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Inoguchi, Jun-ichi
Department of Mathematical Sciences, Yamagata University

Kajiwara, Kenji
Faculty of Mathematics, Kyushu University

Matsuura, Nozomu
Department of Applied Mathematics, Fukuoka University

Ohta, Yasuhiro
Department of Mathematics, Kobe University

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

MOTION AND BÄCKLUND TRANSFORMATIONS OF DISCRETE PLANE CURVES

Jun-ichi Inoguchi[1], Kenji Kajiwara[2], Nozomu Matsuura[3] and Yasuhiro Ohta[4]

[1]: Department of Mathematical Sciences, Yamagata University,
1-4-12 Kojirakawa-machi, Yamagata 990-8560, Japan.
inoguchi@sci.kj.yamagata-u.ac.jp
[2]: Faculty of Mathematics, Kyushu University, 744 Motooka, Fukuoka 819-8581, Japan.
kaji@math.kyushu-u.ac.jp
[3]: Department of Applied Mathematics, Fukuoka University,
Nanakuma, Fukuoka 814-0180, Japan.
nozomu@fukuoka-u.ac.jp
[4]: Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan.
ohta@math.sci.kobe-u.ac.jp

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Abstract

We construct explicit solutions to discrete motion of discrete plane curves which has been introduced by one of the authors recently. Explicit formulas in terms the τ function are presented. Transformation theory of motions of both smooth and discrete curves is developed simultaneously.

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1 Introduction

Differential geometry has a close relationship with the theory of integrable systems. In fact, many integrable differential or difference equations arise as compatibility conditions of some geometric objects. For instance, it is well-known that the compatibility condition of pseudospherical surfaces gives rise to the sine-Gordon equation under the Chebyshev net parametrization. For more information on such connections we refer to a monograph [30] by Rogers and Schief.

The above connection between the differential geometry of surfaces and the integrable systems has been already known in the nineteenth century (although the theory of integrable systems has not yet established). However, it is curious that the link between differential geometry of curves and the integrable systems has been noticed rather recently. Actually G. Lamb [22] and Goldstein–Petrich [11] discovered an interesting connection between integrable systems and differential geometry of plane curves. Namely, they found that the *modified Korteweg-de Vries equation* (mKdV equation in short) appears as the compatibility condition of a certain motion of plane curves. Here a motion of curves means an isoperimetric time evolution of arc-length parametrized plane curves. More precisely, the compatibility condition implies that the curvature function of a motion should satisfy the mKdV equation. As a result, the angle function of the motion satisfies the *potential modified Korteweg-de Vries equation* (potential mKdV equation, in short).

On the other hand, in the theory of integrable systems, discretization of integrable differential equations preserving the integrability has been paid much attention, after the pioneering work of Ablowitz–Ladik [1] and Hirota [12–16]. Later, Date, Jimbo and Miwa developed a unified algebraic approach from the view of so-called the KP theory [5–9, 21, 26]. For other approaches to the discrete integrable systems, see, for example, [27, 31]. Thus one can expect the existence of discretized differential geometric objects governed by the discrete integrable systems. This idea has been realized by the works of Bobenko–Pinkall [3] and Doliwa [10] where the discrete analogue of classical surface theory has been proposed, and it is now actively studied under the name of discrete differential geometry [4].

On the contrary, the discrete analogue of curves has not been studied well in contrast to discrete surfaces. For instance, Hisakado et al proposed a discretization of arc-length parametrized plane curve [19]. They obtained from the compatibility condition of the motion of curves a certain semi-discrete equation (discrete space variable and continuous time variable) which may be considered as a semi-discretization of the mKdV equation. Hoffmann and Kutz [20] considered discretization of the curvature function. By using their discrete curvature function and Möbius geometry, they obtained another semi-discretization of the mKdV equation. However in both works, discretization of time variable of curve motions was not established.

Recently one of the authors of the present paper formulated a full discretization of motion of discrete curves [25], where the discrete potential mKdV equation proposed by Hirota [17] is deduced as the compatibility condition. In the smooth curve theory, the potential function coincides with the angle function of a curve, the primitive function of the curvature. However in the discrete case, the potential function and the angle function become different objects. In this framework, the primal geometric object is the potential function rather than curvature (see [25] and Section 2 of the present paper). Natural and systematic construction of the discrete motion of the curves is expected by using the theory of discrete integrable systems.

The purpose of the paper is to construct explicit solutions to discrete motion of discrete curves by using the theory of τ function. This paper is organized as follows. In Section 2, we prepare fundamental ingredients of plane curve geometry and motions (isoperimetric time evolutions) of plane curves described by the potential mKdV equation. Next we give a brief review of the discrete motion of discrete curves [25]. In Section 3, we shall give a construction of motions for both smooth and discrete curves by the theory of τ function. More precisely we introduce a system of bilinear equations of Hirota type which can be obtained by a certain reduction of the discrete two-dimensional Toda lattice hierarchy [21, 32, 33]. We shall give a representation formula for curve motions in terms of τ function.

One of the central topics in classical differential geometry is the transformation theory of curves and surfaces. The best known example might be the Bäcklund transformations of pseudospherical surfaces. The original Bäcklund transformation was defined as a tangential line congruence satisfying *constant distance property* and *constant normal angle property* (see [30]). In plane curve geometry, Bäcklund transformations on arc-length parametrized plane curves can be defined as arc-length preserving transformations satisfying constant distance property. Such transformations can be extended to transformations on smooth curve motions via the transformation of solutions to the potential mKdV equation. Motivated by this fact, we shall introduce Bäcklund transformations for discrete motion of discrete curves in Section 4. In particular we shall give another type of Bäcklund transformations on motions of both smooth and discrete curves, which is related to the discrete sine-Gordon equation. In Section 5, we shall construct and exhibit some explicit solutions of curve motions, namely, the multi-soliton and multi-breather solutions. We also present some pictures of discrete motions of discrete curves. We finally give some explicit formulas for the Bäcklund transformations of both smooth and discrete curve motions via the τ functions.

2 Motion of plane curves

Let $\gamma(x)$ be an arc-length parametrized curve in Euclidean plane $\mathbb{R}^2$. Then the Frenet equation of γ is

$$\gamma'' = \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix} \gamma'. \tag{2.1}$$

Here $'$ denotes the differentiation with respect to x, and the function κ is the *curvature* of γ. Let us consider the following motion in time t, *i.e.*, isoperimetric time evolution:

$$\frac{\partial}{\partial t}\gamma' = \begin{bmatrix} 0 & \kappa'' + \frac{\kappa^3}{2} \\ -\kappa'' - \frac{\kappa^3}{2} & 0 \end{bmatrix} \gamma'. \tag{2.2}$$

Then the *potential function* $\theta(x,t)$ defined by $\kappa = \theta'$ satisfies the *potential mKdV equation* [11,22]:

$$\theta_t + \frac{1}{2}(\theta_x)^3 + \theta_{xxx} = 0. \tag{2.3}$$

The function θ is called the *angle function* of γ in differential geometry. Note that γ' can be expressed as

$$\gamma' = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}. \tag{2.4}$$

For any nonzero constant λ, the following set of equations

$$\frac{\partial}{\partial x}\left(\frac{\widetilde{\theta}+\theta}{2}\right) = 2\lambda \sin\frac{\widetilde{\theta}-\theta}{2}, \tag{2.5}$$

$$\frac{\partial}{\partial t}\left(\frac{\widetilde{\theta}+\theta}{2}\right) = -\lambda\left\{(\theta_x)^2 + 8\lambda^2\right\}\sin\frac{\widetilde{\theta}-\theta}{2} + 2\lambda\theta_{xx}\cos\frac{\widetilde{\theta}-\theta}{2} + 4\lambda^2\theta_x, \tag{2.6}$$

defines a solution $\widetilde{\theta}$ to the potential mKdV equation [34]. The solution $\widetilde{\theta}$ is called a *Bäcklund transform* of θ.

Definition 2.1 A map $\gamma:\ \mathbb{Z} \to \mathbb{R}^2;\ n \mapsto \gamma_n$ is said to be a *discrete curve* of segment length a_n if

$$\left|\frac{\gamma_{n+1} - \gamma_n}{a_n}\right| = 1. \tag{2.7}$$

We introduce the *angle function* Ψ_n of a discrete curve γ by

$$\frac{\gamma_{n+1}-\gamma_n}{a_n} = \begin{bmatrix} \cos\Psi_n \\ \sin\Psi_n \end{bmatrix}. \tag{2.8}$$

A discrete curve γ satisfies

$$\frac{\gamma_{n+1}-\gamma_n}{a_n} = R(K_n)\,\frac{\gamma_n-\gamma_{n-1}}{a_{n-1}}, \tag{2.9}$$

for $K_n = \Psi_n - \Psi_{n-1}$, where $R(K_n)$ denotes the rotation matrix given by

$$R(K_n) = \begin{pmatrix} \cos K_n & -\sin K_n \\ \sin K_n & \cos K_n \end{pmatrix}. \tag{2.10}$$

Now let us recall the following discrete motion of discrete curve $\gamma_n^m : \mathbb{Z}^2 \to \mathbb{R}^2$ introduced by Matsuura [25]:

$$\left|\frac{\gamma_{n+1}^m-\gamma_n^m}{a_n}\right| = 1, \tag{2.11}$$

$$\frac{\gamma_{n+1}^m-\gamma_n^m}{a_n} = R(K_n^m)\,\frac{\gamma_n^m-\gamma_{n-1}^m}{a_{n-1}}, \tag{2.12}$$

$$\frac{\gamma_n^{m+1}-\gamma_n^m}{b_m} = R(W_n^m)\,\frac{\gamma_{n+1}^m-\gamma_n^m}{a_n}, \tag{2.13}$$

where a_n and b_m are arbitrary functions in n and m, respectively. Compatibility of the system (2.11)–(2.13) implies the existence of the *potential function* Θ_n^m defined by

$$W_m^n = \frac{\Theta_n^{m+1}-\Theta_{n+1}^m}{2}, \quad K_n^m = \frac{\Theta_{n+1}^m-\Theta_{n-1}^m}{2}, \tag{2.14}$$

and it follows that Θ_n^m satisfies the *discrete potential mKdV equation* [17]:

$$\tan\left(\frac{\Theta_{n+1}^{m+1}-\Theta_n^m}{4}\right) = \frac{b_m+a_n}{b_m-a_n}\,\tan\left(\frac{\Theta_n^{m+1}-\Theta_{n+1}^m}{4}\right). \tag{2.15}$$

Note that the angle function Ψ_n^m can be expressed as

$$\Psi_n^m = \frac{\Theta_{n+1}^m+\Theta_n^m}{2}. \tag{2.16}$$

Remark 2.2 The potential discrete mKdV equation (2.15) has been also known as the superposition formula for the modified KdV equation (2.3) [34] and the sine-Gordon equation [2, 30].

