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Morse Reductions for Quiver Complexes and Persistent Homology on the Finite-Type Commutative Ladder Quivers

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Abstract

Persistent homology is a tool in topological data analysis for studying the robust topological features of data. The persistence diagram provides a compact way to summarize the presence, scale, and persistence of these features. In this thesis, we extend the applicability of persistent homology to a wider variety of settings by using bound quivers and their representations.

We give a definition of quiver complexes. This generalizes filtrations and zigzag complexes – settings where persistence analysis is already well developed. We then define the persistent homology $H_q(\mathbb{X})$ of a quiver complex $\mathbb{X}$, which we show to be a representation of a quiver bound by commutativity. Motivated by applications, we focus specifically on the so-called commutative ladder quivers $CL_n(\tau)$.

In this direction, we show that the commutative ladder quivers $CL_n(\tau)$ with length $n \leqslant 4$ are representation-finite by computing their Auslander-Reiten quivers. Moreover, using the Auslander-Reiten quivers, we provide a generalization of the definition of persistence diagrams. In the representation-finite commutative ladder case, we show how to visualize our generalized persistence diagrams. Moreover, some methods and examples for the interpretation of the persistence diagrams in these cases are provided.

We extend the use of discrete Morse theory to our setting of quiver complexes. In particular, given a quiver complex $\mathbb{X}$ and an acyclic matching for $\mathbb{X}$, we show that there is an associated Morse quiver complex $\mathbb{A}$ with the property that $\mathbb{X}$ and $\mathbb{A}$ have isomorphic persistent homology. The Morse quiver complex $\mathbb{A}$ tends to be smaller in size, so that computing the persistent homology from $\mathbb{A}$ instead of $\mathbb{X}$ tends to be less costly.

An algorithm to compute an acyclic matching for a quiver complex $\mathbb{X}$ and the associated Morse quiver complex $\mathbb{A}$ is given. The computation of a persistence diagram follows from the computation of an indecomposable decomposition of a representation. We give an algorithm for this in the case of $CL_n(\tau)$ with $n = 3$.

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Contents

Chapter 1

Introduction

The major theme of this work is the use of representation theory in the development of tools for data analysis. We shall see that some known techniques in representation theory have much to contribute to topological data analysis (TDA) via persistent homology.

Topological data analysis is a fast-growing field that applies various algebraic and topological methods to data analysis. The survey paper [6] discusses several distinguishing characteristics of the TDA point of view. For example, TDA tools are independent of coordinates and particular metrics, and focus on qualitative features such as shape, in the sense of topology. For a detailed explanation of why these characteristics of TDA would be appropriate for certain types of data, we refer the reader to [6].

In particular, we focus on the ideas of persistent homology. First, let us roughly sketch the basic framework. Dealing with topological spaces directly may be difficult. An insight from algebraic topology is that a functor, say from topological spaces to abelian groups, can be used to convert a topological problem into an algebraic one. Of course, some information (such as coordinates and metrics) may be lost, but depending on application this can be desirable. For example, the homology groups of a topological space contain information about certain topological features – connected components, holes, voids, and so on.

As a further refinement of this basic idea, suppose that instead of just one topological space, we have a diagram of topological spaces, and would like to study what topological features are shared (persistent) across different spaces. Applying the homology functor with field coefficients K, we obtain a diagram of homology K-vector spaces and K-linear maps between them. This brings us into the realm of the representation theory of quivers.

This leap to representations of quivers hides and abstracts a lot of detail, so let us retrace its development starting from the persistent homology of filtrations.

Also, before going into specifics, let us note the following. At the level of representation theory, we can ignore the fact that the diagrams of vector spaces and maps are obtained from diagrams of topological spaces, and treat them purely as algebraic objects. This allows the use of powerful known representation-theoretic techniques.

In the following discussion, we shall make some choices about geometric constructions and shapes of diagrams, in order to concretely motivate our study of persistent homology. It should be clear, however, that the framework sketched above applies to a wide variety of constructions. By the observation above, once we go into the representation theory, we have a unified language for treating all the specific examples fitting into this framework.

The first choice that we make is that we approximate a topological space by a simplicial complex – a collection of vertices, edges, triangles, and so on. The homology groups (homology vector spaces) of simplicial complexes can be given similar interpretations as those of topological spaces. In fact, this approximation can be made precise. In the case that the topological space is triangulable, it has a triangulation. A triangulation is a simplicial complex with homology groups isomorphic to the homology groups of the topological space.

As a second choice, we further assume that the input data comes as a *point cloud* – a finite set of points $P \subset \mathbb{R}^N$. There are methods, as discussed in Section 2.2, for constructing simplicial complexes that attempt to give a "shape" to the point cloud data. These methods involve a choice of parameter values. This presents a challenge, as we may not have a priori information about the correct parameter value(s), if one even exists. Moreover, for fixed parameter value, a slight change in the points of P, say from noise, may radically change the structure of the resulting simplicial complex. The information provided by the homology groups alone may not reflect only the relevant features.

Persistent homology [16] allows one to study robust topological features in a filtration – a nested sequence of spaces $X_1, \ldots, X_n$:

$$X_1 \hookrightarrow X_2 \hookrightarrow \ldots \hookrightarrow X_n. \tag{1.1}$$

Applied to the choices above, this is a natural setting for when varying the parameter in the geometric construction results in a nested sequence of spaces. Then, we are freed from the need to choose the parameter value. Of course, persistent homology can be applied to a wide variety of inputs where a filtration can be obtained.

From the above diagram, we compute the diagram of K-vector spaces and maps induced by homology

$$H_q(X_1) \xrightarrow{H_q(\iota_1)} \ldots \xrightarrow{H_q(\iota_{n-1})} H_q(X_n) \tag{1.2}$$

called the persistent homology of the filtration. Here, ι_i are the inclusion maps and $H_q(\iota_i)$ the induced linear maps of homology vector spaces.

A standard structure theorem shows that we can decompose the above persistent homology module into the so-called interval modules. Each interval then tracks the life of a homology generator through the filtration, together with its birth and death indices. This allows us to distinguish between topological features likely to have been created by noise, and those that are robust – those with long lifespans. These information can be presented in a compact format called the persistence diagram (PD).

Let us illustrate this with a toy example. In Fig. 1.1, we show a filtration obtained by progressively enlarging the radii in the union of balls model of a set of points. At $r = 0.1$, the tiny hole on the right is born, and it disappears at $r = \frac{\sqrt{2}}{10} \approx 0.1414$. Then, at radius value $r \approx 0.2588$, the big central hole is born, and lasts until $r \approx 0.9526$.

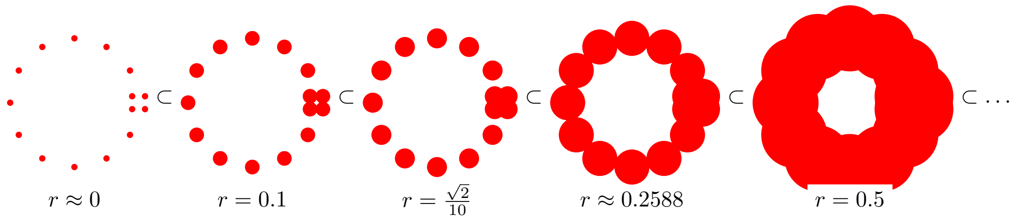


Figure 1.1: A filtration

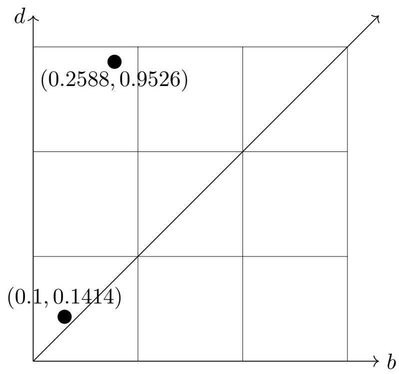


Figure 1.2: Dimension 1 persistence diagrams of the filtration in Fig. 1.1. The persistence diagram separates features into "noisy" features and robust features according to their distances from the diagonal.

The filtration has dimension 1 persistence diagram given in Fig. 1.2. Note that it consists of the persistence intervals $[0.1, 0.1414)$ and $[0.2588, 0.9526)$, corresponding to a short-lived and a persistent feature, respectively. In this easy case, the PD analysis is not really necessary as a visual check suffices for distinguishing between persistent and small-scale features. However, for very complicated or high-dimensional data sets, the computation of the PDs, which can be done by a computer, allows one to automate the detection of persistent features.

Under certain conditions, it can be shown that what we get is a descriptor that is stable [11, 13] under small perturbations of the input data – an essential feature for a data analysis tool. Applied to data, the power of the persistent homology analysis comes from being able to measure the presence, scale, and robustness of certain topological features. For example, persistent homology has been applied to protein structural analysis [22] and amorphous glass [33, 34].

However, there may be other settings where a persistence analysis may provide insights into the input data. For example, suppose instead that we have spaces $X_1, \dots, X_T$ with no natural filtration structure. An example of this is time series data describing the dynamics of some system. To extract the topological features that persist through time, form the diagram of spaces

$$X_1 \hookrightarrow X_1 \cup X_2 \hookleftarrow X_2 \hookrightarrow \dots \hookrightarrow X_{T-1} \cup X_T \hookleftarrow X_T$$

which we call a zigzag complex. Here, the arrows between the spaces represent inclusion maps. We then study the resulting diagram of homology vector spaces and linear maps induced from inclusions

$$H_q(X_1) \longrightarrow H_q(X_1 \cup X_2) \longleftarrow \dots \longrightarrow H_q(X_{T-1} \cup X_T) \longleftarrow H_q(X_T),$$

which is called the (zigzag) persistent homology of the zigzag complex. The theory of zigzag persistence can be found in [7].

Another example is a multifiltration which is a set of spaces nested in two or more "dimensions" or "axes". We illustrate a multifiltration of dimension 2:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \iddots \\
X_{1,3} \arrow[u, hook] \arrow[r, hook] & \dots \arrow[u, hook] \arrow[r, hook] & \dots \arrow[u, hook] \arrow[r, hook] & \dots \\
X_{1,2} \arrow[u, hook] \arrow[r, hook] & X_{2,2} \arrow[u, hook] \arrow[r, hook] & \dots \arrow[u, hook] \arrow[r, hook] & \dots \\
X_{1,1} \arrow[u, hook] \arrow[r, hook] & X_{2,1} \arrow[u, hook] \arrow[r, hook] & X_{3,1} \arrow[u, hook] \arrow[r, hook] & \dots
\end{tikzcd}$$

with corresponding persistent homology of the multifiltration

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \iddots \\
H_q(X_{1,3}) \arrow[u] \arrow[r] & \dots \arrow[u] \arrow[r] & \dots \arrow[u] \arrow[r] & \dots \\
H_q(X_{1,2}) \arrow[u] \arrow[r] & H_q(X_{2,2}) \arrow[u] \arrow[r] & \dots \arrow[u] \arrow[r] & \dots \\
H_q(X_{1,1}) \arrow[u] \arrow[r] & H_q(X_{2,1}) \arrow[u] \arrow[r] & H_q(X_{3,1}) \arrow[u] \arrow[r] & \dots
\end{tikzcd}$$

where the linear maps are induced from the inclusions.

The persistent homology of multifiltrations was treated in the theory of multidimensional persistence [8]. There are algebraic difficulties in this direction, so the classical persistence analysis cannot be generalized to this case without modifications. One observation in [8] is that there is no complete discrete invariant that can parametrize all the indecomposable multidimensional persistence modules.

An insight of Carlsson and de Silva in the paper [7] is that the algebraic foundations of the persistent homology of filtrations can be rephrased in terms of representations of quivers. In this work, a quiver is a finite acyclic and connected directed graph, and a representation of a quiver is a finite-dimensional K-vector space for each vertex and a K-linear map between corresponding spaces for every arrow in the quiver. Immediately, it can be seen that the persistent homology of a filtration (Diagram (1.2)) is a representation of the quiver:

$$\vec{A}_n = \quad \overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \overset{3}{\circ} \longrightarrow \ldots \longrightarrow \overset{n}{\circ} \quad .$$

Once the link to the representation theory of quivers was established, it is easy to consider more general indexing quivers for diagrams of homology vector spaces. For example, the persistent homology of zigzag complexes and multifiltrations can be viewed as representations over the appropriate underlying quivers.

Historically, the link to the representation theory of quivers was not used in the original formulation of the algebraic basis of persistent homology. Rather, the persistent homology of a filtration was first recognized as a graded module over a graded polynomial ring. However, this point of view may be difficult to generalize. For example, it is not immediately clear how to handle arrows that point in different directions.

As another consequence of this change of perspective, the term persistence module can now be understood more generally. Persistence modules were originally defined [37] to mean what we now recognize to be nothing but representations of some quiver. While we prefer the use of the term representation, we follow this convention.

The consideration of a general theory for persistent homology is not just out of a desire for beautiful abstractions. Our use of the representation theory of quivers is also motivated by a practical example.

In the study [33], data consisting of the 3D locations of the atoms, and their radii, of an atomic configuration of amorphous glass was obtained by molecular dynamics simulation. To an atomic configuration is associated its shape by the union of balls model. To study its persistent features, we construct the union of balls filtration obtained by progressively increasing the radii of the balls.

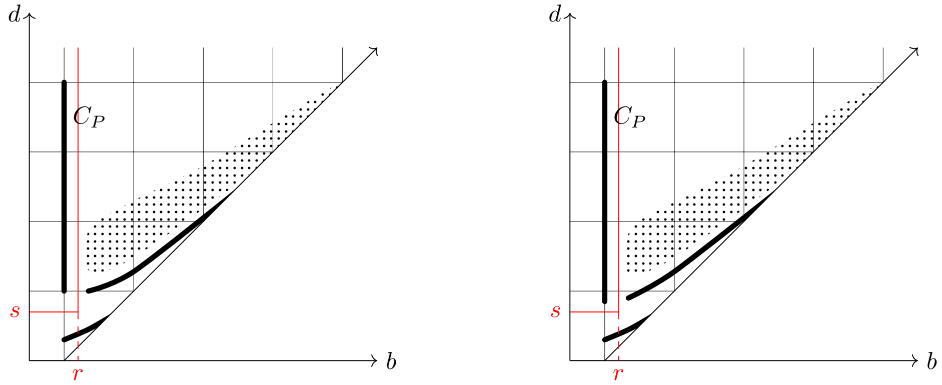
However, the union of balls model is challenging to treat computationally. Instead, we construct the weighted alpha complex filtration [15] of the atomic configuration and compute its persistent homology. The weighted alpha complex has underlying topological space homotopy equivalent to the union of balls space [15]. Thus, by studying the persistent homology of the weighted alpha complex filtration, we can extract information about the persistent topological features in the union of balls filtration.

The PDs of the dimension 1 persistent homology of the weighted alpha complex filtrations contain certain regions which clarifies some of the geometric structure of glass. We display a rough sketch of one such PD in Fig. 1.3a. Here, we cluster the persistence intervals – usually drawn as points on the plane – into regions and curves. In particular, the presence of the vertical region C_P in the PDs represents an important medium-range order in the atomic structure of glass.

Then, a simulation of isotropic pressurization on the amorphous glass was performed. The resulting atomic configuration has a PD that looks very similar to that of the point cloud before pressurization. For comparison, we display the before-and-after PDs in Fig. 1.3.

Let X be the weighted alpha complex filtration of the atomic configuration before pressurization, and Y, after pressurization. In the PDs in Fig. 1.3, the persistence intervals in C_P are characterized by having birth $b \leqslant r$ and death $d > s$, for some fixed numbers r, s. Equivalently, C_P contains features that are persistent in the two-step persistence modules

$$H_q(X_r) \xrightarrow{H_q(\iota)} H_q(X_s) \quad \text{and} \quad H_q(Y_r) \xrightarrow{H_q(\iota)} H_q(Y_s) \ .$$



(a) PD of $H_1(X)$. X is the weighted alpha complex filtration before pressurization.

(b) PD of $H_1(Y)$. Y is the weighted alpha complex filtration after pressurization. Shifts in curves after pressurization are exaggerated.

Figure 1.3: Rough sketches of persistence diagrams of amorphous glass, showing characteristic curves and the band region. The primary curve C_p is important.

So far, we have only used the persistent homology of filtrations. However, from only the presence of the region C_P in the PDs before and after the simulation of pressurization, we cannot conclude that the topological features giving rise to C_P are preserved. It is possible that the features are broken by the pressurization, and coincidentally new and unrelated persistent features described by a similar vertical region are created. To check this, we need to consider the simultaneously robust and common features.

The tool that we propose for studying simultaneously robust and common features is the diagram of spaces

$$\mathbb{X}: \begin{array}{ccccc} X_s & \hookrightarrow & X_s \cup Y_s & \hookleftarrow & Y_s \\ \uparrow & & \uparrow & & \uparrow \\ X_r & \hookrightarrow & X_r \cup Y_r & \hookleftarrow & Y_r \end{array} \tag{1.3}$$

from which we compute the diagram

$$H_q(\mathbb{X}): \begin{array}{ccccc} H_q(X_s) & \longrightarrow & H_q(X_s \cup Y_s) & \longleftarrow & H_q(Y_s) \\ \uparrow & & \uparrow & & \uparrow \\ H_q(X_r) & \longrightarrow & H_q(X_r \cup Y_r) & \longleftarrow & H_q(Y_r) \end{array} \tag{1.4}$$

of homology K-vector spaces and linear maps induced from the inclusions. The left and right sides of $H_q(\mathbb{X})$ in Diagram (1.4) give features persistent in X and Y, respectively. Horizontally, we get features that are common between X and Y. Clearly, $H_q(\mathbb{X})$ is a representation of the quiver

$$L_3(fb): \begin{tikzcd} \circ \arrow[r] & \circ & \circ \arrow[l] \\ \circ \arrow[r] \arrow[u] & \circ \arrow[u] & \circ \arrow[l] \arrow[u] \end{tikzcd}. \tag{1.5}$$

The questions now are, can we decompose the persistent homology $H_q(\mathbb{X})$ in manner a similar to the persistent homology of filtrations, and what are the analogues for the "interval modules" – the building blocks for persistent homology?

Using the language of representation theory, the decomposition is given by a Krull-Schmidt theorem, and the analogues to interval modules are the indecomposable representations. In the above example, the indecomposable summands of $H_q(\mathbb{X})$ isomorphic to

$$\begin{tikzcd} K \arrow[r, "1"] & K & K \arrow[l, "1"'] \\ K \arrow[r, "1"] \arrow[u, "1"] & K \arrow[u, "1"] & K \arrow[l, "1"'] \arrow[u, "1"] \end{tikzcd} \tag{1.6}$$

give the simultaneously robust and common features.

More than studying just the indecomposable summands isomorphic to Eq. (1.6), we give an extension of the definition of persistence diagrams. For this, it is convenient if there were only a finite number of isomorphism classes of indecomposables. A quiver is said to be representation-finite if the number of isomorphism classes of its indecomposable representations is finite. It is representation-infinite otherwise. In the representation-finite case, the persistence diagram is a complete finite discrete invariant which we can use to classify and analyze the persistence modules.

For our purposes, however, it is not enough to consider only quivers and their representations. It turns out that in many cases the whole representation category is too complicated. One known result in this direction is the following.

Theorem 1.0.1 (Gabriel's Theorem [21]). *A connected quiver is representation-finite if and only if its underlying graph is one of the ADE Dynkin diagrams: A_n, D_n, E_6, E_7, E_8.*

The quivers with underlying graph A_n underlie the persistent homology of filtrations and zigzag complexes. The exact forms of the diagrams D_n, E_6, E_7, E_8 are not important here. What we need to know is that the underlying graph of the quiver $L_3(fb)$ is not part of this list. At first glance it seems our study of Diagram (1.4) will be very complicated.

However, there is extra information in Diagram (1.4) that so far we have not used; it is in fact a commutative diagram. This leads us to consider the commutative ladders, which are simply the ladder quivers bound by commutativity, and more generally, quivers bound by relations. See Definition 2.1 for the definition of the commutative ladders $CL_n(\tau)$. In particular, $L_3(fb)$ in Diagram (1.5) with the commutativity relations is $CL_3(fb)$, and Diagram (1.4) is a representation of $CL_3(fb)$.

Note that Theorem 1.0.1 does not say anything about quivers bound by relations. We have the following theorem.

Theorem 1.0.2 (cf. [17, 26]). *Let τ be an arbitrary orientation of length n. The commutative ladder quiver $CL_n(\tau)$ is representation-finite if $n \leqslant 4$ and representation-infinite if $n > 4$.*

We prove the representation-finite part of this theorem via computation of the so-called Auslander-Reiten quivers. An advantage of this technique is that we get lists of the indecomposable representations of $CL_n(\tau)$ for $n \leqslant 4$ via the knitting of the Auslander-Reiten quivers. Moreover, we show that the Auslander-Reiten quiver of $\vec{A}_n$ is related to the persistence diagrams of the persistent homology of filtrations. By this relationship, we generalize the definition of persistence diagrams. In the representation-finite commutative ladder case, we propose methods for interpreting our generalized persistence diagrams.

While a different proof for Theorem 1.0.2 can be provided via the main theorem of [26] about representation-finite triangular matrix algebras, we adopt the proof strategy of computing the Auslander-Reiten quivers because of the advantages above. Nevertheless, we provide a brief discussion of this link to triangular matrix algebras.

Let us take a step back and take the general point of view from the start of the process. Motivated by the quiver-theoretic point of view, we consider diagrams of (simplicial) complexes indexed by a quiver, which we call quiver complexes (Definition 2.2). For example, a filtration is simply a quiver complex over $\vec{A}_n$, and the Diagram (1.3) is a quiver complex over the quiver $L_3(fb)$. In Definition 2.3, we give a definition of the persistent homology $H_q(\mathbb{X})$ of a quiver complex $\mathbb{X}$ which generalizes the persistent homology of a filtration.

The persistent homology $H_q(\mathbb{X})$ of a quiver complex is a representation of a bound quiver, and thus has an indecomposable decomposition, unique up to isomorphism:

$$H_q(\mathbb{X}) \cong \bigoplus_{[I]\in\Gamma_0} I^{m_{[I]}}, \tag{1.7}$$

where Γ_0 is the set of isomorphism classes of indecomposable representations of the underlying bound quiver (and in fact is equal to the set of vertices of its Auslander-Reiten quiver). In the case that the underlying bound quiver is $\vec{A}_n$, Eq. (1.7) gives the decomposition of persistent homology into the intervals. Generally, in some representation-finite case, for example $CL_n(\tau)$ with $n \leqslant 4$, Eq. (1.7) gives a finite set of numbers $m_{[I]}$ that characterizes $H_q(\mathbb{X})$ up to isomorphism.

For applications, it is essential to have algorithms. First, the computation of $H_q(\mathbb{X})$ can be done by combining known methods into our quiver-theoretic perspective. The computation of homology groups is classical. Applied to each vertex i, we can compute the homology K-vector space $H_q(X_i)$ of the complex X_i. For each arrow $\alpha : i \to j$ in Q_1, the computation of the map $H_q(\iota)$ induced from inclusion is simply linear algebra. This involves writing the chosen basis of $H_q(X_i)$ in terms of the basis of $H_q(X_j)$.

To decrease the size of the computation above, we use discrete Morse theory to replace the input quiver complex $\mathbb{X}$ by another quiver complex $\mathbb{A}$ with the property that $H_q(\mathbb{A}) \cong H_q(\mathbb{X})$. The quiver complex $\mathbb{A}$ is called the Morse quiver complex of $\mathbb{X}$ induced by an acyclic matching of $\mathbb{X}$. In general, $\mathbb{A}$ will be smaller in size compared to $\mathbb{X}$, so that the computation of $H_q(\mathbb{A})$ will hopefully be less costly. This process of using discrete Morse theory to replace $\mathbb{X}$ by a smaller quiver complex with isomorphic persistent homology is called Morse reduction.

Discrete Morse theory [20] has been developed as a discrete analogue of Morse theory, with discrete versions of Morse functions and vector fields. A link to the formulation in terms of acyclic matchings is provided in the paper [10]. This combinatorial point of view has been used effectively for efficient computations of the persistent homology of filtrations [30] by Mischaikow and Nanda. We extend these ideas to the quiver complex case.

There are also papers approaching Morse theory from an algebraic point of view [25, 36], where the acyclic matchings are defined on the level of chain complexes. However, we do not use this here.

For the second computational part, we focus on computing indecomposable decompositions only for representations of the representation-finite commutative ladder quivers. We shall see that by computing an indecomposable decomposition of a representation, we also get its persistence diagram.

Now, the computation of indecomposable decompositions of modules over algebras is a well-researched field [12, 27]. Let A be a finite-dimensional K-algebra. One fact that follows from a result in [12] is that, given a module $M \in \operatorname{mod} A$, there is a polynomial time algorithm for computing an indecomposable decomposition of M. While the general case is phrased in terms of modules, a link to representations of bound quivers is provided by Theorem 2.3.1. To summarize it here, there is an equivalence between the categories of representations of a bound quiver and of modules over a certain K-algebra. Thus, a representation can be viewed as a module and the preexisting algorithms can be applied.

We do not use these general methods. Instead, we give a rough sketch of the algorithm in [17] for computing an indecomposable decomposition of a representation V of $CL_3(fb)$. The algorithm involves only elementary linear algebra via changes of bases, in order to get direct sum decompositions. The general strategy of the algorithm can be applied to derive similar algorithms for $CL_n(\tau)$ with $n \leqslant 3$.

We also provide another alternative. It is possible to take advantage of the specific structure of the commutative ladder quivers to obtain algorithms. In particular, the structure of representations of $CL_n(\tau)$ lends itself very well to a reformulation as matrix problems in the sense of [35]. This connection is explored in the final chapter.

1.1 Overview of contributions

The main contributions of this thesis are the following. In Subsection 2.4.1, we provide our definition of quiver complexes and their persistent homology.

1. We use discrete Morse theory to show that given a quiver complex $\mathbb{X}$, it is associated to a Morse quiver complex $\mathbb{A}$ by an acyclic matching of $\mathbb{X}$. Moreover, we show that $H_q(\mathbb{X}) \cong H_q(\mathbb{A})$, in Theorem 3.2.4. (Section 3.2)

2. For use in computations, we also provide an algorithm for computation of a Morse quiver complex $\mathbb{A}$ of an input quiver complex $\mathbb{X}$, by modification of the known algorithm for computing an acyclic matching. Our computational strategy is enabled by Lemma 3.3.4. (Section 3.3)

3. We prove the representation-finite part of Theorem 1.0.2 above by computation of the relevant Auslander-Reiten quivers. (Section 4.2)

4. We show that the representation category $\operatorname{rep} CL_n(\tau)$ is equivalent to $\operatorname{mod} T_2(KA_n(\tau))$. From this we show that Theorem 1.0.2 follows from the main theorem of [26]. (Subsection 4.2.2)

In spite of the proof using [26], we argue that our use of the Auslander-Reiten quivers is valuable, because of next two contributions.

5. By using the Auslander-Reiten quivers, we generalize the definition of persistence diagrams to the representations of any representation-finite bound quiver. (Subsection 4.3.1)

6. Moreover, we give examples of ways to interpret the persistence diagrams of representations of $CL_n(\tau)$ with $n \leqslant 4$, in the context of input data. One example is given by the application to amorphous glass. (Subsection 4.3.3)

7. We provide an algorithm to compute indecomposable decompositions of representations of $CL_3(fb)$. (Section 4.4)

8. In Chapter 5, we link the representation theory of $CL_n(\tau)$ to so-called matrix problems, and show that this may provide a more elegant algorithm for computing indecomposable decompositions. This is an upcoming work.

This thesis primarily takes its results the author's works [17, 18, 19]. However, at places the treatment may vary with slightly different proof strategies and improved exposition. As discussed in the introduction, the work [33] provides one motivation for considering the commutative ladder quivers.

1.2 Organization

In Chapter 2, we review some basic facts and definitions used throughout this work. We start by defining some basic terms from category theory and homological algebra. Then, we move to geometric/combinatorial models in Section 2.2. For generalizing persistent homology, we introduce the concept of quivers and their representations in Section 2.3, which naturally leads to modules over algebras, reviewed in Section 2.5. We place Section 2.4 on persistent homology right after the section on quivers to take advantage of the quiver-theoretic point of view.

In Chapter 3, we discuss Morse reductions for quiver complexes. Our main theorem appears in Section 3.2. In Section 3.1 we provide a quick review of known results that we need for building up to our theorem. In Section 3.3, we provide an algorithm for computing an acyclic matching of a quiver complex, and its resulting Morse quiver complex.

In Chapter 4, we study the representation theory of quivers and its applications in persistence theory. In Section 4.1, we review the necessary algebraic background in the representation theory. In Section 4.2, we discuss the representation theory of the commutative ladders quivers and the computation of the Auslander-Reiten quivers of the finite-type commutative ladder quivers. Section 4.3 focuses on applications of the theoretical results derived in Sections 4.1 and 4.2 to topological data analysis. We place the algorithm for computing an indecomposable decomposition of a representation of $CL_3(fb)$ in Section 4.4.

In Chapter 5, we provide a link from representations of the commutative ladders to matrix problems.

An index of terms is also provided.

Chapter 2

Background

The algebraic structure of persistent homology was first described using graded modules over a polynomial ring. However, to provide a generalizable definition of persistent homology, we use the point of view of quiver representations. This motivates our use of the representation theory of quivers and K-algebras, where K is a field. To build all this algebraic machinery, we first review some basic terminology.

2.1 Basic terminology

2.1.1 Some category theory

We provide definitions for some of the less commonly used terms from category theory that we need. The reader is assumed to be familiar with the basic category theoretic notions of categories and subcategories, functors, equivalences and dualities, direct sums (coproducts) and zero objects. For more details, see [28] or the Appendix of [2].

Let C be a category, with class of objects $\operatorname{Ob} C$. For each pair of objects X, Y, $\operatorname{Hom}_C(X, Y)$ is the set of morphisms from X to Y. We shall use the notation $X \in \operatorname{Ob} C$, or even $X \in C$, to mean that X is an object of C. Moreover, we write $f : X \to Y$ to denote $f \in \operatorname{Hom}_C(X, Y)$.

Let K be a field. A *K-category* C is a category such that for every $X, Y \in \operatorname{Ob} C$, $\operatorname{Hom}_C(X, Y)$ has K-vector space structure and the composition $\circ$ of morphisms is K-bilinear. A K-category is said to be additive if for any finite set $X_1, \ldots, X_n$ of objects, there exists a direct sum $X_1 \oplus \ldots \oplus X_n$ in C.

A covariant functor $F : C \to D$ is said to be K-linear if F preserves direct sums, and for each $X, Y \in \operatorname{Ob} C$, the map

$$F_{XY} : \operatorname{Hom}_C(X, Y) \to \operatorname{Hom}_D(F(X), F(Y))$$

defined by $F_{XY}(f) = F(f)$ is a K-linear map of K-vector spaces. If C and D are K-categories, we simply say that $F : C \to D$ is a functor where we mean K-linear functor.

Let C be a K-category. The *arrow category* of C, denoted $\operatorname{arr}(C)$, is the K-category defined as follows. The objects of $\operatorname{arr}(C)$ are morphisms $(f : X \to$

$Y) \in \mathrm{Hom}_C(X,Y)$ for any X, Y in $\mathrm{Ob}\, C$. The morphisms from $(f : X \to Y)$ to $(g : M \to N)$ in $\mathrm{arr}\, C$ consist of pairs of morphisms $\phi_1 \in \mathrm{Hom}_C(X,M)$, $\phi_2 \in \mathrm{Hom}_C(Y,N)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi_1}\downarrow & & \downarrow{\scriptstyle\phi_2} \\ M & \xrightarrow{g} & N \end{array}$$

is commutative. We denote this morphism by $(\phi_1, \phi_2) : f \to g$. The composition of morphisms in $\mathrm{arr}(C)$ is defined by the following. For $(\phi_1, \phi_2) : f \to g$, $(\psi_1, \psi_2) : g \to h$, define

$$(\psi_1, \psi_2) \circ (\phi_1, \phi_2) = (\psi_1\phi_1, \psi_2\phi_2) : f \to h.$$

The K-vector space structure of the morphisms is defined in the obvious way: for $(\phi_1, \phi_2), (\psi_1, \psi_2) : f \to g$ and $k \in K$, $(\psi_1, \psi_2) + (\phi_1, \phi_2) = (\psi_1 + \phi_1, \psi_2 + \phi_2)$ and $k(\phi_1, \phi_2) = (k\phi_1, k\phi_2)$. It can be checked that if C is additive, then so is $\mathrm{arr}(C)$.

Let C be an additive K-category and I be a class of morphisms in C. Suppose that I satisfies the following properties.

1. For every $X \in \mathrm{Ob}\, C$, the zero morphism $0_X : X \to X$ is in I.

2. If $f, g : X \to Y$ are in I, then $af + bg$ is in I for any $a, b \in K$.

3. For a diagram of objects and morphisms in C:

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

   if $f \in I$ or $g \in I$, then $gf \in I$.

Then we say that I is a *two-sided ideal* of C. For any pair of objects $X, Y \in \mathrm{Ob}\, C$, let $I(X,Y)$ be the collection of morphisms $f : X \to Y$ in I. Clearly, this forms a K-vector subspace of $\mathrm{Hom}_C(X,Y)$. The *quotient category* C/I is the category with objects the same as in C, and morphisms

$$\mathrm{Hom}_{C/I}(X,Y) = \mathrm{Hom}_C(X,Y)/I(X,Y).$$

The quotient category C/I is also an additive K-category.

2.1.2 Algebras and modules

Let K be a field. A *K-algebra* A is a ring with identity A which is simultaneously a K-vector space such that the ring multiplication and the scalar multiplication by K satisfy

$$k(ab) = (ka)b = a(kb)$$

for all $k \in K$ and $a, b \in A$. To be precise, the condition that the ring is simultaneously a K-vector space simply means that the ring $(A, +, \cdot, 1)$ has underlying abelian group $(A, +)$ that is a K-vector space. We do not assume that the product is commutative. A K-algebra is said to be finite-dimensional if its dimension as a K-vector space is finite. We consider only finite-dimensional K-algebras.

A *left ideal* (*right ideal*) of a K-algebra A is a K-vector subspace I of A such that $ax \in I$ ($xa \in I$) for all $x \in I$, $a \in A$. If I is both a left and right ideal of A, it is said to be a *two-sided ideal* of A.

A left A-module is a K-vector space M together with a left multiplication by A, $\lambda : A \times M \to M$, satisfying the following conditions. For $a, b \in A$, $m, n \in M$, $k \in K$,

1. $(a + b)m = am + bm$ and $a(m + n) = am + an$,
2. $(ab)m = a(bm)$, $k(am) = (ka)m = a(km)$
3. $1m = m$,

where we write $\lambda(a, m) = am$, in the more familiar left multiplication notation. As notation, we write ${}_AM$ to indicate that M is a left A-module. A morphism of A-modules is a K-vector space morphism $f : M \to N$ such that $f(am) = af(m)$ for all $a \in A$, $m \in M$. The K-category of left A-modules is denoted $\operatorname{Mod} A$. This category is additive, and in fact is an Abelian K-category. The full subcategory of finitely generated A-modules is denoted by $\operatorname{mod} A$. Since A is finite-dimensional, $\operatorname{mod} A$ is the same as the full subcategory of finite-dimensional A-modules. Moreover, the category $\operatorname{mod} A$ has the Krull-Schmidt property.

Proposition 2.1.1 (Krull-Schmidt Theorem). *Let A be a finite-dimensional K-algebra. Suppose that $M \in \operatorname{mod} A$.*

1. *M has an indecomposable decomposition $M = M_1 \oplus \ldots \oplus M_s$ where M_i are indecomposable for all $i \in \{1, \ldots, s\}$.*
2. *If*
$$M = M_1 \oplus \ldots \oplus M_s = N_1 \oplus \ldots \oplus N_t$$
with M_i, N_j indecomposable for $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, t\}$. Then, $s = t$ and there is a permutation σ of $\{1, \ldots, s\}$ such that $M_i \cong N_{\sigma(i)}$ for all $i \in \{1, \ldots, s\}$.

2.1.3 Chain complexes

Let us review chain complexes of modules and their homologies. We refer the reader to the book [29], for example, for more details on homological algebra. This general point of view provides a convenient way to encapsulate the homology of simplicial complexes and even the persistent homology of quiver complexes, which we shall explain later.

Let K be a field, and A be a finite-dimensional K-algebra. A *chain complex* over $\operatorname{mod} A$ is a sequence of A-modules $\{C_q\}_{q \in \mathbb{Z}}$ in $\operatorname{mod} A$, together with morphisms $\partial_q : C_q \to C_{q-1}$ such that $\partial_q \partial_{q+1} = 0$ for every q. A chain complex is denoted by (C_q, ∂_q). In this work, we consider only chain complexes with

$C_q = 0$ for all $q < 0$. A *chain map* between two chain complexes $C = (C_q, \partial_q)$, $D = (D_q, \partial'_q)$ is a sequence of A-module morphisms $\phi_q : C_q \to D_q$, for $q \in \mathbb{Z}$, so that

$$\begin{tikzcd} C_q \arrow[r, "\partial_q"] \arrow[d, "\phi_q"'] & C_{q-1} \arrow[d, "\phi_{q-1}"] \\ D_q \arrow[r, "\partial'_q"] & D_{q-1} \end{tikzcd}$$

commutes for every $q \in \mathbb{Z}$.

The *homology* $H(C)$ of a chain complex $C = (C_q, \partial_q)$ is the collection of A-modules

$$H_q(C) = \operatorname{Ker} \partial_q / \operatorname{Im} \partial_{q+1}$$

for $q \in \mathbb{Z}$. For a fixed $q \in \mathbb{Z}$, $H_q(C)$ is called the qth *homology module* of C.

Moreover, if $\phi : C \to D$ is a chain map, then ϕ induces a morphism of A-modules

$$H_q(\phi) : H_q(C) \to H_q(D)$$

via $z + \operatorname{Im} \partial_{q+1} \mapsto \phi_q z + \operatorname{Im} \partial'_{q+1}$, for $z \in \operatorname{Ker} \partial_q$. Note that $\partial'_q \phi_q z = \phi_{q-1} \partial_q z = 0$, so that $\phi_q z$ is indeed in $\operatorname{Ker} \partial'_q$. Moreover, if $z - z' \in \operatorname{Im} \partial_{q+1}$, then $z - z' = \partial_{q+1} b$ for some b so that $\phi_q(z - z') = \phi_q \partial_{q+1} b = \partial'_{q+1} \phi_{q+1} b \in \operatorname{Im} \partial'_{q+1}$.

Let $C = (C_q, \partial_q)$ and $D = (D_q, \partial'_q)$ be two chain complexes, and $\theta = (\theta_q : C_q \to D_{q+1})$ be a sequence of morphisms in $\operatorname{mod} A$. Define the morphisms

$$h_q = \theta_{q-1} \partial_q + \partial'_{q+1} \theta_q : C_q \to D_q. \tag{2.1}$$

Then, $h = \{h_q\}$ is a chain map from C to D. To show that $h : C \to D$ is a chain map, it suffices to check that for every q, $\partial'_q h_q = h_{q-1} \partial_q$. We have

$$\begin{aligned} \partial'_q h_q &= \partial'_q \theta_{q-1} \partial_q + \partial'_q \partial'_{q+1} \theta_q \\ &= \partial'_q \theta_{q-1} \partial_q \\ &= \theta_{q-2} \partial_{q-1} \partial_q + \partial'_q \theta_{q-1} \partial_q \\ &= (\theta_{q-2} \partial_{q-1} + \partial'_q \theta_{q-1}) \partial_q \\ &= h_{q-1} \partial_q. \end{aligned}$$

Any chain map $h : C \to D$ satisfying Eq. (2.1) for some θ is said to be homotopic to 0 via the homotopy θ, denoted $h \sim 0$.

Let us show the following property of chain maps homotopic to 0. Let $C = (C_q, \partial_q)$, $D = (D_q, \partial'_q)$, and $E = (E_q, \partial''_q)$. If

$$B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{f} E$$

is a sequence of chain maps with $h \sim 0$, then $fh \sim 0$ and $hg \sim 0$. We have $h_q = \theta_{q-1} \partial_q + \partial'_{q+1} \theta_q$ so that

$$\begin{aligned} f_q h_q &= f_q \theta_{q-1} \partial_q + f_q \partial'_{q+1} \theta_q \\ &= f_q \theta_{q-1} \partial_q + \partial''_{q+1} f_{q+1} \theta_q \\ &= (f_q \theta_{q-1}) \partial_q + \partial''_{q+1} (f_{q+1} \theta_q) \end{aligned}$$

for every q. This shows that $fh \sim 0$ via $\theta' = \{\theta'_q = f_{q+1}\theta_q\}$. A similar argument shows that $hg \sim 0$.

Now, two chain maps $f, g : C \to D$ are said to be *homotopic* via the *homotopy* θ, denoted $f \sim g$, if $f - g \sim 0$ with

$$f_q - g_q = \theta_{q-1}\partial_q + \partial'_{q+1}\theta_q$$

for every q, for some collection of morphisms $\theta = (\theta_q : C_q \to D_{q+1})$. By the above property of chain maps homotopic to 0, it can be checked that given a diagram

$$B \xrightarrow{g} C \underset{h_2}{\overset{h_1}{\rightrightarrows}} D \xrightarrow{f} E$$

of chain complexes and chain maps with $h_1 \sim h_2$, then $fh_1 \sim fh_2$ and $h_1g \sim h_2g$.

Finally, if $f, g : C \to D$ and $f \sim g$, then $H_q(f) = H_q(g)$. This follows from definition, for:

$$\begin{aligned} H_q(f)(z + \operatorname{Im}\partial_{q+1}) &= f_qz + \operatorname{Im}\partial'_{q+1} \\ &= g_qz + \theta_{q-1}\partial_qz + \partial'_{q+1}\theta_qz + \operatorname{Im}\partial'_{q+1} \\ &= g_qz + \operatorname{Im}\partial'_{q+1} \\ &= H_q(g)(z + \operatorname{Im}\partial_{q+1}) \end{aligned}$$

for all $(z + \operatorname{Im}\partial_{q+1}) \in H_q(C)$. The following lemma follows from this observation.

Lemma 2.1.2. *Let C, D be chain complexes. If there exists chain maps $f : C \to D$ and $g : D \to C$ such that $fg \sim 1_D$ and $gf \sim 1_C$, then $H_q(C) \cong H_q(D)$ for every $q \in \mathbb{Z}$.*

In the case where $f : C \to D$ satisfies the hypothesis of the lemma above, we say that $f : C \to D$ is a *chain equivalence*, and that C and D are *chain equivalent*, denoted $C \sim D$.

Lemma 2.1.3. *Suppose that B, C, and D are chain complexes. If $B \sim C$ and $C \sim D$, then $B \sim D$.*

Proof. We have a diagram of complexes $B \underset{g_2}{\overset{g_1}{\rightleftarrows}} C \underset{h_2}{\overset{h_1}{\rightleftarrows}} D$ with $g_2g_1 \sim 1_B$, $g_1g_2 \sim 1_C$ and $h_2h_1 \sim 1_C$, $h_1h_2 \sim 1_D$. By the above discussion,

$$(g_2h_2)(h_1g_1) = g_2(h_2h_1)g_1 \sim g_2g_1 \sim 1_B$$

and

$$(h_1g_1)(g_2h_2) = h_1(g_1g_2)h_2 \sim h_1h_2 \sim 1_D$$

so that $B \sim D$. □

2.2 Combinatorial models

To fully explain the motivation of persistent homology in terms of topological data analysis, it is necessary to discuss the objects whose homology groups we are interested in. For a background in homology groups applied to topology, we refer the reader to the books [23, 32], for example.

2.2.1 Complexes

As before, let K be a field. A *complex* is a pair (X, κ) of a set X of elements called *cells* and an incidence map $\kappa : X \times X \to K$. The set of cells X is a disjoint union $X = \bigsqcup_{q \geqslant 0} X_q$. If $\sigma \in X_q$ we say that σ has dimension q, denoted $\dim \sigma = q$ and that σ is a q-cell. An *incidence map* is a map $\kappa : X \times X \to K$ that satisfies the following properties. If $\kappa(\sigma, \tau) \neq 0$, then $\dim \sigma = \dim \tau + 1$, and for $\rho, \tau \in X$,

$$\sum_{\sigma \in X} \kappa(\rho, \sigma)\kappa(\sigma, \tau) = 0.$$

Here, we only consider complexes with X finite. Where it does not cause any confusion, we suppress writing κ and just write X for a complex (X, κ).

Given a complex X, its q-th chain group is

$$C_q(X) = \left\{ \sum_{\sigma \in X_q} c_\sigma \sigma \;\middle|\; c_\sigma \in K \right\} = KX_q,$$

the free K-module generated by X_q. Since K is a field, this is nothing but the K-vector space generated by X_q.

Define the boundary maps $\partial_q : C_q(X) \to C_{q-1}(X)$, $q \geqslant 1$ by linear extension of

$$\partial_q \sigma = \sum_{\tau \in X} \kappa(\sigma, \tau)\tau$$

for $\sigma \in X_q$. Note that due to the requirement on κ, any τ that contributes a nonzero $\kappa(\sigma, \tau)\tau$ in the summation above will necessarily be in X_{q-1} so that $\partial_q \sigma \in C_{q-1}(X)$. The second requirement on κ shows that $\partial_{q-1}\partial_q = 0$ for all $q \geqslant 1$, where $\partial_0 : C_0(X) \to 0$ is defined to be the 0 morphism. Thus, $C(X) = (C_q(X), \partial_q)$ defines a chain complex over $\operatorname{mod} K$, which we call the *chain complex of the complex* (X, κ). From the general construction in the previous section, we get the homology K-modules of (X, κ) by $H_q(X) = H_q(C(X)) = \operatorname{Ker} \partial_q / \operatorname{Im} \partial_{q+1}$.

Define the *face relation* $<$ on the cells of (X, κ) by transitive extension of the relation $<_\kappa$ defined by $\tau <_\kappa \sigma$ if and only if $\kappa(\sigma, \tau) \neq 0$. If $\tau <_\kappa \sigma$, we say that τ is a *boundary face* of σ. If $\tau < \sigma$, then τ is a *face* of σ.

A *subcomplex* of a complex (X, κ) is another complex (X', κ') such that $X'_q \subset X_q$ for all $q \geqslant 0$, $\kappa' = \kappa|_{X' \times X'}$, and for every $\sigma \in X'$ and $\tau \in X$, if $\kappa(\sigma, \tau) \neq 0$ then $\tau \in X'$. The last condition is equivalent to requiring that the face of any cell in X' is also a cell in X'.

We have the following important lemma.

Lemma 2.2.1. *Let (X', κ') be a subcomplex of (X, κ). The map $\iota : (C_q(X'), \partial'_q) \hookrightarrow (C_q(X), \partial_q)$ induced from the inclusion is a chain map.*

Proof. We check the commutativity of

$$\begin{array}{ccc} C_q(X') & \xrightarrow{\partial'_q} & C_{q-1}(X') \\ \big\downarrow & & \big\downarrow \\ C_q(X) & \xrightarrow{\partial_q} & C_{q-1}(X) \end{array}$$

for every $q \geqslant 0$. It suffices to show this for any $\sigma \in X'_q$, where

$$\begin{aligned} \iota\partial'_q(\sigma) &= \iota \sum_{\tau \in X'} \kappa'(\sigma,\tau)\tau \\ &= \sum_{\tau \in X} \kappa(\sigma,\tau)\tau \\ &= \partial_q \iota(\sigma). \end{aligned}$$

The above equalities can be verified by the following reasoning. Since X' is a subcomplex of X and $\sigma \in X'$, any $\tau \in X$ with $\kappa(\sigma,\tau) \neq 0$ is in X'. Then, for any $\sigma, \tau \in X'$, $\kappa(\sigma,\tau) = \kappa'(\sigma,\tau)$. □

As a consequence, we have the induced map $H_q(\iota) : H_q(X') \to H_q(X)$ of homology modules, for each $q \geqslant 0$.

2.2.2 Simplicial complexes

Let V be a finite set of *vertices*. An *abstract simplicial complex* on V is a set S of nonempty subsets of V such that

1. $s \in S$ implies any nonempty $t \subset s$ is also in S and,
2. for every $v \in V$, $\{v\} \in S$.

An $s \in S$ is a *q-simplex* if the cardinality $|s|$ of s is equal to $q+1$. In this case, the dimension of s is defined to be q, denoted $\dim s = q$. The dimension of the abstract simplicial complex S is defined to be $\dim S = \max_{s \in S} \dim s$. Throughout this work, we shall drop the adjective abstract for simplicial complexes.

Let $q \geqslant 0$. The *q-skeleton* of a simplicial complex S is the simplicial complex

$$S^q = \{s \in S \mid \dim s \leqslant q\}.$$

A *simplicial subcomplex* S' of a simplicial complex S on vertices V is a simplicial complex S' on $V' \subset V$ such that $S' \subset S$ as sets. If S' is a simplicial subcomplex of S, we write $S' \subset S$. Clearly, the skeletons of S are all simplicial subcomplexes of S with the same vertices.

An abstract simplicial complex as defined above is a purely combinatorial structure, but often we give it a geometric interpretation. For example, let $V = \{v_0, v_1, v_2, v_3\}$, with

$$S = \left\{ \begin{array}{l} \{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \\ \{v_0, v_1\}, \{v_1, v_2\}, \{v_0, v_2\}, \{v_2, v_3\}, \{v_0, v_1, v_2\} \end{array} \right\}.$$

The simplicial complex S can be thought of as an abstraction of the combinatorial (connectivity) information of the vertices, edges, and faces of the geometric object given in Fig. 2.1. For example, the 1-simplex $\{v_0, v_1\}$ is identified with the edge e_0 between v_0 and v_1.

For more information concerning the relationship between abstract simplicial complexes and topological spaces, we refer the reader to [23]. We shall be constructing abstract simplicial complexes with vertex set $V \subset \mathbb{R}^n$. Then, the geometric interpretation is fairly straightforward.

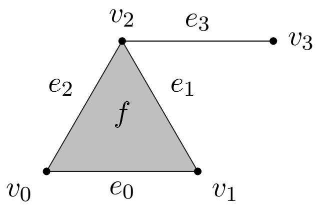


Figure 2.1: A simplicial complex

One may think of abstract simplicial complexes as a higher-dimensional generalization of graphs. Any finite graph may be thought of as a one-dimensional simplicial complex containing a 1-simplex $\{v, v'\}$ if and only if there is an edge between vertices v and v'.

