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Abstract

The UC hierarchy is an extension of the KP hierarchy, which possesses not only an infinite set of positive time evolutions but also that of negative ones. Through a similarity reduction we derive from the UC hierarchy a class of the Schlesinger systems including the Garnier system and the sixth Painlevé equation, which describes the monodromy preserving deformations of Fuchsian linear differential equations with certain spectral types. We also present a unified formulation of the above Schlesinger systems as a canonical Hamiltonian system whose Hamiltonian functions are polynomials in the canonical variables.

1 Introduction

This work is aimed to present a certain connection between infinite-dimensional integrable systems of soliton type and finite-dimensional integrable systems of isomonodromic type. The KP hierarchy is, undoubtedly, the most basic one among the former and is a series of nonlinear partial differential equations in infinitely many independent variables $\boldsymbol{x} = (x_1, x_2, x_3, \dots)$ that are consistent with each other. It literally includes as the first nontrivial member the KP (Kadomtsev–Petviashvili) equation

$$\frac{3}{4}\frac{\partial^2 f}{\partial {x_2}^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_3} - \frac{3}{2} f \frac{\partial f}{\partial x_1} - \frac{1}{4}\frac{\partial^3 f}{\partial {x_1}^3}\right), \tag{1.1}$$

which is a typical soliton equation. If we count the degree of variables as $\deg x_n = n$ and $\deg f = -2$, then both sides of (1.1) are equally homogeneous (of degree -6) as differential polynomials in f with respect to x_n. Every equation of the KP hierarchy is known to be homogeneous, in fact. In this sense we may say that the KP hierarchy forms a homogeneous integrable system equipped with an infinite set of time evolutions of positive degree. The UC hierarchy, introduced in [Tsu04], is an infinite-dimensional integrable system which naturally generalizes the KP hierarchy by taking into account the negative time evolutions besides the positive ones while keeping its homogeneity. The independent variables of the UC hierarchy consist of two sets of infinitely many variables $\boldsymbol{x}$ and $\boldsymbol{y} = (y_1, y_2, y_3, \dots)$ with their degrees given as $\deg x_n = n$ and $\deg y_n = -n$. In this paper we show

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that a similarity reduction of the UC hierarchy yields a broad class of the Schlesinger systems including the Garnier system and the sixth Painlevé equation, which describes the monodromy preserving deformations of Fuchsian linear differential equations with certain spectral types.

We begin by recalling the definition of the UC hierarchy. Let us introduce the commuting pair of linear differential operators (called the *vertex operators*)

$$X^{\pm}(z) = \sum_{n\in\mathbb{Z}} X_n^{\pm} z^n = e^{\pm\xi(\boldsymbol{x}-\widetilde{\partial}_{\boldsymbol{y}},z)} e^{\mp\xi(\widetilde{\partial}_{\boldsymbol{x}},z^{-1})}, \tag{1.2a}$$

$$Y^{\pm}(w) = \sum_{n\in\mathbb{Z}} Y_n^{\pm} w^n = e^{\pm\xi(\boldsymbol{y}-\widetilde{\partial}_{\boldsymbol{x}},w)} e^{\mp\xi(\widetilde{\partial}_{\boldsymbol{y}},w^{-1})}, \tag{1.2b}$$

where we have used the notations

$$\xi(\boldsymbol{x}, z) = \sum_{n=1}^{\infty} x_n z^n \quad \text{and} \quad \widetilde{\partial}_{\boldsymbol{x}} = \left(\frac{\partial}{\partial x_1}, \frac{1}{2}\frac{\partial}{\partial x_2}, \frac{1}{3}\frac{\partial}{\partial x_3}, \ldots \right).$$

Definition 1.1. For an unknown function $\tau = \tau(\boldsymbol{x}, \boldsymbol{y})$, the simultaneous bilinear equation

$$\sum_{i+j=-1} X_i^- \tau \otimes X_j^+ \tau = \sum_{i+j=-1} Y_i^- \tau \otimes Y_j^+ \tau = 0 \tag{1.3}$$

is called the *UC hierarchy*.

The UC hierarchy is homogeneous indeed as it has the following scaling symmetry: if τ is a solution of (1.3) then so is $\tau(cx_1, c^2 x_2, \ldots, c^{-1} y_1, c^{-2} y_2, \ldots)$ for any $c \in \mathbb{C}^{\times}$. The UC hierarchy is regarded as an extension of the KP hierarchy. If τ does not depend on $\boldsymbol{y}$, then the latter equality of (1.3) trivially holds and the former reduces to the bilinear expression of the KP hierarchy, which is due to Date–Jimbo–Kashiwara–Miwa (see [Kac90, MJD00]); for reference the variable transformation toward the original KP equation, (1.1), is given by $f = 2(\partial/\partial x_1)^2 \log \tau$. We always require the solution $\tau = \tau(\boldsymbol{x}, \boldsymbol{y})$, called the *$\tau$-function*, to be an entire function with respect to each independent variable. Note that τ-functions are distinguished up to multiplication by constants, as can be seen from (1.3). Concerning the UC hierarchy there is a counterpart of the Sato theory about the KP hierarchy; cf. [Sat81]. That is, the totality of solutions of the UC hierarchy forms a direct product of two Sato Grassmannians and the action of its transformation group can be realized by means of the vertex operators. For details to [Tsu04]. Of particular interest is its homogeneous polynomial solution, which is a fixed solution with respect to the above scaling symmetry.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{\ell'})$ be a pair of partitions. Consider the following determinant of *twisted* Jacobi–Trudi type:

$$S_{[\lambda,\mu]} = \det \begin{pmatrix} \widetilde{h}_{\mu_{\ell'-i+1}+i-j}, & i \leq \ell' \\ h_{\lambda_{i-\ell'}-i+j}, & i > \ell' + 1 \end{pmatrix}_{1\leq i,j\leq \ell+\ell'}, \tag{1.4}$$

where $h_n = h_n(\boldsymbol{x})$ ($n \in \mathbb{Z}$) is a polynomial in only $\boldsymbol{x}$ and is defined by the generating function

$$e^{\xi(\boldsymbol{x},z)} = \sum_{n\in\mathbb{Z}} h_n z^n,$$

and $\widetilde{h}_n = \widetilde{h}_n(\boldsymbol{y})$ is exactly the same as h_n except replacing $\boldsymbol{x}$ with $\boldsymbol{y}$. If $\mu = \emptyset$ then (1.4) reduces to the (usual) Jacobi–Trudi formula: $S_\lambda = S_{[\lambda,\emptyset]} = \det(h_{\lambda_i-i+j})$, which defines the Schur function

$S_\lambda = S_\lambda(\boldsymbol{x})$. The polynomial $S_{[\lambda,\mu]} = S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ is called the *universal character* and was originally introduced by Koike [Koi89] in the study of classical groups. It is easy to see that $S_{[\lambda,\mu]}$ becomes a homogeneous polynomial whose degree equals the difference $|\lambda| - |\mu|$, where the sum $|\lambda| = \lambda_1 + \cdots + \lambda_\ell$ denotes the weight of a partition λ. A few examples are $S_{[\emptyset,\emptyset]} = 1$, $S_{[(1),\emptyset]} = x_1$, $S_{[(1),(1)]} = x_1 y_1 - 1$, $S_{[(2,1),(1)]} = y_1\left(x_1{}^3/3 - x_3\right) - x_1{}^2$, etc. Remarkably, the set of homogeneous polynomial solutions of the UC hierarchy, (1.3), coincides with that of the universal characters $\{S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})\}_{\lambda,\mu:\text{partitions}}$.

By considering general homogeneous solutions of the UC hierarchy that are not necessarily polynomials, we can find a link to the theory of monodromy preserving deformations. Let us explain it in more detail. First we derive from the original one, (1.3), similar bilinear equations among solutions generated by successive application of vertex operators. Let $\tau_{m,n} = \tau_{m,n}(\boldsymbol{x},\boldsymbol{y})$ denote such a sequence of solutions of the UC hierarchy. A typical example of the bilinear equations is

$$\tau_{m,n} \otimes \tau_{m+1,n+1} = \sum_{i+j=0} X_i^- \tau_{m+1,n} \otimes X_j^+ \tau_{m,n+1}.$$

Next we impose on the sequence $\tau_{m,n}$ of solutions homogeneity

$$E\tau_{m,n} = d_{m,n}\tau_{m,n} \quad (d_{m,n} \in \mathbb{C}) \tag{1.5}$$

and periodicity

$$\tau_{m+L,n} = \tau_{m,n+L} = \tau_{m,n} \tag{1.6}$$

for an integer $L(\geq 2)$ fixed. Here we have used the *Euler operator*

$$E = \sum_{n=1}^{\infty} \left(n x_n \frac{\partial}{\partial x_n} - n y_n \frac{\partial}{\partial y_n} \right),$$

which is a linear differential operator measuring the degree of a homogeneous function; for instance, $ES_{[\lambda,\mu]} = (|\lambda| - |\mu|)S_{[\lambda,\mu]}$. Finally we substitute into each x_n and y_n the 'power sum' of new independent variables $\boldsymbol{t} = (t_0, t_1, \ldots, t_N)$ as

$$x_n = \frac{1}{n}\sum_{i=0}^{N} \theta_i t_i{}^n \quad \text{and} \quad y_n = \frac{1}{n}\sum_{i=0}^{N} \theta_i t_i{}^{-n} \quad (n = 1, 2, \ldots) \tag{1.7}$$

where $\theta_i \in \mathbb{C}$ are constant parameters. In view of the homogeneity (1.5), we may take $t_0 = 1$ without loss of generality. Under the reduction conditions (1.5), (1.6), and (1.7), the UC hierarchy yields a system of nonlinear partial differential equations in N variables, hereafter denoted by $\mathcal{G}_{L,N}$, whose phase space is essentially of $2N(L-1)$ dimension. To sum up the above procedure, we say that $\mathcal{G}_{L,N}$ is a *similarity reduction* of the UC hierarchy. The system $\mathcal{G}_{L,N}$ is a finite-dimensional integrable system of isomonodromic type. For instance $\mathcal{G}_{2,N}$ corresponds to the Garnier system in N variables and $\mathcal{G}_{2,1}$, the first nontrivial case, does the sixth Painlevé equation. From the viewpoint of the UC hierarchy we can clearly understand various aspects of $\mathcal{G}_{L,N}$, e.g., Hirota bilinear relations for τ-functions, Weyl group symmetries, and algebraic solutions expressed in terms of the universal character.