3 τ function representation of plane curves

In this section, we give a representation formula for curve motions in terms of τ function.

Let $\tau_n^m = \tau_n^m(x,t;y)$ be a complex valued function dependent on two discrete variables m and n, three continuous variables x, t and y, which satisfies the following system of bilinear equations:

$$\frac{1}{2}D_xD_y\,\tau_n^m\cdot\tau_n^m = -\left(\tau^{*m}_n\right)^2, \tag{3.1}$$
$$D_x^2\,\tau_n^m\cdot\tau^{*m}_n = 0, \tag{3.2}$$
$$\left(D_x^3 + D_t\right)\,\tau_n^m\cdot\tau^{*m}_n = 0, \tag{3.3}$$
$$D_y\,\tau_{n+1}^m\cdot\tau_n^m = -a_n\tau^{*m}_{n+1}\tau^{*m}_n, \tag{3.4}$$
$$D_y\,\tau_n^{m+1}\cdot\tau_n^m = -b_m\tau^{*m}_{n+1}\tau^{*m}_n, \tag{3.5}$$
$$b_m\tau^{*m+1}_n\tau_{n+1}^m - a_n\tau^{*m}_{n+1}\tau_n^{m+1} + (a_n-b_m)\tau^{*m+1}_{n+1}\tau_n^m = 0. \tag{3.6}$$

Here, * denotes the complex conjugate, and D_x, D_y, D_t are Hirota's *bilinear differential operators* (D-operators) defined by

$$D_x^iD_y^jD_t^k\,f\cdot g = (\partial_x-\partial_{x'})^i(\partial_y-\partial_{y'})^j(\partial_t-\partial_{t'})^k\,f(x,y,t)g(x',y',t')\Big|_{x=x',y=y',t=t'}\,. \tag{3.7}$$

For the calculus of the D-operators, we refer to [18]. In general, the functions satisfying the bilinear equations of Hirota type are called the τ *functions*.

Theorem 3.1 *Let τ_n^m be a solution to eqs.(3.1)–(3.6). Define a real function $\Theta_n^m(x,t;y)$ and an $\mathbb{R}^2$-valued function $\gamma_n^m(x,t;y)$ by*

$$\Theta_n^m(x,t;y) := \frac{2}{\sqrt{-1}}\log\frac{\tau_n^m}{\tau^{*m}_n}, \tag{3.8}$$
$$\gamma_n^m(x,t;y) := \begin{bmatrix} -\dfrac{1}{2}\left(\log\tau_n^m\tau^{*m}_n\right)_y \\ \dfrac{1}{2\sqrt{-1}}\left(\log\dfrac{\tau_n^m}{\tau^{*m}_n}\right)_y \end{bmatrix}. \tag{3.9}$$

(1) *For any $m,n\in\mathbb{Z}$ and $y\in\mathbb{R}$, the functions $\theta(x,t) = \Theta_n^m(x,t;y)$ and $\gamma(x,t) = \gamma_n^m(x,t;y)$ satisfy eqs.(2.1)–(2.3).*

(2) *For any $x,t,y\in\mathbb{R}$, the functions $\Theta_n^m = \Theta_n^m(x,t;y)$ and $\gamma_n^m = \gamma_n^m(x,t;y)$ satisfy eqs.(2.11)–(2.15).*

Proof. (1) Express $\gamma_n^m = {}^t(X_n^m, Y_n^m)$. Then by using eq.(3.1) together with its complex conjugate, we have

$$\begin{aligned}(X_n^m)' &= -\frac{1}{2}\log(\tau^{*m}_n\tau_n^m)_{xy} = -\frac{1}{2}\left[\frac{\frac{1}{2}D_xD_y\,\tau^{*m}_n\cdot\tau^{*m}_n}{(\tau^{*m}_n)^2} + \frac{\frac{1}{2}D_xD_y\,\tau_n^m\cdot\tau_n^m}{(\tau_n^m)^2}\right] \\ &= \frac{1}{2}\left[\left(\frac{\tau_n^m}{\tau^{*m}_n}\right)^2 + \left(\frac{\tau^{*m}_n}{\tau_n^m}\right)^2\right] = \cos\Theta_n^m.\end{aligned}$$

Similarly we obtain $(Y_n^m)' = \sin\Theta_n^m$. Differentiating $(\gamma_n^m)' = {}^t(\cos\Theta_n^m, \sin\Theta_n^m)$ by x and noticing that $\kappa = \Theta'$, we obtain eq.(2.1):

$$(\gamma_n^m)'' = (\Theta_n^m)'\begin{pmatrix} -\sin\Theta_n^m \\ \cos\Theta_n^m \end{pmatrix} = \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix}(\gamma_n^m)'.$$

On the other hand, differentiating $(\gamma_n^m)'$ by t, we have

$$(\gamma_n^m)'_t = (\Theta_n^m)_t\begin{pmatrix} -\sin\Theta_n^m \\ \cos\Theta_n^m \end{pmatrix} = (\Theta_n^m)_t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(\gamma_n^m)'.$$

By using the bilinear equations (3.2) and (3.3), $(\Theta_n^m)_t$ can be rewritten as

$$\begin{aligned}(\Theta_n^m)_t &= \frac{2}{\sqrt{-1}}\frac{D_t\,\tau_n^m\cdot\tau^{*m}_n}{\tau_n^m\tau^{*m}_n} = -\frac{2}{\sqrt{-1}}\frac{D_x^3\,\tau_n^m\cdot\tau^{*m}_n}{\tau_n^m\tau^{*m}_n}\\ &= -\frac{2}{\sqrt{-1}}\left[\left(\log\frac{\tau_n^m}{\tau^{*m}_n}\right)_{xxx} + 3\left(\log\frac{\tau_n^m}{\tau^{*m}_n}\right)_x(\log\tau_n^m\tau^{*m}_n)_{xx} + \left\{\left(\log\frac{\tau_n^m}{\tau^{*m}_n}\right)_x\right\}^3\right]\\ &= -\frac{2}{\sqrt{-1}}\left[\left(\log\frac{\tau_n^m}{\tau^{*m}_n}\right)_{xxx} - 2\left\{\left(\log\frac{\tau_n^m}{\tau^{*m}_n}\right)_x\right\}^3\right] = -\kappa_{xx} - \frac{\kappa^3}{2}\end{aligned} \qquad (3.10)$$

which yields eq.(2.2). Here we have used the relation

$$0 = \frac{D_x^2\,\tau_n^m\cdot\tau^{*m}_n}{\tau_n^m\tau^{*m}_n} = (\log\tau_n^m\tau^{*m}_n)_{xx} + \left(\log\frac{\tau_n^m}{\tau^{*m}_n}\right)_x^2$$

which is a consequence of eq.(3.2). The potential mKdV equation (2.3) follows immediately from eq.(3.10) by noticing that $\kappa = \Theta'$.

(2) From eq. (3.4) and its complex conjugate we have

$$\left(\log\frac{\tau_{n+1}^m}{\tau_n^m}\right)_y = -a_n\frac{\tau^{*m}_{n+1}\tau^{*m}_n}{\tau_{n+1}^m\tau_n^m}, \qquad \left(\log\frac{\tau^{*m}_{n+1}}{\tau^{*m}_n}\right)_y = -a_n\frac{\tau_{n+1}^m\tau_n^m}{\tau^{*m}_{n+1}\tau^{*m}_n}. \qquad (3.11)$$

Adding these two equations we obtain

$$(\log\tau_{n+1}^m\tau^{*m}_{n+1})_y - (\log\tau_n^m\tau^{*m}_n)_y = -a_n\left(\frac{\tau^{*m}_{n+1}\tau^{*m}_n}{\tau_{n+1}^m\tau_n^m} + \frac{\tau_{n+1}^m\tau_n^m}{\tau^{*m}_{n+1}\tau^{*m}_n}\right), \qquad (3.12)$$

which yields

$$\frac{X_{n+1}^m - X_n^m}{a_n} = \cos\Psi_n^m, \quad \Psi_n^m = \frac{1}{\sqrt{-1}}\log\left(\frac{\tau_{n+1}^m\tau_n^m}{\tau^{*m}_{n+1}\tau^{*m}_n}\right) = \frac{\Theta_{n+1}^m + \Theta_n^m}{2}. \qquad (3.13)$$