A different construction of a simplicial complex from a graph is the following. Let G be a graph with vertices G_0. The so-called *clique complex* of G is the abstract simplicial complex S with vertex set G_0, and a subset $\{v_0, \ldots, v_q\}$ of G_0 is a q-simplex of the clique complex if and only if the full subgraph of G spanned by the vertices $v_0, \ldots, v_q$ is the complete graph on $q+1$ vertices. Equivalently, the clique complex of G is largest simplicial complex with 1-skeleton G.

Next, let us prepare to define the chain complex and thus the homology modules of a simplicial complex. First, we recall the definition of an orientation of a simplex. With this construction, we will see later that any simplicial complex can be viewed as a complex as we have defined in the previous section.

Let $s = \{v_0, \ldots, v_q\}$ be a q-simplex in S. We define an equivalence relation on the set of total orderings of $\{0, \ldots, q\}$ by setting two orderings to be equivalent if they differ by an even permutation. An equivalence class of orderings under this relation is called an *orientation of the simplex* s.

An *oriented* q*-simplex* is a q-simplex $s \in S$ together with an orientation of s. We write $\sigma = [v_0, \ldots, v_q]$ to denote the simplex s together with the equivalence class of the total ordering $0 < \ldots < q$. Note that for every q-simplex s with $q \geqslant 1$, there are two oriented q-simplices with underlying q-simplex s: $[v_0, \ldots, v_q]$ and $[v_{\tau(0)}, \ldots, v_{\tau(q)}]$ for any odd permutation τ of $\{0, \ldots, q\}$.

The q-th chain group of S, denoted $C_q(S)$, is defined to be the K-module (K-vector space) generated freely by the oriented q-simplices of S, modulo the relations

$$[v_0, \ldots, v_q] = \operatorname{sgn}(\tau)[v_{\tau(0)}, \ldots, v_{\tau(q)}]$$

for all q-simplices $s = \{v_0, \ldots, v_q\}$ and where τ is any permutation of $\{0, \ldots, q\}$. Here, $\operatorname{sgn}(\tau)$ is sign of the permutation τ and is equal to 1 for τ even and -1 for τ odd. The elements of $C_q(S)$ are formal sums

$$x = \sum c_i \sigma_i$$

modulo the relations given above, where $c_i \in K$ and σ_i are oriented q-simplices.

The boundary maps $\partial_q : C_q(S) \to C_{q-1}(S)$ are defined by linearly extending:

$$\partial_q \sigma = \partial_q [v_0, \ldots, v_q] = \sum_{i=0}^{q} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_q]$$

for oriented q-simplices σ. In the above formula, $\hat{v}_i$ means to exclude the vertex v_i; this gives $\partial_q\sigma \in C_{q-1}(S)$. The above formula is valid for $q \geqslant 1$. For $q = 0$, we simply have $\partial_0 = 0 : C_0(S) \to 0$.

Then, $C_q(S)$ together with ∂_q for $q \geqslant 0$ defines a chain complex over mod K. Checking that $\partial^2 = 0$ follows by straightforward computation. We obtain the homology K-vector spaces of S, $H_q(S) = \operatorname{Ker}\partial_q / \operatorname{Im}\partial_{q+1}$, for $q \geqslant 0$.

The following facts are important for discussing persistent homology. If S' is a simplicial subcomplex of S, then $C_q(S')$ is a K-vector subspace of $C_q(S)$, with an inclusion map $\iota_q : C_q(S') \to C_q(S)$ for each $q \geqslant 0$. It can be shown that $\iota = (\iota_q) : C(S') \to C(S)$ is a chain map. As before, ι induces morphisms of K-modules

$$H_q(\iota) : H_q(S') \to H_q(S)$$

for $q \geqslant 0$.

Note that an abstract simplicial complex S gives rise to a complex (X, κ). For every q-simplex s in S, choose an orientation. Construct X_q as the set of the chosen oriented q-simplices for $q \geqslant 0$ and $X = \bigsqcup_{q\geqslant 0} X_q$. Then, $\kappa : X \times X \to R$ is defined by setting $\kappa(\sigma, \tau)$ to be the coefficient of τ in $\partial_q(\sigma)$ for all $\sigma, \tau \in X$, denoted by $\kappa(\sigma, \tau) = \langle \partial_q(\sigma), \tau \rangle$.

2.2.3 Simplicial complexes from point clouds

A *point cloud* is a nonempty finite set of points in $\mathbb{R}^n$. For example, a point cloud may represent a sampling of points from some unknown manifold embedded in $\mathbb{R}^n$. By the following constructions, we try to construct a simplicial complex approximating a shape that can be inferred from the point cloud.

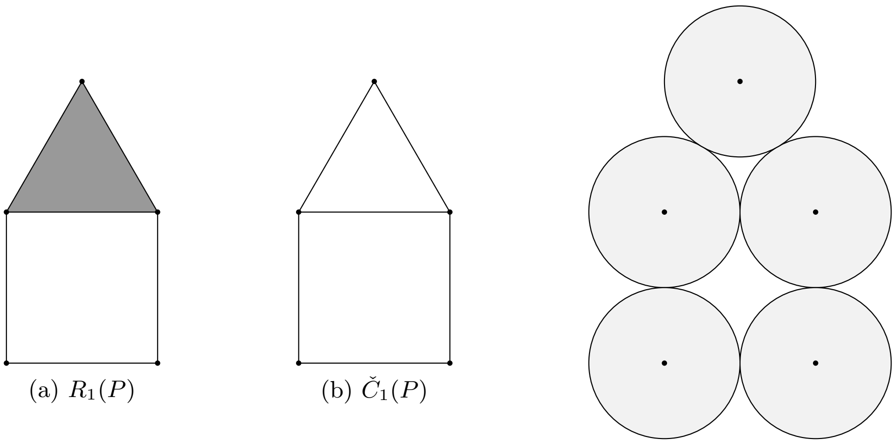
Let $X \subset \mathbb{R}^n$ be a point cloud. The *Vietoris-Rips complex* $R_\epsilon(X)$ with parameter ϵ is the abstract simplicial complex with vertex set X, and simplices defined by the following. Form the graph with vertices X and edges $\{x_0, x_1\} \subset X$ for every $x_0, x_1 \in X$ with $d(x_0, x_1) \leqslant \epsilon$. Then $R_\epsilon(X)$ is the clique complex of this graph. Equivalently, the simplices in $R_\epsilon(X)$ are given by nonempty subsets of X with diameter less than ϵ. The ϵ parameter acts as a "sensitivity" or "resolution" parameter.

The *Čech complex* $\check{C}_\epsilon(X)$ is defined by creating balls with radius $\epsilon/2$ at each point of X, and checking intersections. That is, a simplex $\{x_0, \ldots, x_q\}$ is defined to be in $\check{C}_\epsilon(X)$ if and only if

$$\bigcap_{i=0}^{q} \bar{B}\left(x_i, \frac{\epsilon}{2}\right) \neq \varnothing$$

where $\bar{B}(x, r) \subset \mathbb{R}^n$ is the closed Euclidean ball of radius r and center x.

Figure 2.2 illustrates the difference between the Vietoris-Rips and the Čech construction. Let P be the point cloud that forms the vertices of an equilateral triangle on top of a square, as shown. Suppose that both the triangle and square have side length 1. In $R_1(P)$, all the edges of the triangle are present, so we fill in the triangle face. In $\check{C}_1(P)$, the edges of the triangle are also present, but since the intersection of the balls of radius $\frac{1}{2}$ at the vertices of the triangle is empty, the face is not filled in.



(a) $R_1(P)$ (b) $\check{C}_1(P)$ (c) Union of balls with radius $\frac{\epsilon}{2} = \frac{1}{2}$ centered at the vertices

Figure 2.2: Vietoris-Rips and Čech complexes of the same point cloud.

Note that both the Vietoris-Rips and Čech complex construction satisfy the following nesting property. For any point cloud X, and if $\epsilon < \epsilon'$, then

$$R_\epsilon(X) \subseteq R_{\epsilon'}(X) \text{ and } \check{C}_\epsilon(X) \subseteq \check{C}_{\epsilon'}(X).$$

This property is important for defining the Vietoris-Rips and Čech filtrations, which we shall see later.

Let us give another construction. A *weighted point cloud* is a finite set

$$P = \{(p_i, r_i) \mid p_i \in \mathbb{R}^n, r_i \geqslant 0, i = 1, \ldots, N\}.$$

From a weighted point cloud P, one can construct the so-called weighted alpha complex [15]. First, we assume that P satisfies a general position assumption, to avoid certain degeneracies. For more details, see [15].

Let $(p_i, r_i) \in P$. The weighted Voronoi cell of (p_i, r_i) in P is

$$V_i = \{x \in \mathbb{R}^n \mid d(x, p_i)^2 - r_i^2 \leqslant d(x, p_j)^2 - r_j^2 \ \forall (p_j, r_j) \in P\}.$$

By letting the radius vary by some parameter α, we can study the data at different scales. For a fixed $\alpha \geqslant -\min_i\{r_i^2\}$, we define

$$r_i(\alpha) = \sqrt{r_i^2 + \alpha}.$$

Note that $r_i^2 + \alpha \geqslant 0$ for all $\alpha \geqslant -\min_i\{r_i^2\}$. Define the cut balls to be

$$C_i(\alpha) = V_i \cap \bar{B}(p_i, r_i(\alpha)).$$

The *weighted alpha shape* of P at alpha value α is the union of the cut balls:

$$\bigcup_{i=1}^{N} C_i(\alpha) = \bigcup_{i=1}^{N} \bar{B}(p_i, r_i(\alpha)).$$

The *weighted alpha complex* of P at alpha value α is defined to be the nerve of $\{C_i(\alpha)\}_{i=1,\dots,N}$. This is the abstract simplicial complex $\mathscr{A}_\alpha(P)$ with vertex set

$$V = \{C_i(\alpha) \mid i = 1, \dots N\}$$

and set of simplices

$$\mathscr{A}_\alpha(P) = \left\{ \varnothing \neq s \subset V \;\middle|\; \bigcap_{C \in s} C \neq \varnothing \right\}.$$

Clearly, for $\alpha < \alpha'$,

$$\mathscr{A}_\alpha(P) \subseteq \mathscr{A}_{\alpha'}(P).$$

Let us give an example of an application of weighted point clouds and weighted alpha complexes. Suppose that the input is a weighted point cloud $P = \{(p_i, r_i)\}$ representing an atomic configuration by its list of atoms and their 3D coordinates p_i in space, and radii r_i. In the introduction, this is the setting used for our motivating example concerning amorphous glass. To give a shape to such an atomic configuration, we use the space-filling union of balls model, formed by taking the union of spherical balls centered at the locations of the atoms and with corresponding radii

$$\bigcup_{i=1}^{N} \bar{B}(p_i, r_i(\alpha)).$$

However, the union of balls model is hard to treat computationally. Instead, we use the weighted alpha complex, which is known [15] to have underlying topological space homotopy equivalent to the union of balls space. In this work, we do not define what we mean by homotopy equivalence of topological spaces, and instead refer the reader to books on topology, for example [23]. Suffice to say, the weighted alpha complex will have homology isomorphic to that of the union of balls space, so that we do not lose any homological information by this substitution. In our computations, we use the software library CGAL [9], which has a package for efficient computation of the weighted alpha complexes in the cases where the weighted point cloud has points in dimension $n = 2$ or $n = 3$.

2.3 Quivers

As explained in the introduction, persistent homology is a diagram of K-vector spaces and K-linear maps between them. This general point of view leads one to the theory of quiver representations, and more generally, the theory of representations of K-algebras. We shall use this general theory to expand the scope of persistent homology for data analysis. First, let us review the theoretical background.

2.3.1 Bound quivers and their representations

A *quiver* Q is a directed graph. Formally, it is quadruple (Q_0, Q_1, s, t) of a set of vertices Q_0, arrows Q_1 and two maps $s, t : Q_1 \to Q_0$. For an arrow $\alpha \in Q_1$, we call $s(\alpha)$ its *source* and $t(\alpha)$ its *target*. An arrow is also written as $\alpha : s(\alpha) \to t(\alpha)$. The underlying undirected graph of a quiver Q is denoted $\bar{Q}$; this is the graph with edges defined by ignoring the directions of the arrows.

For any $i, j \in Q_0$, a *path* γ from i to j is a sequence of arrows $\alpha_1, \ldots, \alpha_n$ with $s(\alpha_1) = i$, $t(\alpha_n) = j$, and $s(\alpha_{a+1}) = t(\alpha_a)$ for $1 \leqslant a < n$. This path is denoted by

$$\gamma = (j|\alpha_n \ldots \alpha_1|i).$$

We can also talk about the source and target of a path, defined by $s(\gamma) = s(\alpha_1) = i$ and $t(\gamma) = t(\alpha_n) = j$. The length of a path $l(\gamma)$ is the number of arrows in it. Here, $l(\gamma) = n$. For every $i \in Q_0$, we define the *trivial path* (or *stationary path*) at i, denoted by $e_i = (i||i)$.

The pre-concatenation of the path $(j|\alpha_n \ldots \alpha_1|i)$ by another path $(\ell|\beta_m \ldots \beta_1|k)$ is defined as the path

$$(\ell|\beta_m \ldots \beta_1|k) \circ (j|\alpha_n \ldots \alpha_1|i) = (\ell|\beta_m \ldots \beta_1\alpha_n \ldots \alpha_1|i)$$

only in the case $k = j$. Otherwise, the concatenation is left undefined. This notation places sources on the right side of paths and targets on the left. We compose paths from the left following functional notation, as we wish to think of paths as morphisms.

We consider only finite acyclic connected quivers – that is, both Q_0 and Q_1 are finite sets, there is no nontrivial path γ with $s(\gamma) = t(\gamma)$, and $\bar{Q}$ is a connected graph. In this work, unless specified otherwise the term "quiver" will be taken to mean a finite acyclic connected quiver.

The *path algebra* of Q is the K-algebra KQ generated by paths of Q. The elements of KQ are formal sums

$$\sum_{\substack{a_\gamma \in K, \\ \gamma \text{ path in } Q}} a_\gamma \gamma,$$

with multiplication defined by concatenation of paths. That is, we define

$$(\ell|\beta_m \ldots \beta_1|k)(j|\alpha_n \ldots \alpha_1|i) = \begin{cases} (\ell|\beta_m \ldots \beta_1\alpha_n \ldots \alpha_1|i) & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases}$$

for paths and then extend K-linearly to elements of KQ. An element e of a K-algebra is said to be idempotent if $e^2 = e$. Clearly, the trivial paths e_i for $i \in Q_0$ are idempotents.

The assumption that Q is finite, acyclic, and connected simplifies the theory in many ways. Since Q is finite, KQ has identity element $1 = \sum_{i \in Q_0} e_i$. Since Q is finite and acyclic, given a pair of vertices i, j, the number of distinct paths from i to j is finite. Moreover, $e_j(KQ)e_i$ for $i, j \in Q_0$ treated as a K-vector space, is the vector space of all paths from i to j. This is finite-dimensional as a K-vector space, and thus so is KQ. See [2, Corollary II.1.11] for more details.

A finite-dimensional *representation* M of a quiver Q is a set of a finite-dimensional K-vector space $M(i)$ for each $i \in Q_0$, and a linear map $M(\alpha) : M(i) \to M(j)$ for every arrow $\alpha : i \to j$ in Q_1. In this work, a representation shall be taken to mean a finite-dimensional representation.

A morphism from a representation M to another, N, is a set of linear maps $f_i : M(i) \to N(i)$, one for every $i \in Q_0$, such that for every arrow $\alpha : i \to j$, the diagram

$$\begin{tikzcd} M(i) \arrow[r, "M(\alpha)"] \arrow[d, "f_i"] & M(j) \arrow[d, "f_j"] \\ N(i) \arrow[r, "N(\alpha)"] & N(j) \end{tikzcd}$$

commutes. The category of finite-dimensional representations and their morphisms is denoted by $\operatorname{rep} Q$.

Now, Q also defines the path algebra KQ. We shall see in a bit that there is a very close relationship between finite-dimensional KQ-modules and finite-dimensional representations of Q. Instead of doing this now, we first define bound quivers and exhibit this relationship in more general terms.

Let Q be a quiver. A *relation* σ in Q is a sum of paths $p_1, \dots, p_n$:

$$\sigma = a_1 p_1 + \ldots + a_n p_n \in KQ$$

with $a_i \in K$ such that all the paths p_i have the same source and the same target. We define the source of this relation as the common source of its paths, and its target as the common target.

Let ρ be a finite set of relations of a quiver Q. The pair (Q, ρ) is called a *bound quiver*. We also say that Q is bound by relations ρ. A *representation of the bound quiver* (Q, ρ) is an $M \in \operatorname{rep} Q$ such that $M(\sigma) = 0$ for every $\sigma \in \rho$. Here, $M(\sigma)$ is the *evaluation* of M along the relation σ, defined by the following. The evaluation of M along a path $p = (j|\alpha_m \dots \alpha_1|i)$ is the K-linear map

$$M(p) = M(\alpha_m) \dots M(\alpha_1) : M(i) \to M(j).$$

The evaluation of M along the relation $\sigma = a_1 p_1 + \dots a_n p_n$ is given by

$$M(\sigma) = \sum_{k=1}^{n} a_k M(p_k) : M(i) \to M(j).$$

A morphism $f : M \to N$ of representations of (Q, ρ) is a morphism $f : M \to N$ in $\operatorname{rep}(Q)$. We denote the category of representations of (Q, ρ), together with these morphisms, by $\operatorname{rep}(Q, \rho)$.

Given an $M \in \operatorname{rep}(Q, \rho)$, a subrepresentation $N \subset M$ of M is an $N \in \operatorname{rep}(Q, \rho)$ such that $N(i)$ is a K-vector subspace of $M(i)$ for every $i \in Q_0$, and for each arrow $\alpha : i \to j$, the morphism $N(\alpha) : N(i) \to N(j)$ is defined by the restriction of $M(\alpha)$ to $N(i)$.

Given a morphism $f : M \to N$ in $\operatorname{rep}(Q, \rho)$, $\operatorname{Ker} f$ is the subrepresentation of M with $(\operatorname{Ker} f)(i) = \operatorname{Ker} f_i$ for every vertex $i \in Q_0$. The maps $(\operatorname{Ker} f)(\alpha)$ are of course given by restrictions of $M(\alpha)$ for arrows $\alpha : i \to j$. Likewise,

$(\operatorname{Coker} f)(i) = \operatorname{Coker} f_i$ and $(\operatorname{Coker} f)(\alpha) : N_i/\operatorname{Im} f_i \to N_j/\operatorname{Im} f_j$ for each arrow $\alpha : i \to j$ is defined by mapping $n+\operatorname{Im} f_i$ to $N(\alpha)(n)+\operatorname{Im} f_j$. This is well-defined, by the commutativity of $N(\alpha)$ with f.

Let $M, N \in \operatorname{rep}(Q, \rho)$. The *direct sum* of M and N is the representation $M \oplus N$ given by $(M \oplus N)(i) = M(i) \oplus N(i)$ for every $i \in Q_0$, and $(M \oplus N)(\alpha) = M(\alpha) \oplus N(\alpha) : (M \oplus N)(i) \to (M \oplus N)(j)$ for every arrow $\alpha : i \to j$, where

$$M(\alpha) \oplus N(\alpha) = \begin{bmatrix} M(\alpha) & 0 \\ 0 & N(\alpha) \end{bmatrix}.$$

Suppose that $M_1 \subset M$ is a subrepresentation such that there exists another subrepresentation $M_2 \subset M$ with $M(i) = M_1(i) \oplus M_2(i)$ as K-vector spaces, for each $i \in Q_0$. Then, M is said to be an internal direct sum of M_1 and M_2, denoted by $M = M_1 \oplus M_2$. A representation M of (Q, ρ) is said to be indecomposable if $M = M' \oplus M''$ implies that $M' = 0$ or $M'' = 0$.

The *dimension vector* of M, denoted by $\underline{\dim}\, M$, is the function

$$\begin{array}{cccc} \underline{\dim}\, M & : & Q_0 & \to & \mathbb{N}_0 \\ & & i & \mapsto & \dim M(i), \end{array}$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We shall display dimension vectors in a form that suggests the shape of the underlying quiver. For example, the dimension vector of the representation

$$\begin{tikzcd} K \arrow[r, "1"] & K & K \arrow[l, "1"'] \\ 0 \arrow[u] \arrow[r] & K \arrow[u, "1"] & 0 \arrow[u] \arrow[l] \end{tikzcd}$$

of the quiver in Diagram (2.4) is written as $\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{smallmatrix}$.

Let (Q, ρ) be a bound quiver. Then, ρ generates a two-sided ideal $I = \langle \rho \rangle$ of KQ. Taking the quotient, we get a finite-dimensional K-algebra $A = KQ/I$, called the *algebra of the bound quiver* (Q, ρ). The following theorem is very important.

Theorem 2.3.1 (See [2, Theorem III.1.6])**.** *Let Q be a finite acyclic quiver, ρ a finite set of relations of Q, and $A = KQ/I$ the algebra of the bound quiver (Q, ρ). There is an equivalence of categories* $\operatorname{rep}(Q, \rho) \cong \operatorname{mod} KQ/I$.

Proof. First, let us define a functor $F : \operatorname{rep}(Q, \rho) \to \operatorname{mod} KQ/I$. For $M = (M(i), M(\alpha))_{i \in Q_0, \alpha \in Q_1} \in \operatorname{rep}(Q, \rho)$, construct the KQ/I-module $F(M)$ by the following. Let

$$F(M) = \bigoplus_{i \in Q_0} M(i)$$

as a K-vector space. To define its KQ/I-structure, we first give $F(M)$ a KQ-module structure. Let

$$pm_i = \begin{cases} M(p)(m_i) & \text{if } s(p) = i, \\ 0 & \text{otherwise,} \end{cases}$$

for p a path in Q, $m_i \in M(i)$. Note that $pm_i \in M(t(p))$. For $m = (m_i)_{i \in Q_0} \in F(M)$, define pm to be the image of $\sum_{i \in Q_0} pm_i$ under the natural inclusion $M(t(p)) \hookrightarrow$

$F(M)$. By K-linear extension to arbitrary elements of KQ, this gives a KQ-module structure on $F(M)$.

For any relation σ in ρ and $m \in F(M)$, it can be checked that $\sigma m = 0$, since the evaluation $M(\sigma)$ equals 0. This shows that $F(M)$ is a KQ/I-module.

For $f = (f_i)_{i \in Q_0} : M \to N$ in $\operatorname{rep}(Q, \rho)$, we define $F(f) : F(M) \to F(N)$ as

$$f' = F(f) = \bigoplus_{i \in Q_0} f_i : \bigoplus_{i \in Q_0} M(i) \to \bigoplus_{i \in Q_0} N(i).$$

To check that this is KQ/I-linear, we check that for every path $\bar{p} = p+I$ in KQ/I, we have $f'(\bar{p}m) = \bar{p}f'(m)$ for all $m \in F(M)$. It suffices to check this equality for the image of any $m_i \in M(i)$ under the natural inclusion $M(i) \hookrightarrow F(M)$, for each $i \in Q_0$.

In the case $s(p) \neq i$, $\bar{p}m_i = 0$ and $\bar{p}f'(m_i) = 0$ so there is nothing to check. Otherwise,

$$f_j(\bar{p}m_i) = f_j(M(p)m_i) = N(p)f_i(m_i) = \bar{p}f_i(m_i),$$

where $j = t(p)$ and where the second equality follows from the definition of morphisms of representations.

In the opposite direction, we define $G : \operatorname{mod} KQ/I \to \operatorname{rep}(Q, \rho)$ by the following. Let $M \in \operatorname{mod} KQ/I$, then $G(M) = (V(i), V(\alpha))$ is the representation of (Q, ρ) with $V(i) = \bar{e}_i M$, where $\bar{e}_i = e_i + I = (i||i) + I$, for $i \in Q_0$. Moreover, the morphism $V(\alpha) : V(i) \to V(j)$ for an arrow $\alpha : i \to j$ is defined by

$$V(\alpha)(\bar{e}_i m) = \bar{e}_j \bar{\alpha} \bar{e}_i m \in V(j)$$

for $\bar{e}_i m \in \bar{e}_i M = V(i)$. Clearly, $G(M)$ is a representation of the bound quiver since if $\sigma \in \rho$ is a relation, then the evaluation $V(\sigma)$ is defined by the action of $\bar{\sigma} = \sigma + I = 0 + I$.

For a morphism $f : M \to N$ in $\operatorname{mod} A$, we have $G(M) = (V(i), V(\alpha))$ and $G(N) = (W(i), W(\alpha))$. We define $G(f) = (f_i)_{i \in Q_0} : G(M) \to G(N)$ by restriction

$$\begin{array}{rccl} f_i : & \bar{e}_i M & \to & \bar{e}_i N \\ & \bar{e}_i m & \mapsto & f(\bar{e}_i m) = \bar{e}_i f(m). \end{array}$$

We have to check the commutativity of:

$$\begin{tikzcd} \bar{e}_i M \arrow[r, "G(M)(\alpha)"] \arrow[d, "f_i"] & \bar{e}_j M \arrow[d, "f_j"] \\ \bar{e}_i N \arrow[r, "G(N)(\alpha)"] & \bar{e}_j N \end{tikzcd}$$

for each arrow $\alpha : i \to j$. For $\bar{e}_i m \in \bar{e}_i M$,

$$\begin{array}{rcl} f_j(G(M)(\alpha)(\bar{e}_i m)) & = & f_j(\bar{e}_j \bar{\alpha} \bar{e}_i m) \\ & = & \bar{e}_j \bar{\alpha} f(\bar{e}_i m) \\ & = & G(N)(\alpha)(f_i(\bar{e}_i m)) \end{array}$$

confirms the commutativity.

It is easy to check that F and G are functors, and are quasi-inverses of each other. $\square$

With this theorem, we shall interchangeably use the terms module and representation. Also, with ρ being the empty set, Theorem 2.3.1 provides the equivalence $\operatorname{rep} Q \cong \operatorname{mod} KQ$.

2.3.2 Useful quivers

Let us give some examples of quivers that we will use often in this work.

An *orientation* τ is a sequence of symbols f and b, standing for "forwards" and "backwards". We write an orientation by $\tau = \tau_1, \ldots, \tau_{n-1}$, where the ith symbol of τ is τ_i, and we say that τ has length n.

Let τ be an orientation of length n. The *linear quiver* $A_n(\tau)$ is the quiver

$$A_n(\tau) = \quad \overset{1}{\circ} \overset{\alpha_1}{\longleftrightarrow} \overset{2}{\circ} \overset{\alpha_2}{\longleftrightarrow} \overset{3}{\circ} \overset{\alpha_3}{\longleftrightarrow} \cdots \overset{\alpha_{n-1}}{\longleftrightarrow} \overset{n}{\circ} \quad , \tag{2.2}$$

where the direction of the ith arrow is determined by τ_i, for $i \in \{1, \ldots, n-1\}$. That is, if $\tau_i = f$, the arrow α_i is $\alpha_i : i \to i+1$, otherwise it is $\alpha_i : i+1 \to i$. With $\tau = f \ldots f$ we get the quiver

$$\vec{A}_n = A_n(f \ldots f) = \quad \overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \overset{3}{\circ} \longrightarrow \cdots \longrightarrow \overset{n}{\circ} \quad .$$

Let τ be an orientation of length n. The *ladder quiver* of type τ is given by

$$L_n(\tau) = \begin{tikzcd} \overset{1'}{\circ} \arrow[r, leftrightarrow] & \overset{2'}{\circ} \arrow[r, leftrightarrow] & \overset{3'}{\circ} \arrow[r, leftrightarrow] & \cdots \arrow[r, leftrightarrow] & \overset{n'}{\circ} \\ \underset{1}{\circ} \arrow[u] \arrow[r, leftrightarrow] & \underset{2}{\circ} \arrow[u] \arrow[r, leftrightarrow] & \underset{3}{\circ} \arrow[u] \arrow[r, leftrightarrow] & \cdots \arrow[r, leftrightarrow] & \underset{n}{\circ} \arrow[u] \end{tikzcd} \tag{2.3}$$

where for every $i \in \{1, \ldots, n-1\}$, the pair of horizontal arrows point forwards $i \to i+1$, $i' \to (i+1)'$ if $\tau_i = f$, and points backwards $i \leftarrow i+1$, $i' \leftarrow (i+1)'$ otherwise. In a sense, it is a "product" of an $\vec{A}_2$ quiver in the vertical direction and an $A_n(\tau)$ quiver in the horizontal direction.

The ladder quiver with orientation $\tau = fb$ is

$$\begin{tikzcd} \overset{4}{\circ} \arrow[r, "\alpha_{54}"] & \overset{5}{\circ} & \overset{6}{\circ} \arrow[l, "\alpha_{56}"'] \\ \underset{1}{\circ} \arrow[u, "\alpha_{41}"] \arrow[r, "\alpha_{21}"] & \underset{2}{\circ} \arrow[u, "\alpha_{52}"] & \underset{3}{\circ} \arrow[l, "\alpha_{23}"'] \arrow[u, "\alpha_{63}"] \end{tikzcd} \quad . \tag{2.4}$$

Here, the arrows are labeled by $\alpha_{ts} : s \to t$. This quiver, together with the commutative relations

$$\rho = \{\alpha_{52}\alpha_{21} - \alpha_{54}\alpha_{41}, \alpha_{52}\alpha_{23} - \alpha_{56}\alpha_{63}\}$$

forms the bound quiver which we call the *commutative triple ladder*. As an example, Diagram (1.4), which we propose to use for studying simultaneously robust and common topological features, is a representation of the commutative triple ladder.

In general, let us define the commutativity relations of $L_n(\tau)$. For any $i \in \{1, \ldots, n-1\}$, let $j = i+1$ and $j' = (i+1)'$. If $\tau_i = f$, set

$$w_i = \alpha_{j',j}\alpha_{j,i} - \alpha_{j',i'}\alpha_{i',i},$$

and if $\tau_i = b$, then

$$w_i = \alpha_{i',i}\alpha_{i,j} - \alpha_{i',j'}\alpha_{j',j},$$

for $i \in \{1, \ldots, n-1\}$. The set $c = \{w_i\}_{i=1}^{n-1}$ is defined to be the set of commutativity relations of $L_n(\tau)$. This requires the commutativity in each small square

$$\begin{tikzcd} \overset{i'}{\circ} \arrow[r, "\alpha_{j',i'}"] & \overset{j'}{\circ} \\ \underset{i}{\circ} \arrow[u, "\alpha_{i',i}"] \arrow[r, "\alpha_{j,i}"'] & \underset{j}{\circ} \arrow[u, "\alpha_{j',j}"'] \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \overset{i'}{\circ} & \overset{j'}{\circ} \arrow[l, "\alpha_{i',j'}"'] \\ \underset{i}{\circ} \arrow[u, "\alpha_{i',i}"] & \underset{j}{\circ} \arrow[u, "\alpha_{j',j}"'] \arrow[l, "\alpha_{i,j}"] \end{tikzcd}$$

in the ladder quiver, with i, i', j, j' as above.

Definition 2.1. The bound quiver $CL_n(\tau) = (L(\tau), c)$ is called the *commutative ladder quiver* with length n and orientation τ.

2.4 Persistent homology

In this section, we explain the basics of persistent homology, which forms the foundation and motivation for much of what we do in this work.

We start with our definition of the persistent homology of a quiver complex, which provides a generalization of the persistent homology of filtrations and zigzag complexes.

2.4.1 Persistent homology of quiver complexes

Recall that all quivers we consider are assumed to be finite, connected and acyclic. To generalize the filtrations used for persistent homology, we introduce the concept of a quiver complex.

Definition 2.2. Let Q be a quiver. A *quiver complex* over Q is a set of complexes (X_i, κ_i), one for each vertex $i \in Q_0$ so that whenever there is an arrow $\alpha : i \to j$ in Q_1, (X_i, κ_i) is a subcomplex of (X_j, κ_j).

We denote a quiver complex by $\mathbb{X} = (X_i, \kappa_i)_{i \in Q_0}$. We only consider finite quiver complexes: for each $i \in Q_0$, (X_i, κ_i) is a finite complex.

Given a quiver complex $\mathbb{X}$, we construct its chain complex $C(\mathbb{X})$ below. For each $q \geqslant 0$, consider the representation $C_q(\mathbb{X})$ of Q obtained by associating to each vertex i the K-vector space $C_q(X_i)$, and for every arrow $\alpha : i \to j$ the inclusion $\iota_\alpha : C_q(X_i) \hookrightarrow C_q(X_j)$.

We define the *commutativity relations* of Q to be the set c of relations of the form $w = p - p'$, where p and p' are any two unequal paths from i to j, for all pairs i, j with $i \neq j \in Q_0$. Clearly, c is a finite set of relations, since there are only a finite number of pairs $i \neq j$, and for each pair $i \neq j$, the number of paths from i to j is finite.

Now, $C_q(\mathbb{X})$ is a representation of (Q, c). To prove this, let $V = C_q(\mathbb{X})$ as a representation of Q, and let $w = p - p'$ be a commutativity relation from vertex i to vertex j. Then, the evaluations $V(p) : C_q(X_{s(p)}) \to C_q(X_{t(p)})$ and $V(p') : C_q(X_{s(p')}) \to C_q(X_{t(p')})$ are both equal to the inclusion $C_q(X_i) \hookrightarrow C_q(X_j)$ so that the evaluation $V(w) = V(p) - V(p') = 0$. Thus, $V = C_q(\mathbb{X}) \in \mathrm{rep}(Q, c)$.

Moreover, we define morphisms $\partial_q : C_q(\mathbb{X}) \to C_{q-1}(\mathbb{X})$ by combining over all $i \in Q_0$ the boundary maps $\partial_{q,i} : C_q(X_i) \to C_{q-1}(X_i)$. For each vertex i, the boundary maps $\partial_{q,i} : C_q(X_i) \to C_{q-1}(X_i)$ are the boundary maps of the chain complex of (X_i, κ_i), as defined in Section 2.2.1. To check that this is really a morphism of representations, we only need to check the commutativity of

$$\begin{tikzcd} C_q(X_i) \arrow[r, "\iota"] \arrow[d, "\partial_{q,i}"'] & C_q(X_j) \arrow[d, "\partial_{q,j}"] \\ C_{q-1}(X_i) \arrow[r, "\iota"] & C_{q-1}(X_j) \end{tikzcd}$$

for every arrow $\alpha : i \to j$. This result follows from the fact that $(C_q(X_i), \partial_{q,i}) \hookrightarrow (C_q(X_j), \partial_{q,j})$ is a chain map by Lemma 2.2.1. Moreover, $\partial_q \partial_{q+1} = 0$ follows from the fact that $\partial_{q,i}\partial_{q+1,i} = 0$ for every $i \in Q_0$. Thus, $C(\mathbb{X}) = (C_q(\mathbb{X}), \partial_q)$ is a chain complex over $\mathrm{rep}(Q, c)$.

Let us collect these facts below.

Lemma 2.4.1. *Let $\mathbb{X}$ be a quiver complex over Q, with c the set of commutativity relations of Q. For each $q \geqslant 0$, $C_q(\mathbb{X})$ is a representation of (Q, c), and $C(\mathbb{X})$ is a chain complex over* $\mathrm{mod}\, KQ/I \cong \mathrm{rep}(Q, c)$*, where KQ/I is the algebra of (Q, c).*

Definition 2.3 (Persistent homology of a quiver complex)**.** Let $\mathbb{X}$ be a quiver complex with chain complex $C(\mathbb{X})$. The qth homology module of $C(\mathbb{X})$:

$$H_q(C(\mathbb{X})) = \mathrm{Ker}\, \partial_q / \mathrm{Im}\, \partial_{q+1}$$

is also called the qth persistent homology of the quiver complex $\mathbb{X}$. To simplify the notation, we denote this by $H_q(\mathbb{X})$.

In the definition above, $H_q(\mathbb{X})$ is a KQ/I-module and is obtained by computing the quotient of the kernel and image, at the KQ/I-module level. It can be shown that $H_q(\mathbb{X})$ corresponds to the representation of (Q, c) with $H_q(X_i)$ at each vertex $i \in Q_0$ and induced maps $H_q(\iota) : H_q(X_i) \to H_q(X_j)$ for every arrow $\alpha : i \to j$. This is the "slice-wise" (vertex-wise) point of view.

2.4.2 Persistent homology of filtrations

Let $\vec{A}_n = A_n(f \dots f)$ as above. A *filtration* is a nested sequence of (simplicial) complexes:

$$\mathbb{X} : \ X_1 \hookrightarrow \dots \hookrightarrow X_n. \tag{2.5}$$

This is a quiver complex on $\vec{A}_n$.

Let us give some examples of how filtrations arise in practice. For a point cloud P and some $\epsilon \geqslant 0$, consider the Vietoris-Rips complex $R_\epsilon(P)$, or the Čech complex $\check{C}_\epsilon(P)$. On the other hand, given a weighted point cloud P, consider the weighted alpha complex $\mathscr{A}_\alpha(P)$. These are all simplicial complexes X_a that vary by some parameter a, where a is equal to ϵ for the Vietoris-Rips and Čech complexes and α for the weighted alpha complex. Moreover, for $a < a'$, $X_a \subset X_{a'}$.

However, the parameter a varies through the real numbers. We can easily remap the unique X_a in the sequence to a finite set of indices $1, \ldots, n$ by the following argument. In all cases above, X_a is an abstract simplicial complex on a fixed set of vertices $V = P$. Thus, each X_a is a subset of the power set of V, $X_a \subset \mathcal{P}(V)$. Since the sequence $\{X_a\}_a$ is nondecreasing, we obtain a filtration

$$X_{a_1} \subset \ldots \subset X_{a_n}$$

consisting of the finite number of unequal complexes X_{a_i} in $\{X_a\}_a$. This construction provides the Vietoris-Rips filtration, Čech complex filtration, and the weighted alpha complex filtration, respectively.

Going back to the general case, the chain complex $(C_q(\mathbb{X}), \partial_q)$ of a filtration $\mathbb{X}$ is given by the following. For a fixed $q \geqslant 0$, $C_q(\mathbb{X})$ is the representation of $\vec{A}_n$:

$$C_q(\mathbb{X}): \ C_q(X_1) \xrightarrow{\iota_1} C_q(X_2) \xrightarrow{\iota_2} \ldots \xrightarrow{\iota_{n-1}} C_q(X_n)$$

where the ι_i are inclusions, and the boundary map $\partial_q = (\partial_{q,i})_i : C_q(\mathbb{X}) \to C_{q-1}(\mathbb{X})$ is a morphism of representations. This can be given as a commutative diagram

$$\begin{tikzcd}
C_q(X_1) \arrow[r] \arrow[d, "\partial_{q,1}"] & C_q(X_2) \arrow[r] \arrow[d, "\partial_{q,2}"] & \ldots \arrow[r] & C_q(X_n) \arrow[d, "\partial_{q,n}"] \\
C_{q-1}(X_1) \arrow[r] & C_{q-1}(X_2) \arrow[r] & \ldots \arrow[r] & C_{q-1}(X_n)
\end{tikzcd}$$

where $\partial_q = (\partial_{q,i})_{i\in Q_0}$ is defined "slice-wise". The chain complex $C(\mathbb{X})$ has qth homology module

$$H_q(\mathbb{X}): \ H_q(X_1) \xrightarrow{H_q(\iota_1)} \ldots \xrightarrow{H_q(\iota_{n-1})} H_q(X_n), \tag{2.6}$$

which is the qth *persistence homology* of the filtration [16, 37]. By Theorem 2.3.1, we also view $H_q(\mathbb{X})$ as a $K\vec{A}_n$-module.

In the introduction, we roughly sketched the use of the persistence diagram, which is obtained by a decomposition of persistent homology into intervals. These intervals correspond to pairs of numbers (b, d) representing the lifespans of homology generators. Let us give a precise treatment of these ideas.

First, let us define the representation $\mathbb{I}[b,d] \in \mathrm{rep}(\vec{A}_n)$, with $1 \leqslant b \leqslant d \leqslant n$, by

$$0 \longrightarrow \ldots \longrightarrow 0 \longrightarrow K \xrightarrow{1} \ldots \xrightarrow{1} K \longrightarrow 0 \longrightarrow \ldots \longrightarrow 0$$

with $\mathbb{I}[b,d](i) = K$ if $b \leqslant i \leqslant d$, and $\mathbb{I}[b,d](i) = 0$ otherwise, and identity maps between the nonzero K-vector spaces. These are also called the *interval representations*. Again by Theorem 2.3.1, $\mathbb{I}[b,d]$ can be treated as a $K\vec{A}_n$-module, which we call an *interval module*.

As one of the consequences of Gabriel's theorem [21], it is known that the interval modules $\mathbb{I}[b,d]$ for $1 \leqslant b \leqslant d \leqslant n$ gives a complete list of indecomposable $K\vec{A}_n$-modules, up to isomorphism. For an elementary proof, see the book [4, Theorem 2.14]. By this fact and by Proposition 2.1.1, there is an indecomposable decomposition

$$H_q(\mathbb{X}) = \bigoplus_{i=1}^{m} V_i \cong \bigoplus_{1 \leqslant b \leqslant d \leqslant n} \mathbb{I}[b,d]^{m_{b,d}} \tag{2.7}$$

of $H_q(\mathbb{X})$, unique up to isomorphism and permutations of terms, into the interval modules. In the above decomposition, each direct summand V_i is an indecomposable module and is isomorphic to some $\mathbb{I}[b,d]$.

This indecomposable decomposition can be viewed as simultaneous changes of K-vector space bases for each of the $H_q(X_i)$, $i \in \{1, \ldots, n\}$ that tracks how the homology generators are mapped through the sequence. More precisely, an indecomposable direct summand V_i isomorphic to $\mathbb{I}[b,d]$ represents a homology generator born at index b, and is mapped by identity (in the chosen bases) to the homology vector spaces at indices $b+1, \ldots, d$. Then, it is mapped to 0, or dies, after index d.

The decomposition into indecomposable representations shows the "birth" and "death" indices of homology generators. To summarize this information, the persistence diagram can be defined to be the multiset of pairs (b, d), where the (b, d) occurs with multiplicity $m_{b,d}$, determined from Eq. (2.7). For our purposes, however, we use the equivalent definition of the *persistence diagram* as the map

$$\begin{array}{rccc} D(H_q(\mathbb{X})): & \Gamma_0 & \to & \mathbb{N}_0 \\ & \mathbb{I}[b,d] & \mapsto & m_{b,d} \end{array}$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and Γ_0 is the set of interval representations of $\vec{A}_n$.

The persistence diagram can be visualized by drawing the corresponding multiset of points on the plane. For example, suppose that some quiver complex $\mathbb{X}$ on $\vec{A}_5$ (a filtration with $n = 5$) has

$$H_q(\mathbb{X}) \cong \mathbb{I}[1,2] \oplus \mathbb{I}[1,4]^2 \oplus \mathbb{I}[2,5] \oplus \mathbb{I}[3,3].$$

We display the persistence diagram of $H_q(\mathbb{X})$ in Fig. 2.3, where the point $(1, 4)$ occurs with multiplicity 2. Here, we have explicitly written out the multiplicities of the intervals. Of course, there are other ways of visualizing a persistence diagram. Other methods to indicate the multiplicities include coloring the points according to some color scale, or via a 3D histogram.

For later use in studying the category $\operatorname{mod} K\vec{A}_n$, we also need all the morphisms between the interval modules of $K\vec{A}_n$. Given two fixed indecomposables $\mathbb{I}[a,b]$, $\mathbb{I}[c,d]$, let us describe all morphisms $f : \mathbb{I}[a,b] \to \mathbb{I}[c,d]$. First of all, note that if $b < c$ or $d < a$, there is no nonzero morphism $f : \mathbb{I}[a,b] \to \mathbb{I}[c,d]$.

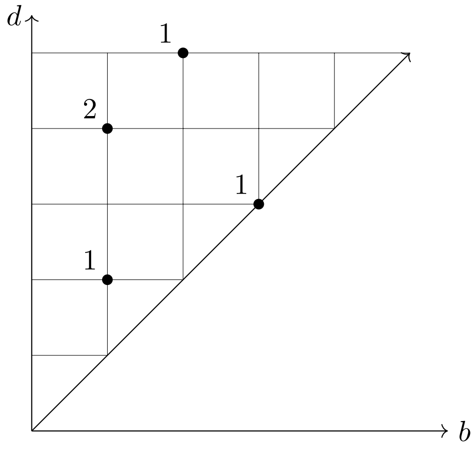


Figure 2.3: An example of a persistence diagram.

Suppose that $a < c$. Then, looking at the relevant part of the morphism:

$$\begin{tikzcd}
\mathbb{I}[a,b] \arrow[d, "f"] & \dots \arrow[r] & K \arrow[r, "1"] \arrow[d, "f_{c-1}=0"'] & K \arrow[r, "1"] \arrow[d, "f_c"] & \dots \\
\mathbb{I}[c,d] & \dots \arrow[r] & 0 \arrow[r, "0"] & K \arrow[r, "1"] & \dots
\end{tikzcd},$$

we deduce that $f_c = 0$ due to the commutativity requirement. Further application of the commutativity shows that $f_i = 0$ for all $i \in \{1, \dots, n\}$, so that $f = 0$. A similar argument shows that if $d > b$, there is no nonzero morphism $f : \mathbb{I}[a,b] \to \mathbb{I}[c,d]$.

Now, consider the case $c \leqslant a \leqslant d \leqslant b$, and suppose that $f : \mathbb{I}[a,b] \to \mathbb{I}[c,d]$ is a morphism. For any vertex i outside the intersection of intervals, $[a, b] \cap [c, d]$, f_i is necessarily 0. Choose any $i \in [a, b] \cap [c, d] = [a, d]$. If $f_i = k \in \mathrm{Hom}_K(K, K) \cong K$, then $f_j = k$ for all $j \in [a, b] \cap [c, d]$ by the commutativity requirement. We illustrate this in the diagram below:

$$\begin{tikzcd}
\mathbb{I}[a,b] \arrow[d, "f"] & \dots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & K \arrow[r, "1"] \arrow[d, "k"'] & \dots \arrow[r, "1"] \arrow[d, "f_i=k"] & K \arrow[r, "1"] \arrow[d, "k"] & K \arrow[r] \arrow[d, "0"] & \dots \\
\mathbb{I}[c,d] & \dots \arrow[r] & K \arrow[r, "1"] & K \arrow[r, "1"] & \dots \arrow[r, "1"] & K \arrow[r] & 0 \arrow[r] & \dots \\
& & & a & & d & &
\end{tikzcd}.$$

Thus,

$$\mathrm{Hom}_{K\vec{A}_n}(\mathbb{I}[a,b], \mathbb{I}[c,d]) = \begin{cases} K f_{a,b}^{c,d}, & c \leqslant a \leqslant d \leqslant b, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\left(f_{a,b}^{c,d}\right)_\ell = \begin{cases} 1_K, & a \leqslant \ell \leqslant d, \\ 0, & \text{otherwise.} \end{cases}$$

The choice of $f_{a,b}^{c,d}$ is a choice of basis for each of the nonzero homomorphism spaces above. Moreover, this choice of morphisms has the nice property that if for a triple of pairs $(a,b),(a',b'),(a'',b'')$ the morphisms

$$\begin{tikzcd} \mathbb{I}[a,b] \arrow[r, "f_{a,b}^{a',b'}"] \arrow[rr, bend right, "f_{a,b}^{a'',b''}"'] & \mathbb{I}[a',b'] \arrow[r, "f_{a',b'}^{a'',b''}"] & \mathbb{I}[a'',b''] \end{tikzcd} ,$$

are defined and nonzero, then

$$f_{a,b}^{a'',b''} = f_{a',b'}^{a'',b''} f_{a,b}^{a',b'}. \tag{2.8}$$

2.4.3 Zigzag persistence

We recall zigzag persistence [7]. For some orientation τ of length n, a *zigzag complex of type* τ is defined to be a quiver complex over the quiver $A_n(\tau)$. A zigzag complex of type τ:

$$\mathbb{X}: \; X_1 \longleftrightarrow X_2 \longleftrightarrow \ldots \longleftrightarrow X_n$$

has (zigzag) persistent homology

$$H_q(\mathbb{X}): \; H_q(X_1) \longleftrightarrow H_q(X_2) \longleftrightarrow \ldots \longleftrightarrow H_q(X_n)$$

which is a representation of $A_n(\tau)$, where the linear maps between the homology vector spaces $H_q(X_i)$ are induced from the respective inclusions.

Again by Gabriel's theorem [21], the indecomposable representations of $A_n(\tau)$ are the interval representations $\mathbb{I}[a,b]$, given by:

$$0 \longleftrightarrow \ldots \longleftrightarrow 0 \longleftrightarrow K \xleftrightarrow{1} \ldots \xleftrightarrow{1} K \longleftrightarrow 0 \longleftrightarrow \ldots \longleftrightarrow 0 \;,$$

where of course the directions of the arrows are determined by τ. In the special case where $\tau = f \ldots f$, we get the interval representations discussed in the previous subsection.

Thus,

$$H_q(\mathbb{X}) \cong \bigoplus_{1 \leqslant i \leqslant j \leqslant n} \mathbb{I}[i,j]^{m_{i,j}}$$

and this indecomposable decomposition of $H_q(\mathbb{X})$ is unique up to isomorphism and permutation of summands, by Proposition 2.1.1. From this it is easy to define the corresponding persistence diagram.

An example is already provided in the introduction, which we repeat here. Suppose that we have spaces $X_1, \ldots, X_T$, with no natural filtration structure. A way to detect common features would be to form the diagram

$$X_1 \longrightarrow X_1 \cup X_2 \longleftarrow X_2 \longrightarrow \ldots \longrightarrow X_{T-1} \cup X_T \longleftarrow X_T$$

with alternating arrows and then study

$$H_q(X_1) \longrightarrow H_q(X_1 \cup X_2) \longleftarrow \ldots \longrightarrow H_q(X_{T-1} \cup X_T) \longleftarrow H_q(X_T)$$

which is a representation of $A_m(\tau)$ with $\tau = fb \ldots fb$ and $m = 2T - 1$. This allows us to extract the topological features that persist across the different spaces.

2.5 Modules and representations

In this section, we collect various algebraic background for the rest of this work.

