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Abstract

In algebraic geometry, a number of invariants for classifying algebraic varieties are obtained from the cohomology groups of coherent sheaves. Some typical algorithms to compute the dimensions of the cohomology groups have been proposed by Decker and Eisenbud, and their algorithms have been implemented over compute algebra systems such as Macaulay2 and Magma. On the other hand, M. Maruyama showed an alternative method to compute the dimensions in his textbook. However, Maruyama's method was not described in an algorithmic format, and it has not been implemented yet. In this paper, we give an explicit algorithm of his method to compute the dimensions and bases of the cohomology groups of coherent sheaves. We also analyze the complexity of our algorithm, and implemented it over Magma. By our implementation, we examine the computational practicality of our algorithm. Moreover, we give some possible applications of our algorithm in algebraic geometry over fields of positive characteristics.

Key words— Computer algebra Gröbner bases Algebraic geometry Cohomology groups

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Contents

1 Introduction

For a positive integer r, let $S = K[X_0, \ldots, X_r]$ be the polynomial ring of $(r+1)$ variables over a field K. The polynomial ring S can be represented as the graded ring $S = \bigoplus_{d \geq 0} S_d$, by taking S_d for each $d \geq 0$ to be the set of all linear combinations of monomials of total degree d in $X_0, \ldots, X_r$. Let $\mathbb{P}_K^r = \mathrm{Proj}\,(S)$ denote the projective r-space, and $\mathcal{O}_{\mathbb{P}_K^r}$ the structure sheaf on $\mathbb{P}_K^r$. Given a coherent sheaf $\mathcal{F}$ on $\mathbb{P}_K^r$ and $q \in \mathbb{Z}$, denote by $H^q\,(\mathbb{P}_K^r, \mathcal{F})$ its q-th cohomology group. It is important to compute the cohomology groups of coherent sheaves since the groups are used to compute a number of invariants such as Hilbert functions, Euler characteristics, and arithmetic genera of algebraic varieties. Thus the computation of the cohomolgy groups is one of the most crucial topics in algebraic geometry.

J.-P. Serre [11] theoretically showed a possibility to compute the dimension $\dim_K H^q\,(\mathbb{P}_K^r, \mathcal{F})$ of the cohomology group $H^q\,(\mathbb{P}_K^r, \mathcal{F})$ as a K-vector space. After that, some algorithms to compute $\dim_K H^q\,(\mathbb{P}_K^r, \mathcal{F})$ have been proposed by Smith [12] and Decker-Eisenbud [4]. Their algorithms are based on computational techniques of commutative algebra such as Gröbner bases for free modules, free resolutions, exterior algebra and Tate resolutions. In particular, algorithms proposed by Decker-Eisenbud have been implemented over computer algebra systems such as Macaulay2 [6] and Magma [1]. On the other hand, M. Maruyama showed an alternative method[1] to compute $\dim_K H^q\,(\mathbb{P}_K^r, \mathcal{F})$ in his textbook [9] (unfortunately, it is written in Japanese). Different from Decker-Eisenbud's algorithms, Maruyama's method enables to directly compute $H^q\,(\mathbb{P}_K^r, \mathcal{F})$ by computing projective resolutions and the Čech cohomology via Gröbner bases for free modules and linear algebra. We can also compute an explicit basis of $H^q\,(\mathbb{P}_K^r, \mathcal{F})$ by Maruyama's method. As we will see in Section 5.2, this basis is useful to compute important objects in algebraic geometry. For example, for given finitely generated graded S-modules M, N and a graded homomorphism $\psi : M \longrightarrow N$ of degree zero, the induced K-linear map $H^q(\widetilde{\psi}) : H^q(\mathbb{P}_K^r, \widetilde{M}) \longrightarrow H^q(\mathbb{P}_K^r, \widetilde{N})$ can be computed via the basis for $q \geq 1$, where $\widetilde{M}$ (resp. $\widetilde{N}$) is the coherent sheaf associated with M (resp. N) on $\mathbb{P}_K^r$.

In this paper, we focus on Maruyama's method to compute the cohomology group $H^q\,(\mathbb{P}_K^r, \mathcal{F})$ for $q \geq 1$ (in Appendix, we also introduce a method by Maruyama of the computation in the case of $q = 0$). His method is not described in an algorithmic format, and it has not been implemented yet over computer algebra systems. Then our main contributions are as follows:

(1) We write down Maruyama's method as an explicit algorithm (Algorithm 3.2.1 in Subsection 3.2) to compute the dimension and a basis of the K-vector space $H^q(\mathbb{P}_K^r, \mathcal{F})$ for $q \geq 1$. We also implemented our algorithm over Magma as a new function "`CohomologyBasis`". (cf. Magma has the function "`CohomologyDimension`"[2], which outputs only the dimension but not any basis of $H^q\,(\mathbb{P}_K^r, \mathcal{F})$). In particular, our algorithm enables to compute a representation matrix of a morphism of the cohomology groups.

(2) We analyze the complexity of our algorithm. We also examine the efficiency of our algorithm by experiments.

(3) We present some applications of our algorithm, which are useful to study algebraic geometry over fields of positive characteristics. In particular, as a typical application of our algorithm,

[1] In [13, Chapter 8], only an example is computed by using a similar method to the method while Maruyama gave the method in a general style.

[2] The source code of our implementation and the computation results are available at `http://www2.math.kyushu-u.ac.jp/~m-kudo/` (our source code is in the file `CohomologyBasis.txt`).

we show a method to compute the action of Frobenius to varieties defined over fields of positive characteristics (we briefly describe the application below).

One of the most valuable applications of our algorithm is to compute the action of Frobenius to varieties defined over a field K of positive characteristic p. Let $f_1, \ldots f_s \in S = K[X_0, \ldots, X_r]$ be homogeneous polynomials. We set X as the locus $V(f_1, \ldots, f_s) \subset \mathbb{P}^r_K$ of the zeros of the polynomials $f_1, \ldots, f_s$. Let $\mathcal{O}_X$ be the structure sheaf on X. Then the computation of the action of Frobenius $H^q(X, \mathcal{O}_X) \to H^q(X, \mathcal{O}_X)$ can be reduced to the computation of the morphism $H^q\left(X^{(p)}, \mathcal{O}_{X^{(p)}}\right) \to H^q(X, \mathcal{O}_X)$ induced by the canonical inclusion $X \hookrightarrow X^{(p)}$, where $X^{(p)}$ denotes $V(f_1^p, \ldots, f_s^p)$. As we will describe in Section 5, our algorithm gives a useful tool to compute a representation matrix of the morphism $H^q\left(X^{(p)}, \mathcal{O}_{X^{(p)}}\right) \to H^q(X, \mathcal{O}_X)$ of K-vector spaces, by which the structure of the action of Frobenius $H^q(X, \mathcal{O}_X) \to H^q(X, \mathcal{O}_X)$ can be obtained explicitly.

The rest of this paper is organized as follows: In Section 2, we introduce Maruyama's method to compute the dimensions and bases of the cohomology groups of coherent sheaves on the projective r-space $\mathbb{P}^r_K$. In Section 3, we give an explicit algorithm of Maruyama's method, and analyze its complexity. In Section 4, we show experimental results obtained from our implementation of our algorithm over Magma. We examine the computational behavior of our algorithm, by which it is conclude that our algorithm and implementation are practical. In Section 5, we discuss possible applications of our function `CohomologyBasis`. In Section 6, we conclude our work, and give our future works.

Notation

- $\bigoplus_{j=1}^t M_j$: the direct sum of R-modules $M_1, \ldots, M_t$,
- $M(m)$: the m-th twisted graded R-module $\bigoplus_{t\in\mathbb{Z}} M_{m+t}$ of a graded R-module $M = \bigoplus_{t\in\mathbb{Z}} M_t$, where each M_t is the homogeneous part with degree t of M,
- $\mathcal{O}_Y$: the structure sheaf on a scheme Y,
- $\widetilde{M}$: the sheaf associated with an S-module M,
- $\mathcal{F}(m)$: the m-th Serre twist of a sheaf $\mathcal{F}$ of $\mathcal{O}_Y$-modules,
- $H^q(Y, \mathcal{F})$: the q-th cohomology group of a sheaf $\mathcal{F}$ on a scheme Y,
- $R_{(f)}$: the localization of a ring R by an element $f \in R$,
- R_d : the homogeneous part with degree d of a graded ring R,
- $\binom{m}{n}$: the binomial coefficient of two non-negative integers m and n with $m \geq n$.

2 Preliminaries

In this section, we introduce Maruyama's method given in [9, Chapter 6] to compute the dimensions of the cohomology groups of coherent sheaves on a projective space.

2.1 Fundamental properties of the cohomology groups of coherent sheaves

In this subsection, we review some general facts on the cohomology groups of coherent sheaves on projective schemes (see [7, Chapter 3] for details). These properties are necessary to describe Maruyama's method in the next subsection.

Theorem 2.1.1 ([7], Theorem 5.1) *Let K be a field. Let $\mathbb{P}_K^r = \mathrm{Proj}\,(S)$ be the projective r-space $\mathbb{P}_K^r$ with $S = K[X_0, \ldots, X_r]$, and $\mathcal{O}_{\mathbb{P}_K^r}$ the structure sheaf on $\mathbb{P}_K^r$. Let $S = \bigoplus_{d \geq 0} S_d$ denote the graded ring, where S_d is the set of all linear combinations of monomials of total degree d in $X_0, \ldots, X_r$ for $d \geq 0$. Let $S\,(m)$ denote the m-th twisted graded ring $\bigoplus_{t \in \mathbb{Z}} S_{m+t}$ of S for $m \in \mathbb{Z}$. We have the following results:*

(1) *For all $m \in \mathbb{Z}$, there exist isomorphisms of K-vector spaces as follows:*

$$H^0\left(\mathbb{P}_K^r, \mathcal{O}_{\mathbb{P}_K^r}\,(m)\right) \cong \begin{cases} S_m & \text{for } m \geq 0, \\ 0 & \text{for } m < 0. \end{cases}$$

In other words, for every $m \geq 0$, the set

$$\{X_0^{l_0} \cdots X_r^{l_r} \ ; \ l_i \geq 0 \text{ for } 0 \leq i \leq r, \text{ and } l_0 + \cdots + l_r = m\}$$

of monomials of total degree m is a basis of $H^0\left(\mathbb{P}_K^r, \mathcal{O}_{\mathbb{P}_K^r}\,(m)\right)$.

(2) *$H^q\left(\mathbb{P}_K^r, \mathcal{O}_{\mathbb{P}_K^r}\,(m)\right) = 0$ for $0 < q < r$ and arbitrary m.*

(3) *Let $\left(S\,(m)_{(X_0 \cdots X_r)}\right)_0$ denote the homogeneous part with degree 0 of the localization $S\,(m)_{(X_0 \cdots X_r)}$ by $X_0 \cdots X_r$. Note that $\left(S\,(m)_{(X_0 \cdots X_r)}\right)_0$ is the K-vector space spanned by the set*

$$\{aX_0^{l_0} \cdots X_r^{l_r} \ ; \ a \in K, \ l_i \in \mathbb{Z} \text{ for } 0 \leq i \leq r, \text{ and } l_0 + \cdots + l_r = m\}.$$

Let W be the K-vector subspace of $\left(S\,(m)_{(X_0 \cdots X_r)}\right)_0$ spanned by

$$\{X_0^{l_0} \cdots X_r^{l_r} \ ; \ l_i \geq 0 \text{ for some } i, \text{ and } l_0 + \cdots + l_r = m\}.$$

Then we have the following isomorphism of K-vector spaces:

$$H^r\left(\mathbb{P}_K^r, \mathcal{O}_{\mathbb{P}_K^r}\,(m)\right) \ \cong \ \left(S\,(m)_{(X_0 \cdots X_r)}\right)_0 \ / \ W. \tag{2.1.1}$$

Hence for every $m < 0$, the set

$$\{{X_0}^{l_0} \cdots {X_r}^{l_r} \ ; \ l_i < 0 \text{ for } 0 \leq i \leq r, \text{ and } l_0 + \cdots + l_r = m\}$$

gives rise to a basis of the K-vector space $H^r\left(\mathbb{P}_K^r, \mathcal{O}_{\mathbb{P}_K^r}\,(m)\right)$ via the above isomorphism (2.1.1).

Corollary 2.1.2 ([7], Theorem 5.1) *For all $m \in \mathbb{Z}$, we have the following:*

$$\dim_K H^0\left(\mathbb{P}_K^r, \mathcal{O}_{\mathbb{P}_K^r}\,(m)\right) = \begin{cases} \binom{m+r}{r} & \text{for } m \geq 0, \\ 0 & \text{for } m < 0. \end{cases}$$

$$\dim_K H^r\left(\mathbb{P}_K^r, \mathcal{O}_{\mathbb{P}_K^r}\,(m)\right) = \begin{cases} \binom{-m-1}{r} & \text{for } m \leq -r-1, \\ 0 & \text{for } m > -r-1. \end{cases}$$

We also have $H^q\left(\mathbb{P}_K^r, \mathcal{O}_{\mathbb{P}_K^r}\,(m)\right) = 0$ for $q \neq 0, r$ and $m \in \mathbb{Z}$.

Theorem 2.1.3 ([9], Theorem 4.78) *Let $X \subset \mathbb{P}^r_K$ be a projective scheme over a field K. Let $\mathcal{F}$ be a coherent sheaf on X. Then for all $q > r$,* we have $H^q(X, \mathcal{F}) = 0$.

Theorem 2.1.4 ([7], Chapter 3, Theorem 5.2) *Let $X \subset \mathbb{P}^r_K$ be a projective scheme over a field K. Let $\mathcal{F}$ be a coherent sheaf on X.*

(1) *For each $q \geq 0$, the q-th cohomology group $H^q(X, \mathcal{F})$ is a finite-dimensional K-vector space.*

(2) *There exists an integer m_0, depending on $\mathcal{F}$, such that $H^q(X, \mathcal{F}(m)) = 0$ for each $q > 0$ and $m \geq m_0$.*

2.2 Maruyama's method

In this subsection, we introduce a result (Theorem 2.2.1) by Maruyama. The result gives us a method to compute the cohomology groups of coherent sheaves on a projective space.

Let $\mathbb{P}^r_K = \mathrm{Proj}(S)$ be the projective r-space with $S = K[X_0, \ldots, X_r]$, and $\mathcal{O}_{\mathbb{P}^r_K}$ the structure sheaf on $\mathbb{P}^r_K$. To simplify the notations, we denote $\mathbb{P}^r_K$ and $\mathcal{O}_{\mathbb{P}^r_K}$ by $\mathbb{P}^r$ and $\mathcal{O}_{\mathbb{P}^r}$, respectively. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r$. By the definition of coherent sheaves on $\mathbb{P}^r$, there exists an exact sequence

$$0 \to \bigoplus_{j=1}^{t_{r+2}} \mathcal{O}_{\mathbb{P}^r}\left(m_j^{(r+2)}\right) \overset{f_{r+1}}{\to} \cdots \overset{f_1}{\to} \bigoplus_{j=1}^{t_1} \mathcal{O}_{\mathbb{P}^r}\left(m_j^{(1)}\right) \overset{f_0}{\to} \mathcal{F} \to 0 \tag{2.2.1}$$

for some t_i and $m_j^{(i)}$ with $1 \leq i \leq r+2$ and $1 \leq j \leq t_i$. For an index i with $t_i = 0$, we identify $\bigoplus_{j=1}^{t_i} \mathcal{O}_{\mathbb{P}^r}\left(m_j^{(i)}\right) = 0$. Put

$$\begin{aligned} \mathcal{G}_{i+1} &:= \bigoplus_{j=1}^{t_{i+1}} \mathcal{O}_{\mathbb{P}^r}\left(m_j^{(i+1)}\right), \\ \mathcal{K}_i &:= \mathrm{Ker}(f_i) \ \text{ for } 0 \leq i \leq r+1, \quad \mathcal{K}_{-1} := \mathcal{F}. \end{aligned} \tag{2.2.2}$$

In general, for a sheaf $\mathcal{F}$ on a topological space, the cohomology groups are defined by a (canonical) injective resolution of $\mathcal{F}$ and they are computed by the injective resolution (see [7, Chapter 3] for details). However, in Maruyama's method, we have a *projective* resolution (2.2.1) for a coherent sheaf $\mathcal{F}$ on the projective space $\mathbb{P}^r$. This implies that the cohomology groups are computed by the projective resolution without computing an injective resolution. In this case, it requires the following result to compute the cohomology groups by using the projective resolution.

