denote the free group of rank $r$ on $x_1, \ldots, x_r$. Let $x_{r+1}$ be the element of $F_r$ defined by $x_1 \cdots x_r x_{r+1} = 1$ so that $F_r$ has the presentation $F_r = \langle x_1, \ldots, x_r, x_{r+1} | x_1 \cdots x_r x_{r+1} = 1 \rangle$. Let $\mathfrak{F}_r$ denote the pro-$l$ completion of $F_r$. Let $\mathrm{Aut}(\mathfrak{F}_r)$ (resp. $\mathrm{Int}(\mathfrak{F}_r)$) denote the group of topological automorphisms (resp. inner-automorphisms) of $\mathfrak{F}_r$ with compact-open topology. We note that any abstract automorphism of $\mathfrak{F}_r$ is bicontinuous ([**DDMS**, Corollary 1.22]) and that $\mathrm{Aut}(\mathfrak{F}_r)$ is virtually a pro-$l$ group ([**DDMS**, Theorem 5.6]). Let $H_{\mathbb{Z}_l}$ be the abelianization of $\mathfrak{F}_r$, $H_{\mathbb{Z}_l} := \mathfrak{F}_r/[\mathfrak{F}_r, \mathfrak{F}_r]$, and let $\pi : \mathfrak{F}_r \to H_{\mathbb{Z}_l}$ be the abelianization homomorphism. For $f \in \mathfrak{F}_r$, we let $[f] := \pi(f)$. We set $X_i := [x_i]$ ($1 \leqslant i \leqslant r+1$) for simplicity so that $H_{\mathbb{Z}_l}$ is the free $\mathbb{Z}_l$-module with basis $X_1, \ldots, X_r$ and we have $X_1 + \cdots + X_r + X_{r+1} = 0$. Each $\varphi \in \mathrm{Aut}(\mathfrak{F}_r)$ induces an automorphism of the $\mathbb{Z}_l$-module $H_{\mathbb{Z}_l}$ which is denoted by $[\varphi] \in \mathrm{GL}(H_{\mathbb{Z}_l})$.

Let $\overline{\mathbb{Q}}$ be the field of algebraic numbers in $\mathbb{C}$. Let $S$ be a given set of ordered $r+1$ $\overline{\mathbb{Q}}$-rational points $P_1, \ldots, P_{r+1}$ on the projective line $\mathbb{P}^1_{\mathbb{Q}}$ and we suppose that $P_1 = 0, P_2 = 1$ and $P_{r+1} = \infty$. Let $k := \mathbb{Q}(S \setminus \{\infty\})$, the finite algebraic number field generated over $\mathbb{Q}$ by coordinates of $P_1, \ldots, P_r$, so that all $P_i$'s are $k$-rational points of $\mathbb{P}^1$. Let $\mathrm{Gal}_k := \mathrm{Gal}(\overline{\mathbb{Q}}/k)$ be the absolute Galois group of $k$ equipped with the Krull topology. Note that $\mathrm{Gal}_k$ is the étale fundamental group $\pi_1^{\text{ét}}(\mathrm{Spec}\, k)$ with the base point $\mathrm{Spec}\, \overline{\mathbb{Q}} \to \mathrm{Spec}\, k$. Let $\pi_1^{\text{pro-}l}(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus S)$ denote the maximal pro-$l$ quotient of the étale fundamental group of $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus S$ with a base point $\mathrm{Spec}\, \overline{\mathbb{Q}} \to \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus S$ which lifts $\mathrm{Spec}\, \overline{\mathbb{Q}} \to \mathrm{Spec}\, k$. By [**Gr**, XII, Corollaire 5.2], $\pi_1^{\text{pro-}l}(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus S)$ is the pro-$l$ completion of the topological fundamental group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S)$. We fix an isomorphism $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S) \simeq F_r$ obtained by associating to each $x_i$ the homotopy class of a small loop around $P_i$ and hence an identification of $\pi_1^{\text{pro-}l}(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus S)$ with $\mathfrak{F}_r$.

The absolute Galois group $\mathrm{Gal}_k = \pi_1^{\text{ét}}(\mathrm{Spec}\, k)$ acts, as the monodromy, on the geometric fiber $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus S$ of the fibration $\mathbb{P}^1_k \setminus S \to \mathrm{Spec}\, k$ and hence acts continuously on the pro-$l$ fundamental group $\pi_1^{\text{pro-}l}(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus S) = \mathfrak{F}_r$. The effect of changing a base point of $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus S$ is given as an inner automorphism of $\mathfrak{F}_r$. Thus we have the continuous outer Galois representation

$$\Phi_S : \mathrm{Gal}_k \longrightarrow \mathrm{Out}(\mathfrak{F}_r) := \mathrm{Aut}(\mathfrak{F}_r)/\mathrm{Int}(\mathfrak{F}_r). \tag{3.1.2}$$

In terms of the field extensions, the representation $\Phi_S$ is described as follows. Let $t$ be a variable over $k$. We regard $\mathbb{P}^1$ as the $t$-line and so the function field $K$ of $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ is the rational function field $\overline{\mathbb{Q}}(t)$. The $k$-rational points $P_i$ are identified with places of $K/\overline{\mathbb{Q}}$. Let $M$ be the maximal pro-$l$ extension of $K$ unramfied outside $P_i$ ($1 \leqslant i \leqslant r+1$). Then we have an isomorphism $\iota : \mathfrak{F}_r \xrightarrow{\sim} \mathrm{Gal}(M/K)$ such that each $\iota(x_i)$ is a topological generator of the inertia group of an extension $P_i^M$ of $P_i$ to a place of $M$. Since $P_i$'s are $k$-rational, $M/k(t)$ is a Galois extension and so we have the exact sequence

$$1 \to \mathfrak{F}_r \simeq \mathrm{Gal}(M/K) \to \mathrm{Gal}(M/k(t)) \to \mathrm{Gal}(K/k(t)) = \mathrm{Gal}_k \to 1.$$

For $g \in \mathrm{Gal}_k$, choose $\tilde{g} \in \mathrm{Gal}(M/k(t))$ which lifts $g$. Consider the action of $\mathrm{Gal}_k$ on $\mathrm{Gal}(M/K)$ defined by $f \mapsto \tilde{g}f\tilde{g}^{-1}$ and regard it as an automorphism of $\mathfrak{F}_r$ via the isomorphism $\iota$. The effect of changing a lift $\tilde{g}$ is given as an inner automorphism of $\mathfrak{F}_r$. Thus we obtain the representation $\Phi_S$. Note further that $g \circ P_i^M \circ \tilde{g}^{-1}$ is a place of $M$ which coincides with $P_i^M$ on $K$ ($1 \leqslant i \leqslant r+1$). So we have $g \circ P_i^M \circ \tilde{g}^{-1} \circ h = P_i^M$ for some $h \in \mathrm{Gal}(M/K)$ so that $h^{-1}\tilde{g}x_i\tilde{g}^{-1}h$ is a topological generator of the inertia group of $P_i^M$. Hence $\tilde{g}x_i\tilde{g} \sim x_i^{c_i}$ for some $c_i$ in $\mathbb{Z}_l$, the ring of $l$-adic integers. We pass to the abelianization $H_{\mathbb{Z}_l}$. Applying the conjugate by $\tilde{g}$ on the equality $X_1 + \cdots + X_{r+1} = 0$ in $H_{\mathbb{Z}_l}$, we have $c_1X_1 + \cdots + c_{r+1}X_{r+1} = 0$. From these equations, we have $c_1 = \cdots = c_{r+1}$. Therefore the action of $\mathrm{Gal}_k$ on $\mathfrak{F}_r$ gives an element of the subgroup $\widetilde{P}(\mathfrak{F}_r)$ of $\mathrm{Aut}(\mathfrak{F}_r)$ defined by

$$\widetilde{P}(\mathfrak{F}_r) := \{\varphi \in \mathrm{Aut}(\mathfrak{F}_r) \mid \varphi(x_i) \sim x_i^{N(\varphi)} \ (1 \leqslant i \leqslant r+1) \text{ for some } N(\varphi) \in \mathbb{Z}_l^\times\}.$$

Here the exponent $N(\varphi)$, called the *norm* of $\varphi$, gives a homomorphism $N : \mathrm{Aut}(\mathfrak{F}_r) \to \mathbb{Z}_l^\times$. So each $\varphi \in \widetilde{P}(\mathfrak{F}_r)$ acts on the abelianization $H_{\mathbb{Z}_l}$ by the multiplication by $N(\varphi)$, $[\varphi](X_i) = N(\varphi)X_i$ for $1 \leqslant i \leqslant r$. It is easy to see $\mathrm{Int}(\mathfrak{F}_r) \subset \widetilde{P}(\mathfrak{F}_r)$. Thus we have the outer Galois representation (3.1.2)

$$\Phi_S : \mathrm{Gal}_k \longrightarrow \widetilde{P}(\mathfrak{F}_r)/\mathrm{Int}(\mathfrak{F}_r). \tag{3.1.3}$$

**3.1.3. The Ihara representation.** We will lift $\Phi_S$ to a representation in $\mathrm{Aut}(\mathfrak{F}_r)$. For this, consider the subgroup $P(\mathfrak{F}_r)$ of $\widetilde{P}(\mathfrak{F}_r)$ defined by

(3.1.4)

$$P(\mathfrak{F}_r) := \left\{ \varphi \in \mathrm{Aut}(\mathfrak{F}_r) \,\middle|\, \begin{array}{l} \varphi(x_i) \sim x_i^{N(\varphi)} \ (1 \leqslant i \leqslant r-1) \ \varphi(x_r) \approx x_r^{N(\varphi)}, \\ \varphi(x_{r+1}) = x_{r+1}^{N(\varphi)} \text{ for some } N(\varphi) \in \mathbb{Z}_l^\times \end{array} \right\},$$

where $\approx$ denotes conjugacy by an element of the subgroup $\mathfrak{K}$ of $\mathfrak{F}_r$ generated by $[\mathfrak{F}_r, \mathfrak{F}_r]$ and $x_1, \ldots, x_{r-2}$. We denote by $P^1(\mathfrak{F}_r)$ the kernel of $N|_{\widetilde{P}(\mathfrak{F}_r)}$:

$$P^1(\mathfrak{F}_r) := \left\{ \varphi \in \mathrm{Aut}(\mathfrak{F}_r) \,\middle|\, \begin{array}{l} \varphi(x_i) \sim x_i \ (1 \leqslant i \leqslant r-1) \ \varphi(x_r) \approx x_r, \\ \varphi(x_{r+1}) = x_{r+1} \end{array} \right\}.$$

The following proposition was proved in [**Ih1**, Proposition 3, page 55] for the case $r = 2$ and stated in [**Ih3**, page 252] for the general case.

Proposition 3.1.5. *The natural homomorphism* $\mathrm{Aut}(\mathfrak{F}_r) \to \mathrm{Aut}(\mathfrak{F}_r)/\mathrm{Int}(\mathfrak{F}_r)$ *induces the isomorphism* $P(\mathfrak{F}_r) \simeq \widetilde{P}(\mathfrak{F}_r)/\mathrm{Int}(\mathfrak{F}_r)$. The representatives in $P(\mathfrak{F}_r)$ of $\widetilde{P}(\mathfrak{F}_r)/\mathrm{Int}(\mathfrak{F}_r)$ are called *Belyĭ's lifts*.

Proof. The proof is similar to that for $r = 2$. First, we note that the centralizer of $x_i$ in $\mathfrak{F}_r$ is $\langle x_i \rangle = x_i^{\mathbb{Z}_l}$ for $1 \leqslant i \leqslant r+1$.
Injectivity: We must show $P(\mathfrak{F}_r) \cap \mathrm{Int}(\mathfrak{F}_r) = \{1\}$. Suppose $\varphi \in P(\mathfrak{F}_r)$ and $\varphi = \mathrm{Int}(f)$ with $f \in \mathfrak{F}_r$. Then $fx_{r+1}f^{-1} = x_{r+1}^{N(\varphi)}$. Passing to $H_{\mathbb{Z}_l}$, we see $N(\varphi) = 1$ and so $f$ is in the centralizer of $x_{r+1}$. Hence $f = x_{r+1}^a$ for some $a \in \mathbb{Z}_l$. Since $\varphi \in P(\mathfrak{F}_r)$, $fx_rf^{-1} = \varphi(x_r) = gx_rg^{-1}$ for some $g \in \mathfrak{K}$. Therefore we have $g^{-1}fx_r(g^{-1}f)^{-1} = x_r$ and so $g^{-1}f = x_r^b$ for some $b \in \mathbb{Z}_l$. Passing to the abelianization $H_{\mathbb{Z}_l}$, we have $-[g] + aX_{r+1} = bX_r$. Since $[g] \in \mathbb{Z}_lX_1 + \cdots + \mathbb{Z}_lX_{r-2}$, we

have $a = b = 0$. Hence $f = g = 1$ and so $\varphi = 1$.
Surjectivity: We must show $P(\mathfrak{F}_r)\mathrm{Int}(\mathfrak{F}_r)/\mathrm{Int}(\mathfrak{F}_r) = \widetilde{P}(\mathfrak{F}_r)/\mathrm{Int}(\mathfrak{F}_r)$. Take $\varphi \in \widetilde{P}(\mathfrak{F}_r)$. Multiplying $\varphi$ by an element of $\mathrm{Int}(\mathfrak{F}_r)$, we may assume $\varphi(x_{r+1}) = x_{r+1}^{N(\psi)}$. Set $\varphi(x_r) = g x_r g^{-1}$ with $g \in \mathfrak{F}_r$. Write $[g] = c_1 X_1 + \cdots c_r X_r$ in $H_{\mathbb{Z}_l}$ ($c_i \in \mathbb{Z}_l$). Let $\varphi_1 := \mathrm{Int}(x_{r-1}^{-c_{r-1}} x_r^{-c_r}) \circ \varphi$. Then $\varphi_1(x_r) = g_1 x_r g_1^{-1}$ and $g_1 := x_{r-1}^{-c_{r-1}} x_r^{-c_r} g \in \mathfrak{K}$. Hence $\varphi_1 \in P(\mathfrak{F}_r)$ and $\varphi \equiv \varphi_1 \bmod \mathrm{Int}(\mathfrak{F}_r)$. □

By Proposition 3.1.5, we can lift $\Phi_S$ of (3.1.3) to the representation in $P(\mathfrak{F}_r)$, denoted by $\mathrm{Ih}_S$:

$$\mathrm{Ih}_S : \mathrm{Gal}_k \longrightarrow P(\mathfrak{F}_r), \tag{3.1.6}$$

which we call the *Ihara representation* associated to $S$. Let $\Omega_S$ denote the subfield of $\overline{\mathbb{Q}}$ corresponding to the kernel of $\mathrm{Ih}_S$ so that $\mathrm{Ih}_S$ factors through the Galois group $\mathrm{Gal}(\Omega_S/k)$:

$$\mathrm{Ih}_S : \mathrm{Gal}(\Omega_S/k) \longrightarrow P^1(\mathfrak{F}_r). \tag{3.1.7}$$

We recall some arithmetic properties on the ramification in the Galois extension $\Omega_S/k$. For this, let us prepare some notations. Let $\zeta_{l^n}$ be a primitive $l^n$-th root of unity for a positive integer $n$ such that $(\zeta_{l^{n+1}})^l = \zeta_{l^n}$ for $n \geqslant 1$. We set $k(\zeta_{l^\infty}) := \cup_{n\geqslant 1} k(\zeta_{l^n})$. The *$l$-cyclotomic character* $\chi_l : \mathrm{Gal}_k \to \mathbb{Z}_l^\times$ is defined by $g(\zeta_{l^n}) = \zeta_{l^n}^{\chi_l(g)}$ for $g \in \mathrm{Gal}_k$. Finally we define the set $R_S$ of finite primes of $k$ associated to $S$ as follows: Let $s_i$ be the coordinate of $P_i$ for $1 \leqslant i \leqslant r$, and let $\mathcal{O}_S$ be the integral closure of $\mathbb{Z}[l^{-1}, (s_i - s_j)^{-1}(1 \leqslant i \neq j \leqslant r)]$ in $k$. We then define $R_S$ by the maximal spectrum

$$R_S := \mathrm{Spm}\,\mathcal{O}_S. \tag{3.1.8}$$

Theorem 3.1.9. *Notations being as above, the following assertions hold:*
(1) ([**Ih1**, Proposition 2, page 53]). *$N \circ \mathrm{Ih}_S : \mathrm{Gal}_k \to \mathbb{Z}_l^\times$ coincides with $\chi_l$. In particular, the restriction of $\varphi_S$ to $\mathrm{Gal}_{k(\zeta_{l^\infty})} := \mathrm{Gal}(\overline{\mathbb{Q}}/k(\zeta_{l^\infty}))$, denoted by $\mathrm{Ih}_S^1$, gives the representation*

$$\mathrm{Ih}_S^1 : \mathrm{Gal}_{k(\zeta_{l^\infty})} \longrightarrow P^1(\mathfrak{F}_r)$$

*and we have $k(\zeta_{l^\infty}) \subset \Omega_S$.*
(2) ([**AI**, Proposition 2.5.2, Theorem 3]). *The Galois extension $\Omega_S/k$ is unramified over $R_S$ and $\Omega_S/k(\zeta_l)$ is a pro-$l$ extension.*

Remark 3.1.10. Recall that he Artin representation $\mathrm{Ar}_r$ of the pure braid group $PB_r$ in Section 1.1.2 is given by

$$\mathrm{Ar}_r : PB_r \xrightarrow{\sim} \mathrm{Aut}_0(F_r).$$

So the representation

$$\mathrm{Ih}_S^1 : \mathrm{Gal}_{k(\zeta_{l^\infty})} \longrightarrow P^1(\mathfrak{F}_r)$$

(resp. $\mathrm{Ih}_S : \mathrm{Gal}_k \to P(\mathfrak{F}_r)$) may be seen as an (resp. extended) arithmetic analogue of the Artin representation $\mathrm{Ar}_r$.

## 3.2. $l$-adic Milnor invariants and pro-$l$ link groups

**3.2.1. Pro-$l$ Magnus expansions.** Let $\{\Gamma_n\mathfrak{F}_r\}_{n\geqslant 1}$ be the lower central series of $\mathfrak{F}_r$ defined by

$$\Gamma_1\mathfrak{F}_r := \mathfrak{F}_r,\ \ \Gamma_{n+1}\mathfrak{F}_r := [\Gamma_n\mathfrak{F}_r, \mathfrak{F}_r]\ \ (n \geqslant 1).$$

Note that each $\Gamma_n\mathfrak{F}_r$ is a closed normal subgroup of $\mathfrak{F}_r$ so that $\Gamma_n\mathfrak{F}_r/\Gamma_{n+1}\mathfrak{F}_r$ is central in $\mathfrak{F}_r/\Gamma_{n+1}\mathfrak{F}_r$, and that each $\Gamma_n\mathfrak{F}_r$ is a finitely generated pro-$l$ group ([**DDMS**, 1.7, 1.14]). As in Section 3.1, let $H_{\mathbb{Z}_l}$ denote the abelianization of $\mathfrak{F}_r$:

$$H_{\mathbb{Z}_l} := \mathrm{gr}_1(\mathfrak{F}_r) = H_1(\mathfrak{F}_r, \mathbb{Z}_l) = H_1(F_r, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_l,$$

which is the free $\mathbb{Z}_l$-module with basis $X_1, \dots, X_r$, where $X_i$ is the image of $x_i$ in $H_{\mathbb{Z}_l}$. Let $T(H_{\mathbb{Z}_l})$ be the tensor algebra of $H_{\mathbb{Z}_l}$ over $\mathbb{Z}_l$ defined by $\bigoplus_{n\geqslant 0} H_{\mathbb{Z}_l}^{\otimes n}$, where $H_{\mathbb{Z}_l}^{\otimes 0} := \mathbb{Z}_l$ and $H_{\mathbb{Z}_l}^{\otimes n} := H_{\mathbb{Z}_l} \otimes_{\mathbb{Z}_l} \cdots \otimes_{\mathbb{Z}_l} H_{\mathbb{Z}_l}$ ($n$ times) for $n \geqslant 1$. It is nothing but the non-commutative polynomial algebra $\mathbb{Z}_l\langle X_1, \dots, X_r\rangle$ over $\mathbb{Z}_l$ with variables $X_1, \dots, X_r$:

$$T(H_{\mathbb{Z}_l}) = \bigoplus_{n\geqslant 0} H_{\mathbb{Z}_l}^{\otimes n} = \mathbb{Z}_l\langle X_1, \dots, X_r\rangle.$$

Let $\widehat{T}(H_{\mathbb{Z}_l})$ be the completion of $T(H_{\mathbb{Z}_l})$ with respect to the $\mathfrak{m}_T$-adic topology, where $\mathfrak{m}_T$ is the maximal two-sided ideal of $T(H_{\mathbb{Z}_l})$ generated by $X_1, \dots, X_r$ and $l$. It is the infinite product $\prod_{n\geqslant 0} H_{\mathbb{Z}_l}^{\otimes n}$, which is nothing but the *Magnus algebra* $\mathbb{Z}_l\langle\langle X_1, \dots, X_r\rangle\rangle$ over $\mathbb{Z}_l$, namely, the algebra of non-commutative formal power series (called *Magnus power series*) over $\mathbb{Z}_l$ with variables $X_1, \dots, X_r$:

$$\widehat{T}(H_{\mathbb{Z}_l}) = \prod_{n\geqslant 0} H_{\mathbb{Z}_l}^{\otimes n} = \mathbb{Z}_l\langle\langle X_1, \dots, X_r\rangle\rangle.$$

For $n \geqslant 0$, we set $\widehat{T}(n) := \prod_{m\geqslant n} H_{\mathbb{Z}_l}^{\otimes m}$. The *degree* of a Magnus power series $\Phi$, denoted by $\deg(\Phi)$, is defined to be the minimum $n$ such that $\Phi \in \widehat{T}(n)$. We note that $H_{\mathbb{Z}_l}^{\otimes n}$ is the free $\mathbb{Z}_l$-module on monomials $X_{i_1}\cdots X_{i_n}$ $(1 \leqslant i_1, \dots, i_n \leqslant r)$ of degree $n$ and $\widehat{T}(n)$ consists of Magnus power series of degree $\geqslant n$.

Let $\mathbb{Z}_l[[\mathfrak{F}_r]]$ be the complete group algebra of $\mathfrak{F}_r$ over $\mathbb{Z}_l$ and let $\epsilon_{\mathbb{Z}_l[[\mathfrak{F}_r]]} : \mathbb{Z}_l[[\mathfrak{F}_r]] \to \mathbb{Z}_l$ be the augmentation homomorphism with the augmentation ideal $I_{\mathbb{Z}_l[[\mathfrak{F}_r]]} := \mathrm{Ker}(\epsilon_{\mathbb{Z}_l[[\mathfrak{F}_r]]})$. The correspondence $x_i \mapsto 1 + X_i$ $(1 \leqslant i \leqslant r)$ gives rise to the isomorphism of topological $\mathbb{Z}_l$-algebras

$$\Theta : \mathbb{Z}_l[[\mathfrak{F}_r]] \ \xrightarrow{\sim} \widehat{T}(H_{\mathbb{Z}_l}), \tag{3.2.1}$$

which we call the *pro-$l$ Magnus isomorphism.* Here $I_{\mathbb{Z}_l[[\mathfrak{F}_r]]}^n$ corresponds, under $\Theta$, to $\widehat{T}(n)$ for $n \geqslant 0$. For $\alpha \in \mathbb{Z}_l[[\mathfrak{F}_r]]$, $\Theta(\alpha)$ is called the *pro-$l$ Magnus expansion* of $\alpha$. In the following, for a multi-index $I = (i_1 \cdots i_n)$, $1 \leqslant i_1, \dots, i_n \leqslant r$, we set

$$|I| := n \text{ and } X_I := X_{i_1} \cdots X_{i_n}.$$

We call the coefficient of $X_I$ in $\Theta(\alpha)$ the *$l$-adic Magnus coefficient* of $\alpha$ for $I$ and denote it by $\mu(I; \alpha)$. So we have

$$\Theta(\alpha) = \epsilon_{\mathbb{Z}_l[[\mathfrak{F}_r]]}(\alpha) + \sum_{|I|\geqslant 1} \mu(I; \alpha) X_I. \tag{3.2.2}$$

Restricting $\Theta$ to $\mathfrak{F}_r$, we have an injective group homomorphism, denoted by the same $\Theta$,

$$\Theta : \mathfrak{F}_r \hookrightarrow 1 + \widehat{T}(1), \tag{3.2.3}$$

which we call the *pro-$l$ Magnus embedding* of $\mathfrak{F}_r$ into $1 + \widehat{T}(1)$.

Here are some basic properties of $l$-adic Magnus coefficients:
For $\alpha, \beta \in \mathbb{Z}_l[[\mathfrak{F}_r]]$ and a multi-index $I$, we have

$$\mu(I; \alpha\beta) = \sum_{I=AB} \mu(A;\alpha)\mu(B;\beta), \tag{3.2.4}$$

where the sum ranges over all pairs $(A, B)$ of multi-indices such that $AB = I$, and we understand that $\mu(A; \alpha) = \epsilon_{\mathbb{Z}_l[[\mathfrak{F}_r]]}(\alpha)$ (resp. $\mu(B;\beta) = \epsilon_{\mathbb{Z}_l[[\mathfrak{F}_r]]}(\beta)$) if $|A| = 0$ (resp. $|B| = 0$).
(*Shuffle relation*) For $f \in \mathfrak{F}_r$ and multi-indices $I, J$ with $|I|, |J| \geqslant 1$, we have

$$\mu(I; f)\mu(J; f) = \sum_{A \in \mathrm{Sh}(I,J)} \mu(A; f), \tag{3.2.5}$$

where $\mathrm{Sh}(I, J)$ denotes the set of the results of all shuffles of $I$ and $J$ ([**CFL**]).
For $f \in \mathfrak{F}_r$ and $d \geqslant 2$, we have

$$\begin{aligned} \mu(I; f) = 0 \text{ for } |I| < d \ \text{ i.e., } \deg(\Theta(f-1)) \geqslant d \quad &\Longleftrightarrow \ f \in \Gamma_d \mathfrak{F}_r \\ &\Longleftrightarrow \ f - 1 \in I^d_{\mathbb{Z}_l[[\mathfrak{F}_r]]}. \end{aligned} \tag{3.2.6}$$

An automorphism $\varphi$ of the topological $\mathbb{Z}_l$-algebra $\mathbb{Z}_l[[\mathfrak{F}_r]]$ (resp. $\widehat{T}(H_{\mathbb{Z}_l})$) is said to be *filtration-preserving* if $\varphi(I^n_{\mathbb{Z}_l[[\mathfrak{F}_r]]}) = I^n_{\mathbb{Z}_l[[\mathfrak{F}_r]]}$ (resp. $\varphi(\widehat{T}(n)) = \widehat{T}(n)$) for all $n \geqslant 1$. Let $\mathrm{Aut}^{\mathrm{fil}}(\mathbb{Z}_l[[\mathfrak{F}_r]])$ (resp. $\mathrm{Aut}^{\mathrm{fil}}(\widehat{T}(H_{\mathbb{Z}_l}))$) be the group of filtration-preserving automorphisms of the topological $\mathbb{Z}_l$-algebras $\mathbb{Z}_l[[\mathfrak{F}_r]]$ (resp. $\widehat{T}(H_{\mathbb{Z}_l})$). The pro-$l$ Magnus isomorphism $\Theta$ in (3.2.1) induces the isomorphism

$$\mathrm{Aut}^{\mathrm{fil}}(\mathbb{Z}_l[[\mathfrak{F}_r]]) \xrightarrow{\sim} \mathrm{Aut}^{\mathrm{fil}}(\widehat{T}(H_{\mathbb{Z}_l})); \ \ \varphi \mapsto \Theta \circ \varphi \circ \Theta^{-1}. \tag{3.2.7}$$

In the following we set

$$\varphi^* := \Theta \circ \varphi \circ \Theta^{-1}. \tag{3.2.8}$$

We note by (3.2.6) that any automorphism $\varphi$ of $\mathfrak{F}_r$ can be extended uniquely to a filtration-preserving topological automorphism of $\mathbb{Z}_l[[\mathfrak{F}_r]]$, which is also denoted by $\varphi$. It is easy to see by (3.2.7) that for $\varphi \in \mathrm{Aut}^{\mathrm{fil}}(\mathbb{Z}_l[[\mathfrak{F}_r]]), \alpha \in \mathbb{Z}_l[[\mathfrak{F}_r]]$, we have

$$\Theta(\varphi(\alpha)) = \varphi^*(\Theta(\alpha)). \tag{3.2.9}$$

**3.2.2. $l$-adic Milnor invariants.** Let $\mathrm{Ih}_S : \mathrm{Gal}_k \to P(\mathfrak{F}_r)$ be the Ihara representation associated to $S$ in (3.1.6).