First, let us give the following general comment. As shown in Theorem 2.3.1, the representation category of a bound quiver is equivalent to the module category of its algebra. There is a close relationship between the bound quivers and algebras themselves. From an Artin K-algebra A, we can construct a quiver Q_A, called the quiver of A. It is known that if K is algebraically closed, then $A \cong KQ_A/I$ for some ideal I of KQ_A. For example, see [5, Prop. 4.1.7]. If we weaken the requirement to K being a perfect field, there is an analogous result [5, Cor. 4.1.11] involving so-called valued quivers.

For the purpose of computation, we avoid requiring K to be algebraically closed. For example, with $K = \mathbb{Z}_2$ (binary) or $K = \mathbb{Q}$, one aspect of implementing algorithms will potentially be simplified, since we can avoid the need to implement a custom number type. For our purposes, it is enough to be able to construct the algebra of a bound quiver.

2.5.1 Modules

Let A be a finite dimensional K-algebra. A module $M \in \operatorname{mod} A$ is said to be *simple* if its submodules are only 0 and itself. A module is said to be *semisimple* if it is the direct sum of simple modules. The module generated by all simple submodules of M is the *socle* of M, denoted $\operatorname{Soc} M$. It is known that $\operatorname{Soc} M$ is semisimple. See [1, Proposition 9.7]

The *radical* $\operatorname{Rad} M$ of an A-module M is the intersection of its maximal submodules. Similarly, the *radical* of A, denoted r_A, is defined to be the intersection of maximal left ideals of A. The radical r_A is equal to the intersection of maximal right ideals, and is a two-sided ideal of A. An algebra A can be considered as a module over itself, ${}_AA$, with radical $\operatorname{Rad} {}_AA$. Moreover, $r_A = \operatorname{Rad} {}_AA$, so we simply write $\operatorname{Rad} A$ for the radical of A.

A ring with unity R is said to be *local* if the noninvertible elements of R forms a two-sided ideal. It is known that if R is local, the $\operatorname{Rad} R$ is equal to the set of noninvertible elements of R.

For $M \in \operatorname{mod} A$, the endomorphism ring $\operatorname{End}_A(M)$ can be given a K-algebra structure as well.

Lemma 2.5.1 ([2, Corollary I.4.8]). *Let A be a finite-dimensional K-algebra and $M \in \operatorname{mod} A$.*

1. *M is indecomposable if and only if $\operatorname{End}_A(M)$ is local.*
2. *If M is indecomposable, then any noninvertible $f \in \operatorname{End}_A(M)$ is nilpotent.*

A module P is said to be *projective* if it satisfies the following "lifting" property. For every morphism $g : P \to N$ and every epimorphism $f : M \to N$, there exists an $h : P \to M$ such that $fh = g$, as in the diagram:

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[dr, "g"] & & \\ M \arrow[rr, "f"] & & N \arrow[r] & 0. \end{tikzcd}$$

Dually, an module I is said to be *injective* if for every morphism $g : M \to I$ and every monomorphism $f : M \to N$, there is a morphism $h : N \to I$ such that $hf = g$, as:

$$\begin{tikzcd} & & I & \\ 0 \arrow[r] & M \arrow[ur, "g"] \arrow[rr, "f"] & & N. \arrow[ul, dashed, "h"'] \end{tikzcd}$$

The *opposite algebra* A^{op} of A is the K-algebra with the same elements as A and operation $\cdot$ defined by $a \cdot b = ba$ for $a, b \in A^{\text{op}}$.

Define the contravariant functor:

$$D(-) = \operatorname{Hom}_K(-, K) : \operatorname{mod} A \to \operatorname{mod} A^{\text{op}}.$$

That is, for $M \in \operatorname{mod} A$, $D(M) = \operatorname{Hom}_K(M, K)$ is defined by treating M as a K-vector space and taking its K-dual. This has an A^{op}-module structure, and thus a right A-module structure, by the following. For $f \in \operatorname{Hom}_K(M, K)$ and $a \in A$, define $fa \in \operatorname{Hom}_K(M, K)$ by $(fa)(m) = f(am)$ for all $m \in M$. See the Prop. 4.4 of the book [1]. Let $f : M \to N$ be a morphism in $\operatorname{mod} A$. Then, $D(f) : D(N) \to D(M)$ is defined by $D(f)(g) = gf \in D(M)$ for $(g : N \to K) \in D(N)$.

It is known that $D(-)$ is a duality, with quasi-inverse given by

$$\operatorname{Hom}_K(-, K) : \operatorname{mod} A^{\text{op}} \to \operatorname{mod} A.$$

which we also denote by $D(-)$.

We also need the contravariant functor

$$(-)^t : \operatorname{Hom}_A(-, A) : \operatorname{mod} A \to \operatorname{mod} A^{\text{op}}$$

that takes M to $\operatorname{Hom}_A(M, A)$ and $f : M \to N$ to $\operatorname{Hom}_A(f, A) : \operatorname{Hom}_A(N, A) \to \operatorname{Hom}_A(M, A)$. Here, $\operatorname{Hom}_A(M, A)$ is given the structure of a right A-module by $(fa)(m) = f(m)a$ for $f \in \operatorname{Hom}_A(M, A)$, $a \in A$, and $m \in M$.

However, $(-)^t$ is not a duality in general. It is known [3, Prop. II.4.3] that if we restrict $(-)^t$ to the full subcategory of finitely generated projective modules $\operatorname{proj} A$, we get a duality

$$(-)^t : \operatorname{proj} A \to \operatorname{proj} A^{\text{op}}.$$

A submodule N of M is said to be *superfluous* in M if whenever $N + X = M$ for some submodule X implies that $X = M$. A *projective cover* of a module M is an epimorphism $f : P \to M$ where P is projective, and $\operatorname{Ker} f$ is superfluous in P. Below, let us list some properties of projective covers.

Proposition 2.5.2. *Let A be a finite-dimensional K-algebra.*

1. *$M \in \operatorname{mod} A$ has a projective cover $f : P \to M$ in $\operatorname{mod} A$.*

2. *If $f_1 : P_1 \to M$ and $f_2 : P_2 \to M$ are projective covers of $M \in \operatorname{mod} A$, then there exists an isomorphism $h : P_1 \to P_2$ such that $f_1 = f_2 h$.*

3. *Given a finite family $\{f_i : P_i \to M_i\}_{i=1}^n$ of epimorphisms with P_i projective modules, $\bigoplus_{i=1}^n f_i : \bigoplus_{i=1}^n P_i \to \bigoplus_{i=1}^n M_i$ is a projective cover if and only if each $f_i : P_i \to M_i$ is a projective cover.*

4. *For each $P \in \operatorname{proj} A$, the epimorphism $P \to P/\operatorname{Rad} P$ is a projective cover.*

For a proof, see Theorems I.4.2, I.4.4 and Proposition I.4.3 of [3].

Dually, $N \subset M \in \operatorname{mod} A$, N is said to be *essential* in M if $X \cap N \neq 0$ for any nonzero submodule X of M. A monomorphism $f : M \to I$ with I injective is said to be an *injective envelope* of M if $\operatorname{Im} f$ is essential in I.

Proposition 2.5.3. *Let A be a finite-dimensional K-algebra. For $M \in \operatorname{mod} A$,*

1. *$f : M \to I$ is an injective envelope if and only if the induced map $\operatorname{Soc} M \to I$ is an injective envelope.*

2. *Suppose that M is a semisimple module, with a projective cover $P \to M$ in $\operatorname{mod} A$. Then, $D(P^t)$ is an injective envelope of M.*

See Propositions II.4.1 and II.4.6 of [3].

Thus, to compute an injective envelope of $M \in \operatorname{mod} A$, we do the following. First, we compute $\operatorname{Soc} M$ and try to find an injective envelope I for $\operatorname{Soc} M$. This induces an injective envelope for M, by Prop 2.5.3, part *1*. Since $\operatorname{Soc} M$ is semisimple, a projective cover for $\operatorname{Soc} M$, say P, provides an injective envelope $I = D(P^t)$ for $\operatorname{Soc} M$, by Prop. 2.5.3, part *2*. Using Prop. 2.5.2 part *3*, and since $\operatorname{Soc} M$ is a direct sum of simple modules, it suffices to find projective covers for each of its summands, which are all simple modules.

In the above procedure, we have not described how to compute $\operatorname{Soc} M$, and projective covers for the simple modules. In the case where the algebra A is given by $A = KQ/I$ as the algebra of a bound quiver (Q, ρ), a way to compute $\operatorname{Soc} M$ is provided in the next subsection. Moreover, we can get a complete list (up to isomorphism) of simple modules S_i and indecomposable projective modules P_i, together with projective covers $P_i \to P_i/\operatorname{Rad} P_i = S_i$.

2.5.2 Representations of a bound quiver

In this subsection, we collect some useful facts concerning the representation of bound quivers. In particular, we will focus on results that enable one to do computations.

First, let us reformulate the duality $D(-)$ above. Given a quiver $Q = (Q_0, Q_1)$, the *opposite quiver* is the quiver Q^{op} with vertices Q_0, and an arrow $\alpha^{\text{op}} : j \to i$ for every arrow $\alpha : i \to j$ in Q_1. We can identify $(KQ)^{\text{op}} = K(Q^{\text{op}})$.

Let $p = (j|\alpha_m, \ldots, \alpha_1|i)$ be a path in Q. Then, the opposite path is $p^{\text{op}} = (i|\alpha_1^{\text{op}}, \ldots, \alpha_m^{\text{op}}|j)$ in $K(Q^{\text{op}})$. If ρ is a set of relations of Q, ρ induces a set of relations ρ^{op} of Q^{op} by taking the opposite paths. Let I^{op} be the two-sided ideal of $K(Q^{\text{op}})$ generated by ρ^{op}. We can identify the algebras

$$(KQ/I)^{\text{op}} = K(Q^{\text{op}})/I^{\text{op}}.$$

In the case where the algebra $A = KQ/I$ is defined by (Q, ρ), where I is the two-sided ideal of KQ generated by ρ, Theorem 2.3.1 provides an equivalence $\mathrm{rep}(Q, \rho) \cong \mathrm{mod}\, A$. Likewise, $\mathrm{rep}(Q^{\mathrm{op}}, \rho^{\mathrm{op}}) \cong \mathrm{mod}\, A^{\mathrm{op}}$. Thus, $D(-) : \mathrm{mod}\, A \to \mathrm{mod}\, A^{\mathrm{op}}$ induces a duality

$$D(-) : \mathrm{rep}(Q, \rho) \to \mathrm{rep}(Q^{\mathrm{op}}, \rho^{\mathrm{op}})$$

also denoted by $D(-)$.

Furthermore, this duality $D(-)$ on quiver representations can be given explicitly by the following computation. It is easy to check that $D(M)$ is the representation that has the K-vector space $D(M_i)$ at each vertex $i \in Q_0$, and $D(f_\alpha) : D(M_j) \to D(M_i)$ for every arrow $\alpha : i \to j$ in Q, where $D(-)$ is the usual duality for finite-dimensional K-vector spaces. Likewise, for $f : M \to N$ a morphism of representations, $D(f) : D(N) \to D(M)$ can be computed by taking the dual map at every vertex.

Next, we give explicit formulations of the indecomposable projective, injective, and simple representations. Recall that an element $e \in A$ is said to be *idempotent* if $e^2 = e$. The idempotents $0, 1 \in A$ are called the trivial idempotents. Two idempotents e_1, e_2 are said to be *orthogonal* if $e_1 e_2 = 0 = e_2 e_1$. An idempotent is said to be *primitive* if it cannot be written as a sum of two nontrivial orthogonal idempotents $e = e_1 + e_2$.

A *complete set of primitive orthogonal idempotents* is a set $\{e_1, \ldots, e_n\}$ of primitive and pairwise orthogonal idempotents such that $1 = e_1 + \ldots + e_n$. It is known that

Lemma 2.5.4 (cf. [3, Prop. I.4.8]). *Let A be a finite-dimensional K-algebra. Then:*

1. *There is a complete set of primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$ and $A = Ae_1 \oplus \ldots \oplus Ae_n$ (as left A-modules),*

2. *Given $e \in A$ an idempotent, e is a primitive idempotent if and only if Ae is an indecomposable projective A-module.*

For some choice of a complete set of primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$, we write $P_i = Ae_i$ for $i \in \{1, \ldots, n\}$. This gives a complete list of isomorphism classes of indecomposable projectives. Dually, $I_i = D(e_i A)$ for $i \in \{1, \ldots, n\}$ gives a complete list of isomorphism classes of indecomposable injectives. Moreover, $S_i = P_i / \mathrm{Rad}\, P_i$ gives all the indecomposable simple modules, up to isomorphism.

In particular, we are interested in the algebra of a bound quiver (Q, ρ). In this case, recall that the stationary paths in KQ are defined to be $e_i = (i||i)$ for $i \in Q_0$. It can be checked that the set of stationary paths modulo I

$$\{\bar{e}_i = e_i + I \mid i \in Q_0\}$$

is a complete set of primitive orthogonal idempotents of the algebra A of (Q, ρ). We have the following.

Lemma 2.5.5 ([2, Lemma III.2.4]). *Let (Q, ρ) be a bound quiver, with complete set of primitive orthogonal idempotents $\{\bar{e}_i = e_i + I \mid i \in Q_0\}$ for its algebra $A = KQ/I$.*

1. *$P_i = A\bar{e}_i$ is the representation of (Q, ρ) with K-vector space $P_i(j)$ at the vertex j generated by $p + I$ for p paths from i to j in Q. For every arrow $\alpha : j \to k$, the map $P_i(\alpha) : P_i(j) \to P_i(k)$ is induced by multiplication of $\bar{\alpha}$.*

2. *$I_i = D(\bar{e}_i A)$ is the representation of (Q, ρ) with $I_i(j)$ the dual of the K-vector space generated by $p + I$, for p paths from j to i in Q. For every arrow $\alpha : j \to k$, $I_i(\alpha) : I_i(j) \to I_i(k)$ is the dual of the map induced by multiplication of $\bar{\alpha}$.*

3. *$S_i = P_i/\operatorname{Rad} P_i$ is the representation of (Q, ρ) with*

$$S_i(j) = \begin{cases} K\bar{e}_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

   and all maps $S_i(\alpha)$ equal to 0 for each arrow α.

Proof. By the equivalence in Theorem 2.3.1, the representation P_i corresponding to the indecomposable projective module P_i has K-vector space $\bar{e}_j P_i = \bar{e}_j A\bar{e}_i$ at vertex $j \in Q_0$. Clearly, this is the K-vector space generated by paths modulo I from i to j in Q, as claimed. The statement for the indecomposable injectives I_i follows by similar arguments. The computation for S_i follows easily from $S_i = P_i/\operatorname{Rad} P_i$ and the next lemma. □

The following lemma is also useful.

Lemma 2.5.6 ([2, Lemma III.2.2]). *Let $V = (V(i), V(\alpha))$ be a representation of (Q, ρ).*

1. *The radical of V, $\operatorname{Rad} V$ is the representation $(W(i), W(\alpha))$ with*

$$W(i) = \sum_{\alpha : j \to i} \operatorname{Im} V(\alpha)$$

   for every vertex i that is not a source, and $W(i) = 0$ otherwise. The map $W(\alpha) : W(i) \to W(j)$ for every arrow $\alpha : i \to j$ is given by the restriction of $V(\alpha)$ to $W(i)$.

2. *The socle of V, $\operatorname{Soc} V$ is the representation $(W(i), W(\alpha))$ with*

$$W(i) = \bigcap_{\alpha : i \to j} \operatorname{Ker} V(\alpha)$$

   for every vertex i that is not a sink, and $W(i) = V(i)$ otherwise. $W(\alpha) = 0$ for every arrow $\alpha : i \to j$.

2.5.3 Exact sequences and extensions

A sequence of A-modules and morphisms

$$\dots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \dots$$

is said to be *exact* if for every i, $\operatorname{Im} f_{i-1} = \operatorname{Ker} f_i$.

Let $M \in \operatorname{mod} A$ be a module. A *projective resolution* of M is an exact sequence

$$\dots \longrightarrow P_n \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

such that all the P_i's are projective modules. A projective resolution is said to be finite and have length n if P_n is nonzero and $P_i = 0$ for all $i > n$. If $M \in \operatorname{mod} A$ has a projective resolution with finite length, then the *projective dimension* $\operatorname{pd}_A M$ of M is the minimal length of all of its finite projective resolutions.

The (left) global dimension of an algebra A is defined by

$$\operatorname{gl.dim} A = \sup\{\operatorname{pd}_A M \mid M \in \operatorname{mod} A\}.$$

The "left" comes from the fact that we are looking at left modules of A. There is of course a "right" version to this definition, but here we consider only the left global dimension.

An algebra A is said to be *left hereditary* if all of its left ideals are projective. We simply call left hereditary algebras hereditary.

Lemma 2.5.7 (cf. [3, Cor. I.5.2])**.** *The following are equivalent for a finite-dimensional K-algebra A.*

1. *A is hereditary.*

2. $\operatorname{Rad} A$ *is a projective A-module.*

3. $\operatorname{gl.dim} A \leqslant 1$.

It is known that if A is hereditary, the submodules of projective A-modules are themselves projective. In particular, $\operatorname{Rad} P_i \subset P_i$ is projective for each indecomposable projective P_i.

Given a diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"] & N \\ L, & \end{tikzcd}$$

in $\operatorname{mod} A$, recall that a *pushout* of this diagram is an $X \in \operatorname{mod} A$ together with maps $g' : N \to X$, and $f' : L \to X$ such that $f'f = g'g$ and satisfying the following universal property. If there is a $Y \in \operatorname{mod} A$ with maps $k : N \to Y$ and

$\ell : L \to Y$ such that $kg = \ell f$, then there is a map $h : X \to Y$ such that $hg' = k$ and $hf' = \ell$.

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"] & N \arrow[d, "g'"] \arrow[ddr, "k"] & \\ L \arrow[r, "f'"] \arrow[drr, "\ell"'] & X \arrow[dr, dashed, "h"] & \\ & & Y. \end{tikzcd}$$

In $\mathrm{mod}\, A$ there is an explicit construction of pushouts. Let

$$Q = \langle (g(m), -f(m)) \mid m \in M \rangle$$

be the submodule of $N \oplus L$ generated by elements of the form $(g(m), -f(m))$. Define $X = (N \oplus L)/Q$ together with $g' : N \to X$ and $f' : L \to X$ via the obvious inclusions then the projection. For $m \in M$, $f'fm = (0, fm) + Q$ while $g'gm = (gm, 0) + Q$ so that $f'f = g'g$.

Let M and N be in $\mathrm{mod}\, A$. An extension of M by N is a short exact sequence

$$E : \; 0 \longrightarrow M \xrightarrow{g} L \xrightarrow{f} N \longrightarrow 0$$

of A-modules. Note that if $M, N \in \mathrm{mod}\, A$, any extension of M by N will have $L \in \mathrm{mod}\, A$ as well.

Let $x : M \to M'$, and let E be an extension of M by N. The extension xE of M' by N is the bottom row in:

$$\begin{tikzcd} E: & 0 \arrow[r] & M \arrow[r, "g"] \arrow[d, "x"] & L \arrow[r, "f"] \arrow[d, "y"] & N \arrow[r] \arrow[d, "1"] & 0 \\ xE: & 0 \arrow[r] & M' \arrow[r, "g'"] & L' \arrow[r, "f'"] & N \arrow[r] & 0 \end{tikzcd},$$

where $M' \xrightarrow{g'} L' \xleftarrow{y} L$ is the pushout of $M' \xleftarrow{x} M \xrightarrow{g} L$ constructed above, and where $f' : L' \to N$ is defined by

$$\begin{array}{rcl} f' : L' = (L \oplus M')/Q & \to & N \\ (l, m) + Q & \mapsto & f(l). \end{array}$$

This is well-defined, since if $(l, m') - (l', m'') \in Q$, then $l - l' = g(m)$ for some $m \in M$. Thus,

$$f'((l, m') + Q) - f'((l', m'') + Q) = f(l) - f(l') = fg(m) = 0.$$

The other required properties can be easily checked.

Concerning short exact sequences, the following lemma is useful.

Lemma 2.5.8 (Short Five Lemma)**.** *Given a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "g"] \arrow[d, "\mu"] & L \arrow[r, "f"] \arrow[d, "\lambda"] & N \arrow[r] \arrow[d, "\nu"] & 0 \\ 0 \arrow[r] & M' \arrow[r, "g'"] & L \arrow[r, "f'"] & N' \arrow[r] & 0 \end{tikzcd}$$

with both rows short exact, if both μ and ν are isomorphisms, then so is λ.

Chapter 3

Morse Reductions for Quiver Complexes

In this chapter, we generalize the use of Morse reductions for filtrations [30] to quiver complexes. The main goal can be explained quite simply. Given a quiver complex $\mathbb{X}$, we construct a smaller quiver complex $\mathbb{A}$, but with the property that their persistent homology modules are isomorphic: $H_q(\mathbb{X}) \cong H_q(\mathbb{A})$. Such a reduction can be used as a preprocessing step.

We first review the basic ideas of discrete Morse theory in Section 3.1. The discussion of discrete Morse theory for quiver complexes is in Section 3.2. In Section 3.3, we present an algorithm for computing an acyclic matching for an input quiver complex $\mathbb{X}$, and thus for computing $\mathbb{A}$. This is done by a modification of the algorithm presented in the paper [30].

The content here is an expanded version of the works [18, 19], and we have adopted a slightly different proof strategy to prove the main theorem, Theorem 3.2.4. Also, we have simplified our exposition of the algorithm by the use of Lemma 3.3.4. It is hoped that this will clarify the main idea behind our modifications to the algorithm.

3.1 Morse reduction for a complex

First, let us review discrete Morse theory [20] and Morse reductions for a complex (X, κ). We follow the presentation in [30].

A *partial matching* for a complex (X, κ) is a partition of X into sets $\mathcal{A}, \mathcal{B}, \mathcal{D}$, together with a bijection $w : \mathcal{B} \to \mathcal{D}$ such that for every $\beta \in \mathcal{B}$, $\kappa(w(\beta), \beta)$ is nonzero. We denote a partial matching by $(\mathcal{A}, w : \mathcal{B} \to \mathcal{D})$. Recall that τ is said to be a boundary face of σ if $\kappa(\sigma, \tau) \neq 0$. Given a partial matching, for every $\beta \in \mathcal{B}$, β is a boundary face of $w(\beta)$ and $\dim w(\beta) = \dim \beta + 1$.

A partial matching $(\mathcal{A}, w : \mathcal{B} \to \mathcal{D})$ induces a relation $<_w$ on $\mathcal{B}$ by setting $\beta <_w \beta'$ if β is a boundary face of $w(\beta')$ and extending transitively. The relation $<_w$ is reflexive and transitive by definition. If in addition it is antisymmetric, then we say that the partial matching $(\mathcal{A}, w : \mathcal{B} \to \mathcal{D})$ is an *acyclic matching*. Where it is not likely to cause confusion, we refer to an acyclic matching simply by w.

Note that an acyclic matching always exists for any complex (X, κ), given by the *empty acyclic matching*. This is the acyclic matching $(\mathcal{A}, w : \mathcal{B} \to \mathcal{D})$ with $\mathcal{A} = X$, $\mathcal{B} = \mathcal{D} = \varnothing$, and $w = \varnothing$.

A *gradient path* p is an alternating sequence

$$(\beta_1, w(\beta_1), \beta_2, w(\beta_2), \ldots, \beta_n, w(\beta_n))$$

of cells $\beta_i \in \mathcal{B}$ and $w(\beta_i) \in \mathcal{B}$ such that β_{i+1} is a boundary face of $w(\beta_i)$ for every $i \in \{1, \ldots, n-1\}$. The cells $\beta_i \in \mathcal{B}$ in a gradient path all have the same dimension. This can be verified by the fact that both β_{i+1} and β_i are boundary faces of $w(\beta_i)$, the former by definition of a gradient path and the latter by that of a partial matching. For simplicity of notation, we shall denote the gradient path $(\beta_1, w(\beta_1), \ldots, \beta_n, w(\beta_n))$ by $p = (\beta_1, \ldots, \beta_n)$.

Let $\sigma, \tau \in \mathcal{A}$. A gradient path $p = (\beta_1, \ldots, \beta_n)$ is said to be a *connection* from σ to τ, denoted $p : \sigma \rightsquigarrow \tau$, if β_1 is a boundary face of σ and τ is a boundary face of $w(\beta_n)$. The multiplicity of a connection $p : \sigma \rightsquigarrow \tau$ is defined to be

$$\epsilon(p) = \kappa(\sigma, \beta_1) i(p) \kappa(w(\beta_n), \tau)$$

where $i(p)$ is its index as a gradient path:

$$i(p) = \frac{\prod_{i=1}^{n-1} \kappa(w(\beta_i), \beta_{i+1})}{\prod_{i=1}^{n} -\kappa(w(\beta_i), \beta_i)}.$$

The multiplicity $\epsilon(p) \in K$ of a connection can be checked to be nonzero. For $\sigma, \tau \in \mathcal{A}$, if there is a connection $p : \sigma \rightsquigarrow \tau$, then $\dim \sigma = \dim \tau + 1$. This follows from the fact that $\dim \beta_1 = \dim \beta_n$, $\dim \tau = \dim \beta_n$, and $\dim \sigma = \dim \beta_1 + 1$.

The cells in $\mathcal{A}$ are called the *critical cells* of the acyclic matching w. In fact, we can create a complex from the critical cells $\mathcal{A}$ by setting

$$\mathcal{A}_q = \mathcal{A} \cap X_q$$

so that $\mathcal{A} = \bigsqcup_{q \geqslant 0} \mathcal{A}_q$ gives a grading on $\mathcal{A}$ by dimension. Then, define a new incidence map $\kappa_w : \mathcal{A} \times \mathcal{A} \to K$ by

$$\kappa_w(\sigma, \tau) = \kappa(\sigma, \tau) + \sum_{p : \sigma \rightsquigarrow \tau} \epsilon(p), \tag{3.1}$$

where the summation is taken over all connections $p : \sigma \rightsquigarrow \tau$ in X.

Theorem 3.1.1 ([30, Theorem 2.4])**.** *Let (X, κ) be a complex with an acyclic matching $(\mathcal{A}, w : \mathcal{B} \to \mathcal{D})$.*

1. *If κ_w is the incidence map defined in Eq.* (3.1)*, then $(\mathcal{A}, \kappa_w)$ is a complex, called the* Morse complex *of (X, κ) associated to the acyclic matching $(\mathcal{A}, w : \mathcal{B} \to \mathcal{D})$.*

2. *If $(\mathcal{A}, \kappa_w)$ is the Morse complex associated to $(\mathcal{A}, w : \mathcal{B} \to \mathcal{D})$, then the chain complexes of (X, κ) and $(\mathcal{A}, \kappa_w)$ are chain equivalent and thus $H_q(X) \cong H_q(\mathcal{A})$ for every $q \geqslant 0$.*

To show Theorem 3.1.1, [30] uses one-step reduction, which we explain below. This technique will also be used in the next section for quiver complexes, so it bears repeating here.

For any arbitrary $\beta \in \mathcal{B}$, let us consider the pair (X', κ'), with $X' = X\backslash\{\beta, w(\beta)\}$ and incidence function $\kappa' : X' \times X' \to K$ defined by

$$\kappa'(\sigma, \tau) = \kappa(\sigma, \tau) + \frac{\kappa(\sigma, \beta)\kappa(w(\beta), \tau)}{-\kappa(w(\beta), \beta)} \tag{3.2}$$

for any $\sigma, \tau \in X'$. Lemma 3.1.2 shows that (X', κ') is a complex, and we call (X', κ') the *complex induced by removal of* $\{\beta, w(\beta)\}$. By comparing Eq. (3.2) and Eq. (3.1), it can be checked that (X', κ') is the Morse complex associated to the acyclic matching $(X', \mu : \{\beta\} \to \{w(\beta)\})$.

Lemma 3.1.2. *Given a complex (X, κ) with acyclic matching $(\mathcal{A}, w : \mathcal{B} \to \mathcal{D})$ and $\beta \in \mathcal{B}$, (X', κ') as defined above is a complex.*

Proof. Let $\sigma, \tau \in X'$. If $\kappa'(\sigma, \tau) \neq 0$, then either $\kappa(\sigma, \tau) \neq 0$ or $\kappa(\sigma, \beta)\kappa(w(\beta), \tau) \neq 0$. In the first case, $\dim \sigma = \dim \tau + 1$ follows from definition. In the second case, $\kappa(\sigma, \beta) \neq 0$ and $\kappa(w(\beta), \tau) \neq 0$, together with the fact that $\kappa(w(\beta), \beta) \neq 0$ by definition, shows that $\dim \sigma = \dim \tau + 1$.

Next, we show that

$$\sum_{\sigma \in X'} \kappa'(\rho, \sigma)\kappa'(\sigma, \tau) = 0$$

for any $\rho, \tau \in X'$. Denote the summation on the left by S. Expanding S, we get

$$\begin{aligned} S = \sum_{\sigma \in X'} \kappa(\rho, \sigma)\kappa(\sigma, \tau) \quad &+ \sum_{\sigma \in X'} \kappa(\rho, \sigma)\tfrac{\kappa(\sigma,\beta)\kappa(w(\beta),\tau)}{-\kappa(w(\beta),\beta)} \\ &+ \sum_{\sigma \in X'} \tfrac{\kappa(\rho,\beta)\kappa(w(\beta),\sigma)}{-\kappa(w(\beta),\beta)}\kappa(\sigma, \tau) \\ &+ \sum_{\sigma \in X'} \tfrac{\kappa(\rho,\beta)\kappa(w(\beta),\sigma)}{-\kappa(w(\beta),\beta)}\tfrac{\kappa(\sigma,\beta)\kappa(w(\beta),\tau)}{-\kappa(w(\beta),\beta)} \end{aligned} \tag{3.3}$$

The last summand in Eq. (3.3) is always zero. If not, then there exists a $\sigma \in X'$ such that $\kappa(\rho, \beta)\kappa(w(\beta), \sigma) \neq 0$ and $\kappa(\sigma, \beta)\kappa(w(\beta), \tau) \neq 0$. This implies that σ is a boundary face of $w(\beta)$, and β is a boundary face of σ, leading to $\dim w(\beta) = \dim \sigma + 1 = (\dim \beta + 1) + 1$, contradicting the fact that $\dim w(\beta) = \dim \beta + 1$.

To simplify the remaining summations, we use the fact that for any $x, y \in X'$,

$$\sum_{\sigma \in X'} \kappa(x, \sigma)\kappa(\sigma, y) = -\kappa(x, \beta)\kappa(\beta, y) - \kappa(x, w(\beta))\kappa(w(\beta), y),$$

which follows from $\sum_{\sigma \in X} \kappa(x, \sigma)\kappa(\sigma, y) = 0$ and $X = X' \sqcup \{\beta, w(\beta)\}$. Using this equality one can show that $S = 0$. ∎

Next, given the complex (X', κ') induced by removal of $\{\beta, w(\beta)\}$ from (X, κ), let us construct chain equivalences between $C(X)$ and $C(X')$. Since we will use the chain equivalences in the next section, let us give the definitions of the chain equivalences ψ, ϕ between $C(X)$ and $C(X')$, together with the homotopy θ between $\phi\psi$ and $1_{C(X)}$.

We first define ψ, ϕ, and θ as sequences of morphisms. Lemma 3.1.3 then shows that these are the required chain equivalences and the homotopy, respectively. Let ψ be the sequence of morphisms $\psi = (\psi_q : C_q(X) \to C_q(X'))$ defined by linear extension of the formula

$$\psi_q(x) = \begin{cases} 0 & \text{if } x = w(\beta) \\ -\sum\limits_{\sigma \in X'} \frac{\kappa(w(\beta),\sigma)}{\kappa(w(\beta),\beta)}\sigma & \text{if } x = \beta \\ x & \text{otherwise,} \end{cases} \tag{3.4}$$

for $x \in X_q$, for each $q \geqslant 0$. Note that this is well-defined, because if $\dim x = q$, then the formula for $\psi_q(x)$ above is a q-chain in $C_q(X')$. Similarly, define the sequence of morphisms $\phi = (\phi_q : C_q(X') \to C_q(X))$ by linear extension of

$$\phi_q(x) = x - \frac{\kappa(x, \beta)}{\kappa(w(\beta), \beta)} w(\beta), \tag{3.5}$$

for $x \in X'_q$ and $q \geqslant 0$. Again, $\dim x = q$ implies that $\phi_q(x) \in C_q(X)$. Finally, $\theta = (\theta_q : C_q(X) \to C_{q+1}(X))$ is defined by K-linear extension of

$$\theta_q(x) = \begin{cases} \frac{1}{\kappa(w(\beta),\beta)} w(\beta) & \text{if } x = \beta \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

for $x \in X_q$, and for $q \geqslant 0$.

Lemma 3.1.3 (cf. [30, Lemma 2.5]). *The collection of maps $\psi : C(X) \to C(X')$ and $\phi : C(X') \to C(X)$ as defined above are chain maps. Moreover, $\psi\phi = 1_{C(X')}$, and $\phi\psi \sim 1_{C(X)}$. Thus, ψ and ϕ are chain equivalences.*

The proof of Lemma 3.1.3, which we skip here, is by straightforward computations to check that required identities are satisfied.

A final ingredient is needed to show Theorem 3.1.1.

Definition 3.1. Given a complex (X, κ) and an acyclic matching $w = (\mathcal{A}, w : \mathcal{B} \to \mathcal{D})$ on (X, κ), let (X', κ') be the complex induced by removal of $\{\beta, w(\beta)\}$ for some $\beta \in \mathcal{B}$.

The acyclic matching $w' = (\mathcal{A}, w' : \mathcal{B}' \to \mathcal{D}')$ on (X', κ') induced from w by removal of $\{\beta, w(\beta)\}$ is defined by $\mathcal{B}' = \mathcal{B} \backslash \{\beta\}$, $\mathcal{D}' = \mathcal{D} \backslash \{w(\beta)\}$, and $w'(b) = w(b)$ for each $b \in \mathcal{B}'$.

We can thus iterate this process of removing pairs $\{\beta, w(\beta)\}$ from the induced complexes. Lemma 3.1.4 ensures that in the end, the repeated one-step removals of pairs $\{\beta, w(\beta)\}$ gives an induced complex that is the same as $(\mathcal{A}, \kappa_w)$.

Lemma 3.1.4 ([30, Prop. 2.6]). *Let (X, κ) be a complex with an acyclic matching $(\mathcal{A}, w : \mathcal{B} \to \mathcal{D})$. Fix a $\beta \in \mathcal{B}$, and let (X', κ'), $(\mathcal{A}, w' : \mathcal{B}' \to \mathcal{D}')$ be the complex and acyclic matching, respectively, induced by removal of $\{\beta, w(\beta)\}$, as above. Then,*

$$\kappa_w = (\kappa')_{w'}$$

For a proof, see the paper [30].

Proof of Theorem 3.1.1. Put some arbitrary ordering on the cells in $\mathcal{B}$, say $\beta_1, \ldots, \beta_N$. Then, let $(X^{\beta_1}, \kappa^{\beta_1})$ be the complex induced by removal of $\{\beta_1, w(\beta_1)\}$ from (X, κ). For every $i \in \{2, \ldots, N\}$, let $(X^{\beta_i}, \kappa^{\beta_i})$ be induced from removal of $\{\beta_i, w(\beta_i)\}$ from $(X^{\beta_{i-1}}, \kappa^{\beta_{i-1}})$. Similarly, for each i, let $(\mathcal{A}, w^{\beta_i} : \mathcal{B}^{\beta_i} \to \mathcal{D}^{\beta_i})$ be the induced acyclic matching on $(X^{\beta_i}, \kappa^{\beta_i})$.

Then, by repeated application of Lemma 3.1.4,

$$\begin{aligned} \kappa_w &= (\kappa^{\beta_1})_{w^{\beta_1}} \\ &= (\kappa^{\beta_2})_{w^{\beta_2}} \\ &= \ldots \\ &= (\kappa^{\beta_N})_{w^{\beta_N}} \\ &= \kappa^{\beta_N}. \end{aligned}$$

The last step follows since we have removed all the paired cells $\mathcal{B}, \mathcal{D}$. Thus,

$$(\mathcal{A}, \kappa_w) = (X^{\beta_N}, \kappa^{\beta_N})$$

and $(\mathcal{A}, \kappa_w)$ is a complex by Lemma 3.1.2. This shows the first part of Theorem 3.1.1. Moreover, by repeated application of Lemma 3.1.3, we get a sequence of chain equivalences

$$C(X) \sim C(X^{\beta_1}) \sim \ldots \sim C(X^{\beta_N}) = C(\mathcal{A}),$$

showing the second part of the theorem. □

In the next section we extend Theorem 3.1.1 to the quiver complex case.

3.2 Morse quiver complexes

Recall that a quiver complex over a quiver Q is a set of complexes, consisting of a complex (X_i, κ_i) for each $i \in Q_0$, such that whenever there is an arrow $\alpha : i \to j$ in Q_1, (X_i, κ_i) is a subcomplex of (X_j, κ_j).

In this section we expand the results reviewed in the previous section to the setting of quiver complexes. In particular, we give a definition of acyclic matchings of quiver complexes. We show that Theorem 3.1.1 has a natural analogue in this case. The main idea is to collect the chain equivalences given in the previous section "vertex-wise". By our definition of the acyclic matchings of quiver complexes, these collections then form the necessary chain equivalences.

Definition 3.2. Let $\mathbb{X} = (X_i, \kappa_i)_{i \in Q_0}$ be a quiver complex on the quiver Q. An *acyclic matching* of $\mathbb{X}$ is a collection $(\mathcal{A}_i, w_i : \mathcal{B}_i \to \mathcal{D}_i)_{i \in Q_0}$ of an acyclic matching of (X_i, κ_i) for every vertex $i \in Q_0$ such that for every arrow $\alpha : i \to j$, the conditions $\mathcal{A}_i \subset \mathcal{A}_j$, $\mathcal{B}_i \subset \mathcal{B}_j$, $\mathcal{D}_i \subset \mathcal{D}_j$ and $w_i(\sigma) = w_j(\sigma)$ for every $\sigma \in \mathcal{B}_i$ are satisfied.

Intuitively speaking, we require the acyclic matchings to agree across the inclusions. Our definition of an acyclic matching for a quiver complex is inspired by the definition of filtration-subordinate acyclic matchings used in [30]. To abbreviate the notation, we simply write w for the acyclic matching $(\mathcal{A}_i, w_i : \mathcal{B}_i \to \mathcal{D}_i)_{i \in Q_0}$.

The following equivalent characterization is convenient.

Lemma 3.2.1. *Let $\mathbb{X} = (X_i, \kappa_i)_{i \in Q_0}$ be a quiver complex on Q, and $w = (\mathcal{A}_i, w_i : \mathcal{B}_i \to \mathcal{D}_i)_{i \in Q_0}$ a collection of acyclic matchings, one for each (X_i, κ_i). The following are equivalent.*

1. *w is an acyclic matching of $\mathbb{X}$.*

2. *For every arrow $\alpha : i \to j$, we have $\mathcal{A}_i = \mathcal{A}_j \cap X_i$, $\mathcal{B}_i = \mathcal{B}_j \cap X_i$, $\mathcal{D}_i = \mathcal{D}_j \cap X_i$ and $w_i(\sigma) = w_j(\sigma)$ for every $\sigma \in \mathcal{B}_i$.*

Proof.

1→2 Let w be an acyclic matching of $\mathbb{X}$. We only show the proof for $\mathcal{A}_i = \mathcal{A}_j \cap X_i$. The others are exactly the same in form. Clearly, $\mathcal{A}_i \subset \mathcal{A}_j$ and $\mathcal{A}_i \subset X_i$, so that $\mathcal{A}_i \subset \mathcal{A}_j \cap X_i$.

Now if $\sigma \in \mathcal{A}_j \cap X_i$ but $\sigma \notin \mathcal{A}_i$, then σ is in $\mathcal{B}_i$ or $\mathcal{D}_i$ since $\mathcal{A}_i, \mathcal{B}_i, \mathcal{D}_i$ is a partition of X_i by definition. It follows that σ is in $\mathcal{B}_j$ or $\mathcal{D}_j$, a contradiction, since σ is in $\mathcal{A}_j$ and $\mathcal{A}_j, \mathcal{B}_j, \mathcal{D}_j$ is a partition of X_j. Thus $\mathcal{A}_i \supset \mathcal{A}_j \cap X_i$.

2→1 This follows directly from the definition.

□

Let $\mathbb{X} = (X_i, \kappa_i)_{i \in Q_0}$ be a quiver complex, and $w = (\mathcal{A}_i, w_i : \mathcal{B}_i \to \mathcal{D}_i)_{i \in Q_0}$ an acyclic matching of $\mathbb{X}$. For every vertex $i \in Q_0$, associated to the acyclic matching w_i of (X_i, κ_i) is the Morse complex $(\mathcal{A}_i, (\kappa_i)_{w_i})$, by part *1* of Theorem 3.1.1. To simplify the notation, we write $\tilde{\kappa}_i$ for $(\kappa_i)_{w_i}$.

Definition 3.3 (Morse quiver complex)**.** In the setting given above, the collection of complexes $\mathbb{A} = (\mathcal{A}_i, \tilde{\kappa}_i)_{i \in Q_0}$ is called the *Morse quiver complex* of $\mathbb{X}$ associated to the acyclic matching w.

Lemma 3.2.3 shows that $\mathbb{A}$ is indeed a quiver complex. To prove Lemma 3.2.3, let us first show the following technical lemma.

Lemma 3.2.2. *Let $\mathbb{X} = ((X_1, \kappa_1) \to (X_2, \kappa_2))$ be a quiver complex over the quiver $\vec{A}_2$ and*

$$(\mathcal{A}_i, w_i : \mathcal{B}_i \to \mathcal{D}_i)_{i=1,2}$$

an acyclic matching of $\mathbb{X}$. Suppose that $\sigma, \tau \in \mathcal{A}_2 \subset X_2$ with a connection $p = (\beta_1, \ldots, \beta_n) : \sigma \rightsquigarrow \tau$ in X_2. If $\sigma \in \mathcal{A}_1$, then τ is also in $\mathcal{A}_1$ and moreover $\beta_i \in X_1$ for all $i \in \{1, \ldots, n\}$.

Proof. Suppose that $\sigma \in \mathcal{A}_1 \subset X_1$. Since β_1 is a boundary face of σ, $\beta_1 \in X_1$ and so $\beta_1 \in X_1 \cap \mathcal{B}_2 = \mathcal{B}_1$.

If $\beta_i \in \mathcal{B}_1$, then $w_2(\beta_i) = w_1(\beta_i) \in \mathcal{D}_1 \subset X_1$. Furthermore, β_{i+1} is a boundary face of $w_2(\beta_i)$ by definition, and so $\beta_{i+1} \in X_1$ and thus $\beta_{i+1} \in X_1 \cap \mathcal{B}_2 = \mathcal{B}_1$. Inductively, we get that $\beta_i \in \mathcal{B}_1$ for all $i \in \{1, \ldots, n\}$. The final step is to note that τ is a boundary face of $w_2(\beta_n) = w_1(\beta_n) \in \mathcal{D}_1 \subset X_1$, so $\tau \in X_1$. Since τ is also in $\mathcal{A}_2$, we get that $\tau \in \mathcal{A}_1$. □

The following justifies calling the Morse quiver complex as such, since it is in fact a quiver complex.

Lemma 3.2.3. *Let $\mathbb{X}$ be a quiver complex over Q, and w be an acyclic matching of $\mathbb{X}$, as above. Then, for every arrow $\alpha : i \to j$ in Q, $(\mathcal{A}_i, \tilde{\kappa}_i)$ is a subcomplex of $(\mathcal{A}_j, \tilde{\kappa}_j)$. Thus, $\mathbb{A}$ is a quiver complex.*

Proof. By definition, for each arrow $\alpha : i \to j$ in Q_1, $\mathcal{A}_i \subset \mathcal{A}_j$ as sets. To show that $(\mathcal{A}_i, \tilde{\kappa}_i)$ is a subcomplex of $(\mathcal{A}_j, \tilde{\kappa}_j)$, we need to check the following two conditions.

1. $\tilde{\kappa}_j|_{\mathcal{A}_i \times \mathcal{A}_i} = \tilde{\kappa}_i$.
2. For any $\sigma, \tau \in \mathcal{A}_j$, if $\sigma \in \mathcal{A}_i$ and τ is a face of σ (in $(\mathcal{A}_j, \tilde{\kappa}_j)$), then $\tau \in \mathcal{A}_i$.

Let $\sigma, \tau \in \mathcal{A}_i \subset \mathcal{A}_j$. By definition,

$$\tilde{\kappa}_j(\sigma, \tau) = \kappa_j(\sigma, \tau) + \sum_{p:\sigma \rightsquigarrow \tau \text{ in } X_j} \epsilon_j(p),$$

as given in Eq. (3.1). Since (X_i, κ_i) is a subcomplex of (X_j, κ_j),

$$\kappa_i(\sigma, \tau) = \kappa_j(\sigma, \tau)$$

follows from definition. Furthermore, since $\sigma \in \mathcal{A}_i \subset X_i$, Lemma 3.2.2 shows that $\tau \in \mathcal{A}_i \subset X_i$, that all the cells in any connection $p : \sigma \rightsquigarrow \tau$ are in X_i, and thus $\epsilon_j(p) = \epsilon_i(p)$. From this, we conclude that

$$\tilde{\kappa}_j(\sigma, \tau) = \tilde{\kappa}_i(\sigma, \tau),$$

and the first condition holds.

It suffices to show the second condition for τ a boundary face of σ. So suppose that $\tilde{\kappa}_j(\sigma, \tau) \neq 0$ for some $\sigma \in \mathcal{A}_i$ and $\tau \in \mathcal{A}_j$. Then, either $\kappa_j(\sigma, \tau) \neq 0$, or there is a connection $p : \sigma \rightsquigarrow \tau$. In the first case we get $\tau \in X_i$ since (X_i, κ_i) is a subcomplex of (X_j, κ_j). Because $\tau \in \mathcal{A}_j$, it is also in $\mathcal{A}_i$. In the second case, it follows from Lemma 3.2.2 that $\tau \in \mathcal{A}_i$. □

Let a quiver complex $\mathbb{X}$ over a quiver Q be given. Recall that by Lemma 2.4.1, for any $q \geqslant 0$, $C_q(\mathbb{X})$ is a representation of (Q, c), where c is the set of commutativity relations of Q. Moreover, the chain complex of $\mathbb{X}$, $C(\mathbb{X}) = (C_q(\mathbb{X}), \partial_q)_q$ is a chain complex over $\operatorname{mod} KQ/I \cong \operatorname{rep}(Q, c)$, where KQ/I is the algebra of the (Q, c).

We state our main theorem in this section.

Theorem 3.2.4. *Let $\mathbb{X}$ be a quiver complex on a quiver Q, and let $w = (\mathcal{A}_i, w_i : \mathcal{B}_i \to \mathcal{D}_i)_{i \in Q_0}$ be an acyclic matching on $\mathbb{X}$, with associated Morse quiver complex $\mathbb{A}$. Then, the chain complexes $C(\mathbb{X})$ and $C(\mathbb{A})$ are chain equivalent. Thus, for all $q \geqslant 0$, the KQ/I-modules $H_q(\mathbb{X})$ and $H_q(\mathbb{A})$ are isomorphic.*

The rest of this section is devoted to proving Theorem 3.2.4 above. Before giving the proof, let us first explain the proof technique. Similar to the proof of Theorem 3.1.1, we shall use a one-step reduction, by defining the quiver complex $\mathbb{X}^\beta$ (in Lemma 3.2.6) induced by the removal of a pair $\{\beta, w_\ell(\beta)\}$ for some $\beta \in \mathcal{B}_\ell$, for some $\ell \in Q_0$.

Before going into details, first note that our definition of a quiver complex and its acyclic matching is local at each vertex, requiring consistency only across the arrows. Thus, when we define $\mathbb{X}^\beta$, we should be careful that we are using only the local information.

Suppose that $\beta \in \mathcal{B}_j$ for some $j \in Q_0$. Let us illustrate an example where trying to remove $\{\beta, w_i(\beta)\}$ from all (a global view) of the complexes (X_i, κ_i) with $\beta \in X_i$ may lead to inconsistencies.

Set the base field to be $K = \mathbb{Z}_2$ and consider $Q = A_3(bf)$, with a quiver complex $\mathbb{X} = (X_i, \kappa_i)_{i=1}^3$ as follows:

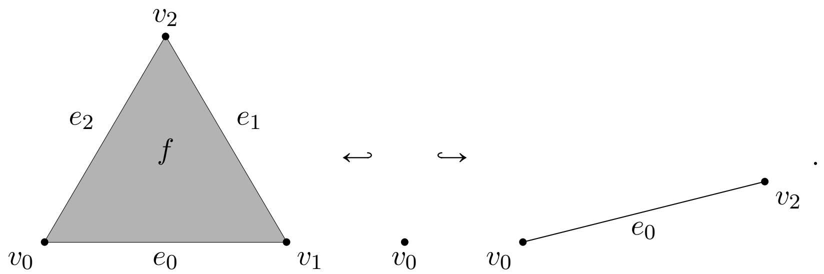
$$\begin{cases} X_1 = \{v_0, v_1, v_2, e_0, e_1, e_2, f\} \\ \partial_{2,1} f = e_0 + e_1 + e_2 \\ \partial_{1,1} e_0 = v_1 + v_0 \\ \partial_{1,1} e_1 = v_2 + v_1 \\ \partial_{1,1} e_2 = v_0 + v_2 \\ \partial_{0,1} = 0 \end{cases}, \quad \begin{cases} X_2 = \{v_0\} \\ \partial_{q,2} = 0 \end{cases}, \text{ and } \begin{cases} X_3 = \{v_0, v_2, e_0\} \\ \partial_{1,3} e_0 = v_0 + v_2 \\ \partial_{0,3} = 0 \end{cases}$$

where we have abbreviated the definitions of $\kappa_i(\cdot, \cdot)$ by instead specifying the values of the boundary maps

$$\partial_{q,i} \sigma = \sum_{\tau \in X_i} \kappa_i(\sigma, \tau) \tau$$

for $\sigma \in X_i$ with $\dim \sigma = q$. For example, $\partial_{2,1} f = e_0 + e_1 + e_2$ means $\kappa_1(f, e_0) = \kappa_1(f, e_1) = \kappa_1(f, e_2) = 1$ and $\kappa_1(f, x) = 0$ for $x \neq e_0, e_1, e_2$ in X_1.