As analogous to the case of the KP hierarchy, the UC hierarchy (1.3) generates the linear equations for unknown functions (called the *wave functions*)

$$\psi_{m,n} = \psi_{m,n}(\boldsymbol{x},\boldsymbol{y},k) = \frac{\tau_{m,n-1}(\boldsymbol{x} - [k^{-1}], \boldsymbol{y} - [k])}{\tau_{m,n}(\boldsymbol{x},\boldsymbol{y})} e^{\xi(\boldsymbol{x},k)},$$

where $[k] = (k, k^2/2, k^3/3, \ldots)$. Through the reduction procedure they induce an auxiliary system of linear differential equations; one of which is a Fuchsian system of rank L in the *spectral variable* $z = k^L$ with $N + 3$ poles on the Riemann sphere, and the others govern its monodromy preserving deformations. The nonlinear system $\mathcal{G}_{L,N}$ can be reformulated as a compatibility condition of this auxiliary linear system (Lax formalism). Remark here that the compatibility itself is *a priori* established because all the linear equations originate from the same bilinear equation (1.3).

The *spectral type* of the Fuchsian system under consideration is given by the $(N + 3)$-tuple

$$\underbrace{(L-1,1),\ldots,(L-1,1)}_{N+1},(1,1,\ldots,1),(1,1,\ldots,1)$$

of partitions of L, which indicates how the characteristic exponents overlap at each of the $N + 3$ singularities. Thus we conclude that $\mathcal{G}_{L,N}$ is equivalent to a particular case of the Schlesinger systems specified by this spectral type. We also present a unified description of $\mathcal{G}_{L,N}$ for any L and N as a canonical Hamiltonian system, denoted by $\mathcal{H}_{L,N}$, whose Hamiltonian functions are polynomials in the canonical variables.

In the next section we derive some difference (and differential) equations from the UC hierarchy as a preliminary. In Sect. 3, we construct a sequence of homogeneous solutions of the UC hierarchy and present its Weyl group symmetry of type A. In Sect. 4, we consider a similarity reduction of the UC hierarchy by requiring its solutions to satisfy the homogeneity and periodicity. As a result we obtain a nonlinear system $\mathcal{G}_{L,N}$ of partial differential equations, which provides an extension of both the Garnier system and the sixth Painlevé equation. The universal characters $S_{[\lambda,\mu]}$ are homogeneous solutions of the UC hierarchy and thereby consistent with the similarity reduction. Hence, as described in Sect. 5, it is immediate to obtain particular solutions of $\mathcal{G}_{L,N}$ expressed in terms of $S_{[\lambda,\mu]}$. The subject of Sect. 6 is the Lax formalism of the systems $\mathcal{G}_{L,N}$, which reveals that they constitute a class of the Schlesinger systems. We show that the auxiliary linear problem of $\mathcal{G}_{L,N}$ arises naturally from the linear equations satisfied by the wave functions of the UC hierarchy. In Sect. 7, we transform $\mathcal{G}_{L,N}$ into the canonical Hamiltonian system $\mathcal{H}_{L,N}$ with polynomial Hamiltonian functions. Section 8 is devoted to the birational symmetries. We observe that the Weyl group actions, discussed in Sect. 3, give rise to birational canonical transformations of $\mathcal{H}_{L,N}$. In the appendix we briefly indicate a relationship between our polynomial Hamiltonian structure and that given by Kimura and Okamoto [KO84] for the Garnier system, i.e., the case where $L = 2$.

2 Method for generating a 'closed' functional equation

Unlike in the case of the KP hierarchy, every differential equation of the UC hierarchy with respect to the original variables $\boldsymbol{x}$ and $\boldsymbol{y}$ is of infinite order. In this section we show how to overcome this difficulty, i.e., a method for generating a 'closed' functional equation from the UC hierarchy; cf. [DJM82].

We first recall that if $\tau = \tau(\boldsymbol{x}, \boldsymbol{y})$ is a solution of (1.3) then so are $X^+(a)\tau$ and $Y^+(b)\tau$ for any $a, b \in \mathbb{C}^\times$. With this fact in mind, let us take our interest in bilinear equations for a sequence of solutions generated by successive application of vertex operators. Suppose $\tau_{0,0} = \tau(\boldsymbol{x}, \boldsymbol{y})$ to be a solution of the UC hierarchy, (1.3). Define a sequence $\tau_{m,n}$ of solutions by

$$\tau_{m,n} = \prod_{i=0}^{m-1} X^+(a_i) \prod_{j=0}^{n-1} Y^+(b_j)\tau_{0,0}, \tag{2.1}$$

where we write as

$$\prod_{i=0}^{m-1} X^+(a_i) = X^+(a_{m-1}) \cdots X^+(a_1)X^+(a_0).$$

Then we can derive similar bilinear equations from the UC hierarchy, the original one (1.3).

Lemma 2.1. *For integers $m, n \geq 0$, it holds that*

$$\sum_{i+j=-m-1} X_i^- \tau_{0,0} \otimes X_j^+ \tau_{m,n} = \sum_{i+j=-n-1} Y_i^- \tau_{0,0} \otimes Y_j^+ \tau_{m,n} = 0, \tag{2.2}$$

$$\tau_{0,0} \otimes \tau_{1,n} - \sum_{i+j=0} X_i^- \tau_{1,0} \otimes X_j^+ \tau_{0,n} = \sum_{i+j=-n-1} Y_i^- \tau_{1,0} \otimes Y_j^+ \tau_{0,n} = 0, \tag{2.3}$$

$$\sum_{i+j=-m-1} X_i^- \tau_{0,1} \otimes X_j^+ \tau_{m,0} = \tau_{0,0} \otimes \tau_{m,1} - \sum_{i+j=0} Y_i^- \tau_{0,1} \otimes Y_j^+ \tau_{m,0} = 0. \tag{2.4}$$

Proof. Notice that the operators $X_i^\pm$ ($i \in \mathbb{Z}$) satisfy the *fermionic* relations: $X_i^\pm X_j^\pm + X_{j-1}^\pm X_{i+1}^\pm = 0$ and $X_i^+ X_j^- + X_{j+1}^- X_{i-1}^+ = \delta_{i+j,0}$. The same relations hold also for $Y_i^\pm$. Moreover, $X_i^\pm$ and $Y_j^\pm$ mutually commute. See [Tsu04]. By virtue of the above relations, applying $1 \otimes \prod_{i=0}^{m-1} X^+(a_i) \prod_{j=0}^{n-1} Y^+(b_j)$, $X^+(a_0) \otimes \prod_{j=0}^{n-1} Y^+(b_j)$, and $Y^+(b_0) \otimes \prod_{i=0}^{m-1} X^+(a_i)$ to (1.3), we obtain (2.2), (2.3), and (2.4), respectively. □

We shall look closely at (2.2), which corresponds to the original UC hierarchy (1.3) when $m = n = 0$. It can be rewritten equivalently into

$$\frac{1}{2\pi\sqrt{-1}} \oint z^m e^{\xi(\boldsymbol{x}-\boldsymbol{x}',z)} \mathrm{d}z \, \tau_{0,0}(\boldsymbol{x}' + [z^{-1}], \boldsymbol{y}' + [z])\tau_{m,n}(\boldsymbol{x} - [z^{-1}], \boldsymbol{y} - [z]) = 0, \tag{2.5a}$$

$$\frac{1}{2\pi\sqrt{-1}} \oint w^n e^{\xi(\boldsymbol{y}-\boldsymbol{y}',w)} \mathrm{d}w \, \tau_{0,0}(\boldsymbol{x}' + [w], \boldsymbol{y}' + [w^{-1}])\tau_{m,n}(\boldsymbol{x} - [w], \boldsymbol{y} - [w^{-1}]) = 0 \tag{2.5b}$$

with $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{x}'$, and $\boldsymbol{y}'$ being arbitrary parameters, where $\oint \frac{\mathrm{d}z}{2\pi\sqrt{-1}}$ means taking the coefficient of $1/z$ of the integrand as a (formal) Laurent series expansion in z. If we try to write down a differential equation *naively* after the case of the KP hierarchy, namely if we consider the Taylor series expansion of (2.5a) around $\{\boldsymbol{x}' = \boldsymbol{x}, \boldsymbol{y}' = \boldsymbol{y}\}$, then we have an infinite set of differential equations of infinite order; see [Tsu04]. This result reflects the fact that the integrand of (2.5a) under the substitution $\boldsymbol{x}' = \boldsymbol{x}$ and $\boldsymbol{y}' = \boldsymbol{y}$ may admit an essential singularity not only at $z = 0$ but also at $z = \infty$. However, we can construct a functional equation in a closed expression by taking an appropriate choice of parameters $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{x}'$, and $\boldsymbol{y}'$ instead.

Let $I, J \subset \mathbb{Z}$ be a disjoint pair of finite indexing sets. By specializing the parameters in (2.5) as

$$\boldsymbol{x}' = \boldsymbol{x} - \sum_{j\in I}[t_j] + \sum_{j\in J}[t_j], \quad \boldsymbol{y}' = \boldsymbol{y} - \sum_{j\in I}[t_j^{-1}] + \sum_{j\in J}[t_j^{-1}],$$

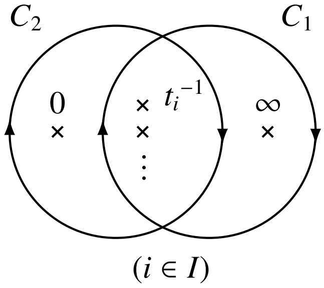

Figure 1: Contours of integration and singularities of $F(z)$.

we obtain

$$\Omega_1 := z^m e^{\xi(\boldsymbol{x}-\boldsymbol{x}',z)}\mathrm{d}z = z^m \frac{\prod_{j\in J}(1-t_jz)}{\prod_{j\in I}(1-t_jz)}\mathrm{d}z \quad (|t_jz|<1),$$

$$\Omega_2 := w^n e^{\xi(\boldsymbol{y}-\boldsymbol{y}',w)}\mathrm{d}w = w^n \frac{\prod_{j\in J}(1-w/t_j)}{\prod_{j\in I}(1-w/t_j)}\mathrm{d}w \quad (|w/t_j|<1).$$

Here we have used the Taylor expansion, $\log(1-u) = -\sum_{k=1}^{\infty} u^k/k$ valid for $|u|<1$. Suppose $zw = 1$. Then we observe that

$$\Omega_2 = z^{-n}\frac{\prod_{j\in J}(1-1/t_jz)}{\prod_{j\in I}(1-1/t_jz)}\left(-\frac{\mathrm{d}z}{z^2}\right) = -z^{|I|-|J|-m-n-2}\frac{\prod_{j\in I}(-t_j)}{\prod_{j\in J}(-t_j)}\Omega_1.$$

Consequently, both integrands of (2.5a) and (2.5b) coincide up to constant multiplication if the condition $|I|-|J| = m+n+2$ is fulfilled. In this case the integrand of (2.5a) reads

$$F(z) = z^m \frac{\prod_{j\in J}(1-t_jz)}{\prod_{j\in I}(1-t_jz)}\tau_{0,0}(\boldsymbol{x}'+[z^{-1}],\boldsymbol{y}'+[z])\tau_{m,n}(\boldsymbol{x}-[z^{-1}],\boldsymbol{y}-[z]).$$

Since $\tau_{0,0}(\boldsymbol{x},\boldsymbol{y})$ and $\tau_{m,n}(\boldsymbol{x},\boldsymbol{y})$ are entire, $F(z)$ has the $|I|+2$ singularities: $z = 1/t_i$ (simple poles) for $i \in I$ and $z = 0,\infty$ (which may be essential singularities). Hence (2.5) becomes

$$\int_{C_1} F(z)\mathrm{d}z = \int_{C_2} F(z)\mathrm{d}z = 0, \tag{2.6}$$

where the integration contour C_1 (resp. C_2) is a positively oriented small circle around $z = 0$ (resp. $z = \infty$) such that all the other singularities are exterior to it; see Figure 1. We verify through the Cauchy–Goursat theorem that

$$\sum_{i\in I}\operatorname*{Res}_{z=1/t_i} F(z)\mathrm{d}z = 0 \tag{2.7}$$

by canceling contribution of residues at $z = \infty$ and $z = 0$ respectively to the first and second integrals in (2.6). In other words, we have successfully avoided the residue calculus at possible essential singularities $z = 0,\infty$ thanks to the presence of *two* bilinear equations (2.5a) and (2.5b).