Theorem 2.2.1 ([9], Chapter 6) *Let $\mathbb{P}^r_K = \mathrm{Proj}(S)$ be the projective r-space with $S = K[X_0, \ldots, X_r]$, and $\mathcal{O}_{\mathbb{P}^r_K}$ the structure sheaf on $\mathbb{P}^r_K$. To simplify the notations, we denote $\mathbb{P}^r_K$ and $\mathcal{O}_{\mathbb{P}^r_K}$ by $\mathbb{P}^r$ and $\mathcal{O}_{\mathbb{P}^r}$, respectively. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r$. Recall that the coherent sheaf $\mathcal{F}$ has a projective resolution of the form* (2.2.1)*. Put $\mathcal{G}_i$ and $\mathcal{K}_i$ as in* (2.2.2)*. Then there exist the following isomorphisms of K-vector spaces:*

(1) $H^q(\mathbb{P}^r, \mathcal{F}) \cong \mathrm{Ker}(H^r(f_{r-q})) / \mathrm{Im}(H^r(f_{r-q+1}))$ *for $1 \leq q \leq r-1$,*

(2) $H^r(\mathbb{P}^r, \mathcal{F}) \cong \mathrm{Coker}(H^r(f_1))$,

where $H^r(f_i)$ denotes the morphism $H^r(\mathbb{P}^r, \mathcal{G}_{i+1}) \longrightarrow H^r(\mathbb{P}^r, \mathcal{G}_i)$ induced by f_i for $1 \leq i \leq r+1$.

Proof. As only a sketch of a proof is given in [9], we give a complete proof here. To simplify the notations, we denote $H^q(\mathbb{P}^r, \mathcal{H})$ by $H^q(\mathcal{H})$ for a coherent sheaf $\mathcal{H}$ on $\mathbb{P}^r$ in this proof.

First we show the second statement $H^r(\mathcal{F}) \cong \mathrm{Coker}(H^r(f_1))$. The sequence of coherent $\mathcal{O}_{\mathbb{P}^r}$-modules $\mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{F} \to 0$ is exact, and the functor $H^r(\cdot)$ is right exact. Hence the sequence

$$H^r(\mathcal{G}_2) \xrightarrow{H^r(f_1)} H^r(\mathcal{G}_1) \xrightarrow{H^r(f_0)} H^r(\mathcal{F}) \longrightarrow 0$$

is exact, and thus $H^r(\mathcal{F}) \cong \mathrm{Coker}(H^r(f_1))$.

Next we show the first statement

$$H^q(\mathcal{F}) \cong \mathrm{Ker}(H^r(f_{r-q})) / \mathrm{Im}(H^r(f_{r-q+1})) \text{ for } 1 \le q \le r-1.$$

For every $1 \le i \le r+2$, we have the following short exact sequence of coherent sheaves:

$$0 \to \mathcal{K}_{i-1} \to \mathcal{G}_i \to \mathcal{K}_{i-2} \to 0. \tag{E_i}$$

Thus there exists a long exact sequence of cohomology groups

$$\begin{array}{ccccccccc} 0 & \to & H^0(\mathcal{K}_{i-1}) & \to & H^0(\mathcal{G}_i) & \to & H^0(\mathcal{K}_{i-2}) & & \\ & \to & H^1(\mathcal{K}_{i-1}) & \to & H^1(\mathcal{G}_i) & \to & H^1(\mathcal{K}_{i-2}) & & \\ & \to & \cdots & & & & & & \\ & \to & H^{r-1}(\mathcal{K}_{i-1}) & \to & H^{r-1}(\mathcal{G}_i) & \to & H^{r-1}(\mathcal{K}_{i-2}) & & \\ & \to & H^r(\mathcal{K}_{i-1}) & \to & H^r(\mathcal{G}_i) & \to & H^r(\mathcal{K}_{i-2}) & \to & 0 \end{array} \tag{L_i}$$

for each $1 \le i \le r+2$.

Then we claim $H^q(\mathcal{G}_i) = 0$ for $1 \le q \le r-1$. In fact, by its definition in (2.2.2), it follows that

$$H^q(\mathcal{G}_i) = H^q\left(\bigoplus_{j=1}^{t_i} \mathcal{O}_{\mathbb{P}^r}\left(m_j^{(i)}\right)\right) \cong \bigoplus_{j=1}^{t_i} H^q\left(\mathcal{O}_{\mathbb{P}^r}\left(m_j^{(i)}\right)\right)$$

since $\mathbb{P}^r$ is a noetherian topological space, and the cohomology commutes with arbitrary direct sums on a noetherian topological space in general. By Theorem 2.1.1 (2), we have $H^q\left(\mathcal{O}_{\mathbb{P}^r}\left(m_j^{(i)}\right)\right) = 0$ for all indexes i, j and $1 \le q \le r-1$. Hence we have $H^q(\mathcal{G}_i) = 0$ for $1 \le q \le r-1$.

From the long exact sequences (L_i) for $1 \le i \le r+2$, it follows that

$$H^q(\mathcal{F}) \cong H^{q+1}(\mathcal{K}_0) \cong \cdots \cong \quad H^{r-1}(\mathcal{K}_{r-q-2}).$$

Recall that the sequence

$$0 \to H^q(\mathcal{F}) \cong H^{r-1}(\mathcal{K}_{r-q-2}) \to H^r(\mathcal{K}_{r-q-1}) \to H^r(\mathcal{G}_{r-q})$$

is exact. We have $H^q(\mathcal{F}) \cong \mathrm{Ker}(\sigma_q)$, where σ_q denotes the K-linear map $H^r(\mathcal{K}_{r-q-1}) \to H^r(\mathcal{G}_{r-q})$ in the above exact sequence. Note that the following diagram of morphisms of coherent sheaves commutes

$$\begin{tikzcd} \mathcal{G}_{r-q+2} \arrow[r, "f_{r-q+1}"] & \mathcal{G}_{r-q+1} \arrow[r] \arrow[dr, "f_{r-q}"'] & \mathcal{K}_{r-q-1} \arrow[r] \arrow[d] & 0 \\ & & \mathcal{G}_{r-q} & \end{tikzcd}$$

where the horizontal sequence is exact. Since the functor $H^r(\cdot)$ is right exact, the horizontal sequence of the following commutative diagram is also exact:

$$\begin{tikzcd}
H^r\left(\mathcal{G}_{r-q+2}\right) \arrow[r, "H^r(f_{r-q+1})"] & H^r\left(\mathcal{G}_{r-q+1}\right) \arrow[r] \arrow[dr, "H^r(f_{r-q})"'] & H^r\left(\mathcal{K}_{r-q-1}\right) \arrow[r] \arrow[d, "\sigma_q"] & 0 \\
& & H^r\left(\mathcal{G}_{r-q}\right) &
\end{tikzcd}$$

Thus we have $H^q(\mathcal{F}) \cong \mathrm{Ker}(\sigma_q) \cong \mathrm{Ker}\left(H^r\left(f_{r-q}\right)\right)/\mathrm{Im}\left(H^r\left(f_{r-q+1}\right)\right)$ as K-vector spaces. □

By Theorem 2.2.1, we have the following explicit formulae.

Corollary 2.2.2 *The notations are same as in* Theorem 2.2.1. *Then we have*

$$\dim_K H^q(\mathcal{F}) \quad = \quad \dim_K H^r\left(\mathcal{G}_{r-q+1}\right) - \mathrm{rank} H^r\left(f_{r-q}\right) - \mathrm{rank} H^r\left(f_{r-q+1}\right)$$

for $1 \le q \le r-1$, *and*

$$\dim_K H^r(\mathcal{F}) \quad = \quad \dim_K H^r\left(\mathcal{G}_1\right) - \mathrm{rank} H^r\left(f_1\right).$$

Remark 2.2.3 In a similar way to the proof of Theorem 2.2.1, it is possible to give an explicit formula for computing the dimension of the global section $\Gamma\left(\mathbb{P}_K^r, \mathcal{F}\right) = H^0\left(\mathbb{P}_K^r, \mathcal{F}\right)$. We shall introduce the formula in Appendix A.

3 Explicit algorithm of Maruyama's method

In this section, we give an algorithm (Algorithm 3.2.1) of Maruyama's method to compute a basis of the cohomology groups of coherent sheaves on the projective r-space $\mathbb{P}_K^r$ over a field K.

3.1 Interpretation of Maruyama's method

In this subsection, we give an interpretation of Maruyama's key results (Theorem 2.2.1 and Corollary 2.2.2) to compute the cohomology groups of coherent sheaves. As in the previous section, let $\mathbb{P}_K^r = \mathrm{Proj}(S)$ be the projective r-space over a field K with $S = K[X_0, \ldots, X_r]$, and $\mathcal{O}_{\mathbb{P}_K^r}$ the structure sheaf on $\mathbb{P}_K^r$. For a coherent sheaf $\mathcal{H}$ on $\mathbb{P}_K^r$, let $H^q(\mathbb{P}_K^r, \mathcal{H})$ denote the q-th cohomology group of $\mathcal{H}$. To simplify the notations, we denote $\mathbb{P}_K^r$, $\mathcal{O}_{\mathbb{P}_K^r}$ and $H^q(\mathbb{P}_K^r, \mathcal{H})$ by $\mathbb{P}^r$, $\mathcal{O}_{\mathbb{P}^r}$ and $H^q(\mathcal{H})$, respectively. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r$. Let M be a finitely generated graded S-module corresponding to $\mathcal{F}$, that is, $\mathcal{F} = \widetilde{M}$.

For an integer n, we now describe Maruyama's method to compute the cohomology groups $H^q(\mathcal{F}(n))$ for $1 \le q \le r$ for the n-th twisted coherent sheaf $\mathcal{F}(n)$. We note that $\mathcal{F}(n) = \widetilde{M(n)}$, where $M(n)$ is the n-th twisted graded S-module of M. As $\mathcal{F}$ has the projective resolution (2.2.1), the S-module M has the (minimal) graded resolution of length at most $r+1$

$$0 \to \bigoplus_{j=1}^{t_{r+2}} S\left(-d_j^{(r+2)}\right) \overset{\varphi_{r+1}}{\to} \cdots \overset{\varphi_1}{\to} \bigoplus_{j=1}^{t_1} S\left(-d_j^{(1)}\right) \overset{\varphi_0}{\to} M \to 0 \tag{3.1.1}$$

for some integers t_i and $d_j^{(i)}$ ($1 \le i \le r+2$ and $1 \le j \le t_i$), where $S(m)$ denotes the m-th twisted graded ring $\bigoplus_{t \in \mathbb{Z}} S_{m+t}$ of S for an integer $m \in \mathbb{Z}$. If the resolution (3.1.1) has length ℓ, we set $t_i = 0$ for $\ell + 2 \le i \le r+2$ (i.e., we may assume that the resolution always has length $r+1$). Note that each morphism φ_i is a graded homomorphism of degree zero between two free S-modules $\bigoplus_{j=1}^{t_{i+1}} S(-d_j^{(i+1)})$ and $\bigoplus_{j=1}^{t_i} S(-d_j^{(i)})$. Thus, for each $0 \le i \le r+1$, we can represent each morphism φ_i as a $t_{i+1} \times t_i$-matrix

$$A_i := \begin{bmatrix} g_{1,1}^{(i)} & \cdots & g_{1,t_i}^{(i)} \\ \vdots & & \vdots \\ g_{t_{i+1},1}^{(i)} & \cdots & g_{t_{i+1},t_i}^{(i)} \end{bmatrix}, \tag{3.1.2}$$

where the (k,ℓ)-entry $g_{k,\ell}^{(i)} \in S$ is homogeneous of degree $\left(d_\ell^{(i)} - d_k^{(i+1)}\right)$. For the twisted sheaf $\mathcal{F}(n)$, by the resolution (3.1.1), we have an exact sequence of coherent $\mathcal{O}_{\mathbb{P}^r}$-modules

$$0 \to \bigoplus_{j=1}^{t_{r+2}} \mathcal{O}_{\mathbb{P}^r}\left(n - d_j^{(r+2)}\right) \overset{\widetilde{\varphi_{r+1}(n)}}{\to} \cdots \overset{\widetilde{\varphi_1(n)}}{\to} \bigoplus_{j=1}^{t_1} \mathcal{O}_{\mathbb{P}^r}\left(n - d_j^{(1)}\right) \overset{\widetilde{\varphi_0(n)}}{\to} \mathcal{F}(n) \to 0, \tag{3.1.3}$$

where each twisted morphism $\widetilde{\varphi_i(n)}$ is a morphism induced by $\varphi_i(n)$. Here $\varphi_i(n)$ denotes the morphism

$$\bigoplus_{j=1}^{t_{i+1}} S\left(n - d_j^{(i+1)}\right) \longrightarrow \bigoplus_{j=1}^{t_i} S\left(n - d_j^{(i)}\right) \ ; \ \mathbf{u} \mapsto \mathbf{u} \cdot A_i. \tag{3.1.4}$$

Note that each $\varphi_i(n)$ is also represented by A_i as well as φ_i (but the graded S-modules of the domains and codomains of φ_i and $\varphi_i(n)$ are different if $n \neq 0$). The exact sequence (3.1.3) gives a projective resolution for the twisted sheaf $\mathcal{F}(n)$. With the projective resolution, we can compute cohomology groups $H^q(\mathcal{F}(n))$ for $1 \le q \le r$ by Theorem 2.2.1.

Specifically, we compute the dimensions of the cohomology groups as follows: As in the previous section, put

$$\begin{gathered} \mathcal{G}_{i+1} := \bigoplus_{j=1}^{t_{i+1}} \mathcal{O}_{\mathbb{P}^r}\left(n - d_j^{(i+1)}\right), \ f_i := \widetilde{\varphi_i(n)}, \\ \mathcal{K}_i := \mathrm{Ker}\,(f_i) \quad \text{for } 0 \le i \le r+1, \ \mathcal{K}_{-1} := \mathcal{F}. \end{gathered} \tag{3.1.5}$$

Then we can apply Corollary 2.2.2 to compute $\dim_K H^q(\mathcal{F}(n))$. In fact, we have

$$\begin{aligned} \dim_K H^r(\mathcal{G}_i) &= \dim_K \bigoplus_{j=1}^{t_i} H^r\left(\mathcal{O}_{\mathbb{P}^r}\left(n - d_j^{(i)}\right)\right) \\ &= \sum_{j=1}^{t_i} \dim_K H^r\left(\mathcal{O}_{\mathbb{P}^r}\left(n - d_j^{(i)}\right)\right). \end{aligned} \tag{3.1.6}$$

Moreover, for $d = d_j^{(i)}$, we have

$$\dim_K H^r(\mathcal{O}_{\mathbb{P}^r}(n-d)) = \begin{cases} \binom{-1-n+d}{r} & \text{for } n - d \le -r-1, \\ 0 & \text{for } n - d > -r-1 \end{cases} \tag{3.1.7}$$

by Corollary 2.1.2. Note that the morphism $H^r(f_i)$ can be represented by the matrix A_i. Since K-bases of $H^r\left(\mathcal{O}_{\mathbb{P}^r_K}\left(n-d_j^{(i)}\right)\right)$ for $1 \le j \le t_i$ can be obtained by Theorem 2.1.1 (3), it is possible to compute $\mathrm{rank} H^r(f_{r-q})$ and $\mathrm{rank} H^r(f_{r-q+1})$, and thus $\dim_K H^q(\mathcal{F}(n))$ can be computed. As a summary, once

$$t_i, d_j^{(i)} \text{ and } A_i = \left(g_{k,\ell}^{(i)}\right)_{\substack{1\le k\le t_{i+1},\\ 1\le \ell\le t_i}} \text{ for } 1 \le i \le r+2 \text{ and } 1 \le j \le t_i \tag{3.1.8}$$

are determined, we can easily compute $\dim_K H^q(\mathcal{F}(n))$ for $1 \le q \le r$.

3.2 Algorithm based on Maruyama's method

In this subsection, we present an algorithm to compute a basis of cohomology groups of coherent sheaves on the projective r-space $\mathbb{P}^r_K$ over a field K. Before we give our algorithm, we first give an outline of our algorithm in order to make procedures of our algorithm clear. Our algorithm consists of some sub-procedures, which shall be explained precisely in Subsection 3.3.

3.2.1 Outline of our algorithm

Let $\mathbb{P}^r_K = \mathrm{Proj}(S)$ be the projective r-space over a field K with the polynomial ring $S = K[X_0, \ldots, X_r]$, and $\mathcal{O}_{\mathbb{P}^r_K}$ the structure sheaf of $\mathbb{P}^r_K$. For a coherent sheaf $\mathcal{H}$ on $\mathbb{P}^r_K$, let $H^q(\mathbb{P}^r_K, \mathcal{H})$ denote the q-th cohomology group of $\mathcal{H}$. To simplify the notations, as in Subsection 3.1, we denote $\mathbb{P}^r_K$, $\mathcal{O}_{\mathbb{P}^r_K}$ and $H^q(\mathbb{P}^r_K, \mathcal{H})$ by $\mathbb{P}^r$, $\mathcal{O}_{\mathbb{P}^r}$ and $H^q(\mathcal{H})$, respectively.

Given a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$ and an integer n, we give an algorithm based on Maruyama's method to compute a basis of the cohomology group $H^q(\mathcal{F}(n))$ for $1 \le q \le r$. As in Subsection 3.1, let M denote a finitely generated graded S-module corresponding to $\mathcal{F}$. Our algorithm has the following two main procedures Steps A and B, and Step B consists of four sub-procedures (B-1)–(B-4):

Step A. Given a set of explicit S-generators of M, we first compute a (minimal projective) resolution of the form (3.1.1) for M. Specifically, we compute all elements of (3.1.8), which are determined from the resolution (3.1.1).