This can be visualized as the following diagram of simplicial complexes:



In particular, note that e_0 in (X_1, κ_1) has boundary $v_0 + v_1$, but its boundary in (X_3, κ_3) is $v_0 + v_2$. While this may seem wrong, nothing in our definition of a quiver complex prevents this!

Now consider the acyclic matching w on $\mathbb{X}$ given by:

$$\mathcal{A}_1 = \{v_0, v_1, v_2, e_1, e_2\};\ \mathcal{B}_1 = \{e_0\};\ \mathcal{D}_1 = \{f\};\ w_1 : e_0 \mapsto f,$$

and empty acyclic matchings on (X_i, κ_i) for $i = 2, 3$. The e_0 at vertex 1 is matched to f, but the e_0 at vertex 3 is matched to nothing. Thus when we remove the pair $\{e_0, f\}$, we should take care that vertex 3 is not affected.

The above example motivates the following definition. It will be used frequently enough that it needs to be highlighted.

Definition 3.4. Let $\mathbb{X} = (X_i, \kappa_i)_{i \in Q_0}$ be a quiver complex. Suppose that $\sigma \in X_\ell$ for some fixed $\ell \in Q_0$. The *locus* of σ in ℓ is the set of vertices $L(\sigma, \ell) \subset Q_0$ with i in $L(\sigma, \ell)$ if and only if there is a (possibly stationary) path in $\bar{Q}$ from ℓ to i such that for every vertex k on that path, σ is a cell in (X_k, κ_k).

In the definition above, recall that $\bar{Q}$ is the underlying graph of Q so that the paths considered above are undirected paths. Note that the locus $L(\sigma, \ell)$ depends on both ℓ and σ. We also warn that $\sigma \in X_k$ does not imply that $k \in L(\sigma, \ell)$, though the converse is true.

In this section, we will primarily be dealing with the loci of $\beta \in \mathcal{B}_\ell$ when given an acyclic matching $(\mathcal{A}_i, w_i : \mathcal{B}_i \to \mathcal{D}_i)_{i \in Q_0}$. Using Lemma 3.2.1, it can be checked that if $i \in L(\beta, \ell)$, then $\beta \in \mathcal{B}_i$ and that $w_k(\beta) = w_{k'}(\beta)$ for every $k, k' \in L(\beta, \ell)$. We denote the common matched cell by δ.

The following technical lemma is useful for treating vertices on the boundary of the locus.

Lemma 3.2.5. *Let $\beta \in \mathcal{B}_\ell$ and let $L(\beta, \ell)$ be its locus. Suppose that there is an arrow $\alpha : i \to j$ in Q_0*

1. *If $i \in L(\beta, \ell)$, then $j \in L(\beta, \ell)$.*

2. *If $i \notin L(\beta, \ell)$ and $j \in L(\beta, \ell)$, then β and $w_j(\beta)$ are not in X_i.*

Proof.

1. Since $i \in L(\beta, \ell)$, by the definition, there exists a path p from ℓ to i in $\bar{Q}$ such that for all vertices k on this path, $\beta \in X_k$. In particular, $\beta \in X_i$, so that $\beta \in X_j$ since $X_i \subset X_j$. Extend the path p to a path p' from ℓ to j by appending the underlying edge of $\alpha : i \to j$. From this we conclude that $j \in L(\beta, \ell)$.

2. Since $j \in L(\beta, \ell)$, there exists a path p from ℓ to j in $\bar{Q}$ such that for all vertices k on this path, $\beta \in X_k$.

   If $\beta \in X_i$, then extending the path p via the underlying edge of the arrow $\alpha : i \to j$, we get a path p' from ℓ to i in $\bar{Q}$ such that for all vertices k on this path, $\beta \in X_k$. This shows that $i \in L(\beta, \ell)$, a contradiction. Thus $\beta \notin X_i$.

Suppose that $w_j(\beta) \in X_i$. Since $k_j(w_j(\beta), \beta) \neq 0$ by definition of an acyclic matching, we conclude that $\beta \in X_i$ because (X_i, κ_i) is a subcomplex of (X_j, κ_j). This is a contradiction to what we have just proved.

□

Finally we can give the definition of the quiver complex $\mathbb{X}^\beta$. Let us call $\mathbb{X}^\beta$ obtained in Lemma 3.2.6 the quiver complex induced by removal of $\{\beta, w_\ell(\beta)\}$ from $L(\beta, \ell)$ in $\mathbb{X}$. Clearly, $\mathbb{X}^\beta$ is dependent not only on β but also on ℓ.

Lemma 3.2.6. *Given a quiver complex $\mathbb{X}$ with acyclic matching w, fix a vertex $\ell \in Q_0$ and a $\beta \in \mathcal{B}_\ell$. Then, the collection of complexes:*

$$(\mathbb{X}^\beta)_i = \begin{cases} (X_i^\beta, \kappa_i^\beta) & \text{if } i \in L(\beta, \ell), \\ (X_i, \kappa_i) & \text{otherwise,} \end{cases}$$

forms a quiver complex $\mathbb{X}^\beta$. Here, each $(X_i^\beta, \kappa_i^\beta)$ is the complex induced by removal of $\{\beta, w_i(\beta)\}$ from (X_i, κ_i), as in the previous section.

Proof. To show that this is a quiver complex, let us first define $\mu = (\mathcal{A}'_i, \mu_i : \mathcal{B}' \to \mathcal{D}')_{i \in Q_0}$, a collection of acyclic matchings, one for each (X_i, κ_i).

Recall that for every $k, k' \in L(\beta, \ell)$, $w_k(\beta) = w_{k'}(\beta)$ and that this common matched cell is denoted by δ. For $i \in L(\beta, \ell)$, let

$$\begin{aligned} \mathcal{A}'_i &= X_i \backslash \{\beta, \delta\}, \\ \mathcal{B}'_i &= \{\beta\}, \\ \mathcal{D}'_i &= \{\delta\}, \\ \mu_i &: \beta \mapsto \delta. \end{aligned}$$

Note that this is well-defined, since $i \in L(\beta, \ell)$ implies that $\beta \in \mathcal{B}_i$ and $\delta = w_i(\beta) \in \mathcal{D}_i$. For $i \notin L(\beta, \ell)$, define μ_i to be the empty acyclic matching on (X_i, κ_i).

In fact, μ is an acyclic matching of $\mathbb{X}$. We need only check the consistency conditions given in Definition 3.2, for each arrow $\alpha : i \to j$. By Lemma 3.2.5, part *1*, it is not possible to have $i \in L(\beta, \ell)$ and $j \notin L(\beta, \ell)$, so there are only three cases to check.

Case 1: $i, j \in L(\beta, \ell)$. The inclusions

$$\begin{aligned} \mathcal{A}'_i = X_i \backslash \{\beta, \delta\} &\subset \mathcal{A}'_j = X_j \backslash \{\beta, \delta\}, \\ \mathcal{B}'_i = \{\beta\} &\subset \mathcal{B}'_j = \{\beta\}, \\ \mathcal{D}'{}_i = \{\delta\} &\subset \mathcal{D}'_j = \{\delta\} \end{aligned}$$

hold, and μ_i and μ_j are in fact the same maps.

Case 2: $i \notin L(\beta, \ell)$ and $j \in L(\beta, \ell)$. Here,

$$\begin{aligned} \mathcal{A}'_i = X_i &\subset \mathcal{A}'_j = X_j \backslash \{\beta, \delta\}, \\ \mathcal{B}'_i = \varnothing &\subset \mathcal{B}'_j = \{\beta\}, \\ \mathcal{D}'{}_i = \varnothing &\subset \mathcal{D}'_j = \{\delta\}, \end{aligned}$$

where the first inclusion above follows from the fact that β and $\delta = w_j(\beta)$ are not in X_i, by the second part of Lemma 3.2.5. The fact that $\mu_i = \varnothing$ is equal to μ_j restricted to $\mathcal{B}'_i = \varnothing$ is vacuously true.
Case 3: $i, j \notin L(\beta, \ell)$. This case is trivial.

By definition, $\mathbb{X}^\beta$ is the Morse quiver complex of $\mathbb{X}$ associated to the acyclic matching μ. From Lemma 3.2.3 we conclude that $\mathbb{X}^\beta$ is a quiver complex. □

Since $\mathbb{X}^\beta$ is a quiver complex on Q, its chain complex $C(\mathbb{X}^\beta)$ is a chain complex over $\text{rep}(Q, c)$ by Lemma 2.4.1. Next, we show that $C(\mathbb{X})$ and $C(\mathbb{X}^\beta)$ are chain equivalent. First, we need to construct the appropriate collections of morphisms. For each $q \geqslant 0$, $i \in Q_0$, define

$$\psi^\beta_{q,i} : C_q(\mathbb{X}_i) = C_q(X_i) \to C_q((\mathbb{X}^\beta)_i)$$

to be

$$\psi^\beta_{q,i} = \begin{cases} \psi_{q,i} : C_q(X_i) \to C_q(X_i^\beta) & \text{if } i \in L(\beta, \ell), \\ 1 : C(X_i) \to C(X_i) & \text{otherwise,} \end{cases}$$

where each $\psi_{q,i}$ is the map induced by the removal of $\{\beta, w(\beta)\}$ from (X_i, κ_i) at fixed vertex i, as defined in Eq. (3.4). Similarly, define

$$\psi^\beta_{q,i} : C_q((\mathbb{X}^\beta)_i) \to C_q(\mathbb{X}_i) = C_q(X_i)$$

by

$$\phi^\beta_{q,i} = \begin{cases} \phi_{q,i} : C_q(X_i^\beta) \to C_q(X_i) & \text{if } i \in L(\beta, \ell), \\ 1 : C(X_i) \to C(X_i) & \text{otherwise,} \end{cases}$$

where $\phi_{q,i}$ is induced from the removal of $\{\beta, w(\beta)\}$ from (X_i, κ_i) at fixed vertex i, as in Eq. (3.5). We apologize to the reader for the proliferation of indices.

For each $q \geqslant 0$, we form the collections

$$\psi^\beta_q = (\psi^\beta_{q,i})_{i \in Q_0} \text{ and } \phi^\beta_q = (\phi^\beta_{q,i})_{i \in Q_0}$$

and define $\psi^\beta = (\psi^\beta_q)_{q \geqslant 0}$ and $\phi^\beta = (\phi^\beta_q)_{q \geqslant 0}$.

To visualize ψ^β, we provide the following. For any arrow $\alpha : i \to j$ in Q_1 and for any $q \geqslant 1$, we have a diagram

$$\begin{tikzcd}
C_q(X_i) \arrow[rr, "\psi^\beta_{q,i}"] \arrow[dr, "\partial_{q,i}"'] \arrow[dd, "\iota"'] & & C_q((\mathbb{X}^\beta)_i) \arrow[dr, "\partial^\beta_{q,i}"] \arrow[dd, "\iota"'] & \\
& C_{q-1}(X_i) \arrow[rr, "\psi^\beta_{q-1,i}"] \arrow[dd, "\iota"'] & & C_{q-1}((\mathbb{X}^\beta)_i) \arrow[dd, "\iota"] \\
C_q(X_j) \arrow[rr, "\psi^\beta_{q,j}"] \arrow[dr, "\partial_{q,j}"'] & & C_q((\mathbb{X}^\beta)_j) \arrow[dr, "\partial^\beta_{q,j}"] & \\
& C_{q-1}(X_j) \arrow[rr, "\psi^\beta_{q-1,j}"] & & C_{q-1}((\mathbb{X}^\beta)_j)
\end{tikzcd} \tag{3.7}$$

that we have not yet shown to be entirely commutative. Note that the spaces on the right face depend on the membership of i and j in the locus of β in ℓ, $L(\beta,\ell)$. The commutativity of left and right faces follow from the fact that (X_i,κ_i) is a subcomplex of (X_j,κ_j) and $(X_i^\beta,\kappa_i^\beta)$ is a subcomplex of $(X_j^\beta,\kappa_j^\beta)$ by Lemma 3.2.6. Then Lemma 2.2.1 can be applied. The next two propositions show that Diagram (3.7) is in fact commutative.

Proposition 3.2.7. *For each $q \geqslant 0$, $\psi_q^\beta = (\psi_{q,i}^\beta)_{i\in Q_0} : C_q(\mathbb{X}) \to C_q(\mathbb{X}^\beta)$ and $\phi_q^\beta = (\phi_{q,i}^\beta)_{i\in Q_0} : C_q(\mathbb{X}^\beta) \to C_q(\mathbb{X})$ as defined above are morphisms of representations of (Q,c).*

Proof. We divide the proof into two parts, one for ψ_q^β and the other for ϕ_q^β.

1. For fixed $q \geqslant 0$, and for any arrow $\alpha : i \to j$, we need to show the commutativity of

$$\begin{array}{ccc} C_q(X_i) & \xrightarrow{\psi_{q,i}^\beta} & C_q((\mathbb{X}^\beta)_i) \\ \big\downarrow & & \big\downarrow \\ C_q(X_j) & \xrightarrow{\psi_{q,i}^\beta} & C_q((\mathbb{X}^\beta)_j). \end{array}$$

As in the proof of Lemma 3.2.6, there are three cases.

Case 1: $i,j \in L(\beta,\ell)$. As discussed above, it can be checked that β is in both $\mathcal{B}_i$ and $\mathcal{B}_j$, and that $w_i(\beta) = w_j(\beta)$. For $x \in X_i$ with $\dim x = q$,

$$\iota\psi_{q,i}^\beta(x) = \begin{cases} 0 & \text{if } x = w_i(\beta), \\ -\sum\limits_{\sigma\in X_i^\beta} \frac{\kappa_i(w_i(\beta),\sigma)}{\kappa_i(w_i(\beta),\beta)}\sigma & \text{if } x = \beta, \\ x & \text{otherwise.} \end{cases}$$

On the other hand,

$$\psi_{q,j}^\beta\iota(x) = \begin{cases} 0 & \text{if } x = w_j(\beta), \\ -\sum\limits_{\sigma\in X_j^\beta} \frac{\kappa_j(w_j(\beta),\sigma)}{\kappa_j(w_j(\beta),\beta)}\sigma & \text{if } x = \beta, \\ x & \text{otherwise.} \end{cases} \tag{3.8}$$

We only need to check the equality in the case that $x = \beta$.

In the summation in Eq. (3.8), suppose that for some $\sigma \in X_j^\beta$, the term $\kappa_j(w_j(\beta),\sigma) \neq 0$ contributes a nonzero summand. Then, we claim that $\sigma \in X_i^\beta$. Since $(X_i^\beta,\kappa_i^\beta)$ is a subcomplex of $(X_j^\beta,\kappa_j^\beta)$ and $w_j(\beta) = w_i(\beta) \in X_i$, it follows that $\sigma \in X_i$. Since $\sigma \in X_j^\beta$ by assumption, $\sigma \in X_i^\beta$, as required.

Thus the summation in Eq. (3.8) can be taken over $\sigma \in X_i^\beta$. Moreover, $\kappa_j(w_j(\beta),\sigma) = \kappa_i(w_i(\beta),\sigma)$ since both σ and $w_i(\beta) = w_j(\beta)$ are in X_i. Similarly, $\kappa_j(w_j(\beta),\beta) = \kappa_i(w_i(\beta),\beta)$. This shows the desired equality.

Case 2: $i \notin L(\beta,\ell)$ and $j \in L(\beta,\ell)$. By definition, $(\mathbb{X}^\beta)_i = X_i$ and $\psi_{q,i}^\beta$ is the identity.

For all $x \in X_i$ with $\dim x = q$,

$$\iota\psi^{\beta}_{q,i}(x) = x = \psi^{\beta}_{q,j}\iota(x)$$

where the equality on the right follows from definition of $\psi_{q,j}$ and the fact that $x \in X_i$ cannot be equal to β nor $w_j(\beta)$, since Lemma 3.2.5, part *2*, asserts that $\beta, w_j(\beta) \notin X_i$.

Case 3: $i, j \notin L(\beta, \ell)$. This case is trivial, for both $\psi^{\beta}_{q,i}$ and $\phi^{\beta}_{q,i}$ are identity morphisms.

2. The proof for ϕ^{β}_{q} is similar. For every arrow $\alpha : i \to j$, we need to show the commutativity of

$$\begin{tikzcd} C_q(X_i) \arrow[d, hook] & C_q((\mathbb{X}^{\beta})_i) \arrow[l, "\phi^{\beta}_{q,i}"'] \arrow[d, hook] \\ C_q(X_j) & C_q((\mathbb{X}^{\beta})_j). \arrow[l, "\phi^{\beta}_{q,j}"'] \end{tikzcd}$$

Case 1: $i, j \in L$. By definition, β is in both X_i and X_j. Then, for every $x \in (\mathbb{X}^{\beta})_i = X^{\beta}_i$ with $\dim x = q$,

$$\iota\phi^{\beta}_{q,i}(x) = x - \frac{\kappa_i(x, \beta)}{\kappa_i(w_i(\beta), \beta)} w_i(\beta)$$

and

$$\phi^{\beta}_{q,j}\iota(x) = x - \frac{\kappa_j(x, \beta)}{\kappa_j(w_j(\beta), \beta)} w_j(\beta).$$

These are clearly equal.

Case 2: $i \notin L(\beta, \ell)$, $j \in L(\beta, \ell)$. By definition, $\phi^{\beta}_{q,i}$ is the identity morphism.

We claim if $x \in X_i$, then $\kappa_j(x, \beta) = 0$. If this were not the case, then $\beta \in X_i$ since (X_i, κ_i) is a subcomplex of (X_j, κ_j). This contradicts Lemma 3.2.5, part *2*. Thus, for $x \in X_i$ with $\dim x = q$,

$$\phi^{\beta}_{q,j}\iota(x) = x - \frac{\kappa_j(x, \beta)}{\kappa_j(w_j(\beta), \beta)} w_j(\beta) = x = \iota\phi^{\beta}_{q,i}(x)$$

as required.

Case 3: $i, j \notin L$. This case is trivial.

□

Incidentally, Proposition 3.2.7 shows the commutativity of the front and back faces of the cube in Diagram (3.7). We have the following analogue of Lemma 3.1.3.

Proposition 3.2.8.

1. *The collection of morphisms* $\psi^\beta : C(\mathbb{X}) \to C(\mathbb{X}^\beta)$ *and* $\phi^\beta : C(\mathbb{X}^\beta) \to C(\mathbb{X})$ *are chain maps.*

2. *Moreover,* $\psi^\beta\phi^\beta = 1_{C(\mathbb{X}^\beta)}$ *and* $\phi^\beta\psi^\beta \sim 1_{C(\mathbb{X})}$.

Thus $C(\mathbb{X}) \sim C(\mathbb{X}^\beta)$.

Proof.

1. To show that ψ^β and ϕ^β are chain maps, we need to check that for all $q \geqslant 0$:

   (a) ψ^β_q and ϕ^β_q are morphisms of representations, and

   (b) $\partial^\beta_q\psi^\beta_q = \psi^\beta_{q-1}\partial_q$ and $\partial_q\phi^\beta_q = \phi^\beta_{q-1}\partial^\beta_q$

   Part (a) is shown in Proposition 3.2.7. It suffices to check the equalities in (b) for each vertex $i \in Q_0$. In other words, we only need to show the commutativity of the top face in Diagram (3.7) for any vertex $i \in Q_0$.

   For a fixed $i \in L(\beta, \ell)$, Lemma 3.1.3 shows that the collection $\psi^\beta_{q,i}$ over all $q \geqslant 0$ is a chain map, so that

$$\partial^\beta_{q,i}\psi^\beta_{q,i} = \psi^\beta_{q-1,i}\partial_{q,i}$$

   as required. Otherwise, if $i \notin L(\beta, \ell)$, $(\psi^\beta_{q,i})_{q\geqslant 0}$ is the identity morphism and the above equality is automatically satisfied.

   This shows that ψ^β is a chain map. The proof for ϕ^β being a chain map is similar.

2. Since $\psi^\beta_{q,i}\phi^\beta_{q,i} = 1_{C_q((\mathbb{X}^\beta)_i)}$ for every $i \in L(\beta, \ell)$ by Lemma 3.1.3, and for $i \notin L(\beta, \ell)$, $\psi^\beta_{q,i}\phi^\beta_{q,i} = 1_{C_q((\mathbb{X}^\beta)_i)} = 1_{C_q(X_i)}$ by definition,

$$\psi^\beta\phi^\beta = 1_{C(\mathbb{X}^\beta)}.$$

   Let us now prove that $\phi^\beta\psi^\beta \sim 1_{C(\mathbb{X})}$ by constructing the required homotopy. For each vertex $i \in L(\beta, \ell)$, let $\phi^\beta_i = (\phi^\beta_{q,i})_{q\geqslant 0}$ and $\psi^\beta_i = (\psi^\beta_{q,i})_{q\geqslant 0}$. We already know that $\phi^\beta_i\psi^\beta_i \sim 1_{C(X_i)}$, via Lemma 3.1.3. For every $i \in L(\beta, \ell)$,

$$\begin{array}{rcccl} \theta_{q,i} & : & C_q(X_i) & \to & C_{q+1}(X_i) \\ & & x & \mapsto & \begin{cases} \frac{1}{\kappa_i(w_i(\beta),\beta)} w_i(\beta) & \text{if } x = \beta \\ 0 & \text{otherwise} \end{cases} \end{array} \tag{3.9}$$

   provides a homotopy between $\phi^\beta_i\psi^\beta_i$ and $1_{C(X_i)}$.

   Form the collection $\theta_q = (\theta_{q,i})_{i\in Q_0}$, where $\theta_{q,i}$ is as defined in Eq. (3.9) for $i \in L(\beta, \ell)$, and $\theta_{q,i} = 0$ otherwise. For each $q \geqslant 0$, θ_q is a morphism of representations $\theta_q : C_q(\mathbb{X}) \to C_{q+1}(\mathbb{X})$. This can be checked by showing the commutativity of

$$\begin{tikzcd} C_q(X_i) \arrow[r, "\theta_{q,i}"] \arrow[d, "\iota"] & C_{q+1}(X_i) \arrow[d, "\iota"] \\ C_q(X_j) \arrow[r, "\theta_{q,j}"] & C_{q+1}(X_j) \end{tikzcd}$$

for each $q \geqslant 0$, for each arrow $\alpha : i \to j$. Let $x \in X_i$ with $\dim x = q$. If x is not β, then $\iota\theta_{q,i}x = 0 = \theta_{q,j}\iota x$ independent of the membership of i and j in $L(\beta, \ell)$. Thus we only need to check the cases for when $x = \beta$.

Case 1: $i, j \in L(\beta, \ell)$. For $x = \beta \in X_i \subset X_j$ and if $\dim x = q$, it is clear that

$$\iota\theta_{q,i}\beta = \frac{1}{\kappa_i(w_i(\beta), \beta)} = \frac{1}{\kappa_j(w_j(\beta), \beta)} = \theta_{q,j}\iota\beta.$$

Case 2: $i \notin L(\beta, \ell)$, $j \in L(\beta, \ell)$. By part *2* of Lemma 3.2.5, $\beta \notin X_i$ so there is nothing to check.

Case 3: $i, j \notin L(\beta, \ell)$. This case is still trivial.

Finally,

$$\phi_q^\beta \psi_q^\beta - 1_{C_q(\mathbb{X})} = \theta_{q-1}\partial_q + \partial_{q+1}\theta_q$$

can be checked "vertex-wise", using Lemma 3.1.3 for vertices $i \in L(\beta, \ell)$, and the fact that $1 \cdot 1 - 1 = 0\partial_{q,i} + \partial_{q-1,i}0$ for vertices $i \notin L(\beta, \ell)$.

This shows that $\theta = (\theta_q)$ is provides a homotopy between $\phi^\beta\psi^\beta$ and $1_{C(\mathbb{X})}$.

□

Finally, we can provide the proof for Theorem 3.2.4

Proof of Theorem 3.2.4. If all acyclic matchings w_i in w were empty, then there is nothing to do. Suppose that there a vertex $\ell \in Q_0$ with a nonempty acyclic matching w_ℓ. Arbitrarily choose a $\beta \in \mathcal{B}_\ell$.

By the construction above, we let $\mathbb{X}^\beta$ be induced by removal of the pair $\{\beta, w(\beta)\}$ from $L(\beta, \ell)$ in $\mathbb{X}$ as given in Lemma 3.2.6. From Proposition 3.2.8, it follows that $C(\mathbb{X}) \sim C(\mathbb{X}^\beta)$.

To iterate this procedure, we need to show that the acyclic matching w of $\mathbb{X}$ induces an acyclic matching w' of $\mathbb{X}^\beta$. For each vertex $i \in Q_0$, w_i induces an acyclic matching w'_i on $(\mathbb{X}^\beta)_i$, as in Definition 3.1. Let us explicitly write down these acyclic matchings. In the case that $i \notin L(\beta, \ell)$, w'_i is the same as $w_i = (\mathcal{A}_i, w_i : \mathcal{B}_i \to \mathcal{D}_i)$. Otherwise, it is $w'_i = (\mathcal{A}_i, w'_i : \mathcal{B}'_i \to \mathcal{D}'_i)$ where $\mathcal{B}'_i = \mathcal{B}_i \backslash \{\beta\}$, $\mathcal{D}'_i = \mathcal{D}_i \backslash \{w_i(\beta)\}$, and $w'_i(b) = w_i(b)$ for each $b \in \mathcal{B}'_i$.

Then, to show that the collection $w' = (w'_i)_{i \in Q_0}$ defines an acyclic matching of $\mathbb{X}^\beta$, we need to check the consistency across arrows $\alpha : i \to j$.

Case 1: $i, j \in L(\beta, \ell)$. Note that $w_i(\beta) = w_j(\beta)$. The inclusions

$$\begin{array}{rcl} \mathcal{A}_i & \subset & \mathcal{A}_j, \\ \mathcal{B}'_i = \mathcal{B}_i \backslash \{\beta\} & \subset & \mathcal{B}'_j = \mathcal{B}_j \backslash \{\beta\}, \\ \mathcal{D}'_i = \mathcal{D}_i \backslash \{w_i(\beta)\} & \subset & \mathcal{D}'_j = \mathcal{D}_j \backslash \{w_j(\beta)\} \end{array}$$

are satisfied. Moreover, for any $b \in \mathcal{B}'_i$, $w'_j(b) = w_j(b) = w_i(b) = w'_i(b)$.
Case 2: $i \notin L(\beta, \ell)$ and $j \in L(\beta, \ell)$. By part *2* of Lemma 3.2.5, $\beta, w_j(\beta) \notin X_i$. Hence,

$$\begin{array}{rcl} \mathcal{A}_i & \subset & \mathcal{A}_j, \\ \mathcal{B}_i & \subset & \mathcal{B}'_j = \mathcal{B}_j \backslash \{\beta\}, \\ \mathcal{D}_i & \subset & \mathcal{D}'_j = \mathcal{D}_j \backslash \{w_j(\beta)\}, \end{array}$$

and for $b \in \mathcal{B}_i$, $b \neq \beta$ so that $w'_j(b) = w_j(b) = w_i(b)$, as required.
Case 3: $i, j \notin L(\beta, \ell)$. In this case, $w'_i = w_i$ and $w'_j = w_j$ and so the required conditions hold because w is an acyclic matching.

Repeated application of the above construction of the induced quiver complex and induced acyclic matching gives us the result. Note that at each step, the number defined as

$$\sum_{i \in Q_0} |\mathcal{B}_i|,$$

the total cardinality of the sets $\mathcal{B}_i$, strictly decreases. After a finite number of iterations we are left with an empty acyclic matching. At this point, the induced quiver complex is the same as $\mathbb{A}$, which can be checked vertex-wise and applying Lemma 3.1.4 as in the previous section.

Thus, $C(\mathbb{X}) \sim C(\mathbb{A})$ and the theorem is proved.

□

3.3 Algorithm and numerical examples

In this section, we give an algorithm for computing an acyclic matching for an input quiver complex $\mathbb{X} = (X_i, \kappa_i)_{i \in Q_0}$. We make the simplifying assumption that there is a complex (X, κ) such that (X_i, κ_i) is a subcomplex of (X, κ) for every $i \in Q_0$. This (X, κ) does not necessarily have to be a complex in $\mathbb{X}$. The assumption above will allow us to talk of cells in a global manner. For example, given the quiver complex

$$\begin{tikzcd} X_s \arrow[r, hook] & X_s \cup Y_s & Y_s \arrow[l, hook'] \\ X_r \arrow[r, hook] \arrow[u, hook] & X_r \cup Y_r \arrow[u, hook] & Y_r \arrow[l, hook'] \arrow[u, hook] \end{tikzcd},$$

on the commutative triple ladder, each complex is a subcomplex of $X_s \cup Y_s$.

Of course in the general case, this assumption may not hold. Instead, given a quiver complex $\mathbb{X}$, let us construct a complex (X, κ). We then show that we can rename the cells in each (X_i, κ_i) to get complexes $(\hat{X}_i, \hat{\kappa}_i)$ with the property $(\hat{X}_i, \hat{\kappa}_i)$ is a subcomplex of (X, κ) for each $i \in Q_0$. We identify the complexes (X_i, κ_i) in $\mathbb{X}$ with the complexes $(\hat{X}_i, \hat{\kappa}_i)$ consisting of the renamed cells. Let us show this construction below.

Let $\bigsqcup_{i \in Q_0} X_i = \{(\sigma, i) | \sigma \in X_i, i \in Q_0\}$ be the disjoint union of the complexes of $\mathbb{X}$. Define an equivalence relation on $\bigsqcup_{i \in Q_0} X_i$ by $(\sigma, i) \sim (\tau, j)$ if and only if

$\sigma = \tau$ and there is a path p in $\bar{Q}$ from i to j such that for all vertices ℓ on the path p, $\sigma \in X_\ell$. That $\sim$ is an equivalence relation can be easily checked.

Let $\sigma \in X_i$. Recall that the locus $L(\sigma, i)$ of σ at i is defined to be the set of vertices j in Q_0 such that there is a (possibly stationary) path p in $\bar{Q}$ from i to j satisfying the property that $\sigma \in X_k$ for any vertex k on the path p. Note that $(\sigma, i) \sim (\tau, j)$ if and only if $\sigma = \tau$, and $j \in L(\sigma, i)$ or $i \in L(\tau, j)$. Moreover, if $m \in L(\sigma, i)$, then $L(\sigma, i) = L(\sigma, m)$.

Let $X = \bigsqcup_{i \in Q_0} X_i / \sim$ and denote the equivalence class of (σ, i) by $[\sigma, i] \in X$. The set X is given a grading from the gradings on X_i. That is, $X_q = \{[\sigma, i] \mid \dim \sigma = q\}$.

In order to define an incidence map $\kappa : X \times X \to K$, let us show that the incidence maps $\kappa_\ell(\sigma, \tau)$ are consistent over $L(\sigma, i)$. First, let us show the following technical lemma.

Lemma 3.3.1. *Let $\sigma \in X_i$, $\tau \in X_j$. If there exists some $\ell \in Q_0$ such that $\ell \in L(\sigma, i) \cap L(\tau, j)$ and $\kappa_\ell(\sigma, \tau) \neq 0$, then, $i \in L(\tau, j)$ and $k_i(\sigma, \tau) = k_\ell(\sigma, \tau)$.*

Proof. This is equivalent to showing that $\tau \in X_i$, $(\tau, j) \sim (\tau, i)$, and $k_i(\sigma, \tau) = k_\ell(\sigma, \tau)$. By definition, $\ell \in Q_0$ has the property that $(\sigma, i) \sim (\sigma, \ell)$ and $(\tau, j) \sim (\tau, \ell)$, and so there is a path p in $\bar{Q}$ from i to ℓ such that $\sigma \in X_k$ for all vertices k in p.

Let the vertices of the path p from i to ℓ be

$$k_N = i, k_{N-1}, \ldots, k_1, \ell = k_0,$$

in that order. By induction over $s = 0, \ldots, N$, let us show that for all $s = \{0, \ldots, N\}$, $\tau \in X_{k_s}$ and $(\tau, j) \sim (\tau, k_s)$ and $\kappa_{k_s}(\sigma, \tau) = \kappa_\ell(\sigma, \tau)$. The case $s = 0$ is true. In this case, $k_s = \ell$ and the statement follows from definition.

Now suppose that the statement is true for some $0 \leqslant s < N$. Since k_{s+1} and k_s are adjacent vertices on a path in $\bar{Q}$, then either there is an arrow $k_{s+1} \to k_s$ or there is an arrow $k_{s+1} \leftarrow k_s$ in Q. In the first case, $X_{k_{s+1}}$ is a subcomplex of X_{k_s}. Since $\sigma \in X_{k_{s+1}}$, and $\kappa_{k_s}(\sigma, \tau) \neq 0$, then $\tau \in X_{k_{s+1}}$. In the second case, $\tau \in X_{k_{s+1}}$ follows from the fact that X_{k_s} is a subset of $X_{k_{s+1}}$.

In either case, $\tau \in X_{k_{s+1}}$, and $k_{s+1}(\sigma, \tau) = k_s(\sigma, \tau) = k_\ell(\sigma, \tau)$ is clear. By inductive hypothesis, $(\tau, j) \sim (\tau, k_s)$ and so there is a path path p' in $\bar{Q}$ from j to k_s such that $\tau \in X_m$ for all vertices m in p'. Extend the path p' by underlying edge of the arrow between k_{s+1} and k_s. This shows that $(\tau, j) \sim (\tau, k_{s+1})$.

By induction, the statement is true for all $s = 0, \ldots, N$ and in particular true for $s = N$, $k_N = i$. □

Lemma 3.3.2. *Let $\sigma \in X_i$, $\tau \in X_j$. If there exists some $\ell \in Q_0$ such that $\ell \in L(\sigma, i) \cap L(\tau, j)$ and $\kappa_\ell(\sigma, \tau) \neq 0$, then $L(\sigma, i) \subset L(\tau, j)$ and $\kappa_m(\sigma, \tau) = \kappa_\ell(\sigma, \tau)$ for all $m \in L(\sigma, i)$.*

Proof. Note that for any $m \in L(\sigma, i)$, $L(\sigma, m) = L(\sigma, i)$. Substituting, $\ell \in L(\sigma, m) \cap L(\tau, j)$ and $\kappa_\ell(\sigma, \tau) \neq 0$. By the previous lemma, $m \in L(\tau, j)$ and $\kappa_m(\sigma, \tau) = \kappa_\ell(\sigma, \tau)$. This also shows that $L(\sigma, i) \subset L(\tau, j)$. □

Given a filtration, recall that the birth index of a cell σ is the smallest index i such that $\sigma \in X_i$. The above result (roughly) states that the locus of a boundary face τ of σ should contain the locus of σ. This result is analogous to the fact that in the filtration case, the birth index of a boundary face of σ should be no larger than the birth index of σ. Indeed, in the filtration case, $L(\sigma, \ell) = \{i | b(\sigma) \leqslant i \leqslant n\}$, where $b(\sigma)$ is the birth index of σ.

Now, define a map $\kappa : X \times X \to K$ by

$$\kappa([\sigma, i], [\tau, j]) = \begin{cases} \kappa_\ell(\sigma, \tau) & \text{if } \exists \ell \in Q_0 \text{ such that } \ell \in L(\sigma, i) \cap L(\tau, j) \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

By Lemma 3.3.2, this definition is well-defined. If $\kappa_\ell(\sigma, \tau)$ above is nonzero, Lemma 3.3.2 shows that this is equal to $\kappa_m(\sigma, \tau)$ for any $m \in L(\sigma, i) = L(\sigma, i) \cap L(\tau, j)$.

Lemma 3.3.3. *Given (X, κ) as defined above, (X, κ) is a complex.*

Proof. We need to show that κ defined in Eq. (3.10) is an incidence map. The condition that $\kappa([\sigma, i], [\tau, j]) \neq 0$ implies $\dim[\sigma, i] = \dim[\tau, j] + 1$ obviously holds. Let us show that the summation

$$\sum_{[\sigma, j] \in X} \kappa([\rho, i], [\sigma, j]) \kappa([\sigma, j], [\tau, k])$$

is zero for any fixed $[\rho, i], [\tau, k] \in X$.

Consider only $[\sigma, j] \in X$ contributing nonzero summands. By definition of κ and together with Lemma 3.3.2, $L(\rho, i) \subset L(\sigma, j)$,

$$\kappa([\rho, i], [\sigma, j]) = \kappa_i(\rho, \sigma)$$

and similarly, $L(\sigma, j) \subset L(\tau, k)$ so that $i \in L(\rho, i) \subset L(\sigma, j) \subset L(\tau, k)$,

$$\kappa([\sigma, j], [\tau, k]) = \kappa_i(\sigma, \tau).$$

Thus,

$$\begin{aligned} \sum_{[\sigma, j] \in X} \kappa([\rho, i], [\sigma, j]) \kappa([\sigma, j], [\tau, k]) &= \sum_{[\sigma, i] \in X} \kappa([\rho, i], [\sigma, i]) \kappa([\sigma, i], [\tau, i]) \\ &= \sum_{\sigma \in X_i} \kappa_i(\rho, \sigma) \kappa_i(\sigma, \tau) \\ &= 0, \end{aligned}$$

as claimed. □

Finally, rename the cells of all (X_i, κ_i) by replacing σ by $[\sigma, i]$. This gives the same complex. Strictly speaking, we want to define a category of complexes and identify (X_i, κ_i) with the complex $(\hat{X}_i, \hat{\kappa_i})$ containing the renamed cells by an isomorphism in that category. We skip this category-theoretic complication.

Let us show that for every $i \in Q_0$, $(\hat{X}_i, \hat{\kappa_i})$ is a subcomplex of (X, κ). By definition, $\hat{X}_i \subset X$ as sets. For any $[\sigma, i], [\tau, i] \in \hat{X}_i$ corresponding to $\sigma, \tau \in X_i$,

$$\kappa([\sigma, i], [\tau, i]) = \kappa_i(\sigma, \tau) = \hat{\kappa_i}([\sigma, i], [\tau, i])$$

by definition. Moreover, if $\kappa([\sigma, i], [\tau, j]) \neq 0$ for some $[\sigma, i], [\tau, j] \in X$, then by Lemma 3.3.1 we have $i \in L(\tau, j)$ and so $\tau \in X_i$ and $[\tau, j] = [\tau, i]$. Thus, $[\tau, i] \in \hat{X}_i$ corresponding to $\tau \in X_i$.

Next, we give the following lemma, which we will later use to explain the strategy behind the algorithm for computing an acyclic matching for $\mathbb{X}$.

Lemma 3.3.4. *Let (X, κ) be a complex with an acyclic matching $(\mathcal{A}, w : \mathcal{B} \to \mathcal{D})$.*

1. *Let (X', κ') be a subcomplex of (X, κ). If for every $\beta \in \mathcal{B}$, $\beta \in X'$ if and only if $w(\beta) \in X'$, then w induces an acyclic matching $(\mathcal{A}', w' : \mathcal{B}' \to \mathcal{D}')$ on X' defined by:*
$$\begin{aligned} \mathcal{B}' &= \mathcal{B} \cap X' \\ \mathcal{D}' &= \mathcal{D} \cap X' \\ \mathcal{A}' &= X' \backslash (\mathcal{B}' \cup \mathcal{D}') \\ w'(\beta) &= w(\beta) \text{ for } \beta \in \mathcal{B}'. \end{aligned}$$

2. *With the hypothesis of part 1, let $(\mathcal{A}, \tilde{\kappa})$ be the Morse complex of (X, κ) associated to w, and $(\mathcal{A}', \tilde{\kappa}')$ the Morse complex of (X', κ') associated to w'. Then for every $\sigma, \tau \in \mathcal{A}'$,*
$$\tilde{\kappa}'(\sigma, \tau) = \tilde{\kappa}(\sigma, \tau).$$

3. *Let $\mathbb{X} = (X_i, \kappa_i)_{i \in Q_0}$ be a quiver complex over Q such that for every $i \in Q_0$, (X_i, κ_i) is a subcomplex of (X, κ). Suppose that $(\mathcal{A}, w : \mathcal{B} \to \mathcal{D})$ is an acyclic matching of (X, κ). If for every $\beta \in \mathcal{B}$ and for each $i \in Q_0$, $\beta \in X_i$ if and only if $w(\beta) \in X_i$, then the collection of acyclic matchings w_i' on (X_i, κ_i) induced by w, as above, forms an acyclic matching (w_i') of $\mathbb{X}$.*

Proof.

1. The required properties are easy to check.

2. In this setting, $(X', \kappa') \hookrightarrow (X, \kappa)$ can be viewed as a quiver complex on the quiver $\vec{A}_2 : \circ \longrightarrow \circ$. Moreover, by definition of w', the pair of acyclic matchings w', w forms an acyclic matching of this quiver complex. It follows from Lemma 3.2.3 that
$$(\mathcal{A}', \tilde{\kappa}') \longrightarrow (\mathcal{A}, \tilde{\kappa})$$
is a quiver complex. In particular, $(\mathcal{A}', \tilde{\kappa}')$ is a subcomplex of $(\mathcal{A}, \tilde{\kappa})$ so that for every $\sigma, \tau \in \mathcal{A}'$,
$$\tilde{\kappa}'(\sigma, \tau) = \tilde{\kappa}(\sigma, \tau).$$

3. For every i, w induces an acyclic matching w_i' on (X_i, κ_i) by the first part

of this lemma. For any arrow $\alpha : i \to j$ in Q_1, it can be checked that

$$\begin{array}{rcl} \mathcal{B}_i & = & \mathcal{B} \cap X_i = \mathcal{B} \cap (X_j \cap X_i) \\ & = & \mathcal{B}_j \cap X_i, \\ \mathcal{D}_i & = & \mathcal{D} \cap X_i = \mathcal{D} \cap (X_j \cap X_i) \\ & = & \mathcal{D}_j \cap X_i \\ \mathcal{A}_i & = & X_i \backslash (\mathcal{B}_i \cup \mathcal{D}_i) \\ & = & [X_j \backslash (\mathcal{B}_j \cup \mathcal{D}_j)] \cap X_i \\ & = & \mathcal{A}_j \cap X_i \\ w_i'(\beta) & = & w(\beta) \\ & = & w_j'(\beta) \text{ for } \beta \in \mathcal{B}_i. \end{array}$$

It follows from Lemma 3.2.1 that $w' = (w_i')$ is an acyclic matching of $\mathbb{X}$.

□

The strategy of the algorithm is as follows. Let $\mathbb{X} = (X_i, \kappa_i)_{i \in Q_0}$ be a quiver complex and (X, κ) a complex such that for each $i \in Q_0$, (X_i, κ_i) is a subcomplex of (X, κ). We compute an acyclic matching w for (X, κ) satisfying the condition that for every $\beta \in \mathcal{B}$, $\beta \in X_i$ if and only if $w(\beta) \in X_i$. By Lemma 3.3.4, part 3, (w_i') induced from w is an acyclic matching of $\mathbb{X}$. Moreover, by computing the incidence map $\tilde{\kappa}$ of $(\mathcal{A}, \tilde{\kappa})$, we also get the incidence maps $\tilde{\kappa}_i$ for each $(\mathcal{A}_i, \tilde{\kappa}_i)$ in $\mathbb{A}$.

We use the algorithm given in [30], with some modifications. We have to be careful that the acyclic matching we produce satisfies the hypothesis in Lemma 3.3.4. To this end, define the birth indicator function of a cell $\sigma \in X$, $b(\sigma) : Q_0 \to \{0, 1\}$, by

$$b(\sigma)(i) = \begin{cases} 1 & \text{if } \sigma \in X_i, \\ 0 & \text{otherwise.} \end{cases}$$

At initialization, place all the cells of (X, κ) into a set U of the unprocessed cells. We also need the following definitions. The boundary of a cell σ, relative to the current state of the unprocessed cells, is

$$\partial^U \sigma = \sum_{\tau \in U} \kappa(\sigma, \tau) \tau,$$

while its coboundary is

$$\mathrm{cb}_U(\sigma) = \{\rho \in U \mid \kappa(\rho, \sigma) \neq 0\}.$$

The algorithm will iterate through the cells in U. While the set U of unprocessed cells in not empty, we take a cell of minimal dimension and make it critical. At the end, the set of critical cells will be the cells of the Morse complex $\mathcal{A}$. The rest of the cells are paired up by w.

Recall that an elementary coreduction pair [31] (relative to U) is a pair (β, δ) of cells, such that $\partial^U \delta = u\beta$, for some $u \neq 0$ in K. The algorithm looks for elementary coreduction pairs to remove. We define the acyclic matching by setting $w(\beta) = \delta$, for every pair (β, δ) sent to RemovePair. However, we also require that all pairs $(\beta, \delta = w(\beta))$ so extracted satisfy $b(\delta) = b(\beta)$.

Now, the removal of a cell (either by being declared critical, or being removed as part of a pair) may cause its coboundary cells to become part of some elementary coreduction pair (β, δ). We insert the coboundary cells into a queue $\mathcal{Q}$ that keeps track of candidate δ cells. The queue structure should have a guard in place to ensure that each cell gets queued no more than once into $\mathcal{Q}$ for each iteration of the outer while loop.

We also define the gradient chain $g(\cdot)$ for every cell. At initialization, we set $g(\sigma) = 0$. As we progressively remove cells, the values of $g(\sigma)$ will change, reflecting the changes made to the incidence map. Then, when a cell A is made critical, $g(A)$ contains the boundary of A in the resulting Morse complex, $\partial^{\mathcal{A}}\sigma$.

---

procedure UPDATEGRADIENTCHAIN(σ)
  for $\rho \in \mathrm{cb}_U(\sigma)$ **do**
    if $\sigma = A \in \mathbb{A}$ **then**
      $g(\rho) \leftarrow g(\rho) + \kappa(\rho, A)A$
    else
      $g(\rho) \leftarrow g(\rho) + \kappa(\rho, \sigma)g(\sigma)$

---

procedure REMOVEPAIR(β, δ, d)
  remove: δ from U
  enqueue: $\mathrm{cb}_U(\beta)$ in $\mathcal{Q}$
  if $\dim \beta = d$ **then**
    $g(\beta) \leftarrow -\frac{g(\delta)}{u}$
    UPDATEGRADIENTCHAIN(β)
  remove: β from U

procedure MAKECRITICAL
  choose: $A \in U$ of minimal dimension
  add: A to $\mathcal{A}$
  UPDATEGRADIENTCHAIN(A)
  remove: A from U
  $\partial^{\mathcal{A}}A \leftarrow g(A)$
  return A

---

procedure MORSEREDUCE(U, κ, b)
  while $U \neq \varnothing$ **do**
    $A \leftarrow$ MAKECRITICAL()
    $\mathcal{Q} \leftarrow$ **new** Queue
    enqueue: $\mathrm{cb}_U(A)$ in $\mathcal{Q}$
    while $\mathcal{Q} \neq \varnothing$ **do**
      dequeue: ξ from $\mathcal{Q}$
      if $\partial^U \xi = 0$ **then**
        enqueue: $\mathrm{cb}_U(\xi)$ in $\mathcal{Q}$
      else if $\partial^U(\xi) = u \cdot \eta$ with $b(\xi) = b(\eta)$, $u \neq 0$ **then**
        REMOVEPAIR(η, ξ, $\dim A$)
  return $\mathcal{A}$, $\partial^{\mathcal{A}}$

The following theorem is simply Theorem 5.1 and Proposition 5.2 of [30], applied to the trivial filtration of (X, κ).

Theorem 3.3.5 ([30, Theorem 5.1]). *The algorithm* MORSEREDUCE *terminates* *with an acyclic matching* $(\mathcal{A}, w : \mathcal{B} \to \mathcal{D})$ *of* (X, κ) *defined by*

$$\mathcal{B} = \{\beta \mid (\beta, \delta) \text{ was sent to } \textsc{RemovePair}.\}$$

$$\mathcal{D} = \{\delta \mid (\beta, \delta) \text{ was sent to } \textsc{RemovePair}.\}$$

and $w(\beta) = \delta$ *for all* (β, δ) *sent to* RemovePair*. Moreover, for cells* $A, A' \in \mathcal{A}$*:*

$$\tilde{\kappa}(A, A') = \langle \partial^{\mathcal{A}} A, A' \rangle.$$

Finally, we state the following theorem, which shows that the algorithm computes an acyclic matching for the input quiver complex $\mathbb{X}$.

Theorem 3.3.6. *Given an input quiver complex* $\mathbb{X}$ *on* Q *and complex* (X, κ) *such that* $(X_i, \kappa_i) \subset (X, \kappa)$ *for all vertices* $i \in Q_0$*, the algorithm* MorseReduce *gives an acyclic matching of* $\mathbb{X}$*,* $(w'_i) = (\mathcal{A}'_i, w'_i : \mathcal{B}'_i \to \mathcal{D}'_i)_{i \in Q_0}$ *induced from the acyclic matching* w *of* (X, κ) *as defined in Theorem 3.3.5. Moreover, for every vertex* $i \in Q_0$ *and cells* $A, A' \in \mathcal{A}_i$*,*

$$\tilde{\kappa}_i(A, A') = \langle \partial^{\mathcal{A}} A, A' \rangle.$$

Proof. By construction, $b(\beta) = b(w(\beta))$ for every $\beta \in \mathcal{B}$. Thus, $\beta \in X_i$ if and only if $w(\beta) \in X_i$, for every vertex $i \in Q_0$. The result then follows immediately from Theorem 3.3.5 and Lemma 3.3.4. □

Let us give the numerical results appearing in [18, 19]. We use $K = \mathbb{Z}_2$, the finite field with two elements, and compute qth persistent homology modules with $q = 1$. The underlying quiver Q of the quiver complexes will either be $A_n(\tau)$, where $n = 8$ and τ is randomized, or $CL_3(fb)$. In either setting, the procedures performed are the same. As the first procedure, we start with a quiver complex $\mathbb{X}$, then compute $H_q(\mathbb{X})$ and an indecomposable decomposition of $H_q(\mathbb{X})$.

In the case $Q = A_n(\tau)$, the computation of the indecomposable decomposition uses the algorithm for zigzag persistence provided in [7]. On the other hand, for the case $Q = CL_3(fb)$, we use the algorithm in [17], which we will also discuss in the next chapter, in Section 4.4.

To compare the use of the Morse reduction algorithm we have described above, we do the following as a second procedure.

1. Compute a Morse quiver complex $\mathbb{A}$ of $\mathbb{X}$ by MorseReduce, then
2. compute $H_q(\mathbb{A})$, and an indecomposable decomposition of $H_q(\mathbb{A})$.