LEMMA 3.2.10. *Let $g \in \mathrm{Gal}_k$. For each $1 \leqslant i \leqslant r$, there exists uniquely $y_i(g) \in \mathfrak{F}_r$ satisfying the following properties:*
(1) $\mathrm{Ih}_S(g)(x_i) = y_i(g) x_i^{\chi_l(g)} y_i(g)^{-1}$*, where $\chi_l$ is the $l$-cyclotomic character,*
(2) *In the expression* $[y_i(g)] = c_1^{(i)} X_1 + \cdots + c_r^{(i)} X_r$ $(c_j^{(i)} \in \mathbb{Z}_l)$, $c_i^{(i)} = 0$.

PROOF. Existence: By the definition (3.1.4) of $P(\mathfrak{F}_r)$ and Theorem 3.1.9 (1), there is $z_i \in \mathfrak{F}_r$ such that $\mathrm{Ih}_S(g)(x_i) = z_i x_i^{\chi_l(g)} z_i^{-1}$ for each $i$. Let $[z_i] = a_1^{(i)} X_1 + \cdots + a_r^{(i)} X_r$ ($a_j^{(i)} \in \mathbb{Z}_l$). We set $y_i := z_i x_i^{-a_i^{(i)}}$. Then the conditions (1) and (2) are satisfied for $y_i$.

Uniqueness: Suppose that $y_i$ and $z_i$ in $\mathfrak{F}_r$ satisfy the conditions (1) and (2). Since $z_i^{-1} y_i$ is in the centralizer of $x_i^{\chi_l(g)}$, $z_i^{-1} y_i = x_i^{b_i}$ for some $b_i \in \mathbb{Z}_l$. Comparing the coefficients of $X_i$ in $[z_i^{-1} y_i]$ and $[x_i^{b_i}]$, we have $b_i = 0$ and hence $y_i = z_i$. □

We call $y_i(g) \in \mathfrak{F}_l$ in Lemma 3.2.10 the $i$-th (*preferred*) *longitude* of $g \in \mathrm{Gal}_k$ for $S$. By Lemma 3.2.10, $\mathrm{Ih}_S(g)$ for $g \in \mathrm{Gal}_k$ is determined by the $l$-cyclotomic value $\chi_l(g)$ and the $r$-tuple $\mathbf{y}(g) := (y_1(g), \ldots, y_r(g))$ of longitudes of $g$ for $S$. We note that $\mathrm{Ih}_S(g)$ acts on the abelianization $H_{\mathbb{Z}_l}$ of $\mathfrak{F}_r$ by the multiplication by $\chi_l(g)$, $[\mathrm{Ih}_S(g)](X_i) = \chi_l(g) X_i$ for $1 \leqslant i \leqslant r$. We also note that $y_i : \mathrm{Gal}_k \to \mathfrak{F}_r$ is continuous, since $\mathrm{Ih}_S$ is continuous.

Following the case for pure braids as in Chapter 1, we will define the $l$-adic Milnor numbers of $g \in \mathrm{Gal}_k$ by the $l$-adic Magnus coefficients of the $i$-th longitude $y_i(g)$: Let $I = (i_1 \cdots i_n)$ be a multi-index, where $1 \leqslant i_1, \ldots, i_n \leqslant r$ and $|I| = n \geqslant 1$. The *$l$-adic Milnor number* or *$l$-adic Milnor $\mu$ invariant* of $g \in \mathrm{Gal}_k$ for $I$, denoted by $\mu(g; I) = \mu(g; i_1 \cdots i_n)$, is defined by the $l$-adic Magnus coefficient of $y_{i_n}(g)$ for $I' := (i_1 \cdots i_{n-1})$:

$$\mu(g; I) := \mu(I'; y_{i_n}(g)). \tag{3.2.11}$$

Here we set $\mu(g; I) := 0$ if $|I| = 1$. We note that the map $\mu(\ ; I) : \mathrm{Gal}_k \to \mathbb{Z}_l$ is continuous for each $I$, since $y_i : \mathrm{Gal}_k \to \mathfrak{F}_r$ is continuous. We define $\mathfrak{a}(g)$ to be the ideal of $\mathbb{Z}_l$ generated by $\chi_l(g) - 1$. Note that $\mathfrak{a}(g) = 0$ when $g \in \mathrm{Gal}_{k(\zeta_{l^\infty})}$. We then define the *indeterminacy* $\Delta(g; I)$ by

(3.2.12)

$$\begin{array}{rl} \Delta(g; I) & := \text{the ideal of } \mathbb{Z}_l \text{ generated by } \mathfrak{a}(g) \text{ and } \mu(J; y_j(g)), \text{ where } J \\ & \qquad\qquad \text{ranges over proper subsequence } I' \text{ and } j = i_n \text{ or } j \text{ is in } J \end{array}$$

We also write $\Delta(I'; y_{i_n}(g))$ for $\Delta(g; I)$ for the convenience later. We then set

$$\overline{\mu}(g; I) := \mu(g; I) \bmod \Delta(g; I), \tag{3.2.13}$$

which we call the *$l$-adic Milnor $\bar{\mu}$ invariant* of $g \in \mathrm{Gal}_k$ for $I$.

We will show that the $l$-adic Milnor invariant $\overline{\mu}(g; I)$ for $g \in \mathrm{Gal}_k$ is unchanged when $g$ is replaced by its conjugate in $\mathrm{Gal}_k$. To prove this, we prepare some lemmas.

LEMMA 3.2.14. *For $g, h \in \mathrm{Gal}_k$ and $1 \leqslant i \leqslant r$, we have*
(1)$y_i(h^{-1}) = \mathrm{Ih}_S(h^{-1})(y_i(h)^{-1})$,
(2)$y_i(hg) = \mathrm{Ih}_S(h)(y_i(g)) y_i(h)$,
(3)$y_i(hgh^{-1}) = \mathrm{Ih}_S(hg)(y_i(h^{-1})) \mathrm{Ih}_S(h)(y_i(g)) y_i(h)$.

PROOF. (1) By Lemma 3.2.10, we have

$$\begin{aligned} x_i &= \mathrm{Ih}_S(h^{-1}) \mathrm{Ih}_S(h)(x_i) \\ &= \mathrm{Ih}_S(h^{-1})(y_i(h) x_i^{\chi_l(h)} y_i(h)^{-1}) \\ &= \mathrm{Ih}_S(h^{-1})(y_i(h)) y_i(h^{-1}) x_i y_i(h^{-1})^{-1} \mathrm{Ih}_S(h^{-1})(y_i(h)^{-1}), \end{aligned}$$

from which we find $\mathrm{Ih}_S(h^{-1})(y_i(h)) y_i(h^{-1}) = x_i^{a_i}$ for some $a_i \in \mathbb{Z}_l$. Passing to the abelianization $H_{\mathbb{Z}_l}$ of $\mathfrak{F}_r$ and comparing the coefficients of $X_i$, we find $a_i = 0$ and

hence we obtain (1).
(2) By Lemma 3.2.10, we have

$$\mathrm{Ih}_S(hg)(x_i) = y_i(hg) x_i^{\chi_l(hg)} y_i(hg)^{-1}. \tag{3.2.15}$$

On the other hand, we have

$$\begin{aligned} \mathrm{Ih}_S(hg)(x_i) &= \mathrm{Ih}_S(h)\mathrm{Ih}_S(g)(x_i) \\ &= \mathrm{Ih}_S(h)(y_i(g)x_i^{\chi_l(g)}y_i(g)^{-1}) \\ &= \mathrm{Ih}_S(h)(y_i(g))\mathrm{Ih}_S(h)(x_i^{\chi_l(g)})\mathrm{Ih}_S(h)(y_i(g)^{-1}) \\ &= \mathrm{Ih}_S(h)(y_i(g))y_i(h)x_i^{\chi_l(hg)}y_i(h)^{-1}\mathrm{Ih}_S(h)(y_i(g)^{-1}). \end{aligned} \tag{3.2.16}$$

Comparing (3.2.10) and (3.2.16), we have $y_i(hg)^{-1}\mathrm{Ih}_S(h)(y_i(g))y_i(h) = x_i^{b_i}$ for some $b_i \in \mathbb{Z}_l$. Passing to the abelianization and comparing the coefficients of $X_i$, we find $b_i = 0$ and hence we obtain (2).
(3) By (2), we have

$$y_i(hgh^{-1}) = \mathrm{Ih}_S(hg)(y_i(h^{-1}))y_i(hg) = \mathrm{Ih}_S(hg)(y_i(h^{-1}))\mathrm{Ih}_S(h)(y_i(g))y_i(h).$$

□

For $\rho \in \mathrm{Gal}_k$ and a multi-index $J$ with $|J| \geqslant 1$, we define $\Theta_J(\rho)$ by

$$\Theta_J(\rho) := \mathrm{Ih}_S(\rho)^*(X_J) - \chi_l(\rho)^{|J|}X_J. \tag{3.2.17}$$

Since $\mathrm{Ih}_S(\rho)^*$ is filtration-preserving, we note $\deg(\Theta_J(\rho)) \geqslant |J|$.

Lemma 3.2.18. *Notations being as above, the following assertions hold.*
(1) *$\Theta_J(\rho)$ is a Magnus power series $\sum_{|A|\geqslant|J|} m_A(J;\rho)X_A$ satisfying the following properties:*
(*i*) *if $m_A(J;\rho) \neq 0$, then $A$ contains $J$ as a proper subsequence. So we may write $\Theta_J(\rho) = \sum_{J \subsetneq A} m_A(J;\rho)X_A$.*
(*ii*) *any coefficient $m_A(J;\rho)$ is a multiple of $\mu(B; y_j(\rho))$ by an $l$-adic integer, where $B$ is some proper subsequence of $A$ and $j$ is in $J$.*
(2) *For $y \in \mathfrak{F}_r$, we have*

$$\begin{aligned} \Theta(\mathrm{Ih}_S(\rho)(y)) &= 1 + \sum_{|J|\geqslant 1} \chi_l(\rho)^{|J|}\mu(J;y)X_J + \sum_{|J|\geqslant 1}\mu(J;y)\Theta_J(\rho) \\ &\equiv \Theta(y) + \sum_{|J|\geqslant 1}\mu(J;y)\Theta_J(\rho) \mod \mathfrak{a}(\rho). \end{aligned}$$

Proof. (1) Let $1 \leqslant j \leqslant r$ and write $\Theta(y_j(\rho)) = 1 + Y_j(\rho)$. By (3.2.9) and Lemma 3.2.10, we have

$$\begin{aligned} \mathrm{Ih}_S(\rho)^*(X_j) &= \mathrm{Ih}_S(\rho)^*(\Theta(x_j - 1)) \\ &= \Theta(\mathrm{Ih}_S(\rho)(x_j - 1)) \\ &= \Theta(y_j(\rho)x_j^{\chi_l(\rho)}y_j(\rho)^{-1}) - 1 \\ &= \Theta(y_j(\rho))\Theta(x_j)^{\chi_l(\rho)}\Theta(y_j(\rho)^{-1}) - 1 \\ &= (1 + Y_j(\rho))(1 + X_j)^{\chi_l(\rho)}(1 - Y_j(\rho) + Y_j(\rho)^2 - \cdots) - 1 \\ &= \chi_l(\rho)X_j + \Theta_j(\rho), \end{aligned} \tag{3.2.19}$$

where $\Theta_j(\rho)$ is the sum of terms of the form $uY_j(\rho)^a X_j^b Y_j(\rho)^c$ for some $a, c \geqslant 0$ with $a + c \geqslant 1$, $b \geqslant 1$ and $u \in \mathbb{Z}_l$. Write $\Theta_j(\rho) = \sum_{|A|\geqslant 2} m_A(j;\rho)X_A$. Then it is easy to see that if $m_A(j;\rho) \neq 0$, then $A$ must contain $j$. Moreover, since

$Y_j(\rho) = \sum_{|B|\geqslant 1} \mu(B; y_j(\rho))X_B$, $m_A(j;\rho)$ is a multiple of $\mu(B; y_j(\rho))$ by an $l$-adic integer, where $B$ is some proper subsequence $B$ of $A$. Let $J = (j_1 \cdots j_n)$. By (3.2.19), we have

$$\begin{aligned}\textstyle\sum_{|A|\geqslant|J|} m_A(J;\rho)X_A \quad &:= \Theta_J(\rho)\\ &:= \mathrm{Ih}_S(\rho)^*(X_J) - \chi_l(\rho)^{|J|}X_J\\ &= \mathrm{Ih}_S(\rho)^*(X_{j_1})\cdots \mathrm{Ih}_S(\rho)^*(X_{j_n}) - \chi_l(\rho)^{|J|}X_J\\ &= (\chi_l(\rho)X_{j_1} + \Theta_{j_1}(\rho))\cdots(\chi_l(\rho)X_{j_n} + \Theta_{j_n}(\rho)) - \chi_l(\rho)^{|J|}X_J\\ &= \Phi_{j_1}(\rho)\cdots\Phi_{j_n}(\rho),\end{aligned}$$

where $\Phi_j(\rho)$ is $\chi_l(\rho)X_j$ or $\Theta_j(\rho)$ and at least one $\Theta_j(\rho)$ is involved for some $j$. Hence, by the properties of coefficients of $\Theta_j(\rho) = \sum_{|A|\geqslant 2} m_A(j;\rho)X_A$ proved above, we obtain the properties (i) and (ii).
(2) By (3.2.9) and (3.2.17), we have

$$\begin{aligned}\Theta(\mathrm{Ih}_S(\rho)(y)) \quad &= \mathrm{Ih}_S(\rho)^*(\Theta(y))\\ &= \mathrm{Ih}_S(\rho)^*(1 + \sum_{|J|\geqslant 1}\mu(J;y)X_J)\\ &= 1 + \sum_{|J|\geqslant 1}\mu(J;y)\mathrm{Ih}_S(\rho)^*(X_J)\\ &= 1 + \sum_{|J|\geqslant 1}\mu(J;y)(\chi_l(\rho)^{|J|}X_J + \Theta_J(\rho))\\ &= 1 + \sum_{|J|\geqslant 1}\chi_l(\rho)^{|J|}\mu(J;y)X_J + \sum_{|J|\geqslant 1}\mu(J;y)\Theta_J(\rho)\\ &\equiv \Theta(y) + \sum_{|J|\geqslant 1}\mu(J;y)\Theta_J(\rho) \mod \mathfrak{a}(\rho).\end{aligned}$$

□

We are ready to prove the following.

THEOREM 3.2.20. *For a multi-index $I$, the $l$-adic Milnor invariant $\overline{\mu}(g;I)$ for $g \in \mathrm{Gal}_k$ is unchanged when $g$ is replaced with its conjugate by an element of $\mathrm{Gal}_{k(\zeta_{l^\infty})}$. To be precise, let $I$ be a multi-index with $|I| \geqslant 1$. Let $g \in \mathrm{Gal}_k$ and $h \in \mathrm{Gal}_{k(\zeta_{l^\infty})}$. Then we have $\Delta(hgh^{-1};I) = \Delta(g;I)$ and*

$$\overline{\mu}(hgh^{-1};I) = \overline{\mu}(g;I)$$

.

PROOF. Let $I$ be a multi-index with $|I| \geqslant 1$ and $1 \leqslant i \leqslant r$. For $g, h \in \mathrm{Gal}_k$, we will show

$$\mu(I; y_i(hgh^{-1})) \equiv \mu(I; y_i(g)) \bmod \Delta(I; y_i(g)). \tag{3.2.21}$$

By Lemma 3.2.14 (3), we have

$$\Theta(y_i(hgh^{-1})) = \Theta(\mathrm{Ih}_S(hg)(y_i(h^{-1})))\Theta(\mathrm{Ih}_S(h)(y_i(g)))\Theta(y_i(h)). \tag{3.2.22}$$

For simplicity, we set, for a multi-index $J$ with $|J| \geqslant 1$,
(3.2.23)

$$a_J := \mu(J; \mathrm{Ih}_S(hg)(y_i(h^{-1}))), b_J := \mu(J; \mathrm{Ih}_S(h)(y_i(g))), c_J := \mu(J; y_i(h)).$$

Then, from (3.2.22) or (3.2.4), we have

(3.2.24)
$$\begin{aligned}&\mu(I;y_i(hgh^{-1}))\\ &= a_I + b_I + c_I + \sum_{AB=I} a_A b_B + \sum_{BC=I} b_B c_C + \sum_{AC=I} a_A c_C + \sum_{ABC=I} a_A b_B c_C,\end{aligned}$$

where $A, B, C$ are multi-indices with $|A|, |B|, |C| \geqslant 1$.

First, we look at $b_B$ for a subsequence $B$ of $I$. By Lemma 3.2.18 (1), (2) and by $h \in \mathrm{Gal}_{k(\zeta_{l^\infty})}$, we have

$$b_B = \mu(B; y_i(g)) + \mu(J; y_i(g))(\text{an } l\text{-adic integer})$$

for some proper subsequence $J$ of $B$. Therefore, by (3.2.24) and the definition of $\Delta(I; y_i(g))$, we have

$$(3.2.25) \quad \mu(I;y_i(hgh^{-1})) - \mu(I;y_i(g)) \equiv a_I + c_I + \sum_{AC=I} a_A c_C \mod \Delta(I;y_i(g)).$$

Here we note that the right hand side of (3.2.25) is the coefficient of $X_I$ of $\Theta(\mathrm{Ih}_S(hg)(y_i(h^{-1})))\Theta(y_i(h))$.

So, next, we look at $\Theta(\mathrm{Ih}_S(hg)(y_i(h^{-1})))\Theta(y_i(h))$. By (3.2.9), Lemma 3.2.14 (1) and Lemma 3.2.18 (2), we have

(3.2.26)
$$\begin{aligned}\Theta(\mathrm{Ih}_S(hg)(y_i(h^{-1}))) &= \mathrm{Ih}_S(hg)^*(\Theta(y_i(h^{-1})))\\ &= \mathrm{Ih}_S(h)^*\mathrm{Ih}_S(g)^*(\Theta(y_i(h^{-1})))\\ &\equiv \mathrm{Ih}_S(h)^*(\Theta(y_i(h^{-1})) + \sum_{|J|\geqslant 1} \mu(J;y_i(h^{-1}))\Theta_J(g)) \pmod{\mathfrak{a}(g)}\\ &= \Theta(\mathrm{Ih}_S(h)(y_i(h^{-1}))) + \sum_{|J|\geqslant 1} \mu(J;y_i(h^{-1}))\mathrm{Ih}_S(h)^*(\Theta_J(g))\\ &= \Theta(y_i(h)^{-1}) + \sum_{|J|\geqslant 1} \mu(J;y_i(h^{-1}))\mathrm{Ih}_S(h)^*(\Theta_J(g)).\end{aligned}$$

Here let us write $\mathrm{M}_J(g) = \sum_{J \subsetneq A} m_A(J;g)X_A$ as in Lemma 3.2.18 (1). Then we have, by $h \in \mathrm{Gal}_{k(\zeta_{l^\infty})}$,

$$(3.2.27) \quad \begin{aligned}\mathrm{Ih}_S(h)^*(\Theta_J(g)) &= \sum_{J\subsetneq A} m_A(J;g)\mathrm{Ih}_S(h)^*(X_A)\\ &= \sum_{J\subsetneq A} m_A(J;g)(X_A + \Theta_A(h)) \pmod{\mathfrak{a}(g)}\\ &= \sum_{J\subsetneq A} m_A(J;g)(X_A + \sum_{A\subsetneq A'} m_{A'}(A;h)X_{A'}).\end{aligned}$$

By (3.2.26) and (3.2.27), we have

(3.2.28)
$$\begin{aligned}&\Theta(\mathrm{Ih}_S(hg)(y_i(h^{-1})))\\ &\equiv \Theta(y_i(h)^{-1}) + \sum_{|J|\geqslant 1}\sum_{J\subsetneq A} \mu(J;y_i(h^{-1}))m_A(J;g)(X_A + \sum_{A\subsetneq A'} m_{A'}(A;h)X_{A'})\\ &\hfill \mod \mathfrak{a}(g)\end{aligned}$$

and hence

(3.2.29)
$$\begin{aligned}&\Theta(\mathrm{Ih}_S(hg)(y_i(h^{-1}))\Theta(y_i(h))\\ &\equiv 1+\sum_{|J|\geqslant 1}\sum_{J\subsetneqq A}\mu(J;y_i(h^{-1}))m_A(J;g)(X_A+\sum_{A\subsetneqq A'}m_{A'}(A;h)X_{A'})\Theta(y_i(h))\end{aligned}$$
$$\bmod\ \mathfrak{a}(g).$$

Here we note by Lemma 3.2.18 (2) that $m_A(J;g)$ is a multiple of $\mu(B;y_j(g))$ by an $l$-adic integer for some proper subsequence $B$ of $A$ and $j$ in $J$. By the definition (3.2.12) of $\Delta(I;y_i(g))$, the coefficient of $X_I$ in the right hand side of (3.2.29) must be congruent to 0 mod $\Delta(I;y_i(g))$. By (3.2.25), we obtain (3.2.21).

Finally, we show that $\Delta(I;y_i(hgh^{-1}))=\Delta(I;y_i(g))$ by induction on $|I|$. When $|I|=1$, this is obvioulу true (both sides are $\mathfrak{a}(g)=\mathfrak{a}(hgh^{-1})$) by the definition. Assume that $\Delta(I;y_i(hgh^{-1}))=\Delta(I;y_i(g))$ for all $I$ with $|I|\leqslant n$ ($n\geqslant 1$). Then, by (3.2.21), we have, for all $I$ with $|I|\leqslant n$ and $1\leqslant i\leqslant r$,

(3.2.30) $$\mu(I;y_i(hgh^{-1}))\equiv\mu(I;y_i(g)) \bmod \Delta(I;y_i(g)))(=\Delta(I;y_i(hgh^{-1}))).$$

Using (3.2.30) and the definition (3.2.12) of $\Delta(I;y_i(\rho))$ for $\rho=hgh^{-1},g$, we have $\Delta(I;y_i(hgh^{-1}))=\Delta(I;y_i(g))$ for $I$ with $|I|=n+1$. □

Remark 3.2.31. It is known that a braid $\beta$ and its conjugate $\gamma\beta\gamma^{-1}$ give rise to the same link as their closures ($\beta\mapsto\gamma\beta\gamma^{-1}$ is one of Markov's transforms. cf. [**Bi1**, 2.2], [**MK**, Chapter 9]). In particular, they have the same Milnor invariants. So Theorem 3.2.20 may be seen as an arithmetic analogue of this known fact for braids.

As a property of $l$-adic Milnor invariants, we have the following shuffle relation.

Proposition 3.2.32. *Let $g\in\mathrm{Gal}_k$. For multi-indices $I,J$ with $|I|,|J|\geqslant 1$ and $1\leqslant i\leqslant r$, we have*
$$\sum_{H\in\mathrm{PSh}(I,J)}\overline{\mu}(g;Hi)\equiv 0\ \mathit{mod}\ g.c.d\{\Delta(Hi)\mid H\in\mathrm{PSh}(I,J)\},$$
*where* $\mathrm{PSh}(I,J)$ *denotes the set of results of all proper shuffles of $I$ and $J$* ([**CFL**]).

Proof. By (3.2.5), we have
$$\mu(g;Ii)\mu(g;Ji)=\sum_{A\in\mathrm{Sh}(I,J)}\mu(g;Ai).$$
Taking mod g.c.d$\{\Delta(Hi)\mid H\in\mathrm{PSh}(I,J)\}$, the left hand side is congruent to 0 and any term $\mu(g;Ai)$ with $A\notin\mathrm{PSh}(I,J)$ is also congruent to 0. So the assertion follows. □

Let $R_S^\infty$ be the set of primes of $k(\zeta_{l^\infty})$ lying over $R_S$ in (3.1.8). For $\mathfrak{p}_\infty\in R_S^\infty$, choose a prime $\mathfrak{P}$ of $\Omega_S$ lying over $\mathfrak{p}_\infty$. Since $\mathfrak{P}$ is unramified in the Galois extension $\Omega_S/k$ by Theorem 3.1.9 (2), we have the Frobenius automorphism $\sigma_{\mathfrak{P}}\in\mathrm{Gal}(\Omega_S/k)$ of $\mathfrak{P}$. By Theorem 3.2.20, $\overline{\mu}(\sigma_{\mathfrak{P}};I)$ is independent of the choice of $\mathfrak{P}$ lying over $\mathfrak{p}_\infty$. So we define the *$l$-adic Milnor invariant* of $\mathfrak{p}_\infty$ for a multi-index $I$ by

(3.2.33) $$\overline{\mu}(\mathfrak{p}_\infty;I):=\overline{\mu}(\sigma_{\mathfrak{P}};I).$$

We also set $\Delta(\mathfrak{p}_\infty; I) := \Delta(\sigma_{\mathfrak{P}}; I)$ so that $\overline{\mu}(\mathfrak{p}_\infty; I) \in \mathbb{Z}_l/\Delta(\mathfrak{p}_\infty; I)$. Let $\mathfrak{p}$ be the prime of $k$ lying below $\mathfrak{p}_\infty$. Since $\chi_l(\sigma_{\mathfrak{P}}) = \mathrm{N}\mathfrak{p}$ (the norm of $\mathfrak{p}$), in order to have $\mathbb{Z}_l/\Delta(\mathfrak{p}_\infty; I) \neq 0$, it is necessary for us to consider only primes $\mathfrak{p}_\infty$ in $R_S^\infty$ lying over

$$R_S^1 := \{\mathfrak{p} \in R_S \mid \mathrm{N}\mathfrak{p} \equiv 1 \bmod l\}.$$

For $\mathfrak{p} \in R_S^1$, let $e(\mathfrak{p})$ denote the maximal integer such that

$$\mathrm{N}\mathfrak{p} \equiv 1 \bmod l^{e(\mathfrak{p})}.$$

It means that $\mathfrak{p}$ is completely decomposed in $k(\zeta_{l^{e(\mathfrak{p})}})/k$ and any prime of $k(\zeta_{l^{e(\mathfrak{p})}})$ lying over $\mathfrak{p}$ is inert in $k(\zeta_{l^\infty})/k(\zeta_{l^{e(\mathfrak{p})}})$. Hence $\sigma_{\mathfrak{P}} \in \mathrm{Gal}(\Omega_S/k(\zeta_{l^{e(\mathfrak{p})}}))$. Then the indeterminacy $\Delta(\mathfrak{p}_\infty; I)$ is a quotient ring of $\mathbb{Z}/l^{e(\mathfrak{p})}\mathbb{Z}$. We note that if $\mu(\sigma_{\mathfrak{P}}; I) \equiv 0 \bmod l^{e(\mathfrak{p})}$ for all $|I| \leqslant n$, then $\overline{\mu}(\mathfrak{p}_\infty; I)$ is well defined in $\mathbb{Z}/l^{e(\mathfrak{p})}\mathbb{Z}$ for $|I| = n+1$.

Remark 3.2.34. In [**Ms1**] and [**Ms2**, Chapter 8], the arithmetic Milnor invariants for certain primes of a number field were introduced as multiple generalizations of power residue symbols and the Rédei triple symbol ([**R**]). They are arithmetic analogues for primes of Milnor invariants of links. We see that Milnor invariants for a pure braid coincide with those for the link obtained by closing the pure braid as in Chapter 1. Recently, we found a relation between $l$ -adic Milnor invariants, Wojtkowiak ' s $l$-adic iterated integrals and $l$-adic polylogarithms ([**NW**], [**W1**] [**W4**]) and multiple power residue symbols (in particular, Rédei symbols), which will be discussed in the forthcoming paper.

Finally, we introduce a filtration on $\mathrm{Gal}_k$ using $l$-adic Milnor numbers. We set $\mathrm{Gal}_k^{\mathrm{Mil}}[0] := \mathrm{Gal}_k$. For each integer $n \geqslant 1$, we define a subset $\mathrm{Gal}_k^{\mathrm{Mil}}[n]$ of $\mathrm{Gal}_k$ by

$$\begin{aligned} \mathrm{Gal}_k^{\mathrm{Mil}}[n] &:= \{g \in \mathrm{Gal}_{k(\zeta_{l^\infty})} \mid \mu(g; I) = 0 \text{ for } |I| \leqslant n\} \\ &= \{g \in \mathrm{Gal}_{k(\zeta_{l^\infty})} \mid \deg(\Theta(y_i(g)) - 1) \geqslant n \text{ for } 1 \leqslant i \leqslant r\}. \end{aligned} \tag{3.2.35}$$

We then have the descending series

$$\mathrm{Gal}_k = \mathrm{Gal}_k^{\mathrm{Mil}}[0] \supset \mathrm{Gal}_k^{\mathrm{Mil}}[1] \supset \cdots \supset \mathrm{Gal}_k^{\mathrm{Mil}}[n] \supset \cdots \tag{3.2.36}$$

and we call it the *Milnor filtration* of $\mathrm{Gal}_k$.