Subtracting the second equation from the first equation in eq.(3.11) we have

$$\frac{Y_{n+1}^m - Y_n^m}{a_n} = \sin\Psi_n^m.$$

Therefore we obtain

$$\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n} = \begin{pmatrix}\cos\Psi_n^m \\ \sin\Psi_n^m\end{pmatrix}. \qquad (3.14)$$

which gives eq.(2.11). Next, from eq.(3.14) we see that

$$\frac{\gamma_{n+1}^m-\gamma_n^m}{a_n}=R(\Psi_n^m-\Psi_{n-1}^m)\,\frac{\gamma_n^m-\gamma_{n-1}^m}{a_{n-1}},\qquad \Psi_n^m-\Psi_{n-1}^m=\frac{\Theta_{n+1}^m-\Theta_{n-1}^m}{2}=K_n^m, \tag{3.15}$$

which is nothing but eq.(2.12). Similarly, starting from eq.(3.5) and its complex conjugate we obtain

$$\frac{\gamma_n^{m+1}-\gamma_n^m}{b_m}=\begin{pmatrix}\cos\Phi_n^m\\ \sin\Phi_n^m\end{pmatrix},\qquad \Phi_n^m=\frac{1}{\sqrt{-1}}\log\left(\frac{\tau_n^{m+1}\tau_n^m}{\tau^{*m+1}_n\tau^{*m}_n}\right)=\frac{\Theta_n^{m+1}+\Theta_n^m}{2}, \tag{3.16}$$

which yields

$$\frac{\gamma_n^{m+1}-\gamma_n^m}{b_m}=R(\Phi_n^m-\Psi_n^m)\,\frac{\gamma_{n+1}^m-\gamma_n^m}{a_n},\qquad \Phi_n^m-\Psi_n^m=\frac{\Theta_n^{m+1}-\Theta_{n+1}^m}{2}=W_n^m. \tag{3.17}$$

This is equivalent to eq.(2.13).

Finally let us derive the discrete potential mKdV equation (2.15). Dividing eq.(3.6) and its complex conjugate by $\tau^{*m}_{n+1}\tau^{*m+1}_n$ we have

$$\begin{aligned} b_m\exp\left(\frac{\sqrt{-1}\,\Theta_{n+1}^m}{2}\right)-a_n\exp\left(\frac{\sqrt{-1}\,\Theta_n^{m+1}}{2}\right)&=-(a_n-b_m)\frac{\tau^{*m+1}_{n+1}\tau_n^m}{\tau^{*m+1}_n\tau^{*m}_{n+1}},\\ b_m\exp\left(\frac{\sqrt{-1}\,\Theta_n^{m+1}}{2}\right)-a_n\exp\left(\frac{\sqrt{-1}\,\Theta_{n+1}^m}{2}\right)&=-(a_n-b_m)\frac{\tau^{m+1}_{n+1}\tau^{*m}_n}{\tau^{*m+1}_n\tau^{*m}_{n+1}}, \end{aligned} \tag{3.18}$$

respectively. Dividing these two equations we obtain

$$\frac{b_m\exp\left(\frac{\sqrt{-1}\,\Theta_{n+1}^m}{2}\right)-a_n\exp\left(\frac{\sqrt{-1}\,\Theta_n^{m+1}}{2}\right)}{b_m\exp\left(\frac{\sqrt{-1}\,\Theta_n^{m+1}}{2}\right)-a_n\exp\left(\frac{\sqrt{-1}\,\Theta_{n+1}^m}{2}\right)}=\exp\left[-\frac{\sqrt{-1}\,\left(\Theta_{n+1}^{m+1}-\Theta_n^m\right)}{2}\right], \tag{3.19}$$

which is easily verified to be equivalent to eq.(2.15). Thus we have completed the proof of Theorem 3.1. □

Corollary 3.2 **(Representation Formula)**
γ_n^m can be expressed in terms of the potential function Θ_n^m as follows:

$$\gamma_n^m(x,t;y)=\begin{bmatrix}\displaystyle\int^x\cos\Theta_n^m(x',t;y)\,dx'\\ \displaystyle\int^x\sin\Theta_n^m(x',t;y)\,dx'\end{bmatrix}=\begin{bmatrix}\displaystyle\sum_{n'}^{n-1}a_{n'}\cos\left(\frac{\Theta_{n'}^m(x,t;y)+\Theta_{n'+1}^m(x,t;y)}{2}\right)\\ \displaystyle\sum_{n'}^{n-1}a_{n'}\sin\left(\frac{\Theta_{n'}^m(x,t;y)+\Theta_{n'+1}^m(x,t;y)}{2}\right)\end{bmatrix}. \tag{3.20}$$

Proof. The first equation is a consequence of

$$\frac{\partial}{\partial x}\gamma_n^m(x,t;y)=\begin{bmatrix}\cos\Theta_n^m(x,t;y)\\ \sin\Theta_n^m(x,t;y)\end{bmatrix}, \tag{3.21}$$

and the second equation follows from eq.(3.14). □

It should be noted here that the bilinear equations (3.1)–(3.6) are derived from the reduction of the following equations,

$$\frac{1}{2}D_xD_y\ \tau_n^m(s)\cdot\tau_n^m(s) = -\tau_n^m(s+1)\tau_n^m(s-1), \tag{3.22}$$

$$\left(D_x^2 - D_z\right)\ \tau_n^m(s+1)\cdot\tau_n^m(s) = 0, \tag{3.23}$$

$$\left(D_x^3 + D_t + 3D_xD_z\right)\ \tau_n^m(s+1)\cdot\tau_n^m(s) = 0, \tag{3.24}$$

$$D_y\ \tau_{n+1}^m(s)\cdot\tau_n^m(s) = -a_n\tau_{n+1}^m(s+1)\tau_n^m(s-1), \tag{3.25}$$

$$D_y\ \tau_n^{m+1}(s)\cdot\tau_n^m(s) = -b_m\tau_{n+1}^m(s+1)\tau_n^m(s-1), \tag{3.26}$$

$$b_m\tau_n^{m+1}(s+1)\tau_{n+1}^m(s) - a_n\tau_{n+1}^m(s+1)\tau_n^{m+1}(s) + (a_n-b_m)\tau_{n+1}^{m+1}(s+1)\tau_n^m(s) = 0, \tag{3.27}$$

for $\tau_n^m(s) = \tau_n^m(x,z,t;y;s)$, which are included in the discrete two-dimensional Toda lattice hierarchy [21,32,33]. In fact, imposing the condition

$$\frac{\partial}{\partial z}\tau_n^m(s) = B\,\tau_n^m(s), \qquad \tau_n^m(s+1) = C\,\tau^{*m}_n(s), \qquad B,C\in\mathbb{R}, \tag{3.28}$$

and denoting $\tau_n^m = \tau_n^m(0)$, then eqs.(3.22)–(3.27) yield eqs.(3.1)–(3.6), respectively.

4 Bäcklund transformations

We start with the following fundamental fact on plane curves.

Proposition 4.1 *Let $\gamma(x)$ be an arc-length parametrized curve with angle function $\theta(x)$. Take a nonzero constant λ and a solution $\widetilde{\theta}(x)$ to*

$$\left(\frac{\widetilde{\theta}+\theta}{2}\right)' = 2\lambda\sin\frac{\widetilde{\theta}-\theta}{2}. \tag{4.1}$$

Then

$$\widetilde{\gamma}(x) = \gamma(x) + \frac{1}{\lambda}R\left(\frac{\widetilde{\theta}(x)-\theta(x)}{2}\right)\gamma'(x) \tag{4.2}$$

is an arc-length parametrized curve with angle function $\widetilde{\theta}(x)$. In other words, if $\gamma(x)$ is a solution to eq.(2.1), then $\widetilde{\gamma}(x)$ is another solution to eq.(2.1) with $\widetilde{\kappa}(x) = \widetilde{\theta}'(x)$. The curve $\widetilde{\gamma}$ is called a Bäcklund transform of γ.

Proposition 4.1 can be verified easily by direct computation. We next extend the Bäcklund transformation to those of motion of curve.

Proposition 4.2 *Let $\gamma(x,t)$ be a motion of arc-length parametrized curve determined by eqs.(2.2) and (2.3). Take a Bäcklund transform $\widetilde{\theta}(x,t)$ defined by eqs.(2.5) and (2.6) of $\theta(x,t)$. Then*

$$\widetilde{\gamma}(x,t) = \gamma(x,t) + \frac{1}{\lambda}R\left(\frac{\widetilde{\theta}(x,t)-\theta(x,t)}{2}\right)\gamma'(x,t) \tag{4.3}$$

is a motion of arc-length parametrized curve with the angle function $\widetilde{\theta}(x,t)$.

Proof. By the preceding Proposition, $\widetilde{\gamma}$ satisfies the isoperimetric condition $|\widetilde{\gamma}'| = 1$. Computing the t-derivative of $\widetilde{\gamma}$ by using (2.6), we can show that $\widetilde{\gamma}$ satisfies eq.(2.2) with $\widetilde{\kappa} = \widetilde{\theta}'$ □

Now we introduce a Bäcklund transformation of discrete curve.

Proposition 4.3 *Let γ_n be a discrete curve of segment length a_n. Let Θ_n be the potential function defined by*

$$\frac{\gamma_{n+1} - \gamma_n}{a_n} = \begin{bmatrix} \cos \Psi_n \\ \sin \Psi_n \end{bmatrix}, \quad \Psi_n = \frac{\Theta_{n+1} + \Theta_n}{2}. \tag{4.4}$$

For a nonzero constant λ, take a solution $\widetilde{\Theta}_n$ to the following equation

$$\tan\left(\frac{\widetilde{\Theta}_{n+1} - \Theta_n}{4}\right) = \frac{\frac{1}{\lambda} + a_n}{\frac{1}{\lambda} - a_n} \tan\left(\frac{\widetilde{\Theta}_n - \Theta_{n+1}}{4}\right), \tag{4.5}$$

then

$$\widetilde{\gamma}_n = \gamma_n + \frac{1}{\lambda} R\left(\frac{\widetilde{\Theta}_n - \widetilde{\Theta}_{n+1}}{2}\right) \frac{\gamma_{n+1} - \gamma_n}{a_n} \tag{4.6}$$

is a discrete curve with the potential function $\widetilde{\Theta}_n$.