Now we prepare some notations. For a function $f = f(\boldsymbol{x},\boldsymbol{y})$, we define a shift operator T_i by $T_i(f) = f(\boldsymbol{x}-[t_i],\boldsymbol{y}-[t_i^{-1}])$. We also write $T_{\{i_1,i_2,\dots,i_r\}}(f) = T_{i_1}\circ T_{i_2}\circ\cdots\circ T_{i_r}(f)$ for brevity. Then (2.7) takes the following form:

$$\sum_{i\in I} t_i^{\,n}\frac{\prod_{j\in J}(t_i-t_j)}{\prod_{j\in I\setminus\{i\}}(t_i-t_j)}T_{I\setminus\{i\}}(\tau_{0,0})T_{J\cup\{i\}}(\tau_{m,n}) = 0,$$

which can be regarded as a difference equation with each t_i being the difference interval. Along the same lines as (2.2), also (2.3) and (2.4) generate similar difference equations. Summarizing above we have the

Proposition 2.2. *The following difference equations hold.*

1. *If* $|I| - |J| = m + n + 2$ *and* $m, n \geq 0$*, then*

$$\sum_{i \in I} t_i{}^n \frac{\prod_{j \in J}(t_i - t_j)}{\prod_{j \in I \setminus \{i\}}(t_i - t_j)} T_{I \setminus \{i\}}(\tau_{0,0}) T_{J \cup \{i\}}(\tau_{m,n}) = 0. \tag{2.8}$$

2. *If* $|I| - |J| = n + 1$ *and* $n \geq 0$*, then*

$$T_I(\tau_{0,0}) T_J(\tau_{1,n}) = \sum_{i \in I} \frac{\prod_{j \in J}(1 - t_j/t_i)}{\prod_{j \in I \setminus \{i\}}(1 - t_j/t_i)} T_{I \setminus \{i\}}(\tau_{1,0}) T_{J \cup \{i\}}(\tau_{0,n}). \tag{2.9}$$

3. *If* $|I| - |J| = m + 1$ *and* $m \geq 0$*, then*

$$T_I(\tau_{0,0}) T_J(\tau_{m,1}) = \sum_{i \in I} \frac{\prod_{j \in J}(1 - t_i/t_j)}{\prod_{j \in I \setminus \{i\}}(1 - t_i/t_j)} T_{I \setminus \{i\}}(\tau_{0,1}) T_{J \cup \{i\}}(\tau_{m,0}). \tag{2.10}$$

Example 2.3. Consider the case $m = n = 0$, $I = \{1, 2, 3\}$, and $J = \{4\}$. Write $\tau = \tau_{0,0}$. Then (2.8) reduces to the equation

$$\begin{aligned}(t_1 - t_2)(t_3 - t_4)T_{1,2}(\tau)T_{3,4}(\tau) + (t_2 - t_3)(t_1 - t_4)T_{2,3}(\tau)T_{1,4}(\tau) \\ + (t_3 - t_1)(t_2 - t_4)T_{1,3}(\tau)T_{2,4}(\tau) = 0,\end{aligned}$$

which was found by Ohta [Oht07] as a quadratic relation for the universal character.

Let $m = 1$, $n = 0$, $I = \{1, 2, 3\}$, and $J = \emptyset$. Then (2.8) reduces to

$$(t_1 - t_2)T_{1,2}(\tau_{0,0})T_3(\tau_{1,0}) + (t_2 - t_3)T_{2,3}(\tau_{0,0})T_1(\tau_{1,0}) + (t_3 - t_1)T_{1,3}(\tau_{0,0})T_2(\tau_{1,0}) = 0. \tag{2.11}$$

Let $n = 1$, $I = \{1, 2\}$, and $J = \emptyset$. Then (2.9) reduces to

$$(t_1 - t_2)T_{1,2}(\tau_{0,0})\tau_{1,1} = t_1 T_1(\tau_{0,1})T_2(\tau_{1,0}) - t_2 T_2(\tau_{0,1})T_1(\tau_{1,0}). \tag{2.12}$$

The above difference equations, (2.11) and (2.12), were introduced in a study of the connection between the universal character and q-Painlevé equations; see [Tsu05b, Tsu09a].

Furthermore, we can obtain a functional equation that involves derivative terms from the difference equations through a limit process causing a confluence of the poles $z = 1/t_i$. For instance let us take the limit $t_3 \to t_1$ in (2.11). Rewrite $(t, s) = (t_1, t_2)$ and shift the variables as $\boldsymbol{x} \mapsto \boldsymbol{x} + [t]$ and $\boldsymbol{y} \mapsto \boldsymbol{y} + [t^{-1}]$. Then we find

$$\begin{aligned}&\left(D_{\delta_t} + \frac{t}{s - t}\right)\tau_{0,0}(\boldsymbol{x} - [s], \boldsymbol{y} - [s^{-1}]) \cdot \tau_{1,0}(\boldsymbol{x}, \boldsymbol{y}) \\ &\quad + \frac{t}{t - s}\tau_{0,0}(\boldsymbol{x} - [t], \boldsymbol{y} - [t^{-1}])\tau_{1,0}(\boldsymbol{x} + [t] - [s], \boldsymbol{y} + [t^{-1}] - [s^{-1}]) = 0.\end{aligned} \tag{2.13}$$

Here we have introduced the vector fields

$$\delta_t = \sum_{n=1}^{\infty}\left(t^n\frac{\partial}{\partial x_n} - t^{-n}\frac{\partial}{\partial y_n}\right) \quad \text{and} \quad \tilde{\delta}_t = \sum_{n=1}^{\infty}\left(nt^n\frac{\partial}{\partial x_n} + nt^{-n}\frac{\partial}{\partial y_n}\right),$$

and let $D_{\boldsymbol{v}}$ denote the Hirota differential with respect to a vector field $\boldsymbol{v}$. If we take continuously the limit $s \to t$ in (2.13) with divided by $t-s$, then we obtain

$$\left({D_{\delta_t}}^2 - D_{\delta_t} + D_{\tilde{\delta}_t}\right)\tau_{0,0}(\boldsymbol{x}-[t],\boldsymbol{y}-[t^{-1}])\cdot\tau_{1,0}(\boldsymbol{x},\boldsymbol{y}) = 0. \tag{2.14}$$

In this manner we can produce various functional equations from the UC hierarchy. We list the ones relevant to the following sections.

Proposition 2.4. *The following difference* (*and differential*) *equations hold*:

$$\begin{aligned}
&(t-s)\tau_{m,n}(\boldsymbol{x}-[t]-[s],\boldsymbol{y}-[t^{-1}]-[s^{-1}])\tau_{m+1,n+1}(\boldsymbol{x},\boldsymbol{y})\\
&\quad - t\tau_{m,n+1}(\boldsymbol{x}-[t],\boldsymbol{y}-[t^{-1}])\tau_{m+1,n}(\boldsymbol{x}-[s],\boldsymbol{y}-[s^{-1}])\\
&\quad + s\tau_{m,n+1}(\boldsymbol{x}-[s],\boldsymbol{y}-[s^{-1}])\tau_{m+1,n}(\boldsymbol{x}-[t],\boldsymbol{y}-[t^{-1}]) = 0, &(2.15)\\
&(D_{\delta_t}-1)\,\tau_{m,n+1}(\boldsymbol{x},\boldsymbol{y})\cdot\tau_{m+1,n}(\boldsymbol{x},\boldsymbol{y})\\
&\quad + \tau_{m,n}(\boldsymbol{x}-[t],\boldsymbol{y}-[t^{-1}])\tau_{m+1,n+1}(\boldsymbol{x}+[t],\boldsymbol{y}+[t^{-1}]) = 0, &(2.16)\\
&\left(D_{\delta_t}+\frac{t}{s-t}\right)\tau_{m,n}(\boldsymbol{x}-[s],\boldsymbol{y}-[s^{-1}])\cdot\tau_{m+1,n}(\boldsymbol{x},\boldsymbol{y})\\
&\quad + \frac{t}{t-s}\tau_{m,n}(\boldsymbol{x}-[t],\boldsymbol{y}-[t^{-1}])\tau_{m+1,n}(\boldsymbol{x}+[t]-[s],\boldsymbol{y}+[t^{-1}]-[s^{-1}]) = 0. &(2.17)
\end{aligned}$$

Proof. Clearly (2.15) and (2.17) are equivalent to (2.12) and (2.13), respectively. Taking the limit $s \to t$ in (2.15) leads to (2.16). □

3 Homogeneous τ-sequence and its Weyl group symmetry

This section is concerned with a sequence of homogeneous solutions of the UC hierarchy, connected by vertex operators. We show that such a sequence naturally admits a commutative pair of Weyl group actions of type A generated by a permutation of two serial vertex operators.

We first introduce partial differential operators $V_X(c)$ and $V_Y(c')$ ($c, c' \in \mathbb{C}$) defined by

$$V_X(c) = \int_\gamma X^+(z)z^{-c-1}\mathrm{d}z \quad \text{and} \quad V_Y(c') = \int_{\gamma'} Y^+(z^{-1})z^{c'-1}\mathrm{d}z,$$

where the integration paths $\gamma, \gamma' : [0,1] \to \mathbb{C}$ is taken such that $[X^+(z)z^{-c}]_{\gamma(0)}^{\gamma(1)} = [Y^+(z^{-1})z^{c'}]_{\gamma'(0)}^{\gamma'(1)} = 0$. For instance γ and γ' can be chosen to be cycles. Note that in general γ and γ' may depend on c and c', respectively. It is easy to see that $V_X(c)$ and $V_Y(c')$ mutually commute.