Step B. Given the elements of (3.1.8), we next compute a basis of $H^q(\mathcal{F}(n))$ for each $1 \le q \le r$. This step can be divided in the following four steps:

- (B-1) Given the elements of (3.1.8), we compute a basis of $H^r(\mathcal{G}_i)$ for $r-q \le i \le r-q+2$ by Theorem 2.1.1 (3), where $H^r(\mathcal{G}_i)$ denotes the cohomology group of $\mathcal{G}_i$ given in (3.1.5).
- (B-2) From bases of $H^r(\mathcal{G}_i)$ for $r-q \le i \le r-q+2$ and A_i for $r-q \le i \le r-q+1$, we compute the representation matrices of the maps $H^r(f_{r-q})$ and $H^r(f_{r-q+1})$, where

  $$H^r(f_i) : H^r(\mathcal{G}_{i+1}) \longrightarrow H^r(\mathcal{G}_i)$$

  is a K-linear map given by $v \mapsto vA_i$ for $i = r-q$ and $r-q+1$.
- (B-3) We next compute bases of the K-vector spaces $\mathrm{Ker}(H^r(f_{r-q}))$ and $\mathrm{Im}(H^r(f_{r-q+1}))$.
- (B-4) Finally, we compute a basis of

  $$\mathrm{Ker}(H^r(f_{r-q}))/\mathrm{Im}(H^r(f_{r-q+1})) \cong H^q(\mathcal{F}(n)).$$

In Step A, we utilize theory of Gröbner bases. For Step B, we utilize partition and sorting techniques in (B-1), and linear algebra techniques in (B-2)–(B-4). In particular, in (B-4), we utilize a linear algebra technique to extend a given basis of a vector space to a basis of a higher dimensional space, and to compute a basis of a quotient vector space $\mathrm{Ker}(H^r(f_{r-q}))/\mathrm{Im}(H^r(f_{r-q+1}))$ over K.

Remark 3.2.1 In Step A, we compute a (minimal) free resolution for the input module. Several computational methods for free resolutions are proposed and implemented in computer algebra systems. Such computations for free resolutions can be done in exponential time in general. However, objects such as the cohomology groups should be determined by mathematical invariants of input structures. From this, in our complexity analysis of Subsection 3.5, we set such mathematical invariants obtained from the form of free resolutions as inputs.

3.2.2 Procedures in Step B

Computation of Step A is well-known in theory of Gröbner bases (see [3, Chapter 6] for details), and we skip Step A. Here we give an algorithm of Step B. We also describe an outline of sub-producers in Step B (details of the sub-procedures shall be described in Subsection 3.3).

Recall that we obtain a minimal free resolution of length $\ell = r+1$ for the graded S-module M in Step A. Then we assume that we have the following:

$$0 \to \bigoplus_{j=1}^{t_{\ell+1}} S\left(-d_j^{(\ell)}\right) \overset{\varphi_\ell}{\to} \cdots \overset{\varphi_1}{\to} \bigoplus_{j=1}^{t_1} S\left(-d_j^{(1)}\right) \overset{\varphi_0}{\to} M \to 0. \tag{3.2.1}$$

Put

$$\begin{aligned} &M_i := \begin{cases} \bigoplus_{j=1}^{t_i} S\left(-d_j^{(i)}\right) & (0 \le i \le \ell+1), \\ M & (i=0), \end{cases} \qquad t := [\ t_1, \ldots, t_{\ell+1}\ ], \\ &d^{(i)} := \left[\ d_1^{(i)}, \ldots, d_{t_i}^{(i)}\ \right] \ (1 \le i \le \ell+1), \quad d := \left[\ d^{(1)}, \ldots, d^{(\ell+1)}\ \right]. \end{aligned} \tag{3.2.2}$$

Recall that φ_i is represented by the matrix A_i given in (3.1.2) for each $0 \le i \le \ell$. In Step A, all entries $g_{k,\ell}^{(i)}$ of A_i are also determined explicitly.

In Algorithm 3.2.1, we show our algorithm to compute a basis of the cohomology group $H^q(\mathcal{F}(n))$ for $1 \le q \le r$. In our algorithm, we take M, $F(M) := \left((M_i, \varphi_i, t_i, d^{(i)}, A_i)\right)_{0 \le i \le \ell+1}$, q and $n \in \mathbb{Z}$ as inputs. In Subsection 3.3, we shall prove the correctness of Algorithm 3.2.1 to output a basis of $H^q(\mathcal{F}(n))$.

Algorithm 3.2.1 includes several sub-procedures as subroutines. In the following, we describe each of the sub-procedures (detailed description of the sub-procedures shall be given in Subsection 3.3):

(1) `CohomologyOfStructureSheafSum`: Given n, t and $d = [d_1, \ldots, d_t]$, this sub-procedure computes a basis of $H^r(\mathcal{G})$, where $\mathcal{G} := \bigoplus_{j=1}^t \mathcal{O}_{\mathbb{P}^r}(n - d_j)$. In Step (B-1) of Algorithm 3.2.1, we apply this sub-procedure to compute a basis of $H^r(\mathcal{G}_i)$ for $r-q \le i \le r-q+2$, where $\mathcal{G}_i$ is defined by (3.1.5).

(2) `RepresentationMatrix`: Let φ be a homomorphism defined by

$$\varphi\ :\ \bigoplus_{j=1}^{t} S(m_j) \longrightarrow \bigoplus_{j=1}^{t'} S\left(m_j'\right)\ ;\ \mathbf{u} \mapsto \mathbf{u} \cdot A, \tag{3.2.3}$$

where A is a $(t \times t')$ matrix such that the (k, ℓ)-entry $g_{k,\ell}$ of the matrix A is a homogeneous polynomial of degree $(m'_\ell - m_k)$ in S for each k and ℓ. Let

$$\mathcal{G} := \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}^r}(m_j) \text{ and } \mathcal{G}' := \bigoplus_{j=1}^{t'} \mathcal{O}_{\mathbb{P}^r}(m'_j) \tag{3.2.4}$$

denote the coherent sheaves associated with $\bigoplus_{j=1}^{t} S(m_j)$ and $\bigoplus_{j=1}^{t'} S(m'_j)$, respectively. Given φ and bases of $H^r(\mathcal{G})$ and $H^r(\mathcal{G}')$, this sub-procedure computes the representation matrix of the induced morphism

$$H^r(\widetilde{\varphi}) \ : \ H^r(\mathbb{P}^r, \mathcal{G}) \longrightarrow H^r\left(\mathbb{P}^r, \mathcal{G}'\right) \ ; \ w \mapsto w \cdot A. \tag{3.2.5}$$

In our implementation for this sub-procedure, we minimize the representation matrix for efficiency. We use this sub-procedure in Step (B-2) of Algorithm 3.2.1. This sub-procedure calls the following two functions:

- `Action`: For each element v of a basis of $H^r(\mathcal{G})$, this function computes the element $H^r(\widetilde{\varphi})(v) \in H^r(\mathcal{G}')$.
- `ColumnOfRepresentationMatrix`: This function computes a column of a representation matrix of $H^r(\widetilde{\varphi})$.

(3) `QuatientSpaceBasisMatrix`: Given a K-vector space V with a (row) basis matrix A and its subspace $W \subseteq V$ with a (row) basis matrix B, this sub-procedure computes a basis matrix C of the quotient vector space V/W. We use this sub-procedure in Step (B-4) of Algorithm 3.2.1.

In Table 1, we give an outline of our algorithm (Algorithm 3.2.1) to compute a basis $H^q(\mathcal{F}(n))$ for $1 \leq q \leq r$.

3.3 Detailed description on sub-procedures in Step B

In this subsection, we give a precise description on the sub-procedures given in Subsection 3.2.2. Recall that the following sub-procedures are used as subroutines in Algorithm 3.2.1:

(1) `CohomologyOfStructureSheafSum`,

(2) `RepresentationMatrix`,

- `Action`,
- `ColumnOfRepresentationMatrix`, and

(3) `QuatientSpaceBasisMatrix`.

Let $\mathbb{P}^r_K = \operatorname{Proj}(S)$ be the projective r-space with $S = K[X_0, \ldots, X_r]$, and $\mathcal{O}_{\mathbb{P}^r_K}$ the structure sheaf on $\mathbb{P}^r_K$. Let $H^q(\mathbb{P}^r_K, \mathcal{H})$ denote the q-th cohomology group of a coherent sheaf $\mathcal{H}$ on $\mathbb{P}^r_K$. To simplify the notations, we denote $\mathbb{P}^r_K$ and $\mathcal{O}_{\mathbb{P}^r_K}$ by $\mathbb{P}^r$ and $\mathcal{O}_{\mathbb{P}^r}$, respectively, and denote $H^q(\mathbb{P}^r, \mathcal{H})$ by $H^q(\mathcal{H})$ for a coherent sheaf $\mathcal{H}$ on $\mathbb{P}^r$.

Table 1: Outline of our algorithm (Algorithm 3.2.1) of Maruyama's method to compute $H^q(\mathcal{F}(n))$ for $1 \leq q \leq r$

Procedures			Techniques
Step A	Compute a (minimal) free resolution of M		Gröbner basis
Step B	(B-1)	Compute a basis of $H^r(\mathcal{G}_i)$ for each $r-q \leq i \leq r-q+2$ (`CohomologyOfStructureSheafSum`).	Partition, Sorting
	(B-2)	Compute representation matrices of $H^r(f_{r-q})$ and $H^r(f_{r-q+1})$ (`RepresentationMatrix`, `ColumnOfRepresentationMatrix` and `Action`).	Linear algebra
	(B-3)	Compute bases of $\mathrm{Ker}(H^r(f_{r-q}))$ and $\mathrm{Im}(H^r(f_{r-q+1}))$ (Solve a linear system and compute an echelon form of a matrix)	
	(B-4)	Compute a basis of $\mathrm{Ker}(H^r(f_{r-q}))/\mathrm{Im}(H^r(f_{r-q+1}))$ (`QuatientSpaceBasisMatrix`)	

3.3.1 Description on `CohomologyOfStructureSheafSum`

Recall that given n, t and $d = [d_1, \ldots, d_t]$, this sub-procedure computes a basis of the K-vector space $H^r(\mathcal{G})$, where $\mathcal{G} := \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}^r}(n-d_j)$. In Algorithm 3.3.1, we show an algorithm for this sub-procedure.

Proposition 3.3.1 *Given n, t and $d = [d_1, \ldots, d_t]$,* Algorithm 3.3.1 *outputs a basis of the K-vector space $H^r(\mathcal{G})$, where $\mathcal{G} := \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}^r}(n-d_j)$. In particular, if $H^r(\mathcal{G}) = 0$,* Algorithm 3.3.1 *outputs $\emptyset$.*

Proof. It is sufficient to consider the case of $t \geq 1$. We have $H^r(\mathcal{G}) = \bigoplus_{j=1}^{t} H^r(\mathcal{O}_{\mathbb{P}^r}(n-d_j))$. For an element $v \in H^r(\mathcal{O}_{\mathbb{P}^r}(n-d_j))$, we denote by $\iota_j(v)$ an element $(0, \ldots, 0, v, 0, \ldots, 0)$ in $H^r(\mathcal{G})$, where we let ι_j denote the embedding

$$\iota_j : H^r(\mathcal{O}_{\mathbb{P}^r}(n-d_j)) \hookrightarrow H^r(\mathcal{G}) = \bigoplus_{j=1}^{t} H^r(\mathcal{O}_{\mathbb{P}^r}(n-d_j)) \tag{3.3.1}$$

for $1 \leq j \leq t$. Put $R_j := \mathrm{Im}(\iota_j) = \{\iota_j(v) \; ; \; v \in H^r(\mathcal{O}_{\mathbb{P}^r}(n-d_j))\}$ for $1 \leq j \leq t$. For each $1 \leq j \leq t$, the set

$$\{{X_0}^{l_0} \cdots {X_r}^{l_r} \; ; \; l_i < 0 \text{ for } 0 \leq i \leq r, \text{ and } l_0 + \cdots + l_r = n - d_j\} \tag{3.3.2}$$

is a basis of the K-vector space $H^r(\mathcal{O}_{\mathbb{P}^r}(n-d_j))$ if $n - d_j \leq -r-1$ by Theorem 2.1.1 (3). Thus

$$\left\{\iota_j\left({X_0}^{l_0} \cdots {X_r}^{l_r}\right); l_i < 0 \text{ for } 0 \leq i \leq r, \text{ and } l_0 + \cdots + l_r = n - d_j\right\} \tag{3.3.3}$$

Algorithm 3.2.1 `CohomologyBasis`$(M, F(M), q, n)$

Input: A finitely generated graded S-module M, the minimal free resolution $F(M) := \left((M_i, \varphi_i, t_i, d^{(i)}, A_i)\right)_{0 \leq i \leq r+2}$ of M, an integer $q \in \{1, \ldots, r\}$ and $n \in \mathbb{Z}$

Output: A basis of $H^q(\mathbb{P}^r_K, \mathcal{F}(n))$, where $\mathcal{F} := \widetilde{M}$

/*Step A has been finished*/
$q' \leftarrow r - q + 1$
/***Step (B-1)***/
for $i = q' - 1$ **to** $q' + 1$ **do**
  $\mathcal{V}_i \leftarrow$ `CohomologyOfStructureSheafSum`$\left(t_i, d^{(i)}, n\right)$ /*Basis of $H^r(\mathcal{G}_i)$*/
  $k_i \leftarrow \#(\mathcal{V}_i)$ /*$\dim_K H^r(\mathcal{G}_i)$*/
end for
if $\mathcal{V}_{q'} = \emptyset$ **then**
  return $\emptyset$
else
  /*Case of $H^r\left(\mathcal{G}_{q'}\right) \neq 0$*/
  $\{v_1, \ldots, v_{k_{q'}}\} \leftarrow \mathcal{V}_{q'}$
  /***Step (B-2)***/
  for $i = q' - 1$ **to** q' **do**
    $R_i \leftarrow$ `RepresentationMatrix`$(\mathcal{V}_{i+1}, \mathcal{V}_i, A_i)$ /*Representation matrix of $H^r(\widetilde{\varphi}_i)$*/
  end for
  /***Step (B-3)***/
  /*Solve the linear system $\mathbf{v} \cdot {}^t R_{q'-1} = \mathbf{0}$ over K*/
  $B_{\mathrm{Ker}} \leftarrow$ (basis matrix of $\left\{\mathbf{v} \in K^{k_{q'}} \ ; \ \mathbf{v} \cdot {}^t R_{q'-1} = \mathbf{0}\right\} \subseteq K^{k_{q'}}$)
  /*Compute the reduced row echelon form of the matrix $R_{q'}$*/
  $B_{\mathrm{Im}} \leftarrow$ (basis matrix of $\left\{\mathbf{u} \cdot {}^t R_{q'} \ ; \ \mathbf{u} \in K^{k_{q'+1}}\right\} \subseteq K^{k_{q'}}$)
  /***Step (B-4)***/
  $B_{\mathrm{coh}} \leftarrow$ `QuatientSpaceBasisMatrix`$(B_{\mathrm{Ker}}, B_{\mathrm{Im}})$
  return (the set of the row vectors of the matrix $B_{\mathrm{coh}} \cdot {}^t[v_1, \ldots, v_{k_{q'}}]$)
end if

is a basis of the K-vector space R_j if $n - d_j \leq -r - 1$. We also note that $H^r(\mathcal{O}_{\mathbb{P}^r}(n - d_j)) = 0$ if $n - d_j > -r - 1$. Thus the output basis $\mathcal{V}$ of Algorithm 3.3.1 is a basis of the K-vector space $H^r(\mathcal{G})$ if there exists an index j such that $n - d_j \leq r - 1$. In the case of $n - d_j > r - 1$ for all $1 \leq j \leq t$, we have $H^r(\mathcal{G}) = 0$. In this case, Algorithm 3.3.1 outputs $\emptyset$. $\square$

3.3.2 Description on `RepresentationMatrix`

Let φ be a homomorphism defined by (3.2.3). Let $\mathcal{G}$ and $\mathcal{G}'$ be coherent sheaves as in (3.2.4). Given φ and bases of $H^r(\mathcal{G})$ and $H^r(\mathcal{G}')$, this sub-procedure computes the representation matrix of $H^r(\widetilde{\varphi})$, defined in (3.2.5). In Algorithm 3.3.2, we give an algorithm for this sub-procedure.