Proof. It suffices to show that

$$\frac{\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n}{a_n} = \begin{bmatrix} \cos \widetilde{\Psi}_n \\ \sin \widetilde{\Psi}_n \end{bmatrix}, \quad \widetilde{\Psi}_n = \frac{\widetilde{\Theta}_{n+1} + \widetilde{\Theta}_n}{2} \tag{4.7}$$

for $\widetilde{\gamma}_n$ defined by eq.(4.6). This follows from eqs.(4.4) and (4.5). □

We next extend the Bäcklund transformation to those of motion of discrete curve. In order to do so, we first present the Bäcklund transformation to the discrete potential mKdV equation:

Lemma 4.4 *Let Θ_n^m be a solution to the discrete potential mKdV equation (2.15). A function $\widetilde{\Theta}_n^m$ satisfying the following system of equations*

$$\tan\left(\frac{\widetilde{\Theta}_{n+1}^m - \Theta_n^m}{4}\right) = \frac{\frac{1}{\lambda} + a_n}{\frac{1}{\lambda} - a_n} \tan\left(\frac{\widetilde{\Theta}_n^m - \Theta_{n+1}^m}{4}\right), \tag{4.8}$$

$$\tan\left(\frac{\widetilde{\Theta}_n^{m+1} - \Theta_n^m}{4}\right) = \frac{\frac{1}{\lambda} + b_m}{\frac{1}{\lambda} - b_m} \tan\left(\frac{\widetilde{\Theta}_n^m - \Theta_n^{m+1}}{4}\right), \tag{4.9}$$

gives another solution to eq.(2.15). We call $\widetilde{\Theta}_n^m$ a Bäcklund transform of Θ_n^m.

Proof. First note that eq.(2.15) is equivalent to

$$e^{U_n^{m+1} + U_n^m} - e^{U_{n+1}^{m+1} + U_{n+1}^m} = \frac{a_n}{b_m}\left(e^{U_{n+1}^m + U_n^m} - e^{U_{n+1}^{m+1} + U_n^{m+1}}\right), \tag{4.10}$$

where we put $\frac{\sqrt{-1}\Theta_n^m}{2} = U_n^m$ for notational simplicity. Similarly, eqs.(4.8) and (4.9) are rewritten as

$$e^{\widetilde{U}_n^m + U_n^m} - e^{\widetilde{U}_{n+1}^m + U_{n+1}^m} = \lambda a_n\left(e^{U_{n+1}^m + U_n^m} - e^{\widetilde{U}_{n+1}^m + \widetilde{U}_n^m}\right), \tag{4.11}$$

$$e^{\widetilde{U}_n^m + U_n^m} - e^{\widetilde{U}_n^{m+1} + U_n^{m+1}} = \lambda b_m\left(e^{U_n^{m+1} + U_n^m} - e^{\widetilde{U}_n^{m+1} + \widetilde{U}_n^m}\right), \tag{4.12}$$

respectively, where $\frac{\sqrt{-1}\widetilde{\Theta}_n^m}{2} = \widetilde{U}_n^m$. Subtracting eq.(4.12) from eq.(4.11), we have

$$e^{\widetilde{U}_n^{m+1}+U_n^{m+1}} - e^{\widetilde{U}_{n+1}^m+U_{n+1}^m} = \lambda\left(a_n e^{U_{n+1}^m+U_n^m} - b_m e^{U_n^{m+1}+U_n^m}\right) - \lambda\left(a_n e^{\widetilde{U}_{n+1}^m+\widetilde{U}_n^m} - b_m e^{\widetilde{U}_n^{m+1}+\widetilde{U}_n^m}\right). \tag{4.13}$$

Similarly, subtracting eq.$(4.12)_{n\to n+1}$ from eq.$(4.11)_{m\to m+1}$, we get

$$e^{\widetilde{U}_n^{m+1}+U_n^{m+1}} - e^{\widetilde{U}_{n+1}^m+U_{n+1}^m} = \lambda\left(a_n e^{U_{n+1}^{m+1}+U_n^{m+1}} - b_m e^{U_{n+1}^{m+1}+U_{n+1}^m}\right) - \lambda\left(a_n e^{\widetilde{U}_{n+1}^{m+1}+\widetilde{U}_n^{m+1}} - b_m e^{\widetilde{U}_{n+1}^{m+1}+\widetilde{U}_{n+1}^m}\right). \tag{4.14}$$

Subtracting eq. (4.14) from eq.(4.13) yields

$$\begin{aligned} &a_n\left(e^{\widetilde{U}_{n+1}^m+\widetilde{U}_n^m} - e^{\widetilde{U}_{n+1}^{m+1}+\widetilde{U}_n^{m+1}}\right) - b_m\left(e^{\widetilde{U}_n^{m+1}+\widetilde{U}_n^m} - e^{\widetilde{U}_{n+1}^{m+1}+\widetilde{U}_{n+1}^m}\right)\\ &= a_n\left(e^{U_{n+1}^m+U_n^m} - e^{U_{n+1}^{m+1}+U_n^{m+1}}\right) - b_m\left(e^{U_n^{m+1}+U_n^m} - e^{U_{n+1}^{m+1}+U_{n+1}^m}\right). \end{aligned} \tag{4.15}$$

Now we see that the right hand side of eq.(4.15) vanishes since it is equivalent to eq.(4.10). Then the left hand side gives eq.(2.15) for $\widetilde{\Theta}_n^m$. □.

Proposition 4.5 *Let γ_n^m be a discrete motion of discrete curve. Take a Bäcklund transform $\widetilde{\Theta}_n^m$ of Θ_n^m defined in Lemma 4.4. Then*

$$\widetilde{\gamma}_n^m = \gamma_n^m + \frac{1}{\lambda} R\left(\frac{\widetilde{\Theta}_n^m - \Theta_{n+1}^m}{2}\right) \frac{\gamma_{n+1}^m - \gamma_n^m}{a_n} \tag{4.16}$$

is a discrete motion of discrete curve with potential function $\widetilde{\Theta}_n^m$. We call $\widetilde{\gamma}_n^m$ a Bäcklund transform of γ_n^m.

Proof. It suffices to show that $\widetilde{\gamma}_n^m$ satisfies eqs.(2.11)–(2.13) with potential function $\widetilde{\Theta}_n^m$, but eqs.(2.11) and (2.12) follow from Proposition 4.3 immediately. Noticing the symmetry in n and m, similar calculations to those in Proposition 4.3 yield

$$\frac{\widetilde{\gamma}_n^{m+1} - \widetilde{\gamma}_n^m}{b_m} = \begin{bmatrix} \cos\left(\frac{\widetilde{\Theta}_n^{m+1}+\widetilde{\Theta}_n^m}{2}\right) \\ \sin\left(\frac{\widetilde{\Theta}_n^{m+1}+\widetilde{\Theta}_n^m}{2}\right) \end{bmatrix} \tag{4.17}$$

by using eq.(4.9). Comparing eqs.(4.7) and (4.17) we obtain

$$\frac{\widetilde{\gamma}_n^{m+1} - \widetilde{\gamma}_n^m}{b_m} = R\left(\frac{\widetilde{\Theta}_n^{m+1} - \widetilde{\Theta}_{n+1}^m}{2}\right) \frac{\widetilde{\gamma}_{n+1}^m - \widetilde{\gamma}_n^m}{a_n}, \tag{4.18}$$

which implies eq.(2.13). □

It is possible to construct another type of Bäcklund transformations for motions of both smooth and discrete curves by using the symmetry of the potential modified KdV equation (2.3) and discrete potential modified KdV equation (2.15). In fact, if $\theta(x,t)$ is a solution to eq.(2.3), then $-\theta(x,t)$ satisfies the same equation. Combining this symmetry and the Bäcklund transformation defined by eqs.(2.5) and (2.6), we have the following Bäcklund transformation:

Lemma 4.6 *Let $\theta(x,t)$ be a solution to the potential modified KdV equation (2.3). For any nonzero constant λ, a function $\overline{\theta}(x,t)$ satisfying the following set of equations*

$$\frac{\partial}{\partial x}\left(\frac{\overline{\theta}-\theta}{2}\right) = 2\lambda \sin\frac{\overline{\theta}+\theta}{2}, \tag{4.19}$$

$$\frac{\partial}{\partial t}\left(\frac{\overline{\theta}-\theta}{2}\right) = -\lambda\left\{(\theta_x)^2 + 8\lambda^2\right\}\sin\frac{\overline{\theta}+\theta}{2} - 2\lambda\theta_{xx}\cos\frac{\overline{\theta}+\theta}{2} - 4\lambda^2\theta_x, \tag{4.20}$$

gives another solution to eq.(2.3).

Lemma 4.6 immediately yields the following Bäcklund transformation for $\gamma(x,t)$:

Proposition 4.7 *Let $\gamma(x,t)$ be a motion of arc-length parametrized curve determined by eqs.(2.2) and (2.3). Take a Bäcklund transform $\overline{\theta}(x,t)$ of $\theta(x,t)$ defined in Lemma 4.6. Then*

$$\overline{\gamma}(x,t) = S\left[\gamma(x,t) + \frac{1}{\lambda}R\left(-\frac{\overline{\theta}(x,t)+\theta(x,t)}{2}\right)\gamma'(x,t)\right],\quad S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{4.21}$$

is a motion of arc-length parametrized curve with angle function $\overline{\theta}(x,t)$.