Suppose $\tau_{0,0} = \tau_{0,0}(\boldsymbol{x},\boldsymbol{y})$ to be a solution of the UC hierarchy (1.3) satisfying the homogeneity $E\tau_{0,0} = d_{0,0}\tau$. Instead of (2.1), let us consider a sequence $\{\tau_{m,n}\}_{m,n\geq 0}$ determined recursively by

$$\tau_{m+1,n} = V_X(c_m)\tau_{m,n} \quad \text{and} \quad \tau_{m,n+1} = V_Y(c'_n)\tau_{m,n}$$

for arbitrary constant parameters $c_m, c'_n \in \mathbb{C}$ given. Since the UC hierarchy (1.3) takes the form of bilinear equations, it can be verified in exactly the same way as (2.1) that each $\tau_{m,n}$ gives a solution of (1.3). Furthermore, they all obey the homogeneity

$$E\tau_{m,n} = d_{m,n}\tau_{m,n}$$

with $d_{m+1,n} = d_{m,n} + c_m$ and $d_{m,n+1} = d_{m,n} - c'_n$, as a consequence of the formulae $[E, V_X(c)] = cV_X(c)$ and $[E, V_Y(c')] = -c'V_Y(c')$; cf. [Tsu09b, Lemma 2.4]. Hence the balancing condition

$$d_{m,n} + d_{m+1,n+1} = d_{m,n+1} + d_{m+1,n}$$

is fulfilled. We call the above sequence of homogeneous solutions of the UC hierarchy a *homogeneous τ-sequence*. Obviously, any functional equation in Sect. 2 still remains valid for the homogeneous τ-sequence $\{\tau_{m,n}\}$; we may call also $V_X(c)$ and $V_Y(c')$ vertex operators.

Example 3.1. If we take $c = c' = n$ to be an integer and each γ and γ' a positively oriented small circle around the origin $z = 0$, then $V_X(n) = 2\pi\sqrt{-1}X^+_n$ and $V_Y(n) = 2\pi\sqrt{-1}Y^+_n$ according to (1.2). Recall now that these operators play roles of raising operators for the universal characters; namely,

$$S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}) = X^+_{\lambda_1} \dots X^+_{\lambda_\ell} Y^+_{\mu_1} \dots Y^+_{\mu_{\ell'}}.1$$

for any pair of partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_{\ell'})$; see [Tsu04, Theorem 1.2]. Starting from a trivial solution $\tau(\boldsymbol{x}, \boldsymbol{y}) = S_{[\emptyset,\emptyset]}(\boldsymbol{x}, \boldsymbol{y}) \equiv 1$ of the UC hierarchy, we thus obtain a homogeneous τ-sequence expressed in the universal characters by successive application of X^+_n and Y^n_n.

Next we consider the Weyl group symmetry of the homogeneous τ-sequence. Fix a positive integer k. Let us look at $m = k$ cites in the (m, n)-lattice and interchange the $(k-1)$th and kth operations of the vertex operators V_X in view of the fermionic relation $V_X(a)V_X(b) + V_X(b-1)V_X(a+1) = 0$. To be more precise, we transform the original sequence

$$\cdots \xrightarrow{V_X(c_{k-2})} \tau_{k-1,n} \xrightarrow{V_X(c_{k-1})} \tau_{k,n} \xrightarrow{V_X(c_k)} \tau_{k+1,n} \xrightarrow{V_X(c_{k+1})} \cdots$$

into a new one

$$\cdots \xrightarrow{V_X(c_{k-2})} \tau_{k-1,n} \xrightarrow{V_X(c_k+1)} \hat{\tau}_{k,n} \xrightarrow{V_X(c_{k-1}-1)} \tau_{k+1,n} \xrightarrow{V_X(c_{k+1})} \cdots$$

that is identical with the original one except $\tau_{k,n}$ is replaced by

$$\hat{\tau}_{k,n} = V_X(c_k + 1)\tau_{k-1,n}.$$

Besides, the degree of $\hat{\tau}_{k,n}$ reads

$$\hat{d}_{k,n} = d_{k-1,n} + c_k + 1 = d_{k-1,n} - d_{k,n} + d_{k+1,n} + 1.$$

We refer to the above permutation of vertex operators as r_k. Put

$$\alpha_k = \hat{d}_{k,n} - d_{k,n} = d_{k-1,n} - 2d_{k,n} + d_{k+1,n} + 1, \tag{3.1}$$

which is a quantity that does not depend on n. The operation r_k induces the transformation

$$r_k(\alpha_k) = -\alpha_k, \quad r_k(\alpha_{k\pm1}) = \alpha_{k\pm1} + \alpha_k, \quad \text{and} \quad r_k(\alpha_\ell) = \alpha_\ell \quad (\ell \neq k, k \pm 1).$$

Therefore α_k can be regarded as a root variable of the Weyl group of type A, and $\langle r_k \rangle$ indeed fulfills its fundamental relations

$$r_k{}^2 = 1, \quad r_k r_{k\pm1} r_k = r_{k\pm1} r_k r_{k\pm1}, \quad \text{and} \quad r_k r_\ell = r_\ell r_k \quad (\ell \neq k, k \pm 1).$$

Along the same lines we can derive from a permutation of operators V_Y another action of the Weyl group of type A, which commutes with the previous one. As demonstrated in Sect 8, this kind of Weyl group actions gives rise to a group of birational canonical transformations of the Hamiltonian system $\mathcal{H}_{L,N}$.

We conclude this section with some formulae that will be employed later.

Lemma 3.2. *It holds that*

$$\tau_{k-1,n} \otimes \tau_{k+1,n} - \sum_{i+j=-1} X_i^- \hat{\tau}_{k,n} \otimes X_j^+ \tau_{k,n} = \sum_{i+j=-1} Y_i^- \hat{\tau}_{k,n} \otimes Y_j^+ \tau_{k,n} = 0, \tag{3.2}$$

$$\tau_{k-1,n+1} \otimes \tau_{k+1,n} - \sum_{i+j=-1} X_i^- \hat{\tau}_{k,n+1} \otimes X_j^+ \tau_{k,n} = \hat{\tau}_{k,n} \otimes \tau_{k,n+1} - \sum_{i+j=0} Y_i^- \hat{\tau}_{k,n+1} \otimes Y_j^+ \tau_{k,n} = 0. \tag{3.3}$$

Proof. First we have

$$\sum_{i+j=-2} X_i^- \tau_{k-1,n} \otimes X_j^+ \tau_{k,n} = \sum_{i+j=-1} Y_i^- \tau_{k-1,n} \otimes Y_j^+ \tau_{k,n} = 0, \tag{3.4}$$

which is equivalent to (2.2) with $m = 1$ and $n = 0$. Applying $V_X(c_k + 2) \otimes 1$ and $V_X(c_k + 1) \otimes 1$ respectively to the first and second equalities in (3.4) leads to (3.2). We deduce (3.3) from (3.2) by applying $V_Y(c_n') \otimes 1$. □

Lemma 3.3. *The following difference (and differential) equations hold:*

$$D_{\delta_t} \hat{\tau}_{k,n}(\boldsymbol{x}, \boldsymbol{y}) \cdot \tau_{k,n}(\boldsymbol{x}, \boldsymbol{y}) - t\tau_{k-1,n}(\boldsymbol{x} - [t], \boldsymbol{y} - [t^{-1}])\tau_{k+1,n}(\boldsymbol{x} + [t], \boldsymbol{y} + [t^{-1}]) = 0, \tag{3.5}$$

$$\begin{aligned} &t\tau_{k-1,n+1}(\boldsymbol{x} - [t], \boldsymbol{y} - [t^{-1}])\tau_{k+1,n}(\boldsymbol{x}, \boldsymbol{y}) \\ &\quad - \hat{\tau}_{k,n+1}(\boldsymbol{x}, \boldsymbol{y})\tau_{k,n}(\boldsymbol{x} - [t], \boldsymbol{y} - [t^{-1}]) + \hat{\tau}_{k,n}(\boldsymbol{x} - [t], \boldsymbol{y} - [t^{-1}])\tau_{k,n+1}(\boldsymbol{x}, \boldsymbol{y}) = 0. \end{aligned} \tag{3.6}$$

Proof. The verification can be done along the same argument as Proposition 2.2. First we shall regard the symbol $f \otimes g$ as a product of two functions $f(\boldsymbol{x}', \boldsymbol{y}')g(\boldsymbol{x}, \boldsymbol{y})$ in distinct indeterminates $(\boldsymbol{x}', \boldsymbol{y}')$ and $(\boldsymbol{x}, \boldsymbol{y})$. Taking the variables in (3.2) as $\boldsymbol{x} - \boldsymbol{x}' = 2[t]$ and $\boldsymbol{y} - \boldsymbol{y}' = 2[t^{-1}]$ thus leads to (3.5). Similarly, we deduce (3.6) from (3.3) with $\boldsymbol{x} - \boldsymbol{x}' = [t]$ and $\boldsymbol{y} - \boldsymbol{y}' = [t^{-1}]$. □

4 Similarity reduction of UC hierarchy

In this section we consider a reduction of the UC hierarchy by requiring certain homogeneity and periodicity. As a result we derive a finite-dimensional integrable system of partial differential equations, denoted by $\mathcal{G}_{L,N}$, which provides an extension of both the Garnier system and the sixth Painlevé equation $P_{\rm VI}$.