Algorithm 3.3.2 calls `Action` and `ColumnOfRepresentationMatrix` as subroutines. In the following, let us first describe each function:

Algorithm 3.3.1 `CohomologyOfStructureSheafSum` (t, d, n)

Input: An integer $t \in \mathbb{Z}_{\geq 0}$, a sequence $d = [d_1, \ldots, d_t]$ of integers, and $n \in \mathbb{Z}$
Output: A basis of the K-vector space $H^r(\mathbb{P}^r, \mathcal{G})$, or $\emptyset$, where $\mathcal{G} := \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}^r}(n - d_j)$

$\mathcal{V} \leftarrow \emptyset$
if $t \geq 1$ **then**
  for $j = 1$ **to** t **do**
    if $n - d_j \leq -r - 1$ **then**
      $\mathcal{V}_j \leftarrow \{\iota_j\left({X_0}^{l_0} \cdots {X_r}^{l_r}\right) \ ; \ l_i < 0 \text{ for } 0 \leq i \leq r, \text{ and } l_0 + \cdots + l_r = n - d_j\}$
      $\mathcal{V} \leftarrow \mathcal{V} \cup \mathcal{V}_j$
    end if
  end for
end if
return $\mathcal{V}$

Algorithm 3.3.2 `RepresentationMatrix` $(\mathcal{V}, \mathcal{V}', A)$

Input: Ordered bases $\mathcal{V} = \{v_1, \ldots, v_s\}$ and $\mathcal{V}' = \{v'_1, \ldots, v'_{s'}\}$ of the K-vector spaces $H^r\left(\bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}^r}(m_j)\right)$ and $H^r\left(\bigoplus_{j=1}^{t'} \mathcal{O}_{\mathbb{P}^r}(m'_j)\right)$, respectively, and a $(t \times t')$ matrix A representing a given homomorphism $\varphi : \bigoplus_{j=1}^{t} S(m_j) \to \bigoplus_{j=1}^{t'} S(m'_j)$
Output: The $(s' \times s)$ matrix R such that $[H^r(\widetilde{\varphi})(v_1), \ldots, H^r(\widetilde{\varphi})(v_s)] = [v'_1, \ldots, v'_{s'}] \cdot R$

$R \leftarrow$ (the $(s' \times s)$ zero matrix over K)
for $i = 1$ **to** s **do**
  $\mathrm{Im}_i \leftarrow$ `Action` (v_i, A) /*Compute $H^r(\widetilde{\varphi})(v_i)$*/
  $\mathrm{Cols}_i \leftarrow$ `ColumnOfRepresentationMatrix` $(\mathrm{Im}_i, \mathcal{V}')$ /*Compute the i-th column of R*/
  Replace the i-th column of R by ${}^t(\mathrm{Cols}_i)$
end for
return R

Description on `Action`: Recall that the function `Action` computes the element $H^r(\widetilde{\varphi})(v) \in H^r(\mathcal{G}')$ for each element v of a basis of $H^r(\mathcal{G})$. In Algorithm 3.3.3, we give an algorithm for this function.

Proposition 3.3.2 *Let φ be a graded homomorphism of degree zero of free S-modules defined by* (3.2.3). *Fix $1 \leq i \leq t$, and let $\iota_i({X_0}^{l_0} \cdots {X_r}^{l_r})$ be an element of a basis of the K-vector space $H^r\left(\mathbb{P}^r, \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}^r}(m_j)\right)$, where ι_i is defined by* (3.3.1). *Let $H^r(\widetilde{\varphi})$ denote the K-linear map given in* (3.2.5). *For $1 \leq j \leq t'$, let h_j denote the j-th entry of $H^r(\widetilde{\varphi})(\iota_i(X_0^{l_0} \cdots X_r^{l_r}))$, namely,*

$$H^r(\widetilde{\varphi})(\iota_i(X_0^{l_0} \cdots X_r^{l_r})) = [h_1, \ldots, h_{t'}].$$

Then Algorithm 3.3.3 *outputs the set*

$$\left\{\iota_j(h_{j,k}) \ ; \ 1 \leq j \leq t', \ h_{j,k} \in \{\text{term of } h_j\} \ \right\}$$

with $\sum_{j=1}^{t'} \sum_k \iota_j(h_{j,k}) = H^r(\widetilde{\varphi})\left(\iota_i({X_0}^{l_0} \cdots {X_r}^{l_r})\right)$.

Algorithm 3.3.3 `Action` $\left(\iota_i({X_0}^{l_0}\cdots {X_r}^{l_r}), A\right)$

Input: An element $\iota_i\left({X_0}^{l_0}\cdots {X_r}^{l_r}\right)$ of a basis of the K-vector space $H^r\left(\mathbb{P}^r, \bigoplus_{i=1}^{t}\mathcal{O}_{\mathbb{P}^r}(m_j)\right)$ and a $(t\times t')$ matrix A representing a given homomorphism $\varphi : \bigoplus_{j=1}^{t} S(m_j)\to\bigoplus_{j=1}^{t'} S(m'_j)$

Output: The set $\{\iota_j(h_{j,k})\ ;\ 1\le j\le t',\ h_{j,k}\in\{\text{term of } h_j\}\ \}$ or $\emptyset$, where each h_j is the j-th entry of the vector $H^r(\widetilde{\varphi})\left(\iota_i\left({X_0}^{l_0}\cdots {X_r}^{l_r}\right)\right)$

Im $\leftarrow\emptyset$
for $j=1$ **to** t' **do**
  $g_{i,j}\leftarrow$ (the (i,j)-entry of A)
  $N_{i,j}\leftarrow$ (the number of the terms of $g_{i,j}$) /*$g_{i,j}$ is a homogeneous polynomial*/
  if $N_{i,j}\neq 0$ **then**
    $T\leftarrow$ (the (lexicographical) ordered set of the terms of $g_{i,j}$)
    for $k=(k_0,\ldots,k_r)\in T$ **do**
      $h_{j,k}\leftarrow 1_K$
      $a_k\leftarrow$ (the coefficient of $X_0^{k_0}\cdots X_r^{k_r}$ in $g_{i,j}$)
      $\mathit{break}(i')\leftarrow 0$
      for $i'=0$ **to** r **do**
        if $l_{i'}+k_{i'}\le -1$ **then**
          $h_{j,k}\leftarrow h_{j,k}\cdot X_{i'}^{(l_{i'}+k_{i'})}$
        else
          $\mathit{break}\,(i')\leftarrow 1$
          break i /*Means $a_k{X_0}^{k_0+l_0}\cdots {X_r}^{k_r+l_r}=0$ in $H^r(\mathbb{P}^r,\mathcal{O}_{\mathbb{P}^r}(m'_j))$*/
        end if
      end for
      if $\mathit{break}\,(i')\neq 0$ **then**
        $h_{j,k}\leftarrow a_k\cdot h_{j,k}$
        Im $\leftarrow$ Im $\cup\{\iota_j(h_{j,k})\}$
      end if
    end for
  end if
end for
return Im

Proof. Let $A=(g_{i,j})_{i,j}$ denote the representation matrix of φ defined in (3.2.3). Note that each $g_{i,j}$ is a homogeneous polynomial in S. We denote by $N_{i,j}$ the number of the terms of each $g_{i,j}$. Suppose $N_{i,j}\ge 1$ (the case $N_{i,j}$ is trivial). Then we can write

$$g_{i,j}=\sum a_k X_0^{k_0}\cdots X_r^{k_r}\in S$$

for some $a_k\in K\smallsetminus\{0\}$ with $k=(k_0,\ldots,k_r)\in(\mathbb{Z}_{\ge 0})^{r+1}$. Let T be the lexicographical ordered set consisting of the terms of $g_{i,j}$. The j-th entry of the vector $H^r(\widetilde{\varphi})\left(\iota_i({X_0}^{l_0}\cdots {X_r}^{l_r})\right)$ is

$${X_0}^{l_0}\cdots {X_r}^{l_r}g_{i,j}=\sum a_k{X_0}^{l_0+k_0}\cdots X_r^{l_r+k_r}.$$

If $l_{i'}+k_{i'}\ge 0$ for some $0\le i'\le r$, then the term $a_k{X_0}^{l_0+k_0}\cdots X_r^{l_r+k_r}$ can be regarded as 0 via the isomorphism (2.1.1) in Theorem 2.1.1 (3). Thus $\iota_i(h_{j,k})$ is equal to $a_k{X_0}^{l_0+k_0}\cdots X_r^{l_r+k_r}$ if

$l_{i'} + k_{i'} \leq -1$ for all i'. It also follows that $\{\iota_i(h_{j,k})\}_k$ is the set of the terms of the j-th entry of the vector $H^r(\widetilde{\varphi})\left(\iota_i({X_0}^{l_0}\cdots {X_r}^{l_r})\right)$ for each $1 \leq j \leq t'$. Then Proposition 3.3.2 holds. $\square$

Description on `ColumnOfRepresentationMatrix`: Recall that this function computes a column of a representation matrix of $H^r(\widetilde{\varphi})$. In Algorithm 3.3.4, we give an algorithm for this function.

Algorithm 3.3.4 `ColumnOfRepresentationMatrix` $(\mathrm{Im}, \mathcal{W})$

Input: A finite subset $\mathrm{Im} = \{\iota_{i(k)}(a_k X_0^{k_0} \cdots X_r^{k_r})\ ;\ k_i < 0 \text{ for } 0 \leq i \leq r, \text{and } \sum_i k_i = m_{i(k)}\}_k$ indexed by $k = (k_0, \ldots, k_r)$'s of the K-vector space $W := H^r\left(\mathbb{P}^r, \bigoplus_{j=1}^t \mathcal{O}_{\mathbb{P}^r}(m_j)\right)$ and its basis $\mathcal{W} = \{\iota_j({X_0}^{l_0} \cdots {X_r}^{l_r})\ ;\ 1 \leq j \leq t,\ l_i < 0 \text{ for } 0 \leq i \leq r, \text{ and } l_0 + \cdots + l_r = m_j\}$, where $t \in \mathbb{Z}_{>0}$ and m_j $(1 \leq j \leq t)$ are given

Output: The vector $[b_1, \ldots, b_s] \in K^s$ such that $\sum_{h \in \mathrm{Im}} h = \sum_{j=1}^s b_j w_j$, where $s = \#(\mathcal{W})$ and $\mathcal{W} = \{w_1, \ldots, w_s\}$

b $\leftarrow [0_K, \ldots, 0_K] \in K^s$
$N_{\mathrm{Im}} \leftarrow$ (the (ordered) finite set of the indexes $k = (k_0, \ldots, k_r)$ of Im)
for $k = (k_0, \ldots, k_r) \in N_{\mathrm{Im}}$ **do**
  $\iota_{i(k)}(a_k {X_0}^{k_0} \cdots {X_r}^{k_r}) \leftarrow$ (the element of Im corresponding to the index k)
  $\mathcal{W}_k \leftarrow \mathcal{W}$
  $\{\iota_{i(j)}({X_0}^{j_0} \cdots {X_r}^{j_r})\}_j \leftarrow \mathcal{W}_k$
  $\mathrm{Ind}(\mathcal{W}_k) \leftarrow$ (the set of the indexes $j = (j_0, \ldots, j_r)$ of $\mathcal{W}_k$)
  for $j = (j_0, \ldots, j_r) \in \mathrm{Ind}(\mathcal{W}_k)$ **do**
    $\iota_{i(j)}({X_0}^{j_0} \cdots {X_r}^{j_r}) \leftarrow$ (the element of $\mathcal{W}_k$ corresponding to the index j)
    if $(k_0, \ldots, k_r, i(k)) = (j_0, \ldots, j_r, i(j))$ **then**
      Replace by a_k the entry of **b** corresponding to $\iota_{i(j)}\left({X_0}^{j_0} \cdots {X_r}^{j_r}\right)$
      $\mathcal{W}_k \leftarrow \mathcal{W}_k \setminus \{\iota_{i(j)}\left({X_0}^{j_0} \cdots {X_r}^{j_r}\right)\}$
      break j
    end if
  end for
end for
return b

Proposition 3.3.3 *Let*

$$\mathrm{Im} = \left\{ \iota_{i(k)}\left(a_k X_0^{k_0} \cdots X_r^{k_r}\right)\ ;\ k_i < 0 \text{ for } 0 \leq i \leq r, \text{ and } \sum_i k_i = m_{i(k)} \right\}_k$$

be a (ordered) finite subset indexed by $k = (k_0, \ldots, k_r)$'s of the K-vector space $H^r\left(\mathbb{P}^r, \bigoplus_{j=1}^t \mathcal{O}_{\mathbb{P}^r}(m_j)\right)$ with the basis

$$\mathcal{W} = \{\iota_j({X_0}^{l_0} \cdots {X_r}^{l_r})\ ;\ 1 \leq j \leq t,\ l_i < 0 \text{ for } 0 \leq i \leq r, \text{ and } l_0 + \cdots + l_r = m_j\},$$

where ι_j is defined by (3.3.1). *We put $s := \#(\mathcal{W})$. Let $\{w_1, \ldots, w_s\}$ be the (ordered) set of all elements of $\mathcal{W}$. Then* Algorithm 3.3.4 *outputs the vector $[b_1, \ldots, b_s] \in K^s$ such that $\sum_{h \in \mathrm{Im}} h = \sum_{j=1}^s b_j w_j$.*

Proof. The iteration starts with some $k = (k_0, \ldots, k_r) \in N_{\mathrm{Im}}$, where N_{Im} denotes the (ordered) finite set of the indexes $k = (k_0, \ldots, k_r)$ of Im. We set $\mathcal{W}_k := \mathcal{W}$. Since $\mathcal{W}_k$ is a basis of the K-vector space $H^r\left(\mathbb{P}^r, \bigoplus_{j=1}^t \mathcal{O}_{\mathbb{P}^r}(m_j)\right)$, there exists unique $w_{j(k')} \in \mathcal{W}$ such that $\iota_{i(k)}({X_0}^{k_0} \cdots {X_0}^{k_r}) = w_{j(k')}$. Then we set $b_{j(k')} := a_k$. Note that $\iota_{i(l)}({X_0}^{l_0} \cdots {X_0}^{l_r}) \neq w_{j(k')}$ for each $l \neq k$. Thus we obtain $b_{j(k')} \in K$ such that $\iota_{i(k)}(a_k {X_0}^{k_0} \cdots {X_0}^{k_r}) = b_{j(k')} w_{j(k')}$ for each $k \in N_{\mathrm{Im}}$. As a result, we have $\sum_{h \in \mathrm{Im}} h = \sum_{j=1}^s b_j w_j$, where we set $b_j := 0$ if $\iota_{i(k)}(a_k {X_0}^{k_0} \cdots {X_0}^{k_r}) \neq b_j w_j$ for each $k \in N_{\mathrm{Im}}$. $\square$

Correctness of `RepresentationMatrix`: Here we prove the correctness of the function `RepresentationMatrix` (Algorithm 3.3.2).

Proposition 3.3.4 *Let φ be a homomorphism defined by* (3.2.3). *Let $\mathcal{G}$ and $\mathcal{G}'$ be coherent sheaves as in* (3.2.4). *Given φ and bases $\mathcal{V}$ and $\mathcal{W}$ of $H^r(\mathcal{G})$ and $H^r(\mathcal{G}')$,* Algorithm 3.3.2 *outputs the representation matrix of $H^r(\widetilde{\varphi})$, defined in* (3.2.5) *via the bases.*

Proof. Each element $v \in \mathcal{V}$ can be written as $v = \iota_i({X_0}^{l_0} \cdots {X_r}^{l_r})$ for some i and $l_0, \ldots, l_r$. By Proposition 3.3.2, Algorithm 3.3.3 outputs the set

$$\left\{\iota_j(h_{j,k})\ ;\ 1 \leq j \leq t',\ h_{j,k} \in \{\text{term of } h_j\}\ \right\}$$

such that $\sum_{j=1}^{t'} \sum_k \iota_j(h_{j,k}) = H^r(\widetilde{\varphi})\left(\iota_i({X_0}^{l_0} \cdots {X_r}^{l_r})\right)$, where each h_j is an element in $H^r(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m_j))$. Put $s := \#(\mathcal{W})$. By Proposition 3.3.3, Algorithm 3.3.4 outputs the vector $[b_1, \ldots, b_s] \in K^s$ such that $\sum_{j=1}^{t'} \sum_k \iota_j(h_{j,k}) = \sum_{j=1}^s b_j w_j$, and thus we have $H^r(\widetilde{\varphi})\left(\iota_i({X_0}^{l_0} \cdots {X_r}^{l_r})\right) = \sum_{j=1}^s b_j w_j$. Consequently Algorithm 3.3.2 outputs the representation matrix of $H^r(\widetilde{\varphi})$ via $\mathcal{V}$ and $\mathcal{W}$. $\square$

3.3.3 Description on `QuotientSpaceBasisMatrix`

Recall that given a K-vector space V with a (row) basis matrix A and its subspace $W \subseteq V$ with a (row) basis matrix B, this sub-procedure computes a basis matrix C of the quotient vector space V/W. In Algorithm 3.3.5, we show an algorithm for this sub-procedure.