Note that eqs.(4.19) and (4.20) can be derived from eqs.(2.5) and (2.6) simply by putting $\widetilde{\theta}(x,t) = -\overline{\theta}(x,t)$. Moreover, noticing eq.(2.4) and Proposition 4.2, we have

$$\gamma(x,t) + \frac{1}{\lambda}R\left(-\frac{\overline{\theta}(x,t)+\theta(x,t)}{2}\right)\gamma'(x,t) = \begin{bmatrix} \cos(-\overline{\theta}(x,t)) \\ \sin(-\overline{\theta}(x,t)) \end{bmatrix} = \begin{bmatrix} \cos\overline{\theta}(x,t) \\ -\sin\overline{\theta}(x,t) \end{bmatrix}, \tag{4.22}$$

which implies Proposition 4.7.

Similarly, if Θ_n^m is a solution to eq.(2.15), then $-\Theta_n^m$ satisfies the same equation. Therefore Lemma 4.4 and Proposition 4.5 lead to the following Bäcklund transformations:

Lemma 4.8 *Let Θ_n^m be a solution to the discrete potential mKdV equation (2.15). A function $\overline{\Theta}_n^m$ satisfying the following system of equations*

$$\tan\left(\frac{\overline{\Theta}_{n+1}^m + \Theta_n^m}{4}\right) = \frac{\frac{1}{\lambda}+a_n}{\frac{1}{\lambda}-a_n}\tan\left(\frac{\overline{\Theta}_n^m + \Theta_{n+1}^m}{4}\right), \tag{4.23}$$

$$\tan\left(\frac{\overline{\Theta}_n^{m+1} + \Theta_n^m}{4}\right) = \frac{\frac{1}{\lambda}+b_m}{\frac{1}{\lambda}-b_m}\tan\left(\frac{\overline{\Theta}_n^m + \Theta_n^{m+1}}{4}\right), \tag{4.24}$$

gives another solution to eq.(2.15).

Proposition 4.9 *Let γ_n^m be a discrete motion of discrete curve. Take a Bäcklund transform $\overline{\Theta}_n^m$ of Θ_n^m defined in Lemma 4.8. Then*

$$\overline{\gamma}_n^m = S\left[\gamma_n^m + \frac{1}{\lambda}R\left(-\frac{\overline{\Theta}_n^m + \Theta_{n+1}^m}{2}\right)\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n}\right],\quad S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{4.25}$$

is a discrete motion of discrete curve with potential function $\overline{\Theta}_n^m$.

Remark 4.10

(1) It may be interesting to point out that eq.(4.23) and eq.(4.24) are rewritten as

$$\sin\left(\frac{\overline{\Theta}^m_{n+1}-\Theta^m_{n+1}-\overline{\Theta}^m_n+\Theta^m_n}{4}\right)=\lambda a_n\sin\left(\frac{\overline{\Theta}^m_{n+1}+\Theta^m_{n+1}+\overline{\Theta}^m_n+\Theta^m_n}{4}\right), \tag{4.26}$$

$$\sin\left(\frac{\overline{\Theta}^{m+1}_{n}-\Theta^{m+1}_{n}-\overline{\Theta}^m_n+\Theta^m_n}{4}\right)=\lambda b_m\sin\left(\frac{\overline{\Theta}^{m+1}_{n}+\Theta^{m+1}_{n}+\overline{\Theta}^m_n+\Theta^m_n}{4}\right), \tag{4.27}$$

respectively, which are essentially equivalent to the discrete sine-Gordon equation [14].

(2) The Bäcklund transformations described in Propositions 4.2 and 4.5 satisfy "constant distance property", *i.e.*, $|\widetilde{\gamma}-\gamma|\equiv 1/\lambda$ or $|\widetilde{\gamma}^m_n-\gamma^m_n|\equiv 1/\lambda$. These transformations may be regarded as one-dimensional analogue of the original Bäcklund transformations of the pseudospherical surface [30]. On the other hand, the Bäcklund transformations proposed in Propositions 4.7 and 4.9 are characterized by the property $|\widetilde{\gamma}-S\gamma|=1/\lambda$.

5 Explicit Solutions

5.1 Solitons and Breathers

For $N\in\mathbb{Z}_{\geq 0}$ we define a function $\tau^m_n(s)=\tau^m_n(x,t;y,z;s)$ by

$$\tau^m_n(s)=\exp\left[-\left(x+\sum_{n'}^{n-1}a_{n'}+\sum_{m'}^{m-1}b_{m'}\right)y\right]\ \det\left(f^{(i)}_{s+j-1}\right)_{i,j=1,\dots,N}, \tag{5.1}$$

for $(x,t;y,z)\in\mathbb{R}^4$ and $(m,n,s)\in\mathbb{Z}^3$. Here $f^{(i)}_s=f^{(i)}_s(x,t;y,z;m,n)$ $(i=1,\dots,N)$ satisfies the following linear equations

$$\frac{\partial f^{(i)}_s}{\partial x}=f^{(i)}_{s+1},\quad \frac{\partial f^{(i)}_s}{\partial z}=f^{(i)}_{s+2},\quad \frac{\partial f^{(i)}_s}{\partial t}=-4f^{(i)}_{s+3},\quad \frac{\partial f^{(i)}_s}{\partial y}=f^{(i)}_{s-1}, \tag{5.2}$$

$$\frac{f^{(i)}_s(m,n)-f_s(m,n-1)}{a_n}=f^{(i)}_{s+1}(m,n),\quad \frac{f^{(i)}_s(m,n)-f_s(m-1,n)}{b_m}=f^{(i)}_{s+1}(m,n). \tag{5.3}$$

For $N=0$, we set $\det(f^{(i)}_{s+j-1})_{i,j=1\dots,N}=1$. A typical example for $f^{(i)}_s$ is given by

$$f^{(i)}_s=e^{\eta_i}+e^{\mu_i}, \tag{5.4}$$

$$\begin{cases} e^{\eta_i}=\alpha_i p_i^s\prod_{n'}^{n-1}(1-a_{n'}p_i)^{-1}\prod_{m'}^{m-1}(1-b_{m'}p_i)^{-1}e^{p_ix+p_i^2z-4p_i^3t+\frac{1}{p_i}y}, \\ e^{\mu_j}=\beta_i q_i^s\prod_{n'}^{n-1}(1-a_{n'}q_i)^{-1}\prod_{m'}^{m-1}(1-b_{m'}q_i)^{-1}e^{q_ix+q_i^2z-4q_i^3t+\frac{1}{q_i}y}, \end{cases} \tag{5.5}$$

where p_i, q_i, α_i and β_i are arbitrary complex constants.

We note that τ_n^m and $f_s^{(i)}$ are functions of continuous variables x, y, z, t and discrete variables m, n, s, but we will indicate only the relevant variables according to the context, for notational simplicity. Then it is well-known that $\tau_n^m(s)$ satisfies the bilinear equations(3.22)–(3.27) [18, 21, 23, 24, 28, 29, 33]. Actually by using the linear relations (5.2) and (5.3), eqs.(3.22)–(3.27) are reduced to the Plücker relations which are quadratic identities of determinants.

It is possible to construct the solutions to the bilinear equations (3.1)–(3.6) by imposing the reduction condition (3.28) on $\tau_n^m(s)$ in eq.(5.1). Those conditions are realized by putting restrictions on parameters of solutions. As an example, we present the multi-soliton and multi-breather solutions:

Proposition 5.1 *Consider the τ function*

$$\tau_n^m = \exp\left[-\left(x + \sum_{n'}^{n-1} a_{n'} + \sum_{m'}^{m-1} b_{m'}\right)y\right] \det\left(f_{j-1}^{(i)}\right)_{i,j=1,\dots,N}, \tag{5.6}$$

$$f_s^{(i)} = e^{\eta_i} + e^{\mu_i}, \tag{5.7}$$

$$\left\{\begin{aligned} e^{\eta_i} &= \alpha_i p_i^s \prod_{n'}^{n-1}(1 - a_{n'}p_i)^{-1} \prod_{m'}^{m-1}(1 - b_{m'}p_i)^{-1} e^{p_i x - 4p_i^3 t + \frac{1}{p_i}y}, \\ e^{\mu_j} &= \beta_i(-p_i)^s \prod_{n'}^{n-1}(1 + a_{n'}p_i)^{-1} \prod_{m'}^{m-1}(1 + b_{m'}p_i)^{-1} e^{-p_i x + 4p_i^3 t - \frac{1}{p_i}y}. \end{aligned}\right. \tag{5.8}$$

(1) *Choosing the parameters as*

$$p_i,\ \alpha_i \in \mathbb{R}, \quad \beta_i \in \sqrt{-1}\mathbb{R} \quad (i = 1, \dots, N), \tag{5.9}$$

then τ_n^m satisfies the bilinear equations (3.1)–(3.6). This gives the N-soliton solution to eqs. (2.3) and (2.15).

(2) *Taking $N = 2M$, and choosing the parameters as*

$$\begin{aligned} &p_i,\ \alpha_i,\ \beta_i \in \mathbb{C} \quad (i = 1, \dots, 2M), \quad p_{2k} = p_{2k-1}^* \quad (k = 1, \dots, M), \\ &\alpha_{2k} = \alpha_{2k-1}^*, \quad \beta_{2k} = -\beta_{2k-1}^* \quad (k = 1, \dots, M), \end{aligned} \tag{5.10}$$

then τ_n^m satisfies the bilinear equations (3.1)–(3.6). This gives the M-breather solution to eqs. (2.3) and (2.15).

