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A Note on Discrete Bruschi-Ragnisco Lattice

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Discretization of the Bruschi-Ragnisco lattice is investegated by singularity confinement test. The equation is linearized by the Cole-Hopf like transformation.

1. Introduction

Some interesting completely integrable Hamiltonian systems, invariant under the Poincaré group, have been discovered by Ruijsenaars[1)]. The properties of these systems have been studied by various authors and the discrete analogues were presented by Suris[2)].

Bruschi and Ragnisco introduced a new relativistic Hamiltonian system with nearest-neighbor interaction. The Bruschi-Ragnisco lattice is defined by the Hamiltonian

$$H = \sum_{j=1}^{N} \exp(p_j)(q_{j+1} - q_j). \tag{1}$$

Therefore we have the following Hamiltonian equations for canonical variables (p, q),

$$\dot{q}_n = \frac{\partial H}{\partial p_n} = \exp(p_n)(q_{n+1} - q_n), \tag{2}$$

$$\dot{p}_n = -\frac{\partial H}{\partial q_n} = \exp(p_n) - \exp(p_{n-1}), \tag{3}$$

and an equation of motion(Bruschi-Ragnisco lattice equation[3)])

$$\ddot{q}_n = \frac{\dot{q}_{n+1}\dot{q}_n}{q_{n+1} - q_n} - \frac{\dot{q}_n \dot{q}_{n-1}}{q_n - q_{n-1}}. \tag{4}$$

Bruschi-Ragnisco(BR) lattice equation (4) is transformed into

$$\dot{b}_n = b_{n+1}c_n - b_n c_{n-1}, \tag{5a}$$

$$\dot{c}_n = c_n(c_n - c_{n-1}), \tag{5b}$$

through

$$b_n = q_n - q_{n-1}, \quad c_n = \exp(p_n) = \frac{\dot{q}_n}{q_{n+1} - q_n}. \tag{6}$$

This system is interesting because it is one of the few completely integrable Hamiltonian systems with nearest-neighbor nonlinear interaction. This system is also completely integrable in the Arnold-Liouville sense and linearizable by the change of variable[3)].

In this note, the discrete analogue of BR lattice equation presented by Suris is investigated by using singularity confinement test which is a powerful tool in judging integrability for discrete systems and in constructing solutions[6-9)].

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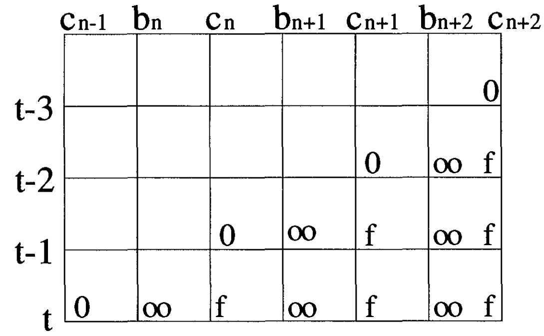

Fig.1 Singularity pattern 1 of BR lattice (f:finite)

2. Singularity confinement test

Suris presented discrete Bruschi-Ragnisco lattice equation[4,5],

$$\frac{q_n^{t+1}-q_n^t}{q_n^t-q_n^{t-1}}=\frac{q_{n+1}^t-q_n^t}{q_{n+1}^{t-1}-q_n^t}\frac{q_n^t-q_{n-1}^{t+1}}{q_n^t-q_{n-1}^t}. \tag{7}$$

If we perform singularity confinement test for eq.(7), we don't have divergence and eq.(7) passes singularity confinement test. However we cannot obtain the dependent variable transformation because of lack of information of singularity pattern.

Eq.(7) is transformed into

$$b_n^{t+1}(1+hc_{n-1}^{t+1})=b_n^t+hb_{n+1}^tc_n^t, \tag{8a}$$

$$c_n^{t+1}=c_n^t\frac{1+hc_n^{t+1}}{1+hc_{n-1}^{t+1}}, \tag{8b}$$

through the transformation

$$b_n^t=q_n^t-q_{n-1}^t,\, c_n^{t+1}=\exp(p_n^{t+1})=\frac{q_n^{t+1}-q_n^t}{h(q_{n+1}^t-q_n^{t+1})}. \tag{9}$$

Through the independent variable transformation $n-t\to n$, discrete BR lattice equation is transformed into

$$b_n^t=\frac{b_{n-2}^{t+1}(1+hc_{n-3}^{t+1})-b_{n-1}^t}{hc_{n-1}^t}, \tag{10a}$$

$$c_n^t=c_{n-1}^{t+1}\frac{1+hc_{n-2}^{t+1}}{1+hc_{n-1}^{t+1}}. \tag{10b}$$

Let us perform the singularity confinement test for this system. There are two sources of singularity. One is the vanishing of the denominator of the first equation and the other that of the second equation.