Fix integers $L \geq 2$ and $N \geq 1$. Let $\tau_{m,n} = \tau_{m,n}(\boldsymbol{x}, \boldsymbol{y})$ be a sequence of the solutions satisfying (2.15)–(2.17) in Proposition 2.4. Suppose that $\tau_{m,n}$ are homogeneous of degree $d_{m,n} \in \mathbb{C}$, i.e.,

$$E\tau_{m,n} = d_{m,n}\tau_{m,n} \quad \text{with} \quad E = \sum_{n=1}^{\infty} \left(nx_n \frac{\partial}{\partial x_n} - ny_n \frac{\partial}{\partial y_n} \right),$$

and fullfil the periodic condition: $\tau_{m+L,n} = \tau_{m,n+L} = \tau_{m,n}$ (up to multiplication by constants). Remark here that the relation $d_{m,n} + d_{m+1,n+1} = d_{m,n+1} + d_{m+1,n}$ necessarily holds; cf. Sect. 3. Let us replace the independent variables x_n and y_n respectively with the nth and $(-n)$th power sum of new ones $\boldsymbol{t} = (t_0, t_1, \dots, t_N)$ as

$$x_n = \frac{1}{n} \sum_{i=0}^{N} \theta_i t_i^{\,n} \quad \text{and} \quad y_n = \frac{1}{n} \sum_{i=0}^{N} \theta_i t_i^{\,-n}. \tag{4.1}$$

Consequently we have

$$t_i \frac{\partial}{\partial t_i} = t_i \sum_{n=1}^{\infty} \left(\frac{\partial x_n}{\partial t_i} \frac{\partial}{\partial x_n} + \frac{\partial y_n}{\partial t_i} \frac{\partial}{\partial y_n} \right) = \theta_i \sum_{n=1}^{\infty} \left(t_i^{\,n} \frac{\partial}{\partial x_n} - t_i^{\,-n} \frac{\partial}{\partial y_n} \right) = \theta_i \delta_{t_i}, \tag{4.2}$$

$$E = \sum_{i=0}^{N} t_i \frac{\partial}{\partial t_i} = \sum_{i=0}^{N} \theta_i \delta_{t_i}. \tag{4.3}$$

In view of the homogeneity, no generality is lost by taking $t_0 = 1$. Set $\sigma_{m,n}(\boldsymbol{\theta}, \boldsymbol{t}) = \tau_{m,n}(\boldsymbol{x}, \boldsymbol{y})$ under the above conditions. For notational simplicity we shall use the abbreviation $\sigma_{m,n}(\theta_i \pm 1)$ to mean that among the constant parameters $\boldsymbol{\theta} = (\theta_0, \theta_1, \dots, \theta_N)$ only the indicated one θ_i is shifted by ± 1 while all the others are unchanged. Then we have the

Proposition 4.1. *The functions $\sigma_{m,n} = \sigma_{m,n}(\boldsymbol{\theta}, \boldsymbol{t})$ satisfy the bilinear equations*

$$\begin{aligned} &(t_i - t_j)\sigma_{m,n}\sigma_{m+1,n+1}(\theta_i + 1, \theta_j + 1) \\ &\qquad = t_i \sigma_{m+1,n}(\theta_i + 1)\sigma_{m,n+1}(\theta_j + 1) - t_j \sigma_{m+1,n}(\theta_j + 1)\sigma_{m,n+1}(\theta_i + 1), \end{aligned} \tag{4.4a}$$

$$(t_i D_i + \theta_i)\, \sigma_{m+1,n} \cdot \sigma_{m,n+1} = \theta_i \sigma_{m,n}(\theta_i - 1)\sigma_{m+1,n+1}(\theta_i + 1), \tag{4.4b}$$

$$\Big((t_j - t_i)D_i + \theta_i\Big) \sigma_{m,n}(\theta_j - 1) \cdot \sigma_{m+1,n} = \theta_i \sigma_{m,n}(\theta_i - 1)\sigma_{m+1,n}(\theta_i + 1, \theta_j - 1), \tag{4.4c}$$

together with the homogeneity constraint

$$\sum_{i=0}^{N} t_i \frac{\partial \sigma_{m,n}}{\partial t_i} = d_{m,n}\sigma_{m,n}. \tag{4.4d}$$

Here D_i denotes the Hirota differential with respect to $\partial/\partial t_i$.

Proof. It is immediate to obtain (4.4a) from (2.15) with $(t, s) = (t_i, t_j)$. Using (4.2) we verify (4.4b) and (4.4c) from (2.16) and (2.17), respectively. □

Next we shall write down nonlinear differential equations for appropriately chosen dependent variables. Let us introduce the functions $f^{(i)}_{m,n} = f^{(i)}_{m,n}(\boldsymbol{\theta}, \boldsymbol{t})$ and $g^{(i)}_{m,n} = g^{(i)}_{m,n}(\boldsymbol{\theta}, \boldsymbol{t})$ defined by

$$f^{(i)}_{m,n} = \frac{\sigma_{m,n-1}(\theta_i + 1)\sigma_{m-1,n-1}}{\sigma_{m-1,n}(\theta_i + 1)\sigma_{m,n-2}}, \tag{4.5}$$

$$g^{(i)}_{m,n} = \frac{t_i D_i \sigma_{m,n-1} \cdot \sigma_{m-1,n}}{\sigma_{m,n-1}\sigma_{m-1,n}} + \theta_i = \theta_i \frac{\sigma_{m-1,n-1}(\theta_i - 1)\sigma_{m,n}(\theta_i + 1)}{\sigma_{m,n-1}\sigma_{m-1,n}} \tag{4.6}$$

for $i = 0, 1, \ldots, N$. Note that the second equality in (4.6) is a consequence of (4.4b). We have the conservation law

$$\prod_{j=1}^{L} f^{(i)}_{m+j,n-j} = 1 \quad \text{and} \quad \sum_{j=1}^{L} g^{(i)}_{m+j,n-j} = L\theta_i. \tag{4.7}$$

In addition we prepare auxiliary variables $U^{(i,j)}_{m,n}$ and $V^{(i,j)}_{m,n}$ ($i \neq j$) defined by

$$U^{(i,j)}_{m,n} = \frac{\theta_i t_j}{t_i - t_j} \frac{\sigma_{m,n-1}(\theta_i - 1, \theta_j + 1)\sigma_{m,n}(\theta_i + 1)}{\sigma_{m,n-1}\sigma_{m,n}(\theta_j + 1)},$$
$$V^{(i,j)}_{m,n} = \frac{\theta_i t_i}{t_i - t_j} \frac{\sigma_{m-1,n}(\theta_i - 1, \theta_j + 1)\sigma_{m,n}(\theta_i + 1)}{\sigma_{m-1,n}\sigma_{m,n}(\theta_j + 1)}.$$

Then we have the following relations among the dependent variables.

Lemma 4.2. *For $i \neq j$, it holds that*

$$V^{(i,j)}_{m,n} - U^{(i,j)}_{m,n} = g^{(i)}_{m,n}, \tag{4.8}$$
$$\frac{U^{(i,j)}_{m-1,n}}{V^{(i,j)}_{m,n-1}} = \frac{t_j f^{(j)}_{m,n}}{t_i f^{(i)}_{m,n}}, \tag{4.9}$$
$$V^{(i,j)}_{m,n-1} - U^{(i,j)}_{m-1,n} = g^{(i)}_{m,n}(\theta_j + 1), \tag{4.10}$$
$$\frac{U^{(i,j)}_{m,n}}{V^{(i,j)}_{m,n}} = \frac{t_j f^{(j)}_{m,n}(\theta_i - 1)}{t_i f^{(i)}_{m,n}(\theta_i - 1)}. \tag{4.11}$$

Proof. Clearly (4.9) and (4.11) are direct consequences of the definition of $f^{(i)}_{m,n}$, (4.5). We obtain (4.8) and (4.10) from the bilinear equation (4.4a). □

Solving the linear equations (4.8) and (4.9) for $U^{(i,j)}_{m,n}$ and $V^{(i,j)}_{m,n}$ with the aid of the (L, L)-periodicity, we conclude that

$$U^{(i,j)}_{m,n} = \frac{1}{\left(\frac{t_i}{t_j}\right)^L - 1} \sum_{b=1}^{L} g^{(i)}_{m-b+1,n+b-1} \prod_{a=1}^{b-1} \frac{t_i f^{(i)}_{m-a+1,n+a}}{t_j f^{(j)}_{m-a+1,n+a}},$$
$$V^{(i,j)}_{m,n} = \frac{1}{\left(\frac{t_i}{t_j}\right)^L - 1} \sum_{b=1}^{L} g^{(i)}_{m-b,n+b} \prod_{a=0}^{b-1} \frac{t_i f^{(i)}_{m-a,n+a+1}}{t_j f^{(j)}_{m-a,n+a+1}}.$$

In fact $U^{(i,j)}_{m,n}$ and $V^{(i,j)}_{m,n}$ can be expressed as polynomials in $f^{(i)}_{m,n}$ and $g^{(i)}_{m,n}$ via (4.7).

Theorem 4.3. *The functions $f^{(i)}_{m,n}$ and $g^{(i)}_{m,n}$ satisfy the system of nonlinear differential equations*

$$t_i\frac{\partial f^{(i)}_{m,n}}{\partial t_i} = \left(\kappa_{m,n} - g^{(i)}_{m,n-1} + \sum_{j\neq i}\left(U^{(j,i)}_{m-1,n} - V^{(j,i)}_{m,n-1}\right)\right)f^{(i)}_{m,n}, \tag{4.12a}$$

$$t_j\frac{\partial f^{(i)}_{m,n}}{\partial t_j} = \left(-g^{(i)}_{m,n-1} - U^{(j,i)}_{m-1,n} + V^{(j,i)}_{m,n-1}\right)f^{(i)}_{m,n} \quad (i \neq j), \tag{4.12b}$$

$$t_i\frac{\partial g^{(i)}_{m,n}}{\partial t_i} = -\sum_{j\neq i}\left(U^{(i,j)}_{m,n}g^{(j)}_{m,n} + V^{(j,i)}_{m,n}g^{(i)}_{m,n}\right), \tag{4.12c}$$

$$t_j\frac{\partial g^{(i)}_{m,n}}{\partial t_j} = U^{(i,j)}_{m,n}g^{(j)}_{m,n} + V^{(j,i)}_{m,n}g^{(i)}_{m,n} \quad (i \neq j), \tag{4.12d}$$

where

$$\kappa_{m,n} = d_{m,n-1} - d_{m-1,n} + \sum_{i=0}^{N}\theta_i = \sum_{i=0}^{N} g^{(i)}_{m,n} \in \mathbb{C} \tag{4.13}$$

are constant parameters.

For each (m,n) fixed the system (4.12) is closed with respect to the $2LN$-tuple of dependent variables $g^{(i)}_{m+j,n-j}$ and $f^{(i)}_{m+j,n-j+1}/f^{(0)}_{m+j,n-j+1}$, where $i = 1, 2, \ldots, N$ and $j \in \mathbb{Z}/L\mathbb{Z}$. Moreover, it possesses the $2N$ conserved quantities; recall (4.7). Accordingly the dimension of the phase space is essentially $2N(L-1)$. If $L = 2$ then it is in fact equivalent to the Garnier system in N variables, whose phase space is $2N$-dimensional; see also the appendix. Let $\mathcal{G}_{L,N}$ denote the nonlinear system (4.12). As shown in Sect. 7, the system $\mathcal{G}_{L,N}$ can be transformed into a canonical Hamiltonian system with polynomial Hamiltonian functions.