Proof. It is sufficient to show that the conditions in eq.(3.28) are satisfied. We first impose the two-periodicity in s, *i.e.*, $\tau_n^m(s+2) = \text{const.} \times \tau_n^m(s)$. For $\tau_n^m(s)$ in eq.(5.1) with entries given by eqs.(5.4) and (5.5), putting

$$q_i = -p_i, \tag{5.11}$$

we have

$$f_{s+2}^{(i)} = p_i^2 f_s^{(i)}, \tag{5.12}$$

which implies

$$\tau_n^m(s+2) = A_N\, \tau_n^m(s), \quad A_N = \prod_{i=1}^{N} p_i^2. \tag{5.13}$$

Note that the condition

$$\frac{\partial \tau_n^m(s)}{\partial z} = B_N\, \tau_n^m(s), \quad B_N = \sum_{i=1}^{N} p_i^2, \tag{5.14}$$

is also satisfied simultaneously. Now we consider case (1) and (2) separately:
Case (1). We see from eqs.(5.7) and (5.8) together with eq.(5.9) that

$$f_1^{(i)} = p_i\, f_0^{(i)*} \tag{5.15}$$

and so

$$\tau_n^m(1) = C_N\, \tau^{*m}_n(0), \quad C_N = \prod_{i=1}^{N} p_i \in \mathbb{R}. \tag{5.16}$$

Case (2). We see from eqs.(5.7) and (5.8) together with eq.(5.10) that

$$f_1^{(2k)} = p_{2k-1}^*\, f_0^{(2k-1)*}, \quad f_1^{(2k-1)} = p_{2k}^*\, f_0^{(2k)*}, \tag{5.17}$$

and so

$$\tau_n^m(1) = C_N\, \tau^{*m}_n(0), \quad C_N = (-1)^M \prod_{i=1}^{M} |p_i|^2 \in \mathbb{R}. \tag{5.18}$$

Therefore we have verified that the conditions in eq.(3.28) are satisfied for both cases. Then putting $\tau_n^m = \tau_n^m(0)$, we obtain the desired result. □

We present some pictures of motions of the discrete curves. Figure 1 shows the simplest example of curve which corresponds to 1-soliton solution (loop soliton). The next example illustrated

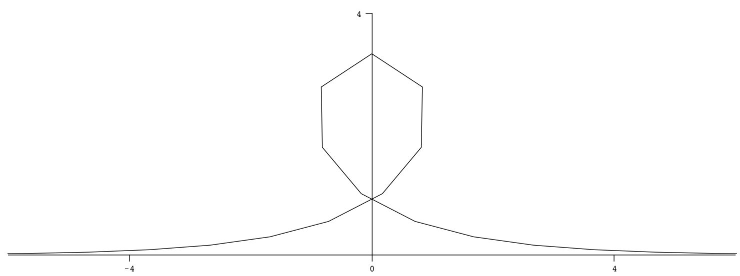

Figure 1. Parameters in eqs.(5.6), (5.7) and (5.8): $N = 1$, $x = 0$, $y = 0$, $\alpha_1 = -1$, $\beta_1 = \sqrt{-1}$, $p_1 = 0.3$, $a_n = 1$, $b_m = 0.5$.

in Figure 2 describes the interaction of two loops which corresponds to the 2-soliton solution. Figures 3 and 4 show the motions which correspond to the 1-breather and 2-breather solutions, respectively.

5.2 Solutions via Bäcklund transformations

In the theory of integrable systems, the Bäcklund transformations are obtained from the shift of a certain discrete independent variable, which also applies to our geometric transformations. We

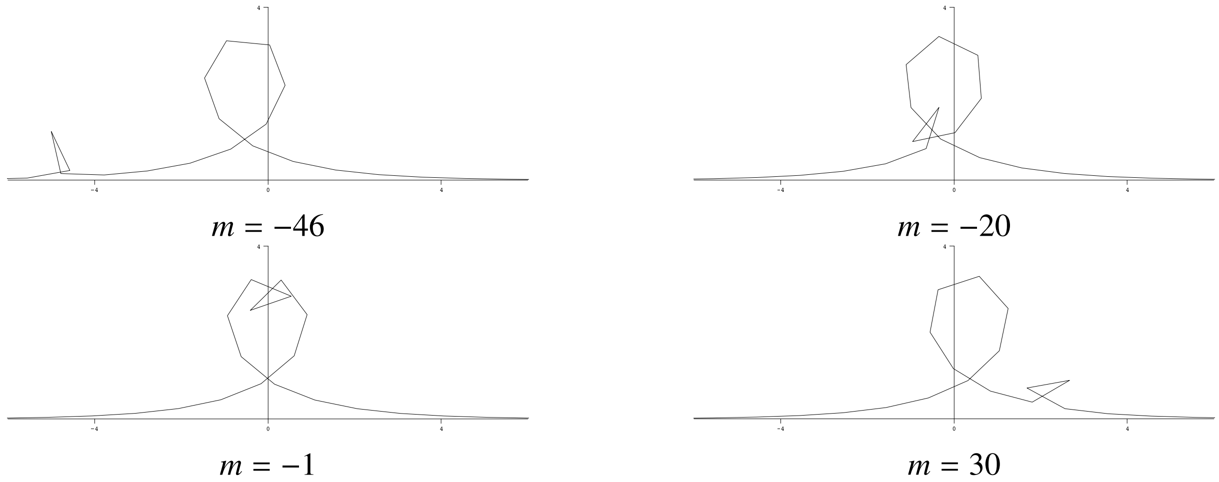

Figure 2. Parameters in eqs.(5.6), (5.7) and (5.8): $N = 2$, $x = 0$, $y = 0$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = -\beta_2 = \sqrt{-1}$, $p_1 = 0.3$, $p_2 = 0.9$, $a_n = 1$, $b_m = 0.5$.

first introduce discrete variables k, l, and regard the determinant size N as an additional discrete variable. We then extend the τ function as $\tau_n^m(k,l,N) = \tau_n^m(x,t;y;k,l,N)$ in the following way:

$$\tau_n^m(k,l,N) = \exp\left[-\left(x + \sum_{n'}^{n-1} a_{n'} + \sum_{m'}^{m-1} b_{m'} + \sum_{k'}^{k-1} c_{k'} + \sum_{l'}^{l-1} \frac{1}{d_{l'}}\right) y\right] \det\left(f_{j-1}^{(i)}\right)_{i,j=1,\ldots,N}, \tag{5.19}$$

$$f_s^{(i)} = e^{\eta_i} + e^{\mu_i}, \tag{5.20}$$

$$\left\{\begin{aligned} e^{\eta_i} &= \alpha_i p_i^s \prod_{n'}^{n-1} (1 - a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 - b_{m'} p_i)^{-1} \prod_{k'}^{k-1} (1 - c_{k'} p_i)^{-1} \prod_{l'}^{l-1} \left(1 - \frac{d_{l'}}{p_i}\right)^{-1} \\ &\qquad \times e^{p_i x - 4p_i^3 t + \frac{1}{p_i} y}, \\ e^{\mu_j} &= \beta_i (-p_i)^s \prod_{n'}^{n-1} (1 + a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 + b_{m'} p_i)^{-1} \prod_{k'}^{k-1} (1 + c_{k'} p_i)^{-1} \prod_{l'}^{l-1} \left(1 + \frac{d_{l'}}{p_i}\right)^{-1} \\ &\qquad \times e^{-p_i x + 4p_i^3 t - \frac{1}{p_i} y}. \end{aligned}\right. \tag{5.21}$$

Accordingly, we extend the relevant dependent variables such as Θ and γ in the same way.

Proposition 5.2

(1) *For any $k \in \mathbb{Z}$, $\widetilde{\gamma}(x,t) = \gamma(x,t;k+1)$ is a Bäcklund transform of $\gamma(x,t) = \gamma(x,t;k)$ related by eq.(4.3) with $\lambda = \frac{1}{c_k}$.*

(2) *For any $k \in \mathbb{Z}$, $\widetilde{\gamma}_n^m = \gamma_n^m(k+1)$ is a Bäcklund transform of $\gamma_n^m = \gamma_n^m(k)$ related by eq.(4.16) with $\lambda = \frac{1}{c_k}$.*

(3) *For any $N \in \mathbb{Z}_{\geq 0}$, $\widetilde{\gamma}(x,t) = \gamma(x,t;N+1)$ is a Bäcklund transform of $\gamma(x,t) = \gamma(x,t;N)$ related by eq.(4.3) with $\lambda = -p_{N+1}$.*

(4) *For any $N \in \mathbb{Z}_{\geq 0}$, $\widetilde{\gamma}_n^m = \gamma_n^m(N+1)$ is a Bäcklund transform of $\gamma_n^m = \gamma_n^m(N)$ related by eq.(4.16) with $\lambda = -p_{N+1}$.*