Proposition 3.2.37. *For $n \geqslant 0$, $\mathrm{Gal}_k^{\mathrm{Mil}}[n]$ is a closed normal subgroup of $\mathrm{Gal}_k$.*

Proof. We may assume $n \geqslant 1$. Since $\mu(\ ; I) : \mathrm{Gal}_k \to \mathbb{Z}_l$ is continuous for each $I$ and $\mathrm{Gal}_k^{\mathrm{Mil}}[n] = \bigcap_{|I| \leqslant n} \mu(\ ; I)^{-1}(0)$, $\mathrm{Gal}_k^{\mathrm{Mil}}[n]$ is closed in $\mathrm{Gal}_k$. Let $g, h \in \mathrm{Gal}_k^{\mathrm{Mil}}[n]$. Write $\Theta(y_i(\rho)) = 1 + Y_i(\rho)$ for $\rho = g, h$ and each $1 \leqslant i \leqslant r$. Then $\deg(Y_i(\rho)) \geqslant n$. First, we will show $g^{-1} \in \mathrm{Gal}_k^{\mathrm{Mil}}[n]$. By Lemma 3.2.14 (1), we have

$$\begin{aligned} \Theta(y_i(g^{-1})) &= \Theta(\mathrm{Ih}_S(g^{-1})(y_i(g)^{-1})) \\ &= \mathrm{Ih}_S(g^{-1})^*(\Theta(y_i(g)^{-1})) \\ &= 1 + \mathrm{Ih}_S(g^{-1})^*(-Y_i(g) + Y_i(g)^2 - \cdots). \end{aligned} \tag{3.2.38}$$

Since $\mathrm{Ih}_S(g^{-1})$ is filtration-preserving, $\deg(\mathrm{Ih}_S(g^{-1})^*(-Y_i(g) + Y_i(g)^2 - \cdots)) \geqslant n$ and hence $g^{-1} \in \mathrm{Gal}_k^{\mathrm{Mil}}[n]$. Next, we will show $gh \in \mathrm{Gal}_k^{\mathrm{Mil}}[n]$. By Lemma 3.2.14 (2), we have

$$\begin{aligned} \Theta(y_i(gh)) &= \Theta(\mathrm{Ih}_S(h)(y_i(g)))\Theta(y_i(h)) \\ &= \mathrm{Ih}_S(h)^*(\Theta(y_i(g)))\Theta(y_i(h)) \\ &= (1 + \mathrm{Ih}_S(h)^*(Y_i(g)))(1 + Y_i(h)). \end{aligned}$$

Since $\deg(\mathrm{Ih}_S(h)^*(Y_i(g))), \deg(Y_i(h)) \geqslant n$, we see $gh \in \mathrm{Gal}_k^{\mathrm{Mil}}[n]$. Finally, we will show $hgh^{-1} \in \mathrm{Gal}_k^{\mathrm{Mil}}[n]$. By Lemma 3.2.14 (3), we have

$$\begin{aligned}\Theta(y_i(hgh^{-1})) &= \Theta(\mathrm{Ih}_S(hg)(y_i(h^{-1})))\Theta(\mathrm{Ih}_S(h)(y_i(g)))\Theta(y_i(h)) \\ &= \mathrm{Ih}_S(hg)^*(\Theta(y_i(h^{-1})))\mathrm{Ih}_S(h)^*(\Theta(y_i(g)))\Theta(y_i(h)).\end{aligned}$$

As we have just proved above, $\deg(\Theta(y_i(h^{-1}))-1), \deg(\mathrm{Ih}_S(h)^*(\Theta(y_i(g)))-1) \geqslant n$. Hence $hgh^{-1} \in \mathrm{Gal}_k^{\mathrm{Mil}}[n]$. Getting all together, the assertion is proved. □

In Section 3.3, we shall give another proof of Proposition 3.2.37 using the Johnson filtration.

**3.2.3. Pro-$l$ link groups and Massey products.** Following the analogy with the link group of a pure braid link as in Chapter 1, we define the *pro-$l$ link group* of each Galois element $g \in \mathrm{Gal}_k$ associated to $\mathrm{Ih}_S$ by

(3.2.39)

$$\begin{aligned}\Pi_S(g) &:= \langle x_1, \ldots, x_r \,|\, y_1(g)x_1^{\chi_l(g)}y_1(g)^{-1} = x_1, \cdots, y_r(g)x_r^{\chi_l(g)}y_r(g)^{-1} = x_r\rangle \\ &= \langle x_1, \ldots, x_r \,|\, x_1^{1-\chi_l(g)}[x_1^{-1}, y_1(g)^{-1}] = \cdots = x_r^{1-\chi_l(g)}[x_r^{-1}, y_r(g)^{-1}] = 1\rangle \\ &:= \mathfrak{F}_r/\mathfrak{N}_S(g),\end{aligned}$$

where $\mathfrak{N}_S(g)$ denotes the closed subgroup of $\mathfrak{F}_r$ generated normally by the pro-$l$ words $x_1^{1-\chi_l(g)}[x_1^{-1}, y_1(g)^{-1}], \ldots, x_r^{1-\chi_l(g)}[x_r^{-1}, y_r(g)^{-1}]$. We will give a cohomological interpretation of $l$-adic Milnor invariants of $g \in \mathrm{Gal}_k$ by Massey products in the cohomology of the pro-$l$ link group $\Pi_S(g)$. In the following, we let $g \in \mathrm{Gal}_k$ and $\mathfrak{a}$ an ideal of $\mathbb{Z}_l$ such that $\mathfrak{a} \neq \mathbb{Z}_l$ and $\chi_l(g) \equiv 1 \bmod \mathfrak{a}$. We may write $\mathfrak{a} = l^a\mathbb{Z}_l$ for some $1 \leqslant a \leqslant \infty$ ($l^a := 0$ if $a = \infty$). When $g \in \mathrm{Gal}_{k(\zeta_l^\infty)}$, we have $a = \infty$ and $\mathfrak{a} = 0$.

Let $C^i(\Pi_S(g), \mathbb{Z}_l/\mathfrak{a})$ be the $\mathbb{Z}_l/\mathfrak{a}$-module of continuous $i$-cochains ($i \geqslant 0$) of $\Pi_S(g)$ with coefficients in $\mathbb{Z}_l/\mathfrak{a}$, where $\Pi_S(g)$ acts on $\mathbb{Z}_l/\mathfrak{a}$ trivially. We consider the differential graded algebra $(C^\bullet(\Pi_S(g), \mathbb{Z}_l/\mathfrak{a}), d)$, where the product on $C^\bullet(\Pi_S(g), \mathbb{Z}_l/\mathfrak{a}) = \bigoplus_{i\geqslant 0} C^i(\Pi_S(g), \mathbb{Z}_l/\mathfrak{a})$ is given by the cup product and the differential $d$ is the coboundary operator. Then we have the cohomology ring $H^*(\Pi_S(g), \mathbb{Z}_l/\mathfrak{a}) := \bigoplus_{i\geqslant 0} H^i(C^\bullet(\Pi_S(g), \mathbb{Z}_l/\mathfrak{a}))$ of the pro-$l$ group $\Pi_S(g)$ with coefficients in $\mathbb{Z}_l/\mathfrak{a}$. In the following, we deal with only one and two dimensional cohomology groups. For the sign convention, we follow [**Dw**]. For $c_1, \ldots, c_n \in H^1(\Pi_S(g), \mathbb{Z}_l/\mathfrak{a})$, an $n$-th *Massey product* $\langle c_1, \ldots, c_n\rangle$ is said to be *defined* if there is an array

$$W = \{w_{ij} \in C^1(\Pi_S(g), \mathbb{Z}_l/\mathfrak{a}) \mid 1 \leqslant i < j \leqslant n+1, (i,j) \neq (1, n+1)\}$$

such that

$$\begin{cases} [\omega_{i,i+1}] = c_i \quad (1 \leqslant i \leqslant n), \\ dw_{ij} = \displaystyle\sum_{a=i+1}^{j-1} w_{ia} \cup w_{aj} \quad (j \neq i+1). \end{cases}$$

Such an array $W$ is called a *defining system* for $\langle c_1, \ldots, c_n\rangle$. The value of $\langle c_1, \ldots, c_n\rangle$ relative to $W$ is defined by the cohomology class represented by the 2-cocycle

$$\sum_{a=2}^{n} w_{1a} \cup w_{a,n+1},$$

and denoted by $\langle c_1, \ldots, c_n\rangle_W$. A Massey product $\langle c_1, \ldots, c_n\rangle$ itself is taken to be the subset of $H^2(\Pi_S(g), \mathbb{Z}_l/\mathfrak{a})$ consisting of elements $\langle c_1, \ldots, c_n\rangle_W$ for some defining system $W$. By convention, $\langle c\rangle = 0$. The following lemma is a baisc fact ([**Kr**]).

LEMMA 3.2.40. *We have $\langle c_1, c_2\rangle = c_1 \cup c_2$. For $n \geqslant 3$, $\langle c_1, \ldots, c_n\rangle$ is defined and consists of a single element if $\langle c_{j_1}, \ldots, c_{j_a}\rangle = 0$ for all proper subsets $\{j_1, \ldots, j_a\}$ $(a \geqslant 2)$ of $\{1, \ldots, n\}$.*

Next, we recall a relation between Massey products and the Magnus coefficients for our situation. Let $\psi : \mathfrak{F}_r \to \Pi_S(g) = \mathfrak{F}_r/\mathfrak{N}_S(g)$ be the natural homomorphism. We denote by $\gamma_i$ the image of $x_i$ under $\psi$, $\gamma_i := x_i \bmod \mathfrak{N}_S(g)$, for $1 \leqslant i \leqslant r$. By the definition (2.3.1) of $\Pi_S(g)$ and our assumption, $\pi$ induces the isomorphism $\mathfrak{F}_r/(\mathfrak{F}_r)^{l^a}\mathfrak{F}_r(2) \xrightarrow{\sim} \Pi_S(g)/\Pi_S(g)^{l^a}[\Pi_S(g), \Pi_S(g)] \simeq (\mathbb{Z}_l/\mathfrak{a})^{\oplus r}$ and so we have the isomorphism $H^1(\Pi_S(g), \mathbb{Z}_l/\mathfrak{a}) \simeq H^1(\mathfrak{F}_r, \mathbb{Z}_l/\mathfrak{a})$. Therefore the Hochschild-Serre spectral sequence yields the isomorphism

$$\mathrm{tg} : H^1(\mathfrak{N}_S(g), \mathbb{Z}_l/\mathfrak{a})^{\Pi_S(g)} \to H^2(\Pi_S(g), \mathbb{Z}_l/\mathfrak{a}).$$

Here tg is the transgression defined as follows. For $a \in H^1(\mathfrak{N}_S(g), \mathbb{Z}_l/\mathfrak{a})^{\Pi_S(g)}$, choose a 1-cochain $b \in C^1(\mathfrak{F}_r, \mathbb{Z}_l/\mathfrak{a})$ such that $b|_{\mathfrak{N}_S(g)} = a$. Since the value $db(f_1, f_2)$, $f_i \in \mathfrak{F}_r$, depends only on the cosets $f_i \bmod \mathfrak{N}_S(g)$, there is a 2-cocyle $c \in Z^2(\Pi_S(g), \mathbb{Z}_l/\mathfrak{a})$ such that $\psi^*(c) = db$. Then $\mathrm{tg}(a)$ is defined to be the class of $c$. The dual to tg is called the Hopf isomorphism:

$$\mathrm{tg}^\vee : H_2(\Pi_S(g), \mathbb{Z}_l/\mathfrak{a}) \xrightarrow{\sim} H_1(\mathfrak{N}_S(g), \mathbb{Z}_l/\mathfrak{a})_{\Pi_S(g)} = \mathfrak{N}_S(g)/\mathfrak{N}_S(g)^{l^a}[\mathfrak{N}_S(g), \mathfrak{F}_r].$$

Then we have the following proposition (cf. [**St**, Lemma 1.5], [**Ms1**, Theorem 2.2.2]).

PROPOSITION 3.2.41. *Notations being as above, let $c, \ldots, c_n \in H^1(\Pi_S(g), \mathbb{Z}_l/\mathfrak{a})$ and $W = (w_{ij})$ a defining system for the Massey product $\langle c_1, \ldots, c_n\rangle$. Let $f \in \mathfrak{N}_S(g)$ and set $\mathfrak{r} := (\mathrm{tg}^\vee)^{-1}(f \bmod [\mathfrak{N}_S(g), \mathfrak{F}_r])$. Then we have*

$$\begin{aligned}&\langle c_1, \ldots, c_n\rangle_W(\mathfrak{r})\\ &= \sum_{j=1}^{n}(-1)^{j+1} \sum_{e_1+\cdots+e_j=n} \sum_{I=(i_1\cdots i_j)} w_{1,1+e_1}(\gamma_{i_1})\cdots w_{n+1-e_j,n+1}(\gamma_{i_j})\mu(I;f)_{\mathfrak{a}},\end{aligned}$$

*where $e_1, \ldots, e_j$ run over positive integers satisfying $e_1+\cdots+e_j = n$ and $\mu(I;f)_{\mathfrak{a}} := \mu(I;f) \bmod \mathfrak{a}$.*

Now, let $\gamma_1^*, \ldots, \gamma_r^* \in H^1(\Pi_S(g), \mathbb{Z}_l/\mathfrak{a})$ be the Kronecker dual to $\gamma_1, \ldots, \gamma_r$, namely, $\gamma_i^*(\gamma_j) = \delta_{ij}$ for $1 \leqslant i, j \leqslant r$. Let $\mathfrak{r}_i := (\mathrm{tg}^\vee)^{-1}(x_i^{1-\chi_l(g)}[x_i^{-1}, y_i(g)^{-1}]) \bmod [\mathfrak{N}_S(g), \mathfrak{F}_r])$ for $1 \leqslant i \leqslant r$. Let $I = (i_1 \cdots i_n)$ be a multi-index such that $|I| = n \geqslant 2$. Let $g \in \mathrm{Gal}_k$. We assume the following conditions:

$$(3.2.42)\qquad \begin{cases} (1)\ \mu((j_1\cdots j_a); x_i^{1-\chi_l(g)}) \equiv 0 \bmod \mathfrak{a} \text{ for any subset} \\ \qquad \{j_1, \ldots, j_a\} \text{ of } \{i_1, \ldots, i_n\} \text{ and } 1 \leqslant i \leqslant r, \\ (2)\ i_1, \ldots, i_n \text{ are distinct each other, and} \\ \qquad \mu(g; (j_1\cdots j_a)) \equiv 0 \bmod \mathfrak{a} \text{ for any proper subset} \\ \qquad \{j_1, \ldots, j_a\} \text{ of } \{i_1, \ldots, i_n\}. \end{cases}$$

We note that the condition (1) is unnecessary when $g \in \mathrm{Gal}_{k(\zeta_{l^\infty})}$. The following theorem gives a cohomological interpretation of $\mu(g;I)_{\mathfrak{a}} := \mu(g;I) \bmod \mathfrak{a}$ by the Massey product in the cohomology of $\Pi_S(g)$.

THEOREM 3.2.43. *Notations and assumtions being as above, the Massey product* $\langle \gamma_{i_1}^*, \dots, \gamma_{i_n}^* \rangle$ *in* $H^2(\Pi_S(g), \mathbb{Z}_l/\mathfrak{a})$ *is uniquely defined and we have*

$$\mu(g; I)_{\mathfrak{a}} = (-1)^n \langle \gamma_{i_1}^*, \dots, \gamma_{i_n}^* \rangle (\mathfrak{r}_{i_n}).$$

PROOF. First, we compute $\mu(J; x_i^{1-\chi_l(g)}[x_i^{-1}, y_i(g)^{-1}])$ for a multi-index $J = (j_1 \cdots j_a)$, where $\{j_1, \dots, j_a\}$ is a subset of $\{i_1, \dots, i_n\}$. We note that

$$\begin{aligned}&\Theta(x_i^{1-\chi_l(g)}[x_i^{-1}, y_i(g)^{-1}]) \\ &\quad = \Theta(x_i^{1-\chi_l(g)})(1 + \Theta(x_i^{-1})\Theta(y_i(g)^{-1})(\Theta(x_i y_i(g)) - \Theta(y_i(g) x_i))).\end{aligned}$$

By our assumption (3.2.42) (1), we have
(3.2.44)

$$\begin{aligned}\mu(J; x_i^{1-\chi_l(g)}[x_i^{-1}, y_i(g)^{-1}]) \quad &\equiv \mu(J; x_i y_i(g)) - \mu(J; y_i(g) x_i) \\ &\quad + \sum_A (\mu(A; x_i y_i(g)) - \mu(A; y_i(g) x_i))\nu_A \bmod \mathfrak{a},\end{aligned}$$

where $A$ runs over some proper subsequences of $J$ and $\nu_A \in \mathbb{Z}_l$. By the straightforward computation, we have

$$\mu(J; x_i y_i(g)) = \begin{cases} \mu(g; (Ji)) & (i \neq j_1), \\ \mu(g; (J j_1)) + \mu(g; (j_2 \cdots j_a j_1)) & (i = j_1), \end{cases}$$

and

$$\mu(J; y_i(g) x_i) = \begin{cases} \mu(g; (Ji)) & (i \neq j_a), \\ \mu(g; (J j_a)) + \mu(g; J) & (i = j_a). \end{cases}$$

Hence we have
(3.2.45)

$$\mu(J; x_i y_i(g)) - \mu(J; y_i(g) x_i) = \begin{cases} \mu(g; (j_2 \cdots j_a j_1)) - \delta_{j_1, j_a}\mu(g; J) & (i = j_1), \\ \mu(g; (j_2 \cdots j_a j_1))\delta_{j_1, j_a} - \mu(g; J) & (i = j_a), \\ 0 & (\text{otherwise}). \end{cases}$$

Now, let $n = 2$. Then we have $\langle \gamma_{i_1}^*, \gamma_{i_2}^* \rangle = \gamma_{i_1}^* \cup \gamma_{i_2}^*$. By Proposition 3.2.41, (3.2.42) (2), (3.2.44) and (3.2.45), we have

$$\langle \gamma_{i_1}^*, \gamma_{i_2}^* \rangle (\mathfrak{r}_{i_2}) = -\mu(I; [x_{i_2}, y_{i_2}(g)])_{\mathfrak{a}} = \mu(g; I)_{\mathfrak{a}}.$$

Suppose $n \geqslant 3$ and let $\{j_1, \dots, j_a\}$ be a proper subset of $\{i_1, \dots, i_n\}$. Then, by our assumption (3.2.42) (2), (3.2.44) and (3.2.45), we have

$$\mu(J; x_i^{1-\chi_l(g)}[x_i^{-1}, y_i(g)^{-1}]) \equiv 0 \bmod \mathfrak{a}$$

for $J = (j_1 \cdots j_a)$ and $1 \leqslant i \leqslant r$. So, by Proposition 3.2.41, we have

$$\langle \gamma_{j_1}^*, \dots, \gamma_{j_a}^* \rangle (\mathfrak{r}_i) = 0$$

for $1 \leqslant i \leqslant r$. Since $H_2(\Pi(g), \mathbb{Z}_l/\mathfrak{a})$ is generated by $x_i^{1-\chi_l(g)}[x_i, y_i(g)]$ for $1 \leqslant i \leqslant r$, we have

$$\langle c_{j_1}, \dots, c_{j_a} \rangle = 0.$$

Therefore, by Lemma 3.2.40, the Massey product $\langle c_{i_1}, \dots, c_{i_n} \rangle$ is uniquely defined. By Proposition 3.2.41, (3.2.42) (2), (3.2.44) and (3.2.45) again, we have

$$\langle \gamma_{i_1}^*, \dots, \gamma_{i_n}^* \rangle (\mathfrak{r}_{i_n}) = (-1)^{n+1}\mu(I; x_n^{1-\chi_l(g)}[x_{i_n}, y_{i_n}(g)])_{\mathfrak{a}} = (-1)^n \mu(g; I)_{\mathfrak{a}}. \qquad \square$$

## 3.3. Pro-$l$ Johnson homomorphisms

**3.3.1. Some algebras associated to lower central series.** For each integer $n \geqslant 1$, we let

$$\mathrm{gr}_n(\mathfrak{F}_r) := \Gamma_n\mathfrak{F}_r/\Gamma_{n+1}\mathfrak{F}_r,$$

which is a free $\mathbb{Z}_l$-module whose rank $\ell_r(n)$ is given by the Witt formula ([**MKS**, 5.6, Theorem 5.11], [**Se**, Ch. IV, 4, 6]):

$$\ell_r(n) = \frac{1}{n}\sum_{d|n}\mu(d)r^{n/d},$$

where $\mu(d)$ is the Möbius function. The graded $\mathbb{Z}_l$-module

$$\mathrm{gr}(\mathfrak{F}_r) := \bigoplus_{n\geqslant 1}\mathrm{gr}_n(\mathfrak{F}_r)$$

has the structure of a graded free Lie algebra over $\mathbb{Z}_l$: For $a = s \bmod \Gamma_{m+1}\mathfrak{F}_r \in \mathrm{gr}_m(\mathfrak{F}_r)$ and $b = t \bmod \Gamma_{n+1}\mathfrak{F}_r \in \mathrm{gr}_n(\mathfrak{F}_r)$ $(s \in \Gamma_m\mathfrak{F}_r, t \in \Gamma_n\mathfrak{F}_r)$, the Lie bracket on $\mathrm{gr}(\mathfrak{F}_r)$ is defined by

$$[a,b] := [s,t] \bmod \mathfrak{F}_r(m+n+1).$$

We consider the graded associative algebra over $\mathbb{Z}_l$ defined by

$$\mathrm{gr}(\mathbb{Z}_l[[\mathfrak{F}_r]]) := \bigoplus_{n\geqslant 0}\mathrm{gr}_n(\mathbb{Z}_l[[\mathfrak{F}_r]]),\quad \mathrm{gr}_n(\mathbb{Z}_l[[\mathfrak{F}_r]]) := I^n_{\mathbb{Z}_l[[\mathfrak{F}_r]]}/I^{n+1}_{\mathbb{Z}_l[[\mathfrak{F}_r]]}.$$

The map $f \mapsto f-1$ $(f \in \mathfrak{F}_r(n))$ defines an injective $\mathbb{Z}_l$-linear map

$$\mathrm{gr}_n(\mathfrak{F}_r) \hookrightarrow \mathrm{gr}_n(\mathbb{Z}_l[[\mathfrak{F}_r]]) \tag{3.3.1}$$

for $n \geqslant 1$ and the injective Lie algebra homomorphism over $\mathbb{Z}_l$

$$\mathrm{gr}(\mathfrak{F}_r) \hookrightarrow \mathrm{gr}(\mathbb{Z}_l[[\mathfrak{F}_r]]),$$

where $\mathrm{gr}(\mathbb{Z}[[\mathfrak{F}_r]])$ is shown to be the universal enveloping algebra of the Lie algebra $\mathrm{gr}(\mathfrak{F}_r)$. Moreover, by the correspondence $x_i - 1 \bmod I^2_{\mathbb{Z}_l[[\mathfrak{F}_r]]} \in \mathrm{gr}_1(\mathbb{Z}_l[[\mathfrak{F}_r]]) \mapsto X_i \in H_{\mathbb{Z}_l}$, we have the isomorphism of $\mathbb{Z}_l$-modules

$$\Theta_n :\ \mathrm{gr}_n(\mathbb{Z}_l[[\mathfrak{F}_r]]) \ \simeq\ H_{\mathbb{Z}_l}^{\otimes n} \tag{3.3.2}$$

for each $n \geqslant 0$ and so $\mathrm{gr}(\mathbb{Z}_l[[\mathfrak{F}_r]])$ is identified with the tensor algebra $T(H_{\mathbb{Z}_l})$:

$$\mathrm{gr}(\mathbb{Z}_l[[\mathfrak{F}_r]]) = T(H_{\mathbb{Z}_l}) = \mathbb{Z}_l\langle X_1, \ldots, X_r\rangle.$$

The composition of the map of (3.3.1) with $\Theta_n$ in (3.3.2), denoted also by $\Theta_n : \mathrm{gr}_n(\mathfrak{F}_r) \hookrightarrow H_{\mathbb{Z}_l}^{\otimes n}$, is the degree $n$ part of the pro-$l$ Magnus embedding in (3.2.3):

$$\Theta_n = (\Theta - 1)|_{\mathfrak{F}_n} \bmod \widehat{T}(n+1). \tag{3.3.3}$$

Here we may note that $\Theta$ is multiplicative, $\Theta(f_1f_2) = \Theta(f_1)\Theta(f_2)$ for $f_1, f_2 \in \mathfrak{F}_r$, while $\Theta_n$ is additive, $\Theta_n([f_1f_2]) = \Theta_n([f_1]+[f_2]) = \Theta_n([f_1]) + \Theta_n([f_2])$, where $[\,\cdot\,]$ stands for the class mod $\mathfrak{F}_r(n+1)$.

Let $S(H_{\mathbb{Z}_l})$ be the symmetric algebra of $H_{\mathbb{Z}_l}$ over $\mathbb{Z}_l$ and let $q : T(H_{\mathbb{Z}_l}) \to S(H_{\mathbb{Z}_l})$ be the natural map. We let $S^m(H_{\mathbb{Z}_l}) := q(H_{\mathbb{Z}_l}^{\otimes m})$ and $T_i := q(X_i)$ for $1 \leqslant i \leqslant r$ so that $S(H_{\mathbb{Z}_l})$ is the graded algebra $\bigoplus_{m\geqslant 0} S^m(H_{\mathbb{Z}_l})$ which is noting but the commutative polynomial algebra over $\mathbb{Z}_l$ of variables $T_1, \ldots, T_r$:

$$S^m(H_{\mathbb{Z}_l}) = \bigoplus_{m\geqslant 0} S^m(H_{\mathbb{Z}_l}) = \mathbb{Z}_l[T_1, \ldots, T_r].$$

**3.3.2. The pro-$l$ Johnson map.** For $\varphi \in \mathrm{Aut}^{\mathrm{fil}}(\widehat{T}(H_{\mathbb{Z}_l}))$, we denote by $[\varphi]$ the induced $\mathbb{Z}_l$-endomorphism of $H_{\mathbb{Z}_l} = \widehat{T}(1)/\widehat{T}(2) = \mathbb{Z}_l^{\oplus r}$. We firstly recall the following lemma due to Kawazumi ([**Ka**]).

LEMMA 3.3.4. *A $\mathbb{Z}_l$-algebra endomorphism $\varphi$ of $\widehat{T}(H_{\mathbb{Z}_l})$ is a filtration-preserving automorphism of $\widehat{T}(H_{\mathbb{Z}_l})$, $\varphi \in \mathrm{Aut}^{\mathrm{fil}}(\widehat{T}(H_{\mathbb{Z}_l}))$, if and only if the following conditions are satisfied:*
(1) $\varphi(\widehat{T}(n)) \subset \widehat{T}(n)$ *for all* $n \geqslant 0$.
(2) *the induced $\mathbb{Z}_l$-linear map $[\varphi]$ on $\widehat{T}(1)/\widehat{T}(2) = H_{\mathbb{Z}_l}$ is an isomorphism.*

PROOF. Suppose $\varphi \in \mathrm{Aut}^{\mathrm{fil}}(\widehat{T}(H_{\mathbb{Z}_l}))$. Since $\varphi$ is filtration-preserving, the condition (1) holds. To show the condition (2), consider the following commutative diagram for vector spaces over $\mathbb{Z}_l$ with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{T}(2) & \longrightarrow & \widehat{T}(1) & \longrightarrow & H_{\mathbb{Z}_l} & \longrightarrow & 0 \\ & & \downarrow \varphi|_{\widehat{T}(2)} & & \downarrow \varphi|_{\widehat{T}(1)} & & \downarrow [\varphi] & & \\ 0 & \longrightarrow & \widehat{T}(2) & \longrightarrow & \widehat{T}(1) & \longrightarrow & H_{\mathbb{Z}_l} & \longrightarrow & 0. \end{array}$$

Since $\varphi(\widehat{T}(n)) = \widehat{T}(n)$ for all $n \geqslant 0$, we have $\mathrm{Coker}(\varphi|_{\widehat{T}(i)}) = 0$ for $i = 1, 2$, in particular. Since $\varphi$ is an automorphism, we have $\mathrm{Ker}(\varphi) = 0$, in particular, $\mathrm{Ker}(\varphi|_{\widehat{T}(i)}) = 0$ for $i = 1, 2$. By snake lemma applied to the above diagram, we obtain $\mathrm{Ker}([\varphi]) = 0$ and $\mathrm{Coker}([\varphi]) = 0$, hence the condition (2).