Proposition 3.3.5 *Given a K-vector space V with a* (*row*) *basis matrix A and its subspace $W \subseteq V$ with a* (*row*) *basis matrix B,* Algorithm 3.3.5 *outputs a basis matrix of the K-quotient space V/W.*

Proof. We denote by $\mathbf{a}_i = [a_{i,1}, \ldots, a_{i,m}]$ and $\mathbf{b}_j = [b_{j,1}, \ldots, b_{j,m}]$ the i-th row vector of the matrix A and the j-th row vector of the matrix B, respectively. Since the K-vector space W is a subspace of the K-vector space V, there exist unique $u_{1,j}, \ldots, u_{m_1,j} \in K$ such that $\mathbf{b}_j = u_{1,j}\mathbf{a}_1 + \cdots + u_{m_1,j}\mathbf{a}_{m_1}$ for each $1 \leq j \leq m_2$. Note that $\mathbf{u}_j := [u_{1,j}, \ldots, u_{m_1,j}]$ is the j-th row of U in Algorithm 3.3.5 for each $1 \leq j \leq m_2$, where U is the $(m_2 \times m_1)$ matrix such that $B = U \cdot A$. Let U' be a basis matrix of K^{m_1} obtained by extending the basis of the row vectors of the matrix U. We denote by $\mathbf{u}'_j = [u'_{1,j}, \ldots, u'_{m_1,j}]$ the j-th row vector of the matrix U' for each $m_2 + 1 \leq j \leq m_1$. Let U'' be the $((m_1 - m_2) \times m_1)$-matrix such that its j-th row vector is $\mathbf{u}'_{j+m_2}$ for each $1 \leq j \leq m_1 - m_2$. The $m_1 - m_2$ elements $u'_{1,j}\mathbf{a}_1 + \cdots + u'_{m_1,j}\mathbf{a}_{m_1} \in V$ for $m_2 + 1 \leq j \leq m_1$ are linearly independent since U'' is a full-rank matrix. In addition, the dimension of the K-vector space V/W is precisely equal to $m_1 - m_2$. Thus the matrix $U'' \cdot A$ can be regarded as a basis matrix of the K-quotient space V/W. $\square$

Algorithm 3.3.5 `QuotientSpaceBasisMatrix` (A, B)

Input: A basis matrix A of a K-vector subspace V of K^m and a basis matrix B of a K-vector subspace $W \subseteq V$ of K^m

Output: A basis matrix C of the K-quotient space V/W

if B is a zero matrix **then**
  return A
else
  $m_1 \leftarrow$ (the number of the rows of A (the dimension of V))
  $m_2 \leftarrow$ (the number of the rows of B (the dimension of W))
  $U \leftarrow$ (the $(m_2 \times m_1)$ matrix such that $B = U \cdot A$)
  $U' \leftarrow$ (a basis matrix of K^{m_1} obtained by extending the basis of the row vectors of U)
  $U'' \leftarrow U'$
  for $j = 1$ **to** m_2 **do**
    $\mathbf{b}_j \leftarrow$ (the j-th row vector of U)
    Remove $\mathbf{b}_j$ from U'' and update U''
  end for
  return $U'' \cdot A$
end if

3.4 Correctness of Algorithm 3.2.1

In Subsection 3.3, we gave a precise description on the sub-procedures of Algorithm 3.2.1. In this subsection, we prove the correctness of Algorithm 3.2.1.

Let $\mathbb{P}_K^r = \mathrm{Proj}\,(S)$ be the projective r-space over a field K with the polynomial ring $S = K[X_0, \ldots, X_r]$. Let $\mathcal{O}_{\mathbb{P}_K^r}$ be the structure sheaf of $\mathbb{P}_K^r$. To simplify the notations, we denote $\mathbb{P}_K^r$ and $\mathcal{O}_{\mathbb{P}_K^r}$ by $\mathbb{P}^r$ and $\mathcal{O}_{\mathbb{P}^r}$, respectively, and denote $H^q(\mathbb{P}^r, \mathcal{H})$ by $H^q(\mathcal{H})$ for a coherent sheaf $\mathcal{H}$ on $\mathbb{P}^r$. Recall that for a given coherent sheaf $\mathcal{F}$, integers $q \geq 1$ and n, our aim is to compute a basis of the K-vector space $H^q(\mathcal{F}(n))$. As in the previous subsection, let M denote a finitely generated graded S-module corresponding to $\mathcal{F}$. Recall that M has a (minimal) free resolution of length $\ell = r + 1$ of the form (3.1.1). We also recall that the input parameters of Algorithm 3.2.1 are M, $F(M) := \left(\left(M_i, \varphi_i, t_i, d^{(i)}, A_i\right)\right)_{0 \leq i \leq \ell+1}$, $q \geq 1$ and $n \in \mathbb{Z}$, where M_i, t_i, $d^{(i)}$ and A_i are given in (3.1.2) and (3.2.2) for $0 \leq i \leq \ell + 1$.

Theorem 3.4.1 *The notations are same as above. For input parameters M, $F(M) = \left(\left(M_i, \varphi_i, t_i, d^{(i)}, A_i\right)\right)_{0 \leq i \leq \ell+1}$ $q \geq 1$ and $n \in \mathbb{Z}$,* Algorithm 3.2.1 *outputs a basis of the K-vector space $H^q(\mathcal{F}(n))$.*

Proof. Recall that by Theorem 2.2.1 (1), the cohomology group $H^q(\mathcal{F}(n))$ is isomorphic to the K vector space $\mathrm{Ker}\,(H^r(f_{r-q}))\,/\mathrm{Im}\,(H^r(f_{r-q+1}))$ via the following complex:

$$H^r(\mathcal{G}_{r-q+2}) \xrightarrow{H^r(f_{r-q+1})} H^r(\mathcal{G}_{r-q+1}) \xrightarrow{H^r(f_{r-q})} H^r(\mathcal{G}_{r-q}).$$

Put $q' := r - q + 1$. Let $\mathcal{V}_i = \{v_1^{(i)}, \ldots, v_{k_i}^{(i)}\}$ be the ordered set output by Algorithm 3.3.1 for inputs t_i, $d^{(i)}$ and n. By Proposition 3.3.1, $\mathcal{V}_i$ is a basis of the K-vector space $H^r(\mathcal{G}_i)$ for $q' - 1 \leq i \leq q' + 1$. If $H^r\left(\mathcal{G}_{q'}\right) = 0$ (i.e., $\mathcal{V}_{q'} = \emptyset$), then $\mathrm{Ker}\left(H^r\left(f_{q'-1}\right)\right) = \mathrm{Im}\left(H^r\left(f_{q'}\right)\right) = 0$ and $H^q(\mathcal{F}(n)) = 0$. Thus the output $\emptyset$ by Algorithm 3.2.1 is correct if $\mathcal{V}_{q'} = \emptyset$.

Suppose $\mathcal{V}_{q'} \neq \emptyset$. Put $k_{q'} := \#\left(\mathcal{V}_{q'}\right) = \dim_K H^r\left(\mathcal{G}_{q'}\right)$. Let R_i denote the matrix output by Algorithm 3.3.2 for inputs $\mathcal{V}_{i+1}$, $\mathcal{V}_i$ and A_i for $i = q'-1$ and q'. By Propositions 3.3.4, R_i is the representation matrix of $H^r\left(f_i\right)$ with respect to the ordered bases $\mathcal{V}_{i+1}$ and $\mathcal{V}_i$ for $i = q'-1$ and q'. By the construction of B_{Ker} (resp. B_{Im}), the matrix $B_{\mathrm{Ker}} \cdot {}^t[v_1^{(q')}, \ldots, v_{k_{q'}}^{(q')}]$ (resp. $B_{\mathrm{Im}} \cdot {}^t[v_1^{(q')}, \ldots, v_{k_{q'}}^{(q')}]$) is clearly a basis matrix of $\mathrm{Ker}\left(H^r\left(f_{q'-1}\right)\right)$ (resp. $\mathrm{Im}\left(H^r\left(f_{q'}\right)\right)$). Let B_{coh} be the matrix output by Algorithm 3.3.5 for inputs B_{Ker} and B_{Im}. By Proposition 3.3.5, $B_{\mathrm{coh}} \cdot {}^t[v_1^{(q')}, \ldots, v_{k_{q'}}^{(q')}]$ is a basis matrix of $\mathrm{Ker}\left(H^r\left(f_{q'-1}\right)\right)/\mathrm{Im}\left(H^r\left(f_{q'}\right)\right)$. Hence Theorem 3.4.1 holds. $\square$

3.5 Complexity analysis

In this subsection, we investigate the complexity of Algorithm 3.2.1. Recall that from Remark 3.2.1, we estimated the complexity of our algorithm for mathematical invariants of the input object since the cohomology groups in algebraic geometry should be determined by such invariants.

The notations are same as in Algorithm 3.2.1. Let $\mathbb{P}_K^r = \mathrm{Proj}(S)$ be the projective r-space with $S = K[X_0, \ldots, X_r]$, and $\mathcal{O}_{\mathbb{P}_K^r}$ the structure sheaf on $\mathbb{P}_K^r$. Let $H^q(\mathbb{P}_K^r, \mathcal{H})$ denote the q-th cohomology group of a coherent sheaf $\mathcal{H}$ on $\mathbb{P}_K^r$. To simplify the notations, we denote $\mathbb{P}_K^r$ and $\mathcal{O}_{\mathbb{P}_K^r}$ by $\mathbb{P}^r$ and $\mathcal{O}_{\mathbb{P}^r}$, respectively, and denote $H^q(\mathbb{P}^r, \mathcal{H})$ by $H^q(\mathcal{H})$ for a coherent sheaf $\mathcal{H}$ on $\mathbb{P}^r$.

First we recall the input and output parameters and the objects to compute in Algorithm 3.2.1 (cf. Step. B in Table 1 of Subsection 3.2.2). Let M be a finitely generated graded S-module. The module M has a (minimal) free resolution of the form (3.2.1). The input parameters of Algorithm 3.2.1 are M, $F(M) := ((M_i, \varphi_i, t_i, d^{(i)}, A_i))_{0 \leq i \leq \ell+1}$, $1 \leq q \leq r$ and $n \in \mathbb{Z}$, where M_i, φ_i, t_i, $d^{(i)}$ and A_i are given in (3.1.2) and (3.2.2), q is the degree of the output cohomology group $H^q(\mathcal{F}(n))$, and n is the twist number of the sheaf $\mathcal{F} = \widetilde{M}$ associated with the module M on $\mathbb{P}^r$. Throughout this section, we set

$$q' := r - q + 1. \tag{3.5.1}$$

The output of Algorithm 3.2.1 is a basis of the K-vector space $H^q(\mathcal{F}(n))$. In Algorithm 3.2.1, the objects to compute in Step B are bases of the K-vector spaces $H^r(\mathcal{G}_i)$ for $r-q \leq i \leq r-q+2$, where each sheaf $\mathcal{G}_i$ is defined in (3.1.5). From Step B-2 to Step B-4 of Algorithm 3.2.1, the objects to compute are

(B-2) The representation matrices R_i of the K-linear maps $H^r(f_i)$ via the bases obtained in Step B-1 for $i = r-q$ and $r-q+1$, where $H^r(f_i) = H^r\left(\widetilde{\varphi_i(n)}\right)$ is given as in (3.2.5) for each i,

(B-3) The basis matrix B_{Ker} (resp. B_{Im}) of the K-vector space $\mathrm{Ker}(R_{r-q}) := \{\mathbf{v} \in K^{k_{q'}} \ ; \ \mathbf{v} \cdot {}^tR_{q'-1} = \mathbf{0}\}$ (resp. $\mathrm{Im}(R_{r-q+1}) := \{\mathbf{v} \cdot {}^tR_{q'} \ ; \ \mathbf{v} \in K^{k_{q'+1}}\}$), where $k_i := \dim_K H^r(\mathcal{G}_i)$ for $r-q \leq i \leq r-q+2$,

(B-4-1) The basis matrix B_{coh} of the K-quotient space $\mathrm{Ker}(R_{r-q})/\mathrm{Im}(R_{r-q+1})$,

(B-4-2) The matrix $B_{\mathrm{coh}} \cdot {}^t[v_1^{(q')}, \ldots, v_{k_{q'}}^{(q')}]$.

Here $\mathcal{V}_i = \{v_1^{(i)}, \ldots, v_{k_i}^{(i)}\}$ is a basis of the K-vector space $H^r(\mathcal{G}_i)$ for each $q'-1 \leq i \leq q'+1$. For fixed r and q, all of the above objects to compute are determined by t_i, $d^{(i)}$, A_i and n for $q'-1 \leq i \leq q'+1$. From this, we analyze the complexity of Algorithm 3.2.1 according to the parameters n, $t^{(\max)} := \max\{t_i \ ; \ q'-1 \leq i \leq q'+1\}$ and $d^{(\max)} := \max\{d^{(i,\ \max)} \ ; \ q'-1 \leq i \leq q'+1\}$, where $d^{(i,\ \max)} := \max\{d_j^{(i)}\ ;\ 1 \le j \le t_i\}$ for $q'-1 \le i \le q'+1$. For a simplicity, suppose that $H^r(\mathcal{G}_i) \neq 0$ for all $q'-1 \le i \le q'+1$.

Proposition 3.5.1 *We use the same notations as above.* Algorithm 3.2.1 (*not counting the computation of a basis of the K-vector space $H^r(\mathbb{P}^r, \mathcal{G}_i)$ for $r-q \le i \le r-q+1$*) *runs in*

$$O\left(\left(t^{(\max)}(d^{(\max)}-n)^r\right)^4\right) \tag{3.5.2}$$

arithmetic operations over $S = K[X_0, \ldots, X_r]$.

Proof. We determine the complexity based on a theory of linear algebra (see [10] for details). Let $\mathcal{V}_i = \left\{v_1^{(i)}, \ldots, v_{k_i}^{(i)}\right\}$ denote the basis of the K-vector space $H^r(\mathcal{G}_i)$ in Algorithm 3.2.1 for $q'-1 \le i \le q'+1$.

(B-2) *The representation matrices R_i of the K-linear maps $H^r(f_i)$ for $i = r-q$ and $r-q+1$:* In this part, we first compute the image of $\left\{v_1^{(i)}, \ldots, v_{k_{(i)}}^{(i)}\right\}$ by the K-linear map $H^r(f_{i-1})$. Recall that by (3.1.6) and (3.1.7), the dimension $k_i := \#(\mathcal{V}_i)$ of the K-vector space $H^r(\mathcal{G}_i)$ is

$$k_i = \dim_K H^r(\mathcal{G}_i) = O\left(t^{(\max)}(d^{(\max)}-n)^r\right)$$

for $q'-1 \le i \le q'+1$. We compute the vector $v_j^{(i)} \cdot A_{i-1}$ for $1 \le j \le k_i$. Note that A_{i-1} is a $(t_i \times t_{i-1})$-matrix over $K[X_0, \ldots, X_r]$. Thus the computation runs in

$$O\left((t^{(\max)})^2(d^{(\max)}-n)^r\right) \tag{3.5.3}$$

arithmetic operations over the polynomial ring $K[X_0, \ldots, X_r]$. We then obtain $O\left(t^{(\max)}(d^{(\max)}-n)^r\right)$ elements in $H^r(\mathcal{G}_{i-1})$. By comparing the elements with the ordered basis $\left\{v_1^{(i-1)}, \ldots, v_{k_{i-1}}^{(i-1)}\right\}$, we obtain the representation matrix R_{i-1} in

$$O\left((t^{(\max)})^2(d^{(\max)}-n)^{2r}\right) \tag{3.5.4}$$

arithmetic operations over $K[X_0, \ldots, X_r]$. Thus the arithmetic complexity in this step is given by (3.5.4).

(B-3) *The basis matrix B_{Ker} (resp. B_{Im}) of the K-vector space* $\mathrm{Ker}\,(R_{r-q})$ (*resp.* $\mathrm{Im}\,(R_{r-q+1})$): We assume that the computation is done by the Gaussian elimination over the field K. In this case, since R_i is a matrix over K and the number of the rows and the columns of R_i is bounded by $t^{(\max)}(d^{(\max)}-n)^r$, the naive computation gives

$$O\left((t^{(\max)})^3(d^{(\max)}-n)^{3r}\right) \tag{3.5.5}$$

arithmetic operations over K.

(B-4-1) *The basis matrix* B_{coh} *of the* K*-quotient space* $\mathrm{Ker}\,(R_{r-q})\,/\mathrm{Im}\,(R_{r-q+1})$: In this step, we first compute the matrix U such that $B_{\mathrm{Im}} = U \cdot B_{\mathrm{Ker}}$. We solve the linear system $\mathbf{b}_i = \mathbf{u} \cdot B_{\mathrm{Ker}}$ by the Gaussian elimination over K for each i, where $\mathbf{b}_i$ is the i-th row of the matrix B_{Im}. Thus the above U is determined in

$$\begin{aligned} & O\left(\left(t^{(\max)}(d^{(\max)}-n)^r\right)\left((t^{(\max)})^3(d^{(\max)}-n)^{3r}\right)\right) \\ = \; & O\left(\left((t^{(\max)})^4(d^{(\max)}-n)^{4r}\right)\right) \end{aligned} \tag{3.5.6}$$

arithmetic operations over K because the number of the rows of both of the matrices B_{Im} and B_{Ker} are $O\left(t^{(\max)}(d^{(\max)}-n)^r\right)$. We then compute the square matrix of full rank by extending the basis of the row vectors of U. Since U is a matrix over K and the number of the rows and the columns of U is bounded by $t^{(\max)}(d^{(\max)}-n)^r$, this computation can be done in

$$O\left((t^{(\max)})^2(d^{(\max)}-n)^{2r}\right) \tag{3.5.7}$$

arithmetic operations over K.