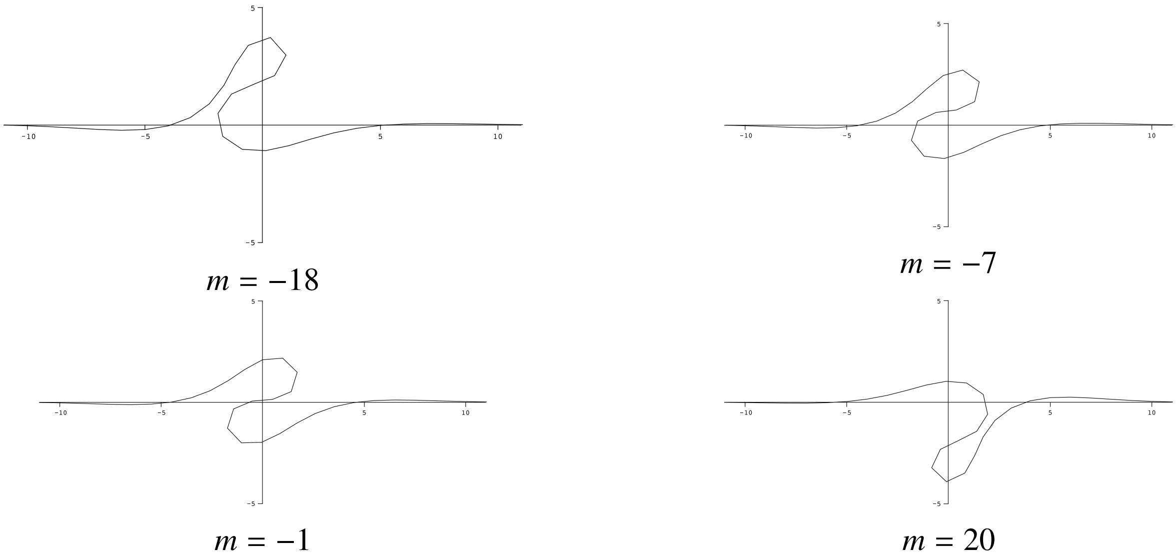

Figure 3. Parameters in eqs.(5.6), (5.7) and (5.8): $N = 2$, $x = 0$, $y = 0$, $\alpha_1 = \alpha_2^* = 1$, $\beta_1 = -\beta_2^* = 1$, $p_1 = p_2^* = 0.2 - 0.2\sqrt{-1}$, $a_n = 1$, $b_m = 1.5$.

(5) *For any* $l \in \mathbb{Z}$, $\overline{\gamma}(x,t) = \gamma(x,t;l+1)$ *is a Bäcklund transform of* $\gamma(x,t) = \gamma(x,t;l)$ *related by eq.(4.21) with* $\lambda = d_l$.

(6) *For any* $l \in \mathbb{Z}$, $\overline{\gamma}_n^m = \gamma_n^m(l+1)$ *is a Bäcklund transform of* $\gamma_n^m = \gamma_n^m(l)$ *related by eq.(4.25) with* $\lambda = d_l$.

Proof. We first prove (1) and (2). It follows from eq.(3.4) that the τ function satisfies the bilinear equation

$$D_y\, \tau_n^m(k+1) \cdot \tau_n^m(k) = -c_k \tau^{*m}_n(k+1)\tau^{*m}_n(k), \tag{5.22}$$

because of the symmetry with respect to the discrete variables m, n and k in eqs.(5.19)–(5.21). Then by the argument similar to that in proof of Theorem 3.1, we see that

$$\frac{\gamma(k+1) - \gamma(k)}{c_k} = \begin{pmatrix} \cos\left(\frac{\theta(k+1)+\theta(k)}{2}\right) \\ \sin\left(\frac{\theta(k+1)+\theta(k)}{2}\right) \end{pmatrix}. \tag{5.23}$$

From eq.(2.4), we have eq.(4.3) with $\widetilde{\gamma} = \gamma(k+1)$ and $\widetilde{\theta} = \theta(k+1)$

$$\frac{\gamma(k+1) - \gamma(k)}{c_k} = R\left(\frac{\theta(k+1) - \theta(k)}{2}\right)\gamma'(k). \tag{5.24}$$

Similarly from eq.(3.14), we obtain eq.(4.16) with $\widetilde{\gamma}_n^m = \gamma_n^m(k+1)$ and $\widetilde{\Theta}_n^m = \Theta_n^m(k+1)$

$$\frac{\gamma_n^m(k+1) - \gamma_n^m(k)}{c_k} = R\left(\frac{\Theta_n^m(k+1) - \Theta_{n+1}^m(k)}{2}\right)\frac{\gamma_{n+1}^m(k) - \gamma_n^m(k)}{a_n}, \tag{5.25}$$

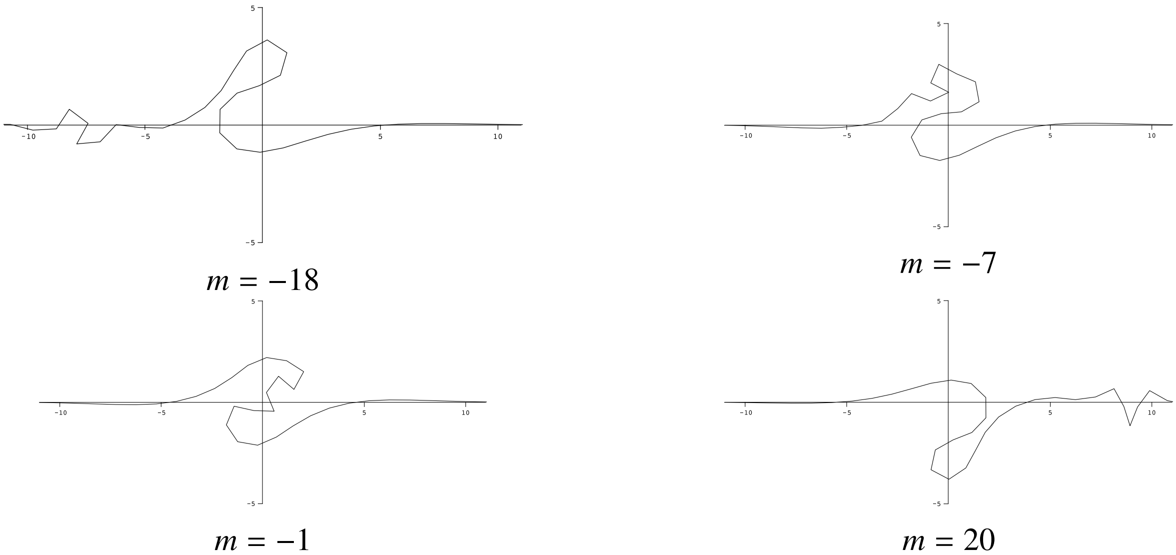

Figure 4. Parameters in eqs.(5.6), (5.7) and (5.8): $N = 4$, $x = 0$, $y = 0$, $\alpha_1 = \alpha_2^* = 1$, $\alpha_3 = \alpha_4^* = \sqrt{-1}$, $\beta_1 = -\beta_2^* = 1$, $\beta_3 = -\beta_4^* = \sqrt{-1}$, $p_1 = p_2^* = 0.2 - 0.2\sqrt{-1}$, $p_3 = p_4^* = 0.8 + 0.8\sqrt{-1}$, $a_n = 1$, $b_m = 1.5$.

which proves (1) and (2). The statements (3)–(4) and (5)–(6) can be proved in much the same way as (1)–(2), by using the bilinear equations

$$D_y\, \tau_n^m(N+1) \cdot \tau_n^m(N) = \frac{1}{p_{N+1}} \tau^{*m}_n(N) \tau^{*m}_n(N+1), \tag{5.26}$$

$$D_y\, \tau_n^m(l+1) \cdot \tau^{*m}_n(l) = -\frac{1}{d_l} \tau^{*m}_n(l+1) \tau_n^m(l), \tag{5.27}$$

respectively. These bilinear equations will be proved in Appendix. □

Remark 5.3 Here we give a physical interpretations of Bäcklund transformations described above. The Bäcklund transforms in (1)–(2) and (5)–(6) correspond to changing the phase of solitons (loops), in other words, positions of solitons. On the other hand, the Bäcklund transforms in (3)–(4) correspond to increasing the number of solitons (loops).

Computing the potential functions of the Bäcklund transforms of the curves, one can verify the following result:

Corollary 5.4

(1) *For any* $k \in \mathbb{Z}$, $\widetilde{\theta}(x,t) = \theta(x,t;k+1)$ *is a Bäcklund transform of* $\theta(x,t) = \theta(x,t;k)$ *related by eqs.(2.5) and (2.6) with* $\lambda = \frac{1}{c_k}$.

(2) *For any* $k \in \mathbb{Z}$, $\widetilde{\Theta}_n^m = \Theta_n^m(k+1)$ *is a Bäcklund transform of* $\Theta_n^m = \Theta_n^m(k)$ *related by eqs.(4.8) and (4.9) with* $\lambda = \frac{1}{c_k}$.

(3) *For any* $N \in \mathbb{Z}_{\geq 0}$, $\widetilde{\theta}(x,t) = \theta(x,t;N+1)$ *is a Bäcklund transform of* $\theta(x,t) = \theta(x,t;N)$ *related by eqs.(2.5) and (2.6) with* $\lambda = -p_{N+1}$.

(4) *For any* $N \in \mathbb{Z}_{\geq 0}$, $\widetilde{\Theta}_n^m = \Theta_n^m(N+1)$ *is a Bäcklund transform of* $\Theta_n^m = \Theta_n^m(N)$ *related by eqs.(4.8) and (4.9) with* $\lambda = -p_{N+1}$.

(5) *For any* $l \in \mathbb{Z}$, $\overline{\theta}(x,t) = \theta(x,t;l+1)$ *is a Bäcklund transform of* $\theta(x,t) = \theta(x,t;l)$ *related by eqs.(4.19) and (4.20) with* $\lambda = d_l$.

(6) *For any* $l \in \mathbb{Z}$, $\overline{\Theta}_n^m = \Theta_n^m(l+1)$ *is a Bäcklund transform of* $\Theta_n^m = \Theta_n^m(l)$ *related by eqs.(4.23) and (4.24) with* $\lambda = d_l$.