Suppose that a $\mathbb{Z}_l$-algebra endomorphism $\varphi$ of $\widehat{T}$ satisfies the conditions (1) and (2). Let $z = (z_m)$ be any element of $\widehat{T}$ with $z_m \in H_{\mathbb{Z}_l}^{\otimes m}$ for $m \geqslant 0$. To show that $\varphi$ is an automorphism, we have only to prove that there exists uniquely $y = (y_m) \in \widehat{T}$ such that

$$z = \varphi(y). \tag{3.3.5}$$

Note by the condition (1) and (2) that $\varphi$ induces a $\mathbb{Z}_l$-linear automorphism of $\widehat{T}(m)/\widehat{T}(m+1) = H_{\mathbb{Z}_l}^{\otimes m}$, which is nothing but $[\varphi]^{\otimes m}$. Then, writing $\varphi(y_i)_j$ for the component of $\varphi(y_i)$ in $H_{\mathbb{Z}_l}^{\otimes j}$ for $i < j$, the equation (3.3.5) is equivalent to the following system of equations:

$$\left\{\begin{array}{l} z_0 = \varphi(y_0) = y_0, \\ z_1 = [\varphi](y_1), \\ z_2 = [\varphi]^{\otimes 2}(y_2) + \varphi(y_1)_2, \\ \cdots \\ z_m = [\varphi]^{\otimes m}(y_m) + \varphi(y_1)_m + \cdots + \varphi(y_{m-1})_m, \\ \cdots \end{array}\right. \tag{3.3.6}$$

Since $[\varphi]^{\otimes m}$ is an automorphism, we can find the unique solution $y = (y_m)$ of (3.3.6) from the lower degree. Therefore $\varphi$ is an $\mathbb{Z}_l$-algebra automorphism. Furthermore, we can see easily that if $z_0 = \cdots = z_{n-1} = 0$, then $y_0 = \cdots = y_{n-1} = 0$ for $n \geqslant 1$. This means that $\varphi^{-1}(\widehat{T}(n)) \subset \widehat{T}(n)$ and so $\varphi$ is filtration-preserving. $\square$

By Lemma 3.3.4, each $\varphi \in \mathrm{Aut}^{\mathrm{fil}}(\widehat{T}(H_{\mathbb{Z}_l}))$ induces a $\mathbb{Z}_l$-linear automorphism $[\varphi]$ of $H_{\mathbb{Z}_l} = \widehat{T}(1)/\widehat{T}(2)$ and so we have a group homomorphism

$$[\ ]\ :\ \mathrm{Aut}^{\mathrm{fil}}(\widehat{T}(H_{\mathbb{Z}_l})) \longrightarrow \mathrm{GL}(H_{\mathbb{Z}_l}),$$

where $\mathrm{GL}(H_{\mathbb{Z}_l})$ denotes the group of $\mathbb{Z}_l$-linear automorphisms of $H_{\mathbb{Z}_l}$. We then define the *induced automorphism group* of $\widehat{T}$ by

$$\begin{aligned}\mathrm{IA}(\widehat{T}(H_{\mathbb{Z}_l})) &:= \mathrm{Ker}([\ \ ])\\ &= \{\varphi \in \mathrm{Aut}(\widehat{T}(H_{\mathbb{Z}_l})) \mid \varphi(h) \equiv h \bmod \widehat{T}(2) \text{ for any } h \in H_{\mathbb{Z}_l}\}.\end{aligned}$$

We note that there is a natural splitting $s : \mathrm{GL}(H_{\mathbb{Z}_l}) \to \mathrm{Aut}^{\mathrm{fil}}(\widehat{T}(H_{\mathbb{Z}_l}))$ of $[\ \ ]$, which is defined by

$$s(P)((z_n)) := (P^{\otimes n}(z_n)) \ \text{ for } P \in \mathrm{GL}(H_{\mathbb{Z}_l}).$$

In the following, we also regard $[P] \in \mathrm{GL}(H_{\mathbb{Z}_l})$ as an element of $\mathrm{Aut}^{\mathrm{fil}}(\widehat{T})$ through the splitting $s$. Thus we have the following

Lemma 3.3.7. *We have a semi-direct decomposition*

$$\mathrm{Aut}^{\mathrm{fil}}(\widehat{T}(H_{\mathbb{Z}_l})) \ = \ \mathrm{IA}(\widehat{T}(H_{\mathbb{Z}_l})) \rtimes \mathrm{GL}(H_{\mathbb{Z}_l}); \ \ \varphi = (\varphi \circ [\varphi]^{-1}, [\varphi]).$$

Let $\varphi \in \mathrm{IA}(\widehat{T}(H_{\mathbb{Z}_l}))$. Then we have $\varphi(h) - h \in \widehat{T}(2)$ for any $h \in H_{\mathbb{Z}_l}$, and so we have a map

$$E \ : \ \mathrm{IA}(\widehat{T}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, \widehat{T}(2)); \ \varphi \mapsto \varphi|_{H_{\mathbb{Z}_l}} - \mathrm{id}_{H_{\mathbb{Z}_l}}, \tag{3.3.8}$$

where $\mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, \widehat{T}(2))$ denotes the group of $\mathbb{Z}_l$-linear maps $H_{\mathbb{Z}_l} \to \widehat{T}(2)$. The following Proposition will play a key role in our discussion.

Proposition 3.3.9. *The map $E$ is bijective.*

Proof. Injectivity: Suppose $E(\varphi) = E(\varphi')$ for $\varphi, \varphi' \in \mathrm{IA}(\widehat{T}(H_{\mathbb{Z}_l}))$. Then we have $\varphi|_{H_{\mathbb{Z}_l}} = \varphi'|_{H_{\mathbb{Z}_l}}$. Since an $\mathbb{Z}_l$-algebra endomorphism of $\widehat{T}(H_{\mathbb{Z}_l})$ is determined by its restriction on $H_{\mathbb{Z}_l}$, we have $\varphi = \varphi'$.
Surjectivity: Take any $\phi \in \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, \widehat{T}(2))$. We can extend $\phi + \mathrm{id}_{H_{\mathbb{Z}_l}} : H_{\mathbb{Z}_l} \to \widehat{T}(2)$ uniquely to a $\mathbb{Z}_l$-algebra endomorphism $\varphi$ of $\widehat{T}(H_{\mathbb{Z}_l})$. Then we have obviously $\varphi(\widehat{T}(n)) \subset \widehat{T}(n)$ for all $n \geqslant 0$. Since $\widehat{T}(1)/\widehat{T}(2) = H_{\mathbb{Z}_l}$ and we see that

$$[\varphi](h \bmod \widehat{T}(2)) = \varphi(h) \bmod \widehat{T}(2) = h + \phi(h) \bmod \widehat{T}(2) = h \bmod \widehat{T}(2),$$

we have $[\varphi] = \mathrm{id}_{H_{\mathbb{Z}_l}}$. By Lemma 3.2.1, we have $\varphi \in \mathrm{IA}(\widehat{T})$ and $E(\varphi) = \phi$. □

By Lemma 3.3.7 and Proposition 3.3.9, we have the following.

Corollary 3.3.10. *We have a bijection*

$$\hat{E} \ : \ \mathrm{Aut}^{\mathrm{fil}}(\widehat{T}(H_{\mathbb{Z}_l})) \simeq \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, \widehat{T}(2)) \times \mathrm{GL}(H_{\mathbb{Z}_l})$$

*defined by* $\hat{E}(\varphi) = (E(\varphi \circ [\varphi]^{-1}), [\varphi])$.

Now, let $\mathrm{Ih}_S : \mathrm{Gal}_k \to P(\mathfrak{F}_r)$ be the Ihara representation associated to $S$ in (3.1.6). We recall that the correspondence $\varphi \mapsto \varphi^* := \Theta \circ \varphi \circ \Theta^{-1}$ in (3.2.8) gives the injective homomorphism $\mathrm{Aut}(\mathfrak{F}_r) \to \mathrm{Aut}^{\mathrm{fil}}(\widehat{T}(H_{\mathbb{Z}_l}))$ and hence the inclusion $P(\mathfrak{F}_r) \hookrightarrow \mathrm{Aut}^{\mathrm{fil}}(\widehat{T}(H_{\mathbb{Z}_l}))$ which satisfies $[\varphi] = [\varphi^*]$ in $\mathrm{GL}(H_{\mathbb{Z}_l})$. Composing $\mathrm{Ih}_S$ with this inclusion, we have the homomorphism $\hat{\eta}_S : \mathrm{Gal}_k \to \mathrm{Aut}^{\mathrm{fil}}(\widehat{T}(H_{\mathbb{Z}_l}))$ defined by

$$\hat{\eta}_S(g) := \mathrm{Ih}_S(g)^* = \Theta \circ \mathrm{Ih}_S(g) \circ \Theta^{-1}.$$

We then define the map $\eta_S : \mathrm{Gal}_k \to \mathrm{IA}(\widehat{T}(H_{\mathbb{Z}_l}))$ by composing $\hat{\eta}_S$ with the projection on $\mathrm{IA}(\widehat{T}(H_{\mathbb{Z}_l}))$:
(3.3.11)
$$\eta_S(g) := \hat{\eta}_S(g) \circ [\mathrm{Ih}_S(g)]^{-1} = \mathrm{Ih}_S(g)^* \circ [\mathrm{Ih}_S(g)]^{-1} = \Theta \circ \mathrm{Ih}_S(g) \circ \Theta^{-1} \circ [\mathrm{Ih}_S(g)]^{-1}.$$
Thus, we have $\hat{\eta}_S(g) = (\eta_S(g), [\mathrm{Ih}_S(g)])$ for $g \in \mathrm{Gal}_k$ under the semi-direct decomposition $\mathrm{Aut}^{\mathrm{fil}}(\widehat{T}(H_{\mathbb{Z}_l})) = \mathrm{IA}(\widehat{T}(H_{\mathbb{Z}_l})) \rtimes \mathrm{GL}(H_{\mathbb{Z}_l})$ of Lemma 3.3.7.

Now, we define the *pro-$l$ Johnson map*
$$\tau_S \ : \ \mathrm{Gal}_k \longrightarrow \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, \widehat{T}(2))$$
by the composing $\eta_S$ with $E$ in (3.3.8), and define the *extended pro-$l$ Johnson map*
$$\hat{\tau}_S \ : \ \mathrm{Gal}_k \longrightarrow \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, \widehat{T}(2)) \rtimes \mathrm{GL}(H_{\mathbb{Z}_l})$$
by composing $\widehat{\kappa}_S$ with $\hat{E}$ of Corollary 3.3.10. So we have, for $g \in \mathrm{Gal}_k$,
(3.3.12)
$$\begin{aligned} \tau_S(g) \ &:= E(\eta_S(g)) = \eta_S(g)|_{H_{\mathbb{Z}_l}} - \mathrm{id}_{H_{\mathbb{Z}_l}} \\ &= \mathrm{Ih}_S(g)^* \circ [\mathrm{Ih}_S(g)]^{-1}|_{H_{\mathbb{Z}_l}} - \mathrm{id}|_{H_{\mathbb{Z}_l}} = \Theta \circ \mathrm{Ih}_S(g) \circ \Theta^{-1} \circ [\mathrm{Ih}_S(g)]^{-1}|_{H_{\mathbb{Z}_l}} - \mathrm{id}|_{H_{\mathbb{Z}_l}}, \\ \hat{\tau}_S(g) &:= (\tau_S(g), [\mathrm{Ih}_S(g)]). \end{aligned}$$
For $m \geqslant 1$, we define the *$m$-th pro-$l$ Johnson map*
$$\tau_S^{(m)} \ : \ \mathrm{Gal}_k \longrightarrow \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, H_{\mathbb{Z}_l}^{\otimes(m+1)})$$
by the $m$-th component of $\tau_S$:
$$\tau_S(g) := \sum_{m \geqslant 1} \tau_S^{(m)}(g) \quad (g \in \mathrm{Gal}_k). \tag{3.3.13}$$

We note that the pro-$l$ Johnson map $\tau_S$ is no longer a homomorphism. In fact, we have the following

Proposition 3.3.14. *For $g_1, g_2 \in \mathrm{Gal}_k$, we have*
$$\eta_S(g_1g_2) = \eta_S(g_1) \circ [\mathrm{Ih}_S(g_1)] \circ \eta_S(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1}.$$
Proof. By (3.3.11), we have
$$\begin{aligned} \eta_S(g_1g_2) \ &= \mathrm{Ih}_S(g_1g_2)^* \circ [\mathrm{Ih}_S(g_1g_2)]^{-1} \\ &= \Theta \circ (\mathrm{Ih}_S(g_1g_2) \circ \Theta^{-1} \circ [\mathrm{Ih}_S(g_1g_2)]^{-1} \\ &= \Theta \circ \mathrm{Ih}_S(g_1) \circ \mathrm{Ih}_S(g_2) \circ \Theta^{-1} \circ [\mathrm{Ih}_S(g_2)]^{-1} \circ [\mathrm{Ih}_S(g_1)]^{-1} \\ &= \Theta \circ \mathrm{Ih}_S(g_1) \circ \Theta^{-1} \circ [\mathrm{Ih}_S(g_1)]^{-1} \circ [\mathrm{Ih}_S(g_1)] \circ \Theta \circ \mathrm{Ih}_S(g_2) \circ \Theta^{-1} \\ &\quad \circ [\mathrm{Ih}_S(g_2)]^{-1} \circ [\mathrm{Ih}_S(g_1)]^{-1} \\ &= \eta_S(g_1) \circ [\mathrm{Ih}_S(g_1)] \circ \eta_S(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1}. \end{aligned}$$
□

Proposition 3.3.14 yields coboundary relations among $\tau_S^{(m)}$. Here we give the formulas only for $\tau_S^{(1)}$ and $\tau_S^{(2)}$.

Proposition 3.3.15. *For $g_1, g_2 \in \mathrm{Gal}_k$, we have*
$$\begin{aligned} \tau_S^{(1)}(g_1g_2) \ &= \tau_S^{(1)}(g_1) + [\mathrm{Ih}_S(g_1)]^{\otimes 2} \circ \tau_S^{(1)}(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1}, \\ \tau_S^{(2)}(g_1g_2) \ &= \tau_S^{(2)}(g_1) + (\tau_S^{(1)}(g_1) \otimes \mathrm{id}_{H_{\mathbb{Z}_l}} + \mathrm{id}_{H_{\mathbb{Z}_l}} \otimes \tau_S^{(1)}(g_1)) \circ [\mathrm{Ih}_S(g_1)]^{\otimes 2} \\ &\quad \circ \tau_S^{(1)}(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1} + [\mathrm{Ih}_S(g_1)]^{\otimes 3} \circ \tau_S^{(2)}(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1}. \end{aligned}$$

PROOF. By definition (3.3.13), we have

$$\tau_S(g_1g_2) = \sum_{m\geqslant 1} \tau_S^{(m)}(g_1g_2). \tag{3.3.16}$$

On the other hand, by Proposition 3.3.14 and (3.3.12), we have, for $h \in H_{\mathbb{Z}_l}$,

$$\begin{aligned}
\tau_S(g_1g_2)(h) &= -h + \eta_S(g_1g_2)(h) \\
&= -h + (\eta_S(g_1) \circ [\mathrm{Ih}_S(g_1)] \circ \eta_S(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1})(h) \\
&= -h + (\eta_S(g_1) \circ [\mathrm{Ih}_S(g_1)] \circ (\mathrm{id}_{H_{\mathbb{Z}_l}} + \tau_S(g_2)))([\mathrm{Ih}_S(g_1)]^{-1}(h)) \\
&= -h + (\eta_S(g_1) \circ [\mathrm{Ih}_S(g_1)]) \left( [\mathrm{Ih}_S(g_1)]^{-1}(h) + \sum_{m\geqslant 1} (\tau_S^{(m)}(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1})(h) \right) \\
&= -h + \eta_S(g_1) \left( h + \sum_{m\geqslant 1} ([\mathrm{Ih}_S(g_1)]^{\otimes m+1} \circ \tau_S^{(m)}(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1})(h) \right) \\
&= -h + \eta_S(g_1)(h) \\
&\qquad + \eta_S(g_1)(([\mathrm{Ih}_S(g_1)]^{\otimes 2} \circ \tau_S^{(1)}(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1})(h)) \\
&\qquad + \eta_S(g_1)(([\mathrm{Ih}_S(g_1)]^{\otimes 3} \circ \tau_S^{(2)}(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1})(h)) \mod \widehat{T}(4).
\end{aligned}$$

We note that

$$\eta_S(g)|_{H_{\mathbb{Z}_l}^{\otimes m}} = (\mathrm{id}_{H_{\mathbb{Z}_l}} + \tau_S(g))^{\otimes m} : H_{\mathbb{Z}_l}^{\otimes m} \longrightarrow H_{\mathbb{Z}_l} \times \widehat{T}(2m)$$

for any $g \in \mathrm{Gal}_k$ and so we have the following congruences mod $\widehat{T}(4)$:

$$\begin{aligned}
&\eta_S(g_1)(h) \equiv h + \tau_S^{(1)}(g_1)(h) + \tau_S^{(2)}(g_1)(h), \\
&\eta_S(g_1)(([\mathrm{Ih}_S(g_1)]^{\otimes 2} \circ \tau_S^{(1)}(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1})(h)) \\
&\qquad \equiv ([\mathrm{Ih}_S(g_1)]^{\otimes 2} \circ \tau_S^{(1)}(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1})(h) \\
&\qquad\quad + ((\tau_S^{(1)}(g_1) \otimes \mathrm{id}_{H_{\mathbb{Z}_l}} + \mathrm{id}_{H_{\mathbb{Z}_l}} \otimes \tau_S^{(1)}(g_1)) \circ [\mathrm{Ih}_S(g_1)]^{\otimes 2} \circ \tau_S^{(1)}(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1})(h), \\
&\eta_S(g_1)(([\mathrm{Ih}_S(g_1)]^{\otimes 3} \circ \tau_S^{(1)}(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1})(h)) \equiv ([\mathrm{Ih}_S(g_1)]^{\otimes 3} \circ \tau_S^{(2)}(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1})(h).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\tau_S(g_1g_2)(h) \\
&\equiv \tau_S^{(1)}(g_1)(h) + \tau_S^{(2)}(g_1)(h) \\
&\quad + ([\mathrm{Ih}_S(g_1)]^{\otimes 2} \circ \tau_S^{(1)}(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1})(h) \\
&\quad + ((\tau_S^{(1)}(g_1) \otimes \mathrm{id}_{H_{\mathbb{Z}_l}} + \mathrm{id}_{H_{\mathbb{Z}_l}} \otimes \tau_S^{(1)}(g_1)) \circ [\mathrm{Ih}_S(g_1)]^{\otimes 2} \circ \tau_S^{(1)}(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1})(h) \\
&\quad + ([\mathrm{Ih}_S(g_1)]^{\otimes 3} \circ \tau_S^{(2)}(g_2) \circ [\mathrm{Ih}_S(g_1)]^{-1})(h) \mod \widehat{T}(4).
\end{aligned} \tag{3.3.17}$$

Comparing (3.3.16) and (3.3.17), we obtain the assertions. □

**3.3.3. Pro-$l$ Johnson homomorphisms.** For $n \geqslant 0$, let $\pi_n : \mathfrak{F}_r \to \mathfrak{F}_r/\Gamma_{n+1}\mathfrak{F}_r$ be the natural homomorphism. Since each $\Gamma_n\mathfrak{F}_r$ is a characteristic subgroup of $\mathfrak{F}_r$, $\pi_n$ induces the natural homomorphism $\pi_{n*} : P(\mathfrak{F}_r) \hookrightarrow \mathrm{Aut}(\mathfrak{F}_r) \to \mathrm{Aut}(\mathfrak{F}_r/\Gamma_{n+1}\mathfrak{F}_r)$. Let $\mathrm{Ih}_S^{(n)}$ denote the composite of $\mathrm{Ih}_S$ with $\pi_{n*}$:

$$\mathrm{Ih}_S^{(n)} : \mathrm{Gal}_k \longrightarrow \mathrm{Aut}(\mathfrak{F}_r/\Gamma_{n+1}\mathfrak{F}_r).$$

In particular, $\mathrm{Ih}_S^{(1)}(g) = [\mathrm{Ih}_S(g)]$ for $g \in \mathrm{Gal}_k$. Let $\mathrm{Gal}_k^{\mathrm{Joh}}[n]$ denote the kernel of $\mathrm{Ih}_S^{(n)}$:

$$\begin{aligned} \mathrm{Gal}_k^{\mathrm{Joh}}[n] &:= \mathrm{Ker}(\mathrm{Ih}_S^{(n)}) \\ &= \{g \in \mathrm{Gal}_k \,|\, \mathrm{Ih}_S(g)(f)f^{-1} \in \Gamma_{n+1}\mathfrak{F}_r \text{ for all } f \in \mathfrak{F}_r\}. \end{aligned} \tag{3.3.18}$$

We then have the descending series of closed normal subgroups of $\mathrm{Gal}_k$:

$$\mathrm{Gal}_k = \mathrm{Gal}_k^{\mathrm{Joh}}[0] \supset \mathrm{Gal}_k^{\mathrm{Joh}}[1] \supset \cdots \supset \mathrm{Gal}_k^{\mathrm{Joh}}[n] \supset \cdots$$

and we call it the *Johnson filtration* of $\mathrm{Gal}_k$ associated to the Ihara representation $\varphi_S$ (cf.[**Aa**], [**Joh1**]). We note by Theorem 3.1.9 (1)

$$\mathrm{Gal}_k^{\mathrm{Joh}}[1] = \mathrm{Ker}(\mathrm{Ih}_S^{(1)} : \mathrm{Gal}_k \to \mathrm{GL}(H_{\mathbb{Z}_l})) = \mathrm{Gal}_{k(\zeta_{l^\infty})}. \tag{3.3.19}$$

The relation with the Milnor filtration defined in (3.2.35) is given as follows.

Proposition 3.3.20. *The Johnson filtration coincides with the Milnor filtration, namely, for each $n \geqslant 0$, we have*

$$\mathrm{Gal}_k^{\mathrm{Joh}}[n] = \mathrm{Gal}_k^{\mathrm{Mil}}[n].$$

Proof. We may assume $n \geqslant 1$ and hence $g \in \mathrm{Gal}_{k(\zeta_{l^\infty})}$. Then we have

$$\begin{aligned} g \in \mathrm{Gal}_k^{\mathrm{Joh}}[n] \quad &\Leftrightarrow \mathrm{Ih}_S(g)(x_i)x_i^{-1} \in \Gamma_{n+1}\mathfrak{F}_r \text{ for all } 1 \leqslant i \leqslant r \\ &\Leftrightarrow y_i(g)x_i y_i(g)^{-1}x_i^{-1} \in \Gamma_{n+1}\mathfrak{F}_r \text{ for all } 1 \leqslant i \leqslant r \\ &\Leftrightarrow y_i(g) \in \mathfrak{F}_r(n) \text{ for all } 1 \leqslant i \leqslant r \\ &\Leftrightarrow \deg(\Theta(y_i(g) - 1)) \geqslant n \text{ for all } 1 \leqslant i \leqslant r \\ &\Leftrightarrow g \in \mathrm{Gal}_k^{\mathrm{Mil}}[n] \end{aligned}$$

□

Note that Proposition 3.3.20 yields Proposition 3.2.37. In the following, we simply write $\mathrm{Gal}_k[n]$ for the $n$-th term of the Johnson (or Milnor) filtration for $n \geqslant 0$ and we denote by $k[n]$ the Galois subextension of $k$ in $\overline{\mathbb{Q}}$ corresponding to $\mathrm{Gal}_k[n]$. By (3.3.19), we have $k[1] = k(\zeta_{l^\infty})$.

We give some basic properties of the Johnson filtration.

Lemma 3.3.21. *For $g \in \mathrm{Gal}_k[m]$ $(m \geqslant 0)$ and $f \in \Gamma_n\mathfrak{F}_r$ $(n \geqslant 1)$, we have*

$$\mathrm{Ih}_S(g)(f)f^{-1} \in \Gamma_{m+n}\mathfrak{F}_r.$$

Proof. We fix $m \geqslant 0$ and $g \in \mathrm{Gal}_k[m]$. We prove the assertion by induction on $n$. For $n = 1$, the assertion $\mathrm{Ih}_S(g)(f)f^{-1} \in \Gamma_{m+1}\mathfrak{F}_r$ is true by definition (3.3.18) of $\mathrm{Gal}_k[m]$. Assume that

$$\mathrm{Ih}_S(g)(f)f^{-1} \in \Gamma_{m+i}\mathfrak{F}_r \text{ if } f \in \Gamma_i\mathfrak{F}_r \text{ and } 1 \leqslant i \leqslant n. \tag{3.3.22}$$

Let $[\Gamma_n\mathfrak{F}_r, \mathfrak{F}_r]_{\mathrm{abst}}$ denote the abstract group generated by $[a, b]$ $(a \in \Gamma_n\mathfrak{F}_r, b \in \mathfrak{F}_r)$. Since $\mathrm{Ih}_S(g)$ is continuous and $[\Gamma_n\mathfrak{F}_r, \mathfrak{F}_r]_{\mathrm{abst}}$ is dense in $\Gamma_{n+1}\mathfrak{F}_r$, it suffices to show that

$$\mathrm{Ih}_S(g)(f)f^{-1} \in \Gamma_{m+n+1}\mathfrak{F}_r \text{ for } f \in [\Gamma_n\mathfrak{F}_r, \mathfrak{F}_r]_{\mathrm{abst}}.$$

For this, since any element $f$ of $[\Gamma_n\mathfrak{F}_r, \mathfrak{F}_r]_{\mathrm{abst}}$ can be written in the form

$$f = [b_1, c_1]^{e_1} \cdots [b_q, c_q]^{e_q},$$

where $b_j \in \Gamma_n \mathfrak{F}_r, c_j \in \mathfrak{F}_r$ for each $j$ $(1 \leqslant j \leqslant q)$ and $e_j$'s are integers, and we have

$$\mathrm{Ih}_S(g)(f)f^{-1} = \mathrm{Ih}_S(g)([b_1,c_1])^{e_1}\cdots \mathrm{Ih}_S(g)([b_q,c_q])^{e_q}[b_q,c_q]^{-e_q}\cdots[b_1,c_1]^{-e_1},$$

we have only to show

$$\mathrm{Ih}_S(g)([b,c])[b,c]^{-1} \in \mathfrak{F}_r(m+n+1) \text{ if } b \in \Gamma_n\mathfrak{F}_r, c \in \mathfrak{F}_r. \tag{3.3.23}$$

For simplicity, we shall use the notation: $[\varphi, x] := \psi(x)x^{-1}$ and $[x,\varphi] := x\varphi(x)^{-1}$ for $x \in \mathfrak{F}_r$ and $\varphi \in \mathrm{Aut}(\mathfrak{F}_r)$. By the "three subgroup lemma" and the induction hypothesis (3.3.22), we have

$$\begin{aligned}
\mathrm{Ih}_S(g)([b,c])[b,c]^{-1} &= [\mathrm{Ih}_S(g),[b,c]] \\
&\in [\mathrm{Ih}_S(g),[\Gamma_n\mathfrak{F}_r,\mathfrak{F}_r]] \\
&\subset [[\mathrm{Ih}_S(g),\Gamma_n\mathfrak{F}_r],\mathfrak{F}_r][[\mathfrak{F}_r,\mathrm{Ih}_S(g)],\Gamma_n\mathfrak{F}_r] \\
&\subset [\Gamma_{m+n}\mathfrak{F}_r,\mathfrak{F}_r][\Gamma_{m+1}\mathfrak{F}_r,\Gamma_n\mathfrak{F}_r] \\
&= \Gamma_{m+n+1}\mathfrak{F}_r
\end{aligned}$$

and our claim (3.3.23) follows. □

Lemma 3.3.21 yields the following

PROPOSITION 3.3.24. *For $m, n \geqslant 0$, we have*

$$[\mathrm{Gal}_k[m], \mathrm{Gal}_k[n]] \subset \mathrm{Gal}_k[m+n] \ \ \textit{for} \ \ m,n \geqslant 0.$$

*In particular, the Johnson (or Milnor) filtration is a central series.*

PROOF. We use the same notation as in the proof of (3.3.23). By Lemma 3.3.21, we have

$$\begin{aligned}
[[\mathrm{Gal}_k[n],\mathfrak{F}_r],\mathrm{Gal}_k[m]] &\subset [\Gamma_{n+1}\mathfrak{F}_r,\mathrm{Gal}_k[m]] \subset \Gamma_{m+n+1}\mathfrak{F}_r, \\
[[\mathfrak{F}_r,\mathrm{Gal}_k[m]],\mathrm{Gal}_k[n]] &\subset [\Gamma_{m+1}\mathfrak{F}_r,\mathrm{Gal}_k[n]] \subset \Gamma_{m+n+1}\mathfrak{F}_r.
\end{aligned}$$

By the three subgroup lemma, we have

$$\begin{aligned}
[[\mathrm{Gal}_k[m],\mathrm{Gal}_k[n]],\mathfrak{F}_r] &\subset [\mathrm{Gal}_k[n],\mathfrak{F}_r],\mathrm{Gal}_k[m]][[\mathfrak{F}_r,\mathrm{Gal}_k[m]],\mathrm{Gal}_k[n]] \\
&\subset \Gamma_{m+n+1}\mathfrak{F}_r.
\end{aligned}$$

By definition (3.3.18), we obtain the assertion. □

For $n \geqslant 0$, let

$$\mathrm{gr}_n(\mathrm{Gal}_k) := \mathrm{Gal}_k[n]/\mathrm{Gal}_k[n+1].$$

Then, by Proposition 3.3.24, the graded $\mathbb{Z}_l$-module

$$\mathrm{gr}(\mathrm{Gal}_k) := \bigoplus_{n\geqslant 0} \mathrm{gr}_n(\mathrm{Gal}_k)$$

has the structure of a graded Lie algebra over $\mathbb{Z}_l$, where the Lie bracket is defined by the commutator: For $a = g \bmod \mathrm{Gal}_k[m+1]$, $b = h \bmod \mathrm{Gal}_k[n+1]$ $(g \in \mathrm{Gal}_k[m], h \in \mathrm{Gal}_k[n])$,

$$[a,b] := [g,h] \bmod \mathrm{Gal}_k[m+n+1].$$

Now, for $m \geqslant 1$, we let $\tau_S^{[m]}$ denote the restriction of the $m$-th $l$-adic Johnson map $\tau_S^{(m)}$ in (3.3.13) to $\mathrm{Gal}_k[m]$:

$$\tau_S^{[m]} := \tau_S^{(m)}|_{\mathrm{Gal}_k[m]} : \mathrm{Gal}_k[m] \longrightarrow \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, H_{\mathbb{Z}_l}^{\otimes(m+1)}).$$

The following theorem asserts that $\tau_S^{[m]}$ describes the action of $\mathrm{Gal}_k[m]$ on $\mathfrak{F}_r/\Gamma_{m+2}\mathfrak{F}_r$.