Proof of Theorem 4.3. We shall demonstrate only (4.12a) here because the others can be done in quite a similar manner. By virtue of the homogeneity (4.4d) we see that

$$\sum_{i=0}^{N}\left(g^{(i)}_{m,n} - \theta_i\right) = \sum_{i=0}^{N}\frac{t_iD_i\sigma_{m.n-1}\cdot\sigma_{m-1,n}}{\sigma_{m.n-1}\sigma_{m-1,n}} = d_{m,n-1} - d_{m-1,n}.$$

Therefore (4.13) certainly holds. By combining (4.10) with (4.13) we have also

$$\begin{aligned} g^{(i)}_{m,n}(\theta_i+1) &= \kappa_{m,n} + 1 - \sum_{j\neq i} g^{(j)}_{m,n}(\theta_i+1) \\ &= \kappa_{m,n} + 1 + \sum_{j\neq i}\left(U^{(j,i)}_{m-1,n} - V^{(j,i)}_{m,n-1}\right). \end{aligned} \tag{4.14}$$

Taking the logarithmic derivative of $f^{(i)}_{m,n}$ shows that

$$\begin{aligned} \frac{t_i}{f^{(i)}_{m,n}}\frac{\partial f^{(i)}_{m,n}}{\partial t_i} &= \frac{t_iD_i\sigma_{m,n-1}(\theta_i+1)\cdot\sigma_{m-1,n}(\theta_i+1)}{\sigma_{m,n-1}(\theta_i+1)\sigma_{m-1,n}(\theta_i+1)} - \frac{t_iD_i\sigma_{m,n-2}\cdot\sigma_{m-1,n-1}}{\sigma_{m,n-2}\sigma_{m-1,n-1}} \\ &= g^{(i)}_{m,n}(\theta_i+1) - g^{(i)}_{m,n-1} - 1, \quad \text{using (4.6)}, \\ &= \kappa_{m,n} - g^{(i)}_{m,n-1} + \sum_{j\neq i}\left(U^{(j,i)}_{m-1,n} - V^{(j,i)}_{m,n-1}\right), \quad \text{using (4.14)}. \end{aligned}$$

We have verified (4.12a) as desired. □

Remark 4.4 (Toda equation). We shall derive a differential-difference equation of Toda-type for $\sigma_{m,n}$, associated with the shift $(\theta_i, \theta_j) \mapsto (\theta_i + 1, \theta_j - 1)$ of parameters. First we differentiate with respect to s the equation (2.17) after shifting the variables $(\boldsymbol{x}, \boldsymbol{y})$ to $(\boldsymbol{x} + [s]/2, \boldsymbol{y} + [s^{-1}]/2)$. We thus find that

$$((s-t)D_{\delta_t}D_{\delta_s} - 2sD_{\delta_t} + tD_{\delta_s})\,\tau_{m,n}\left(\boldsymbol{x} - \frac{[s]}{2}, \boldsymbol{y} - \frac{[s^{-1}]}{2}\right) \cdot \tau_{m+1,n}\left(\boldsymbol{x} + \frac{[s]}{2}, \boldsymbol{y} + \frac{[s^{-1}]}{2}\right)$$
$$+ tD_{\delta_s}\tau_{m,n}\left(\boldsymbol{x} - [t] + \frac{[s]}{2}, \boldsymbol{y} - [t^{-1}] + \frac{[s^{-1}]}{2}\right) \cdot \tau_{m+1,n}\left(\boldsymbol{x} + [t] - \frac{[s]}{2}, \boldsymbol{y} + [t^{-1}] - \frac{[s^{-1}]}{2}\right) = 0.$$

Substitution of (4.1) and $(t, s) = (t_i, t_j)$ leads to

$$\Big((t_j - t_i)D_iD_j - (2\theta_j + 1)D_i + \theta_i D_j\Big)\sigma_{m,n} \cdot \sigma_{m+1,n}(\theta_j + 1)$$
$$+ \theta_i D_j \sigma_{m,n}(\theta_i - 1, \theta_j + 1) \cdot \sigma_{m+1,n}(\theta_i + 1) = 0.$$

Hence, with the aid of (4.4c), we verify that

$$\begin{aligned}(t_i - t_j)^2 &\frac{D_iD_j\sigma_{m,n} \cdot \sigma_{m+1,n}(\theta_j + 1)}{\sigma_{m,n}\sigma_{m+1,n}(\theta_j + 1)} \\ &= -\theta_i(2\theta_j + 1) + \theta_i\theta_j \frac{\sigma_{m,n}(\theta_i + 1, \theta_j - 1)\sigma_{m,n}(\theta_i - 1, \theta_j + 1)}{{\sigma_{m,n}}^2} \\ &\quad + \theta_i(\theta_j + 1)\frac{\sigma_{m+1,n}(\theta_i + 1)\sigma_{m+1,n}(\theta_i - 1, \theta_j + 2)}{\sigma_{m+1,n}(\theta_j + 1)^2} \\ &\quad + (t_i - t_j)^2 \left(\frac{D_i\sigma_{m,n} \cdot \sigma_{m+1,n}(\theta_j + 1)}{\sigma_{m,n}\sigma_{m+1,n}(\theta_j + 1)}\right)\left(\frac{D_j\sigma_{m,n} \cdot \sigma_{m+1,n}(\theta_j + 1)}{\sigma_{m,n}\sigma_{m+1,n}(\theta_j + 1)}\right).\end{aligned} \tag{4.15}$$

Next we express (4.4c) in the form

$$(t_i - t_j)\frac{D_i\sigma_{m,n} \cdot \sigma_{m+1,n}(\theta_j + 1)}{\sigma_{m,n}\sigma_{m+1,n}(\theta_j + 1)} = \theta_i - \theta_i \frac{\sigma_{m,n}(\theta_i - 1, \theta_j + 1)\sigma_{m+1,n}(\theta_i + 1)}{\sigma_{m,n}\sigma_{m+1,n}(\theta_j + 1)}.$$

By differentiating this with respect to t_j, we have

$$\begin{aligned}(t_i - t_j)^2 &\frac{\partial}{\partial t_j}\left(\frac{D_i\sigma_{m,n} \cdot \sigma_{m+1,n}(\theta_j + 1)}{\sigma_{m,n}\sigma_{m+1,n}(\theta_j + 1)}\right) \\ &= \theta_i + \theta_i\theta_j \frac{\sigma_{m,n}(\theta_i + 1, \theta_j - 1)\sigma_{m,n}(\theta_i - 1, \theta_j + 1)}{{\sigma_{m,n}}^2} \\ &\quad - \theta_i(\theta_j + 1)\frac{\sigma_{m+1,n}(\theta_i + 1)\sigma_{m+1,n}(\theta_i - 1, \theta_j + 2)}{\sigma_{m+1,n}(\theta_j + 1)^2}.\end{aligned} \tag{4.16}$$

Finally, combining (4.15) and (4.16), we arive at the *Toda equation*:

$$(t_i - t_j)^2 \frac{D_iD_j\sigma_{m,n} \cdot \sigma_{m,n}}{{\sigma_{m,n}}^2} = -2\theta_i\theta_j + 2\theta_i\theta_j \frac{\sigma_{m,n}(\theta_i + 1, \theta_j - 1)\sigma_{m,n}(\theta_i - 1, \theta_j + 1)}{{\sigma_{m,n}}^2}. \tag{4.17}$$

Note that (4.17) is still valid without requiring the homogeneity and periodicity. Such a differential-difference equation of Toda-type has previously been studied for the case of P_{VI}, i.e., $(L, N) = (2, 1)$ by Okamoto [Oka87]; and for the case of the Garnier systems, i.e., $L = 2$ and general $N \geq 1$, refer to [Tsu06].

5 Particular solutions expressed in terms of the universal character

As described in Sect. 4, the system $\mathcal{G}_{L,N}$ is a similarity reduction of the UC hierarchy. Since the universal characters $S_{[\lambda,\mu]} = S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ are homogeneous solutions of the UC hierarchy, they survive through the reduction procedure; recall Example 3.1. Therefore we can immediately construct a solution of $\mathcal{G}_{L,N}$ expressed in terms of the universal character.

First we recall some terminology. A subset $\mathbf{m} \subset \mathbb{Z}$ is said to be a *Maya diagram* if $i \in \mathbf{m}$ (for $i \ll 0$) and $i \notin \mathbf{m}$ (for $i \gg 0$). Each Maya diagram $\mathbf{m} = \{\dots, m_3, m_2, m_1\}$ corresponds to a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ via $m_i - m_{i+1} = \lambda_i - \lambda_{i+1} + 1$. We can associate with a sequence of integers $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_L) \in \mathbb{Z}^L$ a Maya diagram

$$\mathbf{m}(\boldsymbol{\nu}) = (L\mathbb{Z}_{<\nu_1} + 1) \cup (L\mathbb{Z}_{<\nu_2} + 2) \cup \dots \cup (L\mathbb{Z}_{<\nu_L} + L);$$

let $\lambda(\boldsymbol{\nu})$ denote its corresponding partition. Note that $\lambda(\boldsymbol{\nu} + \mathbf{1}) = \lambda(\boldsymbol{\nu})$ where $\mathbf{1} = (\overbrace{1, 1, \dots, 1}^{L})$. We call a partition of the form $\lambda(\boldsymbol{\nu})$ an *L-core* partition. A partition λ is L-core if and only if λ has no hook with length of a multiple of L; see [Nou04, Proposition 7.13]. For example, if $L = 2$ and $\boldsymbol{\nu} = (0, n)$ $(n > 0)$ then the result is a staircase partition $\lambda(\boldsymbol{\nu}) = (n, n-1, \dots, 2, 1)$, thereby two-core.

There is a cyclic chain of the universal characters attached to L-core partitions that is connected by the action of vertex operators; see [Tsu05a, Lemma 2.2].

Lemma 5.1. *It holds that*

$$X^+_{L\nu_m - |\boldsymbol{\nu}|} S_{[\lambda(\boldsymbol{\nu}(m-1)),\mu]} = \pm S_{[\lambda(\boldsymbol{\nu}(m)),\mu]}$$

for arbitrary $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_L) \in \mathbb{Z}^L$ *and partition* μ. *Here* $\boldsymbol{\nu}(m) = \boldsymbol{\nu} + (\overbrace{1, \dots, 1}^{m}, \overbrace{0, \dots, 0}^{L-m})$ *and* $|\boldsymbol{\nu}| = \nu_1 + \nu_2 + \dots + \nu_L$. *A similar formula holds for the operators* Y^+_n *also.*

Hence we are led to the following expression of rational solutions of $\mathcal{G}_{L,N}$ in terms of the universal character attached to a pair of L-core partitions.