A Derivation of bilinear equations (5.26) and (5.27)

In this appendix, we show that the τ function given in eqs.(5.19)–(5.21) actually satisfies the bilinear equations (5.26) and (5.27). For this purpose, we first introduce the generic τ function $\tau_n^m(k,l,N;s) = \tau_n^m(x,t;y,z;k,l,N;s)$ by

$$\tau_n^m(k,l,N;s) = \exp\left[-\left(x + \sum_{n'}^{n-1} a_{n'} + \sum_{m'}^{m-1} b_{m'} + \sum_{k'}^{k-1} c_{k'} + \sum_{l'}^{l-1} \frac{1}{d_{l'}}\right) y\right] \det\left(f_{s+j-1}^{(i)}\right)_{i,j=1,\dots,N}, \tag{A.1}$$

for $(x,t;y,z) \in \mathbb{R}^4$, $(m,n,k,l,s) \in \mathbb{Z}^5$ and $N \in \mathbb{Z}_{\geq 0}$. We require $f_s^{(i)} = f_s^{(i)}(x,t;y,z;m,n;k,l,N)$ $(i = 1,\dots,N)$ to satisfy the linear equations (5.2), (5.3) and

$$\frac{f_s^{(i)}(k,l) - f_s(k-1,l)}{c_k} = f_{s+1}^{(i)}(k,l), \qquad \frac{f_s^{(i)}(k,l) - f_s(k,l-1)}{d_l} = f_{s-1}^{(i)}(k,l). \tag{A.2}$$

A typical example for $f_s^{(i)}$ is given by

$$f_s^{(i)} = e^{\eta_i} + e^{\mu_i}, \tag{A.3}$$

$$\begin{cases} e^{\eta_i} = \alpha_i p_i^s \displaystyle\prod_{n'}^{n-1}(1 - a_{n'}p_i)^{-1} \prod_{m'}^{m-1}(1 - b_{m'}p_i)^{-1} \prod_{k'}^{k-1}(1 - c_{k'}p_i)^{-1} \prod_{l'}^{l-1}\left(1 - \frac{d_{l'}}{p_i}\right)^{-1} e^{p_i x - 4p_i^3 t + \frac{1}{p_i} y}, \\ e^{\mu_j} = \beta_i(-p_i)^s \displaystyle\prod_{n'}^{n-1}(1 - a_{n'}q_i)^{-1} \prod_{m'}^{m-1}(1 - b_{m'}q_i)^{-1} \prod_{k'}^{k-1}(1 - c_{k'}q_i)^{-1} \prod_{l'}^{l-1}\left(1 - \frac{d_{l'}}{q_i}\right)^{-1} \\ \qquad \times\, e^{q_i x - 4q_i^3 t + \frac{1}{q_i} y}, \end{cases} \tag{A.4}$$

where p_i, q_i, α_i and β_i are arbitrary complex constants. We put

$$\sigma_n^m(y;k,l,N;s) = \det\left(f_{s+j-1}^{(i)}\right)_{i,j=1,\dots,N}. \tag{A.5}$$

Proposition A.1 *σ satisfies the following bilinear equations:*

$$D_y\, \sigma_n^m(N+1;s) \cdot \sigma_n^m(N;s) = \sigma_n^m(N;s+1)\sigma_n^m(N+1;s-1), \tag{A.6}$$

$$\left(D_y - \frac{1}{d_l}\right) \sigma_n^m(l+1;s) \cdot \sigma_n^m(l;s+1) = -\frac{1}{d_l}\sigma_n^m(l+1;s+1)\sigma_n^m(k;s). \tag{A.7}$$

We apply the determinantal technique in order to prove Proposition A.1. The bilinear equations are reduced to the Plücker relations which are quadratic identities of determinants whose columns are appropriately shifted. To this end, we construct such formulas that express the determinants in the Plücker relations in terms of derivative or shift of discrete variable of $\sigma_n^m(k, l, N; s)$ by using the linear relations of the entries. For the details of the technique, we refer to [18, 23, 24, 28, 29].

We introduce a notation

$$\sigma_n^m(l, N; s) = |\, 0_l,\ 1_l,\ \cdots,\ N-2_l,\ N-1_l\,|, \tag{A.8}$$

where "j_l" denotes the column vector

$$j_l = \begin{bmatrix} f_{s+j}^{(1)}(l) \\ \vdots \\ f_{s+j}^{(N)}(l) \end{bmatrix}. \tag{A.9}$$

Lemma A.2 *The following formulas hold:*

$$\partial_y \sigma_n^m(l, N; s) = |\, -1,\ 1,\ \cdots,\ N-2,\ N-1\,|, \tag{A.10}$$

$$\sigma_n^m(l+1, N; s) = |\, 0_{l+1},\ 1,\ \cdots,\ N-2,\ N-1\,|, \tag{A.11}$$

$$d_l \sigma_n^m(l+1, N; s) = |\, 1_{l+1},\ 1,\ \cdots,\ N-2,\ N-1\,|, \tag{A.12}$$

$$-\left(d_l \partial_y - 1\right) \sigma_n^m(l+1, N; s) = |\, 0,\ 1_{l+1},\ 2,\ \cdots,\ N-2,\ N-1\,|. \tag{A.13}$$

Note that the subscript of column vectors are shown only when l is shifted for notational simplicity.

Proof. Eq. (A.10) can be verified by direct calculation by using the fourth equation in eq.(5.2). We have

$$\sigma_n^m(l+1, N; s) = |\, 0_{l+1},\ 1_{l+1},\ \cdots,\ N-2_{l+1},\ N-1_{l+1}\,|. \tag{A.14}$$

Adding the $(N-1)$-th column multiplied by d_{l+1} to the N-th column and using eq.(A.2), we have

$$\sigma_n^m(l+1, N; s) = |\, 0_{l+1},\ 1_{l+1},\ \cdots,\ N-2_{l+1},\ N-1_l\,|. \tag{A.15}$$

Similarly, adding the $(i-1)$-th column multiplied by d_{l+1} to the i-th column and using eq.(A.2) for $i = N-1, \ldots, 2$, we obtain

$$\sigma_n^m(l+1, N; s) = |\, 0_{l+1},\ 1,\ \cdots,\ N-2,\ N-1\,|, \tag{A.16}$$

which is eq. (A.11). Multiplying d_{l+1} to the first column of eq. (A.11) and using eq.(A.2), we obtain eq. (A.12). Finally, differentiating eq.(A.12) with respect to y yields

$$\begin{aligned} d_l \partial_y \sigma_n^m(l+1, N; s) &= |\, 0_{l+1},\ 1,\ 2,\ \cdots,\ N-2,\ N-1\,| + |\, 1_{l+1},\ 0,\ 2,\ \cdots,\ N-2,\ N-1\,| \\ &= \sigma_n^m(l+1, N; s) - |\, 0,\ 1_{l+1},\ 2,\ \cdots,\ N-2,\ N-1\,|, \end{aligned} \tag{A.17}$$

which is equivalent to eq.(A.13). This completes the proof. □

Proof of Proposition A.1 Consider the Plücker relation (see, for example, [29]),

$$\begin{aligned}0 =& |-1,0,1\cdots,N-2| \times |1,\cdots,N-2,N-1,\phi| \\ &+ |0,1\cdots,N-2,N-1| \times |-1,1,\cdots,N-2,\phi| \\ &- |0,1\cdots,N-2,\phi| \times |-1,1,\cdots,N-2,N-1|,\end{aligned} \tag{A.18}$$

where ϕ is a column vector given by

$$\phi = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{A.19}$$

By using eqs.(A.8) and (A.10), expanding the determinant with respect to the column ϕ, eq.(A.18) is rewritten as

$$0 = \sigma_n^m(N;s-1)\,\sigma_n^m(N-1;s+1) + \sigma_n^m(N;s)\,\partial_y\sigma_n^m(N-1;s) - \sigma_n^m(N-1;s)\,\partial_y\sigma_n^m(N;s), \tag{A.20}$$

which implies eq.(A.6). Similarly, applying Lemma A.2 on the Plücker relation

$$\begin{aligned}0 =& |-1,0,1,\cdots,N-2| \times |0_{l+1},1,\cdots,N-2,N-1| \\ &- |0_{l+1},0,1,\cdots,N-2| \times |-1,1,\cdots,N-2,N-1| \\ &- |0,1,\cdots,N-2,N-1| \times |-1,0_{l+1},1,\cdots,N-2|,\end{aligned} \tag{A.21}$$

we obtain

$$\begin{aligned}0 =& \sigma_n^m(l;s-1) \times \sigma_n^m(l+1;s) - d_l\sigma_n^m(l+1;s-1) \times \partial_y\sigma_n^m(l;s) \\ &- \sigma_n^m(l;s) \times \left[-(d_l\partial_y - 1)\sigma_n^m(l+1;s-1)\right],\end{aligned} \tag{A.22}$$

which is equivalent to eq.(A.7). This completes the proof. □

From Proposition A.1 and eq.(A.1), we see that $\tau_n^m(k,l,N;s)$ satisfies

$$D_y\,\tau_n^m(N+1;s)\cdot\tau_n^m(N;s) = \tau_n^m(N;s+1)\tau_n^m(N+1;s-1), \tag{A.23}$$

$$D_y\,\tau_n^m(l+1;s)\cdot\tau_n^m(l;s+1) = -\frac{1}{d_l}\tau_n^m(l+1;s+1)\tau_n^m(k;s). \tag{A.24}$$

We finally obtain eqs.(5.26) and (5.27) from eqs.(A.23) and (A.24), respectively, by imposing the reduction condition (3.28).

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