Theorem 3.3.25. *Notations being as above, the following assertions hold.*
(1) *For $g \in \mathrm{Gal}_k[m]$ and $f \in \mathfrak{F}_r$, we have*

$$\tau_S^{[m]}(g)([f]) = \Theta_{m+1}(\mathrm{Ih}_S(g)(f)f^{-1}),$$

*where $\Theta_{m+1} : \mathrm{gr}_{m+1}(\mathfrak{F}_r) \hookrightarrow H_{\mathbb{Z}_l}^{\otimes(m+1)}$ s the degree $(m+1)$-part of the Magnus embedding in* (3.2.3).
(2) *The map $\tau_S^{[m]}$ is a homomorphism and $\mathrm{Ker}(\tau_S^{[m]}) = \mathrm{Gal}_k[m+1]$. Hence $\tau_S^{[m]}$ induces the injective homomorphism $\mathrm{gr}_m(\mathrm{Gal}_k) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, H_{\mathbb{Z}_l}^{\otimes(m+1)})$.*

Proof. (1) We need to show that for $g \in \mathrm{Gal}_k[m]$,

$$\tau_S^{(m)}(g)(X_i) = \Theta_{m+1}(\mathrm{Ih}_S(g)(x_i)x_i^{-1}) \;\; 1 \leqslant i \leqslant r. \tag{3.3.26}$$

By (3.3.12) and $[\mathrm{Ih}_S(g)] = \mathrm{id}_{H_{\mathbb{Z}_l}}$, we have

$$\begin{aligned}\tau_S(g)(X_i) &= (\Theta \circ \mathrm{Ih}_S(g) \circ \Theta^{-1})(\Theta(x_i) - 1) - (\Theta(x_i) - 1)\\ &= \Theta(\mathrm{Ih}_S(g)(x_i)) - \Theta(x_i).\end{aligned}$$

Therefore, by (3.3.13), we have

$$\tau_S^{(m)}(g)(X_i) = \text{ the component in } H_{\mathbb{Z}_l}^{\otimes(m+1)} \text{ of } \Theta(\mathrm{Ih}_S(g)(x_i)) - \Theta(x_i). \tag{3.3.27}$$

On the other hand, since $\mathrm{Ih}_S(g)(x_i)x_i^{-1} \in \Gamma_{m+1}\mathfrak{F}_r$, we have

$$\Theta(\mathrm{Ih}_S(g)(x_i)x_i^{-1}) \equiv 1 + \Theta_{m+1}(\mathrm{Ih}_S(g)(x_i)x_i^{-1}) \mod \widehat{T}(m+2).$$

Multiplying the above equation by $\Theta(x_i)$ from right, we have

$$\Theta(\mathrm{Ih}_S(g)(x_i)) \equiv \Theta(x_i) + \Theta_{m+1}(\mathrm{Ih}_S(g)(x_i)x_i^{-1}) \mod \widehat{T}(m+2). \tag{3.3.28}$$

By (3.3.27) and (3.3.28), we obtain (3.3.26).
(2) By (1), for $g, h \in \mathrm{Gal}_k[m]$ and $f \in \mathfrak{F}_r$, we have

$$\begin{aligned}\tau_S^{[m]}(gh)([f]) &= \Theta_{m+1}(\mathrm{Ih}_S(gh)(f)f^{-1})\\ &= \Theta_{m+1}(\mathrm{Ih}_S(g)(\mathrm{Ih}_S(h)(f))f^{-1})\\ &= \Theta_{m+1}(\mathrm{Ih}_S(g)(\mathrm{Ih}_S(h)(f)f^{-1})\mathrm{Ih}_S(g)(f)f^{-1}).\end{aligned}$$

Since $\mathrm{Ih}_S(h)(f)f^{-1} \in \Gamma_{m+1}\mathfrak{F}_r$, we have $\mathrm{Ih}_S(g)(\mathrm{Ih}_S(h)(f)f^{-1}) \equiv \mathrm{Ih}_S(h)(f)f^{-1} \bmod \Gamma_{2m+1}\mathfrak{F}_r$ by Lemma 3.3.21. Since $\Gamma_{2m+1}\mathfrak{F}_r \subset \Gamma_{m+2}\mathfrak{F}_r$ by $m \geqslant 1$, we have

$$\begin{aligned}\tau_S^{[m]}(gh)([f]) &= \Theta_{m+1}(\mathrm{Ih}_S(g)(f)f^{-1}) + \Theta_{m+1}(\mathrm{Ih}_S(h)(f)f^{-1}).\\ &= (\tau_S^{[m]}(g) + \tau_S^{[m]}(h))([f])\end{aligned}$$

for any $f \in \mathfrak{F}_r$. Hence the former assertion is proved. The latter follows from (1) and (3.3.18). □

By Theorem 3.3.25 (1), $\tau_S^{[m]}$ factors through $\mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, \mathrm{gr}_{m+1}(\mathfrak{F}_r))$

$$\tau_S^{[m]} : \mathrm{Gal}_k[m] \longrightarrow \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, \mathrm{gr}_{m+1}(\mathfrak{F}_r));\; g \mapsto ([f] \mapsto \mathrm{Ih}_S(g)(f)f^{-1})$$

followed by the map $\mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, \mathrm{gr}_{m+1}(\mathfrak{F}_r)) \to \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, H_{\mathbb{Z}_l}^{\otimes(m+1)})$ induced by $\Theta_{m+1}$. We call $\tau_S^{[m]} : \mathrm{Gal}_k[m] \longrightarrow \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, H_{\mathbb{Z}_l}^{\otimes(m+1)})$ $(m \geqslant 1)$ or the induced injective homomorphism $\mathrm{gr}_m(\mathrm{Gal}_k) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, H_{\mathbb{Z}_l}^{\otimes(m+1)})$, denoted by the same $\tau_S^{[m]}$, the *$m$-th pro-$l$ Johnson homomorphism*.

A relation between the $m$-th pro-$l$ Johnson homomorphisms and $l$-adic Milnor numbers in Section 3.2 is given as follows.

THEOREM 3.3.29. *For $g \in \mathrm{Gal}_k[m]$ $(m \geqslant 1)$, we have*

$$\tau_S^{[m]}(g)(X_i) = -\sum_{|J|=m+1} \mu(J) X_J,$$

*where for $J = (j_1 \cdots j_{m+1})$,*

$$\mu(J) = \begin{cases} \mu(g; j_2 \cdots j_{m+1} j_1) - \delta_{j_1, j_{m+1}} \mu(g; J) & (i = j_1), \\ \mu(g; j_2 \cdots j_{m+1} j_1)\delta_{j_1, j_{m+1}} - \mu(g; J) & (i = j_{m+1}), \\ 0 \ \textit{(otherwise)}. \end{cases}$$

PROOF. By Theorem 3.3.25 (1), we have

$$\begin{aligned} \tau_S^{[m]}(g)(X_i) &= \Theta_{m+1}(\varphi_S(g)(x_i) x_i^{-1}) \\ &= \Theta_{m+1}(y_i(g) x_i y_i(g)^{-1} x_i^{-1}) \\ &= -\Theta_{m+1}([x_i, y_i(g)]) \\ &= -\sum_{|J|=m+1} \mu(J; [x_i, y_i(g)]) X_J. \end{aligned} \tag{3.3.30}$$

By the computation in the proof of Theorem 3.2.41, we have, for $|J| = (j_1 \cdots j_{m+1})$,

(3.3.31)

$$\begin{aligned} \mu(J; [x_i, y_i(g)]) &= \mu(J; x_i y_i(g)) - \mu(J; y_i(g) x_i) \\ &= \begin{cases} \mu(g; j_2 \cdots j_{m+1} j_1) - \delta_{j_1, j_{m+1}} \mu(g; J) & (i = j_1), \\ \mu(g; j_2 \cdots j_{m+1} j_1)\delta_{j_1, j_{m+1}} - \mu(g; J) & (i = j_{m+1}). \\ 0 \ \text{(otherwise)}. \end{cases} \end{aligned}$$

By (3.3.30) and (3.3.31), the assertion follows. □

We compute the pro-$l$ Johnson homomorphisms on commutators.

PROPOSITION 3.3.32. *For $g \in \mathrm{Gal}_k[m], h \in \mathrm{Gal}_k[n]$ $(m, n \geqslant 0)$ and $f \in \mathfrak{F}_r$, we have*

$$\begin{aligned} \tau_S^{[m+n]}([g,h])([f]) &= \Theta_{m+n+1}(\mathrm{Ih}_S(g)(\mathrm{Ih}_S(h)(f)f^{-1})(\mathrm{Ih}_S(h)(f)f^{-1})^{-1} \\ &\quad -\mathrm{Ih}_S(h)(\mathrm{Ih}_S(g)(f)f^{-1})(\mathrm{Ih}_S(g)(f)f^{-1})^{-1}). \end{aligned}$$

PROOF. For simplicity, we set $\psi := \mathrm{Ih}_S(g), \phi := \mathrm{Ih}_S(h)$. By a straightforward computation, we obtain

$$\begin{aligned} &[\psi, \phi](f) f^{-1} \\ &= [\psi, \phi]((\phi(f) f^{-1})^{-1}) \cdot (\psi\phi\psi^{-1})((\psi(f) f^{-1})^{-1}) \cdot \psi(\phi(f) f^{-1}) \cdot \psi(f) f^{-1}. \end{aligned}$$

Since $[g, h] \in \mathrm{Gal}_k[m+n]$ by Proposition 3.3.24 and $\psi(f) f^{-1} \in \Gamma_{m+1}\mathfrak{F}_r$ by Lemma 3.3.21, we have

$$[\psi, \phi]((\phi(f) f^{-1})^{-1}) \equiv (\phi(f) f^{-1})^{-1} \mod \Gamma_{m+2n+1}\mathfrak{F}_r.$$

Similarly, we have

$$(\psi\phi\psi^{-1})((\psi(f) f^{-1})^{-1}) \equiv \phi((\psi(f) f^{-1})^{-1}) \mod \Gamma_{2m+n+1}\mathfrak{F}_r.$$

By these three equations together, we have

$$\begin{aligned} &[\psi, \phi](f) f^{-1} \\ &\equiv (\phi(f) f^{-1})^{-1} \cdot \phi((\psi(f)(f^{-1})^{-1}) \cdot \psi(\phi(f) f^{-1}) \cdot \psi(f) f^{-1} \mod \Gamma_{m+n+1}\mathfrak{F}_r. \end{aligned}$$

Since $\psi(f)f^{-1} \in \Gamma_{m+1}\mathfrak{F}_r, \phi(f)f^{-1} \in \Gamma_{n+1}\mathfrak{F}_r)$ and $[\Gamma_{m+1}\mathfrak{F}_r, \Gamma_{n+1}\mathfrak{F}_r] \subset \Gamma_{m+n+2}\mathfrak{F}$, we have

$$\begin{aligned}&[\psi,\phi](f)f^{-1}\\ &\equiv (\phi(f)f^{-1})^{-1}\cdot\psi(\phi(f)f^{-1})\cdot\phi((\psi(f)f^{-1})^{-1})\cdot\psi(f)f^{-1} \mod \Gamma_{m+n+2}\mathfrak{F}_r.\end{aligned}$$

Since we easily see that

$$\begin{cases} (\phi(f)f^{-1})^{-1}\psi(\phi(f)f^{-1}) \equiv \psi(\phi(f)f^{-1})(\phi(f)f^{-1})^{-1} \mod \Gamma_{m+n+2}\mathfrak{F}_r, \\ \phi((\psi(f)f^{-1})^{-1})\cdot\psi(f)f^{-1} \equiv (\phi(\psi(f)f^{-1})\cdot(\psi(f)f^{-1})^{-1})^{-1} \mod \Gamma_{m+n+2}\mathfrak{F}_r, \end{cases}$$

we obtain the assertion. □

By Proposition 3.3.32, the direct sum of Johnson homomorphisms $\tau_S^{[m]}$ over all $m \geqslant 1$ defines a graded Lie algebra homomorphism from $\mathrm{gr}(\mathrm{Gal}_k)$ to the derivation algebra of $\mathrm{gr}(\mathfrak{F}_r)$ as follows. Recall that a $\mathbb{Z}_l$-linear endomorphism of $\mathrm{gr}(\mathfrak{F}_r)$ is called a *derivation* on $\mathrm{gr}(\mathfrak{F}_r)$ if it satisfies

$$\delta([x,y]) = [\delta(x),y] + [x,\delta(y)] \quad (x,y \in \mathrm{gr}(\mathfrak{F}_r)).$$

Let $\mathrm{Der}(\mathrm{gr}(\mathfrak{F}_r))$ denote the associative $\mathbb{Z}_l$-algebra of all derivations on $\mathrm{gr}(\mathfrak{F}_r)$ which has a Lie algebra structure over $\mathbb{Z}_l$ with the Lie bracket defined by $[\delta,\delta'] := \delta\circ\delta' - \delta'\circ\delta$ for $\delta,\delta' \in \mathrm{Der}(\mathrm{gr}(\mathfrak{F}_r))$. For $m \geqslant 0$, we define the subspace $\mathrm{Der}_m(\mathrm{gr}(\mathfrak{F}_r))$ of $\mathrm{Der}(\mathrm{gr}(\mathfrak{F}_r))$, the degree $m$ part, by

$$\mathrm{Der}_m(\mathrm{gr}(\mathfrak{F}_r)) := \{\delta \in \mathrm{Der}(\mathrm{gr}(\mathfrak{F}_r)) \,|\, \delta(\mathrm{gr}_n(\mathfrak{F}_r)) \subset \mathrm{gr}_{m+n}(\mathfrak{F}_r) \text{ for } n \geqslant 1\}$$

so that $\mathrm{Der}(\mathrm{gr}(\mathfrak{F}_r))$ is a graded Lie algebra over $\mathbb{Z}_l$:

$$\mathrm{Der}(\mathrm{gr}(\mathfrak{F}_r)) = \bigoplus_{m\geqslant 0} \mathrm{Der}_m(\mathrm{gr}(\mathfrak{F}_r)).$$

A derivation $\delta \in \mathrm{Der}_m(\mathrm{gr}(\mathfrak{F}_r))$ is called a *special derivation* if there are $Y_i \in \mathrm{gr}_m(\mathfrak{F}_r)$ such that

$$\delta(X_i) = [Y_i, X_i] \quad (1 \leqslant i \leqslant r)$$

and moreover if the condition

$$\sum_{i=1}^{r}[Y_i, X_i] = 0$$

is satisfied, a special derivation is said to be *normalized* ([**Ih4**, §2]). It is easy to see that normalized special derivations form a graded Lie subalgebra

$$\mathrm{Der}^{\mathrm{n.s}}(\mathrm{gr}(\mathfrak{F}_r)) = \bigoplus_{m\geqslant 0} \mathrm{Der}^{\mathrm{n.s}}_m(\mathrm{gr}(\mathfrak{F}_r))$$

of $\mathrm{Der}(\mathrm{gr}(\mathfrak{F}_r))$. Since a derivation on $\mathrm{gr}(\mathfrak{F}_r)$ is determined by its restriction on $H_{\mathbb{Z}_l} = \mathrm{gr}_1(\mathfrak{F}_r)$, we have a natural inclusion, for each $m \geqslant 1$,

$$\mathrm{Der}_m(\mathrm{gr}(\mathfrak{F}_r)) \subset \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, \mathrm{gr}_{m+1}(\mathfrak{F}_r)); \;\; \delta \mapsto \delta|_{H_{\mathbb{Z}_l}}.$$

Hence we have the inclusions

$$\mathrm{Der}^{\mathrm{n.s}}_+(\mathrm{gr}(\mathfrak{F}_r)) \subset \mathrm{Der}_+(\mathrm{gr}(\mathfrak{F}_r)) \subset \bigoplus_{m\geqslant 1} \mathrm{Hom}_{\mathbb{Z}_p}(H_{\mathbb{Z}_l}, \mathrm{gr}_{m+1}(\mathfrak{F}_r)),$$

where $\mathrm{Der}_+(\mathrm{gr}(\mathfrak{F}_r))$ (resp. $\mathrm{Der}^{\mathrm{n.s}}_+(\mathrm{gr}(\mathfrak{F}_r))$) is the Lie subalgebra of $\mathrm{Der}(\mathrm{gr}(\mathfrak{F}_r))$ (resp. $\mathrm{Der}^{\mathrm{n.s}}(\mathrm{gr}(\mathfrak{F}_r))$) consisting of positive degree part.

PROPOSITION 3.3.33. *The direct sum of $\tau_S^{[m]}$ over $m \geqslant 1$ defines the Lie algebra homomorphism*

$$\mathrm{gr}(\tau) := \bigoplus_{m\geqslant 1} \tau_S^{[m]} \,:\, \mathrm{gr}(\mathrm{Gal}_k) \longrightarrow \mathrm{Der}_+^{\mathrm{n.s}}(\mathrm{gr}(\mathfrak{F}_r)).$$

PROOF. (cf. [**Da**, Proposition 3.18]) By Theorem 3.3.25 (1), it suffices to show that for $g \in \mathrm{Gal}_k[m]$, the map $f \mapsto \mathrm{Ih}_S(g)(f)f^{-1}$ is indeed a special derivation on $\mathrm{gr}(\mathfrak{F}_r)$. Let $s \in \mathrm{Gal}_k[m]$ $(m \geqslant 1)$ and $s \in \Gamma_i\mathfrak{F}_r, h \in \Gamma_j\mathfrak{F}_r$. By using the commutator formulas

$$[ab, c] = a[b, c]a^{-1} \cdot [a, c], \quad [a, bc] = [a, b] \cdot b[a, c]b^{-1} \quad (a, b, c \in G),$$

we obtain

$$\begin{aligned}
&\mathrm{Ih}_S(g)([s,t])[s,t]^{-1} \\
&= [\mathrm{Ih}_S(g)(s), \mathrm{Ih}_S(g)(t)][s,t]^{-1} \\
&= [ss^{-1}\mathrm{Ih}_S(g)(s), \mathrm{Ih}_S(g)(t)t^{-1}t][s,t]^{-1} \\
&= s[s^{-1}\mathrm{Ih}_S(g)(s), \mathrm{Ih}_S(g)(t)t^{-1}] \cdot (\mathrm{Ih}_S(g)(t)t^{-1})[s^{-1}\mathrm{Ih}_S(g)(s), t](\mathrm{Ih}_S(g)(t)t^{-1})^{-1}s^{-1} \\
&\quad \cdot [s, \mathrm{Ih}_S(g)(t)t^{-1}](\mathrm{Ih}_S(g)(t)t^{-1})[s,t](\mathrm{Ih}_S(g)(t)t^{-1})^{-1}[s,t]^{-1} \\
&= s[s^{-1}\mathrm{Ih}_S(g)(s), \mathrm{Ih}_S(g)(t)t^{-1}] \cdot (\mathrm{Ih}_S(g)(t)t^{-1})[s^{-1}\mathrm{Ih}_S(g)(s), t](\mathrm{Ih}_S(g)(t)t^{-1})^{-1}s^{-1} \\
&\quad \cdot [s, \mathrm{Ih}_S(g)(t)t^{-1}][\mathrm{Ih}_S(g)(t)t^{-1}, [s,t]].
\end{aligned}$$

Since $s^{-1}\mathrm{Ih}_S(g)(s) \in \Gamma_{i+m}\mathfrak{F}_r, \mathrm{Ih}_S(g)(t)t^{-1} \in \Gamma_{j+m}\mathfrak{F}_r$ by Lemma 4.3.21, we have

$$[s^{-1}\mathrm{Ih}_S(g)(s), \mathrm{Ih}_S(g)(t)t^{-1}] \in \Gamma_{i+j+2m}\mathfrak{F}_r.$$

Similarly, we have

$$[\mathrm{Ih}_S(g)(t)t^{-1}, [s,t]] \in \Gamma_{i+2j+m}\mathfrak{F}_r.$$

By these three claims together, we have

$$\begin{aligned}
&\mathrm{Ih}_S(g)([s,t])[s,t]^{-1} \\
&\equiv s\mathrm{Ih}_S(g)(t)t^{-1}[s^{-1}\mathrm{Ih}_S(g)(s), t](s\mathrm{Ih}_S(g)(t)t^{-1})^{-1}[s, \mathrm{Ih}_S(g)(t)t^{-1}] \mod \Gamma_{i+j+m+1}\mathfrak{F}_r.
\end{aligned}$$

Noting $x[s^{-1}\mathrm{Ih}_S(g)(s), t]x^{-1} \equiv [s^{-1}\mathrm{Ih}_S(g)(s), t] \mod \Gamma_{i+j+m+1}\mathfrak{F}_r$ for $x \in \mathfrak{F}_r$, we proved that $f \mapsto \mathrm{Ih}_S(g)(f)f^{-1}$ is indeed a derivation. That it is special and normalized follows from $\mathrm{Ih}_S(g)(x_i) = y_i(g)x_iy_i(g)^{-1}$ $(1 \leqslant i \leqslant r)$ and $\mathrm{Ih}_S(g)(x_1 \cdots x_r) = x_1 \cdots x_r$ for $g \in \mathrm{Gal}_k[m]$ $(m \geqslant 1)$ and $1 \leqslant i \leqslant r$. □

Finally we introduce an analogue of the Morita trace map ([**Mt1**, 6]). For each $m \geqslant 1$, we identify $\mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, H_{\mathbb{Z}_l}^{\otimes(m+1)})$ with $H_{\mathbb{Z}_l}^* \otimes_{\mathbb{Z}_l} H_{\mathbb{Z}_l}^{\otimes(m+1)}$, where $H_{\mathbb{Z}_l}^* := \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, \mathbb{Z}_l)$ is the dual $\mathbb{Z}_l$-module, and let

$$c_{m+1} : \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, H_{\mathbb{Z}_l}^{\otimes(m+1)}) = H_{\mathbb{Z}_l}^* \otimes_{\mathbb{Z}_l} H_{\mathbb{Z}_l}^{\otimes(m+1)} \longrightarrow H_{\mathbb{Z}_l}^{\otimes m}$$

be the contraction at $(m+1)$-component defined by

$$c_{m+1}(\phi \otimes h_1 \otimes \cdots \otimes h_{m+1}) := \phi(h_{m+1})h_1 \otimes \cdots \otimes h_m \tag{3.3.34}$$

for $\phi \in H_{\mathbb{Z}_l}^*, h_i \in H_{\mathbb{Z}_l}$. We then define the *$m$-th pro-$l$ Morita trace map*

$$\mathrm{Tr}^{[m]} : \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, H_{\mathbb{Z}_l}^{\otimes(m+1)}) \longrightarrow S^m(H_{\mathbb{Z}_l}) \tag{3.3.35}$$

by the composite map $q \circ c_{m+1}$.

CHAPTER 4

# Pro-$l$ reduced Gassner representation and Ihara power series

In this chapter, we study the arithmetic analogue of Chapter 2. More precisely, we define the pro-$l$ reduced Gassner representation for the absolute Galois group of a number field and give a formula in terms of the $l$-adic Milnor invariant. Moreover, regarding Ihara power series as a special case of the pro-$l$ Gassner repesentation, we give an arithmetic topological interpretation of Jacobi sums and give a formula that relates $l$-adic Milnor invariants and Soulé characters. This chapter is based on [**KMT**, Sections 4 and 5].

## 4.1. Pro-$l$ Magnus-Gassner cocycles

**4.1.1. Pro-$l$ Fox free derivation.** The *pro-$l$ Fox free derivative* $\frac{\partial}{\partial x_j} : \mathbb{Z}_l[[\mathfrak{F}_r]] \to \mathbb{Z}_l[[\mathfrak{F}_r]]$ $(1 \leqslant j \leqslant r)$ is a continuous $\mathbb{Z}_l$-linear map satisfying the following property: For any $\alpha \in \mathbb{Z}_l[[\mathfrak{F}_r]]$,

$$\alpha = \epsilon_{\mathbb{Z}_l[[\mathfrak{F}_r]]}(\alpha) + \sum_{j=1}^{r} \frac{\partial \alpha}{\partial x_j}(x_j - 1). \tag{4.1.1}$$

We note by (4.1.1) that $\dfrac{\partial \alpha}{\partial x_j} \in I^{n-1}_{\mathbb{Z}_l[[\mathfrak{F}_r]]}$ if $\alpha - \epsilon_{\mathbb{Z}_l[[\mathfrak{F}_r]]}(\alpha) \in I^{n}_{\mathbb{Z}_l[[\mathfrak{F}_r]]}$ for $n \geqslant 1$.

Here are some basic rules for the pro-$l$ free calculus:

(i) $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$.

(ii) $\dfrac{\partial \alpha\beta}{\partial x_j} = \dfrac{\partial \alpha}{\partial x_j}\epsilon_{\mathbb{Z}_l[[\mathfrak{F}_r]]}(\beta) + \alpha\dfrac{\partial \beta}{\partial x_j} \quad (\alpha, \beta \in \mathbb{Z}_l[[\mathfrak{F}_r]])$.

(iii) $\dfrac{\partial f^{-1}}{\partial x_j} = -f^{-1}\dfrac{\partial f}{\partial x_j} \quad (f \in \mathfrak{F}_r)$.

(iv) $\dfrac{\partial f^{\alpha}}{\partial x_j} = \beta\dfrac{\partial f}{\partial x_j}$ $(f \in \mathfrak{F}_r, \alpha \in \mathbb{Z}_l[[\mathfrak{F}_r]])$, where $\beta$ is any element of $\mathbb{Z}_l[[\mathfrak{F}_r]]$ such that $\beta(f-1) = f^{\alpha} - 1$ if exists.

(v) $\dfrac{\partial \varphi(\alpha)}{\partial \varphi(x_j)} = \varphi(\dfrac{\partial \alpha}{\partial x_j}) \quad (\varphi \in \mathrm{Aut}(\mathfrak{F}_r), \alpha \in \mathbb{Z}_l[[\mathfrak{F}_r]])$. (Note that $\varphi(x_1), \ldots, \varphi(x_r)$ are free generators of $\mathfrak{F}_r$.)

(vi) If $\mathfrak{F}'$ is an open free subgroup of $\mathfrak{F}_r$ with free generators $y_1, \cdots, y_s$, we have the chain rule: $\dfrac{\partial \alpha}{\partial x_j} = \displaystyle\sum_{i=1}^{s} \dfrac{\partial \alpha}{\partial y_i}\dfrac{\partial y_i}{\partial x_j} \quad (\alpha \in \mathbb{Z}_l[[\mathfrak{F}']])$.