Note that the only difference between the two procedures is whether or not we do Morse reduction as a preprocessing step. By Theorem 3.2.4, $H_q(\mathbb{X}) \cong H_q(\mathbb{A})$ and both procedures above give isomorphic output.

We summarize the time taken in seconds for the computations in the table below. The column under t_{without} (first procedure) contains the total times taken for computing without using Morse reductions and working with $H_q(X)$, while t_{with} (second procedure) gives the total times taken for following the steps above. This latter entry includes the time taken for first computing the Morse quiver complex $\mathbb{A}$ with the time taken to compute $H_q(\mathbb{A})$ and its indecomposable decomposition. We also provide the sizes of the quiver complexes, $|\mathbb{X}|$ and $|\mathbb{A}|$.

#	Q	$\lvert\mathbb{X}\rvert$	$\lvert\mathbb{A}\rvert$	t_{without}	t_{with}
1	$A_8(\tau)$	2001	977	8.868	0.841
2	$A_8(\tau)$	2000	1012	8.792	0.732
3	$A_8(\tau)$	2004	1076	10.849	1.293
4	$CL_3(fb)$	15,341	2,777	903.31	39.53
5	$CL_3(fb)$	17,626	7,164	3497.55	143.41
6	$CL_3(fb)$	32,540	7,834	5162.12	42.34

As expected, $|\mathbb{A}| < |\mathbb{X}|$. In all cases above, using Morse reduction as a preprocessing step reduces the total computation time needed for the whole procedure.

Now, the algorithm we used for the computation of the indecomposable decomposition of the persistent homology modules may not be the most efficient available. However, in the testing we performed above, the same algorithm has been used for both trials. Thus, any improvements to the persistent homology algorithm should improve both t_{without} and t_{with}.

Chapter 4

Representation Theory of Quivers

Motivated by a practical application, we proposed the study of persistent homology on the commutative ladder quivers $CL_n(\tau)$. More generally, we have seen that the persistent homology $H_q(\mathbb{X})$ of a quiver complex $\mathbb{X}$ is a representation of a quiver bound by commutativity. By the equivalence between representations and modules given in Theorem 2.3.1, we can view $H_q(\mathbb{X})$ as an A-module, where A is the algebra of the bound quiver.

This brings us to the study of modules over K-algebras. We first provide a review in Section 4.1 of the general material concerning the Auslander-Reiten theory of modules over algebras. We then apply these general techniques to the representations of commutative ladder quivers, and interpret the resulting theory from the point of view of topological data analysis.

4.1 Auslander-Reiten theory

In this section, we provide a review of the basics of Auslander-Reiten theory. For more detailed treatments, we refer the reader to the books [2, 3, 5]. We omit some of the proofs and instead provide links to the references. The exposition here is oriented towards applications to persistent homology, and we provide perspectives from this viewpoint.

We use the path algebra $K\vec{A}_n$ of the quiver $\vec{A}_n$ to provide examples. Recall from the introduction and background chapters that a filtration is a quiver complex over $\vec{A}_n$, and that its persistent homology is a representation of $\vec{A}_n$, and thus a $K\vec{A}_n$-module. By our use of $K\vec{A}_n$, we give hints as to how the general theory in this section can be applied to persistent homology on the commutative ladders. This application will be developed in the later sections.

In the first subsection, we discuss several functors that will be used later for computations. In the second subsection, we introduce one of the main objects in Auslander-Reiten theory, the almost split sequences. A similar discussion can be made in general for Artin algebras, but we restrict our attention to finite-dimensional K-algebras over a field K which is not required to be algebraically closed.

4.1.1 Transpose and translations

Let A be a finite-dimensional K-algebra. Recall that $\operatorname{mod} A$ is the category of finite-dimensional A-modules. A *minimal projective presentation* of M is an exact sequence

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0 \tag{4.1}$$

where P_0 and P_1 are projective modules and $p_i : P_i \to \operatorname{Im} p_i$ are projective covers for $i = 0, 1$. By Proposition 2.5.2, it follows that for any $M \in \operatorname{mod} A$, M has a *minimal projective presentation.* Of course, if $M = P$ is projective, then it has a minimal projective presentation

$$0 \longrightarrow P \xrightarrow{1} P \longrightarrow 0.$$

Given an indecomposable nonprojective module M, apply the contravariant functor $(-)^t = \operatorname{Hom}_A(-, A)$ to its minimal projective presentation in Eq. (4.1). Since $(-)^t$ is left exact,

$$0 \longrightarrow M^t \xrightarrow{p_0^t} P_0^t \xrightarrow{p_1^t} P_1^t \longrightarrow \operatorname{Coker} p_1^t \longrightarrow 0$$

is an exact sequence in $\operatorname{mod} A^{\mathrm{op}}$. The *transpose* of M, denoted by $\operatorname{Tr} M$, is defined to be the A^{op}-module $\operatorname{Coker} p_1^t$. Note that $\operatorname{Tr} M$ is unique up to isomorphism, by the uniqueness of minimal projective presentations.

So far, we only describe $M \mapsto \operatorname{Tr} M$ as a map from objects in $\operatorname{mod} A$ to objects in $\operatorname{mod} A^{\mathrm{op}}$. To properly define a functor, we do the following construction. Recall that the arrow category $\operatorname{arr} C$ of a K-category C has objects consisting of the morphisms between objects in C. Let $\operatorname{proj} A$ be the full subcategory of $\operatorname{mod} A$ consisting of projective A-modules.

We define the functor $\operatorname{Coker} : \operatorname{arr} \operatorname{proj} A \to \operatorname{mod} A$. The functor Coker is defined on objects by $\operatorname{Coker}(f : P_1 \to P_0) = \operatorname{Coker} f$. Given a morphism $(g_1, g_2) : f \to f'$ in $\operatorname{arr} \operatorname{proj} A$, $\operatorname{Coker}(g_1, g_2)$ is the unique morphism that makes the diagram

$$\begin{tikzcd}
P_1 \arrow[r, "f"] \arrow[d, "g_1"] & P_0 \arrow[r] \arrow[d, "g_2"] & \operatorname{Coker} f \arrow[r] \arrow[d, "\operatorname{Coker}(g_1,g_2)"] & 0 \\
P_1' \arrow[r, "f'"] & P_0' \arrow[r] & \operatorname{Coker} f' \arrow[r] & 0
\end{tikzcd}$$

commute. Explicitly, for $p + \operatorname{Im} f \in \operatorname{Coker} f$, define $\operatorname{Coker}(g_1, g_2)(p + \operatorname{Im} f) = g_2 p + \operatorname{Im} f'$. To show that this is well-defined, note that if $p - p' \in \operatorname{Im} f$, then $g_2 p - g_2 p' \in g_2(\operatorname{Im} f) \subset \operatorname{Im} f'$ since $g_2 f = f' g_1$. Checking the other claimed properties is also easy.

As defined above, the functor Coker is not an equivalence. One way to get an equivalence is the following. First, we define the two-sided ideal $\mathcal{P}(f, f')$ of $\operatorname{arr}(\operatorname{proj} A)$ to consist of morphisms $(g_1, g_2) : (f : P_1 \to P_0) \to (f' : P_1' \to P_0')$ such that there is some $h : P_0 \to P_1'$ satisfying $f'h = g_2$ or $hf = g_1$, as in the diagram

$$\begin{tikzcd} P_1 \arrow[r, "f"] \arrow[d, "g_1"] & P_0 \arrow[d, "g_2"] \arrow[dl, dashed, "h"'] \\ P'_1 \arrow[r, "f'"] & P'_0. \end{tikzcd}$$

On the other hand, a morphism $f : M \to N$ is said to *factor through a projective module* if there is a projective module P and a factorization $f = hg$ with $h : P \to N$, $g : M \to P$. Let $\mathcal{P}(M,N)$ be the two-sided ideal of $\operatorname{mod} A$ consisting of the morphisms $f : M \to N$ that factor through a projective.

It can be shown that (g_1, g_2) is in $\mathcal{P}(f, f')$ if and only if $\operatorname{Coker}(g_1, g_2) \in \mathcal{P}(M,N)$ where $M = \operatorname{Coker} f$, $M' = \operatorname{Coker} f'$. Let $\underline{\operatorname{mod}}\, A$ be the quotient category $(\operatorname{mod} A)/\mathcal{P}$, called the *projectively stable module category* of A.

Proposition 4.1.1 ([3, Props. IV.1.3, IV.1.6])**.**

1. *The functor* $\operatorname{Coker} : \operatorname{arr} \operatorname{proj} A \to \operatorname{mod} A$ *induces an equivalence* $\operatorname{Coker} : (\operatorname{arr} \operatorname{proj} A)/\mathcal{P} \to \underline{\operatorname{mod}}\, A$.

2. *The duality* $(-)^t : \operatorname{proj} A \to \operatorname{proj} A^{\mathrm{op}}$ *induces a duality* $(\operatorname{arr} \operatorname{proj} A)/\mathcal{P} \to (\operatorname{arr} \operatorname{proj} A^{\mathrm{op}})/\mathcal{P}$ *which induces the duality* $\operatorname{Tr} : \underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, A^{\mathrm{op}}$.

The second part of Prop. 4.1.1 provides the correct setting for defining the Tr duality. Let us also list some properties of the object map

$$\operatorname{Tr} : \operatorname{Ob} \operatorname{mod} A \to \operatorname{Ob} \operatorname{mod} A^{\mathrm{op}}$$

defined from Tr. First, we need the following definition. By the existence and uniqueness up to isomorphism of the indecomposable decomposition of $M \in \operatorname{mod} A$, we can write

$$M = M_{\mathcal{P}} \oplus M'$$

where $M_{\mathcal{P}}$ has no nonzero projective summands and M' is projective. Define $\operatorname{mod}_{\mathcal{P}} A$ to be the full subcategory of $\operatorname{mod} A$ of modules with no projective direct summand. That is, $\operatorname{mod}_{\mathcal{P}} A$ consists of $M \in \operatorname{mod} A$ such that $M = M_{\mathcal{P}}$.

Proposition 4.1.2 ([3, Prop. IV.1.7])**.**

1. $\operatorname{Tr} M = 0$ *if and only if* M *is projective.*

2. $\operatorname{Tr}(\bigoplus_{i=1}^{n} M_i) \cong \bigoplus_{i=1}^{n} \operatorname{Tr}(M_i)$ *for* $M_i \in \operatorname{mod} A$.

3. $\operatorname{Tr} \operatorname{Tr} M \cong M_{\mathcal{P}}$.

4. $M, N \in \operatorname{mod}_{\mathcal{P}} A$ *then* $M \cong N$ *if and only if* $\operatorname{Tr} M \cong \operatorname{Tr} N$.

As an example, let us compute $\operatorname{Tr} M$ for the indecomposable modules M of the path algebra $A = K\vec{A}_n$. As discussed in Section 2.4, the list of interval representations $\mathbb{I}[a,b]$, $1 \leqslant a \leqslant b \leqslant n$, gives the complete list of indecomposable representations up to isomorphism. Here, let us abbreviate $\mathbb{I}[a,b]$ by $[a, b]$. By

Lemma 2.5.5, the indecomposable projective representations are given by $[a, n]$ for $1 \leqslant a \leqslant n$.

Some preparation is needed. First, $\vec{A}_n = A_n(f \dots f)$ has opposite quiver $A_n(b \dots b)$ and thus the opposite of its path algebra, $A^{\mathrm{op}} = (K\vec{A}_n)^{\mathrm{op}}$, can be identified with $KA_n(b \dots b) = K(\vec{A}_n^{\mathrm{op}})$. We similarly define the interval modules for A^{op} and denote them by $[a, b]^{\mathrm{op}}$ to remind us that the arrows are reversed (compared to those in $\vec{A}_n$). An easy computation shows that $D([a, b]) \cong [a, b]^{\mathrm{op}}$.

Next, we compute $([a, n])^t = \mathrm{Hom}_A([a, n], A)$. By the proof of Theorem 2.3.1, the K-vector space associated to the vertex c of the representation corresponding to the module $\mathrm{Hom}_A([a, n], A)$ is given by $\mathrm{Hom}_A([a, n], A)e_c$. Note that $\mathrm{Hom}_A([a, n], A)$ is a right A-module, so we multiply by e_c on the right. This is isomorphic to the K-vector space

$$\mathrm{Hom}_A([a, n], Ae_c) \cong \mathrm{Hom}_A([a, n], [c, n]) \cong \begin{cases} K & \text{if } c \leqslant a, \\ 0 & \text{otherwise.} \end{cases}$$

Then, checking the action of $\alpha \in A$, we get

$$([a, n])^t \cong [1, a]^{\mathrm{op}}.$$

Let us compute $\mathrm{Tr}[a, b]$ for $[a, b]$ indecomposable nonprojective. By assumption, $b \neq n$ and $[a, b]$ has minimal projective presentation

$$[b + 1, n] \xrightarrow{p} [a, n] \longrightarrow [a, b] \longrightarrow 0. \tag{4.2}$$

Applying $(-)^t$, we get

$$[1, a]^{\mathrm{op}} \xrightarrow{p^t} [1, b + 1]^{\mathrm{op}} \longrightarrow \mathrm{Coker}\, p^t \longrightarrow 0$$

with $\mathrm{Tr}[a, b] = \mathrm{Coker}\, p^t = [a + 1, b + 1]^{\mathrm{op}}$, an A^{op}-module. To get back into $\mathrm{mod}\, A$, we can apply $D(-)$ and get $D\,\mathrm{Tr}[a, b] = [a + 1, b + 1]^{\mathrm{op}\,\mathrm{op}} \cong [a+1, b+1]$.

We also phrase Tr in terms of more familiar functors, at least for certain simple cases. Note that Eq. (4.2) above not only provides a minimal projective presentation, but can also be extended to an exact sequence

$$0 \longrightarrow [b + 1, n] \xrightarrow{p} [a, n] \longrightarrow [a, b] \longrightarrow 0.$$

This furnishes a projective resolution of $[a, b]$, from which we can compute $\mathrm{Ext}^1_A([a, b], A)$ by using $\mathrm{Hom}_A(-, A) = (-)^t$ to get the (not necessarily exact) sequence

$$0 \longrightarrow ([a, b])^t \longrightarrow [1, a]^{\mathrm{op}} \xrightarrow{p^t} [1, b + 1]^{\mathrm{op}} \xrightarrow{0} 0,$$

with $\mathrm{Ext}^1_A([a, b], A) \cong \mathrm{Ker}\, 0/\,\mathrm{Im}\, p^t = \mathrm{Coker}\, p^t = \mathrm{Tr}[a, b]$.

This is no accident. First, we note that $K\vec{A}_n$ is a hereditary K-algebra. In fact, a more general result can be given.

Proposition 4.1.3 (cf. [2, Theorem VII.1.7])**.** *If Q is a finite, connected, acyclic quiver, then KQ is a hereditary K-algebra.*

It can be checked that the usual $\mathrm{Ext}^1_A(-,A)$ functor induces a functor on the projectively stable module categories. Then, Cor. IV.1.14 of [3] states that for A a hereditary finite-dimensional K-algebra, $\mathrm{Tr} : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A^{\mathrm{op}}$ and $\mathrm{Ext}^1_A(-,A) : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A^{\mathrm{op}}$ are isomorphic.

In general, of course, Tr may not be isomorphic to $\mathrm{Ext}^1_A(-,A)$. Nevertheless, the above example suggests that the functor $D\,\mathrm{Tr}$ allows us to construct new indecomposables out of previously known ones. We will see this in the next few subsections.

Definition 4.1. The *Auslander-Reiten translations* are defined to be

$$\tau = D\,\mathrm{Tr} \text{ and } \tau^{-1} = \mathrm{Tr}\, D.$$

These translations play a very important role in the Auslander-Reiten theory. Properties similar to those listed in Prop. 4.1.2 for Tr can be inferred for τ and τ^{-1}.

Let us properly write down these translations as functors. Recall that $\mathrm{Tr} : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A^{\mathrm{op}}$ is a duality. Let $\mathcal{I}$ be the two-sided ideal of $\mathrm{mod}\, A$ defined by letting $\mathcal{I}(M,N)$ to consist of the morphisms $f : M \to N$ that factors through an injective. Similar to $\underline{\mathrm{mod}}\, A$, we define the *injectively stable module category* $\overline{\mathrm{mod}}\, A$ to be the quotient category $(\mathrm{mod}\, A)/\mathcal{I}$.

Theorem 4.1.4 ([3, Prop. IV.1.9])**.**

1. *The duality $D(-) : \mathrm{mod}\, A \to \mathrm{mod}\, A^{\mathrm{op}}$ induces a duality $D(-) : \underline{\mathrm{mod}}\, A \to \overline{\mathrm{mod}}\, A^{\mathrm{op}}$.*

2. *The Auslander-Reiten translations $\tau = D\,\mathrm{Tr} : \underline{\mathrm{mod}}\, A \to \overline{\mathrm{mod}}\, A$ and $\tau^{-1} = \mathrm{Tr}\, D : \overline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A$ are equivalences and are inverses of each other.*

Here, we will mainly use τ and τ^{-1} as maps between objects.

For computational purposes, we also define the endofunctor

$$\nu(-) = D\,\mathrm{Hom}_A(-,A) = D((-)^t) : \mathrm{mod}\, A \to \mathrm{mod}\, A$$

called the *Nakayama functor*. It is known that ν restricts to an equivalence

$$\nu(-) : \mathrm{proj}\, A \to \mathrm{inj}\, A,$$

where $\mathrm{inj}\, A$ is the full subcategory of $\mathrm{mod}\, A$ consisting of injective A-modules. This has quasi-inverse

$$\nu^{-1}(-) = \mathrm{Hom}_A(D(A_A), -) : \mathrm{inj}\, A \to \mathrm{proj}\, A.$$

We need the following definition before giving Prop. 4.1.5. Let $M \in \mathrm{mod}\, A$. A minimal injective presentation of M is an exact sequence

$$0 \longrightarrow M \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1$$

where I_0 and I_1 are injective modules and $i_0 : M \to I_0$ and $i_1' : \mathrm{Coker}\, i_0 \to I_1$ are both injective envelopes. Here, i_1' is induced from i_1.

There is a close relationship between the translations τ, τ^{-1} and the Nakayama functors ν, ν^{-1}. The following can be shown by a straightforward calculation.

Proposition 4.1.5 ([2, Prop. IV.2.4])**.** *Let $M \in \operatorname{mod} A$.*

1. *Let $P' \xrightarrow{p_1} P \xrightarrow{p_0} M \longrightarrow 0$ be a minimal projective presentation of M. Then, there is an exact sequence:*
$$0 \longrightarrow \tau M \longrightarrow \nu P' \xrightarrow{\nu p_1} \nu P \xrightarrow{\nu p_0} \nu M \longrightarrow 0.$$

2. *Let $0 \longrightarrow M \xrightarrow{i_0} I \xrightarrow{i_1} I'$ be a minimal injective presentation of M. Then, there is an exact sequence:*
$$0 \longrightarrow \nu^{-1} M \xrightarrow{\nu^{-1} i_0} \nu^{-1} I \xrightarrow{\nu^{-1} i_1} \nu^{-1} I' \longrightarrow \tau^{-1} M \longrightarrow 0.$$

In the Section 2.5 of Chapter 2, we described a method for computing an injective envelope of $M \in \operatorname{mod} A$, and thus minimal injective presentations. Moreover, if $\{e_1, \dots, e_n\}$ is a complete list of primitive orthogonal idempotents for A, then $P_k = Ae_k$, $I_k = D(e_k A)$ for $k \in \{1, \dots, n\}$ gives a complete list of indecomposable projectives and injectives, up to isomorphism. We have $\nu P_k = I_k$ and $\nu^{-1} I_k = P_k$. Since direct summands of projective modules (injective modules) are projective (injective), we can use the additivity of the functors ν and ν^{-1} to then compute the middle terms in Prop 4.1.5. The fact that
$$\nu^{-1} M = \operatorname{Hom}_A(D(A_A), M) \cong \bigoplus_{i=1}^{n} \operatorname{Hom}_A(D(e_i A), M)$$
is also useful.

4.1.2 Almost split sequences

So far we have discussed the computation of the Auslander-Reiten translations τ, τ^{-1} with only small hints as to their importance. Here, we show that there exist certain exact sequences (called almost split sequences) in $\operatorname{mod} A$ that provides a very powerful tool to study $\operatorname{mod} A$. Moreover, the translations τ, τ^{-1} provide a way to compute these almost split sequences.

A morphism $f : M \to N$ is said to be *right minimal* if any endomorphism $h : M \to M$ with $fh = f$ is an isomorphism. Dually, $f : M \to N$ is said to be *left minimal* if any $h : N \to N$ with $hf = f$ is an isomorphism.

Let $f : M \to N$ be a morphism, a morphism $h : X \to N$ is said to factor through f if there is an $x : X \to M$ such that $h = fx$. Similarly, a morphism $h : M \to Y$ is said to factor through f if there is a $y : N \to Y$ such that $h = yf$. These cases are given by the diagrams

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ X \arrow[u, dashed, "x"] \arrow[ur, "h"] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "h"] & N \arrow[d, dashed, "y"] \\ & Y. \end{tikzcd}$$

Recall that a morphism $f : M \to N$ is said to be a *split epimorphism* if the identity $1_N : N \to N$ factors through f. That is, there is some $h : N \to M$ such that $1_N = fh$. Dually, we have the concept of a *split monomorphism* which is a morphism $f : M \to N$ such that 1_M factors through f. The following fact about split morphisms is standard.

Lemma 4.1.6 ([29, Prop. I.4.3]). *The following are equivalent for a short exact sequence in* $\operatorname{mod} A$

$$0 \longrightarrow M \xrightarrow{g} L \xrightarrow{f} N \longrightarrow 0$$

1. *f is a split epimorphism.*

2. *g is a split monomorphism.*

3. *The sequence is isomorphic to*

$$0 \longrightarrow M \xrightarrow{\iota_1} M \oplus N \xrightarrow{\pi_2} N \longrightarrow 0$$

A short exact sequence is said to be *split* if it satisfies any of the equivalent conditions above.

Of interest are the following "almost split" morphisms. A morphism $f : M \to N$ is said to be *right almost split* if the following conditions hold.

1. The morphism f is not a split epimorphism.

2. For any morphism $h : X \to N$ that is not a split epimorphism, h factors through f.

If $f : M \to N$ is a split epimorphism, then there is a morphism $s : N \to M$ such that $fs = 1_N$. Given any $h : X \to N$, we have $fsh = h$ so that h factors through f (via sh). The converse also holds: if $f : M \to N$ has the property that for any $h : X \to N$, h factors through f, then f is a split epimorphism. Thus, f is a split epimorphism if and only if for any morphism $h : X \to N$, h factors through f.

In the right almost split case, we do not require that all morphisms $h : X \to N$ satisfy the factorization property, only those that are not split epimorphisms. In this sense a morphism f that is not a split epimorphism is "almost" split.

Dually, a morphism $f : M \to N$ is said to be a *left almost split morphism* if it satisfies the following conditions.

1. The morphism f is not a split monomorphism.

2. For any morphism $h : M \to X$ that is not a split monomorphism, h factors through f.

Lemma 4.1.7 ([3, Lemma V.1.7]). *Let $f : M \to N$ be a morphism in* $\operatorname{mod} A$.

1. *If f is right almost split, then N is indecomposable.*

2. *If f is left almost split, then M is indecomposable.*

Proof. If N were decomposable, then the inclusions from the summands of N factors through f since f is right almost split. However, this implies that 1_N factors through f, a contradiction to the fact that f is not a split epimorphism. Part *2* can be proved in a similar manner. □

The following characterization is important.

Proposition 4.1.8 ([3, Prop V.1.9])**.** *Given $f : M \to N$, the following are equivalent.*

1. *The morphism f is left almost split.*
2. *The morphism f is not a split monomorphism, and every nonisomorphism $h : M \to Y$ with Y indecomposable factors through f.*

In short, it suffices to check the defining property for left almost split morphisms only for nonisomorphisms $h : M \to Y$ with Y indecomposable. We skip writing down the dual version. See [3, Prop V.1.8].

A typical example of a right almost split morphism is the following. If P is an indecomposable projective module, then the inclusion $\iota : \operatorname{Rad} P \to P$ is right almost split. First of all, ι is clearly not a split epimorphism. Since P is projective, $h : X \to P$ is a split epimorphism if and only if h is an epimorphism. So it suffices to check the second property for all $h : X \to P$ that are not epimorphisms. But this is trivial, since $\operatorname{Im} h$ being a proper submodule of P implies that $\operatorname{Im} h \subset \operatorname{Rad} P$ so that h factors through ι. This shows that $\iota : \operatorname{Rad} P \to P$ is right almost split. In fact, it can be checked that ι is also right minimal.

Let us study morphisms that are both right minimal and right almost split. We call these morphisms *right minimal almost split* (RMAS). Of course, we have the dual concept of *left minimal almost split* (LMAS) morphisms, which are morphisms that are both left minimal and left almost split.

The following gives the RMAS ending at an indecomposable projective P, and the LMAS starting at an indecomposable injective I.

Proposition 4.1.9 ([2, Prop. IV.3.5])**.** *Let P be indecomposable projective. If a morphism $f : X \to P$ is a monomorphism with $\operatorname{Im} f \cong \operatorname{Rad} P$, then f is RMAS. Dually, let I be indecomposable injective. If $g : I \to X$ is an epimorphism with $\operatorname{Ker} g \cong \operatorname{Soc} I$, then g is LMAS.*

The following lemma shows that for any fixed M or N, RMAS morphisms ending at N or LMAS morphisms starting at M are unique up to isomorphism.

Lemma 4.1.10 ([2, Prop. IV.1.2])**.**

1. *If $f : M \to N$ and $f' : M' \to N$ are RMAS, then there exists an isomorphism $h : M \to M'$ such that $f = f'h$.*
2. *If $g : M \to N$ and $g' : M \to N'$ are LMAS, then there exists an isomorphism $h : N \to N'$ such that $g' = hg$.*

Proof. Since f and f' are both right almost split, f factors through f' and f' factors through f. Thus there exists $h : M \to M'$ and $h' : M' \to M$ with $f = f'h$ and $f' = fh'$. We then have $f' = f'hh'$ and $f = fh'h$. Since both f and f' are right minimal, hh' and $h'h$ are automorphisms (of M' and M, respectively). Thus h and h' are isomorphisms with the required property.

The proof for part *2* is similar. □

Definition 4.2. An *almost split sequence* is a short exact sequence

$$0 \longrightarrow M \xrightarrow{\ g\ } L \xrightarrow{\ f\ } N \longrightarrow 0$$

such that g is LMAS and f is RMAS.

To show the properties of almost split sequences, let us first give the following technical lemmas.

Lemma 4.1.11 (cf. [3, Cor. IV.4.4])**.** *Let*

$$0 \longrightarrow M \xrightarrow{\ g\ } L \xrightarrow{\ f\ } N \longrightarrow 0$$

be an exact sequence. Then, for any $Y \in \operatorname{mod} A$, the following are equivalent.

1. *For any $h : \tau^{-1}Y \to N$, h factors through f.*

2. *For any $h : M \to Y$, h factors through g.*

Lemma 4.1.12 (cf. [29, Lemma III.3.1])**.** *Consider the following exact sequences in* $\operatorname{mod} A$*.*

$$\begin{tikzcd}
E: & 0 \arrow[r] & M \arrow[r, "g"] \arrow[d, "x"] & L \arrow[r, "f"] \arrow[d, "y"] & N \arrow[r] \arrow[d, "1"] & 0 \\
xE: & 0 \arrow[r] & M' \arrow[r, "g'"] & L' \arrow[r, "f'"] & N \arrow[r] & 0,
\end{tikzcd}$$

where xE is the short exact sequence induced by taking the pushout $M' \xrightarrow{g'} L' \xleftarrow{y} L$ of $M' \xleftarrow{x} M \xrightarrow{g} L$. The exact sequence xE is split if and only if x factors through g.

Proof. Suppose that x factors through g. There is a $k : L \to M$ such that $kg = x$. At the same time, there is the identity $1 : M' \to M'$. Thus, we have a diagram

$$\begin{tikzcd}
M \arrow[r, "g"] \arrow[d, "x"] & L \arrow[d, "y"'] \arrow[ddr, "k"] & \\
M' \arrow[r, "g'"] \arrow[drr, "1"'] & L' \arrow[dr, dashed, "r"] & \\
& & M'
\end{tikzcd}$$

where the existence of the arrow r is inferred by the property of L' being a pushout. Moreover, $rg' = 1$ so that xE splits.

Now, suppose that xE splits. Then, there is an $r : L' \to M'$ such that $rg' = 1$. Then, $x = rg'x = ryg = (ry)g$. Thus, x factors through g. □

Of course, a dual statement can be given for Lemma 4.1.12, but we do not need it in this work.

Finally we can give some of the important properties of almost split sequences. Also, we see how the Auslander-Reiten translations are related to RMAS and LMAS morphisms. The proof below appears in [3], but we have added in more details.

Proposition 4.1.13 ([3, Prop. V.1.14])**.** *The following are equivalent for a short exact sequence*

$$0 \longrightarrow M \xrightarrow{g} L \xrightarrow{f} N \longrightarrow 0.$$

1. *The exact sequence is almost split.*
2. *f is RMAS.*
3. *g is LMAS.*
4. *$N \cong \tau^{-1}M$ and g is left almost split.*
5. *$M \cong \tau N$ and f is right almost split.*

Proof. We break the proof into several parts. First we show that *5* implies *2*, and then *2* implies *3* and *5*. Dually, it can be shown that *4* implies *3*, and *3* implies *2* and *4*. This shows that *2*, *3*, *4*, and *5* are equivalent. Equivalence with *1* then follows by definition.

5 $\to$ *2*. Since f is right almost split, N is indecomposable by Prop. 4.1.7, and thus $M \cong \tau N$ is also indecomposable. Note that $M \neq 0$, for if otherwise, f is an isomorphism and thus a split epimorphism, which is a contradiction.

Now, let us consider any endomorphism $h : L \to L$ with $fh = f$. We have a diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} f \arrow[r, "\iota"] \arrow[d, "\phi"] & L \arrow[r, "f"] \arrow[d, "h"] & N \arrow[r] \arrow[d, "1"] \arrow[dl, dashed, "k"'] & 0 \\ 0 \arrow[r] & \operatorname{Ker} f \arrow[r] & L \arrow[r, "f"] & N \arrow[r] & 0 \end{tikzcd}$$

where $\operatorname{Ker} f \cong M$, and ϕ is defined by restriction of h to $\operatorname{Ker} f$. Since $fh(x) = f(x) = 0$ for any $x \in \operatorname{Ker} f$, $h|_{\operatorname{Ker} f}(x) \in \operatorname{Ker} f$ so that $\phi \in \operatorname{End}(\operatorname{Ker} f)$.

Now, suppose that ϕ is not an automorphism. Since $\operatorname{Ker} f \cong M$ is indecomposable, $\operatorname{End}(\operatorname{Ker} f)$ is local and ϕ is nilpotent by Lemma 2.5.1. Thus there is some positive integer m such that $\phi^m = 0$, and so $h^m\iota = 0$ where $\iota : \operatorname{Ker} f \hookrightarrow L$ is the inclusion.

This shows that h^m factors through f, so that there is some $k : N \to L$ such that $h^m = kf$. Since f is an epimorphism, we can right-cancel f from $f = fh^m = fkf$. Thus, $1 = fk$ showing that f is a split epimorphism, a contradiction.

The above argument shows that ϕ is an automorphism. The final step is to use the short five lemma (Lemma 2.5.8) to conclude that h is an automorphism and thus f is right minimal.

2 $\rightarrow$ *3*, *5*. For *5*, we only need to show that with f RMAS, $\operatorname{Ker} f \cong \tau N$. Without loss of generality, we identify $M = \operatorname{Ker} f$ in the exact sequence. The proof for *3* requires only a little extra effort, so we combine these two.

First, $\operatorname{Ker} f$ is indecomposable. To see this, suppose that $\operatorname{Ker} f \cong \bigoplus_{i=1}^{n} A_i$ is a nontrivial indecomposable decomposition of $\operatorname{Ker} f$. Since $\iota : \operatorname{Ker} f \rightarrow L$ is not a split monomorphism, there is some A_i so that the projection $p_i : \operatorname{Ker} f \rightarrow A_i$ does not factor through ι. Otherwise, ι will be a split monomorphism, a contradiction.

Via pushout, we construct the following diagram:

$$\begin{tikzcd}
E: & 0 \arrow[r] & \operatorname{Ker} f \arrow[r, "\iota"] \arrow[d, "p_i"] & L \arrow[r, "f"] \arrow[d, "v"] & N \arrow[r] \arrow[d, "1"] & 0 \\
p_iE: & 0 \arrow[r] & A_i \arrow[r] & A \cup_{\operatorname{Ker} f} B \arrow[r, "t"] & N \arrow[r] & 0.
\end{tikzcd}$$

Since p_i does not factor through ι, t is not a split epimorphism, by Lemma 4.1.12.

By construction, $f = tv$. Then, for any $h : X \rightarrow N$ not a split epimorphism, h factors through f, $h = fx$ for some $x : X \rightarrow L$, so that $h = tvx$ and h factors through t. This shows that t is right almost split.

By a similar proof as above (in showing *5* implies *2*) we can conclude that t is also right minimal, since A_i is indecomposable. Thus, t is RMAS. By uniqueness of RMAS morphisms ending at N, the top and bottom rows of the above diagram are isomorphic, and so $\operatorname{Ker} f \cong A_i$, a contradiction. Therefore $\operatorname{Ker} f$ is indecomposable.

Since N is indecomposable and f is not split, g is left minimal. Again, this follows from a proof similar to what we have done above.

Finally, we show that $\operatorname{Ker} f \cong \tau N$, and that g is left almost split. Let us consider all morphisms $h : \operatorname{Ker} f \rightarrow Y$ with Y indecomposable and $Y \ncong \tau N$.

We have two cases to check. The first is that Y is injective. In this case, $Y \ncong \operatorname{Ker} f$ since otherwise, g splits which is a contradiction.

If Y is not injective, then $\tau^{-1}Y$ is nonzero. In this case, since $Y \ncong \tau N$, certainly $\tau^{-1}Y \ncong N$. Now, f being right almost split implies that any $\tau^{-1}Y \rightarrow N$ factors through f. By Lemma 4.1.11, this is equivalent to the statement that any $h : \operatorname{Ker} f \rightarrow Y$ factors through g. Thus, $Y \ncong \operatorname{Ker} f$ since g is not split.

In either case, we have the following. For any indecomposable $Y \ncong \tau N$, $Y \ncong \operatorname{Ker} f$. Since $\operatorname{Ker} f$ is indecomposable, we can infer $\operatorname{Ker} f \cong \tau N$. Also, by Prop. 4.1.8, we conclude that g is left almost split. This shows both *3* and *5*.

$\square$

Finally, we see that in the module category $\operatorname{mod} A$, there are enough almost split sequences.

Proposition 4.1.14 ([3, Prop. V.1.15])**.** *In the case that N is an indecomposable nonprojective, or that M is an indecomposable noninjective, then there is an almost split sequence*

$$0 \longrightarrow M \xrightarrow{g} L \xrightarrow{f} N \longrightarrow 0$$

in $\operatorname{mod} A$.

Since the construction of the almost split sequence is interesting, let us provide the proof here. The book [3] provides a proof for the case where N is indecomposable projective. Here, let us do the dual case and consider M indecomposable noninjective.

Proof. By Prop. 4.1.13, it suffices to show that there is an almost split sequence

$$0 \longrightarrow M \xrightarrow{g} L \xrightarrow{f} \tau^{-1}M \longrightarrow 0$$

when M is indecomposable noninjective.

Clearly, $\tau^{-1}M$ is nonprojective, so there is a nonsplit exact sequence

$$E: \qquad 0 \longrightarrow V \xrightarrow{h} B \xrightarrow{j} \tau^{-1}M \longrightarrow 0$$

Now, there is an $x : V \to M$ that does not factor through h. Otherwise, if every $x : V \to M$ factors through h, then every $y : \tau^{-1}M \to \tau^{-1}M$ factors through j by Lemma 4.1.11. In particular $1_{\tau^{-1}M}$ factors through j and j splits, a contradiction.

Let $\Gamma = \operatorname{End}_A(M)^{\mathrm{op}}$. We have the following sequence of Γ-modules

$$\operatorname{Hom}_A(B, M) \xrightarrow{h^*} \operatorname{Hom}_A(V, M) \longrightarrow \operatorname{Coker} \operatorname{Hom}_A(h, M) \longrightarrow 0$$

where $h^* = \operatorname{Hom}_A(h, M)$. Since $x \in \operatorname{Hom}_A(V, M)$ does not factor through h, $\operatorname{Coker} \operatorname{Hom}_A(h, M)$ is nonzero. Moreover, x can be chosen so that it generates a simple submodule $\Gamma(x + \operatorname{Im} h^*)$ of $\operatorname{Coker} \operatorname{Hom}_A(h, M)$.

We get the short exact sequence xE from E via pushout in the diagram:

$$\begin{tikzcd}
E: & 0 \arrow[r] & V \arrow[r, "h"] \arrow[d, "x"] & B \arrow[r, "j"] \arrow[d] & \tau^{-1}M \arrow[r] \arrow[d, "1"] & 0 \\
xE: & 0 \arrow[r] & M \arrow[r, "g"] & L \arrow[r, "f"] & \tau^{-1}M \arrow[r] & 0.
\end{tikzcd}$$

Let us show that the bottom row xE is the required almost split sequence. By Lemma 4.1.12, the bottom row is not split, since x does not factor through h.

To show that g is left almost split, we show that for any $y : M \to Y$ not a split monomorphism, y factors through g. By Lemma 4.1.12, this is equivalent to showing that the bottom row $y(xE)$ in:

$$\begin{array}{llllllllll}
E: & 0 & \longrightarrow & V & \xrightarrow{h} & B & \xrightarrow{j} & \tau^{-1}M & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle x} & & \downarrow & & \downarrow{\scriptstyle 1} & & \\
xE: & 0 & \longrightarrow & M & \xrightarrow{g} & L & \xrightarrow{f} & \tau^{-1}M & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle y} & & \downarrow & & \downarrow{\scriptstyle 1} & & \\
y(xE): & 0 & \longrightarrow & Y & \xrightarrow{g'} & L' & \xrightarrow{f'} & \tau^{-1}M & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle z} & & & & & & \\
 & & & M & & & & & &
\end{array}$$

is split for any y not a split monomorphism.

To do this, we prove that any $z : Y \to M$ factors through g'. This is equivalent to showing that any morphism $\tau^{-1}M \to \tau^{-1}M$ factors through f' by Lemma 4.1.11. In particular, this shows that $1 : \tau^{-1}M \to \tau^{-1}M$ factors through f' and thus the bottom row is split.

For any $z : Y \to M$, $zy : M \to M$ cannot be an isomorphism since y is not a split monomorphism. Since M is indecomposable, Γ is local so $\operatorname{Rad}\Gamma$ contains the noninvertible endomorphisms of M. Thus,

$$zyx \in (\operatorname{Rad}\Gamma)(\Gamma x).$$

Since the image of x in $\operatorname{Coker}\operatorname{Hom}_A(h, M)$ is a simple Γ-module, and the image of $(\operatorname{Rad}\Gamma)(\Gamma x)$ is a proper submodule of that simple module, the image of zyx in $\operatorname{Coker}\operatorname{Hom}_A(h, M)$ is 0. Thus, zyx is in $\operatorname{Im}\operatorname{Hom}_A(h, M)$ and factors through h.

This shows that there is a map $r : B \to M$ such that $rh = zyx$. Since L is the pushout of $h : V \to B$ and $x : V \to M$, there is an $s :\to M$ such that $sg = zy$. Then, since L' is a pushout, there is a $t : L' \to M$ such that $tg' = z$. Thus, $z : X \to M$ factors through g'.

As noted above, we have shown that g is left almost split. Together with Prop. 4.1.13, this shows that

$$0 \longrightarrow M \xrightarrow{g} L \xrightarrow{f} \tau^{-1}M \longrightarrow 0$$

is an almost split sequence. This completes the proof. □

4.1.3 Auslander-Reiten quivers

We review another concept that we need. If the exact sequence

$$0 \longrightarrow M \xrightarrow{g} L \xrightarrow{f} N \longrightarrow 0$$

is almost split, then M and N are indecomposable. However, the module L may have a nontrivial indecomposable decomposition. The next question is, can we characterize the induced morphisms $g_i : M \to L_i$ and $f_i : L_i \to N$, where L_i is an indecomposable summands of the middle term L? Overall, the approach is to study the indecomposables of $\operatorname{mod} A$ and morphisms among them that satisfy certain minimality conditions.

A morphism $f : M \to N$ is said to be an *irreducible morphism* if it is neither a split epimorphism nor a split monomorphism, and whenever there is a factorization $f = gh$, either h is a split monomorphism or g is a split epimorphism. Note that we do not need to discuss left or right versions. This concept is self-dual, for $f : M \to N$ is irreducible if and only if $D(f) : D(N) \to D(M)$ is irreducible.

Theorem 4.1.15 ([3, Theorem V.5.3]). *Let $M, N \in \operatorname{mod} A$ be indecomposable.*

1. *A morphism $g : M \to L$ is irreducible if and only if $L \neq 0$ and there exists an $L' \in \operatorname{mod} A$ and $g' : M \to L'$ such that $\left[\begin{smallmatrix} g \\ g' \end{smallmatrix}\right] : M \to L \oplus L'$ is LMAS.*

2. *A morphism $f : L \to N$ is irreducible if and only if $L \neq 0$ and there exists an $L' \in \operatorname{mod} A$ and $f' : L' \to N$ such that $\left[\begin{smallmatrix} f & f' \end{smallmatrix}\right] : L \oplus L' \to N$ is RMAS.*

In the definition below, each vertex $[M]$ is the isomorphism class of some $M \in \operatorname{mod} A$ indecomposable. We say that a vertex $[M]$ is a *projective vertex* if M is projective, and that $[M]$ is an *injective vertex* if M is injective.

Definition 4.3. Let A be a finite-dimensional K-algebra and let $\Gamma(A)$ be the quiver with vertices given by all the isomorphism classes of indecomposable A-modules in $\operatorname{mod} A$, with an arrow $[M] \to [N]$ if there is an irreducible map $M \to N$. The functor $\tau = D\operatorname{Tr}$ induces a map from nonprojective to noninjective vertices.

The *Auslander-Reiten quiver* (AR quiver) of A is the quiver $\Gamma(A)$ together with τ.

Using Lemma 4.1.10, we shall talk of "the" LMAS morphism or RMAS morphism or almost split sequence starting or ending at an indecomposable. One should remember that this is only unique up to isomorphism in their respective senses.

In the usual definition, the Auslander-Reiten quiver is a *valued quiver*. That is, each arrow $[M] \to [N]$ is given a valuation (a, b), $a, b \in \mathbb{N}_0$, where a is defined to be the number such that there is an RMAS morphism $E = M^a \oplus X \to N$, where M is not a summand of X. That is, a is the multiplicity of M as a direct summand of E. Dually, b is the multiplicity of N as a direct summand in E, where $M \to E$ is the LMAS morphism starting from M. For simplicity however, we drop the valuations on the arrows. In this work, we do not yet need the extra information contained in the valuations.

Recall that a K-algebra A is said to be *representation-finite* if the number of isomorphism classes of indecomposable A-modules is finite. Otherwise, it is said to be *representation-infinite*. Similarly, a bound quiver (Q, ρ) is said to be representation-finite if its algebra $A = KQ/I$ is representation-finite, and representation-infinite otherwise.

The rest of this section is devoted to a discussion of the computation of the Auslander-Reiten quivers of representation-finite algebras. There is a "knitting" procedure for attempting to compute Auslander-Reiten quivers. For example, see the book [3, pages 233 to 234] or [4]. We roughly follow the presentation of the latter.

First, let us state the following useful proposition. As one consequence, if we obtain a finite connected component of $\Gamma(A)$, we can conclude that A is representation-finite.

Proposition 4.1.16 (cf. [2, Thm. IV.5.4], [4, Thm. 7.7]). *Suppose that there is a finite connected component Γ in the Auslander-Reiten quiver $\Gamma(A)$. Then, $\Gamma(A) = \Gamma$.*

The following gives some guidelines on how to proceed. Note that to some extent the knitting procedure can be automated, as can be seen in the examples later. However, there are still sections that require case-by-case analysis. Below, we take the term irreducible morphism to mean only irreducible morphisms between indecomposable modules.

1. For an indecomposable noninjective Z, construct $\tau^{-1}Z$ (by Prop. 4.1.5, for example). Note that $\tau^{-1}Z$ is necessarily nonprojective. We also compute the almost split sequence starting at Z and ending at $\tau^{-1}Z$:
$$0 \longrightarrow Z \xrightarrow{\ g\ } E \xrightarrow{\ f\ } \tau^{-1}Z \longrightarrow 0.$$
Let $E = \bigoplus E_i^{\ell_i}$ be an indecomposable decomposition of E. Via Theorem 4.1.15, we can check for the existence of an irreducible morphism $h : Z \to X$ based on whether or not X is isomorphic to an E_i. Likewise, we get the irreducible morphisms $h' : X \to \tau^{-1}Z$.

2. For irreducible morphisms ending at an indecomposable projective P or starting at an indecomposable injective I, use Prop. 4.1.9 to construct the RMAS morphism ending at P or the LMAS morphism starting from I. Then apply Theorem 4.1.15.

3. We start the computation by listing all indecomposable projectives P_i of A and their radicals $\operatorname{Rad} P_i$. We assume that $A = KQ/I$ is the algebra of a bound quiver (Q, ρ). Since Q is a finite acyclic connected quiver, Q has a sink vertex j. It is clear that $P_j = S_j$ is a simple projective representation of (Q, ρ), and thus corresponds to a simple projective module of A.

Let us give an example. Suppose that we have computed all the arrows ending at $[X]$, and that for each $[Z]$ immediate predecessor of $[X]$, we have already computed the arrows starting from $[Z]$. From:

$$\begin{tikzcd}
 & {[E_1]} \\
{[Z]} \arrow[ur] \arrow[dr] \arrow[dddr] & \\
 & {[E_2]} \\
\vdots \arrow[dr] & \\
 & {[X]} \\
{[I]} \arrow[ur] &
\end{tikzcd}
\qquad \text{obtain} \qquad
\begin{tikzcd}
 & {[E_1]} \arrow[dr] & \\
{[Z]} \arrow[ur] \arrow[dr] \arrow[dddr] & & {[\tau^{-1}Z]} \\
 & {[E_2]} \arrow[ur] & \\
\vdots \arrow[dr] & & \vdots \\
 & {[X]} \arrow[uuur] \arrow[ur] & \\
{[I]} \arrow[ur] & &
\end{tikzcd}$$

by computing $\tau^{-1}Z$ for every noninjective predecessor $[Z]$ of $[X]$, as in item *1* above. By Theorem 4.1.15, we obtain all the arrows ending at $[\tau^{-1}Z]$ since we computed these arrows from the RMAS $f : E \to \tau^{-1}Z$.

Then, we place an arrow $[X] \to [P]$ for every indecomposable projective P such that X isomorphic to a direct summand of $\operatorname{Rad} P$. Thus, the diagram:

$$\begin{tikzcd}
 & {[E_1]} \arrow[dr] & \\
{[Z]} \arrow[ur] \arrow[dr] \arrow[ddr] & & {[\tau^{-1}Z]} \\
\vdots \arrow[dr] & {[E_2]} \arrow[ur] & \vdots \\
 & {[X]} \arrow[uur] \arrow[ur] \arrow[ddr] & \\
{[I]} \arrow[ur] & & \\
 & & {[P]}
\end{tikzcd}$$

now contains all the arrows starting from $[X]$.

From here, we can continue to the "next column", for example with $[\tau^{-1}Z]$ serving as $[X]$ this time, assuming that we have computed all the immediate successors of all the immediate predecessors of $[\tau^{-1}Z]$ by a similar procedure (one may have to deal with the direct summands of E first, in a similar manner).

For indecomposable projective P, however, it is not clear that we can immediately continue the knitting. Here, $\operatorname{Rad} P$ may have other direct summands other than X, leading to other immediate predecessors of P. In the example above, it may be possible to get:

$$\begin{tikzcd}
 & {[E_1]} \arrow[dr] & \\
{[Z]} \arrow[ur] \arrow[dr] \arrow[ddr] & & {[\tau^{-1}Z]} \\
\vdots \arrow[dr] & {[E_2]} \arrow[ur] & \vdots \\
 & {[X]} \arrow[uur] \arrow[ur] \arrow[ddr] & \\
{[I]} \arrow[ur] & & \\
 & & {[P]} \\
 & {[X']} \arrow[ur] &
\end{tikzcd}$$

where $X' \oplus X \cong \operatorname{Rad} P$. Hopefully the vertex $[X']$ either already has appeared in the knitting process, or will appear later, so that we get its predecessors. Similarly, for indecomposable injective I, we use Proposition 4.1.9 to get the LMAS starting from I.

To start the procedure, we note that there is no irreducible morphism ending at a simple projective S. Thus, we can initialize the above computation with $[X] = [S]$.

While the knitting of some small examples may give the impression that the knitting procedure is very powerful, in many cases it can be limited. One obvious limitation is that if the algebra is representation-infinite, then the knitting procedure will go on indefinitely. As hinted above, the appearance of indecomposable projectives and indecomposable injectives requires special care. The algebras we consider have rather uncomplicated structures of indecomposable projectives, so this poses no special problems for our computation.

For examples of cases where the knitting procedure may run into problems, see the Section 7.4 of the book [4]. In particular, [4] provides an example of where some ad hoc arguments are required to continue knitting.

One technique we discussed for the computation of $\tau^{-1}Z$ is the use of the Nakayama functor ν^{-1}, via Prop. 4.1.5. However, this computation may be very tedious. One way to mitigate this is to instead compute just dimension vectors $\underline{\dim}$. That is, if we have an exact sequence

$$0 \longrightarrow \nu^{-1}Z \xrightarrow{\nu^{-1}i_0} P \xrightarrow{\nu^{-1}i_1} P' \longrightarrow \tau^{-1}Z \longrightarrow 0,$$

from application of Prop. 4.1.5, then $\underline{\dim}\,\tau^{-1}Z = \underline{\dim}\,P' - \underline{\dim}\,P + \underline{\dim}\,\nu^{-1}Z$. In the general case, there is no guarantee that two indecomposable representations M and N with $\underline{\dim}\,M = \underline{\dim}\,N$ are isomorphic. Nevertheless, the above computation via dimension vectors does provide a quick way to first guess and then check the form of $\tau^{-1}Z$ in simple cases.