(B-4-2) *The matrix* $B_{\mathrm{coh}} \cdot {}^t[v_1^{(q')}, \ldots, v_{k_{q'}}^{(q')}]$: Finally, we compute the basis matrix $B_{\mathrm{coh}} \cdot {}^t[v_1^{(q')}, \ldots, v_{k_{q'}}^{(q')}]$ of the K-vector space $H^q\left(\mathcal{F}(n)\right)$. The number of the rows and the columns of B_{coh} is bounded by $t^{(\max)}(d^{(\max)}-n)^r$. Recall that $k_{q'} = O\left(t^{(\max)}(d^{(\max)}-n)^r\right)$. Thus the multiplication of the matrix B_{coh} and the vector ${}^t[v_1, \ldots, v_{k_{q'}}]$ requires

$$O\left((t^{(\max)})^2(d^{(\max)}-n)^{2r}\right) \tag{3.5.8}$$

arithmetic operations over $K[X_0, \ldots, X_r]$.

Putting all the steps together, namely considering (3.5.3)-(3.5.8), Proposition 3.5.1 holds. □

Corollary 3.5.2 *It is possible to estimate the complexity of* Algorithm 3.2.1 *over* K*. The notations are same as in* Proposition 3.5.1*. Let* α *be the maximum of the number of the terms of the components of* A_i *for* $q'-1 \leq i \leq q'+1$*. The arithmetic complexity of* Algorithm 3.2.1 *over* K *(not counting the computation of a basis of the* K*-vector space* $H^r(\mathcal{G}_i)$ *for* $q'-1 \leq i \leq q'+1$*) is*

$$O\left(\left(t^{(\max)}(-n+d^{(\max)})^r\right)^4 + \alpha^2\left(t^{(\max)}(-n+d^{(\max)})^r\right)^2\right). \tag{3.5.9}$$

The value

$$D := \max\{\dim_K H^r(\mathcal{G}_i)\ ;\ q'-1 \leq i \leq q'+1\} \tag{3.5.10}$$

is appropriate as an asymptotic parameter of Algorithm 3.2.1. We describe the reason why the parameter D is appropriate as an asymptotic parameter of Algorithm 3.2.1. Recall that we have

$$\dim_K H^r(\mathcal{G}_i) = \sum_{j=1}^{t_i} \binom{n-d_j^{(i)}}{r} \text{ for } q'-1 \leq i \leq q'+1 \tag{3.5.11}$$

and thus $\dim_K H^r(\mathcal{G}_i) = O\left(t^{(\max)}(d^{(\max)} - n)^r\right)$. The values t_i and $d_j^{(i)}$ are uniquely determined for the input module M since the form of the projective resolution of M is uniquely determined. Thus each value $\sum_{j=1}^{t_i} \binom{n-d_j^{(i)}}{r}$ is also uniquely determined by M and n. From this we can take D as an asymptotic parameter of Algorithm 3.2.1. In a similar way to Corollary 3.5.2, the arithmetic complexity of Algorithm 3.2.1 with respect to D over K is as follows.

Corollary 3.5.3 *The notations are same as in* Proposition 3.5.1. *We set* $D := \max\{\dim_K H^r(\mathcal{G}_i)\ ;\ q'-1 \leq i \leq q'+1\}$ *as in* (3.5.10). *Then the arithmetic complexity of* Algorithm 3.2.1 *over* K *is*

$$O(D^4 + \alpha^2 D^2), \tag{3.5.12}$$

where α *is same as in* Corollary 3.5.2.

4 Implementation and experiments

In this section, we show experimental results on our implementation of Algorithm 3.2.1. Our aim is to confirm the performance of our implementation, and we observe that our implementation performs more efficiently than the complexity estimated in Subsection 3.5 for several benchmark examples. This behavior is considered as a result of our computational improvement and its details are given in Observation 4.1.2. We use a computer with 2.60GHz CPU (Intel Corei5) and 8GB memory. The OS is Windows 8.1 Pro 64bit. We implemented Algorithm 3.2.1 over Magma V2.20-10 [1]. The source code of our implementation and the computation results in this section are available at `http://www2.math.kyushu-u.ac.jp/~m-kudo/` (our source code is in the file `CohomologyBasis.txt`).

4.1 Experiments on the performance of our implementation

Let $\mathbb{P}^r_{\mathbb{Q}} = \mathrm{Proj}(S)$ be the projective r-space on the field $\mathbb{Q}$ of rational numbers, where $S := \mathbb{Q}[X_0, \ldots, X_r]$. For given m and T, we choose a set $\mathcal{C}$ of 10 tuples of homogeneous polynomials $f_1, \ldots, f_m \in S$ randomly chosen so that it satisfies the following:

(1) For each tuple $(f_1, \ldots, f_m) \in \mathcal{C}$,

  (a) $\langle f_1, \ldots, f_m \rangle \neq S$, where $\langle f_1, \ldots, f_m \rangle$ denotes the ideal of S generated by $f_1, \ldots, f_m$.

  (b) The number of the terms of f_i is equal to T for each $1 \leq i \leq m$.

  (c) All coefficients of f_i are equal to 1 for each $1 \leq i \leq m$.

  (d) $\sqrt{\langle f_i \rangle} \not\subset \sqrt{\langle f_j \rangle}$ for $1 \leq i < j \leq m$, where $\sqrt{I}$ denotes the radical of an ideal $I \subseteq S$.

  (e) The polynomial f_i is irreducible over $\mathbb{Q}$ for each $1 \leq i \leq m$.

(2) For two different tuples $(f_1, \ldots, f_m)$ and $(f'_1, \ldots, f'_m)$ in $\mathcal{C}$, $\langle f_1, \ldots, f_m \rangle \neq \langle f'_1, \ldots, f'_m \rangle$ as ideals.

(3) For two tuples $(f_1, \ldots, f_m)$ and $(f'_1, \ldots, f'_m)$ in $\mathcal{C}$, two graded S-modules $M = S/\langle f_1, \ldots, f_m \rangle$ and $M' := S/\langle f'_1, \ldots, f'_m \rangle$ have the same form of minimal free resolutions, that is, they have the minimal free resolutions

$$0 \to \bigoplus_{j=1}^{t_{r+2}} S\left(-d_j^{(r+2)}\right) \xrightarrow{\varphi_{r+1}} \cdots \xrightarrow{\varphi_1} \bigoplus_{j=1}^{t_1} S\left(-d_j^{(1)}\right) \xrightarrow{\varphi_0} M \to 0 \tag{4.1.1}$$

and

$$0 \to \bigoplus_{j=1}^{t_{r+2}} S\left(-d_j^{(r+2)}\right) \overset{\varphi'_{r+1}}{\to} \cdots \overset{\varphi'_1}{\to} \bigoplus_{j=1}^{t_1} S\left(-d_j^{(1)}\right) \overset{\varphi'_0}{\to} M' \to 0. \tag{4.1.2}$$

Note that $\varphi_i \neq \varphi'_i$ for $0 \leq i \leq r+1$ in general.

For each set $\mathcal{C}$ and each tuple $(f_1, \ldots, f_m) \in \mathcal{C}$, we compute bases of the K-vector spaces $H^q\left(\mathbb{P}^r_{\mathbb{Q}}, \mathcal{F}(n)\right)$ for some $n \in \mathbb{Z}$ and $1 \leq q \leq r$ by Algorithm 3.2.1, where $\mathcal{F}$ is the coherent sheaf associated with $M := \mathbb{Q}[X_0,\ldots,X_r]/\langle f_1,\ldots,f_m\rangle$. Let $\mathcal{O}_{\mathbb{P}^r_{\mathbb{Q}}}$ be the structure sheaf on $\mathbb{P}^r_{\mathbb{Q}}$. We fix the following notations:

$$\begin{aligned}
q' &:= r-q+1, \quad t^{(\max)} := \max\{t_i \,; q'-1 \leq i \leq q'+1\},\\
d^{(i,\,\max)} &:= \max\{d_j^{(i)} \,;\, 1 \leq j \leq t_i\} \text{ for } q'-1 \leq i \leq q'+1,\\
d^{(\max)} &:= \max\{d^{(i,\,\max)} \,;\, q'-1 \leq i \leq q'+1\},\\
\mathcal{G}_i &:= \bigoplus_{j=1}^{t_i} \mathcal{O}_{\mathbb{P}^r_{\mathbb{Q}}}\left(n - d_j^{(i)}\right) \text{ for } q'-1 \leq i \leq q'+1,\\
D &:= \max\{\dim_K H^r\left(\mathbb{P}^r_{\mathbb{Q}}, \mathcal{G}_i\right) \,;\, q'-1 \leq i \leq q'+1\}.
\end{aligned}$$

To simplify the notations, we denote by $H^q(\mathcal{H})$ the q-th cohomology group $H^q(\mathbb{P}^r_{\mathbb{Q}}, \mathcal{H})$ of a coherent sheaf $\mathcal{H}$ on $\mathbb{P}^r_{\mathbb{Q}}$. Let $\mathbf{f} = (f_1 \ldots, f_m) \in \mathcal{C}$. For the representation matrix $A_i = \left(g_{k,\ell}^{(i)}\right)_{1\leq k\leq t_{i+1}, 1\leq \ell\leq t_i}$ of each homomorphism φ_i in the resolution (4.1.1), we denote by $\alpha_{i,\mathbf{f}}$ the maximum of the number of the terms of $g_{k,\ell}^{(i)}$ for $1 \leq k \leq t_{i+1}$ and $1 \leq \ell \leq t_i$. Then let $\alpha_{\mathbf{f}}$ be the maximum of $\alpha_{r-q,\mathbf{f}}$ and $\alpha_{r-q+1,\mathbf{f}}$. Moreover, let $\alpha^{(\mathrm{bound})}$ be the maximum of $\alpha_{\mathbf{f}}$ for $\mathbf{f} \in \mathcal{C}$.

Remark 4.1.1 As the twist number n decreases, the asymptotic parameter D increases by (3.5.10) and (3.5.11). Thus it is possible to investigate the performance of our implementation by decreasing n.

In the following, we describe the form of the minimal free resolutions in our experiments.

Case 1 Put $r := 3$, $m := 2$, $T := 3$ and $q := 1$. We choose a pair of two homogeneous polynomials f_1 and f_2 in $\mathbb{Q}[X_0, X_1, X_2, X_3]$ (uniformly) at random so that the following conditions hold:

(a) The minimal free resolution of $M := S/\langle f_1, f_2\rangle$ forms

$$0 \to S\,(-6) \overset{\varphi_2}{\to} \bigoplus_{j=1}^{2} S\,(-3) \overset{\varphi_1}{\to} S \overset{\varphi_0}{\to} M \to 0.$$

(b) The number of the terms of f_i is equal to 3 for $i = 1$ and 2.

In this case, it follows that

$$t = [\ 1, 2, 1, 0, 0\ ], \quad d^{(1)} = [0], \quad d^{(2)} = [3, 3], \quad d^{(3)} = [6].$$

We compute bases of $H^1\left(\mathbb{P}^r_{\mathbb{Q}}, \mathcal{F}\,(n)\right)$ for $n = 0$, -5, -10 and -15. Note that $q' = r-q+1 = 3$, $t^{(\max)} = 2$ and $d^{(\max)} = 6$.

Case 2 Put $r := 3$, $m := 2$, $T := 3$ and $q := 1$. We choose a pair of two homogeneous polynomials f_1 and f_2 in $\mathbb{Q}[X_0, X_1, X_2, X_3]$ (uniformly) at random so that the following conditions hold:

(a) The minimal free resolution of $M := S/\langle f_1, f_2\rangle$ forms

$$0 \to S(-8) \xrightarrow{\varphi_2} \bigoplus_{j=1}^{2} S(-4) \xrightarrow{\varphi_1} S \xrightarrow{\varphi_0} M \to 0.$$

(b) The number of the terms of f_i is equal to 4 for $i = 1$ and 2.

In this case, it follows that

$$t = [\ 1, 2, 1, 0, 0\ ], \quad d^{(1)} \ = \ [0], \quad d^{(2)} = [4, 4], \quad d^{(3)} = [8].$$

We compute bases of $H^1\left(\mathbb{P}^r_{\mathbb{Q}}, \mathcal{F}(n)\right)$ for $n = 0, -5, -10$ and -15. Note that $q' = r-q+1 = 3$, $t^{(\max)} = 2$ and $d^{(\max)} = 8$.

Case 3 Put $r := 3$, $m := 3$, $T := 3$ and $q := 1$. We choose a tuple of three homogeneous polynomials f_1, f_2 and f_3 in $\mathbb{Q}[X_0, X_1, X_2, X_3]$ (uniformly) at random so that the following conditions hold:

(a) The minimal free resolution of $M := S/\langle f_1, f_2, f_3\rangle$ forms

$$0 \to S(-9) \xrightarrow{\varphi_3} \bigoplus_{j=1}^{3} S(-6) \xrightarrow{\varphi_2} \bigoplus_{j=1}^{3} S(-3) \xrightarrow{\varphi_1} S \xrightarrow{\varphi_0} M \to 0.$$

(b) The number of the terms of f_i is equal to 3 for $1 \le i \le 3$.

In this case, it follows that

$$t = [\ 1, 3, 3, 1, 0\ ], \quad d^{(1)} = [0], \quad d^{(2)} = [3, 3, 3],$$
$$d^{(3)} = [6, 6, 6], \quad d^{(4)} = [9].$$

We compute bases of $H^1\left(\mathbb{P}^r_{\mathbb{Q}}, \mathcal{F}(n)\right)$ for $n = 0, -5, -10$ and -15. Note that $q' = r-q+1 = 3$, $t^{(\max)} = 3$ and $d^{(\max)} = 9$.

Case 4 Put $r := 5$, $m := 4$, $T := 3$ and $q := 1$. We choose a tuple of four homogeneous polynomials f_1, f_2, f_3 and f_4 in $\mathbb{Q}[X_0, X_1, X_2, X_3, X_4, X_5]$ (uniformly) at random so that the following conditions hold:

(a) Put $I := \langle f_1, f_2, f_3, f_4\rangle$. The minimal free resolution of $M := S/I$ forms

$$0 \to S(-8) \xrightarrow{\varphi_4} \bigoplus_{j=1}^{4} S(-6) \xrightarrow{\varphi_3} \bigoplus_{j=1}^{6} S(-4) \xrightarrow{\varphi_2} \bigoplus_{j=1}^{4} S(-2) \xrightarrow{\varphi_1} S \xrightarrow{\varphi_0} M \to 0.$$

(b) The number of the terms of f_i is equal to 3 for $1 \le i \le 4$.

In this case, it follows that

$$t = [\ 1,4,6,4,1,0,0\ ]\,,\quad d^{(1)} = [0]\,,\quad d^{(2)} = [2,2,2,2]\,,$$
$$d^{(3)} = [4,4,4,4,4,4]\,,\quad d^{(4)} = [6,6,6,6]\,,\quad d^{(5)} = [8]\,.$$

We compute bases of $H^1\left(\mathbb{P}^r_{\mathbb{Q}}, \mathcal{F}(n)\right)$ for $n = 0, -2, -4$ and -6. Note that $q' = r-q+1 = 5$, $t^{(\max)} = 4$ and $d^{(\max)} = 8$.

Case 5 Put $r := 7$, $m := 5$, $T := 2$ and $q := 2$. We choose a tuple of five homogeneous polynomials $f_1, \ldots, f_5$ in $\mathbb{Q}[X_0, \ldots, X_7]$ (uniformly) at random so that the following conditions hold:

(a) Put $I := \langle f_1, \ldots, f_5 \rangle$. The minimal free resolution of $M := S/I$ forms

$$0 \to S(-10) \overset{\varphi_5}{\to} \bigoplus_{j=1}^{5} S(-8) \overset{\varphi_4}{\to} \bigoplus_{j=1}^{10} S(-6) \overset{\varphi_3}{\to} \bigoplus_{j=1}^{10} S(-4)$$
$$\overset{\varphi_2}{\to} \bigoplus_{j=1}^{5} S(-2) \overset{\varphi_1}{\to} S \overset{\varphi_0}{\to} M \to 0.$$

(b) The number of the terms of f_i is equal to 2 for $1 \leq i \leq 5$.

In this case, it follows that

$$t = [\ 1,5,10,10,5,1,0,0,0\ ]\,,$$
$$d^{(1)} = [0]\,,\quad d^{(2)} = [2,2,2,2,2]\,,\quad d^{(3)} = [4,4,4,4,4,4,4,4,4,4]\,,$$
$$d^{(4)} = [6,6,6,6,6,6,6,6,6,6]\,,\quad d^{(5)} = [8,8,8,8,8]\,,\quad d^{(6)} = [10]\,.$$

We compute bases of $H^2\left(\mathbb{P}^r_{\mathbb{Q}}, \mathcal{F}(n)\right)$ for $n = 0, -2, -4$ and -5. Note that $q' = r-q+1 = 6$, $t^{(\max)} = 5$ and $d^{(\max)} = 10$.