The higher derivatives are defined inductively and the $l$-adic Magnus coefficient $\mu(I;\alpha)$ of $\alpha \in \mathbb{Z}_l[[\mathfrak{F}_r]]$ for $I = (i_1 \cdots i_n)$ is expressed by

$$\mu(I;\alpha) = \epsilon_{\mathbb{Z}_l[[\mathfrak{F}_r]]}\left(\frac{\partial^n \alpha}{\partial x_{i_1} \cdots \partial x_{i_n}}\right)$$

so that the pro-$l$ Magnus expansion (3.2.2) is written as

$$\Theta(\alpha) = \epsilon_{\mathbb{Z}_l[[\mathfrak{F}_r]]}(\alpha) + \sum_{1 \leqslant i_1, \ldots, i_n \leqslant r} \epsilon_{\mathbb{Z}_l[[\mathfrak{F}_r]]}\left(\frac{\partial^n \alpha}{\partial x_{i_1} \cdots \partial x_{i_n}}\right) X_{i_1} \cdots X_{i_n}.$$

**4.1.2. Pro-$l$ Magnus cocycles.** Let $\mathrm{Ih}_S : \mathrm{Gal}_k \to P(\mathfrak{F}_r) \subset \mathrm{Aut}(\mathfrak{F}_r)$ be the Ihara representation associated to $S$ in (3.1.6). Let $^- : \mathbb{Z}_l[[\mathfrak{F}_r]] \to \mathbb{Z}_l[[\mathfrak{F}_r]]$ denote the anti-automorphism induced by the involution $\mathfrak{F}_r \ni f \mapsto f^{-1} \in \mathfrak{F}_r$. We define the *pro-l Magnus cocycle* $\mathrm{M}_S : \mathrm{Gal}_k \to \mathrm{M}(r; \mathbb{Z}_l[[\mathfrak{F}_r]])$ associated to $\mathrm{Ih}_S$ by

$$\mathrm{M}_S(g) := \left(\overline{\frac{\partial \mathrm{Ih}_S(g)(x_j)}{\partial x_i}}\right) \tag{4.1.2}$$

for $g \in \mathrm{Gal}_k$. In fact, we have the following

Lemma 4.1.3. *The map $\mathrm{M}_S$ is a 1-cocycle of $\mathrm{Gal}_k$ with coefficients in $\mathrm{GL}(r; \mathbb{Z}_l[[\mathfrak{F}_r]])$ with respect to the action $\mathrm{Ih}_S$. To be precise, for $g, h \in \mathrm{Gal}_k$, we have*

$$\mathrm{M}_S(gh) = \mathrm{M}_S(g)\mathrm{Ih}_S(g)(\mathrm{M}_S(h)),$$

*where $\mathrm{Ih}_S(g)(\mathrm{M}_S(h))$ is the matrix obtained by applying $\mathrm{Ih}_S(g)$ to each enty of $\mathrm{M}_S(h)$.*

Proof. Let $y_j := \mathrm{Ih}_S(h)(x_j)$ for $1 \leqslant j \leqslant r$. Then we have

$$\frac{\partial \mathrm{Ih}_S(gh)(x_j)}{\partial x_i} = \frac{\partial \mathrm{Ih}_S(g)(y_j)}{\partial x_i}. \tag{4.1.4}$$

Using the basic rules (v), (vi) of the pro-$l$ Fox derivatives, we have

$$\begin{aligned} \frac{\partial \mathrm{Ih}_S(g)(y_j)}{\partial x_i} &= \sum_{a=1}^{r} \frac{\partial \mathrm{Ih}_S(g)(y_j)}{\partial \mathrm{Ih}_S(g)(x_a)} \frac{\partial \mathrm{Ih}_S(g)(x_a)}{\partial x_i} \\ &= \sum_{a=1}^{r} \mathrm{Ih}_S(g)\left(\frac{\partial y_j}{\partial x_a}\right) \frac{\partial \mathrm{Ih}_S(g)(x_a)}{\partial x_i}. \end{aligned} \tag{4.1.5}$$

By (4.1.4) and (4.1.5), we have

$$\overline{\frac{\partial \mathrm{Ih}_S(gh)(x_j)}{\partial x_i}} = \sum_{a=1}^{r} \overline{\frac{\partial \mathrm{Ih}_S(g)(x_a)}{\partial x_i}} \cdot \overline{\mathrm{Ih}_S(g)\left(\frac{\partial y_j}{\partial x_a}\right)}.$$

Since $\mathrm{Ih}_S(g)$ and $^-$ are commutative operators, we obtain the desired equality of the matrices. Taking $h = g^{-1}$, we see that $\mathrm{M}_S(g) \in \mathrm{GL}(r; \mathbb{Z}_l[[\mathfrak{F}_r]])$ for $g \in \mathrm{Gal}_k$. □

For $m \geqslant 1$, we let $\mathrm{M}_S^{[m]}$ be the composite of $\mathrm{M}_S$ restricted to $\mathrm{Gal}_k[m]$ with the natural homomorphism $\mathrm{GL}(r; \mathbb{Z}_l[[\mathfrak{F}_r]]) \to \mathrm{GL}(r; \mathbb{Z}_l[[\mathfrak{F}_r]]/I_{\mathbb{Z}_l[[\mathfrak{F}_r]]}^{m+1})$

$$\mathrm{M}_S^{[m]} : \mathrm{Gal}_k[m] \longrightarrow \mathrm{GL}(r; \mathbb{Z}_l[[\mathfrak{F}_r]]/I_{\mathbb{Z}_l[[\mathfrak{F}_r]]}^{m+1}).$$

A relation between $\mathrm{M}_S^{[m]}$ and the $m$-th pro-$l$ Johnson homomorphism is given as follows. First, recall the identification $\Theta_n : \mathrm{gr}_n(\mathfrak{F}_r) \simeq H_{\mathbb{Z}_l}^{\otimes n}$ by the degree $n$ part of the Magnus isomorphism in (3.2.1). We then have a matrix representation of $\mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, H_{\mathbb{Z}_l}^{\oplus(m+1)})$ for $m \geqslant 1$

$$|| \ \ || : \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, H_{\mathbb{Z}_l}^{\oplus(m+1)}) \longrightarrow \mathrm{M}(r; \mathrm{gr}_m(\mathbb{Z}_l[[\mathfrak{F}_r]]))$$

by associating to each element $\tau \in \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, H_{\mathbb{Z}_l}^{\oplus(m+1)})$ the matrix

$$(4.1.6) \qquad ||\tau|| := \left( \frac{\partial(\Theta_{m+1}^{-1} \circ \tau)(X_j)}{\partial x_i} \right) \in \mathrm{M}(r; \mathrm{gr}_m(\mathbb{Z}_l[[\mathfrak{F}_r]])).$$

Proposition 4.1.7. *For $g \in \mathrm{Gal}_k[m]$, we have*

$$\mathrm{M}_S^{[m]}(g) = I + \overline{||\tau_S^{[m]}(g)||}.$$

Proof. By Theorem 3.3.6, we have

$$(\Theta_{m+1}^{-1} \circ \tau_S^{[m]}(g))(X_j) = \mathrm{Ih}_S(g)(x_j)x_j^{-1}$$

and so

$$\begin{aligned} \frac{\partial(\Theta_{m+1}^{-1} \circ \tau_S^{[m]})(X_j)}{\partial x_i} &= \frac{\partial \mathrm{Ih}_S(g)(x_j)x_j^{-1}}{\partial x_i} \\ &= \frac{\partial \mathrm{Ih}_S(g)(x_j)}{\partial x_i} - \mathrm{Ih}_S(g)(x_j)x_j^{-1}\delta_{ij}. \end{aligned}$$

Since $\mathrm{Ih}_S(g)(x_j)x_j^{-1} \in \Gamma_{m+1}\mathfrak{F}_r$, we have $\mathrm{Ih}_S(g)(x_j)x_j^{-1}\delta_{ij} \equiv \delta_{ij} \bmod I_{\mathbb{Z}_l[[\mathfrak{F}_r]]}^{m+1}$ and hence the assertion is proved. □

In terms of $|| \cdot ||$, the $m$-th pro-$l$ Morita trace $\mathrm{Tr}^{[m]}(\tau)$ in (3.3.35) is, in fact, written as the trace of the matrix $||\tau||$ .

Proposition 4.1.8. *For $m \geqslant 1$ and $, \tau \in \mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, H_{\mathbb{Z}_l}^{\otimes(m+1)})$, we have*

$$\mathrm{Tr}^{[m]}(\tau) = q_m(\mathrm{tr}(\Theta_m(||\tau||))),$$

*where $q_m : H_{\mathbb{Z}_l}^{\otimes m} \to S^m(H_{\mathbb{Z}_l})$ is the natural map.*

Proof. We identify $\mathrm{Hom}_{\mathbb{Z}_l}(H_{\mathbb{Z}_l}, H_{\mathbb{Z}_l}^{\otimes(m+1)})$ with $H_{\mathbb{Z}_l}^* \otimes H_{\mathbb{Z}_l}^{\otimes m}$. Let $\tau = \phi \otimes X_{i_1} \otimes \cdots \otimes X_{i_{m+1}}$ ($\phi \in H_{\mathbb{Z}_l}^*$). By (4.1.6), we have

$$(4.1.9) \quad \mathrm{tr}(||\tau||) = \sum_{i=1}^{r} \frac{\partial(\Theta_{m+1}^{-1} \circ \tau)(X_i)}{\partial x_i} = \sum_{i=1}^{r} \phi(X_i) \frac{\partial \Theta_{m+1}^{-1}(X_{i_1} \otimes \cdots \otimes X_{i_{m+1}})}{\partial x_i}.$$

We note that any element $Y$ of $H_{\mathbb{Z}_l}^{\otimes(m+1)}$ can be written uniquely as

$$Y = Y_1 \otimes X_1 + \cdots + Y_r \otimes X_r, \ \ Y_i \in H_{\mathbb{Z}_l}^{\otimes m}$$

and then we have, by (4.1.1),

$$\frac{\partial \Theta_{m+1}^{-1}(Y)}{\partial x_i} = \Theta_m^{-1}(Y_i).$$

Therefore we have

$$\frac{\partial \Theta_{m+1}^{-1}(X_{i_1} \otimes \cdots \otimes X_{i_{m+1}})}{\partial x_i} = \delta_{i, i_{m+1}} X_{i_1} \otimes \cdots \otimes X_{i_m}$$

and hence, by (4.1.9),

$$\mathrm{tr}(\Theta_m(||\tau||)) = \phi(X_{i_{m+1}}) X_{i_1} \otimes \cdots \otimes X_{i_m},$$

where the right-hand side is $c_{m+1}(\tau)$ by (3.3.34). By (3.3.35), the assertion proved. □

Now, for some application later on, we extend the construction of the pro-$l$ Magnus cocycle to a relative situation. Let $\mathfrak{G}$ be a pro-$l$ group and let $\psi : \mathfrak{F}_r \to \mathfrak{G}$ be a continuous surjective homomorphism. We also denote by $\psi$ the induced surjective homomorphism $\mathbb{Z}_l[[\mathfrak{F}_r]] \to \mathbb{Z}_l[[\mathfrak{G}]]$ of complete group algebras over $\mathbb{Z}_l$. Let $\mathfrak{N} := \mathrm{Ker}(\psi)$ so that $\mathfrak{F}_r/\mathfrak{N} \simeq \mathfrak{G}$. We assume that $\mathfrak{N}$ is stable under the action of $\mathrm{Gal}_k$ through $\mathrm{Ih}_S$, namely $\mathrm{Ih}_S(g)(\mathfrak{N}) \subset \mathfrak{N}$ for all $g \in \mathrm{Gal}_k$ (This is certainly satisfied if $\mathfrak{N}$ is a characteristic subgroup of $\mathfrak{F}_r$). Then we have a homomorphism $\mathrm{Ih}_{S,\psi} : \mathrm{Gal}_k \to \mathrm{Aut}(\mathbb{Z}_l[[\mathfrak{G}]])$ defined by

$$\text{(4.1.10)} \qquad \mathrm{Ih}_{S,\psi}(g)(\psi(\alpha)) := \psi(\mathrm{Ih}_S(g)(\alpha)) \ \ (\alpha \in \mathbb{Z}_l[[\mathfrak{F}_r]]).$$

Let $\mathrm{Gal}_k[\psi]$ be the subgroup of $\mathrm{Gal}_k$ defined by

$$\text{(4.1.11)} \qquad \begin{aligned} \mathrm{Gal}_k[\psi] &:= \mathrm{Ker}(\mathrm{Ih}_{S,\psi}) \\ &= \{g \in \mathrm{Gal}_k \mid \psi \circ \mathrm{Ih}_S(g) = \psi\} \end{aligned}$$

and let $k[\psi]$ denote the subfield of $\overline{\mathbb{Q}}/k$ corresponding to $\mathrm{Gal}_k[\psi]$. Now we define the *pro-l Magnus cocycle* $\mathrm{M}_{S,\psi} : \mathrm{Gal}_k \to \mathrm{GL}(r; \mathbb{Z}_l[[\mathfrak{G}]])$ associated to $\mathrm{Ih}_S$ and $\psi$ by

$$\mathrm{M}_{S,\psi}(g) := \psi(\mathrm{M}_S(g)) \quad (g \in \mathrm{Gal}_k),$$

where the right hand side is the matrix obtained by applying $\psi$ to each entry of $\mathrm{M}_S(g)$. For $m \geqslant 1$, let $\mathrm{M}^{[m]}_{S,\psi}$ be the composite of $\mathrm{M}^{[m]}_S$ with the natural homomorphism $\mathrm{GL}(r; \mathbb{Z}_l[[\mathfrak{F}_r]]/I^{m+1}_{\mathbb{Z}_l[[\mathfrak{F}_r]]}) \to \mathrm{GL}(r; \mathbb{Z}_l[[\mathfrak{G}]]/I^{m+1}_{\mathbb{Z}_l[[\mathfrak{G}]]})$ induced by $\psi$. Lemma 4.1.3 and Proposition 4.1.7 are extended to the following.

Proposition 4.1.12. *Notations being as above, the following assertions hold.*
(1) *For $g, h \in \mathrm{Gal}_k$, we have*

$$\mathrm{M}_{S,\psi}(gh) = \mathrm{M}_{S,\psi}(g)\mathrm{Ih}_{S,\psi}(g)(\mathrm{M}_{S,\psi}(h)).$$

(2) *For $g \in \mathrm{Gal}_k$, we have*

$$\mathrm{M}^{[m]}_{S,\psi}(g) = I + \psi(\overline{||\tau^{[m]}_S(g)||}).$$

(3) *The restriction of $\mathrm{M}_{S,\psi}$ to $\mathrm{Gal}_k[\psi]$, denoted by the same $\mathrm{M}_{S,\psi}$,*

$$\mathrm{M}_{S,\psi} : \mathrm{Gal}_k[\psi] \longrightarrow \mathrm{GL}(r; \mathbb{Z}_l[[\mathfrak{G}]]/I^{m+1}_{\mathbb{Z}_l[[\mathfrak{G}]]})$$

*is a homomorphism and factors through the Galois group $\mathrm{Gal}(\Omega_S/k[\psi])$, where $\Omega_S$ is the subfield of $\overline{\mathbb{Q}}$ corresponding to $\mathrm{Ker}(\mathrm{Ih}_S)$ as in* (3.1.7). We call it the *pro-l Magnus representation* of $\mathrm{Gal}_k[\psi]$ associated to $\mathrm{Ih}_S$ and $\psi$.

Proof. (1) The formula is obtained by applying $\psi$ to the both sides of the formula in Lemma 4.1.3. (2) This is also obtained by applying $\psi$ to the matrices of the both sides of the formula in Proposition 4.1.7. (3) Suppose $g, h \in \mathrm{Gal}_{k,\psi}$. Since $\psi \circ \mathrm{Ih}_S(g) = \psi$, we have $\mathrm{Ih}_{S,\psi}(g)(\mathrm{M}_{S,\psi}(h)) = \mathrm{M}_{S,\psi}(h)$ and so $\mathrm{M}_{S,\psi}(gh) = \mathrm{M}_{S,\psi}(g)\mathrm{M}_{S,\psi}(h)$. Since $\mathrm{M}_{S,\psi}(g) = I$ for $g \in \mathrm{Ker}(\mathrm{Ih}_S)$, we have $\mathrm{Ker}(\mathrm{M}_{S,\psi}) \supset \mathrm{Ker}(\mathrm{Ih}_S)$ and hence $\mathrm{M}_{S,\psi}$ factors through $\mathrm{Gal}(\Omega_S/k[\psi])$. □

For $n \geqslant 0$, let $\pi_n : \mathfrak{F}_r \to \mathfrak{F}_r/\Gamma_{n+1}\mathfrak{F}_r$ be the natural homomorphism. We consider the case that $\psi = \pi_n$ and so $\mathrm{Ih}_{S,\psi} = \mathrm{Ih}_S^{(n)}$. By (3.3.18) and Lemma 3.3.21, we have

$$\begin{aligned}\mathrm{Gal}_k[\pi_n] &= \{g \in \mathrm{Gal}_k \mid \pi_n \circ \mathrm{Ih}_S(g) = \pi_n\} \\ &= \{g \in \mathrm{Gal}_k \mid \mathrm{Ih}_S(g)(f) \equiv f \bmod \Gamma_{n+1}\mathfrak{F}_r \text{ for all } f \in \mathfrak{F}_r\} \\ &= \mathrm{Gal}_k[n].\end{aligned}$$

Then we have a family of pro-$l$ Magnus cocycles

$$\mathrm{M}_{S,\pi_n} : \mathrm{Gal}_k \longrightarrow \mathrm{GL}(r; \mathbb{Z}_l[[\mathfrak{F}_r/\Gamma_{n+1}\mathfrak{F}_r]]), \tag{4.1.13}$$

and the pro-$l$ Magnus representation

$$\mathrm{M}_{S,\pi_n} : \mathrm{Gal}_k[n] \longrightarrow \mathrm{GL}(r; \mathbb{Z}_l[[\mathfrak{F}_r/\Gamma_{n+1}\mathfrak{F}_r]]) \tag{4.1.14}$$

associated to $\mathrm{Ih}_S$ and $\pi_n$ for $n \geqslant 0$.

**4.1.3. Pro-$l$ Gassner cocycles.** The pro-$l$ Gassner cocycle is defined by $\mathrm{M}_{S,\pi_1}$ in (4.1.13). To be precise, let $\widehat{\Lambda}_r := \mathbb{Z}_l[[T_1, \ldots, T_r]]$ denote the algebra of commutative formal power series over $\mathbb{Z}_l$ of variables $T_1, \ldots, T_r$, called the *Iwasawa algebra* of $r$ variables. The correspondence $x_i \bmod \Gamma_2\mathfrak{F}_r \mapsto 1 + T_i$ $(1 \leqslant i \leqslant r)$ gives the abelianized pro-$l$ Magnus isomorphism

$$\Theta^{\mathrm{ab}} : \mathbb{Z}_l[[\mathfrak{F}_r/\Gamma_2\mathfrak{F}_r]] \xrightarrow{\sim} \widehat{\Lambda}_r.$$

We let $\pi := \pi_1$ and

$$\chi_{\widehat{\Lambda}_r} := \mathrm{Ih}_{S,\Theta^{\mathrm{ab}}\circ\pi} : \mathrm{Gal}_k \to \mathrm{Aut}(\widehat{\Lambda}_r), \tag{4.1.15}$$

which is defined by (4.1.10) with $\psi = \Theta^{\mathrm{ab}} \circ \pi$. In fact, by Lemma 3.2.10, $\chi_{\widehat{\Lambda}_r}$ is given by

$$\chi_{\widehat{\Lambda}_r}(g)(T_i) = (\Theta^{\mathrm{ab}} \circ \pi)(\mathrm{Ih}_S(g)(x_i - 1)) = (1 + T_i)^{\chi_l(g)} - 1 \quad (1 \leqslant i \leqslant r). \tag{4.1.16}$$

Then the *pro-$l$ Gassner cocycle* of $\mathrm{Gal}_k$ associated to $\mathrm{Ih}_S$

$$\mathrm{Gass}_S : \mathrm{Gal}_k \longrightarrow \mathrm{GL}(r; \widehat{\Lambda}_r)$$

is defined by

$$\mathrm{Gass}_S(g) := \left((\Theta^{\mathrm{ab}} \circ \pi)\left(\frac{\partial \mathrm{Ih}_S(g)(x_j)}{\partial x_i}\right)\right) \quad (g \in \mathrm{Gal}_k), \tag{4.1.17}$$

where we note that we do not need to take the anti-automorphism $^{-}$ in (4.1.17) to obtain the 1-cocycle relation

$$\mathrm{Gass}_S(gh) = \mathrm{Gass}_S(g)\chi_{\widehat{\Lambda}_r}(g)(\mathrm{Gass}_S(h)) \quad (g, h \in \mathrm{Gal}_k),$$

since $\widehat{\Lambda}_r$ is commutative. Here $\chi_{\widehat{\Lambda}_r}(g)(\mathrm{Gass}_S(h))$ is the matrix obtained by applying $\chi_{\widehat{\Lambda}_r}(g)$ to each entry of $\mathrm{Gass}_S(h)$. We can express $\mathrm{Gass}_S(g)$ in terms of $l$-adic Milnor numbers as follows.

PROPOSITION 4.1.18. *The $(i,j)$-entry of* $\mathrm{Gass}_S(g)$ $(g \in \mathrm{Gal}_k)$ *is expressed by*

$$\mathrm{Gass}_S(g)_{ij} = \begin{cases} \dfrac{\chi_{\widehat{\Lambda}_r}(g)(T_i)}{T_i}\left(1 + \displaystyle\sum_{n \geqslant 1} \sum_{\substack{1 \leqslant i_1, \ldots, i_n \leqslant r \\ i_n \neq i}} \mu(g; i_1 \cdots i_n i) T_{i_1} \cdots T_{i_n}\right) & (i = j), \\ -\chi_{\widehat{\Lambda}_r}(g)(T_j)\left(\mu(g; ij) + \displaystyle\sum_{n \geqslant 1} \sum_{1 \leqslant i_1, \ldots, i_n \leqslant r} \mu(g; i_1 \cdots i_n ij) T_{i_1} \cdots T_{i_n}\right) & (i \neq j). \end{cases}$$

PROOF. By Lemma 3.2.10 and a straightforward computation, we have

$$\begin{aligned} \frac{\partial \mathrm{Ih}_S(g)(x_j)}{\partial x_i} &= \frac{\partial y_j(g) x_j^{\chi_l(g)} y_j(g)^{-1}}{\partial x_i} \\ &= y_j(g) \frac{x_j^{\chi_l(g)} - 1}{x_j - 1} \delta_{ij} + (1 - y_j(g) x_j^{\chi_l(g)} y_j(g)^{-1}) \frac{\partial y_j(g)}{\partial x_i} \end{aligned}$$

and hence, by (4.1.16),

(4.1.19)

$$\begin{aligned} &(\Theta^{\mathrm{ab}} \circ \pi)\left(\frac{\partial \mathrm{Ih}_S(g)(x_j)}{\partial x_i}\right) \\ &= (\Theta^{\mathrm{ab}} \circ \pi)(y_j(g)) \frac{(1 + T_j)^{\chi_l(g)} - 1}{T_j} \delta_{ij} + (1 - (1 + T_j)^{\chi_l(g)})(\Theta^{\mathrm{ab}} \circ \pi)\left(\frac{\partial y_j(g)}{\partial x_i}\right) \\ &= \frac{\chi_{\widehat{\Lambda}_r}(g)(T_j)}{T_j}(\Theta^{\mathrm{ab}} \circ \pi)(y_j(g))\delta_{ij} - \chi_{\widehat{\Lambda}_r}(g)(T_j)(\Theta^{\mathrm{ab}} \circ \pi)\left(\frac{\partial y_j(g)}{\partial x_i}\right). \end{aligned}$$

Here we have

$$(\Theta^{\mathrm{ab}} \circ \pi)(y_j(g)) = 1 + \sum_{|I| \geqslant 1} \mu(g; Ij) T_I, \tag{4.1.20}$$

where we set $T_I := T_{i_1} \cdots T_{i_n}$ for $I = (i_1 \cdots i_n)$, and (4.1.1) yields

$$(\Theta^{\mathrm{ab}} \circ \pi)\left(\frac{\partial y_j(g)}{\partial x_i}\right) = \sum_{|I| \geqslant 0} \mu(g; Iij) T_I. \tag{4.1.21}$$

By (4.1.17), (4.1.19), (4.1.20) and (4.1.21), we have

$$\begin{aligned} \mathrm{Gass}_S(g) &= (\Theta^{\mathrm{ab}} \circ \pi)\left(\frac{\partial \mathrm{Ih}_S(g)(x_j)}{\partial x_i}\right) \\ &= \delta_{ij} \frac{\chi_{\widehat{\Lambda}_r}(g)(T_j)}{T_j}\left(1 + \sum_{|I| \geqslant 1} \mu(g; Ij) T_I\right) - \chi_{\widehat{\Lambda}_r}(g)(T_j) \sum_{|I| \geqslant 0} \mu(g; Iij) T_I. \end{aligned}$$

By $\mu(g; ii) = 0$ and a simple observation, we obtain the assertion. □

By (4.1.14), when $\mathrm{Gass}_S$ is restricted to $\mathrm{Gal}_k[1]$, we have a representation

$$\mathrm{Gass}_S : \mathrm{Gal}_k[1] \longrightarrow \mathrm{GL}_r(\widehat{\Lambda}_r),$$

which we call the *pro-$l$ Gassner representation* of $\mathrm{Gal}_k[1]$ associated to $\mathrm{Ih}_S$. It factors through the Galois group $\mathrm{Gal}(\Omega_S/k[1])$ by Theorem 4.1.12 (3).

In the following, for simplicity, we let

$$\mathfrak{F}'_r := \Gamma_2 \mathfrak{F}_r, \mathfrak{F}''_r := [\mathfrak{F}'_r, \mathfrak{F}'_r], \text{ and } \mathfrak{L}_r := \mathfrak{F}'_r / \mathfrak{F}''_r = H_1(\mathfrak{F}'_r, \mathbb{Z}_l).$$

We consider $\mathfrak{L}$ as a $\widehat{\Lambda}_r$-module by conjugation: For $f \in \mathfrak{F}_r$ and $f' \in \mathfrak{F}'_r$, we set

$$[f].(f' \bmod \mathfrak{F}''_r) := ff'f^{-1} \bmod \mathfrak{F}''_r.$$

and extend it by the $\mathbb{Z}_l$-linearity and continuity. The structure of the $\widehat{\Lambda}_r$-module $\mathfrak{L}_r$ can be described by means of the pro-$l$ Crowell exact sequence ([Ms2; Chapter 9]). Attached to the surjective homomorphism $\pi : \mathfrak{F}_r \longrightarrow \mathfrak{F}_r/\mathfrak{F}'_r$, the *pro-$l$ Crowell exact sequence* reads as the exact sequence of $\widehat{\Lambda}_r$-modules:

$$0 \longrightarrow \mathfrak{L}_r \xrightarrow{\nu_1} \widehat{\Lambda}_r^{\oplus r} \xrightarrow{\nu_2} I_{\widehat{\Lambda}_r} \longrightarrow 0,$$

where $I_{\widehat{\Lambda}_r}$ is the (augmentation) ideal of $\widehat{\Lambda}_r$ generated by $T_1, \dots, T_r$ and $\nu_1, \nu_2$ are $\widehat{\Lambda}_r$-homomorphisms defined by

$$\text{(4.1.22)} \quad \nu_1(f' \bmod \mathfrak{F}''_r) := ((\Theta^{\mathrm{ab}} \circ \pi)\left(\frac{\partial f'}{\partial x_i}\right)) \ \ (f' \in \mathfrak{F}'_r); \quad \nu_2((\lambda_i)) := \sum_{i=1}^{r} \lambda_i T_i.$$

(*Convention*: An element $(\lambda_i)$ of $\widehat{\Lambda}_r^{\oplus r}$ is understood as a column vector.) Hence we have the isomorphism of $\widehat{\Lambda}_r$-modules induced by $\nu_1$, called the *Blanchfield-Lyndon isomorphism*:

$$\text{(4.1.23)} \qquad \nu_1 : \mathfrak{L}_r \xrightarrow{\sim} \{(\lambda_i) \in \widehat{\Lambda}_r^{\oplus r} \mid \sum_{i=1}^{r} \lambda_i T_i = 0\}.$$

We define the action $\mathrm{Meta}_S$ of $\mathrm{Gal}_k$ on $\mathfrak{L}_r$ through the Ihara representation $\mathrm{Ih}_S$: For $g \in \mathrm{Gal}_k$ and $f' \in \mathfrak{F}'_r$,

$$\mathrm{Meta}_S(g)(f' \bmod \mathfrak{F}''_r) := \mathrm{Ih}_S(g)(f') \bmod \mathfrak{F}''_r.$$

It is easy to see that $\mathrm{Meta}_S(g)$ is a $\chi_{\widehat{\Lambda}_r}$-linear automorphism of $\mathfrak{L}_r$, namely, a $\mathbb{Z}_l$-linear automorphism and satisfies

$$\mathrm{Meta}_S(g)(\lambda.(f' \bmod \mathfrak{F}''_r)) = \chi_{\widehat{\Lambda}_r}(g)(\lambda).(f' \bmod \mathfrak{F}''_r)$$

for $\lambda \in \widehat{\Lambda}_r$ and $f' \in \mathfrak{F}'_r$. When $\mathrm{Meta}_S$ is restricted to $\mathrm{Gal}_k[1]$, we have the representation, which we call the *pro-$l$ meta-abelian representation* of $\mathrm{Gal}_k[1]$ associated to $\varphi_S$,

$$\mathrm{Meta}_S : \mathrm{Gal}_k[1] \longrightarrow \mathrm{GL}_{\widehat{\Lambda}_r}(\mathfrak{L}_r),$$

where $\mathrm{GL}_{\widehat{\Lambda}_r}(\mathfrak{L}_r)$ is the group of $\widehat{\Lambda}_r$-module automorphisms of $\mathfrak{L}_r$. Regarding $\mathfrak{L}_r$ as a $\widehat{\Lambda}_r$-submodule of $\widehat{\Lambda}_r^{\oplus r}$ by the isomorphism (4.1.23), $\mathrm{Meta}_S$ and $\mathrm{Gass}_S$ has the following relation.