4.1.4 Auslander-Reiten quiver of $\vec{A}_n$

As an example, let us compute the Auslander-Reiten quivers of $K\vec{A}_n$. Other than using the knitting procedure, it is also possible to compute the AR quiver of $K\vec{A}_n$ using elementary methods. In the paper [17], we provide a computation using the knitting of the Auslander-Reiten quiver.

In the background, we have stated that the interval modules $\mathbb{I}[a,b]$, $1 \leqslant a \leqslant b \leqslant n$ provides a complete list, up to isomorphism, of indecomposable $K\vec{A}_n$-modules. Moreover, we have computed all the homomorphism spaces between these interval modules:

$$\operatorname{Hom}_{K\vec{A}_n}(\mathbb{I}[a,b],\mathbb{I}[c,d]) = \begin{cases} Kf_{a,b}^{c,d}, & c \leqslant a \leqslant d \leqslant b, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\left(f_{a,b}^{c,d}\right)_\ell = \begin{cases} 1_K, & a \leqslant \ell \leqslant d, \\ 0, & \text{otherwise.} \end{cases}$$

To simplify the notation, let us write $[a,b]$ for $\mathbb{I}[a,b]$.

Fix $[a,b]$ nonprojective. Let us check by using Eq. (2.8) that the (K-multiples of) the morphisms

$$f_{a+1,b}^{a,b} : [a+1,b] \to [a,b] \text{ or } f_{a,b+1}^{a,b} : [a,b+1] \to [a,b]$$

give all irreducible morphisms from some indecomposable module to fixed $[a, b]$.

Consider $f_{i,j}^{a,b} : [i, j] \to [a, b]$. The case where $(i, j) = (a, b)$ does not give us irreducibles, for in this case $f_{a,b}^{a,b} = 1$ is an isomorphism and cannot be irreducible. In the case where $(i, j) \neq (a + 1, b)$ and $(i, j) \neq (a, b + 1)$, Eq. (2.8) provides a factorization $f_{i,j}^{a,b} = gh$ where simultaneously h is not a split monomorphism and g is not a split epimorphism.

Finally, suppose that $f = f_{a+1,b}^{a,b}$ or $f = f_{a,b+1}^{a,b}$. Clearly f is not a split epimorphism nor a split monomorphism. It can be checked that if $f = gh$, then either g is a split epimorphism or h is a split monomorphism.

In a previous subsection, we have already computed $\tau[a, b] = [a + 1, b + 1]$ for nonprojective indecomposables $[a, b]$. Similarly, $\tau^{-1}[a, b] = [a - 1, b - 1]$ for noninjective indecomposables $[a, b]$. Thus, the following are the almost split sequences for $[a, b]$ nonprojective:

$$0 \longrightarrow \tau[a, b] = [a + 1, b + 1] \longrightarrow [a + 1, b] \oplus [a, b + 1] \longrightarrow [a, b] \longrightarrow 0.$$

Similarly, for $[a, b]$ noninjective, we get almost split sequences:

$$0 \longrightarrow [a, b] \longrightarrow [a, b - 1] \oplus [a - 1, b] \longrightarrow [a - 1, b - 1] = \tau^{-1}[a, b] \longrightarrow 0.$$

Using the above results, the Auslander-Reiten quiver of $K\vec{A}_n$ is:

$$\begin{array}{ccccccccccccccccccccc}
 & & & & & & & & & & [1,n] & & & & & & & & & & \\
 & & & & & & & & & \nearrow & & \searrow & & & & & & & & & \\
 & & & & & & & & [2,n] & & & & [1,n-1] & & & & & & & & \\
 & & & & & & & \nearrow & & \searrow & & \nearrow & & \searrow & & & & & & & \\
 & & & & & & [3,n] & & & & [2,n-1] & & & & [1,n-2] & & & & & & \\
 & & & & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & & & & \\
 & & & & \cdots & & & & \cdots & & & & \cdots & & & & \cdots & & & & \\
 & & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & & \\
 & & [n-1,n] & & & & \cdots & & & & \cdots & & & & \cdots & & & & [1,2] & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
[n,n] & & & & [n-1,n-1] & & & & \cdots & & & & \cdots & & & & [2,2] & & & & [1,1].
\end{array} \tag{4.3}$$

We follow the convention in the representation theory literature of placing the projectives on the left hand side and $[\tau^{-1}M]$ to the right of a vertex $[M]$ for M indecomposable. It is also customary to draw a dotted arrow from $[N]$ to $[\tau N]$. In this work, we choose not to draw these dotted arrows.

Applying a 45° clockwise rotation around $[1, 1]$ and the reflecting about the axis of injective vertices, we get the diagram

$$\begin{array}{ccccccccc}
[1,n] & \leftarrow & [2,n] & \leftarrow & [3,n] & \leftarrow & \ldots & \leftarrow [n-1,n] \leftarrow & [n,n] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
[1,n-1] & \leftarrow & [2,n-1] & \leftarrow & \ldots & \leftarrow & \ldots & \leftarrow [n-1,n-1] & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
[1,n-2] & \leftarrow & \ldots & \leftarrow & \ldots & \leftarrow & \ldots & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\ldots & \leftarrow & \ldots & \leftarrow & \ldots & & & & \\
\downarrow & & \downarrow & & & & & & \\
[1,2] & \leftarrow & [2,2] & & & & & & \\
\downarrow & & & & & & & & \\
[1,1] & & & & & & & &
\end{array} \tag{4.4}$$

Displayed this way, it is easy to see that the vertices of the AR quiver corresponds to the domain of a (classical) persistence diagram. Using this relationship to the Auslander-Reiten quivers, we extend the definition of persistence diagrams, in a later section. Before that, in the next section we show the computation of the Auslander-Reiten quivers for the representation-finite commutative ladders $CL_n(\tau)$, so that the extended definition can actually be used.

4.2 Representations of the commutative ladders

Motivated by extending the use of persistent homology to be able to extract simultaneously common and robust topological features, we have introduced persistent homology on the commutative ladder quivers, and on quiver complexes in general. In this section, let us apply the general theory reviewed in the previous section to the algebras of the commutative ladder quivers.

Recall that the ladder quiver is given by

$$L_n(\tau) = \begin{array}{ccccccccc}
\overset{1'}{\circ} & \longleftrightarrow & \overset{2'}{\circ} & \longleftrightarrow & \overset{3'}{\circ} & \longleftrightarrow & \ldots & \longleftrightarrow & \overset{n'}{\circ} \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\underset{1}{\circ} & \longleftrightarrow & \underset{2}{\circ} & \longleftrightarrow & \underset{3}{\circ} & \longleftrightarrow & \ldots & \longleftrightarrow & \underset{n}{\circ}
\end{array}$$

where the directions of the pairs of arrows are determined by the entries of τ, and that $CL_n(\tau)$ is the quiver $L(\tau)$ bound by the commutativity relations. We have the following theorem.

Theorem 4.2.1 (cf. [17]). *Let τ be an arbitrary orientation of length n. The commutative ladder quiver $CL_n(\tau)$ is representation-finite if $n \leqslant 4$ and representation-infinite if $n > 4$.*

While Theorem 4.2.1 can be shown to follow from the main result of [26], we believe that our computation of the AR quivers of $CL_n(\tau)$ with $n \leqslant 4$ is worthwhile. The computation provides a list of all the isomorphism classes of the indecomposable representations, via the AR quiver. We then use the AR quiver to extend the definition of persistence diagrams, and show its use in topological data analysis.

4.2.1 Computation of AR quiver

Let us prove the following statement (the representation-finite part of Theorem 4.2.1) by computation of the relevant AR quivers.

> Let τ be an arbitrary orientation of length n. The commutative ladder quiver $CL_n(\tau)$ is representation-finite if $n \leqslant 4$.

For brevity, we only show the computation for the quiver $CL_3(bf)$:

$$
\begin{tikzcd}
\overset{4}{\circ} & \overset{5}{\circ} \arrow[l] \arrow[r] & \overset{6}{\circ} \\
\underset{1}{\circ} \arrow[u] & \underset{2}{\circ} \arrow[l] \arrow[r] \arrow[u] & \underset{3}{\circ} \arrow[u]
\end{tikzcd}
$$

with length $n = 3$ and orientation $\tau = bf$ via the knitting procedure described in Subsection 4.1.3. The computations of the AR quivers of $CL_n(\tau)$ with $n \leqslant 4$ use the same general principles.

In the computation, we abbreviate representations of $CL_3(bf)$ by their dimension vectors. For example, we may write $P(1) = \begin{smallmatrix}1&0&0\\1&0&0\end{smallmatrix}$, where we actually mean $\underline{\dim}\, P(1) = \begin{smallmatrix}1&0&0\\1&0&0\end{smallmatrix}$. We note that in general, two nonisomorphic indecomposable representations may have the same dimension vector. In certain cases, it is possible to infer from the dimension vector what indecomposable representation is meant (up to isomorphism). However, where this notation may cause ambiguity, we shall take care to write out the indecomposable representation.

The AR quiver of $CL_3(bf)$ is given in Fig. 4.1, where the indecomposable representations with dimension vectors $\begin{smallmatrix}1&2&1\\1&1&1\end{smallmatrix}$ and $\begin{smallmatrix}0&1&0\\1&2&1\end{smallmatrix}$ are the representations

$$
\begin{tikzcd}
K & K^2 \arrow[l, "{[1\ 0]}"'] \arrow[r, "{[0\ 1]}"] & K \\
K \arrow[u, "1"] & K \arrow[l, "1"] \arrow[r, "1"'] \arrow[u, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"] & K \arrow[u, "1"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
0 & K \arrow[l] \arrow[r] & 0 \\
K \arrow[u] & K^2 \arrow[l, "{[0\ 1]}"] \arrow[r, "{[1\ 0]}"'] \arrow[u, "{[1\ 1]}"] & K \arrow[u]
\end{tikzcd},
$$

respectively. The computation starts from the indecomposable projectives in the left and works its way towards the indecomposable injectives, in the right of Fig. 4.1. In the discussion below, we refer to column numbers, corresponding to the columns in Fig. 4.1, to guide the reader as to where we are in the computation. The column numbers are written below each column in Fig. 4.1.

To start the computation, we first list all the indecomposable projectives:

$$
\begin{aligned}
P(4) &= \begin{smallmatrix}1&0&0\\0&0&0\end{smallmatrix}, & P(5) &= \begin{smallmatrix}1&1&1\\0&0&0\end{smallmatrix}, & P(6) &= \begin{smallmatrix}0&0&1\\0&0&0\end{smallmatrix}, \\
P(1) &= \begin{smallmatrix}1&0&0\\1&0&0\end{smallmatrix}, & P(2) &= \begin{smallmatrix}1&1&1\\1&1&1\end{smallmatrix}, & P(3) &= \begin{smallmatrix}0&0&1\\0&0&1\end{smallmatrix},
\end{aligned}
$$

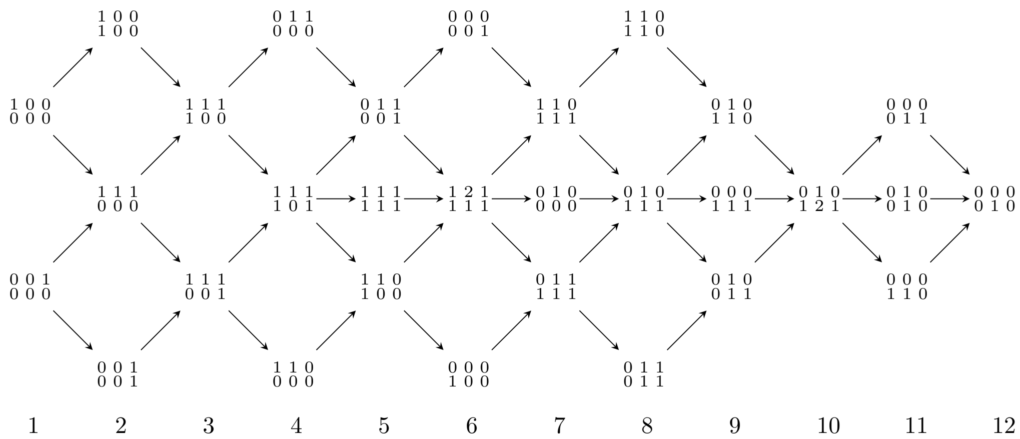


Figure 4.1: The Auslander-Reiten quiver of $CL_3(bf)$.

and their radicals:

$$
\begin{aligned}
\operatorname{Rad} P(1) &= \begin{smallmatrix} 1&0&0\\0&0&0 \end{smallmatrix} = P(4) = S(4),\\
\operatorname{Rad} P(2) &= \begin{smallmatrix} 1&1&1\\1&0&1 \end{smallmatrix},\\
\operatorname{Rad} P(3) &= \begin{smallmatrix} 0&0&1\\0&0&0 \end{smallmatrix} = P(6) = S(6),\\
\operatorname{Rad} P(4) &= 0,\\
\operatorname{Rad} P(5) &= \begin{smallmatrix} 1&0&1\\0&0&0 \end{smallmatrix} = S(4) \oplus S(6),\\
\operatorname{Rad} P(6) &= 0.
\end{aligned}
$$

For convenience, let us also write down the indecomposable injectives:

$$
\begin{aligned}
I(4) &= \begin{smallmatrix} 1&1&0\\1&1&0 \end{smallmatrix}, \quad I(5) = \begin{smallmatrix} 0&1&0\\0&1&0 \end{smallmatrix}, \quad I(6) = \begin{smallmatrix} 0&1&1\\0&1&1 \end{smallmatrix},\\
I(1) &= \begin{smallmatrix} 0&0&0\\1&1&0 \end{smallmatrix}, \quad I(2) = \begin{smallmatrix} 0&0&0\\0&1&0 \end{smallmatrix}, \quad I(3) = \begin{smallmatrix} 0&0&0\\0&1&1 \end{smallmatrix}.
\end{aligned}
$$

We have two simple projectives: $P(4)$ and $P(6)$. Since both $\operatorname{Rad} P(1) = P(4)$ and $\operatorname{Rad} P(5)$ have a direct summand isomorphic to $P(4)$, the almost split sequence starting from $P(4)$ is of the form:

$$
0 \longrightarrow P(4) \longrightarrow P(1) \oplus P(5) \oplus M \longrightarrow \tau^{-1} P(4) \longrightarrow 0 \tag{4.5}
$$

where M is some other module that we need to compute. A minimal injective presentation of $P(4)$ is

$$
0 \longrightarrow P(4) \xrightarrow{\iota} I(4) \longrightarrow I(1) \oplus I(5).
$$

The last term in the above is computed as follows. We have $\operatorname{Coker} \iota = \begin{smallmatrix} 0&1&0\\1&1&0 \end{smallmatrix}$, with socle $S(1) \oplus S(5)$, which has injective envelope $I(1) \oplus I(5)$. Then apply Prop. 2.5.3 part *1* to get an injective envelope for $\operatorname{Coker} \iota$.

Applying the Nakayama functor ν^{-1},

$$0 \longrightarrow \nu^{-1}P(4) \longrightarrow P(4) \longrightarrow P(1) \oplus P(5) \longrightarrow \tau^{-1}P(4) \longrightarrow 0$$

is an exact sequence. Moreover, $\nu^{-1}P(4) = 0$, so that $\tau^{-1}P(4) = \begin{smallmatrix}1&1&1\\1&0&0\end{smallmatrix}$. By a dimension counting argument on the almost split sequence in Eq. (4.5), $M = 0$, and the almost split sequence starting at $P(4)$ is

$$0 \longrightarrow P(4) \longrightarrow P(1) \oplus P(5) \longrightarrow \begin{smallmatrix}1&1&1\\1&0&0\end{smallmatrix} \longrightarrow 0.$$

A similar computation starting from $P(6)$ yields the almost split sequence

$$0 \longrightarrow P(6) \longrightarrow P(3) \oplus P(5) \longrightarrow \begin{smallmatrix}1&1&1\\0&0&1\end{smallmatrix} \longrightarrow 0.$$

This gives the three leftmost columns of Fig. 4.1.

Next, we compute $\tau^{-1}P(1) = \begin{smallmatrix}0&1&1\\0&0&0\end{smallmatrix}$, $\tau^{-1}P(5) = \begin{smallmatrix}1&1&1\\1&0&1\end{smallmatrix}$, and $\tau^{-1}P(3) = \begin{smallmatrix}1&1&0\\0&0&0\end{smallmatrix}$.

We start by computing a minimal injective presentation of $P(1)$. The indecomposable $P(1)$ has socle $S(4)$ with injective envelope $I(4)$, leading to the first term below. Next $\operatorname{Soc}\operatorname{Coker} i_0 = S(5)$, so that the following is a minimal injective presentation of $P(1)$:

$$0 \longrightarrow P(1) \xrightarrow{i_0} I(4) \longrightarrow I(5).$$

From this, we obtain an exact sequence

$$0 \longrightarrow \nu^{-1}P(1) \longrightarrow P(4) \longrightarrow P(5) \longrightarrow \tau^{-1}P(1) \longrightarrow 0.$$

Here, $\nu^{-1}P(1) = 0$, so that $\tau^{-1}P(1)$ has dimension vector $\begin{smallmatrix}0&1&1\\0&0&0\end{smallmatrix}$. Since we already know that there is an irreducible morphism

$$\begin{smallmatrix}1&0&0\\1&0&0\end{smallmatrix} \to \begin{smallmatrix}1&1&1\\1&0&0\end{smallmatrix},$$

the almost split sequence starting from $P(1) = \begin{smallmatrix}1&0&0\\1&0&0\end{smallmatrix}$ is of the form

$$0 \longrightarrow P(1) = \begin{smallmatrix}1&0&0\\1&0&0\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&1&1\\1&0&0\end{smallmatrix} \oplus M \longrightarrow \begin{smallmatrix}0&1&1\\0&0&0\end{smallmatrix} = \tau^{-1}P(1) \longrightarrow 0.$$

By dimension counting it is clear that $M = 0$.

Similar computations give the almost split sequences starting at $P(5)$ and $P(3)$ as:

$$0 \longrightarrow P(5) = \begin{smallmatrix}1&1&1\\0&0&0\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&1&1\\1&0&0\end{smallmatrix} \oplus \begin{smallmatrix}1&1&1\\0&0&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&1&1\\1&0&1\end{smallmatrix} \longrightarrow 0$$

and

$$0 \longrightarrow P(3) = \begin{smallmatrix}0&0&1\\0&0&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&1&1\\0&0&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&1&0\\0&0&0\end{smallmatrix} \longrightarrow 0.$$

From the above arguments, we have obtained the vertices of the four leftmost columns of Fig. 4.1, together with all arrows going into those vertices. Moreover, we are also guaranteed that we have all the arrows starting from the vertices in columns 1 to 3. Next, we compute the arrows starting from the vertices in column 4.

The indecomposable $\begin{smallmatrix}1&1&1\\1&0&1\end{smallmatrix}$ is (isomorphic to) the radical of $P(2) = \begin{smallmatrix}1&1&1\\1&1&1\end{smallmatrix}$. Thus, the inclusion $\begin{smallmatrix}1&1&1\\1&0&1\end{smallmatrix} \to \begin{smallmatrix}1&1&1\\1&1&1\end{smallmatrix}$ is an irreducible morphism. This gives an arrow $\begin{smallmatrix}1&1&1\\1&0&1\end{smallmatrix} \to \begin{smallmatrix}1&1&1\\1&1&1\end{smallmatrix}$. At this stage, we have all the indecomposable projectives, and so we can continue knitting until we get indecomposable injectives.

We compute the following almost split sequences, using the same procedure as above.

1. Up to 5th column, we have almost split sequences

$$0 \longrightarrow \begin{smallmatrix}1&1&1\\1&0&0\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&1&1\\1&0&1\end{smallmatrix} \oplus \begin{smallmatrix}0&1&1\\0&0&0\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&1\\0&0&1\end{smallmatrix} \longrightarrow 0 \ ,$$

$$0 \longrightarrow \begin{smallmatrix}1&1&1\\0&0&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&1&1\\1&0&1\end{smallmatrix} \oplus \begin{smallmatrix}1&1&0\\0&0&0\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&1&0\\1&0&0\end{smallmatrix} \longrightarrow 0 \ ,$$

   and the RMAS

$$\operatorname{Rad} P(2) = \begin{smallmatrix}1&1&1\\1&0&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&1&1\\1&1&1\end{smallmatrix} = P(2) \ .$$

2. Up to the 6th column, the almost split sequences

$$0 \longrightarrow \begin{smallmatrix}0&1&1\\0&0&0\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&1\\0&0&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&0&0\\0&0&1\end{smallmatrix} \longrightarrow 0 \ ,$$

$$0 \longrightarrow \begin{smallmatrix}1&1&0\\0&0&0\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&1&0\\1&0&0\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&0&0\\1&0&0\end{smallmatrix} \longrightarrow 0 \ ,$$

$$0 \longrightarrow \begin{smallmatrix}1&1&1\\1&0&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&1\\0&0&1\end{smallmatrix} \oplus \begin{smallmatrix}1&1&1\\1&1&1\end{smallmatrix} \oplus \begin{smallmatrix}1&1&0\\1&0&0\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&2&1\\1&1&1\end{smallmatrix} \longrightarrow 0$$

   are computed.

Here, we get an indecomposable with a 2-dimension vector space. Let us show how we compute $\tau^{-1}(\begin{smallmatrix}1&1&1\\1&0&1\end{smallmatrix}) = \begin{smallmatrix}1&2&1\\1&1&1\end{smallmatrix}$. We have a minimal injective presentation of $\begin{smallmatrix}1&1&1\\1&0&1\end{smallmatrix}$ given by

$$0 \longrightarrow \begin{smallmatrix}1&1&1\\1&0&1\end{smallmatrix} \xrightarrow{i_0} I(4) \oplus I(6) \xrightarrow{f} I(5) \oplus I(2) \ .$$

The cokernel of the map $i_0 : \begin{smallmatrix}1&1&1\\1&0&1\end{smallmatrix} \to I(4) \oplus I(6)$ has dimension vector given by $\begin{smallmatrix}0&1&0\\0&2&0\end{smallmatrix}$ and is isomorphic to the representation

$$\begin{tikzcd}
0 & K \arrow[l] \arrow[r] & 0 \\
0 \arrow[u] & K^2 \arrow[l] \arrow[r] \arrow[u, "{[1\ 0]}"'] & 0 \arrow[u]
\end{tikzcd} \ ,$$

which is equal to $I(5) \oplus I(2)$. We have $f : I(4) \oplus I(6) \to I(5) \oplus I(2)$ via

$$I(4) \oplus I(6) \longrightarrow \operatorname{Coker} i_0 \cong I(5) \oplus I(2) \ .$$

Thus, we get the exact sequence

$$0 \longrightarrow \nu^{-1}(\begin{smallmatrix}1&1&1\\1&0&1\end{smallmatrix}) \longrightarrow P(4) \oplus P(6) \xrightarrow{\nu^{-1}f} P(2) \oplus P(5) \longrightarrow \tau^{-1}(\begin{smallmatrix}1&1&1\\1&0&1\end{smallmatrix}) \longrightarrow 0$$

by application of ν^{-1}.

In order to get the form of the maps in $\begin{smallmatrix}1&2&1\\1&1&1\end{smallmatrix}$, we must compute $\nu^{-1}f$ and its cokernel. First of all, we write

$$I(4)\oplus I(6) = \begin{tikzcd} K & K^2 \arrow[l, "{[1\ 0]}"'] \arrow[r, "{[0\ 1]}"] & K \\ K \arrow[u, "1"] & K^2 \arrow[l, "{[1\ 0]}"] \arrow[r, "{[0\ 1]}"'] \arrow[u, "{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]}"] & K \arrow[u, "1"] \end{tikzcd},$$

and

$$I(5)\oplus I(2) = \begin{tikzcd} 0 & K \arrow[l] \arrow[r] & 0 \\ 0 \arrow[u] & K^2 \arrow[l] \arrow[r] \arrow[u, "{[1\ 0]}"] & 0 \arrow[u] \end{tikzcd}.$$

The maps of $f : I(4)\oplus I(6) \to I(5)\oplus I(2)$ are given by $f_2 = \left[\begin{smallmatrix}1&-1\\0&1\end{smallmatrix}\right]$, $f_5 = [\,1\ -1\,]$, and $f_i = 0$ elsewhere, in the chosen bases. After a tedious computation, $\nu^{-1}f : P(4)\oplus P(6) \to P(5)\oplus P(2)$ can be shown to be

$$\begin{tikzcd} K \arrow[dddrrr, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"'] & 0 \arrow[l] \arrow[r] & K \arrow[ddrrr, "{\left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]}"] & & & \\ 0 \arrow[u] & 0 \arrow[l] \arrow[r] \arrow[u] & 0 \arrow[u] & & & \\ & & & K^2 & K^2 \arrow[l, "{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]}"'] \arrow[r, "{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]}"'] & K^2 \\ & & & K \arrow[u, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] & K \arrow[l, "1"] \arrow[r, "1"'] \arrow[u, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] & K \arrow[u, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"'] \end{tikzcd}.$$

Computing $\operatorname{Coker}\nu^{-1}f$ and choosing bases, we get

$$\tau^{-1}\left(\begin{smallmatrix}1&1&1\\1&0&1\end{smallmatrix}\right) = \begin{tikzcd} K & K^2 \arrow[l, "{[1\ 0]}"'] \arrow[r, "{[0\ 1]}"] & K \\ K \arrow[u, "1"] & K \arrow[l, "1"] \arrow[r, "1"'] \arrow[u, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"] & K \arrow[u, "1"] \end{tikzcd}.$$

We continue

1. to the 7th column with the almost split sequences

$$0 \longrightarrow \begin{smallmatrix}0&1&1\\0&0&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&0&0\\0&0&1\end{smallmatrix} \oplus \begin{smallmatrix}1&2&1\\1&1&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&1&0\\1&1&1\end{smallmatrix} \longrightarrow 0\ ,$$

$$0 \longrightarrow \begin{smallmatrix}1&1&0\\1&0&0\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&0&0\\1&0&0\end{smallmatrix} \oplus \begin{smallmatrix}1&2&1\\1&1&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&1\\1&1&1\end{smallmatrix} \longrightarrow 0\ ,$$

$$0 \longrightarrow \begin{smallmatrix}1&1&1\\1&1&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&2&1\\1&1&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&0\\0&0&0\end{smallmatrix} \longrightarrow 0\ .$$

2. We get to the 8th column by computing

$$0 \longrightarrow \begin{smallmatrix}0&0&0\\0&0&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&1&0\\1&1&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&1&0\\1&1&0\end{smallmatrix} \longrightarrow 0\ ,$$

$$0 \longrightarrow \begin{smallmatrix}0&0&0\\1&0&0\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&1\\1&1&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&1\\0&1&1\end{smallmatrix} \longrightarrow 0\ ,$$

$$0 \longrightarrow \begin{smallmatrix}1&1&1\\1&2&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&1&0\\1&1&1\end{smallmatrix} \oplus \begin{smallmatrix}0&1&0\\0&0&0\end{smallmatrix} \oplus \begin{smallmatrix}0&1&1\\1&1&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&0\\1&1&1\end{smallmatrix} \longrightarrow 0\ ,$$

3. and then to the 9th column via

$$0 \longrightarrow \begin{smallmatrix}1&1&0\\1&1&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&1&0\\1&1&0\end{smallmatrix} \oplus \begin{smallmatrix}0&1&0\\1&1&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&0\\1&1&0\end{smallmatrix} \longrightarrow 0\ ,$$

$$0 \longrightarrow \begin{smallmatrix}0&1&1\\1&1&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&1\\0&1&1\end{smallmatrix} \oplus \begin{smallmatrix}0&1&0\\1&1&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&0\\0&1&1\end{smallmatrix} \longrightarrow 0\ ,$$

$$0 \longrightarrow \begin{smallmatrix}0&1&0\\0&0&0\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&0\\1&1&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&0&0\\1&1&1\end{smallmatrix} \longrightarrow 0\ .$$

From the 9th column and onwards, we warn that we may get indecomposables M with $\nu^{-1}M$ nonzero.

Going to column 10, note that there are indecomposable injectives at column 8: $I(4) = \begin{smallmatrix}1&1&0\\1&1&0\end{smallmatrix}$ and $I(6) = \begin{smallmatrix}0&1&1\\0&1&1\end{smallmatrix}$. By Prop. 4.1.9, the LMAS starting at $I(4)$ and $I(6)$ are given by $I(4) \to I(4)/\operatorname{Soc} I(4) = \begin{smallmatrix}0&1&0\\1&1&0\end{smallmatrix}$ and $I(6) \to I(6)/\operatorname{Soc} I(6) = \begin{smallmatrix}0&1&0\\0&1&1\end{smallmatrix}$. These are the irreducible morphisms from $I(4)$ and $I(6)$ to indecomposable modules, respectively, and are already accounted for.

So we compute the almost split sequence

$$0 \longrightarrow \begin{smallmatrix}0&1&0\\1&1&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&0\\1&1&0\end{smallmatrix} \oplus \begin{smallmatrix}0&1&0\\1&1&1\end{smallmatrix} \oplus \begin{smallmatrix}0&1&0\\0&1&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&0\\1&2&1\end{smallmatrix} \longrightarrow 0$$

starting at $\begin{smallmatrix}0&1&0\\1&1&1\end{smallmatrix}$. The indecomposable $\begin{smallmatrix}0&1&0\\1&2&1\end{smallmatrix}$ has a dimension vector with entry 2. It can be shown that $\begin{smallmatrix}0&1&0\\1&2&1\end{smallmatrix}$ is isomorphic to the indecomposable representation

$$\begin{tikzcd}
0 & K \arrow[l] \arrow[r] & 0 \\
K \arrow[u] & K^2 \arrow[l, "{[0\ 1]}"] \arrow[r, "{[1\ 0]}"'] \arrow[u, "{[1\ 1]}"] & K \arrow[u]
\end{tikzcd}\ .$$

The remaining few steps are similar. Going to column 11, we compute the almost split sequences

$$0 \longrightarrow \begin{smallmatrix}0&1&0\\1&1&0\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&0\\1&2&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&0&0\\0&1&1\end{smallmatrix} \longrightarrow 0\ ,$$

$$0 \longrightarrow \begin{smallmatrix}0&1&0\\0&1&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&0\\1&2&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&0&0\\1&1&0\end{smallmatrix} \longrightarrow 0\ ,$$

$$0 \longrightarrow \begin{smallmatrix}0&0&0\\1&1&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&0\\1&2&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&1&0\\0&1&0\end{smallmatrix} \longrightarrow 0\ .$$

The final step is to compute the almost split sequence from $\begin{smallmatrix}0&1&0\\1&2&1\end{smallmatrix}$

$$0 \longrightarrow \begin{smallmatrix}0&1&0\\1&2&1\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&0&0\\0&1&1\end{smallmatrix} \oplus \begin{smallmatrix}0&1&0\\0&1&0\end{smallmatrix} \oplus \begin{smallmatrix}0&0&0\\1&1&0\end{smallmatrix} \longrightarrow \begin{smallmatrix}0&0&0\\0&1&0\end{smallmatrix} \longrightarrow 0.$$

At this stage, $I(3) = \begin{smallmatrix}0&0&0\\0&1&1\end{smallmatrix}$, $I(5) = \begin{smallmatrix}0&1&0\\0&1&0\end{smallmatrix}$, $I(1) = \begin{smallmatrix}0&0&0\\1&1&0\end{smallmatrix}$ are injective, with LMAS morphisms $I(3) \to I(2)$, $I(5) \to I(2)$, and $I(1) \to I(2)$ by Prop. 4.1.9. Moreover, $\begin{smallmatrix}0&0&0\\0&1&0\end{smallmatrix} = I(2) = S(2)$ is injective simple and so no LMAS starts at $I(2)$.

Thus, we have found a connected component of $\Gamma(CL_3(bf))$ that is finite. Proposition 4.1.16 shows that this connected component is equal to $\Gamma(CL_3(bf))$, and that $CL_3(bf)$ is representation-finite.

Similar computations show that for $n \leqslant 4$ and any orientation τ, $CL_n(\tau)$ is representation-finite. For fixed n, it is not necessary to check all 2^n possible orientations τ. For example, $CL_3(ff)$ and $CL_3(bb)$ are essentially the same bound quiver after renaming the vertices. Also, $CL_3(fb)$ is isomorphic to the opposite bound quiver $CL_3(bf)^{\mathrm{op}}$. The other AR quivers of the commutative ladders of finite type are given in the Appendix.

4.2.2 Equivalent categories of $\operatorname{rep} CL_n(\tau)$

In this subsection, we take a small detour and rephrase the representation category of the commutative ladder quivers, $\operatorname{rep} CL_n(\tau)$, in terms of other objects. This provides a way to link the commutative ladders to other works in representation theory. Moreover, we provide another proof of Theorem 4.2.1 by [26]. Lemma 4.2.2 is also interesting in its own right, for it serves as the foundation for applying the technique of matrix problems that we will explain in the next chapter.

Lemma 4.2.2. *There is an isomorphism of categories*

$$\operatorname{rep} CL_n(\tau) \cong \operatorname{arr}(\operatorname{rep} A_n(\tau)).$$

Roughly speaking, a representation of $CL_n(\tau)$ can be viewed as a morphism from the bottom row to the top row. The isomorphism should be clear from definition, but let us belabor the point.

Proof. Define a functor $F : \operatorname{rep} CL_n(\tau) \to \operatorname{arr}(\operatorname{rep} A_n(\tau))$ by taking the representation

$$M : \begin{tikzcd} M_1'' \arrow[r, leftrightarrow, "f_1''"] & M_2'' \arrow[r, leftrightarrow, "f_2''"] & \cdots \arrow[r, leftrightarrow, "f_{n-1}''"] & M_n'' \\ M_1' \arrow[u, "g_1"] \arrow[r, leftrightarrow, "f_1'"] & M_2' \arrow[u, "g_2"] \arrow[r, leftrightarrow, "f_2'"] & \cdots \arrow[r, leftrightarrow, "f_{n-1}'"] & M_n' \arrow[u, "g_n"] \end{tikzcd}$$

to the arrow $F(M) = (g : M' \to M'')$, where $M' = (M_i', f_\alpha')$ and $M'' = (M_i'', f_\alpha'')$ are representations of $A_n(\tau)$. Note that $g = (g_i)$ is truly a morphism of representations because of the commutativity relations imposed on M.

Suppose that $\phi : M \to N$ is a morphism in $\operatorname{rep} CL_n(\tau)$ and that $F(M) = (g : M' \to M'')$, $F(N) = (h : N' \to N'')$. Then, ϕ is a collection of maps

$\phi'_i : M'_i \to N'_i$ and $\phi''_i : M''_i \to N''_i$:

$$\begin{tikzcd}
M''_1 \arrow[r, leftrightarrow] \arrow[ddrrr, "\phi''_1"] & M''_2 \arrow[r, leftrightarrow] \arrow[ddrrr, "\phi''_2"] & \cdots \arrow[r, leftrightarrow] & M''_n \arrow[ddrrr, "\phi''_n"] & & & \\
M'_1 \arrow[u, "g_1"] \arrow[r, leftrightarrow] \arrow[ddrrr, "\phi'_1"] & M'_2 \arrow[u, "g_2"] \arrow[r, leftrightarrow] \arrow[ddrrr, "\phi'_2"] & \cdots \arrow[r, leftrightarrow] & M'_n \arrow[u, "g_n"] \arrow[ddrrr, "\phi'_n"] & & & \\
 & & & N''_1 \arrow[r, leftrightarrow] & N''_2 \arrow[r, leftrightarrow] & \cdots \arrow[r, leftrightarrow] & N''_n \\
 & & & N'_1 \arrow[u, "h_1"] \arrow[r, leftrightarrow] & N'_2 \arrow[u, "h_2"] \arrow[r, leftrightarrow] & \cdots \arrow[r, leftrightarrow] & N'_n. \arrow[u, "h_n"]
\end{tikzcd}$$

Define $F(\phi)$ to be the pair $(\phi', \phi'') : F(M) \to F(N)$, where $\phi' = (\phi'_i)$, $\phi'' = (\phi''_i)$. The collections ϕ' and ϕ'' are morphisms of representations of $A_n(\tau)$ since together they form a morphism of representations of $CL_n(\tau)$. That this is a morphism of arrows follows from the fact that

$$\begin{tikzcd}
M'' \arrow[r, "\phi''"] & N'' \\
M' \arrow[u, "g"] \arrow[r, "\phi'"] & N' \arrow[u, "h"']
\end{tikzcd}$$

commutes. The inverse functor G with $FG = 1$ and $GF = 1$ is defined in the obvious way. $\square$

The *triangular matrix algebra* of a finite-dimensional K-algebra A is the algebra

$$T_2(A) = \begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$$

of 2×2 lower triangular matrices with entries in A. Addition and multiplication are defined in the usual way. If $\left[\begin{smallmatrix} a_1 & 0 \\ a_2 & a_3 \end{smallmatrix}\right], \left[\begin{smallmatrix} b_1 & 0 \\ b_2 & b_3 \end{smallmatrix}\right] \in T_2(A)$, then $\left[\begin{smallmatrix} a_1 & 0 \\ a_2 & a_3 \end{smallmatrix}\right] + \left[\begin{smallmatrix} b_1 & 0 \\ b_2 & b_3 \end{smallmatrix}\right] = \left[\begin{smallmatrix} a_1+b_1 & 0 \\ a_2+b_2 & a_3+b_3 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} a_1 & 0 \\ a_2 & a_3 \end{smallmatrix}\right]\left[\begin{smallmatrix} b_1 & 0 \\ b_2 & b_3 \end{smallmatrix}\right] = \left[\begin{smallmatrix} a_1b_1 & 0 \\ a_2b_1+a_3b_2 & a_3b_3 \end{smallmatrix}\right]$. It is clear that $T_2(A)$ is also a finite-dimensional K-algebra.

Then, we have the following.

Proposition 4.2.3. *There is an equivalence of categories*

$$\operatorname{mod} T_2(KA_n(\tau)) \cong \operatorname{rep} CL_n(\tau).$$

Proof. By Lemma 4.2.2 and Theorem 2.3.1,

$$\operatorname{rep} CL_n(\tau) \cong \operatorname{arr}(\operatorname{rep} A_n(\tau)) \cong \operatorname{arr}(\operatorname{mod} KA_n(\tau)).$$

It follows from Proposition III.2.2 in [3] that

$$\operatorname{arr}(\operatorname{mod} A) \cong \operatorname{mod} T_2(A)$$

for any finite-dimensional K-algebra A. Applying this to our case with $A = KA_n(\tau)$, we get the desired result. $\square$

We have rephrased the representation category of the commutative ladders as the module category of a triangular matrix algebra. This gives us the following proof.

Proof of Theorem 4.2.1. By Prop. 4.1.3, $A = KA_n(\tau)$ is a finite-dimensional hereditary K-algebra. By Corollary II.1.11 in [2], A is also a basic algebra. Then, the main theorem of [26] can be applied, which states that $T_2(A)$ is representation-finite if and only if the valued graph of A is a disjoint union of diagrams of the form A_n with $n \leqslant 4$ and B_2. Since $A = KA_n(\tau)$ is given, the valued graph of A is A_n. Thus, $T_2(KA_n(\tau))$ is representation-finite if and only if $n \leqslant 4$. Apply Proposition 4.2.3. □

4.3 Persistence diagrams and AR quivers

4.3.1 Persistence diagram

Recall that any representation V of the quiver $\vec{A}_n$ has an indecomposable decomposition into the interval representations

$$V \cong \bigoplus_{1 \leqslant a \leqslant b \leqslant n} \mathbb{I}[a,b]^{m_{a,b}},$$

which unique up to isomorphism and rearrangement of terms. That is, the collection of numbers $m_{a,b}$, $1 \leqslant a \leqslant b \leqslant n$, is an invariant determining V up to isomorphism. Previously we have considered only $V = H_q(\mathbb{X})$ the persistent homology of some filtration. Here, we take the more general point of view and consider V any representation of $\vec{A}_n$.

As noted before, the persistence diagram of $V \in \operatorname{rep} \vec{A}_n$ is a map

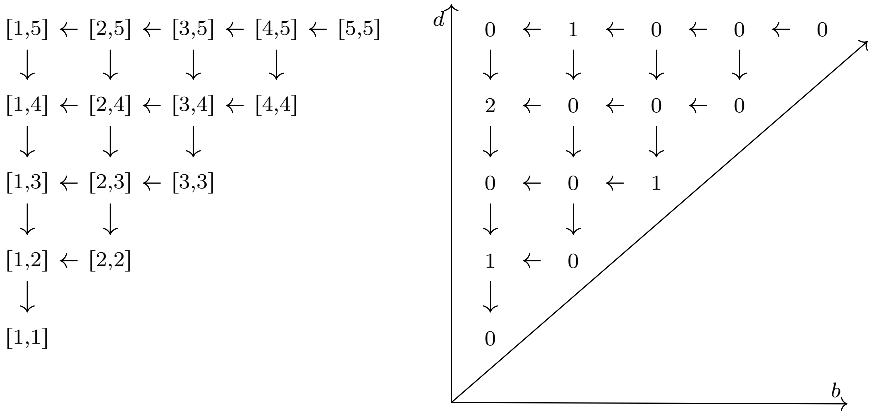
$$\begin{array}{rccc} D_V: & \operatorname{ind} K\vec{A}_n & \to & \mathbb{N}_0 \\ & [a,b] & \mapsto & m_{a,b} \end{array}$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\operatorname{ind} K\vec{A}_n$ is the set of isomorphism classes of indecomposables of $\operatorname{mod} K\vec{A}_n$. Note that $\operatorname{ind} K\vec{A}_n$ is equal to the set of vertices of the Auslander-Reiten quiver $\Gamma(K\vec{A}_n)$.

In Fig. 4.2, we have placed the AR quiver of $K\vec{A}_n$, which is Diagram (4.4) with $n = 5$, side-by-side with the persistence diagram of a representation

$$V \cong \mathbb{I}[1,2] \oplus \mathbb{I}[1,4]^2 \oplus \mathbb{I}[2,5] \oplus \mathbb{I}[3,3],$$

as an example. Compare the presentation in Fig. 4.2b to the way we visualized the persistence diagram in Fig. 2.3.



(a) AR quiver of $K\vec{A}_5$ (b) PD of a representation of $\vec{A}_5$.

Figure 4.2: The AR quiver of $K\vec{A}_5$, and an example persistence diagram of a representation V of $\vec{A}_5$. The domain of the persistence diagram is the vertices of the AR quiver. To emphasize this relationship, we have placed the multiplicity numbers $m_{a,b}$ on the vertices of the AR quiver.

Thus, we propose the following definition.

Definition 4.4 (Persistence Diagram)**.** Let $A = KQ/I$ be the algebra of a bound quiver (Q, ρ) with Auslander-Reiten quiver Γ, and let $V \in \operatorname{rep}(Q, \rho)$. The persistence diagram of V is the map

$$\begin{array}{rccc} D_V : & \Gamma_0 & \to & \mathbb{N}_0 \\ & [I] & \mapsto & m_{[I]}, \end{array}$$

where the numbers $m_{[I]}$ are the multiplicities in an indecomposable decomposition of V,

$$V \cong \bigoplus_{[I]\in\Gamma_0} I^{m_{[I]}}$$

and Γ_0 is the set of vertices of Γ.

This is well-defined, by the uniqueness up to isomorphism of indecomposable decompositions. Moreover, this definition encompasses persistence diagrams of representations of $\vec{A}_n$. Note that the function D_V is an invariant for representations. That is, if $V \cong W$ are two isomorphic representations, then $D_V = D_W$.

Suppose that (Q, ρ) is representation-finite. Similar to how we visualize persistence diagrams of representations of $\vec{A}_n$, we can visualize the persistence diagrams of representations of (Q, ρ) by attaching the multiplicities $m_{[I]}$ to the corresponding vertices $[I]$ in its Auslander-Reiten quiver Γ.

The Auslander-Reiten quiver of $CL_3(fb)$ is shown in Fig. 4.3, where the vertices $\begin{smallmatrix}1&1&1\\1&2&1\end{smallmatrix}$ and $\begin{smallmatrix}1&2&1\\0&1&0\end{smallmatrix}$ correspond to the isomorphism classes of the indecomposable representations

$$\begin{tikzcd} K \arrow[r, "1"] & K & K \arrow[l, "1"'] \\ K \arrow[u, "1"] \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"'] & K^2 \arrow[u, "{[1\ 1]}"] & K \arrow[u, "1"'] \arrow[l, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] \end{tikzcd}, \tag{4.6}$$

and

$$\begin{array}{ccccc} K & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & K^2 & \xleftarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & K \\ {\scriptstyle 0}\uparrow & & {\scriptstyle \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}\uparrow & & {\scriptstyle 0}\uparrow \\ 0 & \xrightarrow[0]{} & K & \xleftarrow[0]{} & 0 \end{array}, \tag{4.7}$$

respectively. The AR quiver $\Gamma(CL_3(fb))$ in Fig. 4.3 can be computed in a similar manner as the computation we have for $\Gamma(CL_3(bf))$ in the previous section.

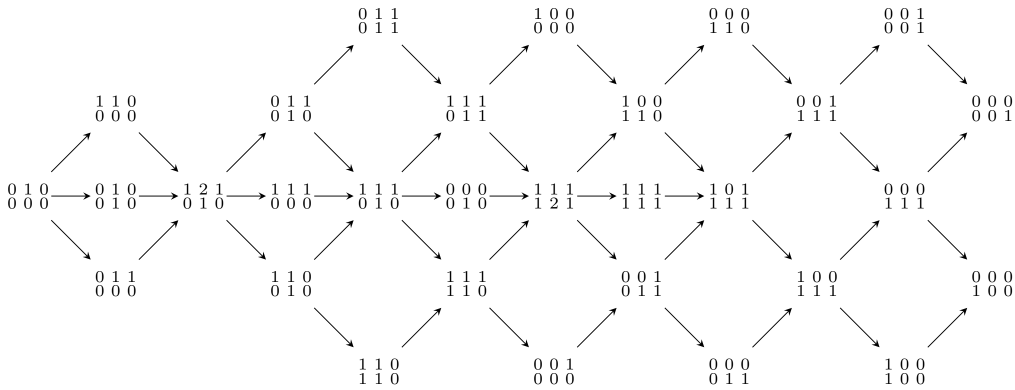


Figure 4.3: The Auslander-Reiten quiver of $CL_3(fb)$.

As an example, the persistence diagram of the representation

$$V \cong \left(\begin{array}{ccccc} K & \xrightarrow{1} & K & \xleftarrow{1} & K \\ {\scriptstyle 1}\uparrow & & {\scriptstyle 1}\uparrow & & {\scriptstyle 1}\uparrow \\ K & \xrightarrow{1} & K & \xleftarrow{1} & K \end{array}\right)^2 \bigoplus \left(\begin{array}{ccccc} K & \xrightarrow{1} & K & \xleftarrow{1} & K \\ {\scriptstyle 1}\uparrow & & {\scriptstyle 1}\uparrow & & {\scriptstyle 0}\uparrow \\ K & \xrightarrow{1} & K & \xleftarrow{0} & 0 \end{array}\right)^3$$

of $CL_3(fb)$ is displayed in Fig. 4.4

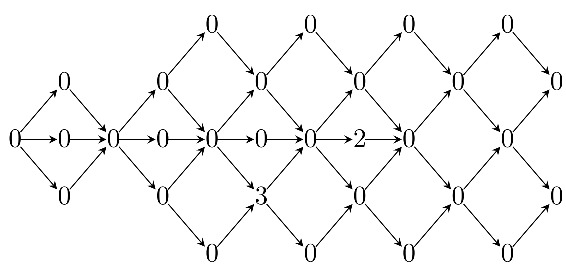


Figure 4.4: An example of the persistence diagram of a representation V of $CL_3(fb)$.

4.3.2 Bottleneck distance

Our definition of the persistence diagram also suggests the following interesting generalization of the ℓ_1-bottleneck distance between persistence diagrams. Let

V be a representation of $\vec{A}_n$, with persistence diagram

$$\begin{array}{rrcl} D_V : & \Gamma_0 & \to & \mathbb{N}_0 \\ & [a,b] & \mapsto & m_{a,b}. \end{array}$$

Equivalently, D_V is a multiset of isomorphism classes of indecomposable representations $[a,b]$ with multiplicity $m_{a,b}$. Define $\bar{D}_V$ to be the multiset D_V together with the elements $[i,i-1]$ of infinite multiplicity for each $i \in \{1,\ldots,n\}$.

Let $V, W \in \operatorname{rep} \vec{A}_n$. The *$\ell_1$-bottleneck distance* between D_V and D_W is defined to be

$$d_B(D_V, D_W) = \inf_{\gamma} \sup_{v \in \bar{D}_V} ||v - \gamma(v)||_1$$

where the infimum is taken over all bijections $\gamma : \bar{D}_V \to \bar{D}_W$ of multisets. For $v = [a_1, b_1]$ and $\gamma(v) = [a_2, b_2]$, define

$$||v - \gamma(v)||_1 = |a_1 - a_2| + |b_1 - b_2|.$$

Let us express the bottleneck distance in terms of the Auslander-Reiten quiver $\Gamma(K\vec{A}_n)$ of $K\vec{A}_n$. In order to have vertices corresponding to the $[i, i-1]$, we create vertices $Z_i = \mathbb{I}[i,i-1]$ and let $\hat{\Gamma}(K\vec{A}_n)$ be the AR quiver $\Gamma(K\vec{A}_n)$ together with additional vertices Z_i and arrows $[i,i] \to Z_i$. The bottleneck distance between D_V and D_W is equal to

$$d_B(D_V, D_W) = \inf_{\gamma} \sup_{v \in \bar{D}_V} d(v, \gamma(v))$$

where $d(v, \gamma(v))$ is defined to be the minimum of the lengths of all undirected paths between the vertices v and $\gamma(v)$ in $\hat{\Gamma}(K\vec{A}_n)$.

Motivated by this formulation, we define the *ℓ_1-bottleneck distance* between persistence diagrams D_V and D_W of $V, W \in \operatorname{rep}(Q, \rho)$, where (Q, ρ) is a representation-finite bound quiver.