In Table 2, we show the results of the computations for each case.

Observation 4.1.2 From the timing ("Average") in Table 2, we see that our implementation of Algorithm 3.2.1 performs more efficiently than the complexity estimated in Subsection 3.5. The timing in Table 2 implies that the practical complexity is $O\left(D^2\right)$ while the complexity estimated in Subsection 3.5 is $O\left(D^4\right)$ for a fixed α. This seems due to that we pruned unnecessary operations in our implementation. More precisely, in Step B-2, our implementation minimizes the representation matrices, by which the linear systems in Step B-3 are solved more efficiently than the estimated complexity $O(D^3)$. Step B-4 can be also terminated more efficiently than the estimated one $O(D^4)$. From this, the practical complexity of our algorithm is considered $O(D^2)$. We conclude that our implementation of Algorithm 3.2.1 has sufficiently practical performance to experimentally investigate some properties of algebraic varieties.

Table 2: Experimental results on Algorithm 3.2.1. We performed experiments for Cases 1-5. Parameters r and q are the dimensions of the projective space and the degree of the cohomology group to compute, respectively. The values of the parameters $t^{(\max)}$, $d^{(\max)}$ and $\alpha^{(\text{bound})}$ are determined from the forms of minimal free resolutions. The parameter D is the asymptotic parameter for the complexity estimated in Subsection 3.5. (The parameter D depends on the value of n in our experiments.) "Average" means the average of time for performing our implemetation.

Case	Parameters fixed in each case		Parameters ($t^{(\max)}$, $d^{(\max)}$ and $\alpha^{(\text{bound})}$ are determined from resolutions)					Experimental results on our algorithm (Algorithm 3.2.1)	
	r	q	n	$t^{(\max)}$	$d^{(\max)}$	$\alpha^{(\text{bound})}$	D	The dimension of the output $H^q(\mathcal{F}(n))$	Average (sec.)
1	3	1	0	2	6	3	10	10	0.01
	3	1	−5	2	6	3	120	54	0.12
	3	1	−10	2	6	3	455	99	1.91
	3	1	−15	2	6	3	1,360	144	14.14
2	3	1	0	2	8	4	35	33	0.02
	3	1	−5	2	8	4	220	112	0.29
	3	1	−10	2	8	4	680	192	3.54
	3	1	−15	2	8	4	1,632	272	22.06
3	3	1	0	3	9	9	56	0	0.03
	3	1	−5	3	9	9	360	0	1.28
	3	1	−10	3	9	9	1,365	0	17.52
	3	1	−15	3	9	9	3,420	0	123.12
4	5	1	0	4	8	11	21	17	0.01
	5	1	−2	4	8	11	126	48	0.19
	5	1	−4	4	8	11	504	80	2.27
	5	1	−6	4	8	11	1,848	112	21.90
5	7	2	0	5	10	4	36	31	0.03
	7	2	−2	5	10	4	330	160	0.58
	7	2	−4	5	10	4	1,716	416	19.51
	7	2	−5	5	10	4	3,960	592	86.19

4.2 Benchmarks

We also show benchmarks by our implementation of Algorithm 3.2.1. The notations are same as in Subsection 4.1. Recall from Subsection 3.5 that main computations of Algorithm 3.2.1 are the following:

Each procedure and its complexity of Algorithm 3.2.1		
Object to compute		Complexity
(B-2)	The representation matrices R_i of the K-linear maps $H^r(f_i)$ for $i = r-q$ and $r-q+1$	$O\left(\alpha^2 D^2\right)$
(B-3)	The basis matrix B_{Ker} (resp. B_{Im}) of the K-vector space $\mathrm{Ker}\left(R_{r-q}\right) := \{\mathbf{v} \in K^{k_{q'}} \ ; \ \mathbf{v} \cdot {}^t R_{q'-1} = \mathbf{0}\}$ (resp. $\mathrm{Im}\left(R_{r-q+1}\right) := \{\mathbf{v} \cdot {}^t R_{q'} \ ; \ \mathbf{v} \in K^{k_{q'+1}}\}$)	$O\left(D^3\right)$
(B-4-1)	The basis matrix B_{coh} of the K-quotient space $\mathrm{Ker}\left(R_{r-q}\right)/\mathrm{Im}\left(R_{r-q+1}\right)$	$O\left(D^4\right)$
(B-4-2)	The matrix $B_{\mathrm{coh}} \cdot {}^t[v_1^{(q')}, \ldots, v_{k_{q'}}^{(q')}]$, where $\mathcal{V} = \{v_1^{(q')}, \ldots, v_{k_{q'}}^{(q')}\}$ is a basis of $H^r(\mathcal{G}_{q'})$	$O\left(\alpha^2 D^2\right)$

Note that the above complexity is estimated over K. Table 3 shows the result of the benchmarks for one sample in Case 4 (see Subsection 4.1). We denote by $C^{(\max)}$ the maximum of the coefficients of the components of the representation matrices A_i (for $q'-1 \leq i \leq q'+1$).

Table 3: The result of the benchmarks for one sample in Case 4 by our implementation of Algorithm 3.2.1. The parameter D is the asymptotic parameter for the complexity estimated in Subsection 3.5. (The parameter D depends on the value of n in our experiments.)

Parameters ($C^{(\max)}$ is determined from resolutions)				Time for each procedure and total time (sec.)				
n	D	α	$C^{(\max)}$	B-2	B-3	B-4-1	B-4-2	Total time (sec.)
-2	126	3	1	0.094	0.000	0.000	0.063	0.157
-4	504	3	1	1.750	0.031	0.000	0.313	2.094
-5	1,848	3	1	18.671	0.359	0.000	1.265	20.295

Observation 4.2.1 It is implied that our implementation follows the complexity determined in Subsection 3.5 for any case. (It does not take much time for the computation of B_{Ker} in certain cases because we use the built-in function over Magma to compute the kernel of a matrix, and because the algorithms implemented in the function include the LU decomposition, which performs faster than the Gaussian elimination.)

5 Applications

In this section, we introduce two possible applications of Algorithm 3.2.1. The first application is to compute the rank of morphisms of the cohomology groups via Maruyama's method. The second one is an algorithm to compute the action of Frobenius to the cohomology groups on algebraic varieties (e.g., algebraic curves). In particular, the rank of the action of Frobenius on a (non-singular) algebraic curve is said to be the *Hasse-Witt rank* of the curve. The Hasse-Witt rank is a very important invariant of such varieties in algebraic geometry over fields of positive characteristics.

5.1 Background

Let K be a field of positive characteristic $\mathrm{char}(K) = p > 0$. We assume that K is a perfect field (e.g., $K = \mathbb{F}_q$ or $\overline{\mathbb{F}_q}$). Let $f_1, \ldots, f_t$ be homogeneous polynomials in the graded ring $S = K[X_0, \ldots, X_r]$. For homogeneous polynomials $f_1, \ldots, f_t \in K[X_0, \ldots, X_r]$, we denote by $V_K(f_1, \ldots, f_t)$ the locus of the zeros in the projective r-space $\mathbb{P}^r_K = \mathrm{Proj}(S)$ of the system of $f_1, \ldots, f_t$. Put $X := V_K(f_1, \ldots, f_t)$ and $X^{(p)} = V_K(f_1^p, \ldots, f_t^p)$. Let $F : X \to X$ be the (absolute) Frobenius morphism on X. In algebraic geometry over fields of positive characteristics, it is important to compute the action of Frobenius $F^* : H^q(X, \mathcal{O}_X) \to H^q(X, \mathcal{O}_X)$ for $1 \leq q \leq r$, where $\mathcal{O}_X$ is the structure sheaf on X and $H^q(X, \mathcal{O}_X)$ denotes the q-th cohomology group of $\mathcal{O}_X$ on X. In algebraic geometry over fields of positive characteristic and arithmetic geometry, it is very important to compute the rank of F^* via a basis of the K-vector space $H^q(X, \mathcal{O}_X)$ The variety X is said to be *superspecial* if $F^* = 0$ and X is non-singular over $\overline{K}$.

Now we describe a basic strategy to compute the action of Frobenius F^*. For a simplicity, we assume that X is a non-singular algebraic curve, and we consider the case of $q = 1$. Let $\mathcal{I} := \widetilde{I}$ and $\mathcal{I}^p := \widetilde{I^p}$ be the ideal sheaves associated with the ideals $I := \langle f_1, \ldots, f_t \rangle_S$ and $I^p := \langle f_1^p, \ldots, f_t^p \rangle_S$, respectively. Let F_1 be the (absolute) Frobenius morphism on $\mathbb{P}^r_K$. The following diagram commutes:

$$\begin{tikzcd}
H^1(X, \mathcal{O}_X) \arrow[rr, "\cong"] \arrow[dr, "\left(F_1|_{X^{(p)}}\right)^*"] \arrow[dd, "F^*"'] & & H^2(\mathbb{P}^r_K, \mathcal{I}) \arrow[d, "F_1^*"] \\
& H^1(X^{(p)}, \mathcal{O}_{X^{(p)}}) \arrow[r, "\cong"] \arrow[dl] & H^2(\mathbb{P}^r_K, \mathcal{I}^p) \arrow[d] \\
H^1(X, \mathcal{O}_X) \arrow[rr, "\cong"] & & H^2(\mathbb{P}^r_K, \mathcal{I})
\end{tikzcd}$$

Here the morphism $H^1(X^{(p)}, \mathcal{O}_{X^{(p)}}) \to H^1(X, \mathcal{O}_X)$ (resp. $H^2(\mathbb{P}^r_K, \mathcal{I}^p) \to H^2(\mathbb{P}^r_K, \mathcal{I})$) is induced by the canonical morphism from $\mathcal{O}_{X^{(p)}}$ to $\mathcal{O}_X$ (resp. $\mathcal{I}^p$ to $\mathcal{I}$) corresponding to the homomorphism $S/I^p \to S/I$ (resp. $I^p \to I$). In other words, $H^1(X^{(p)}, \mathcal{O}_{X^{(p)}}) \to H^1(X, \mathcal{O}_X)$ is the morphism induced by the immersion $X \hookrightarrow X^{(p)}$. In our method, to compute the rank of F^*, it requires to compute

(1) An explicit basis of $H^2(\mathbb{P}^r_K, \mathcal{I})$ and

(2) The representation matrix of the homomorphism $H^2(\mathbb{P}^r_K, \mathcal{I}^p) \to H^2(\mathbb{P}^r_K, \mathcal{I})$ via the basis.

By Algorithm 3.2.1, we can compute a basis of $H^2(\mathbb{P}^r_K, \mathcal{I})$. In the next subsections, we first give an algorithm to compute the representation matrix, after that, we also give an algorithm to compute the rank of F^*.

Remark 5.1.1 Let X be an elliptic curve in $\mathbb{P}^2_K$ defined by a homogeneous polynomial $f \in S = K[x, y, z]$ with $\deg f = 3$. The action of Frobenius F^* is the zero map or a bijective map on

$H^1(X, \mathcal{O}_X)$ since $H^1(X, \mathcal{O}_X)$ is a 1-dimensional K-space and since K is a perfect field with the positive characteristic p. The condition $F^* \neq 0$ (resp. $F^* = 0$) is said that X has *Hasse invariant* 1 (resp. X has *Hasse invariant* 0). In this case, it is easy to determine whether X has Hasse invariant 1 or 0. In fact, $H^2\left(\mathbb{P}^2_K, \mathcal{I}\right)$ always has the canonical basis $\{\frac{1}{xyz}\}$ and $F_1\left(\frac{1}{xyz}\right) = \frac{1}{x^p y^p z^p}$. The morphism $H^2(\mathbb{P}^2_K, \mathcal{I}^p) \to H^2(\mathbb{P}^2_K, \mathcal{I})$ is represented by f^{p-1}. Thus the Hasse invariant of X is determined by the coefficients of $(xyz)^{p-1}$ in f^{p-1} (see [7, Chapter 4] for details).

5.2 Computing morphisms of cohomology groups

Let K be a field (in this subsection, the characteristic $\mathrm{char}(K)$ is not necessary to be positive). Let $f_1, \ldots, f_t$ be homogeneous polynomials in the graded ring $S = K[X_0, \ldots, X_r]$. As in the previous subsection, put $X := V_K(f_1, \ldots, f_t)$ (resp. $X^{(p)} = V_K(f_1^p, \ldots, f_t^p)$), where $V_K(f_1, \ldots, f_t)$ (resp. $V_K(f_1^p, \ldots, f_t^p)$) is the locus of the zeros in the projective r-space $\mathbb{P}^r_K = \mathrm{Proj}(S)$ of the system of $f_1, \ldots, f_t$ (resp. $f_1^p, \ldots, f_t^p$). In the case of $\mathrm{char}(K) > 0$, to compute the action of Frobenius on X, it requires to compute the representation matrix of the morphism $H^2\left(\mathbb{P}^r_K, \mathcal{I}^p\right) \to H^2\left(\mathbb{P}^r_K, \mathcal{I}\right)$ in the previous subsection. Here $\mathcal{I}$ (resp. $\mathcal{I}^p$) is the ideal sheaf induced by the ideal $I := \langle f_1, \ldots, f_t \rangle_S$ (resp. $I^p := \langle f_1^p, \ldots, f_t^p \rangle_S$).

In this subsection, we consider a more general case. In more detail, we consider to compute the morphism $H^q(\Psi) : H^q(X, \mathcal{F}) \to H^q(X, \mathcal{F}')$ induced by a morphism $\Psi : \mathcal{F} \to \mathcal{F}'$ of given coherent sheaves on X. (The characteristic of K does not need to be a positive integer.) We propose an algorithm to compute $H^q(\Psi)$ for $1 \leq q \leq r$ as an application of Algorithm 3.2.1. To simplify the notations, we denote $\mathbb{P}^r_K$, $\mathcal{O}_{\mathbb{P}^r_K}$ and $H^q(\mathbb{P}^r_K, \mathcal{H})$ by $\mathbb{P}^r$, $\mathcal{O}_{\mathbb{P}^r}$ and $H^q(\mathcal{H})$, respectively. Let $\psi : M \to M'$ be a homomorphism of finitely generated graded S-modules. For free resolutions of M and M', there exist ψ_i for $1 \leq i \leq r+2$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{j=1}^{t_{r+2}} S\left(-d_j^{(r+2)}\right) \arrow[r, "\varphi_{r+1}"] \arrow[d, "\psi_{r+2}"] & \cdots \arrow[r, "\varphi_1"] & \bigoplus_{j=1}^{t_1} S\left(-d_j^{(1)}\right) \arrow[r, "\varphi_0"] \arrow[d, "\psi_1"] & M \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \bigoplus_{j=1}^{t'_{r+2}} S\left(-c_j^{(r+2)}\right) \arrow[r, "\varphi'_{r+1}"] & \cdots \arrow[r, "\varphi'_1"] & \bigoplus_{j=1}^{t'_1} S\left(-c_j^{(1)}\right) \arrow[r, "\varphi'_0"] & M' \arrow[r] & 0
\end{tikzcd}$$

Note that it is possible to compute each morphism ψ_i (e.g., see [5, Chapter 15] for details). We set

$$\mathcal{G}_{i+1} := \bigoplus_{j=1}^{t_{i+1}} \mathcal{O}_{\mathbb{P}^r}\left(-d_j^{(i+1)}\right), \ f_i := \widetilde{\varphi}_i, \ \mathcal{K}_i := \mathrm{Ker}(f_i) \text{ for } 0 \leq i \leq r+1, \ \mathcal{K}_{-1} := \mathcal{F},$$

$$\mathcal{G}'_{i+1} := \bigoplus_{j=1}^{t'_{i+1}} \mathcal{O}_{\mathbb{P}^r}\left(-c_j^{(i+1)}\right), \ f'_i := \widetilde{\varphi}'_i, \ \mathcal{K}'_i := \mathrm{Ker}(f'_i) \text{ for } 0 \leq i \leq r+1, \ \mathcal{K}'_{-1} := \mathcal{F}'.$$