First, let us consider the case of vanishing the denominator of the first equation. When we suppose that c_n^t happens to be ϵ, then b_{n+1}^t become ∞. Divergence of b does not disappear in the subsequent steps. In this case the singularity is not confined.(See **Fig.1**.)

However, when we suppose that c_n^t happens to be $-\frac{1}{h}+\epsilon$, the singularity is confined and the singularity pattern is

$$\{c_{n-1}^t,c_n^{t-1},c_{n+1}^{t-1}\}\to\{-\frac{1}{h},\infty,0\}, \tag{11a}$$

$$\{b_{n+1}^{t-1}\}\to\{0\}. \tag{11b}$$

(See **Fig.2**.)

	c_{n-1}	b_n	c_n	b_{n+1}	c_{n+1}	b_{n+2}	c_{n+2}
t-3							
t-2							f
t-1		f	∞	0	0	f	f
t	-1/h	f	f	f	f	f	f

Fig.2 Singularity pattern 2 of BR lattice

Let us assume that the singularity pattern is caused by the fact that a τ function τ_n^t becomes zero at (n,t). We then obtain the dependent variable transformation

$$b_n^t = \alpha\tau_{n-1}^{t+1}, \tag{12a}$$

$$c_n^t = -\frac{1}{h} + \beta\frac{\tau_{n+1}^t}{\tau_n^{t+1}} = \gamma\frac{\tau_{n-1}^{t+1}}{\tau_n^{t+1}}, \tag{12b}$$

and we can get a linear equation

$$-\frac{1}{h}\tau_n^{t+1} + \beta\tau_{n+1}^t = \gamma\tau_{n-1}^{t+1}. \tag{13}$$

Through $n+t \to n$, the following equation is derived:

$$-\frac{1}{h}\tau_{n+1}^{t+1} + \beta\tau_{n+1}^t = \gamma\tau_n^{t+1}. \tag{14}$$

Thus we know that discrete BR lattice equation is integrable, that is, linearizable.

The property of singularity confinement must be satisfied for all conceivable movable singularity. However, in this case the discrete BR lattice is integrable while there is a pattern for which singularity confinement property is not satisfied.

Let us see the discrete BR lattice equation again. This equation is a simultaneous equation for the dependent variables b, c. We must remark that the second equation depends only on c. The dependent variable c is decided by only the second equation, thus the first equation is a linear difference equation that does not have movable singularities. Finally, since the first singularity is non-movable singularity, we do not have to consider this singularity in singularity confinement test. Thus we conclude that discrete BR lattice equation has singularity confinement property.

In the continuous limit, we obtain the dependent transformation

$$b_n = \tau_{n-1}, c_n = \frac{\tau_{n-1}}{\tau_n}. \tag{15}$$

This transformation is Cole-Hopf like transformation. In terms of these new variables the system (5) reduces to

$$\dot{\tau}_n = \tau_n - \tau_{n-1}. \tag{16}$$

This is transformed into

$$\dot{s}_n = s_n(s_n - s_{n-1}), \tag{17}$$

through the variable transformation $s_n = -\frac{d}{dt}\log\tau_n + 1$. From eq.(17), we get the semi-discrete Burgers equation

$$\dot{r}_n = \exp(r_n) - \exp(r_{n-1}), \tag{18}$$

by the variable transformation $r_n = \log s_n$. Therefore the (discrete) BR lattice equation has N-shock wave solution. We remark that the semi-discrete Burgers equation (18) is related with pulses of the FitzHugh-Nagumo equation[10)].

Suris pointed out that there is relation between the relativistic Toda lattice equation and BR lattice equation. However we cannot find such relation in the level of τ function (solution level). We note that the BR lattice is looked upon as the relativistic analogue of discrete Burgers equation.

3. Conclusion

We have shown that discrete BR lattice has singularity confinement property and the linear equation is obtained by using singularity pattern. We notice that singularity confinement test works in linearization of nonlinear equation. Although singularity confinement test has strange result in discrete BR lattice, we can except the strange pattern because of non movable singularity. This result indicates that we must study only movable singularities in singularity confinement test.

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