Some preparation is needed. First, let $\Gamma(A)$ be the Auslander-Reiten quiver of $A = KQ/I$, the algebra of (Q, ρ). Then, suppose that S_i, $i \in \{1,\ldots,n\}$ is a complete list of simple A-modules, up to isomorphism. Each S_i corresponds to a vertex $[S_i]$ in $\Gamma(A)$. Let $\hat{\Gamma}(A)$ be the quiver $\Gamma(A)$ with additional vertices labeled $[Z_i]$ and additional arrows $[S_i] \to [Z_i]$, for $i \in \{1,\ldots,n\}$. Given the persistence diagram D_V of a $V \in \operatorname{rep}(Q, \rho)$, let $\bar{D}_V$ the the multiset D_V together with elements $[Z_i]$, each of infinite multiplicity, for $i = 1,\ldots,n$. The ℓ_1-bottleneck distance between persistence diagrams D_V and D_W is defined to be

$$d_B(D_V, D_W) = \inf_{\gamma} \sup_{v \in \bar{D}_V} d(v, \gamma(v))$$

where $d(v, \gamma(v))$, the graph distance, is the minimum of the lengths of all undirected paths between the vertices v and $\gamma(v)$ in the underlying graph of $\hat{\Gamma}(A)$.

As a future work, it may be interesting to study the properties of this generalized bottleneck distance d_B. Moreover, the definition we have provided ignores the directions of the arrows in $\hat{\Gamma}(A)$ and works with undirected paths. Replacing $d(v, \gamma(v))$ by the directed graph distance could possibly provide more interesting information.

4.3.3 Interpretations of persistence diagrams

From the side of representation theory, the function from indecomposables to multiplicities determined by a representation is nothing new. It simply expresses the representation as a direct sum of $m_{[I]}$ copies of indecomposable representations I.

By looking at D_V from the point of view of topological data analysis, however, we are able to say more. In the case that A is representation-finite, we easily visualize plot D_V and perform an analysis on the persistence diagram, much like in the classical case. For this, though, we need to use the information about where we got the representation V from.

To be concrete, we work with $CL_3(fb)$, and with the following representation in mind. Let X and Y be two families of simplicial complexes on the same set of vertices V, parametrized over $\mathbb{R}$ so that for parameter values $r < s$, $X_r \subseteq X_s$ and $Y_r \subseteq Y_s$. In this subsection, let us fix the quiver complex $\mathbb{X}$ to be

$$\mathbb{X}: \begin{tikzcd} X_s \arrow[r, hook] & X_s \cup Y_s & Y_s \arrow[l, hook'] \\ X_r \arrow[r, hook] \arrow[u, hook] & X_r \cup Y_r \arrow[u, hook] & Y_r \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}, \tag{4.8}$$

for some fixed parameter values r, s, and where it can be checked that the unions $X_s \cup Y_s$, $X_r \cup Y_r$ are simplicial complexes as well. Consider the representation

$$V = H_q(\mathbb{X}): \begin{tikzcd} H_q(X_s) \arrow[r] & H_q(X_s \cup Y_s) & H_q(Y_s) \arrow[l] \\ H_q(X_r) \arrow[r] \arrow[u] & H_q(X_r \cup Y_r) \arrow[u] & H_q(Y_r) \arrow[l] \arrow[u] \end{tikzcd} \tag{4.9}$$

of $CL_3(fb)$ obtained as the qth persistent homology of $\mathbb{X}$, which can be used for detecting simultaneously robust and common features, as explained in the introduction.

Let us review this interpretation. Restricting to the left and right vertical columns in Diagram (4.9), we can extract features that are persistent in the two-step persistent homology modules

$$H_q(X_r) \xrightarrow{H_q(\iota)} H_q(X_s) \quad \text{and} \quad H_q(Y_r) \xrightarrow{H_q(\iota)} H_q(Y_s)\ .$$

As a shorthand, we call these features robust, for they are persistent in the parameter interval $[r, s]$. Whether they are robust or not in the usual sense of the word depends on the parameter values r and s and the input data.

Similarly, the horizontal direction captures common features between X and Y, at parameter values r and s independently. Taken together, we detect the presence, if any, of simultaneously robust and common features via direct summands of V isomorphic to the indecomposable representation

$$\begin{tikzcd} K \arrow[r, "1"] & K & K \arrow[l, "1"'] \\ K \arrow[r, "1"] \arrow[u, "1"] & K \arrow[u, "1"] & K \arrow[l, "1"'] \arrow[u, "1"] \end{tikzcd}.$$

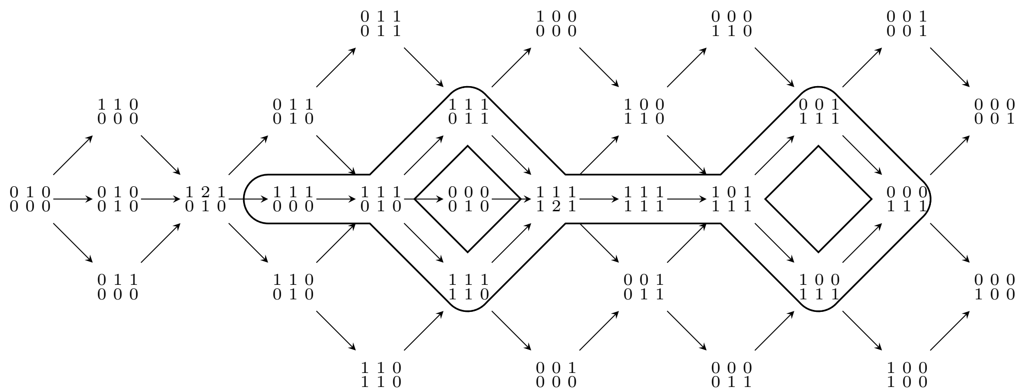


Figure 4.5: Indecomposables with common features between X and Y.

By computing an indecomposable decomposition

$$V \cong \bigoplus_{[I]\in\Gamma_0} I^{m_{[I]}},$$

we get the persistence diagram D_V of V. We postpone the discussion of our algorithm for computing an indecomposable decomposition of V to Section 4.4. Here, we assume we have an indecomposable decomposition and D_V on hand.

Other than counting the number and getting the generators of the simultaneously robust and common features, if any, there is more information we can get using the persistence diagram. In classical persistence, one way of interpreting the intervals that are close to the diagonal is to say that they correspond short-lived topological features and thus are most likely noise. Analogously, we highlight certain regions in $\Gamma(CL_3(fb))$ that deserve more attention.

1. Shared features. In Fig. 4.5 we have marked the region of indecomposables where X and Y share some common feature. Beyond just $\begin{smallmatrix}1&1&1\\1&1&1\end{smallmatrix}$, it may be interesting to study other common features that are less robust.

The indecomposables $\begin{smallmatrix}1&0&1\\1&1&1\end{smallmatrix}$ and $\begin{smallmatrix}1&1&1\\1&2&1\end{smallmatrix}$ are particularly interesting. First, $\begin{smallmatrix}1&0&1\\1&1&1\end{smallmatrix}$ captures the features that were common at parameter value r, but are no longer shared at parameter value s. On the other hand $\begin{smallmatrix}1&1&1\\1&2&1\end{smallmatrix}$ represents features not shared at parameter value r, but come together at s. Indeed, restricting to the lower row, $\begin{smallmatrix}1&2&1\end{smallmatrix}$ decomposes as $\begin{smallmatrix}1&1&0\end{smallmatrix} \oplus \begin{smallmatrix}0&1&1\end{smallmatrix}$.

2. Further decomposition of robust features. Now suppose that we have an understanding of the left space X, and wish to derive some information about Y by comparing it to X. The two-step persistence

$$H_q(X_r) \xrightarrow{H_q(\iota)} H_q(X_s)$$

is a representation of $\vec{A}_2$, with indecomposable decomposition isomorphic to

$$\mathbb{I}[2,2]^{l_2} \oplus \mathbb{I}[1,2]^{l_{12}} \oplus \mathbb{I}[1,1]^{l_1},$$

for some numbers $l_2, l_{12}, l_1 \in \mathbb{N}_0$. Let us relate these persistent features in X to the features of Y, via Fig. 4.6.

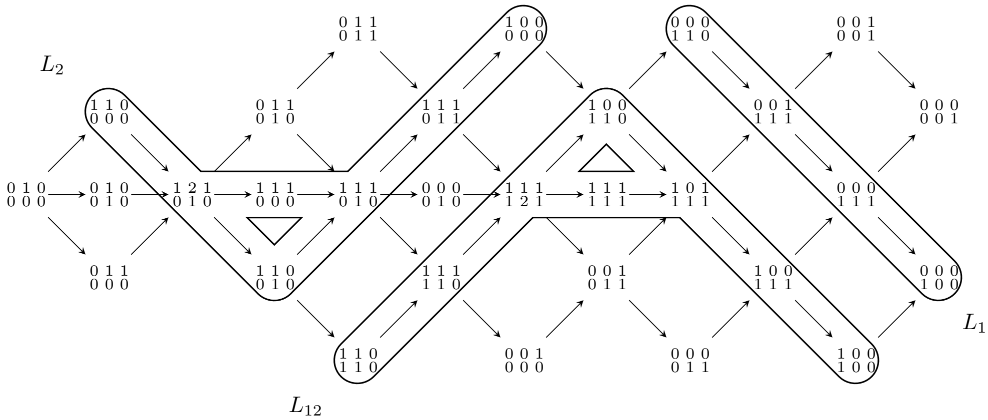


Figure 4.6: Indecomposables with corresponding to different robust features in X

In Fig. 4.6, by looking at the left side of the dimension vectors of the indecomposable representations, we see that the regions L_2, L_{12}, L_1 can be thought of as a further classification of the two-step persistence according to how the persistent features are related to the features of Y. Suppose that $V = H_q(\mathbb{X})$ has indecomposable decomposition

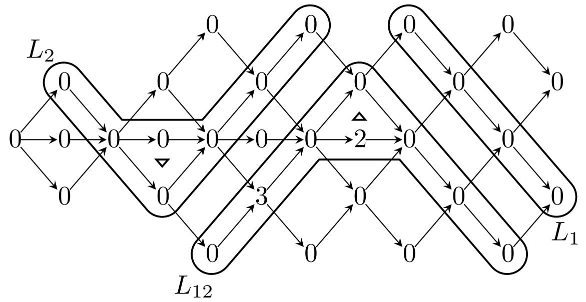
$$V \cong \bigoplus_{[I]\in\Gamma_0} I^{m_{[I]}}.$$

Then, the following equalities hold

$$l_2 = \sum_{[I]\in L_2} m_{[I]}, \quad l_{12} = \sum_{[I]\in L_{12}} m_{[I]}, \quad l_1 = \sum_{[I]\in L_1} m_{[I]}. \tag{4.10}$$

This shows a further classification of the summands $\mathbb{I}[2{,}2]^{l_2}$, $\mathbb{I}[1{,}2]^{l_{12}}$, and $\mathbb{I}[1{,}1]^{l_1}$ according to how they are related to Y.

As an example, suppose that $H_q(\mathbb{X})$ has persistence diagram as given in Fig. 4.4. Let us overlay the marked regions in Fig. 4.6 over the persistence diagram:



By Eq. (4.10), there are 5 robust features in X. As a representation of $\vec{A}_2$, the two-step persistent homology of X decomposes as

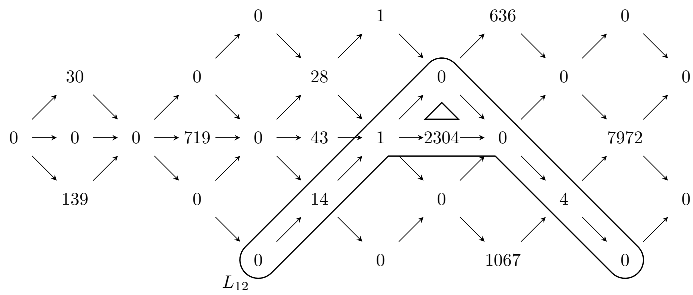
$$\left(\; H_q(X_r) \xrightarrow{H_q(\iota)} H_q(X_s) \; \right) \cong \mathbb{I}[1{,}2]^5.$$

Moreover, by the persistent diagram above, of these 5 persistent features in X, only 2 are common with Y at both parameter values r and s.

As another example, let us apply our newly developed tool to our motivating example of amorphous glass. Recall that the data consists of the atomic configuration of amorphous glass and its atomic configuration after pressurization. We then construct their weighted alpha complex filtrations X and Y, respectively. Note that both atomic configurations contain exactly the same atoms, so that all the simplicial complexes in X and Y are defined on the same set of vertices.

We have argued that the vertical regions C_P in the respective persistence diagrams of X and Y, as shown in Fig. 1.3, contain topological features of interest. The vertical region C_P is then shown to be robust between two parameter values $r = 0$ and $s = 0.36$. The robust features that are preserved under pressurization are given by the simultaneously robust and common features. This brings us to our proposed use of the persistent homology over $CL_3(fb)$.

We compute the qth persistent homology $H_q(\mathbb{X})$ of the quiver complex $\mathbb{X}$ with the given X, Y and $q = 1$. Then, using the algorithm in Section 4.4, we get



as its persistence diagram $D_{H_1(\mathbb{X})}$.

Here, the region L_{12} corresponds to the robust features in X, and corresponds to the features in C_P of X. Using Eq. (4.10), we compute that there are

$$\ell_{12} = 4 + 2304 + 1 + 14 = 2323$$

such features. The vertex $\begin{smallmatrix}1&1&1\\1&1&1\end{smallmatrix}$ corresponds to the simultaneously robust and common features, and in $D_{H_q(\mathbb{X})}$ has multiplicity 2304. Thus, of the 2323 features in C_P, roughly 99.18% ($\approx 2304/2323$) persist under the pressurization.

3. Further decomposition of common features. A similar analysis can be done for the common features. For example, consider the top row of V,

$$V' = \ H_q(X_s) \longrightarrow H_q(X_s \cup Y_s) \longleftarrow H_q(Y_s)\ . \tag{4.11}$$

This restriction gives a representation V' of $A_3(fb)$, which has indecomposable decomposition

$$V' = \bigoplus_I I^{u_I}$$

where I varies over the interval representations of $A_3(fb)$, given by $\mathbb{I}[i,j]$ for $1 \leqslant i \leqslant j \leqslant 3$. We can then give similar equations as in Eq. (4.10). For example,

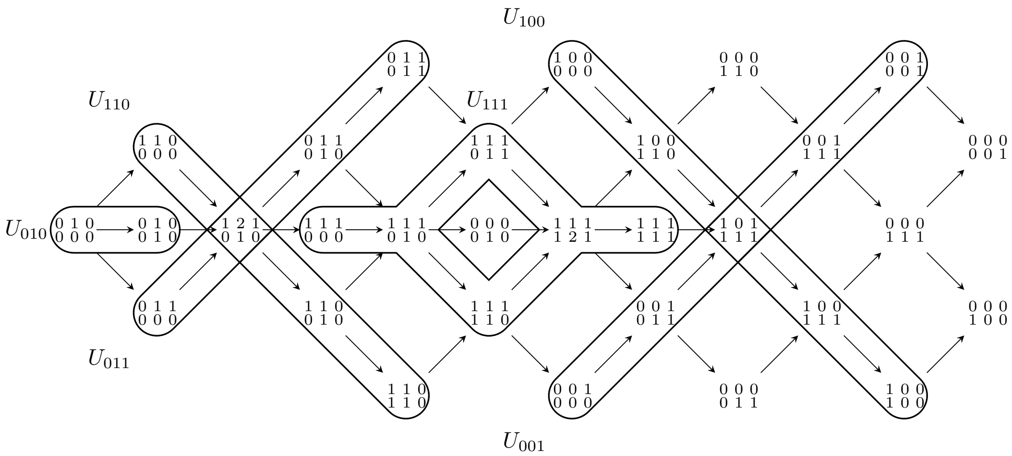


Figure 4.7: Indecomposables corresponding to different types of common features between X and Y at parameter value s

$$u_{\mathbb{I}[1,3]} = \sum_{[I]\in U_{111}} m_{[I]},\ u_{\mathbb{I}[2,3]} = \sum_{[I]\in U_{011}} m_{[I]},\ \text{etc.}$$

Note also that there are overlaps at the vertices $\begin{smallmatrix}1&2&1\\0&1&0\end{smallmatrix}$ and $\begin{smallmatrix}1&0&1\\1&1&1\end{smallmatrix}$. This makes sense, as restricting to the top row,

$$\left(K \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} K^2 \xleftarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} K \right) \cong \left(K \xrightarrow{1} K \xleftarrow{0} 0 \right) \bigoplus \left(0 \xrightarrow{0} K \xleftarrow{1} K \right)$$

and

$$\left(K \xrightarrow{0} 0 \xleftarrow{0} K \right) \cong \left(K \xrightarrow{0} 0 \xleftarrow{0} 0 \right) \bigoplus \left(0 \xrightarrow{0} 0 \xleftarrow{0} K \right)$$

as representations of $A_3(fb)$, so that the vertices $\begin{smallmatrix}1&2&1\\0&1&0\end{smallmatrix}$, $\begin{smallmatrix}1&0&1\\1&1&1\end{smallmatrix}$ each belong to two different corresponding regions.

In the above, we have given three possible ways of interpreting the persistence diagrams of representations of $CL_3(fb)$. Similar analysis can be given for the persistence diagrams of representations of the commutative ladder quivers $CL_n(\tau)$ with $n \leqslant 4$.

Finally, let us note the following. In the discussion above, we highlighted regions in the AR quiver useful for further analysis. Looking only at the persistence diagrams, we only have information about multiplicities of isomorphism classes $[I]$ for I indecomposable, by definition. By referring back to the indecomposable decomposition, it is possible to study the homology generators as well, once we have determined vertices $[I]$ of interest. Suppose that V has an indecomposable decomposition

$$V = \bigoplus_{i=1}^{N} V_i$$

where each V_i is an indecomposable representation. If $[I]$ is a vertex that we have determined to correspond to interesting topological features, we can study the generators of V_i for $V_i \cong I$ in the decomposition above.

4.4 Computation of indecomposable decompositions

In this section, we give our algorithm presented in the paper [17] for computing an indecomposable decomposition, and thus the persistence diagram, of a representation of $CL_3(fb)$. We will skip some of the details. In spite of this, the rest of the algorithm can be filled-in easily, as we discuss its most complicated parts. Moreover, using the same general principles, the strategy of the algorithm can be used to derive similar algorithms for the cases of $CL_n(\tau)$ with $n \leqslant 3$ and arbitrary τ.

Let

$$V: \begin{tikzcd} V^4 \arrow[r, "f_{54}"] & V^5 & V^6 \arrow[l, "f_{56}"'] \\ V^1 \arrow[u, "f_{41}"] \arrow[r, "f_{21}"] & V^2 \arrow[u, "f_{52}"] & V^3 \arrow[l, "f_{23}"'] \arrow[u, "f_{63}"] \end{tikzcd}$$

be a representation of $CL_3(fb)$. We place the indices of the K-vector spaces in V as superscripts. Moreover, contrary to [17], the indices of the morphisms are to be read from right to left. That is,

$$f_{ji} : V^i \to V^j,$$

consistent with our notation for arrows and paths of a quiver.

We compute an indecomposable decomposition

$$V = \bigoplus_{i=1}^{N} M_i \cong \bigoplus_{[I]\in\Gamma_0} I^{m_{[I]}},$$

where each M_i is isomorphic to some I with $[I] \in \Gamma_0$, to get the persistence diagram of V. Here, Γ_0 are the vertices of the AR quiver of $CL_3(fb)$ as given in Fig. 4.3.

Given numbers d_i, $i \in \{1, \dots, 6\}$, let $I(\begin{smallmatrix} d_4 & d_5 & d_6 \\ d_1 & d_2 & d_3 \end{smallmatrix})$ be the (up to isomorphism) indecomposable representation of $CL_3(fb)$ with dimension vector $\begin{smallmatrix} d_4 & d_5 & d_6 \\ d_1 & d_2 & d_3 \end{smallmatrix}$, if one exists. In the case of $CL_3(fb)$, it can be checked by referring to its AR quiver that this notation does cause any ambiguity.

The algorithm is divided into several steps. As a general pattern, each step of the algorithm is then divided into two major parts. The first part extracts a subrepresentation V' specified by a subspace U of a space V^i. In this part, called subspace tracking, we "track" how the subspace is mapped by the maps in the representation. The meaning of this term will become clear as we discuss the algorithm.

The second part, called bases arrangement, chooses the correct bases, so that we get the subrepresentation V' as a direct summand of $V = V' \oplus V''$. We then decompose that direct summand V' into its indecomposable decomposition. This is what we mean by extraction of direct summands. The remaining summand V'' is sent to the next step.

The steps of the algorithm are the following.

1. Track $\operatorname{Ker} f_{21}$ (and $\operatorname{Ker} f_{23}$ by symmetry) and extract the determined direct summand subrepresentation.

2. Track $\operatorname{Ker} f_{41}$ (and $\operatorname{Ker} f_{63}$ by symmetry) and extract the determined direct summand subrepresentation.

3. Track $\operatorname{Ker} f_{54}$ (and $\operatorname{Ker} f_{56}$ by symmetry) and extract the determined direct summand subrepresentation.

4. This step is subdivided into two substeps. Here, we track $\operatorname{Im} f_{63}$, but the indecomposables $I(\begin{smallmatrix} 1&1&1\\1&1&1 \end{smallmatrix})$ and $I(\begin{smallmatrix} 1&1&1\\1&2&1 \end{smallmatrix})$ need to be treated separately.

5. The remaining representation is a representation of the D_4 quiver

$$\begin{tikzcd} \circ \arrow[r] & \circ & \circ \arrow[l] \\ & \circ \arrow[u] & \end{tikzcd} .$$

   A procedure similar to steps 1 to 4 extracts an indecomposable decomposition for the representation.

For ease of notation, we shall identify the maps $f_{ji} : V^i \to V^j$ with their matrix forms in the current basis.

Step 1.

S1. Corresponding to a basis change on V^1, we can write

$$f_{21} = \left[\begin{array}{c|c} \mathbf{0} & * \end{array}\right] : V^1 = U_1^1 \oplus X^1 \to V^2$$

by transforming f_{21} to a column-echelon form. Here, U_1^1 is the kernel of f_{21}, and is the subspace we would like to track.

S2. By basis change on V^4, the matrix of f_{41} becomes

$$f_{41} = \left[\begin{array}{c|c} * & * \\ \hline \mathbf{0} & * \end{array}\right] : U_1^1 \oplus X^1 \to V_{14}^4 \oplus X^4 = V^4,$$

in a row-echelon form. Here, $V_{14}^4 = f_{41}(U_1^1)$ and so the horizontal line is placed after the ℓth row, where ℓ is determined as the largest ℓ such that the left $\dim U_1^1$ entries of the ℓth row are nonzero. There is no need to track further, since $f_{54}(V_{14}^4) = f_{54} f_{41}(U_1^1) = f_{52} f_{21}(U_1^1) = 0$.

Roughly speaking, we have tracked $\operatorname{Ker} f_{21}$ through the spaces V^1 and V^4. In the next part we start choosing proper bases to get a direct summand.

B1. A basis change on U_1^1 gives f_{41} the matrix form

$$f_{41} = \left[\begin{array}{cc|c} \mathbf{1} & \mathbf{0} & * \\ \hline \mathbf{0} & \mathbf{0} & * \end{array}\right] : V_{14}^1 \oplus V_1^1 \oplus X^1 \to V_{14}^4 \oplus X^4,$$

by computing a column-echelon form for the upper-left submatrix. Here, $\mathbf{1}$ is an identity matrix of appropriate size. This basis change does not change the form of f_{21} at all.

Finally, by the basis change of

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & -S \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

on V^1, where S is the upper-right submatrix of f_{41}, f_{41} is now in the form

$$f_{41} = \left[\begin{array}{c|c|c} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & * \end{array}\right] : V_{14}^1 \oplus V_1^1 \oplus Y^1 \to V_{14}^4 \oplus X^4 = V^4,$$

while f_{21} is now in the form

$$f_{21} = \left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & * \end{array}\right] : V^1 = V_{14}^1 \oplus V_1^1 \oplus Y^1 \to V^2.$$

Let $m_1 = \dim V_1^1$ and $m_{1,4} = \dim V_{14}^1 = \dim V_{14}^4$. In the chosen bases, we have the following direct summands of V:

$$M_1 = \begin{tikzcd} 0 \arrow[r] & 0 & 0 \arrow[l] \\ V_1^1 \arrow[u, "\mathbf{0}"] \arrow[r, "\mathbf{0}"] & 0 \arrow[u] & 0 \arrow[l] \arrow[u] \end{tikzcd} \cong I\left(\begin{smallmatrix}0&0&0\\1&0&0\end{smallmatrix}\right)^{m_1}$$

and

$$M_{1,4} = \begin{tikzcd} V_{14}^4 \arrow[r] & 0 & 0 \arrow[l] \\ V_{14}^1 \arrow[u, "\mathbf{1}"] \arrow[r, "\mathbf{0}"] & 0 \arrow[u] & 0 \arrow[l] \arrow[u] \end{tikzcd} \cong I\left(\begin{smallmatrix}1&0&0\\1&0&0\end{smallmatrix}\right)^{m_{1,4}}.$$

By symmetry, we can track $\operatorname{Ker} f_{23}$ and get direct summands

$$M_3 = \begin{tikzcd} 0 \arrow[r] & 0 & 0 \arrow[l] \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] & V_3^3 \arrow[l, "\mathbf{0}"'] \arrow[u, "\mathbf{0}"'] \end{tikzcd} \cong I\left(\begin{smallmatrix}0&0&0\\0&0&1\end{smallmatrix}\right)^{m_3}$$

and

$$M_{3,6} = \begin{tikzcd} 0 \arrow[r] & 0 & V_{36}^6 \arrow[l] \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] & V_{36}^3 \arrow[l, "\mathbf{0}"] \arrow[u, "\mathbf{1}"'] \end{tikzcd} \cong I\left(\begin{smallmatrix}0&0&1\\0&0&1\end{smallmatrix}\right)^{m_{3,6}}.$$

Thus, $V = V' \bigoplus M_1 \bigoplus M_{1,4} \bigoplus M_3 \bigoplus M_{3,6}$, and V' has no direct summands isomorphic to $I\left(\begin{smallmatrix}0&0&0\\1&0&0\end{smallmatrix}\right)$, $I\left(\begin{smallmatrix}1&0&0\\1&0&0\end{smallmatrix}\right)$, $I\left(\begin{smallmatrix}0&0&0\\0&0&1\end{smallmatrix}\right)$, nor $I\left(\begin{smallmatrix}0&0&1\\0&0&1\end{smallmatrix}\right)$. The representation V' gets sent to the next step.

Step 2. At this stage, f_{21} and f_{23} are monomorphisms. Using similar arguments as above, by tracking $\operatorname{Ker} f_{41}$ through the spaces V^1, V^2, V^3, V^6, we get all the direct summands of V isomorphic to

$$I(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{smallmatrix}),\ I(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{smallmatrix}),\ I(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix}).$$

By symmetry, tracking $\operatorname{Ker} f_{23}$ we get all the direct summands of V isomorphic to

$$I(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{smallmatrix}),\ I(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{smallmatrix}),\ I(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{smallmatrix}).$$

Of course, at this stage, the direct summands isomorphic to $I(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{smallmatrix})$ have already been extracted and so will not appear again.

Step 3. Now, f_{21}, f_{23}, f_{41}, and f_{63} are monomorphisms. In this step, we track $\operatorname{Ker} f_{54}$ through the spaces V^4, V^1, V^2, V^3, V^6 (and $\operatorname{Ker} f_{56}$ by symmetry) to get indecomposable direct summands of V isomorphic to

$$I(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}), I(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{smallmatrix}), I(\begin{smallmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix}), I(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix}), \text{ and } I(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}).$$

Step 4a. At this stage, all maps except f_{52} are monomorphisms. Let us track $\operatorname{Im} f_{63}$. This step is split into two parts, because we treat the indecomposables $I(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix})$ and $I(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{smallmatrix})$ separately.

S1. By a change of basis on V^6, we get a row-echelon form on f_{63}:

$$f_{63} = \left[\begin{array}{c} \mathbf{1} \\ \hline \mathbf{0} \end{array}\right] : V^3 \to V^6 = U^6_{36} \oplus X^6.$$

Here, $U^6_{36} = \operatorname{Im} f_{63}$ and is the subspace of interest. We can get an identity submatrix because f_{63} is a monomorphism.

S2. On V^5, row-echelon form:

$$f_{56} = \left[\begin{array}{c|c} \mathbf{1} & * \\ \hline \mathbf{0} & * \end{array}\right] : U^6_{36} \oplus X^6 \to V^5 = U^5_{356} \oplus X^5.$$

Note that f_{56} is a monomorphism, too.

S3. On V^4, column-echelon form:

$$f_{54} = \left[\begin{array}{c|c} * & * \\ \hline \mathbf{0} & * \end{array}\right] : V^4 = U^4_{3456} \oplus X^4 \to U^5_{356} \oplus X^5.$$

S4. On V^1, column-echelon form:

$$f_{41} \left[\begin{array}{c|c} * & * \\ \hline \mathbf{0} & * \end{array}\right] : V^1 = U^1_{13456} \oplus X^1 \to U^4_{3456} \oplus X^4.$$

So far, everything is similar to steps 1 through 3. However, we need to be careful to avoid the indecomposables $I(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix})$ and $I(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{smallmatrix})$.

B1. Perform row operations on f_{41} to change it to the form:

$$f_{41} = \left[\begin{array}{c|c} \mathbf{0} & * \\ \hline \mathbf{1} & * \\ \hline \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{array}\right] : U^1_{13456} \oplus X^1 \to V^4_{23456} \oplus U^4_{13456} \oplus W^4.$$

That is, we do basis changes on U^4_{3456} and X^4, independently of each other. Then, by adding multiples of the $\mathbf{1}$ in the lower-right corner, we can zero out the entries above it, and get:

$$f_{41} = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{array}\right] : U^1_{13456} \oplus X^1 \to V^4_{23456} \oplus U^4_{13456} \oplus Z^4.$$

The effect on the matrix form of f_{54} is that it is now:

$$f_{54} = \left[\begin{array}{c|c|c} S_1 & S_2 & * \\ \hline \mathbf{0} & \mathbf{0} & * \end{array}\right] : V^4 = V^4_{23456} \oplus U^4_{13456} \oplus Z^4 \to U^5_{356} \oplus X^5,$$

where it can be checked that the columns of S_1 and S_2 are linearly independent.

In this step, we do not yet extract the spaces U^*_{13456}, as they are involved in the indecomposables $I(\begin{smallmatrix}1&1&1\\1&1&1\end{smallmatrix})$ and $I(\begin{smallmatrix}1&1&1\\1&2&1\end{smallmatrix})$.

B2. By basis changes on U^5_{356} and X^5, independent of each other, we now have

$$f_{54} = \left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & T_1 \\ \hline \mathbf{1} & \mathbf{0} & T_2 \\ \hline \mathbf{0} & \mathbf{1} & T_3 \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array}\right] : V^4_{23456} \oplus U^4_{13456} \oplus Z^4 \to V^5_{2356} \oplus V^5_{23456} \oplus U^5_{13456} \oplus W^5.$$

Then the basis change by

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -T_1 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & -T_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & -T_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}^{-1}$$

on V^5 gives

$$f_{54} = \left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array}\right] : V^4_{23456} \oplus U^4_{13456} \oplus Z^4 \to V^5_{2356} \oplus V^5_{23456} \oplus U^5_{13456} \oplus Z^5.$$

B3. The matrix form of f_{56} is now

$$f_{56} = \left[\begin{array}{c|c} * & * \\ \hline * & * \\ \hline * & * \\ \hline \mathbf{0} & * \end{array}\right] : U_{36}^6 \oplus X^6 \to V_{2356}^5 \oplus V_{23456}^5 \oplus U_{13456}^5 \oplus Z^5,$$

Note that at step S2 above, we have $U_{36}^6 \cong U_{356}^5$, and $U_{356}^5 \cong V_{2356}^5 \oplus V_{23456}^5 \oplus U_{13456}^5$. Thus, $U_{36}^6 \cong V_{2356}^5 \oplus V_{23456}^5 \oplus U_{13456}^5$ and we can perform a change of basis on V^6 to get:

$$f_{56} = \left[\begin{array}{c|c|c|c} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & * \end{array}\right] : \quad V_{2356}^6 \oplus V_{23456}^6 \oplus U_{13456}^6 \oplus Z^6 \to V_{2356}^5 \oplus V_{23456}^5 \oplus U_{13456}^5 \oplus Z^5.$$

B4. Basis changes on V^3, and then on V^2 turns f_{63} and f_{23} into similar forms. In these chosen bases, the matrices of V are now in the forms:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
V^4 \arrow[r, "\left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & * \end{array}\right]"] & V^5 & V^6 \arrow[l, "\left[\begin{array}{c|c|c|c} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & * \end{array}\right]"'] \\
V^1 \arrow[u, "\left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & * \end{array}\right]"] \arrow[r] & V^2 \arrow[u, "f_{52}"] & V^3 \arrow[l, "\left[\begin{array}{c|c|c} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]"'] \arrow[u, "\left[\begin{array}{c|c|c} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]"']
\end{tikzcd}$$

By commutativity, it can be checked that in the chosen bases,

$$f_{52} = \left[\begin{array}{c|c|c|c} \mathbf{1} & \mathbf{0} & \mathbf{0} & S_1 \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} & S_2 \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{1} & S_3 \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & * \end{array}\right] : \quad V_{2356}^2 \oplus V_{23456}^2 \oplus U_{13456}^2 \oplus X^2 \to V_{2356}^5 \oplus V_{23456}^5 \oplus U_{13456}^5 \oplus Z^5.$$

By a change of basis on V^2, we can obtain the form

$$f_{52} = \left[\begin{array}{c|c|c|c} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & * \end{array}\right] : \quad V_{2356}^2 \oplus V_{23456}^2 \oplus U_{13456}^2 \oplus Z^2 \to V_{2356}^5 \oplus V_{23456}^5 \oplus U_{13456}^5 \oplus Z^5$$

and this does not affect the form of f_{32}. Again by commutativity,

$$f_{21} = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \\ \hline \mathbf{1} & \mathbf{0} \\ \hline * & * \end{array}\right] : U^1_{13456} \oplus X^1 \to V^2_{2356} \oplus V^2_{23456} \oplus U^2_{13456} \oplus Z^2.$$

We do not need U^*_{13456} at this stage. In the chosen bases, the matrices are in the forms:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
V^4 \arrow[r, "\left[\begin{array}{c||c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline\hline \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & * \end{array}\right]"] & V^5 & V^6 \arrow[l, swap, "\left[\begin{array}{c|c||c|c} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline\hline \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & * \end{array}\right]"] \\
V^1 \arrow[u, "\left[\begin{array}{c} \mathbf{0} \\ \hline\hline * \\ \hline * \end{array}\right]"] \arrow[r, swap, "\left[\begin{array}{c} \mathbf{0} \\ \hline \mathbf{0} \\ \hline\hline * \\ \hline * \end{array}\right]"] & V^2 \arrow[u, swap, "\left[\begin{array}{c|c||c|c} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline\hline \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & * \end{array}\right]"] & V^3 \arrow[u, swap, "\left[\begin{array}{c|c||c} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline\hline \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]"] \arrow[l, "\left[\begin{array}{c|c||c} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline\hline \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]"]
\end{tikzcd}$$

where we emphasize the direct summands that we extract by the double lines. These are:

$$\begin{tikzcd}
0 \arrow[r, "0"] & V^5_{2356} & V^6_{2356} \arrow[l, swap, "\mathbf{1}"] \\
0 \arrow[u, "0"] \arrow[r, "0"] & V^2_{2356} \arrow[u, "\mathbf{1}"] & V^3_{2356} \arrow[u, "\mathbf{1}"] \arrow[l, swap, "\mathbf{1}"]
\end{tikzcd} \cong I\left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right)^{m_{2,3,5,6}}$$

and

$$\begin{tikzcd}
V^4_{23456} \arrow[r, "\mathbf{1}"] & V^5_{23456} & V^6_{23456} \arrow[l, swap, "\mathbf{1}"] \\
0 \arrow[u, "0"] \arrow[r, "0"] & V^2_{23456} \arrow[u, "\mathbf{1}"] & V^3_{23456} \arrow[u, "\mathbf{1}"] \arrow[l, swap, "\mathbf{1}"]
\end{tikzcd} \cong I\left(\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right)^{m_{2,3,4,5,6}}.$$

By symmetry, we also get indecomposable summands isomorphic to $I\left(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{smallmatrix}\right)$ and $I\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{smallmatrix}\right)$.

Step 4b. In step 4b, some special arguments are needed to deal with the indecomposables $I\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right)$ and $I\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{smallmatrix}\right)$. Note that all maps except f_{52} are monomorphisms. This allows us to obtain the appropriate identity submatrices, denoted $\mathbf{1}$, in the steps below.

S1. By a change of basis on V^4, we get a row echelon form

$$f_{41} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} : V^1 \to V^4 = U^4_{14} \oplus X^4.$$

S3. Perform a change of basis on V^5 to get

$$f_{54} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & * \end{bmatrix} : V^4 = U^4_{14} \oplus X^4 \to V^5 = U^5_{145} \oplus X^5$$

via row operations.

S3. Perform a change of basis on V^6 (corresponding to the appropriate column operations on f_{56}) to obtain the matrix form

$$f_{56} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & * \end{bmatrix} : V^6 = U^6_{1456} \oplus X^6 \to V^5 = U^5_{145} \oplus X^5.$$

Note that we can obtain a submatrix $\mathbf{1}$ in the upper left of f_{56}, since the direct summands isomorphic to $I(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{smallmatrix})$ have all been removed in the previous step.

S4. Perform a change of basis on V^3 so that we get the form

$$f_{63} = \left[\begin{array}{c} \mathbf{1} \\ \hline \mathbf{0} \end{array}\right] : V^3 \to U^6_{1456} \oplus X^6.$$

Here, we can obtain a submatrix $\mathbf{1}$ in f_{63}, since the direct summands isomorphic to $I(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{smallmatrix})$ have all been removed.

The above matrix forms provides a sequence of isomorphisms

$$V^1 \cong U^4_{14} \cong U^5_{145} \cong U^6_{1456} \cong V^3,$$

showing that $\dim V^1 = \dim V^3$. We denote this dimension by s.

Next, we need to choose the correct bases to extract direct summands isomorphic to $I(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix})$ and $I(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{smallmatrix})$, where the indecomposable representation $I(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{smallmatrix})$ is given in Eq. (4.6). Denote the columns in the matrix forms of f_{21} and f_{23} by

$$f_{21} = \begin{bmatrix} a_1 & \cdots & a_s \end{bmatrix} \text{ and } f_{23} = \begin{bmatrix} b_1 & \cdots & b_s \end{bmatrix},$$

where a_i and b_i are the columns of f_{21} and f_{23}, respectively. By commutativity in $CL_3(fb)$ and the bases chosen in the subspace tracking, it can be checked that $f_{52}(a_i) = f_{52}(b_i) \in V^5$ for $i \in \{1, \dots, s\}$ and that these vectors in V^5 are linearly independent.

Define the matrix

$$C = \begin{bmatrix} a_1 & b_1 & a_2 & b_2 & \cdots & a_s & b_s \end{bmatrix}$$

by interleaving the columns of the matrix forms of f_{21} and f_{23}. In the following induction, we transform C into the form

$$\hat{C} = \left[\begin{array}{ccc|cccc} 1 & & & & & & \\ & \ddots & & & \mathbf{0} & & \\ & & 1 & & & & \\ \hline & & & 1 & 1 & & \\ & \mathbf{0} & & & & \ddots & \\ & & & & & 1 & 1 \\ \hline & \mathbf{0} & & & \mathbf{0} & & \end{array}\right] \tag{4.12}$$

by row operations and certain column operations, with a $2s_1 \times 2s_1$ identity matrix in the upper left and an $s_2 \times 2s_2$ matrix in the middle right for some $s_1, s_2 \in \mathbb{N}_0$ with $s = s_1 + s_2$. In transforming C to the form $\hat{C}$, only column operations on pairs of columns c_i, c_j in $C = [c_1 \ldots c_{2s}]$ with $i = j(\text{mod } 2)$ are permissible. That is, we do not to mix the columns of f_{21} and f_{23}.

Note that a row operation on C corresponds to a basis change on V^2. The above permissible column operations ensure that each column operation corresponds to a basis change for either V^1 or V^3. In the bases chosen via the transformation of C to $\hat{C}$, the matrices of f_{21} and f_{23} will taken on their corresponding forms. That is, f_{21} now has matrix form consisting of the odd-numbered columns of $\hat{C}$, while f_{23} has matrix form consisting of the even-numbered columns of $\hat{C}$.

Of course, the matrix forms of f_{41}, f_{54}, f_{56}, f_{63} already obtained in the subspace tracking in steps S1 to S4 may change under this process. In each induction step, we also perform changes of bases on V^4, V^5, V^6 to restore them to the forms obtained in the subspace tracking.

To compress the notation, define the $nk \times mk$ matrix $D_k(M)$, given an $n \times m$ matrix M, as the matrix

$$D_k(M) = \begin{bmatrix} M & & & \\ & M & & \\ & & \ddots & \\ & & & M \end{bmatrix}$$

obtained by placing k copies of M along the block diagonal. Then,

$$\hat{C} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & D_{s_2}([\begin{array}{cc} 1 & 1 \end{array}]) \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where the $\mathbf{1}$ identity submatrix in the upper left is of size $2s_1$.

To begin each induction step, assume that the matrix C is in the form

$$C = \begin{bmatrix} C' & * \\ \mathbf{0} & * \end{bmatrix}$$

where C' is

$$C' = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & D_{\ell_2}([\begin{array}{cc} 1 & 1 \end{array}]) \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

and the $\mathbf{1}$ identity submatrix in the upper left is of size $2\ell_1$. Clearly, the matrix C' is of size $m \times 2\ell$, where $\ell = \ell_1 + \ell_2$ and $m = 2\ell_1 + \ell_2$.

As the inductive step, we show that the number of columns of C' can be extended from 2ℓ to $2\ell + 2$ without affecting the matrix forms of $f_{41}, f_{54}, f_{56}, f_{63}$. As for the number of rows of C', it either becomes $m + 1$ or $m + 2$.

The column in C adjacent to C' is the column $c_{2\ell+1}$ and corresponds to the $(\ell + 1)$st column $a_{\ell+1}$ of f_{12}. If all the entries $a_{i,\ell+1}$ of the column $a_{\ell+1}$ are 0 for $i > m$, then $a_{\ell+1}$ can be expressed as a linear combination

$$a_{\ell+1} = \sum_{i=1}^{\ell_1} (\alpha_i a_i + \beta_i b_i) + \sum_{i=\ell_1+1}^{\ell} \alpha_i a_i$$

for some $\alpha_i, \beta_i \in K$, due to the current form of the matrix C. Applying f_{52} and by linearity,

$$f_{52}(a_{\ell+1}) = \sum_{i=1}^{\ell_1} (\alpha_i + \beta_i)\, f_{52}(a_i) + \sum_{i=\ell_1+1}^{\ell} \alpha_i f_{52}(a_i).$$

This contradicts the fact that $f_{52}(a_1), \ldots, f_{52}(a_{\ell+1})$ are linearly independent.

Hence there exists a nonzero entry $a_{i,\ell+1}$ for some $i > m$ in the column $a_{\ell+1}$. After suitable row operations, we can transform C to the form

$$C = \begin{array}{cc} \begin{array}{ccc} \qquad & (2\ell+1)\text{st} & \qquad \end{array} & \\ \left[\begin{array}{c|c|c} C' & \mathbf{0} & * \\ \hline \mathbf{0} & 1 & * \\ \hline \mathbf{0} & \mathbf{0} & * \end{array}\right] & \begin{array}{l} \\ (m+1)\text{st} \\ \\ \end{array} \end{array}.$$

The next adjacent column is $c_{2\ell+2}$ of C, corresponding to the column $b_{\ell+1}$ of f_{23}. Here, there are two possible cases, giving the cases where the number of rows of C' either extends to $m + 2$ or $m + 1$, respectively.

Case 1: If the column $b_{\ell+1}$ has a nonzero element $b_{i,\ell+1}$ for some $i > m + 1$, then C can be transformed into

$$C = \begin{array}{cc} \left[\begin{array}{c|cc|c} C' & \mathbf{0} & \mathbf{0} & * \\ \hline \mathbf{0} & 1 & 0 & * \\ \mathbf{0} & 0 & 1 & * \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & * \end{array}\right] & \begin{array}{l} \\ (m+1)\text{st} \\ (m+2)\text{nd} \\ \\ \end{array} \end{array}$$

by suitable row operations. Appropriate column and row permutations finish the induction step.

Case 2: Otherwise, $b_{i,\ell+1} = 0$ for all $i > m + 1$. That is, all row entries in column $b_{\ell+1}$ below row $m + 1$ are 0. Again, by referring to the current form of C, the column vector $b_{\ell+1}$ can be expressed as a linear combination

$$b_{\ell+1} = \sum_{i=1}^{\ell_1}(\alpha_i a_i + \beta_i b_i) + \sum_{i=\ell_1+1}^{\ell} \alpha_i a_i + \alpha_{\ell+1} a_{\ell+1}$$

for some $\alpha_i, \beta_i \in K$. Mapping both sides to V^5 by f_{52} leads to

$$(1-\alpha_{\ell+1})f_{52}(a_{\ell+1}) = \sum_{i=1}^{\ell_1}(\alpha_i+\beta_i)f_{52}(a_i) + \sum_{i=\ell_1+1}^{\ell} \alpha_i f_{52}(a_i).$$

By linear independence, all the coefficients above must be 0, so that

$$\alpha_{\ell+1} = 1, \qquad \alpha_i = \begin{cases} -\beta_i, & i = 1,\dots,\ell_1, \\ 0, & i = \ell_1+1,\dots,\ell. \end{cases}$$

Therefore, the matrix C has the form

$$C = \left[\begin{array}{c|c|c|c} & & \alpha_1 & \\ & & -\alpha_1 & \\ C' & \mathbf{0} & \vdots & * \\ & & \alpha_{\ell_1} & \\ & & -\alpha_{\ell_1} & \\ & & \mathbf{0} & \\ \hline \mathbf{0} & 1 & 1 & * \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & * \end{array}\right]$$

By taking the odd-numbered columns of C for f_{21} and the even-numbered columns of C for f_{23}, we see that f_{21} and f_{23} now have matrix forms

$$f_{21} = \begin{array}{c} \begin{array}{cccc} & \phantom{\mathbf{0}} & {\scriptstyle (\ell+1)\mathrm{st}} & \end{array} \\ \left[\begin{array}{c|c|c|c} D_a & \mathbf{0} & \mathbf{0} & * \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} & * \\ \hline \mathbf{0} & \mathbf{0} & 1 & * \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & * \end{array}\right] \end{array}, \quad f_{23} = \begin{array}{c} \begin{array}{cccc} & \phantom{\mathbf{0}} & {\scriptstyle (\ell+1)\mathrm{st}} & \end{array} \\ \left[\begin{array}{c|c|c|c} D_b & \mathbf{0} & \alpha_\pm & * \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} & * \\ \hline \mathbf{0} & \mathbf{0} & 1 & * \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & * \end{array}\right] \end{array},$$

where $D_a = D_{\ell_1}([\begin{smallmatrix}1\\0\end{smallmatrix}])$ and $D_b = D_{\ell_1}([\begin{smallmatrix}0\\1\end{smallmatrix}])$ are the $2\ell_1 \times \ell_1$ matrices

$$D_a = \begin{bmatrix} 1 & & & \\ 0 & & & \\ & 1 & & \\ & 0 & & \\ & & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix}, \quad D_b = \begin{bmatrix} 0 & & & \\ 1 & & & \\ & 0 & & \\ & 1 & & \\ & & \ddots & \\ & & & 0 \\ & & & 1 \end{bmatrix}$$

and where

$$\alpha_\pm = \begin{bmatrix} \alpha_1 \\ -\alpha_1 \\ \vdots \\ \alpha_{\ell_1} \\ -\alpha_{\ell_1} \end{bmatrix}, \quad \alpha_+ = \begin{bmatrix} \alpha_1 \\ 0 \\ \vdots \\ \alpha_{\ell_1} \\ 0 \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{\ell_1} \end{bmatrix}$$

for convenience of notation.

At this stage, we want to zero out $\alpha_\pm$ in C. Note that we cannot just use the $b_{m+1,\ell+1} = 1$ entry below $\alpha_\pm$ as pivot to zero out $\alpha_\pm$ via row operations because the $(2\ell+1)$st column in C will be affected. Instead, we first use column operations, taking advantage of the symmetry in $\alpha_\pm$.

In f_{23}, the entries containing a 1 to the left of $\alpha_\pm$ are in the even-numbered rows. We perform a basis change on V^3 corresponding to column operations on f_{23}. The even-numbered entries in $\alpha_\pm$ can be zeroed out, giving

$$f_{23} = \begin{array}{c} {\scriptstyle (\ell+1)\text{st}} \\ \left[\begin{array}{c|c|c|c} D_b & \mathbf{0} & \alpha_+ & * \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} & * \\ \hline \mathbf{0} & \mathbf{0} & 1 & * \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & * \end{array}\right] \end{array}$$

via the basis change matrix on V^3

$$R = \left[\begin{array}{c|c|c|c} \mathbf{1} & & \alpha & \\ \hline & \mathbf{1} & \mathbf{0} & \\ \hline & & 1 & \\ \hline & & & \mathbf{1} \end{array}\right] \tag{4.13}$$

where α is on the $(\ell+1)$st column of R. The effect of this change of basis on the matrix form of f_{63} is that it is now

$$f_{63} = \left[\begin{array}{c} \mathbf{1} \\ \hline \mathbf{0} \end{array}\right] R = \left[\begin{array}{c} R \\ \hline \mathbf{0} \end{array}\right].$$

We will see later that f_{63} can be restored to its proper form within the current inductive step.