PROPOSITION 4.1.24. *For $g \in \mathrm{Gal}_k$ and $f' \in \mathfrak{F}'_r$, we have*

$$(\nu_1 \circ \mathrm{Meta}_S(g))(f' \bmod \mathfrak{F}'_r) = \mathrm{Gass}_S(g)(\chi_{\widehat{\Lambda}_r}(g) \circ \nu_1)(f' \bmod \mathfrak{F}''_r).$$

*When $\mathrm{Meta}_S$ and $\mathrm{Gass}_S|_{\mathfrak{L}_r}$ are restricted to $\mathrm{Gal}_k[1]$, they are equivalent representations over $\widehat{\Lambda}_r$.*

Proof. The first assertion follows from the direct computation: By (4.1.15), (4.1.17) and (4.1.22), we have, for any $g \in \mathrm{Gal}_k$ and $f' \in \mathfrak{F}_r'$,

$$\begin{aligned}(\nu_1 \circ \mathrm{Meta}_S(g))(f' \bmod \mathfrak{F}_r'') &= \nu_1(\mathrm{Ih}_S(g)(f') \bmod \mathfrak{F}_r'') \\ &= ((\Theta^{\mathrm{ab}} \circ \pi)\left(\frac{\partial \mathrm{Ih}_S(g)(f')}{\partial x_i}\right)) \\ &= ((\Theta^{\mathrm{ab}} \circ \pi)\left(\sum_{a=1}^{r} \frac{\partial \mathrm{Ih}_S(g)(f')}{\partial \mathrm{Ih}_S(g)(x_a)} \frac{\partial \mathrm{Ih}_S(g)(x_a)}{\partial x_i}\right)) \\ &= (\sum_{a=1}^{r} (\Theta^{\mathrm{ab}} \circ \pi)\left(\frac{\partial \mathrm{Ih}_S(g)(x_a)}{\partial x_i}\right)(\Theta^{\mathrm{ab}} \circ \pi \circ \mathrm{Ih}_S(g))\left(\frac{\partial f'}{\partial x_a}\right)) \\ &= \mathrm{Gass}_S(g)\chi_{\widehat{\Lambda}_r}(g)(\nu_1(f' \bmod \mathfrak{F}_r'')).\end{aligned}$$

When $\mathrm{Meta}_S$ and $\mathrm{Gass}_S$ are restricted to $\mathrm{Gal}_k[1]$, by the first assertion, we have the commutative diagram of $\widehat{\Lambda}_r$-modules for any $g \in \mathrm{Gal}_k[1]$:

$$\begin{array}{ccccc} & \mathfrak{L}_r & \overset{\nu_1}{\hookrightarrow} & \widehat{\Lambda}_r^{\oplus r} & \\ \mathrm{Meta}_S(g) & \downarrow & & \downarrow & \mathrm{Gass}_S(g) \\ & \mathfrak{L}_r & \overset{\nu_1}{\hookrightarrow} & \widehat{\Lambda}_r^{\oplus r}, & \end{array}$$

from which the latter assertion follows. □

Next, following Oda ([**O2**]), we introduce the pro-$l$ reduced Gassner cocycle associated to the Ihara representation $\mathrm{Ih}_S$. For this, we define a certain $\widehat{\Lambda}_r$-submodule $\mathfrak{L}_r^{\mathrm{prim}}$ of $\mathfrak{L}_r$, which Oda calls the primitive part of $\mathfrak{L}$, as follows. For $1 \leqslant i \leqslant r$, let $\mathfrak{N}_i$ be the closed subgroup generated normally by $x_i$ and let $\mathfrak{F}_r^{(i)} := \mathfrak{F}_r/\mathfrak{N}_i$. Let $\widehat{\Lambda}_r^{(i)} := \mathbb{Z}_l[[T_1, \ldots, \hat{T}_i, \ldots, T_r]] \simeq \mathbb{Z}_l[[\mathfrak{F}_r^{(i)}/(\mathfrak{F}_r^{(i)})']]$ ($\hat{T}_i$ means deleting $T_i$) with augmentation ideal $I_{\widehat{\Lambda}_r^{(i)}}$, and let $\delta_i : \widehat{\Lambda}_r \to \widehat{\Lambda}_r^{(i)}$ be the $\mathbb{Z}_l$-algebra homomorphism defined by $\delta_i(T_j) := T_j$ if $j \neq i$ and $\delta_i(T_i) := 0$. Note that any $\widehat{\Lambda}_r^{(i)}$-module is regarded as a $\widehat{\Lambda}_r$-module via $\delta_i$. Let $\mathfrak{L}_r^{(i)} := (\mathfrak{F}_r^{(i)})'/(\mathfrak{F}_r^{(i)})''$ and let $\xi_i : \mathfrak{L}_r \to \mathfrak{L}_r^{(i)}$ be the $\widehat{\Lambda}_r$-homomorphism induced by the natural homomorphism $\mathfrak{F}_r \to \mathfrak{F}_r^{(i)}$. Then the *primitive part* $\mathfrak{L}_r^{\mathrm{prim}}$ of $\mathfrak{L}_r$ is defined by

$$\mathfrak{L}_r^{\mathrm{prim}} := \bigcap_{i=1}^{r} \mathrm{Ker}(\xi_i). \tag{4.1.25}$$

We set $w := T_1 \cdots T_r$.

Theorem 4.1.26. *Notations being as above, the following assertions hold.*
(1) *The Blanchfield-Lyndon isomorphism* $\nu_1$ *in* (4.1.23) *restricted to* $\mathfrak{L}_r^{\mathrm{prim}}$ *induces the following isomorphism of* $\widehat{\Lambda}_r$*-modules*

$$\mathfrak{L}_r^{\mathrm{prim}} \simeq \{(\lambda_j \frac{w}{T_j}) \in \widehat{\Lambda}_r^{\oplus r} \mid \lambda_j \in \widehat{\Lambda}_r,\ \sum_{j=1}^{r} \lambda_j = 0\}.$$

*In particular,* $\mathfrak{L}_r^{\mathrm{prim}}$ *is the free* $\widehat{\Lambda}_r$*-module of rank* $r-1$ *on the basis*

$$\boldsymbol{v}_1 := {}^t(-\frac{w}{T_1}, \frac{w}{T_2}, 0, \ldots, 0), \ldots, \boldsymbol{v}_{r-1} := {}^t(0, \ldots, 0, -\frac{w}{T_{r-1}}, \frac{w}{T_r}).$$

(2) $\mathfrak{L}_r^{\mathrm{prim}}$ *is stable under the action of* $\mathrm{Gal}_k$ *through* $\mathrm{Meta}_S$ *and defines* 1*-cocycle*

$$\mathrm{Gass}_S^{\mathrm{red}} : \mathrm{Gal}_k \longrightarrow \mathrm{GL}_{r-1}(\widehat{\Lambda}_r)$$

*with respect to the basis* $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{r-1}$ *and the action* $\chi_{\widehat{\Lambda}_r}$ *in* (4.1.15).We call $\mathrm{Gass}_S^{\mathrm{red}}$ the *pro-l reduced Gassner cocycle* of $\mathrm{Gal}_k$ associated to $\mathrm{Ih}_S$.

PROOF. (1) (due to Oda) We define the $\widehat{\Lambda}_r$-homomorphism $\tilde{\xi}_i : \widehat{\Lambda}_r^{\oplus r} \to (\widehat{\Lambda}_r^{(i)})^{\oplus(r-1)}$ by

$$\tilde{\xi}_i({}^t(\lambda_1, \ldots, \widehat{\Lambda}_r)) := {}^t(\delta_i(\lambda_1), \ldots, \delta_i(\lambda_{i-1}), \delta_i(\lambda_{i+1}), \ldots, \delta_i(\widehat{\Lambda}_r)).$$

Then we have $\xi_i = \tilde{\xi}_i|_{\mathfrak{L}_r}$ for $1 \leqslant i \leqslant r$ and the commutative diagram of $\widehat{\Lambda}_r$-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{L}_r & \longrightarrow & \widehat{\Lambda}_r^{\oplus r} & \longrightarrow & I_{\widehat{\Lambda}_r} & \longrightarrow & 0 \\ & & \downarrow \xi_i & & \downarrow \tilde{\xi}_i & & \downarrow \delta_i & & \\ 0 & \longrightarrow & \mathfrak{L}_r^{(i)} & \longrightarrow & (\widehat{\Lambda}_r^{(i)})^{\oplus(r-1)} & \longrightarrow & I_{\widehat{\Lambda}_r^{(i)}} & \longrightarrow & 0, \end{array}$$

where two rows are the pro-$l$ Crowell exact sequences. It is easy to see that $\mathrm{Ker}(\tilde{\xi}_i)$ is given by

$$\mathrm{Ker}(\tilde{\xi}) = \{ {}^t(\lambda_1 T_i, \ldots, \lambda_{i-1} T_i, \lambda_i, \lambda_{i+1} T_i, \ldots, \widehat{\Lambda}_r T_i) \mid \lambda_j \in \widehat{\Lambda}_r \ (1 \leqslant j \leqslant r)\}$$

and hence, by (4.1.23) and (4.1.25), we have

$$\mathfrak{L}_r^{\mathrm{prim}} = \{(\lambda_j) \in \widehat{\Lambda}_r^{\oplus r} \mid \sum_{j=1}^r \lambda_j T_j = 0, \ \ \lambda_j \equiv 0 \bmod T_i \text{ if } i \neq j\}.$$

Since $\widehat{\Lambda}_r$ is a regular local ring, it is factorial. Therefore we have the first assertion

$$\mathfrak{L}_r^{\mathrm{prim}} = \{(\lambda_j) \in \widehat{\Lambda}_r^{\oplus r} \mid \sum_{j=1}^r \lambda_j T_j = 0, \ \ \lambda_j \equiv 0 \bmod \frac{w}{T_j} \ (1 \leqslant j \leqslant r)\}.$$

The assertion for a basis of $\mathfrak{L}_r^{\mathrm{prim}}$ is clear.
(2) Since $\mathrm{Ih}_S(g)(x_i)$ is conjugate to $x_i^{\chi_l(g)}$ for $g \in \mathrm{Gal}_k$ and $1 \leqslant i \leqslant r$, the definition (4.1.25) implies that $\mathfrak{L}_r^{\mathrm{prim}}$ is $\mathrm{Gal}_k$-stable under the action $\mathrm{Meta}_S$. So we may write, for $1 \leqslant j \leqslant r-1$,

$$\mathrm{Ih}_S(g)(\boldsymbol{v}_j) = \sum_{i=1}^{r-1} \mathrm{Gass}_S^{\mathrm{red}}(g)_{ij} \boldsymbol{v}_i, \tag{4.1.27}$$

where $\mathrm{Gass}_S^{\mathrm{red}}(g)_{ij} \in \widehat{\Lambda}_r$ is the $(i,j)$-entry of the representation matrix of $\mathrm{Ih}_S(g)$ with respect to $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{r-1}$. Then we have, for $g, h \in \mathrm{Gal}_k$,

$$\begin{aligned} \mathrm{Ih}_S(gh)(\boldsymbol{v}_j) &= \mathrm{Ih}_S(g)(\mathrm{Ih}_S(h)(\boldsymbol{v}_j)) \\ &= \mathrm{Ih}_S(g)\left(\sum_{i=1}^{r-1} \mathrm{Gass}_S^{\mathrm{red}}(h)_{ij} \boldsymbol{v}_i\right) \\ &= \sum_{i=1}^{r-1} \chi_{\widehat{\Lambda}_r}(\mathrm{Gass}_S^{\mathrm{red}}(h)_{ij}) \mathrm{Ih}_S(g)(\boldsymbol{v}_i) \ \ (\text{by } (4.1.15)) \\ &= \sum_{t=1}^{r-1} \left(\sum_{i=1}^{r-1} \mathrm{Gass}_S^{\mathrm{red}}(g)_{ti} \chi_{\widehat{\Lambda}_r}(g)(\mathrm{Gass}_S^{\mathrm{red}}(h)_{ij})\right) \boldsymbol{v}_t, \end{aligned}$$

which means the cocycle relation

$$\mathrm{Gass}_S^{\mathrm{red}}(gh) = \mathrm{Gass}_S^{\mathrm{red}}(g) \chi_{\widehat{\Lambda}_r}(g)(\mathrm{Gass}_S^{\mathrm{red}}(h)).$$

Hence the assertion is proved. □

When we restrict $\mathrm{Gass}_S^{\mathrm{red}}$ to $\mathrm{Gal}_k[1]$, we have a representation

$$\mathrm{Gass}_S^{\mathrm{red}} : \mathrm{Gal}_k[1] \longrightarrow \mathrm{GL}(r-1; \widehat{\Lambda}_r),$$

which we call the *pro-l reduced Gassner representation* of $\mathrm{Gal}[1]$ associated to $\mathrm{Ih}_S$.

Let $\Gamma$ be a free pro-$l$ group of rank 1 generated by $x$ so that $\mathbb{Z}_l[[\Gamma]]$ is identified with the Iwasawa algebra $\widehat{\Lambda} := \mathbb{Z}_l[[T]]$ ($x \leftrightarrow 1+T$). Let $\mathfrak{z} : \mathfrak{F}_r \to \Gamma$ be the homomorphism defined by $\mathfrak{z}(x_i) := x$ for $1 \leqslant i \leqslant r$. Let $\chi_{\widehat{\Lambda}}$ be the action of $\mathrm{Gal}_k$ on $\widehat{\Lambda}$ defined by $\chi_{\widehat{\Lambda}}(g)(u) := (1+T)^{\chi_l(g)} - 1$ for $g \in \mathrm{Gal}_k$ Then we have the pro-$l$ Magnus cocycle associated to $\mathrm{Ih}_S$ and $\mathfrak{z}$

$$\mathrm{Bur}_S : \mathrm{Gal}_k \longrightarrow \mathrm{GL}(r; \widehat{\Lambda}),$$

which we call the *pro-l Burau cocycle* of $\mathrm{Gal}_k$ associated to $\mathrm{Ih}_S$. It is the 1-cocycle of $\mathrm{Gal}_k$ with coefficients in $\mathrm{GL}(r; \widehat{\Lambda})$ with respect to the action $\chi_{\widehat{\Lambda}}$. By definition, we have

$$\mathrm{Bur}_S(g) = \mathrm{Gass}_S(g)|_{T_1 = \cdots = T_r = T}.$$

Similarly, we have the *pro-l reduced Burau cocycle* associated to $\mathrm{Ih}_S$

$$\mathrm{Bur}_S^{\mathrm{red}} : \mathrm{Gal}_k \longrightarrow \mathrm{GL}(r-1; \widehat{\Lambda})$$

defined by

$$\mathrm{Bur}_S^{\mathrm{red}}(g) := \mathrm{Gass}_S^{\mathrm{red}}(g)|_{T_1 = \cdots = T_r = T}.$$

Since $(\mathfrak{z} \circ \mathrm{Ih}_S(g))(x_i) = \mathfrak{z}(y_i(g) x_i y_i(g)^{-1}) = \mathfrak{z}(x_i)$ for $g \in \mathrm{Gal}_k[1]$, we have

$$\mathfrak{z} \circ \mathrm{Ih}_S(g) = \mathfrak{z} \quad (g \in \mathrm{Gal}_k[1]).$$

So, when we restrict $\mathrm{Bur}_S$ and $\mathrm{Bur}_S^{\mathrm{red}}$ to $\mathrm{Gal}_k[1]$, we have representations

$$\mathrm{Bur}_S : \mathrm{Gal}_k[1] \to \mathrm{GL}_r(\widehat{\Lambda}), \quad \mathrm{Bur}_S^{\mathrm{red}} : \mathrm{Gal}_k[1] \to \mathrm{GL}_{r-1}(\widehat{\Lambda}),$$

which are called the *pro-l Burau representation* and the *pro-l reduced Burau representation* of $\mathrm{Gal}_k[1]$ associated to $\mathrm{Ih}_S$, respectively.

### 4.2. $l$-adic Alexander invariants

**4.2.1. Pro-$l$ link modules.** Let $g \in \mathrm{Gal}_k$. As in (3.2.39), let $\Pi_S(g)$ be the pro-$l$ link group of $g$ associated to the Ihara representation $\varphi_S$:

$$\begin{aligned} \Pi_S(g) &= \langle x_1, \ldots, x_r \mid y_1(g) x_1^{\chi_l(g)} y_1(g)^{-1} x_1^{-1} = \cdots = y_r(g) x_r^{\chi_l(g)} y_r(g)^{-1} x_r^{-1} = 1 \rangle \\ &= \mathfrak{F}_r / \mathfrak{N}_S(g), \end{aligned}$$

where $\mathfrak{N}_S(g)$ is the closed subgroup of $\mathfrak{F}_r$ generated normally by the pro-$l$ words $y_1(g) x_1^{\chi_l(g)} y_1(g)^{-1} x_1^{-1}, \ldots, y_r(g) x_r^{\chi_l(g)} y_r(g)^{-1} x_r^{-1}$. Let $\psi : \mathfrak{F}_r \to \Pi_S(g)$ be the natural homomorphism and let $\gamma_i := \psi(x_i)$ ($1 \leqslant i \leqslant r$). Recall that $\mathfrak{a}(g)$ denotes the ideal of $\mathbb{Z}_l$ generated by $\chi_l(g) - 1$. Then we have

$$\Pi_S(g)/\Pi_S(g)' = \mathbb{Z}_l/\mathfrak{a}(g)[\gamma_1] \oplus \cdots \oplus \mathbb{Z}_l/\mathfrak{a}(g)[\gamma_r] \simeq (\mathbb{Z}_l/\mathfrak{a}(g))^{\oplus r},$$

where $[\gamma_i] := \gamma_i \bmod \Pi_S(g)'$ ($1 \leqslant i \leqslant r$). The correspondence $\gamma_i \mapsto T_i$ induces the $\mathbb{Z}_l$-algebra isomorphism

$$\Theta^{\mathrm{ab}}(g) : \mathbb{Z}_l[[\Pi_S(g)/\Pi_S(g)']] \simeq \widehat{\Lambda}_r/((1+T_1)^{\chi_l(g)-1} - 1, \ldots, (1+T_r)^{\chi_l(g)-1} - 1).$$

We denote the right hand side by $\widehat{\Lambda}_r(g)$:

$$\widehat{\Lambda}_r(g) := \widehat{\Lambda}_r/((1+T_1)^{\chi_l(g)-1} - 1, \ldots, (1+T_r)^{\chi_l(g)-1} - 1),$$

and by $I_{\widehat{\Lambda}_r(g)}$ the augmentation ideal of $\widehat{\Lambda}_r(g)$.

We define the *pro-l link module* $\mathfrak{L}_S(g)$ of $g$ associated to $\mathrm{Ih}_S$ by

$$\mathfrak{L}_S(g) := \Pi_S(g)'/\Pi_S(g)'',$$

which is considered as a $\widehat{\Lambda}_r(g) = \mathbb{Z}_l[[\Pi_S(g)/\Pi_S(g)']]$-module. It may be seen as an analogue of the classical link module in link theory (cf. [**Hi**], [**Ms2**, Chapter 9]).

Let $\varpi : \Pi_S(g) \to \Pi_S(g)/\Pi_S(g)'$ be the abelianization map. We define the *pro-l Alexander module* $\mathfrak{A}_S(g)$ of $g$ associated to $\mathrm{Ih}_S$ by the *pro-l differential module* associated to $\varpi$, namely the quotient module of the free $\widehat{\Lambda}_r(g)$-module on symbols $d\gamma$ for $\gamma \in \Pi_S(g)$ by the $\widehat{\Lambda}_r(g)$-submodule generated by $d(\gamma_1\gamma_2) - d\gamma_1 - \varpi(\gamma_1)d\gamma_2$ for $\gamma_1, \gamma_2 \in \Pi_S(g)$ ([**Ms2**, 9.3]):

$$\mathfrak{A}_S(g) := \bigoplus_{\gamma \in \Pi_S(g)} \widehat{\Lambda}_r(g) d\gamma / \langle d(\gamma_1\gamma_2) - d\gamma_1 - \varpi(\gamma_1)d\gamma_2 \ (\gamma_1, \gamma_2 \in \Pi_S(g))\rangle_{\widehat{\Lambda}_r(g)}.$$

We define the *l-adic Alexander matrix* $Q_S(g)$ by the Jacobian matrix of the relators of $\Pi_S(g)$:

$$Q_S(g) := \left( (\Theta^{\mathrm{ab}}(g) \circ \varpi \circ \psi) \left( \frac{\partial\, y_j(g) x_j^{\chi_l(g)} y_j(g)^{-1} x_j^{-1}}{\partial x_i} \right) \right). \tag{4.2.1}$$

Proposition 4.2.2. *Notations being as above, the following assertions hold.*
(1) *The correspondence* $d\gamma \mapsto ((\Theta^{\mathrm{ab}}(g) \circ \varpi \circ \psi) \left( \dfrac{\partial f}{\partial x_i} \right))$ *gives the isomorphism*

$$\mathfrak{A}_S(g) \xrightarrow{\sim} \mathrm{Coker}(Q_S(g) : \widehat{\Lambda}_r(g)^{\oplus r} \to \widehat{\Lambda}_r(g)^{\oplus r}),$$

*where $f$ is any element of $\mathfrak{F}_r$ such that $\gamma = \psi(f)$.*
(2) (Pro-$l$ Crowell exact sequence) *We have the following exact sequence of $\widehat{\Lambda}_r(g)$-modules:*

$$0 \longrightarrow \mathfrak{L}_S(g) \xrightarrow{\nu_1} \mathfrak{A}_S(g) \xrightarrow{\nu_2} I_{\widehat{\Lambda}_r(g)} \longrightarrow 0,$$

*where $\nu_1, \nu_2$ are given by*

$$\nu_1(\gamma' \ mod\, \Pi_S(g)'') := d\gamma \ (\gamma' \in \Pi_S(g)'); \ \ \nu_2(d\gamma) := (\Theta^{\mathrm{ab}}(g) \circ \varpi)(\gamma) - 1 \ (\gamma \in \Pi_S(g)).$$

Proof. We refer to [**Ms2**, Theorems 9.3.6, 9.4.2]. □

Let $\phi_g : \widehat{\Lambda}_r \to \widehat{\Lambda}_r(g)$ be the natural $\mathbb{Z}_l$-algebra homomorphism.

Proposition 4.2.3. *We have*

$$Q_S(g) = \phi_g(\mathrm{Gass}_S(g) - I)$$

*and its $(i, j)$-entry is given by*

$$Q_S(g)_{ij} = \begin{cases} \phi_g \left( \displaystyle\sum_{n \geqslant 1} \sum_{\substack{1 \leqslant i_1, \ldots, i_n \leqslant r \\ i_n \neq i}} \mu(g; i_1 \cdots i_n i) T_{i_1} \cdots T_{i_n} \right) & (i = j), \\ \phi_g \left( -T_j \left( \mu(g; ij) + \displaystyle\sum_{n \geqslant 1} \sum_{1 \leqslant i_1, \ldots, i_n \leqslant r} \mu(g; i_1 \cdots i_n ii) T_{i_1} \cdots T_{i_n} \right) \right) & (i \neq j). \end{cases}$$

Proof. By the definition (4.2.1), we have

$$Q_S(g)_{ij} := (\Theta^{\mathrm{ab}}(g) \circ \varpi \circ \psi) \left( \frac{\partial\, y_j(g) x_j^{\chi_l(g)} y_j(g)^{-1} x_j^{-1}}{\partial x_i} \right).$$

By the basic rules of pro-$l$ Fox free derivatives, we have

$$\frac{\partial\, y_j(g) x_j^{\chi_l(g)} y_j(g)^{-1} x_j^{-1}}{\partial x_i} = \frac{\partial\, y_j(g) x_j^{\chi_l(g)} y_j(g)^{-1}}{\partial x_i} - \delta_{ij} y_j(g) x_j^{\chi_l(g)} y_j(g)^{-1} x_j^{-1}.$$

By (4.1.17) and $\Theta^{\mathrm{ab}}(g) \circ \varpi \circ \psi = \phi_g \circ \Theta^{\mathrm{ab}} \circ \pi$, we have

$$(\Theta^{\mathrm{ab}}(g) \circ \varpi \circ \psi) \left( \frac{\partial\, y_j(g) x_j^{\chi_l(g)} y_j(g)^{-1}}{\partial x_i} \right) = \phi_g(\mathrm{Gass}_S(g)_{ij}),$$

and we also have

$$(\Theta^{\mathrm{ab}}(g) \circ \varpi \circ \psi)(y_j(g) x_j^{\chi_l(g)} y_j(g)^{-1} x_j^{-1}) = \Theta^{\mathrm{ab}}(g)(\gamma_j^{\chi_l(g)-1}) = (1+T_j)^{\chi_l(g)-1} = 1.$$

Therefore we have

$$Q_S(g)_{ij} = \phi_g(\mathrm{Gass}_S(g)_{ij} - \delta_{ij}).$$

The second assetion follows from Proposition 4.1.18 and

$$\phi_g(\chi_{\widehat{\Lambda}_r}(g)(T_j)) = \phi_g((1+T_j)^{\chi_l(g)} - 1) = \phi_g(T_j).$$

□

Corollary 4.2.4. *For $g, h \in \mathrm{Gal}_k[1]$, we have the following isomorphisms of $\widehat{\Lambda}_r$-modules*

$$\mathfrak{A}_S(hgh^{-1}) \simeq \mathfrak{A}_S(g), \quad \mathfrak{L}_S(hgh^{-1}) \simeq \mathfrak{L}_S(g).$$

Proof. Since $\mathrm{Gass}_S : \mathrm{Gal}_k \to \mathrm{GL}(r; \widehat{\Lambda}_r)$ is a representation, we have

$$Q_S(hgh^{-1}) = \phi_g(\mathrm{Gass}_S(hgh^{-1}) - I) = \phi_g(\mathrm{Gass}_S(h)) Q_S(g) \phi_g(\mathrm{Gass}_S(h))^{-1}$$

by Proposition 4.2.3. Then the first assertion follows from Proposition 4.2.2 (1). The second assertion follows from Proposition 4.2.2 (2). □

**4.2.2. $l$-adic Alexander invariants.** For $n \geqslant 0$, we define the *$n$-th $l$-adic Alexander ideal* $\mathfrak{E}_S(g)^{(n)}$ of $g \in \mathrm{Gal}_k$ associated to $\mathrm{Ih}_S$ by the $n$-th Fitting ideal of the pro-$l$ Alexander module $\mathfrak{A}_S(g)$ over $\widehat{\Lambda}_r(g)$. The *$n$-th $l$-adic Alexander invariant* $A_S(g)^{(n)}$ is then defined by a generator of the divisorial hull of $\mathfrak{E}_S(g)^{(n)}$. By Proposition 4.2.2 (1), $\mathfrak{E}_S(g)^{(n)}$ is the ideal generated by all $(r-n)$-minors of $Q_S(g)$ if $r - n > 0$ and $\mathfrak{E}_S(g)^{(n)} := \widehat{\Lambda}_r(g)$ if $r - n \leqslant 0$, and $A_S(g)^{(n)}$ is the greatest common divisor of all $(r-n)$-minors of $Q_S(g)$) if $r - n > 0$ and $A_S(g)^{(n)} := 1$ if $r - n \geqslant 0$:

$$A_S(g)^{(n)} := \begin{cases} \text{g.c.d of all } (r-n)\text{-minors of } Q_S(g) & (r-n > 0), \\ 1 & (r - n \geqslant 0). \end{cases}$$

We note that $A_S(g)^{(n)}$ is defined up to multiplication of a unit of $\widehat{\Lambda}_r(g)$. We write $\mathfrak{E}_S(g)$ (resp. $A_S(g)$) for $\mathfrak{E}_S(g)^{(0)}$ (resp. $A_S(g)^{(0)}$) and call $\mathfrak{E}_S(g)$ (resp. $A_S(g)$) the $l$-adic Alexander ideal (resp. $l$-adic Alexander invariant) of $g$ associated to $\mathrm{Ih}_S$. From Proposition 4.2.3, the following proposition is immediate.

Proposition 4.2.5. *For $g \in \mathrm{Gal}_k$, we have*

$$A_S(g) = \phi_g(\det(\mathrm{Gass}_S(g) - I)).$$

*When $g \in \mathrm{Gal}_k[1]$, $A_S(g) = 0$ if and only if $\mathrm{Gass}_S(g)$ has the eigenvalue 1.*

Moreover, since the $l$-adic Alexander matrix $Q_S(g)$ is described by $l$-adic Milnor numbers as in Proposition 4.2.3, $n$-th $l$-adic Alexander invariants are also described by $l$-adic Milnor numbers (cf. [**Ms2**, Chapter 10], [**Mu**]).

## 4.3. The Ihara power series

In this section, we suppose that $S = \{0, 1, \infty\}$ and so $k = \mathbb{Q}$. In the following, we will omit $S$ in the notations. The Ihara representation in this case is

$$\mathrm{Ih} : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow P(\mathfrak{F}_2),$$

which factors through the Galois group $\mathrm{Gal}(\Omega_l/\mathbb{Q})$ by Theorem 3.1.9 (2), where $\Omega_l$ denotes the maximal pro-$l$ extension of $\mathbb{Q}[1] = \mathbb{Q}(\zeta_{l^\infty})$ unramified outside $l$.

**4.3.1. The Ihara power series.** The following lemma is a restatement of [**Ih1**, Theorem 2 (i)]. Our proof is different from Ihara's.

LEMMA 4.3.1. *We have* $\mathfrak{L}_2 = \mathfrak{L}_2^{\mathrm{prim}}$ *with basis* ${}^t(-T_2, T_1)$ *over* $\widehat{\Lambda}_2$, *and* ${}^t(-T_2, T_1) = \nu_1([x_1, x_2])$.