We denote by Ψ_i the induced morphism $\widetilde{\psi}_i : \mathcal{G}_i \longrightarrow \mathcal{G}'_i$. Since the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}_{i-1} \arrow[r] \arrow[d] & \mathcal{G}_i \arrow[r] \arrow[d, "\Psi_i"] & \mathcal{K}_{i-2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K}'_{i-1} \arrow[r] & \mathcal{G}'_i \arrow[r] & \mathcal{K}'_{i-2} \arrow[r] & 0
\end{tikzcd}$$

commutes, where the horizontal sequences are exact. In a similar way to the proof of Theorem 2.2.1, we have the following commutative diagram:

$$\begin{tikzcd}
H^q(\mathcal{F}) \arrow[r, "\cong"] \arrow[d, "H^q(\Psi)"'] & H^{q+1}(\mathcal{K}_0) \arrow[r, "\cong"] \arrow[d] & \cdots \arrow[r, "\cong"] & H^{r-1}(\mathcal{K}_{r-q-2}) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ker}(\sigma_q) \\
H^q(\mathcal{F}') \arrow[r, "\cong"] & H^{q+1}(\mathcal{K}'_0) \arrow[r, "\cong"] & \cdots \arrow[r, "\cong"] & H^{r-1}(\mathcal{K}'_{r-q-2}) \arrow[r, "\cong"] & \mathrm{Ker}(\sigma'_q)
\end{tikzcd}$$

where σ_q and σ'_q denote the K-linear maps $H^r(\mathcal{K}_{r-q-1}) \to H^r(\mathcal{G}_{r-q})$ and $H^r(\mathcal{K}'_{r-q-1}) \to H^r(\mathcal{G}'_{r-q})$, respectively. Here a morphism between the quotient spaces $\mathrm{Ker}(H^r(f_{r-q}))/\mathrm{Im}(H^r(f_{r-q+1}))$ and $\mathrm{Ker}\left(H^r\left(f'_{r-q}\right)\right)/\mathrm{Im}\left(H^r\left(f'_{r-q+1}\right)\right)$ is induced as follows:

$$\begin{tikzcd}
H^r(\mathcal{G}_{r-q+2}) \arrow[r, "H^r(f_{r-q+1})"] \arrow[d, "H^r(\Psi_{r-q+2})"] & \mathrm{Ker}(H^r(f_{r-q})) \arrow[r] \arrow[d, "H^r(\Psi_{r-q+1})"] & \mathrm{Ker}(H^r(f_{r-q}))/\mathrm{Im}(H^r(f_{r-q+1})) \arrow[d, dotted] \\
H^r(\mathcal{G}'_{r-q+2}) \arrow[r, "H^r(f'_{r-q+1})"] & \mathrm{Ker}(H^r(f'_{r-q})) \arrow[r] & \mathrm{Ker}(H^r(f'_{r-q}))/\mathrm{Im}(H^r(f'_{r-q+1}))
\end{tikzcd}$$

The diagram

$$\begin{tikzcd}
\mathrm{Ker}\left(H^r(f_{r-q})\right)/\mathrm{Im}\left(H^r(f_{r-q+1})\right) \arrow[r, "\cong", "\tau"'] \arrow[d] & \mathrm{Ker}(\sigma_q) \arrow[r] & H^r(\mathcal{K}_{r-q-1}) \arrow[d] \\
\mathrm{Ker}\left(H^r(f'_{r-q})\right)/\mathrm{Im}\left(H^r(f'_{r-q+1})\right) \arrow[r, "\cong", "\tau'"'] & \mathrm{Ker}(\sigma'_q) \arrow[r] & H^r(\mathcal{K}'_{r-q-1})
\end{tikzcd}$$

commutes, where τ and τ' are the K-isomorphisms naturally induced by $H^r(\mathcal{G}_{r-q+1}) \to H^r(\mathcal{K}_{r-q-1})$ and $H^r(\mathcal{G}'_{r-q+1}) \to H^r(\mathcal{K}'_{r-q-1})$, respectively. Hence the following diagram commutes:

$$\begin{tikzcd}
H^q(\mathcal{F}) \arrow[r, "\cong"] \arrow[d, "H^q(\Psi)"'] & H^{r-1}(\mathcal{K}_{r-q-2}) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ker}(H^r(f_{r-q}))/\mathrm{Im}\left(H^r(f_{r-q+1})\right) \arrow[d, "H^r(\Psi_{r-q+1})"] \\
H^q(\mathcal{F}') \arrow[r, "\cong"] & H^{r-1}(\mathcal{K}'_{r-q-2}) \arrow[r, "\cong"] & \mathrm{Ker}(H^r(f'_{r-q}))/\mathrm{Im}\left(H^r(f'_{r-q+1})\right)
\end{tikzcd}$$

Thus it follows that $\mathrm{rank} H^q(\Psi) = \mathrm{rank} H^r(\Psi_{r-q+1})$. From the above commutative diagrams and the equality $\mathrm{rank} H^q(\Psi) = \mathrm{rank} H^r(\Psi_{r-q+1})$, $\mathrm{rank} H^q(\Psi)$ is computable as follows:

(1) Compute bases of the K-vector spaces $H^q(\mathcal{F})$ and $H^q(\mathcal{F}')$ by Algorithm 3.2.1. Let $\mathcal{V}$ and $\mathcal{V}'$ be the bases of the K-vector spaces $H^q(\mathcal{F})$ and $H^q(\mathcal{F}')$, respectively.

(2) Compute the lifting map ψ_{r-q+1} (e.g., see [5, Chapter 15] for details). Let C_{r-q+1} be the representation matrix of ψ_{r-q+1}.

(3) Compute $\mathrm{rank} H^r(\Psi_{r-q+1})$ by $(\mathcal{V}, \mathcal{V}', C_{r-q+1})$, and it.

5.3 Computing the action of Frobenius to the cohomology groups

As in Section 5.1, let K be a field of positive characteristic $\mathrm{char}(K) = p > 0$. Suppose that K is a perfect field (e.g., $K = \mathbb{F}_q$ or $\overline{\mathbb{F}_q}$). Let $f_1, \ldots, f_t$ be homogeneous polynomials in $S = K[X_0, \ldots, X_r]$ with $\gcd(f_i, f_j) = 1$ for $1 \leq i < j \leq t$. Put $X := V_K(f_1, \ldots, f_t)$ and $X^{(p)} = V_K(f_1^p, \ldots, f_t^p)$, where $V_K(f_1, \ldots, f_t)$ (resp. $V_K(f_1^p, \ldots, f_t^p)$) denotes the locus of the zeros in the projective r-space $\mathbb{P}_K^r = \mathrm{Proj}(S)$ of the system of $f_1, \ldots, f_t$ (resp. $f_1^p, \ldots, f_t^p$). Let $F : X \to X$ be the (absolute) Frobenius morphism on X, and $F^* : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ the action of Frobenius for $q = 1$, where $\mathcal{O}_X$ is the structure sheaf on X and $H^1(X, \mathcal{O}_X)$ denotes the 1st cohomology group of $\mathcal{O}_X$ on X.

In this subsection, for given p and $f_1, \ldots, f_t$, we give an algorithm to compute the action of Frobenius F^*. Let $\mathcal{I} := \widetilde{I}$ and $\mathcal{I}^p := \widetilde{I^p}$ be the ideal sheaves associated with the ideals $I := \langle f_1, \ldots, f_t \rangle_S$ and $I^p := \langle f_1^p, \ldots, f_t^p \rangle_S$, respectively. Let $\psi : S/I^p \to S/I$ denote the homomorphism defined by $f + I^p \mapsto f + I$, which corresponds to the immersion $X \hookrightarrow X^{(p)}$. For free resolutions of S/I^p and S/I, there exist ψ_i for $1 \leq i \leq r+2$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & M_{r+2}^{(p)} \arrow[r, "\varphi_{r+1}^{(p)}"] \arrow[d, "\psi_{r+2}"] & \cdots \arrow[r, "\varphi_2^{(p)}"] & \bigoplus_{j=1}^{t_2} S\left(-d_j^{(1)} p\right) \arrow[r, "\varphi_1^{(p)}"] \arrow[d, "\psi_2"] & S \arrow[r, "\varphi_0^{(p)}"] \arrow[d, "\psi_1"] & S/I^p \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & M_{r+2} \arrow[r, "\varphi_{r+1}"] & \cdots \arrow[r, "\varphi_2"] & \bigoplus_{j=1}^{t_2} S\left(-d_j^{(1)}\right) \arrow[r, "\varphi_1"] & S \arrow[r, "\varphi_0"] & S/I \arrow[r] & 0
\end{tikzcd}$$

Here we give an algorithm to compute a representation matrix of F^*.

(1) Compute a basis $\mathcal{V} = \{v_1, \ldots, v_g\}$ of the K-vector space $H^2(\mathbb{P}^r, \mathcal{I})$ by Algorithm 3.2.1.

(2) Compute the the lifting map ψ_r (e.g., see [5, Chapter 15] for details). Let C_r be the representation matrix of ψ_r as a homomorphism of S-modules.

(3) Compute $C_r \cdot (v_i)^p$ for $1 \leq i \leq g$, and the representation matrix of F^* via the basis $\mathcal{V}$.

6 Concluding remarks and future works

In this paper, we introduced and analyzed Maruyama's method to compute the dimensions of the cohomology groups of coherent sheaves on a projective space. Our main contributions are as follows:

(1) We wrote down Maruyama's method as an explicit algorithm (Algorithm 3.2.1 in Subsection 3.2) which compute not only the dimension but also a basis. As mentioned below, this basis is very useful to computes important invariants. We also implemented the algorithm over Magma as a new function "`CohomologyBasis`".

(2) We analyzed the complexity of our algorithm to verify that our implementation has no unnecessary operations. We also examined the efficiency of our algorithm by experiments. In fact, the practical complexity of Algorithm 3.2.1 estimated by our implementation and experiments is $O\left(D^2\right)$ while the complexity estimated theoretically in Subsection 3.5 is $O\left(D^4\right)$, where D is the asymptotic parameter of Algorithm 3.2.1 in our analysis. This is due to apply the pruning unnecessary operations to our implementation. Thus Algorithm 3.2.1 and our implementation are practical to investigate the structures of varieties in algebraic geometry.

(3) As applications of Algorithm 3.2.1, we gave two further algorithms. One is an algorithm to compute the rank of morphisms of cohomology groups via bases obtained by Algorithm 3.2.1. Another one is to compute the rank of the representation matrix of the action of the Frobenius to varieties such as modular curves over fields of positive characteristics. The rank of the representation matrix of the action of the Frobenius is said to be the *Hasse-Witt rank* which is a very important invariant of varieties in algebraic geometry over fields of positive characteristics.

From our contributions of this work, it is concluded that Maruyama's method provides a very computationally useful tool to investigate the structures of varieties in algebraic geometry.

However, our algorithm works well under the assumption that a free resolution of a module has been computed. Thus it is necessary to improve the computation of a free resolution for a finitely generated module. From this, our future works are the following:

- Improve the efficiency of computing syzygies and free resolutions.
- Investigate the behavior of the Hasse-Witt rank of varieties over fields of positive characteristics and find a special class of varieties whose Hasse-Witt ranks take strange behavior.

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A Computation of global sections

Let $S = K[X_0, \ldots, X_r]$ be the polynomial ring with $r+1$ variables over a field K, $\mathbb{P}^r_K = \mathrm{Proj}\,(S)$ the projective r-space on the field K, and $\mathcal{O}_{\mathbb{P}^r_K}$ the structure sheaf on $\mathbb{P}^r_K$. For a coherent sheaf $\mathcal{H}$ on $\mathbb{P}^r_K$, let $H^q(\mathbb{P}^r_K, \mathcal{H})$ denote the q-th cohomology group of $\mathcal{H}$. To simplify the notations, we denote $\mathbb{P}^r_K$, $\mathcal{O}_{\mathbb{P}^r_K}$ and $H^q(\mathbb{P}^r_K, \mathcal{H})$ by $\mathbb{P}^r$, $\mathcal{O}_{\mathbb{P}^r}$ and $H^q(\mathcal{H})$, respectively. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r$, and M a finitely generated graded S-module corresponding to $\mathcal{F}$, that is, $\mathcal{F} = \widetilde{M}$.

In this appendix, we give an algorithm to compute the dimension of the global section $\Gamma(\mathbb{P}^r_K, \mathcal{F}(n)) = H^0(\mathbb{P}^r_K, \mathcal{F}(n))$, where $\mathcal{F}(n)$ denotes the n-th Serre twist of the coherent sheaf $\mathcal{F}$. The coherent sheaf $\mathcal{F} = \widetilde{M}$ (resp. the finitely generated graded S-module M) has a resolution (2.2.1) (resp. a (minimal) free resolution (3.1.1)). In the following, we give an explicit formula of $\dim_K H^0(X, \mathcal{F})$.

Theorem A.0.1 ([9], Chapter 6) *Let $\mathbb{P}^r_K = \mathrm{Proj}(S)$ be the projective r-space with $S = K[X_0, \ldots, X_r]$, and $\mathcal{O}_{\mathbb{P}^r_K}$ the structure sheaf on $\mathbb{P}^r_K$. To simplify the notations, we denote $\mathbb{P}^r_K$ and $\mathcal{O}_{\mathbb{P}^r_K}$ by $\mathbb{P}^r$ and $\mathcal{O}_{\mathbb{P}^r}$, respectively. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r$. Recall that the coherent sheaf $\mathcal{F}$ has a projective resolution in a form* (2.2.1)*. Put $\mathcal{G}_i$ and $\mathcal{K}_i$ as in* (2.2.2)*. Then there exist the following isomorphisms of K-vector spaces*::

(1) $H^0(\mathcal{F}) \cong \left(H^0(\mathcal{G}_1)/\mathrm{Ker}\left(H^0(f_0)\right)\right) \oplus \left(\mathrm{Ker}\left(H^r(f_r)\right)/\mathrm{Im}\left(H^r(f_{r+1})\right)\right)$.

(2) $\mathrm{Ker}\left(H^0(f_0)\right) \cong H^0(\mathcal{K}_0)$.

(3) $H^0(\mathcal{K}_0) \cong \mathrm{Im}\left(H^0(f_1)\right) \oplus \mathrm{Ker}\left(H^r(f_{r+1})\right)$,

where $H^q(f_i)$ denotes the morphism $H^q(X, \mathcal{G}_{i+1}) \longrightarrow H^q(X, \mathcal{G}_i)$ induced by f_i for $1 \leq i \leq r+1$. Thus we have the following formula of $\dim_K H^0(\mathcal{F})$:

$$\dim_K H^0(\mathcal{F}) \;=\; \dim_K H^0(\mathcal{G}_1) - \dim_K H^r(\mathcal{G}_{r+2}) + \dim_K H^r(\mathcal{G}_{r+1}) - \mathrm{rk} H^0(f_1) - \mathrm{rk} H^r(f_r),$$

where we set $\mathrm{rk} H^r(f_i) := \dim_K \mathrm{Im}(H^r(f_i))$.

Proof. As only a sketch of a proof is given in [9], we give a complete proof here.

(1) We have the following exact sequence of cohomology groups:

$$0 \to H^0(\mathcal{K}_0) \to H^0(\mathcal{G}_1) \to H^0(\mathcal{F}) \to H^1(\mathcal{K}_0) \to 0. \tag{A.0.1}$$

Since the K-homomorphism $H^0(\mathcal{F}) \to H^1(\mathcal{K}_0)$ is surjective, it follows that

$$H^0(\mathcal{F}) \cong \mathrm{Im}\left(H^0(f_0)\right) \oplus H^1(\mathcal{K}_0) \cong \left(H^0(\mathcal{G}_1)/\mathrm{Ker}\left(H^0(f_0)\right)\right) \oplus H^1(\mathcal{K}_0).$$

In the same way as $H^q(\mathcal{F})$ $(1 \leq q \leq r)$ (Theorem 2.2.1), we have the isomorphism $H^1(\mathcal{K}_0) \cong \mathrm{Ker}(H^r(f_r))/\mathrm{Im}(H^r(f_{r+1}))$.

(2) The result clearly holds by the exact sequence (A.0.1).

(3) By the exact sequence

$$H^0(\mathcal{G}_2) \to H^0(\mathcal{K}_0) \to H^1(\mathcal{K}_1) \to 0,$$

we have $H^0(\mathcal{K}_0) \cong \mathrm{Im}(\tau) \oplus H^1(\mathcal{K}_1)$, where τ denotes the morphism $H^0(\mathcal{G}_2) \to H^0(\mathcal{K}_0)$. Note that the following diagram commutes:

$$\begin{tikzcd} H^0(X,\mathcal{G}_2) \arrow[rr, "H^0(f_1)"] \arrow[dr, "\tau"'] & & H^0(X,\mathcal{G}_1) \\ & H^0(X,\mathcal{K}_0) \arrow[ur] & \end{tikzcd}$$

Since the morphism $H^0(X,\mathcal{K}_0) \to H^0(X,\mathcal{G}_1)$ is injective, we have $\mathrm{Ker}\left(H^0(f_1)\right) = \mathrm{Ker}(\tau)$. Hence we have

$$\begin{aligned} \mathrm{Im}(\tau) &\cong H^0(X,\mathcal{G}_2)/\mathrm{Ker}(\tau) \\ &\cong H^0(X,\mathcal{G}_2)/\mathrm{Ker}\left(H^0(f_1)\right) \\ &\cong \mathrm{Im}\left(H^0(f_1)\right). \end{aligned}$$

In the same way as $H^q(\mathcal{F})$ $(1 \leq q \leq r)$ (Theorem 2.2.1), we have the isomorphism $H^1(\mathcal{K}_1) \cong \mathrm{Ker}(H^r(f_{r+1}))$ because $\mathcal{K}_r \cong \mathcal{G}_{r+2}$. $\square$

By Theorem A.0.1, in a similar way to the case of $1 \leq q \leq r$ (Section 3), it is possible to give an explicit algorithm to compute the dimension of the global section $\Gamma(\mathbb{P}^r, \mathcal{F}(n)) = H^0(\mathbb{P}^r, \mathcal{F}(n))$ (let us omit to write down it explicitly in this paper).

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