At this stage, we can now use the $b_{m+1,\ell+1} = 1$ entry to zero out α_+ in f_{23} via row operations:

$$f_{23} = \begin{array}{c} {\scriptstyle (\ell+1)\text{st}} \\ \left[\begin{array}{c|c|c|c} D_b & \mathbf{0} & \mathbf{0} & * \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} & * \\ \hline \mathbf{0} & \mathbf{0} & 1 & * \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & * \end{array}\right] \end{array}$$

This is effected by the basis change matrix

$$\begin{array}{c} {\scriptstyle (m+1)\text{st}} \\ \left[\begin{array}{c|c|c|c} \mathbf{1} & & -\alpha_+ & \\ \hline & \mathbf{1} & \mathbf{0} & \\ \hline & & 1 & \\ \hline & & & \mathbf{1} \end{array}\right]^{-1} \end{array}$$

on V^2, which changes the matrix form of f_{21} to

$$f_{21} = \left[\begin{array}{c|c|c|c} \mathbf{1} & & -\alpha_+ & \\ \hline & \mathbf{1} & \mathbf{0} & \\ \hline & & 1 & \\ \hline & & & \mathbf{1} \end{array}\right] \left[\begin{array}{c|c|c|c} D_a & \mathbf{0} & \mathbf{0} & * \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} & * \\ \hline \mathbf{0} & \mathbf{0} & 1 & * \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & * \end{array}\right] = \left[\begin{array}{c|c|c|c} D_a & \mathbf{0} & -\alpha_+ & * \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} & * \\ \hline \mathbf{0} & \mathbf{0} & 1 & * \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & * \end{array}\right].$$

Intuitively speaking, we have transferred α_+ from f_{23} to f_{21} as $-\alpha_+$. Then, $-\alpha_+$ can be zeroed out from f_{21} by column operations corresponding to a change of basis on V^1. We obtain

$$
\begin{array}{rc}
 & \begin{array}{cccc} & \phantom{\mathbf{0}} & {\scriptstyle (\ell+1)\mathrm{st}} & \end{array} \\
f_{21} = & \left[\begin{array}{c|c|c|c} D_a & \mathbf{0} & \mathbf{0} & * \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} & * \\ \hline \mathbf{0} & \mathbf{0} & 1 & * \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & * \end{array}\right]
\end{array}
$$

by a basis change matrix that is of the same form as R in Eq. (4.13). Thus, f_{41} is now in the form

$$
f_{41} = \left[\begin{array}{c} \mathbf{1} \\ \hline \mathbf{0} \end{array}\right] R = \left[\begin{array}{c} R \\ \hline \mathbf{0} \end{array}\right].
$$

It remains to transform the maps $f_{41}, f_{54}, f_{56}, f_{63}$ into the forms obtained in the steps S1 to S4. Currently, we have the matrix forms $f_{41} = \left[\begin{smallmatrix} R \\ \mathbf{0} \end{smallmatrix}\right]$ and $f_{63} = \left[\begin{smallmatrix} R \\ \mathbf{0} \end{smallmatrix}\right]$ where the number of rows in the $\mathbf{0}$ submatrices may be different. By the basis change matrices of the form $\left[\begin{smallmatrix} R & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{smallmatrix}\right]$ on V^4 and V^6, we get the forms

$$
f_{41} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} \text{ and } f_{63} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}.
$$

In these bases, f_{54} and f_{56} will now have matrix forms

$$
f_{54} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & * \end{bmatrix} \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}, \quad f_{56} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & * \end{bmatrix} \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}.
$$

Finally, a change of basis on V^5 by $\left[\begin{smallmatrix} R & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{smallmatrix}\right]$ simultaneously gives f_{54} and f_{56} the matrix forms

$$
f_{54} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & * \end{bmatrix}, \quad f_{56} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & * \end{bmatrix}
$$

as required. This finishes the induction step.

To summarize, we now have the matrix forms

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
V^4 \arrow[r, "\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & * \end{bmatrix}"] & V^5 & V^6 \arrow[l, "\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & * \end{bmatrix}"'] \\
V^1 \arrow[u, "\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}"] \arrow[r, "f_{21}"'] & V^2 \arrow[u, "f_{52}"] & V^3 \arrow[u, "\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}"'] \arrow[l, "f_{23}"]
\end{tikzcd}
$$

where

$$f_{21} = \left[\begin{array}{c|c} \begin{matrix}1&&&\\0&&&\\&1&&\\&0&&\\&&\ddots&\\&&&1\\&&&0\end{matrix} & \mathbf{0} \\ \hline \mathbf{0} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\ \hline \mathbf{0} & \mathbf{0}\end{array}\right], \quad f_{23} = \left[\begin{array}{c|c} \begin{matrix}0&&&\\1&&&\\&0&&\\&1&&\\&&\ddots&\\&&&1\\&&&0\end{matrix} & \mathbf{0} \\ \hline \mathbf{0} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\ \hline \mathbf{0} & \mathbf{0}\end{array}\right]$$

in the chosen bases, by the form of $\hat{C}$. By commutativity, f_{52} now has matrix form

$$f_{52} = \left[\begin{array}{c|c|c} \begin{matrix}1&1&&&\\&&\ddots&&\\&&&1&1\end{matrix} & \mathbf{0} & T_1 \\ \hline \mathbf{0} & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & T_2 \\ \hline \mathbf{0} & \mathbf{0} & *\end{array}\right].$$

Using column operations, corresponding to a basis change on V^2, the submatrices T_1 and T_2 can be zeroed out without affecting the forms of f_{21} and f_{23}.

This gives the forms

$$\begin{tikzcd}[column sep=huge, row sep=huge]
V^4 \arrow[r, "{\left[\begin{array}{cc||c}\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{1}&\mathbf{0}\\\hline\hline\mathbf{0}&\mathbf{0}&*\end{array}\right]}"] & V^5 & V^6 \arrow[l, "{\left[\begin{array}{cc||c}\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{1}&\mathbf{0}\\\hline\hline\mathbf{0}&\mathbf{0}&*\end{array}\right]}"'] \\
V^1 \arrow[u, "{\left[\begin{array}{cc}\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{1}\\\hline\hline\mathbf{0}&\mathbf{0}\end{array}\right]}"] \arrow[r, "{\left[\begin{array}{cc}D_{s_1}(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])&\mathbf{0}\\\mathbf{0}&\mathbf{1}\\\hline\hline\mathbf{0}&\mathbf{0}\end{array}\right]}"'] & V^2 \arrow[u, "{\left[\begin{array}{cc||c}D_{s_1}([1\ 1])&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{1}&\mathbf{0}\\\hline\hline\mathbf{0}&\mathbf{0}&*\end{array}\right]}"'] & V^3 \arrow[l, "{\left[\begin{array}{cc}D_{s_1}(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right])&\mathbf{0}\\\mathbf{0}&\mathbf{1}\\\hline\hline\mathbf{0}&\mathbf{0}\end{array}\right]}"] \arrow[u, "{\left[\begin{array}{cc}\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{1}\\\hline\hline\mathbf{0}&\mathbf{0}\end{array}\right]}"']
\end{tikzcd}$$

where again we have emphasized the direct summands we can extract. This clearly gives all the direct summands of V isomorphic to

$$\begin{tikzcd} K \arrow[r, "1"] & K & K \arrow[l, "1"'] \\ K \arrow[u, "1"] \arrow[r, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"'] & K^2 \arrow[u, "{[1\ 1]}"] & K \arrow[u, "1"] \arrow[l, "\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"] \end{tikzcd}$$

and

$$\begin{tikzcd} K \arrow[r, "1"] & K & K \arrow[l, "1"'] \\ K \arrow[u, "1"] \arrow[r, "1"] & K \arrow[u, "1"] & K \arrow[u, "1"] \arrow[l, "1"'] \end{tikzcd}.$$

Step 5. The remaining representation can be viewed as a representation of D_4, since $V^1 = V^3 = 0$, and f_{21}, f_{23}, f_{41}, and f_{63} will be 0. The rest of the algorithm can be derived in a similar manner as what we have given above, specifically following Steps 1 and 4a as templates. For more details, see the paper [17].

Chapter 5

Matrix Problems

In the previous chapter, we studied the representation theory of $CL_n(\tau)$ and its applications to topological data analysis. Here, we reformulate representations of $CL_n(\tau)$ as certain matrix problems. This is an alternative viewpoint that we expect will provide a more elegant algorithm for computing indecomposable decompositions.

By Lemma 4.2.2, the categories $\operatorname{rep} CL_n(\tau)$ and $\operatorname{arr}(\operatorname{rep} A_n(\tau))$ are isomorphic. As a consequence, we can identify $M \in \operatorname{rep} CL_n(\tau)$ with the arrow $F(M) = (\phi : V \to W)$ as given in the proof of Lemma 4.2.2.

Let such an arrow $\phi : V \to W$ be given. Since $V \in \operatorname{rep} A_n(\tau)$, there is an isomorphism to an indecomposable decomposition

$$\eta_V : V \cong \bigoplus_{1 \leqslant a \leqslant b \leqslant n} \mathbb{I}[a, b]^{m_{ab}},$$

where $\mathbb{I}[a,b]$ are the interval representations of $A_n(\tau)$. A similar isomorphism η_W can be constructed for W. Using these isomorphisms, the arrow ϕ is isomorphic to

$$\eta_W \phi \eta_V^{-1} : \bigoplus_{1 \leqslant a \leqslant b \leqslant n} \mathbb{I}[a, b]^{m_{ab}} \to \bigoplus_{1 \leqslant a \leqslant b \leqslant n} \mathbb{I}[a, b]^{m'_{ab}}.$$

This can be written in a matrix form $\left[\Phi_{a,b}^{a',b'}\right]_{(a,b),(a',b')}$ where each entry

$$\pi(\eta_W \phi \eta_V^{-1})\iota = \Phi_{a,b}^{a',b'} : \mathbb{I}[a, b]^{m_{ab}} \to \mathbb{I}[a', b']^{m'_{a'b'}}$$

is obtained by inclusion from and projection to the appropriate direct summand.

It can be shown that for arbitrary orientations τ, the morphisms between the interval representations of $A_n(\tau)$ satisfy

$$\dim_K \operatorname{Hom}_{KA_n(\tau)}(\mathbb{I}[a, b], \mathbb{I}[a', b']) \leqslant 1,$$

by using the property of commutativity. However, precisely which pairs of pairs $(a, b), (a', b')$ give rise to nonzero homomorphism spaces depends on τ. This computation, however, is not needed in what follows.

For each pair of pairs $(a, b), (a', b')$ with $\operatorname{Hom}_{KA_n(\tau)}(\mathbb{I}[a, b], \mathbb{I}[a', b']) \neq 0$, we define a morphism

$$f_{a,b}^{a',b'} \in \operatorname{Hom}_{KA_n(\tau)}(\mathbb{I}[a, b], \mathbb{I}[a', b'])$$

by the following. Note first that if $\mathrm{Hom}_{KA_n(\tau)}(\mathbb{I}[a,b],\mathbb{I}[a',b']) \neq 0$, then the intersection of the intervals

$$[a,b] \cap [a',b'] = \{i \mid a \leqslant i \leqslant b\} \cap \{i \mid a' \leqslant i \leqslant b'\} \neq \varnothing.$$

There is therefore an index $\ell \in [a,b] \cap [a',b']$. By the commutativity requirement on morphisms, it can be checked that any $f : \mathbb{I}[a,b] \to \mathbb{I}[a',b']$ is determined by its map at index ℓ, $f_\ell : K \to K$. Since $\dim_K \mathrm{Hom}_K(K,K) = 1$, this shows that $\dim_K \mathrm{Hom}_{KA_n(\tau)}(\mathbb{I}[a,b],\mathbb{I}[a',b']) = 1$. Moreover, by this property, we choose $f_{a,b}^{a',b'}$ to be the uniquely defined morphism f determined by $f_\ell = 1_K$. In this case, $f_{\ell'} = 1_K$ for all ℓ' in $[a,b] \cap [a',b']$.

Note that $f_{a,b}^{a',b'}$ is only defined for pairs of intervals with nonzero corresponding homomorphism K-vector space. This choice of morphisms satisfies the following very important property. For any triple of pairs $(a,b),(a',b'),(a'',b'')$ with

$$\begin{aligned}
&\mathrm{Hom}_{KA_n(\tau)}(\mathbb{I}[a,b],\mathbb{I}[a',b']) \neq 0,\\
&\mathrm{Hom}_{KA_n(\tau)}(\mathbb{I}[a',b'],\mathbb{I}[a'',b'']) \neq 0, \text{ and}\\
&\mathrm{Hom}_{KA_n(\tau)}(\mathbb{I}[a,b],\mathbb{I}[a'',b'']) \neq 0,
\end{aligned}$$

the chosen morphisms satisfy the property that

$$f_{a,b}^{a'',b''} = f_{a',b'}^{a'',b''} f_{a,b}^{a',b'}.$$

This property can be checked immediately, from the definition. With the conditions given, the three intervals $[a,b]$, $[a',b']$, $[a'',b'']$ have pairwise nonempty intersections. It can then be checked that the three intervals have a common intersection. On each index in this intersection, the equality above is simply $1_K = 1_K \circ 1_K$.

Then, the above morphisms $\Phi_{a,b}^{a',b'}$ can be factored into $F_{a,b}^{a',b'} f_{a,b}^{a',b'}$, where each $F_{a,b}^{a',b'}$ is a matrix of size $m'_{a'b'} \times m_{ab}$ with entries in K. Contrast this with $\Phi_{a,b}^{a',b'}$, which is a matrix of morphisms in $\mathrm{Hom}_{KA_n(\tau)}(\mathbb{I}[a,b],\mathbb{I}[a',b'])$.

To illustrate this procedure, let us give the example with $\tau = f$. Up to isomorphism, the indecomposable representations of $A_2(f)$ are $\mathbb{I}[1,1]$, $\mathbb{I}[1,2]$, $\mathbb{I}[2,2]$. Given a $CL(f)$-module

$$V = \begin{tikzcd} V_3 \arrow[r, "g_{43}"] & V_4 \\ V_1 \arrow[u, "g_{31}"] \arrow[r, "g_{21}"] & V_2 \arrow[u, "g_{42}"'] \end{tikzcd},$$

we can choose bases on V_i so that the matrix forms of g_{43} and g_{21} are of the form $\left[\begin{smallmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{smallmatrix}\right]$, possibly of different sizes. In the chosen bases, V is

$$\begin{tikzcd} V_3' \oplus V_3'' \arrow[r, "{\left[\begin{smallmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{smallmatrix}\right]}"] & V_4' \oplus V_4'' \\ V_1' \oplus V_1'' \arrow[u, "g_{31}'"] \arrow[r, "{\left[\begin{smallmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{smallmatrix}\right]}"] & V_2' \oplus V_2'' \arrow[u, "g_{42}'"'] \end{tikzcd},$$

where g'_{31} and g'_{42} are the matrix forms of g_{31} and g_{42}.

Let $g'_{31} = \left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]$ and $g'_{42} = \left[\begin{smallmatrix} A_2 & B_2 \\ C_2 & D_2 \end{smallmatrix}\right]$. By commutativity,

$$\left[\begin{smallmatrix} A_1 & B_1 \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}\right] = \left[\begin{smallmatrix} A_2 & \mathbf{0} \\ C_2 & \mathbf{0} \end{smallmatrix}\right]$$

must be satisfied. Now, the representation V is mapped by the equivalence into the category $\operatorname{arr}(\operatorname{rep} A_2(f))$, with image the arrow $g = (g_{31}, g_{42})$. In matrix forms, this arrow is given by

$$g = \left(\left[\begin{smallmatrix} A & \mathbf{0} \\ C_1 & D_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & B_2 \\ \mathbf{0} & D_2 \end{smallmatrix}\right]\right)$$

where $A = A_1 = A_2$.

In this form, we have a decomposition (in $\operatorname{rep} A_2(f)$) of the upper and lower rows, as follows:

$$\begin{array}{c} (\ 0 \xrightarrow{0} V_4''\ ) \oplus (\ V_3' \xrightarrow{\mathbf{1}} V_4'\ ) \oplus (\ V_3'' \xrightarrow{0} 0\ ) \\ g \Big\uparrow \\ (\ 0 \xrightarrow{0} V_2''\ ) \oplus (\ V_1' \xrightarrow{\mathbf{1}} V_2'\ ) \oplus (\ V_1'' \xrightarrow{0} 0\ ). \end{array}$$

Thus, $g = (g_{31}, g_{42})$ is isomorphic to an arrow

$$\begin{array}{c} (\mathbb{I}[2,2]^{m'_{22}}) \oplus (\mathbb{I}[1,2]^{m'_{12}}) \oplus (\mathbb{I}[1,1]^{m'_{11}}) \\ \phi \Big\uparrow \\ (\mathbb{I}[2,2]^{m_{22}}) \oplus (\mathbb{I}[1,2]^{m_{12}}) \oplus (\mathbb{I}[1,1]^{m_{11}}) \end{array}$$

where $m'_{22} = \dim V_4''$, $m'_{11} = \dim V_3''$, and $m'_{12} = \dim V_3' = \dim V_4'$ for the codomain, and $m_{22} = \dim V_2''$, $m_{11} = \dim V_1''$, and $m_{12} = \dim V_1' = \dim V_2'$.

It can be checked that written as a matrix, ϕ is given by:

$$\begin{array}{c c} & \begin{array}{ccc} 2{:}2 & 1{:}2 & 1{:}1 \end{array} \\ \begin{array}{c} 2{:}2 \\ 1{:}2 \\ 1{:}1 \end{array} & \begin{bmatrix} D_2 f_{2,2}^{2,2} & \mathbf{0} & \mathbf{0} \\ B_2 f_{2,2}^{1,2} & A f_{1,2}^{1,2} & \mathbf{0} \\ \mathbf{0} & C_1 f_{1,2}^{1,1} & D_1 f_{1,1}^{1,1} \end{bmatrix} \end{array} : \bigoplus_{1 \leqslant a \leqslant b \leqslant 2} \mathbb{I}[a,b]^{m_{ab}} \to \bigoplus_{1 \leqslant a \leqslant b \leqslant 2} \mathbb{I}[a,b]^{m'_{ab}}.$$

In this notation, we have labeled columns and rows by symbols $a{:}b$, corresponding to the direct summands $\mathbb{I}[a,b]^{m_{ab}}$ for columns and $\mathbb{I}[a,b]^{m'_{ab}}$ for rows.

Suppose that two arrows Φ to Ψ are already in the form given above. By definition of the arrow category, a morphism from Φ to Ψ is a pair of morphisms

(F_1, F_2) in rep $A_2(f)$ such that the diagram

$$\begin{array}{ccc} \bigoplus_{1\leqslant a\leqslant b\leqslant 2} \mathbb{I}[a,b]^{m_{ab}} & \xrightarrow{\Phi} & \bigoplus_{1\leqslant a\leqslant b\leqslant 2} \mathbb{I}[a,b]^{m'_{ab}} \\ \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_2} \\ \bigoplus_{1\leqslant a\leqslant b\leqslant 2} \mathbb{I}[a,b]^{n_{ab}} & \xrightarrow{\Psi} & \bigoplus_{1\leqslant a\leqslant b\leqslant 2} \mathbb{I}[a,b]^{n'_{ab}} \end{array}$$

commutes, or equivalently $\Psi F_1 = F_2 \Phi$. Because their domains and codomains are already expressed as direct sums, we can similarly write F_1 and F_2 in matrix forms.

Let us summarize the above matrix problem. Note that we remove mention of the $f_{a,b}^{a',b'}$, to simplify the presentation. Once the objects, morphisms, and composition rules are known, $f_{a,b}^{a',b'}$ can be hidden.

Example 5.1. Form the category with:

- Objects are
$$M = \begin{bmatrix} M(x_{41}) & & \\ M(x_{51}) & M(x_{52}) & \\ & M(x_{62}) & M(x_{63}) \end{bmatrix}$$
where each block entry $M(x_{ji})$ is a matrix with entries in K, of appropriate sizes.

- Morphisms are $F = (F_1, F_2) : M \to N$ given by
$$\left(\begin{bmatrix} F(w_1) & & \\ F(v_{21}) & F(w_2) & \\ & F(v_{32}) & F(w_3) \end{bmatrix}, \begin{bmatrix} F(w_4) & & \\ F(v_{54}) & F(w_5) & \\ & F(v_{65}) & F(w_6) \end{bmatrix} \right)$$
satisfying $F_2 * M = N * F_1$. The block entries $F(*)$ are matrices with entries in K, of appropriate sizes. The identity morphism of M is given by $\left(\begin{bmatrix} \mathbf{1} & & \\ \mathbf{0} & \mathbf{1} & \\ & \mathbf{0} & \mathbf{1} \end{bmatrix}, \begin{bmatrix} \mathbf{1} & & \\ \mathbf{0} & \mathbf{1} & \\ & \mathbf{0} & \mathbf{1} \end{bmatrix} \right) : M \to M$, where the $\mathbf{1}$'s and $\mathbf{0}$'s are appropriately-sized identity and zero matrices.

- Compositions is given by the following. For $F = (F_1, F_2) : M \to N$, $G = (G_1, G_2) : L \to M$, $FG = (F_1 * G_1, F_2 * G_2)$.

In the above, $*$ is the usual multiplication of block matrices, except that the lower-left block entry is always kept empty.

The matrix problem associated with this category is to describe its indecomposable objects, and given an object M, to find its indecomposable decomposition.

That the multiplication $*$ above keeps the lower-left entry empty is related to the fact that $\dim_K \mathrm{Hom}_{KA_n(\tau)}(\mathbb{I}[2,2], \mathbb{I}[1,1]) = 0$ and that $f_{1,2}^{1,1} f_{2,2}^{1,2} = 0$. Note that in this case, $f_{2,2}^{1,1}$ is not even defined.

At this stage, it is not necessary to understand x_{ji}, v_{ji}, w_i as anything other than indexing symbols for the tuples of matrices that comprise the objects and morphisms. While we do not explain it here, the use of this notation will become clear in the context of generators of bocses and differential biquivers.

The category given above is equivalent to $\operatorname{arr}(\operatorname{rep} A_2(f))$, and thus to $\operatorname{rep} CL_2(f)$. We will do this in general via Theorem 5.0.1, later. For now, let us further expound on the matrix problem by converting it into a problem involving matrices and certain permissible row and column operations.

Given an object M of the category above, we are interested in finding a normal form in the isomorphism class of M. If $M \cong N$, then we have an isomorphism $F = (F_1, F_2) : M \to N$, so that $N = F_2 * M * F_1^{-1}$.

From the restrictions made on the form of morphisms $F = (F_1, F_2)$, only the following operations are permissible.

1. Any elementary row or column operation within the same row or column block.

2. Any addition of a K-multiple of a row in row block 1 to a row in row block 2, and a row in 2 to a row in 3.

3. Any addition of a K-multiple of a column in column block 3 to a column in 2, and a column in 2 to one in 1.

By a row block, we mean the collection of block entries in the matrix sharing the same row in the block matrix structure of M. For example, the second row block of an object M is given by the rows of

$$\begin{bmatrix} M(x_{51}) & M(x_{52}) & \end{bmatrix}.$$

The term column block is defined similarly. Also, it is important to remember that the lower-left block entry is always kept empty – not as a requirement for the row and column operations, but rather that no matter what operation we do, it is always empty.

Let us use the permissible operations to find an indecomposable decomposition for M an object of the category in Example 5.1. We denote by $*$ the nonzero block entries in the matrix. In general, an object M of the category above is:

$$\begin{bmatrix} * & & \\ * & * & \\ & * & * \end{bmatrix}$$

where the arrows to represent permissible operations of type 2 and 3.

By using operations of type 1, we can transform each diagonal block into a Smith normal form $\begin{smallmatrix} E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}$, where E is an identity matrix. In other words, M is isomorphic to an object of the form

$$\begin{array}{c c}
 & \begin{array}{ccc} 2{:}2 & 1{:}2 & 1{:}1 \end{array} \\
\begin{array}{c} 2{:}2 \\[1.2em] 1{:}2 \\[1.2em] 1{:}1 \end{array} &
\left[\begin{array}{ccc}
\begin{smallmatrix} E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix} & & \\[1em]
\begin{smallmatrix} * & * \\ * & * \end{smallmatrix} & \begin{smallmatrix} E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix} & \\[1em]
 & \begin{smallmatrix} * & * \\ * & * \end{smallmatrix} & \begin{smallmatrix} E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}
\end{array}\right]
\end{array}.$$

Using operations of type 2 and 3, we can transform the above matrix into the following form. By pivoting using the entries in each E submatrix, we can zero out entries in some of the blocks sharing the same row or column as that E. Of course, which of these blocks can be zeroed out is determined by what operations are permissible. In this simple case, the entries to the left and below each E can be zeroed out.

$$\begin{array}{c c}
 & \begin{array}{ccc} 2{:}2 & 1{:}2 & 1{:}1 \end{array} \\
\begin{array}{c} 2{:}2 \\[1.2em] 1{:}2 \\[1.2em] 1{:}1 \end{array} &
\left[\begin{array}{ccc}
\begin{smallmatrix} E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix} & & \\[1em]
\begin{smallmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & * \end{smallmatrix} & \begin{smallmatrix} E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix} & \\[1em]
 & \begin{smallmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & * \end{smallmatrix} & \begin{smallmatrix} E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}
\end{array}\right]
\end{array}.$$

We extract the extractable summands as below,

$$\begin{array}{c c}
 & \begin{array}{ccc} 2{:}2 & 1{:}2 & 1{:}1 \end{array} \\
\begin{array}{c} 2{:}2 \\[1.2em] 1{:}2 \\[1.2em] 1{:}1 \end{array} &
\left[\begin{array}{ccc}
\begin{smallmatrix} E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix} & & \\[1em]
\begin{smallmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & * \end{smallmatrix} & \begin{smallmatrix} E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix} & \\[1em]
 & \begin{smallmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & * \end{smallmatrix} & \begin{smallmatrix} E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}
\end{array}\right]
\end{array}. \tag{5.1}$$

For example, the middle cross region gives s indecomposable summands with dimension vector $\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$, if the middle E is an s by s identity matrix. More precisely, in the form given in Eq. (5.1), the matrix is a direct sum

$$\begin{array}{c c} & 1{:}2 \\ 1{:}2 & \left[\begin{array}{c} E \end{array}\right] \end{array}
\bigoplus
\begin{array}{c c}
 & \begin{array}{ccc} 2{:}2 & 1{:}2 & 1{:}1 \end{array} \\
\begin{array}{c} 2{:}2 \\[1.2em] 1{:}2 \\[1.2em] 1{:}1 \end{array} &
\left[\begin{array}{ccc}
\begin{smallmatrix} E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix} & & \\[1em]
\begin{smallmatrix} \mathbf{0} & * \end{smallmatrix} & \mathbf{0} & \\[1em]
 & \begin{smallmatrix} \mathbf{0} \\ * \end{smallmatrix} & \begin{smallmatrix} E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}
\end{array}\right]
\end{array},$$

and the matrix $[E]$ above is the arrow $E : \mathbb{I}[1,2]^s \to \mathbb{I}[1,2]^s$, which corresponds to the representation

$$\left(\begin{array}{ccc} K & \xrightarrow{1} & K \\ {\scriptstyle 1}\uparrow & & {\scriptstyle 1}\uparrow \\ K & \xrightarrow{1} & K \end{array}\right)^{s}$$

of $CL_2(f)$. That is, we obtain s copies of the indecomposable representation of $CL_2(f)$ with dimension vector $\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$. Using similar arguments for the other encircled regions in Eq. (5.1), we also extract direct summands corresponding to the dimension vectors $\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$, $\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$, $\begin{smallmatrix}0&1\\0&0\end{smallmatrix}$, and $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$.

What remains is

$$\begin{array}{cc} & \begin{array}{cc} 2{:}2 & 1{:}2 \end{array} \\ \begin{array}{c} 1{:}2 \\ 1{:}1 \end{array} & \begin{bmatrix} * & \mathbf{0} \\ & * \end{bmatrix} \end{array} \cong \begin{array}{cc} & \begin{array}{cc} 2{:}2 & 1{:}2 \end{array} \\ \begin{array}{c} 1{:}2 \\ 1{:}1 \end{array} & \begin{bmatrix} \begin{smallmatrix} E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix} & \mathbf{0} \\ & \begin{smallmatrix} E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix} \end{bmatrix} \end{array}$$

from which we may immediately extract more summands. We obtain $\begin{smallmatrix}1&1\\0&1\end{smallmatrix}$, $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$, $\begin{smallmatrix}0&0\\0&1\end{smallmatrix}$, $\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$, $\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$, and $\begin{smallmatrix}1&0\\0&0\end{smallmatrix}$.

The remaining matrix is empty, and so the algorithm ends. At this stage, we have accounted for all the indecomposable representations. This can be checked against the AR quiver of $CL_2(f)$ given in Figure A.2, which was derived by the knitting procedure.

Let us also write down the general matrix problem corresponding to $\operatorname{arr}(\operatorname{rep} A_n(\tau)) \cong \operatorname{rep} CL_n(\tau)$. First, note that there are $\Lambda = \frac{n(n+1)}{2}$ isomorphism classes of indecomposables in $\operatorname{rep} A_n(\tau)$, given by the isomorphism classes of the interval representations $\mathbb{I}[a,b]$ for $1 \leqslant a \leqslant b \leqslant n$. Order the indecomposables $\mathbb{I}[a,b]$ arbitrarily as

$$J_1, J_2, \ldots J_\Lambda.$$

To summarize the structure of the morphisms $f_{a,b}^{a',b'}$, we use the following formalism.

Definition 5.1 (The relation $\rightharpoonup$)**.** Let $J_1, J_2, \ldots J_\Lambda$ be some arbitrary ordering of all the interval representations $\mathbb{I}[a,b] \in \operatorname{rep} A_n(\tau)$. Define a relation $\rightharpoonup$ on the set $\{1, \ldots, \Lambda\}$ by $i \rightharpoonup j$ if and only if $\operatorname{Hom}_{KA_n(\tau)}(J_i, J_j) \neq 0$.

Clearly, $i \rightharpoonup i$ for all $i \in \{1, \ldots, \Lambda\}$. It can be checked that $\rightharpoonup$ is antisymmetric: if $i \rightharpoonup j$ and $j \rightharpoonup i$, then $i = j$. However, $\rightharpoonup$ is not transitive. From $i \rightharpoonup j$ and $j \rightharpoonup k$, we cannot conclude that $i \rightharpoonup k$. Thus $\rightharpoonup$ is not a partial order. Moreover, we warn that having a path from J_i to J_j in the AR quiver $\Gamma(A_n(\tau))$ does not guarantee that $i \rightharpoonup j$, although from $i \rightharpoonup j$ we can conclude that there is a path from J_i to J_j.

We need two copies of the indices $\{1, \ldots, \Lambda\}$, one for row indices and another for column indices of objects. We use $\{1, \ldots, \Lambda\}$ for columns and $\{1', \ldots, \Lambda'\}$ for rows. To index the submatrices in our matrix problem, create the symbols $x_{j'i}$, v_{ji}, and $v_{j'i'}$, for all pairs (i, j) with $i \rightharpoonup j$. In the definition below, the subscripts $i \rightharpoonup j$ in the matrices means to vary through all pairs (i, j) with $i \rightharpoonup j$.

Definition 5.2. Form the category $C(\tau)$ with:

- An object is a sequence of numbers $d_i, d_{i'}$ indexed by $i \in \{1, \ldots, \Lambda\}$ and a set of matrices
$$\left[M(x_{j'i})\right]_{i \rightleftharpoons j}$$
where each block entry $M(x_{j'i})$ is a matrix of size $d_{j'} \times d_i$ with entries in K, for pairs (i, j) with $i \rightleftharpoons j$.

- Let $M = (d_i, d_{i'}, M(x_{j'i}))$ and $N = (\hat{d}_i, \hat{d}_{i'}, M(x_{j'i}))$. A morphism is $F = (F_1, F_2) : M \to N$, with
$$(F_1, F_2) = \left(\left[F(v_{ji})\right]_{i \rightleftharpoons j}, \left[F(v_{j'i'})\right]_{i \rightleftharpoons j}\right)$$
where each block entry $F(v_{ji})$ is a K-matrix of size $\hat{d}_j \times d_i$ and each $F(v_{j'i'})$ is a K-matrix of size $\hat{d}_{j'} \times d_{i'}$, satisfying the equalities
$$\sum_{i \rightleftharpoons k \rightleftharpoons j} F(v_{j'k'})M(x_{k'i}) = \sum_{i \rightleftharpoons k \rightleftharpoons j} N(x_{j'k})F(v_{ki}) \tag{5.2}$$
are satisfied for every pair (i, j) with $i \rightleftharpoons j$. The summations are taken over all $k \in \{1, \ldots, \Lambda\}$ such that $i \rightleftharpoons k$ and $k \rightleftharpoons j$.

- Compositions: For $F : M \to N$, $G : L \to M$, $FG : L \to N$
$$(FG)(v_{ji}) = \sum_{i \rightleftharpoons k \rightleftharpoons j} F(v_{jk})G(v_{ki}) \tag{5.3}$$
and
$$(FG)(v_{j'i'}) = \sum_{i \rightleftharpoons k \rightleftharpoons j} F(v_{j'k'})G(v_{k'i'}) \tag{5.4}$$
for every pair (i, j) with $i \rightleftharpoons j$.

Clearly, Example 5.1 is $C(\tau)$ in Definition 5.2 in the case of $n = 2$, $\tau = f$, with indices $\{1, 2, 3, 4 = 1', 5 = 2', 6 = 3'\}$, and symbols $w_i = v_{ii}$ for $i \in \{1, 2, 3, 1', 2', 3'\}$. The above conditions for morphisms and their compositions are simply the defining equations, written entry-wise, of conditions originally written as matrix multiplication: $F_2 * M = N * F_1$, $(FG)_1 = F_1 * G_1$, and $(FG)_2 = F_2 * G_2$.

Theorem 5.0.1. *Given an orientation τ, the category $C(\tau)$ constructed in Definition 5.2 is equivalent to* $\operatorname{arr}(\operatorname{rep} A_n(\tau))$.

Proof. This follows by construction. Let us, however, explicitly construct an equivalence $\Upsilon : C(\tau) \to \operatorname{arr}(\operatorname{rep} A_n(\tau))$.

Given an object $M = (d_i, d_{i'}, \left[M(x_{j'i})\right]_{i \rightleftharpoons j})$, define $\Upsilon(M)$ as the arrow:
$$\left(\Upsilon(M) : \bigoplus_{i=1}^{\Lambda} J_i^{d_i} \to \bigoplus_{i=1}^{\Lambda} J_i^{d_{i'}}\right) \in \operatorname{arr}(\operatorname{rep} A_n(\tau))$$

so that in matrix form, $\Upsilon(M)$ is just M. Precisely speaking, this means that for $i \rightharpoonup j$,

$$\pi\Upsilon(M)\iota = M(x_{j'i})f_{a,b}^{a',b'} : J_i^{d_i} \to J_j^{d_{j'}}$$

where $J_i = \mathbb{I}[a,b]$, $J_j = \mathbb{I}[a',b']$, and π and ι is the projection to $J_j^{d_{j'}}$ and the inclusion from $J_i^{d_i}$, respectively. For $i \not\rightharpoonup j$, $\pi\Upsilon(M)\iota = 0 : J_i^{d_i} \to J_j^{d_{j'}}$.

Let

$$\left(\Upsilon(N) : \bigoplus_{i=1}^{\Lambda} J_i^{\hat{d}_i} \to \bigoplus_{i=1}^{\Lambda} J_i^{\hat{d}_{i'}}\right) \in \operatorname{arr}(\operatorname{rep} A_n(\tau))$$

be likewise given. A morphism $F : M \to N$ in $C(\tau)$, then, is a pair (F_1, F_2) of matrices which we can similarly turn into arrows:

$$\begin{aligned}\Upsilon(F_1) : \quad & \bigoplus_{i=1}^{\Lambda} J_i^{d_i} \to \bigoplus_{i=1}^{\Lambda} J_i^{\hat{d}_i}\\ \Upsilon(F_2) : \quad & \bigoplus_{i=1}^{\Lambda} J_i^{d_i} \to \bigoplus_{i=1}^{\Lambda} J_i^{\hat{d}_{i'}}.\end{aligned}$$

Define $\Upsilon(F) = \Upsilon(F_1, F_2)$ to be the morphism

$$(\Upsilon(F_1), \Upsilon(F_2)) : \Upsilon(M) \to \Upsilon(N)$$

in $\operatorname{arr}(\operatorname{rep} A_n(\tau))$. Let us check that $\Upsilon(F)$ is a morphism $\operatorname{arr}(\operatorname{rep} A_n(\tau))$. We need to show that

$$\begin{tikzcd} \bigoplus_{i=1}^{\Lambda} J_i^{d_i} \arrow[r, "\Upsilon(M)"] \arrow[d, "\Upsilon(F_1)"'] & \bigoplus_{i=1}^{\Lambda} J_i^{d_{i'}} \arrow[d, "\Upsilon(F_2)"] \\ \bigoplus_{i=1}^{\Lambda} J_i^{\hat{d}_i} \arrow[r, "\Upsilon(N)"'] & \bigoplus_{i=1}^{\Lambda} J_i^{\hat{d}_{i'}} \end{tikzcd}$$

commutes. To see this, note that Eq. (5.2) implies

$$\sum_{i \rightharpoonup k \rightharpoonup j} F(v_{j'k'})M(x_{k'i})f_{a,b}^{a'',b''} = \sum_{i \rightharpoonup k \rightharpoonup j} N(x_{j'k})F(v_{ki})f_{a,b}^{a'',b''}$$

for each pair i, j with $i \rightharpoonup j$ and where $J_i = \mathbb{I}[a,b]$, $J_j = \mathbb{I}[a'',b'']$. Using the factorization property of $f_{a,b}^{a'',b''}$, we have

$$\sum_{i \rightharpoonup k \rightharpoonup j} F(v_{j'k'})f_{a',b'}^{a'',b''}M(x_{k'i})f_{a,b}^{a',b'} = \sum_{i \rightharpoonup k \rightharpoonup j} N(x_{j'k})f_{a',b'}^{a'',b''}F(v_{ki})f_{a,b}^{a',b'} \tag{5.5}$$

where in the notation above the numbers a', b' with $J_k = \mathbb{I}[a',b']$ vary with k as we take the summation. By the definition of Υ and multiplication of matrices, Eq. (5.5) is nothing but

$$\pi\Upsilon(F_2)\Upsilon(M)\iota = \pi\Upsilon(N)\Upsilon(F_1)\iota : J_i^{d_i} \to J_j^{\hat{d}_{j'}}$$

where ι is the inclusion from $J_i^{d_i}$ and π is the projection to $J_j^{\hat{d}_{j'}}$. Now, the above equality holds for each pair i, j with $i \rightleftharpoons j$. For $i \not\rightleftharpoons j$,

$$\pi\Upsilon(F_2)\Upsilon(M)\iota = 0 = \pi\Upsilon(N)\Upsilon(F_1)\iota$$

by definition. We conclude that $\Upsilon(F_2)\Upsilon(M) = \Upsilon(N)\Upsilon(F_1)$.

Thus, $\Upsilon(F) : \Upsilon(M) \to \Upsilon(N)$ is a morphism in $\mathrm{arr}(\mathrm{rep}\, A_n(\tau))$. Likewise, it can be checked that $\Upsilon(FG) = \Upsilon(F)\Upsilon(G)$ by using Eqs. (5.3) and (5.4).

That Υ is fully faithful follows from construction. Moreover, Υ is dense since any arrow $(\phi : V \to W)$ is isomorphic to an arrow of the form $\left(\Phi : \bigoplus_{i=1}^{\Lambda} J_i^{\tilde{d}_i} \to \bigoplus_{i=1}^{\Lambda} J_i^{\tilde{d}_{i'}}\right)$, from which we can construct an $M \in C(\tau)$ with $\Upsilon(M) = \Phi$ by the construction at the start of this chapter. This shows that Υ is an equivalence. $\square$

Computing indecomposable decompositions of objects M in $C(\tau) \cong \mathrm{arr}(\mathrm{rep}\, A_n(\tau)) \cong \mathrm{rep}\, CL_n(\tau)$ can be interpreted in terms of finding "normal forms" using permissible operations on matrices, as a matrix problem. However, it is not immediately clear that a finite sequence of permissible operations, similar to what we have described for Example 5.1, can completely solve this problem, even just for the representation-finite case.

These definitions can be rephrased in terms of representations of differential biquivers, and in more generality, in terms of representations of bocses [35]. The paper [14] provides a proof of Drozd's tame and wild theorem using the representation theory of bocses. The proof is essentially algorithmic in nature, and can be applied to compute the normal forms we need. The application of these ideas for the computation of indecomposable decompositions of representations of the representation-finite commutative ladder quivers will be treated in an upcoming work.

Appendix A

Auslander-Reiten quivers of $CL_n(\tau)$, $n \leqslant 4$

A.1 $n = 1$

With $n = 1$, only the empty orientation $\tau = \varnothing$ is possible. Moreover, $CL_1(\varnothing)$ can be identified with $\vec{A}_2$.

$$\begin{tikzcd}
& \begin{smallmatrix}1\\1\end{smallmatrix} \arrow[dr] & \\
\begin{smallmatrix}1\\0\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}0\\1\end{smallmatrix}
\end{tikzcd}$$

Figure A.1: $\Gamma(CL_1(\varnothing))$

A.2 $n = 2$

$$\begin{tikzcd}
& \begin{smallmatrix}1&1\\0&0\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}0&0\\0&1\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}1&0\\1&0\end{smallmatrix} \arrow[dr] & \\
\begin{smallmatrix}0&1\\0&0\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1&1\\0&1\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}1&1\\1&1\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1&0\\1&1\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}0&0\\1&0\end{smallmatrix} \\
& \begin{smallmatrix}0&1\\0&1\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}1&0\\0&0\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}0&0\\1&1\end{smallmatrix} \arrow[ur] &
\end{tikzcd}$$

Figure A.2: $\Gamma(CL_2(f))$

A.3 $n = 3$

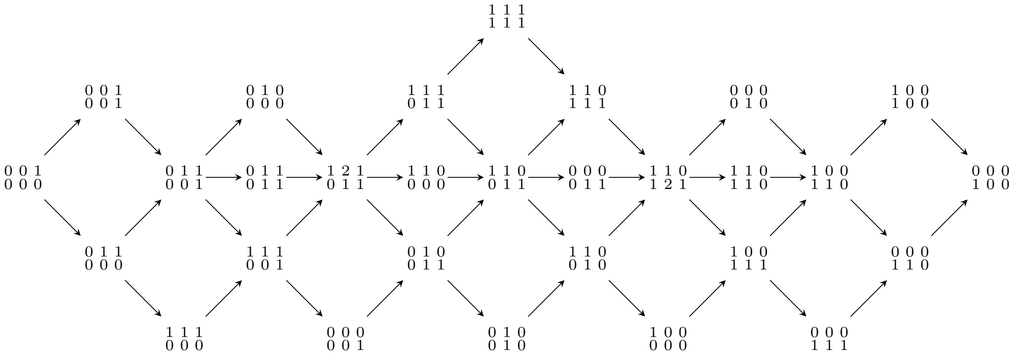


Figure A.3: $\Gamma(CL_3(ff))$

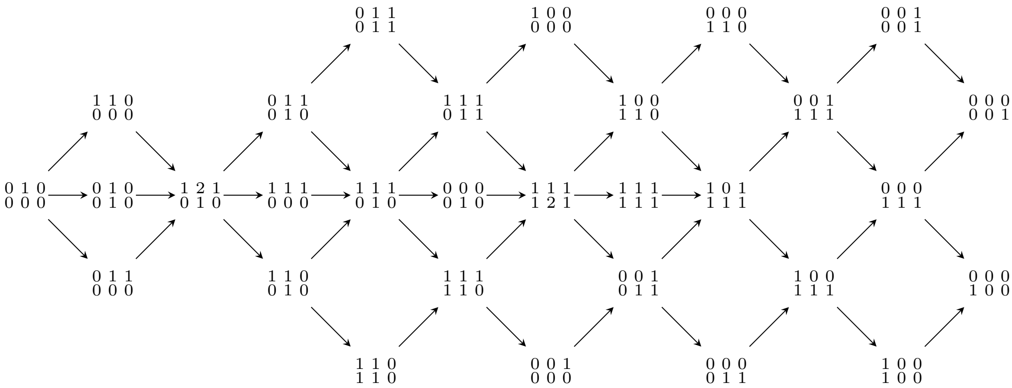


Figure A.4: $\Gamma(CL_3(fb))$

A.4 $n = 4$

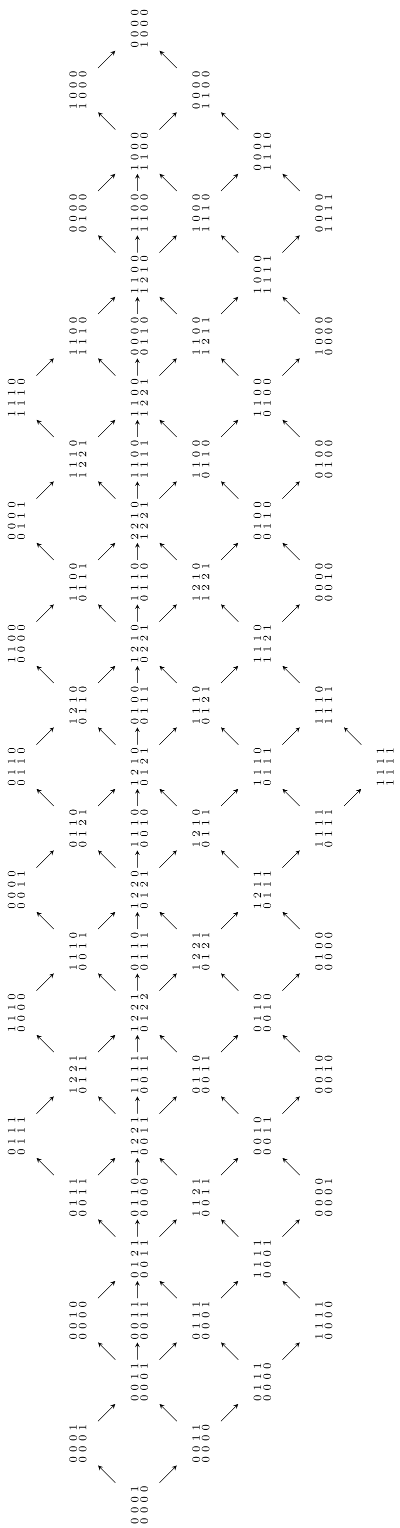


Figure A.5: $\Gamma(CL_4(fff))$

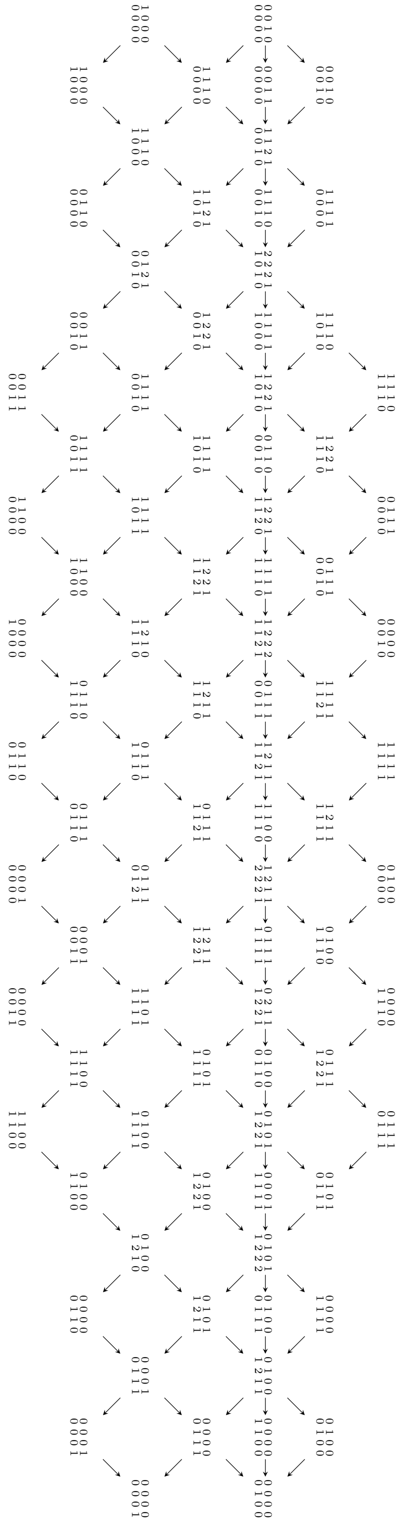


Figure A.6: $\Gamma(CL_4(bfb))$

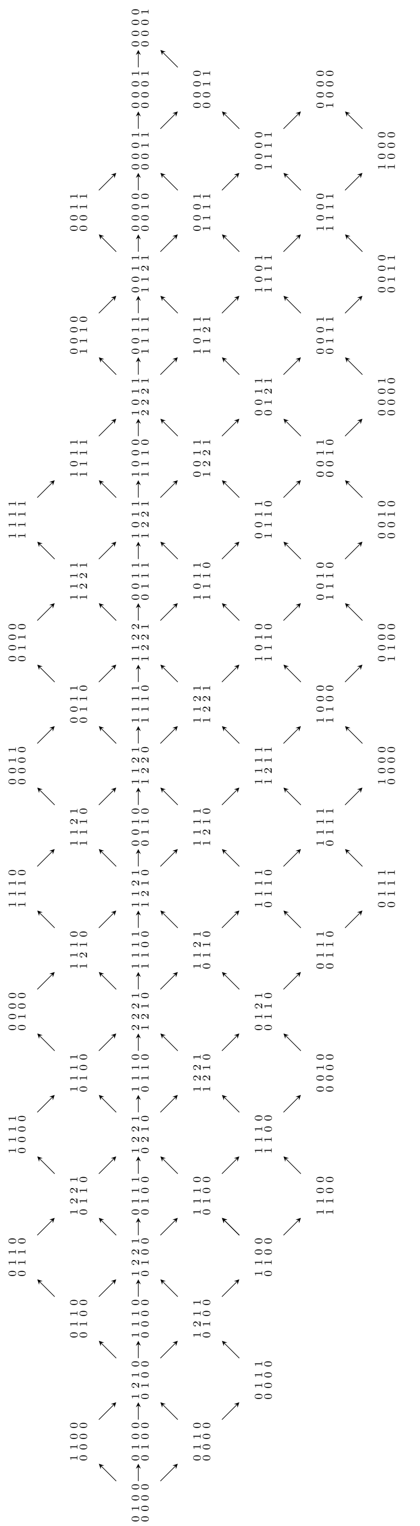


Figure A.7: $\Gamma(CL_4(fbb))$

Index

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