PROOF. By Theorem 4.1.26 (1), $\mathfrak{L}_2^{\mathrm{prim}}$ is the free $\widehat{\Lambda}_2$-module with basis ${}^t(-T_2, T_1)$. On the other hand, we note that $\lambda_1 T_1 + \lambda_2 T_2 = 0$ implies $\lambda_1 = -aT_2, \lambda_2 = aT_1$ for some $a \in \widehat{\Lambda}_2$, because $\widehat{\Lambda}_2$ is U.F.D. Therefore $\mathfrak{L}_2$ is also the free $\widehat{\Lambda}_2$-module with basis ${}^t(-T_2, T_1)$ by (4.1.23). Hence $\mathfrak{L}_2 = \mathfrak{L}_2^{\mathrm{prim}}$. The second assertion follows from

$$(\Theta^{\mathrm{ab}} \circ \pi)\left(\frac{\partial [x_1, x_2]}{\partial x_1}\right) = -T_2, \ (\Theta^{\mathrm{ab}} \circ \pi)\left(\frac{\partial [x_1, x_2]}{\partial x_2}\right) = T_1.$$

□

Thanks to Lemma 4.3.1, Ihara introduced a power series $F_g(T_1, T_2) \in \widehat{\Lambda}_2$, called the *Ihara power series*, by the follwoing equality in $\mathfrak{L}_2$

$$\mathrm{Ih}_S(g)([x_1, x_2]) \equiv F_g(T_1, T_2)[x_1, x_2] \mod \mathfrak{F}_2''. \tag{4.3.2}$$

The following theorem gives an arithmetic topological interpretation of $F_g(T_1, T_2)$. For a multi-index $I = (i_1 \cdots i_n)$ with $i_j = 1$ or 2, we denote by $|I|_1$ (resp. $|I|_2$) the number of $j$'s ($1 \leqslant j \leqslant n$) such that $i_j = 1$ (resp. $i_j = 2$). For integers $n_1, n_2 \geqslant 0$ with $n_1 + n_2 \geqslant 1$ and $g \in \mathrm{Gal}_{\mathbb{Q}}$, we let

$$\mu(g; n_1, n_2) := \sum_{|I|_1 = n_1 - 1, |I|_2 = n_2} \mu(g; I12) + \sum_{|I|_1 = n_1, |I|_2 = n_2 - 1} \mu(g; I21).$$

We recall the pro-$l$ Gassner and the pro-$l$ reduced Gassner cocycles in (4.1.17) and (4.1.27):

$$\mathrm{Gass} : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow \mathrm{GL}(2; \widehat{\Lambda}_2); \ \ \mathrm{Gass}^{\mathrm{red}} : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow \widehat{\Lambda}_2^{\times}.$$

THEOREM 4.3.3. *Notations being as above, we have, for* $g \in \mathrm{Gal}_{\mathbb{Q}}$,

$$\begin{aligned} F_g(T_1,T_2) &= \mathrm{Gass}^{\mathrm{red}}(g) \\ &= \frac{\chi_{\widehat{\Lambda}_2}(g)(T_1T_2)}{T_1T_2}\left(1+\sum_{n\geqslant 1}\sum_{\substack{1\leqslant i_1,\dots,i_n\leqslant 2\\ i_n\neq i_{n+1}}}\mu(g;i_1\cdots i_n i_{n+1})T_{i_1}\cdots T_{i_n}\right) \\ &= \frac{\chi_{\widehat{\Lambda}_2}(g)(T_1T_2)}{T_1T_2}\left(1+\sum_{\substack{n_1,n_2\geqslant 0\\ n_1+n_1\geqslant 1}}\mu(g;n_1,n_2)T_1^{n_1}T_2^{n_2}\right). \end{aligned}$$

PROOF. Applying the $\widehat{\Lambda}_2$-homomorphism $\nu_1$ to (4.3.2), we have, for $g \in \mathrm{Gal}_k$,

$$\nu_1(\mathrm{Ih}(g)([x_1,x_2])) = F_g(T_1,T_2)\nu_1([x_1,x_2]) = F_g(T_1,T_2)\begin{pmatrix} -T_2 \\ T_1 \end{pmatrix}.$$

On the other hand, by the definition of $\mathrm{Gass}_S^{\mathrm{red}}(g)$ (cf. (4.1.27)), we have

$$\nu_1(\mathrm{Ih}(g)([x_1,x_2])) = \mathrm{Gass}^{\mathrm{red}}(g)\begin{pmatrix} -T_2 \\ T_1 \end{pmatrix}.$$

Hence we have

$$F_g(T_1,T_2) = \mathrm{Gass}^{\mathrm{red}}(g).$$

By Proposition 4.1.24 and Lemma 4.3.1, we have

$$\begin{aligned} \nu_1(\mathrm{Ih}(g)([x_1,x_2])) &= \mathrm{Gass}(g)\chi_{\widehat{\Lambda}_2}(g)(\nu_1([x_1,x_2])) \\ &= \mathrm{Gass}(g)\begin{pmatrix} -\chi_{\widehat{\Lambda}_2}(g)(T_2) \\ \chi_{\widehat{\Lambda}_2}(g)(T_1) \end{pmatrix}. \end{aligned}$$

A straightforward calculation using Proposition 4.1.18 yields

$$\begin{aligned} &\mathrm{Gass}(g)\begin{pmatrix} -\chi_{\widehat{\Lambda}_2}(g)(T_2) \\ \chi_{\widehat{\Lambda}_2}(g)(T_1) \end{pmatrix} \\ &= \frac{\chi_{\widehat{\Lambda}_2}(g)(T_1T_2)}{T_1T_2}\left(1+\sum_{n\geqslant 1}\sum_{\substack{1\leqslant i_1,\dots,i_n\leqslant 2\\ i_n\neq i_{n+1}}}\mu(g;i_1\cdots i_n i_{n+1})T_{i_1}\cdots T_{i_n}\right)\begin{pmatrix} -T_2 \\ T_1 \end{pmatrix} \\ &= \frac{\chi_{\widehat{\Lambda}_2}(g)(T_1T_2)}{T_1T_2}\left(1+\sum_{\substack{n_1,n_2\geqslant 0\\ n_1+n_1\geqslant 1}}\mu(g;n_1,n_2)T_1^{n_1}T_2^{n_2}\right)\begin{pmatrix} -T_2 \\ T_1 \end{pmatrix}. \end{aligned}$$

Getting these together, we obtain the assertion. □

Ihara also interpret $\mathfrak{L}_2$ in terms of Fermat Jacobians. For a positive integer $n$, let $C_n$ be the non-singular, projective curve over $\mathbb{Q}$ defined by

$$X^{l^n} + Y^{l^n} = Z^{l^n}$$

and let $\mathrm{Jac}_n$ be the Jacobian variety of $C_n$. Let $\mathrm{T}(\mathrm{Jac}_n)$ be the $l$-adic Tate module of $\mathrm{Jac}_n$:

$$\mathrm{T}(\mathrm{Jac}_n) := \mathrm{Hom}(\mathbb{Q}_l/\mathbb{Z}_l, \mathrm{Jac}_n(\overline{\mathbb{Q}})) \simeq H_1^{\mathrm{sing}}(C_n(\mathbb{C}),\mathbb{Z}) \otimes \mathbb{Z}_l,$$

and let

$$\mathbb{T} := \varprojlim_{n} \mathrm{T}(\mathrm{Jac}_n),$$

where the inverse limit is taken with respect to the maps $\mathrm{T}(\mathrm{Jac}_{n+1}) \to \mathrm{T}(\mathrm{Jac}_n)$ induced by the morphisms $C_{n+1} \to C_n$; $(X,Y,Z) \mapsto (X^l, Y^l, Z^l)$. Let $g_{X,n}, g_{Y,n}$ be the automorphisms of $\overline{C_n} := C_n \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\overline{\mathbb{Q}}$ over $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ defined by

$$g_{X,n} : (X,Y,Z) \mapsto (\zeta_{l^n} X, Y, Z), \quad g_{Y,n} : (X,Y,Z) \mapsto (X, \zeta_{l^n} Y, Z)$$

and set $g_X := \varprojlim g_{X,n}, g_Y := \varprojlim g_{Y,n}$. Then $\mathrm{Gal}(\overline{C_n}/\mathbb{P}^1_{\overline{\mathbb{Q}}}) = (\mathbb{Z}/l^n\mathbb{Z}_l) g_{X,n} \oplus (\mathbb{Z}/l^n\mathbb{Z}) g_{Y,n}$ and so $\varprojlim \mathbb{Z}_l[\mathrm{Gal}(\overline{C_n}/\mathbb{P}^1_{\overline{\mathbb{Q}}})] \simeq \widehat{\Lambda}_2$ by the correspondence $g_X \mapsto 1 + T_1, g_Y \mapsto 1 + T_2$. Thus $\mathbb{T}$ is regarded as a $\widehat{\Lambda}_2$-module. Then we have the isomorphism of $\widehat{\Lambda}_2$-modules

$$\mathfrak{L}_2 \simeq \mathbb{T}.$$

For an explicit construction of the basis of $\mathbb{T}$ corresponding to $[x_1, x_2]$, we consult [**Ae**, §13].

Now, the main results in [**Ih1**] are arithmetic descriptions of
- values of $F_g(T_1, T_2)$ at $l$-powerth roots of unity in terms of the Jacobi sums which arise from the Galois action on $\mathrm{T}(\mathrm{Jac}_n)$, and
- coefficients of $F_g(T_1, T_2)$ in terms of $l$-adic Soulé cocycles which are defined by the Galois action on higher cyclotomic $l$-units.

We will describe these, using Theorem 4.3.3, from the view point of arithmetic topology.

**4.3.2. Values of the Ihara power series.** Let $p$ be a rational prime which is in $R_S$ of (3.1.8) and let $\overline{p}$ be a prime of $\overline{\mathbb{Q}}$ lying over $p$. By Theorem 3.1.9 (2), $\overline{p}$ is unramified in $\overline{\mathbb{Q}}/\mathbb{Q}$ and so we have the Frobenius automorphism $\sigma_{\overline{p}} \in \mathrm{Gal}_{\mathbb{Q}}$. Let $n$ be a fixed positive integer. Let $\mathfrak{p}_n$ be the prime of $\mathbb{Q}(\zeta_{l^n})$ lying below $\overline{p}$ and let $\left(\dfrac{x}{\mathfrak{p}_n}\right)_{l^n}$ denote the $l^n$-th power residue symbol at $\mathfrak{p}_n$ for $x \in (\mathbb{Z}[\zeta_{l^n}]/\mathfrak{p}_n)^\times$. For $a, b \in \mathbb{Z}/l^n\mathbb{Z} \setminus \{0\}$ with $(a,b,l) = 1$, we define the *Jacobi sum* by

$$J_{l^n}(\mathfrak{p}_n)^{(a,b)} = \sum_{\substack{x,y \in (\mathbb{Z}[\zeta_{l^n}]/\mathfrak{p}_n)^\times \\ x+y=-1}} \left(\frac{x}{\mathfrak{p}_n}\right)_{l^n}^a \left(\frac{y}{\mathfrak{p}_n}\right)_{l^n}^b.$$

For $l = 2$, $J_{l^n}(\mathfrak{p}_n)^{(a,b)}$ must be multiplied by $\left(\dfrac{-1}{\mathfrak{p}_n}\right)^a$. Let $f$ be the order of $p$ in $(\mathbb{Z}/l^n\mathbb{Z})^\times$. We note that $\sigma_{\overline{p}}^f \in \mathrm{Gal}_{\mathbb{Q}(\zeta_{l^n})}$. By using Weil's theorem, Ihara showed the following

THEOREM 4.3.4 ([**Ih1**, Theorem 7]). *Let $a, b \in \mathbb{Z}/l^n\mathbb{Z} \setminus \{0\}$ such that $a + b \neq 0$ and $(a, b, a+b, l) = 1$. Then we have*

$$F_{\sigma_{\overline{p}}^f}(\zeta_{l^n}^a - 1, \zeta_{l^n}^b - 1) = J_{l^n}(\mathfrak{p}_n)^{(a,b)}.$$

Combining Theorem 4.3.3 and Theorem 4.3.4, we obtain the following $l$-adic expansion of the Jacobi sum $J_{l^n}(\mathfrak{p}_n)^{(a,b)}$ with coefficients $l$-adic Milnor numbers.

THEOREM 4.3.5. Notations being as above, we have

$$J_{l^n}(\mathfrak{p}_n)^{(a,b)} = 1 + \sum_{\substack{n_1,n_2\geqslant 0\\ n_1+n_2\geqslant 1}} \mu(\sigma_{\tilde{p}}^f; n_1, n_2)(\zeta_{l^n}^a - 1)^{n_1}(\zeta_{l^n}^b - 1)^{n_2}.$$

PROOF. Since we have $\zeta_{l^n}^{\chi_l(\sigma_{\tilde{p}}^f)} = \zeta_{l^n}^{p^f} = \zeta_{l^n}$ by $p^f \equiv 1 \bmod l^n$, the formula follows from Theorem 4.3.3 and Theorem 4.3.4. □

**4.3.3. Coefficients of the Ihara power series.** We will combine Theorem 4.3.3 with the result of Ihara, Kaneko and Yukinari on the Ihara power series ([**IKY**]) and deduce some formulas relating our $l$-adic Milnor numbers with the Soulé cocycles ([**So**]). As in Section 3.1.3, let $\zeta_{l^n}$ be a primitive $l^n$-th root of unity for a positive integer $n$ such that $(\zeta_{l^{n+1}})^l = \zeta_{l^n}$ for $n \geqslant 1$. For $a \in \mathbb{Z}/l^n\mathbb{Z}$, let $\langle a\rangle_{l^n}$ denote the integer such that $0 \leqslant \langle a\rangle_{l^n} < l^n$ and $a = \langle a\rangle_{l^n} \bmod l^n$. For a positive interger $m$, we let

$$\varepsilon_{l^n}^{(m)} := \prod_{a\in(\mathbb{Z}/l^n\mathbb{Z})^\times} (\zeta_{l^n} - 1)^{\langle a^{m-1}\rangle_{l^n}},$$

which is an $l$-unit in $\mathbb{Q}(\zeta_{l^n})$, called a *cyclotomic $l$-unit*. Then we define the *$m$-th $l$-adic Soulé cocycle* $\chi^{(m)} : \mathrm{Gal}_{\mathbb{Q}} \to \mathbb{Z}_l$ by the Kummer cocycle attached to the system of cyclotomic $l$-units $\{\varepsilon_{l^n}^{(m)}\}_{n\geqslant 1}$

$$\zeta_{l^n}^{\chi^{(m)}(g)} = \{(\varepsilon_{l^n}^{(m)})^{1/l^n}\}^{g-1} \ \ (n \geqslant 1, g \in \mathrm{Gal}_{\mathbb{Q}}).$$

It is easy to see the cocycle relation

$$\chi^{(m)}(gh) = \chi^{(m)}(g) + \chi_l(g)\chi^{(m)}(h) \quad (g, h \in \mathrm{Gal}_{\mathbb{Q}})$$

and hence the restriction of $\chi^{(m)}|_{\mathrm{Gal}_{\mathbb{Q}[1]}}$ is a character. Let $\Omega_l^{\mathrm{ab}}$ be the maximal abelian subextension of $\Omega_l/\mathbb{Q}[1]$. Since $\mathbb{Q}(\zeta_{l^n}, (\varepsilon_{l^n}^{(m)})^{1/l^n})$ is a cyclic extension of $\mathbb{Q}(\zeta_{l^n})$ unramified outside $l$, we have $(\varepsilon_{l^n}^{(m)})^{1/l^n} \in \Omega_l^{\mathrm{ab}}$ and so the Soulé character $\chi^{(m)}|_{\mathrm{Gal}_{\mathbb{Q}[1]}}$ factors through the Galois group $\mathrm{Gal}(\Omega_l^{\mathrm{ab}}/\mathbb{Q}[1])$. We note by Theorem 4.1.12 (3) that the pro-$l$ reduced Gassner representation $\mathrm{Gass}^{\mathrm{red}}$ also factors through $\mathrm{Gal}(\Omega_l^{\mathrm{ab}}/\mathbb{Q}[1])$.

We set

$$\kappa_m(g) := \frac{\chi^{(m)}(g)}{1 - l^{m-1}}, \quad (g \in \mathrm{Gal}_{\mathbb{Q}}),$$

and introduce new variables $U_1, U_2$ defined by

$$1 + T_i = \exp(U_i) = \sum_{n=0}^{\infty} \frac{U_i^n}{n!} \ \in \mathbb{Q}_l[[U_i]] \quad (i = 1, 2)$$

and set

$$\mathcal{F}_g(U_1, U_2) := F_g(T_1, T_2)|_{T_i=\exp(U_i)-1}.$$

THEOREM 4.3.6 ([**IKY**, Theorem A$_2$]). *Notations being as above, we have, for $g \in \mathrm{Gal}(\Omega_l^{\mathrm{ab}}/\mathbb{Q}[1])$,*

$$\mathcal{F}_g(U_1, U_2) = \exp\left\{ - \sum_{\substack{m\geqslant 3\\ \text{odd}}} \kappa_m(g) \left( \sum_{\substack{m_1,m_2\geqslant 1\\ m_1+m_2=m}} \frac{U_1^{m_1}U_2^{m_2}}{m_1!\, m_2!} \right) \right\}.$$

Combining Theorem 4.3.3 and Theorem 4.3.6, we can deduce relations between $l$-adic Milnor numbers and $l$-adic Soulé characters. For this, we prepare the following.

Lemma 4.3.7. *Let $a(n_1,n_2)$ and $c(m_1,m_2)$ be given $l$-adic numbers for integers $m_1,m_2,n_1,n_2 \geqslant 0$ with $m_1+m_2, n_1+n_2 \geqslant 1$. Let*

$$A(T_1,T_2) := 1 + \sum_{\substack{n_1,n_2\geqslant 0\\ n_1+n_2\geqslant 1}} a(n_1,n_2)u_1^{n_1}u_2^{n_2} \quad \in \mathbb{Q}_l[[T_1,T_2]]$$

*and set*

$$\begin{aligned} B(U_1,U_2) &:= A(T_1,T_2)|_{T_i=\exp(U_i)-1} \\ &= 1 + \sum_{\substack{N_1,N_2\geqslant 0\\ N_1+N_2\geqslant 1}} b(N_1,N_2)U_1^{N_1}U_2^{N_2} \quad \in \mathbb{Q}_l[[U_1,U_2]]. \end{aligned}$$

*Then we have*

$$b(N_1,N_2) = \sum_{\substack{n_1+n_2\geqslant 1\\ 0\leqslant n_1\leqslant N_1, 0\leqslant n_2\leqslant N_2}} a(n_1,n_2)a_{n_1}(N_1)a_{n_2}(N_2),$$

*where for $j=1,2$,*

$$a_{n_j}(N_j) := \begin{cases} 1 & (n_j=0), \\ \displaystyle\sum_{\substack{e_1,\ldots,e_{n_j}\geqslant 1\\ e_1+\cdots+e_{n_j}=N_j}} \frac{1}{e_1!\cdots e_{n_j}!} & (n_j\geqslant 1). \end{cases}$$

*Let*

$$C(U_1,U_2) := \sum_{\substack{m_1,m_2\geqslant 0\\ m_1+m_2\geqslant 1}} c(m_1,m_2)U_1^{m_1}U_2^{m_2} \quad \in \mathbb{Q}_l[[U_1,U_2]]$$

*and set*

$$\begin{aligned} D(U_1,U_2) &:= \exp(C(U_1,U_2)) \\ &= 1 + \sum_{\substack{N_1,N_2\geqslant 0\\ N_1+N_2\geqslant 1}} d(N_1,N_2)U_1^{N_1}U_2^{N_2} \quad \in \mathbb{Q}_l[[U_1,U_2]]. \end{aligned}$$

*Then we have*

$$d(N_1,N_2) = \sum_{1\leqslant n\leqslant N_1+N_2} \frac{1}{n!}\sum c(m_1^{(1)},m_2^{(1)})\cdots c(m_1^{(n)},m_2^{(n)}),$$

*where the second sum ranges over integers $m_1^{(1)},\ldots,m_1^{(n)},m_2^{(1)},\ldots,m_2^{(n)} \geqslant 0$ satisfying $m_1^{(i)}+m_2^{(i)}\geqslant 1$ $(1\leqslant i\leqslant n)$, $m_1^{(1)}+\cdots+m_1^{(n)}=N_1$ and $m_2^{(1)}+\cdots+m_2^{(n)}=N_2$.*

Proof. Both formulas for $b(N_1,N_2)$ and $d(N_1,N_2)$ follow from straightforward computations. □

We apply Lemma 4.3.7 to the case that $A(u_1,u_2)=\mathrm{Gass}^{\mathrm{red}}(g)$, where

$$a(n_1,n_2)=\mu(g;n_1,n_2)$$

and $C(U_1,U_2)=\log(\mathcal{F}_g(U_1,U_2))$, where

$$c(m_1,m_2) = \begin{cases} -\dfrac{\kappa_{m_1+m_2}(g)}{m_1!\,m_2!} & (m_1+m_2\geqslant 3,\ \text{odd}), \\ 0 & \text{otherwise}. \end{cases}$$

Then, by comparing coefficients of $U_1^{N_1}U_2^{N_2}$ in $\mathrm{Gass}^{\mathrm{red}}(g)|_{T_i=\exp(U_i)-1} = \mathcal{F}_g(U_1, U_2)$, we obtain the following.

THEOREM 4.3.8. *Notations being as above, we have the following equality for* $g \in \mathrm{Gal}_{\mathbb{Q}}[1]$*:*

$$\sum_{\substack{n_1+n_2\geqslant 1\\ 0\leqslant n_1\leqslant N_1, 0\leqslant n_2\leqslant N_2}} \mu(g; n_1, n_2)a_{n_1}(N_1)a_{n_2}(N_2) = \sum_{1\leqslant n\leqslant N_1+N_2} \frac{(-1)^n}{n!} \sum \frac{\kappa_{m_1^{(1)}+m_2^{(1)}}(g)}{m_1^{(1)}!\, m_2^{(1)}!} \cdots \frac{\kappa_{m_1^{(n)}+m_2^{(n)}}(g)}{m_1^{(n)}!\, m_2^{(n)}!},$$

*where the last sum ranges over integers* $m_1^{(1)}, \dots, m_1^{(n)}, m_2^{(1)}, \dots, m_2^{(n)} \geqslant 0$ *satisfying* $m_1^{(i)}+m_2^{(i)} \geqslant 3$; *odd* $(1 \leqslant i \leqslant n)$, $m_1^{(1)}+\cdots+m_1^{(n)} = N_1$ *and* $m_2^{(1)}+\cdots+m_2^{(n)} = N_2$.

For example, lower terms are given by

$$\mu(g; (12)) = \mu(g; (21)) = 0,\ \mu(g; (212)) + \mu(g; (121)) = 0,$$
$$\mu(g; (221)) + \mu(g; (2212)) + \mu(g; (1221)) + \mu(g; (2121)) = -\frac{\kappa_3(g)}{2},$$
$$\mu(g; (112)) + \mu(g; (1121)) + \mu(g; (2112)) + \mu(g; (1212)) = -\frac{\kappa_3(g)}{2}.$$

APPENDIX A

# On definitions of reduced Gassner representations

In this appendix, we prove the equivalence of the two definitions of reduced Gassner representations: the homological one given in this article and the original one as in [**Bi1**].

To begin with, let us recall the original reduced Gassner representation

$$\mathrm{Gass}_r^{\mathrm{O,red}} : PB_r \longrightarrow \mathrm{GL}(r-1;\Lambda_r)$$

where $r$ is a positive integer with $r \geqslant 2$, $PB_r$ denotes the pure braid group with $r$ strings and $\Lambda_r$ denotes the ring of Laurent polynomials $\mathbb{Z}[t_1^{\pm}, \ldots, t_r^{\pm}]$ over $\mathbb{Z}$ with indeterminate $t_1, \ldots, t_r$.

Take a basis set $\mathbf{g} = \{g_1, \ldots, g_r\}$ of $F_r$ where $g_i := x_1 \cdots x_i$. By Proposition 1.1.2, $PB_r$ acts trivially on $g_r = x_1 \cdots x_r$. Hence the $r$-th column of the Gassner representation with respect to the basis $\mathbf{g} = \{g_1, \ldots, g_r\}$

$${}^t\left(\mathrm{ab}\left(\frac{\partial b(g_i)}{\partial g_j}\right)\right)$$

can be written as ${}^t(0, \ldots, 0, 1)$. Hence, the Gassner representation of $PB_r$ is reducible to an $r-1$ dimensional representation. The representation obtained from the Gassner representation with respect to $\mathbf{g} = \{g_1, \ldots, g_r\}$ by eliminating the $r$-th column and $r$-th row is called the *original reduced Gassner representation* and is denoted by $\mathrm{Gass}_r^{\mathrm{O,red}}$. The original reduced Gassner representation can also be obtained from the conjugate $C^{-1}\mathrm{Gass}_r(b)C$ by eliminating the $r$-th column and row, where the matrix $C$ is given by

$$C = {}^t\left(\mathrm{ab}\left(\frac{\partial g_i}{\partial x_j}\right)\right) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & t_1 & t_1 & \cdots & t_1 \\ 0 & 0 & t_1 t_2 & \cdots & t_1 t_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t_1 \cdots t_{r-1} \end{pmatrix}.$$

By direct computation, we have

$$A_{kl}(g_m) = \begin{cases} g_m g_{l-1}^{-1} g_l g_{k-1}^{-1} g_k g_l^{-1} g_{l-1} g_k^{-1} g_{k-1} & (k \leqslant m < l) \\ g_m & (\text{otherwise}) \end{cases}$$

where we understand that $g_0 = 1$. Then, we have the following.

Proposition A.1. *For each generator* $A_{ij} \in PB_r$*, we have*

$$\left(\mathrm{ab}\left(\frac{\partial A_{kl}(g_i)}{\partial g_j}\right)\right) = \begin{cases} t_k t_{k+1} & (k = i = j, l = k+1) \\ 1 - t_k & (k = i, l = k+1 = j) \\ t_k \cdots t_m (1 - t_l) & (k \leqslant i < l, j = k-1) \\ t_k(t_l - 1) + 1 & (k = i < l, j = k) \\ t_k \cdots t_i (t_l - 1) & (k < i < l, j = k) \\ t_k & (i = j = l-1) \\ t_{i+1}^{-1} \cdots t_{l-1}^{-1}(t_k - 1) & (k \leqslant i < l-1, j = l-1) \\ t_{i+1}^{-1} \cdots t_{l-1}^{-1}(1 - t_k) & (k \leqslant i < l, j = l) \\ 1 & (k < i < l, j = i) \\ \delta_{ij} & (\textit{otherwise}). \end{cases}$$

Here, we prove the following theorem.

Theorem A.2. *There exists a group isomorphism* $\mu : \mathrm{Aut}_{\Lambda_n}(L_r^{\mathrm{prim}}) \to \mathrm{GL}(r-1;\Lambda_r)$ *such that the diagram*

$$\begin{array}{ccc} PB_r & \xrightarrow{\quad = \quad} & PB_r \\ \Big\downarrow {\scriptstyle \mathrm{Gass}_r^{\mathrm{red}}} & & \Big\downarrow {\scriptstyle \mathrm{Gass}_r^{\mathrm{O,red}}} \\ \mathrm{Aut}_{\Lambda_r}(L_r^{\mathrm{prim}}) & \xrightarrow{\ \mu\ } & \mathrm{GL}(r-1;\Lambda_r) \end{array}$$

*commutes.*

Proof. In order to prove the theorem, we must first prepare a lemma. A direct computation proves the following,

Lemma A.3. *Let* $F$ *be the matrix,*

$$F := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & t_2 & & 0 \\ 0 & & \ddots & \vdots \\ 0 & 0 & \dots & t_2 t_3 \cdots t_{r-1} \end{pmatrix}$$

*in* $\mathrm{GL}(r-1;\Lambda_r)$*. For any matrix* $A = (a_{ij}) \in \mathrm{GL}(r-1;\Lambda_r)$*, we put* $A' = (a'_{ij}) := F^{-1}AF$ *and* $s_i := t_1 \cdots t_i$*. Then, the following equality holds:*

$$(a'_{ij}) = (s_i s_j^{-1} a_{ij}).$$

We have only to prove the assertion for each generator $A_{ij}$ of $PB_r$. By taking a basis $E_i$ of $L_r^{\mathrm{prim}}$, we can identify $\mathrm{Aut}_{\Lambda_r}(L_r^{\mathrm{prim}})$ with $\mathrm{GL}(r-1;\Lambda_r)$. From Lemma A.3, Proposition 2.1.14, and Proposition A.1, a direct computation leads to

$$F^{-1}\mathrm{Gass}_r^{\mathrm{O,red}}(A_{ij})F = \overline{{}^t\left(\mathrm{Gass}_r^{\mathrm{red}}(A_{ij})^{-1}\right)}.$$

Here, $^- : \Lambda_r \to \Lambda_r$ is the automorphism of $\mathrm{GL}(r-1;\Lambda_r)$ induced by the involution $t_i \mapsto t_i^{-1}$. Hence, an automorphism defined by

$$\mu(A) = F\left({}^t\overline{A^{-1}}\right)F^{-1}.$$

for any $A \in \mathrm{GL}(r-1;\Lambda_r)$ satisfies the condition. This completes the proof.

$\square$

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