Theorem 5.2. *Let* $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \mathbb{Z}^L$ *be arbitrary sequences of integers. Define*

$$\sigma_{m,n}(\boldsymbol{\theta}, \boldsymbol{t}) = S_{[\lambda(\boldsymbol{\nu}(m)),\lambda(\boldsymbol{\nu}'(n))]}(\boldsymbol{x}, \boldsymbol{y})$$

under the substitution

$$x_n = \frac{1}{n} \sum_{i=0}^{N} \theta_i t_i^{\,n} \quad \textit{and} \quad y_n = \frac{1}{n} \sum_{i=0}^{N} \theta_i t_i^{\,-n}.$$

Then the functions $\sigma_{m,n}$ *satisfy the bilinear equations* (4.4a)–(4.4c) *and the homogeneity* (4.4d), *where* $d_{m,n} - d_{m-1,n} = L\nu_m - |\boldsymbol{\nu}|$ *and* $d_{m,n} - d_{m,n-1} = -L\nu'_n + |\boldsymbol{\nu}'|$. *Consequently the functions* $f^{(i)}_{m,n}$ *and* $g^{(i)}_{m,n}$ *defined by* (4.5) *and* (4.6) *give a rational solution of the system* $\mathcal{G}_{L,N}$, (4.12), *with the parameters* $\kappa_{m,n} = L(\nu_m + \nu'_n) - |\boldsymbol{\nu}| - |\boldsymbol{\nu}'| + \sum_{i=0}^{N} \theta_i$.

6 Lax formalism

In this section we derive from the UC hierarchy the auxiliary linear problem whose compatibility condition amounts to the nonlinear system $\mathcal{G}_{L,N}$ (Lax formalism). It is seen that $\mathcal{G}_{L,N}$ describes the monodromy preserving deformations of a Fuchsian system of linear differential equations with a certain spectral type.

We introduce the *wave function*

$$\psi_{m,n}(\boldsymbol{x},\boldsymbol{y},k) = \frac{\tau_{m,n-1}(\boldsymbol{x}-[k^{-1}],\boldsymbol{y}-[k])}{\tau_{m,n}(\boldsymbol{x},\boldsymbol{y})}e^{\xi(\boldsymbol{x},k)},$$

which is a function in $(\boldsymbol{x},\boldsymbol{y}) = (x_1,x_2,\dots,y_1,y_2,\dots)$ equipped with an additional parameter k (the *spectral variable*). Define $\phi_{m,n}(\boldsymbol{\theta},\boldsymbol{t},k) = \psi_{m,n}(\boldsymbol{x},\boldsymbol{y},k)$ under the change of variables (4.1). We then have the

Proposition 6.1. *The wave functions $\phi_{m,n} = \phi_{m,n}(\boldsymbol{\theta},\boldsymbol{t},k)$ satisfy the linear equations*

$$\phi_{m,n} = \frac{1}{f^{(i)}_{m+1,n+1}}\phi_{m,n+1}(\theta_i+1) - t_i k\phi_{m+1,n}(\theta_i+1), \tag{6.1}$$

$$t_i\frac{\partial}{\partial t_i}\phi_{m,n} = \left(g^{(i)}_{m+1,n}-\theta_i\right)\phi_{m,n} + t_i k g^{(i)}_{m+1,n}\phi_{m+1,n}(\theta_i+1), \tag{6.2}$$

$$\left(k\frac{\partial}{\partial k} - \sum_{i=0}^{N} t_i\frac{\partial}{\partial t_i}\right)\phi_{m,n} = (d_{m,n}-d_{m,n-1})\phi_{m,n}. \tag{6.3}$$

Proof. To begin with, we recall the definition of variables $f^{(i)}_{m,n}$ and $g^{(i)}_{m,n}$; see (4.5) and (4.6). Substitution of $(t,s) = (t_i, 1/k)$ in (2.15) and (2.17) produces respectively (6.1) and (6.2), with the aid of (4.2). We deduce from the homogeneity condition $E\tau_{m,n} = d_{m,n}\tau_{m,n}$ that

$$\left(E - k\frac{\partial}{\partial k}\right)\tau_{m,n}(\boldsymbol{x}-[k^{-1}],\boldsymbol{y}-[k]) = d_{m,n}\tau_{m,n}(\boldsymbol{x}-[k^{-1}],\boldsymbol{y}-[k]).$$

On the other hand, we have $(E - k\partial/\partial k)e^{\xi(\boldsymbol{x},k)} = 0$. Hence we are led to the formula

$$\left(E - k\frac{\partial}{\partial k}\right)\psi_{m,n} = (d_{m,n-1}-d_{m,n})\psi_{m,n},$$

which implies (6.3) via (4.3). The proof is now complete. □

Because of the (L,L)-periodicity, the linear equations (6.1) can be solved for $\phi_{m,n}(\theta_i+1)$; thus,

$$\phi_{m,n}(\theta_i+1) = \frac{1}{1-(t_ik)^L}\sum_{b=1}^{L}(t_ik)^{b-1}\left(\prod_{a=1}^{b} f^{(i)}_{m+a,n-a+1}\right)\phi_{m+b-1,n-b}.$$

If we eliminate $\phi_{m+1,n}(\theta_i+1)$ from (6.2) by using the above formula, then we have

$$t_i\frac{\partial}{\partial t_i}\phi_{m,n} = \left(g^{(i)}_{m+1,n}-\theta_i\right)\phi_{m,n} + \frac{g^{(i)}_{m+1,n}}{1-(t_ik)^L}\sum_{b=1}^{L}(t_ik)^b\left(\prod_{a=1}^{b} f^{(i)}_{m+a+1,n-a+1}\right)\phi_{m+b,n-b}. \tag{6.4}$$

Notice that for each m and n fixed (6.4) is closed with respect to $\phi_{m+j,n-j}$ ($j \in \mathbb{Z}/L\mathbb{Z}$). With this fact in mind, we shall write down the linear differential equations satisfied by the vector

$$\Phi = {}^{\mathrm{T}}\left(\phi_{-1,0}, k\phi_{0,-1}, k^2\phi_{1,-2},\dots,k^{L-1}\phi_{L-2,-L+1}\right).$$

Consider the change of variables

$$z = k^L \quad \text{and} \quad u_i = t_i^{-L}. \tag{6.5}$$

We can express (6.4) in the $L \times L$ matrix equation

$$\frac{\partial}{\partial u_i}\Phi = B_i \Phi \tag{6.6}$$

with

$$B_i = \mathrm{diag}\left(\frac{\theta_i}{Lu_i} - v_{n,n}^{(i)}\right)_{0\le n\le L-1} + \frac{1}{z-u_i}\begin{pmatrix} 0 & v_{0,1}^{(i)} & \cdots & v_{0,L-1}^{(i)} \\ & 0 & \ddots & \vdots \\ & & \ddots & v_{L-2,L-1}^{(i)} \\ & & & 0 \end{pmatrix} + \frac{z}{z-u_i}\begin{pmatrix} v_{0,0}^{(i)} & & & O \\ v_{1,0}^{(i)} & v_{1,1}^{(i)} & & \\ \vdots & & \ddots & \\ v_{L-1,0}^{(i)} & v_{L-1,1}^{(i)} & \cdots & v_{L-1,L-1}^{(i)} \end{pmatrix},$$

where

$$v_{n,n+b}^{(i)} = \frac{g_{n,-n}^{(i)}}{L}\prod_{a=1}^{b} t_i f_{n+a,-n-a+1}^{(i)} \tag{6.7}$$

for $0 \le n \le L-1$ and $1 \le b \le L$. Remark that the suffix of each variable should be suitably regarded as an element of $\mathbb{Z}/L\mathbb{Z}$.

Similarly, we obtain from (6.3) the linear differential equation with respect to z:

$$\frac{\partial \Phi}{\partial z} = A\Phi = \sum_{i=0}^{N+1}\frac{A_i}{z-u_i}\Phi, \tag{6.8}$$

where $u_{N+1} = 0$ and the $L \times L$ matrices A_i read

$$A_i = -\begin{pmatrix} 0 & v_{0,1}^{(i)} & \cdots & v_{0,L-1}^{(i)} \\ & 0 & \ddots & \vdots \\ & & \ddots & v_{L-2,L-1}^{(i)} \\ & & & 0 \end{pmatrix} - u_i \begin{pmatrix} v_{0,0}^{(i)} & & & O \\ v_{1,0}^{(i)} & v_{1,1}^{(i)} & & \\ \vdots & & \ddots & \\ v_{L-1,0}^{(i)} & v_{L-1,1}^{(i)} & \cdots & v_{L-1,L-1}^{(i)} \end{pmatrix} \quad (0 \le i \le N),$$

$$A_{N+1} = \begin{pmatrix} e_0 & w_{0,1} & \cdots & w_{0,L-1} \\ & e_1 & \ddots & \vdots \\ & & \ddots & w_{L-2,L-1} \\ & & & e_{L-1} \end{pmatrix}$$

with

$$e_n = \frac{d_{n,-n-1} - d_{n-1,-n-1} + n}{L} \quad \text{and} \quad w_{m,n} = \sum_{i=0}^{N} v_{m,n}^{(i)}.$$

The linear differential equation (6.8) is Fuchsian and has the $N+3$ regular singularities $u_0, u_1, \ldots, u_N$, $u_{N+1} = 0$, $u_{N+2} = \infty$. Observe that every A_i $(0 \le i \le N)$ is not full rank unlike A_{N+1} and

$A_{N+2} = -\sum_{i=0}^{N+1} A_i$. To be specific, if we prepare the column vector $\boldsymbol{b}^{(i)}$ and the row vector $\boldsymbol{c}^{(i)}$ defined by

$$
{}^{\mathrm{T}}\boldsymbol{b}^{(i)} = \left(\frac{-g^{(i)}_{n,-n}}{L{t_i}^n \prod_{m=1}^{n} f^{(i)}_{m,-m+1}} \right)_{0\le n\le L-1} = \frac{-1}{L}\left(g^{(i)}_{0,0}, \frac{g^{(i)}_{1,-1}}{t_i f^{(i)}_{1,0}}, \frac{g^{(i)}_{2,-2}}{{t_i}^2 f^{(i)}_{1,0} f^{(i)}_{2,-1}}, \dots, \frac{g^{(i)}_{L-1,-L+1}}{{t_i}^{L-1} f^{(i)}_{1,0} f^{(i)}_{2,-1} \cdots f^{(i)}_{L-1,2}} \right),
$$

$$
\boldsymbol{c}^{(i)} = \left({t_i}^n \prod_{m=1}^{n} f^{(i)}_{m,-m+1} \right)_{0\le n\le L-1} = \left(1, t_i f^{(i)}_{1,0}, {t_i}^2 f^{(i)}_{1,0} f^{(i)}_{2,-1}, \dots, {t_i}^{L-1} f^{(i)}_{1,0} f^{(i)}_{2,-1} \cdots f^{(i)}_{L-1,2}\right),
$$

then we have indeed

$$
A_i = \boldsymbol{b}^{(i)} \cdot \boldsymbol{c}^{(i)} \quad \text{and} \quad \boldsymbol{c}^{(i)} \cdot \boldsymbol{b}^{(i)} = -\sum_{n=0}^{L-1} \frac{g^{(i)}_{n,-n}}{L} = -\theta_i \in \mathbb{C} \tag{6.9}
$$

for $0 \le i \le N$. The matrix $A_{N+2} = -\sum_{i=0}^{N+1} A_i$ is lower triangular and its diagonal entries are

$$
\sum_{i=0}^{N} u_i v^{(i)}_{n,n} - e_n = \sum_{i=0}^{N} \frac{g^{(i)}_{n,-n}}{L} - e_n = \kappa_n - e_n
$$

for $0 \le n \le L-1$. Here we have used (6.5) and (6.7) and put

$$
\kappa_n = \frac{\kappa_{n,-n}}{L} = \frac{d_{n,-n-1} - d_{n-1,-n} + \sum_{i=0}^{N} \theta_i}{L} = \sum_{i=0}^{N} \frac{g^{(i)}_{n,-n}}{L}; \tag{6.10}
$$

cf. (4.13). Hence the characteristic exponents of (6.8) at each singularity $z = u_i$, i.e., the eigenvalues of each residue matrix A_i, are listed as follows:

Singularity	Exponents
u_i $(0 \le i \le N)$	$(-\theta_i, 0, \dots, 0)$
$u_{N+1} = 0$	$(e_0, e_1, \dots, e_{L-1})$
$u_{N+2} = \infty$	$(\kappa_0 - e_0, \kappa_1 - e_1, \dots, \kappa_{L-1} - e_{L-1})$

(6.11)

Note that the relations

$$
\sum_{n=0}^{L-1} e_n = \frac{L-1}{2} \quad \text{and} \quad \sum_{n=0}^{L-1} \kappa_n = \sum_{i=0}^{N} \theta_i \tag{6.12}
$$

hold among the exponents. The sum of all the exponents certainly equals zero (Fuchs relation).

Compatibility between the above two linear equations, (6.6) and (6.8), is *a priori* established because both originate from the same bilinear equation (1.3). The former, (6.6), governs the monodromy preserving deformation of the latter, (6.8), along a deformation parameter u_i. The nonlinear system $\mathcal{G}_{L,N}$, (4.12), can be recovered from the integrability condition $\left[\frac{\partial}{\partial u_i} - B_i, \frac{\partial}{\partial z} - A\right] = 0$ of the linear system (6.6) and (6.8).

Remark 6.2. In general, we can associate with an $L \times L$ Fuchsian system

$$
\frac{\partial \Phi}{\partial z} = A\Phi = \sum_{i=0}^{N+1} \frac{A_i}{z - u_i} \Phi \tag{6.13}
$$

having $N+3$ regular singularities $u_0, u_1, \dots, u_N, u_{N+1} = 0, u_{N+2} = \infty$ an $(N+3)$-tuple

$$\mathcal{M} = \{(\mu_{0,1}, \mu_{0,2}, \dots, \mu_{0,\ell_0}), (\mu_{1,1}, \mu_{1,2}, \dots, \mu_{1,\ell_1}), \dots, (\mu_{N+2,1}, \mu_{N+2,2}, \dots, \mu_{N+2,\ell_{N+2}})\}$$

of partitions of L in such a way that each residue matrix A_i has the eigenvalues of multiplicity $\mu_{i,j}$; we call $\mathcal{M}$ the *spectral type*. The number of accessory parameters of (6.13) is known to be an even given by

$$(N+1)L^2 - \sum_{i=0}^{N+2} \sum_{j=1}^{\ell_i} {\mu_{i,j}}^2 + 2.$$

We turn now to our case. The spectral type of (6.8) reads the $(N+3)$-tuple

$$\underbrace{(L-1,1), \dots, (L-1,1)}_{N+1}, (1,1,\dots,1), (1,1,\dots,1) \tag{6.14}$$

of partitions of L, according to the table (6.11). Applying the above formula we find the number of accessory parameters to be $2N(L-1)$, which certainly equals the essential dimension of the phase space of $\mathcal{G}_{L,N}$ as was calculated in Sect. 4.

Remark 6.3. Thanks to the algorithm proposed by Oshima [Osh08], Fuchsian systems of the form (6.13) with a fixed number p of accessory parameters can be classified by the spectral types. Let us here take our interest in the Fuchsian systems that have four or more singularities because they admit the monodromy preserving deformations. If $p = 2$ then we have a single fundamental system whose spectral type is $\{(1,1)^4\} = \{(1,1),(1,1),(1,1),(1,1)\}$; and its deformation equation turns out to be $P_{\rm VI}$ $(= \mathcal{G}_{2,1})$. If $p = 4$ then the result is the four Fuchsian systems specified by the spectral types $\{(1,1)^5\}$, $\{(2,1)^2,(1,1,1)^2\}$, $\{(3,1),(2,2)^2,(1,1,1,1)\}$, and $\{(2,2)^3,(2,1,1)\}$. The first one has two deformation parameters and it corresponds to the Garnier system in two variables. The other three cases produce nonlinear ordinary differential equations of fourth order, which have been investigated by Sakai [Sak08] as candidates of the master equations, like $P_{\rm VI}$, among the family of fourth-order Painlevé equations; he clarified the polynomial Hamiltonian structure and coalescence diagram for each. Note that the first and second of the three are equivalent respectively to $\mathcal{G}_{3,1}$ (see Example 7.3) and to the fourth-order Painlevé equation of type $D_6^{(1)}$ introduced by Sasano [Sas06] (see also [FS08]).

7 Polynomial Hamiltonian structure

In this section we present Hamiltonian formalism for the system $\mathcal{G}_{L,N}$ such that Hamiltonian functions are polynomials in the canonical variables.

The *Schlesinger system* is the following system of nonlinear differential equations (see [JMU81, Sch12]):

$$\frac{\partial A_i}{\partial u_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{u_i - u_j}, \quad \frac{\partial A_i}{\partial u_j} = \frac{[A_i, A_j]}{u_i - u_j} \quad (i \neq j) \tag{7.1}$$

for $L \times L$ matrix-valued unknown functions A_i, which describes the monodromy preserving deformations of a Fuchsian system of the form (6.13). Needless to say, $\mathcal{G}_{L,N}$ is equivalent to a particular case of the Schlesinger systems specified by the spectral type (6.14).

Recall first that (7.1) can be written as a Hamiltonian system (see, e.g., [Man99])

$$\frac{\partial A_i}{\partial u_j} = \{A_i, K_j\}$$

with the Hamiltonian functions

$$K_i = \frac{1}{2} \operatorname*{Res}_{z=u_i} \operatorname{tr} A^2 = \sum_{j \neq i} \frac{\operatorname{tr}(A_i A_j)}{u_i - u_j}, \tag{7.2}$$

where the Poisson bracket { , } is given in a standard way by

$$\{(A_i)_{m,n}, (A_j)_{m',n'}\} = \delta_{i,j} \left(\delta_{m,n'} (A_i)_{m',n} - \delta_{m',n} (A_j)_{m,n'} \right). \tag{7.3}$$

Moreover, a method to construct canonical variables for the above Hamiltonian system has been established; see [JMMS80, Appendix 5]. Set $A_i = B^{(i)} C^{(i)}$ and define a Poisson bracket { , } over the space of matrices $B^{(i)}$ and $C^{(i)}$ by

$$\left\{ \left(B^{(i)}\right)_{m,n}, \left(C^{(i)}\right)_{n,m} \right\} = 1 \quad \text{and} \quad \{\text{otherwise}\} = 0.$$

This Poisson bracket coincides with the previous one (7.3), in fact. Hence the Schlesinger system is equivalent to the canonical Hamiltonian system attached with the fundamental 2-form

$$\Gamma = \sum_{i=0}^{N+1} \operatorname{tr} \left(\mathrm{d}C^{(i)} \wedge \mathrm{d}B^{(i)} \right) - \sum_{i=0}^{N+1} \mathrm{d}K_i \wedge \mathrm{d}u_i.$$

However, the above choice of canonical variables is redundant because it is possible to reduce the number of canonical variables to that of accessory parameters of the Fuchsian system (6.13).

Next we shall consider the Hamiltonian formalism of $\mathcal{G}_{L,N}$ and carry out the reduction of canonical variables. In this case the fundamental 2-form reads (see Sect. 6)

$$\Gamma = \sum_{i=0}^{N} \sum_{n=0}^{L-1} \mathrm{d}c_n^{(i)} \wedge \mathrm{d}b_n^{(i)} - \sum_{i=1}^{N} \mathrm{d}K_i \wedge \mathrm{d}u_i \tag{7.4}$$

with

$$b_n^{(i)} = \frac{-g_{n,-n}^{(i)}}{L {t_i}^n \prod_{m=1}^{n} f_{m,-m+1}^{(i)}} \quad \text{and} \quad c_n^{(i)} = {t_i}^n \prod_{m=1}^{n} f_{m,-m+1}^{(i)}$$

for $0 \leq i \leq N$ and $0 \leq n \leq L-1$. Here we have fixed $t_0 = 1$ and thereby $u_0 = 1$. Observe that

$$b_n^{(0)} c_n^{(0)} = -\kappa_n - \sum_{i=1}^{N} b_n^{(i)} c_n^{(i)}, \tag{7.5}$$

which follows from (6.10) by means of $b_n^{(i)} c_n^{(i)} = -g_{n,-n}^{(i)}/L$. Accordingly the first term of (7.4) can be computed as follows:

$$
\begin{aligned}
\sum_{i=0}^{N}\sum_{n=0}^{L-1} \mathrm{d}c_n^{(i)} \wedge \mathrm{d}b_n^{(i)} &= \sum_{i=0}^{N}\sum_{n=1}^{L-1} \mathrm{d}c_n^{(i)} \wedge \mathrm{d}b_n^{(i)}, \quad \text{since } c_0^{(i)} = 1, \\
&= \sum_{i=0}^{N}\sum_{n=1}^{L-1} \mathrm{d}\log c_n^{(i)} \wedge \mathrm{d}\left(b_n^{(i)} c_n^{(i)}\right) \\
&= \sum_{i=1}^{N}\sum_{n=1}^{L-1} \mathrm{d}\log \frac{c_n^{(i)}}{c_n^{(0)}} \wedge \mathrm{d}\left(b_n^{(i)} c_n^{(i)}\right), \quad \text{using (7.5)}, \\
&= \sum_{i=1}^{N}\sum_{n=1}^{L-1} \mathrm{d}\left(\frac{c_n^{(i)}}{c_n^{(0)}}\right) \wedge \mathrm{d}\left(b_n^{(i)} c_n^{(0)}\right) \\
&= \sum_{i=1}^{N}\sum_{n=1}^{L-1} \mathrm{d}\left(-b_n^{(i)} c_n^{(0)}\right) \wedge \mathrm{d}\left(\frac{c_n^{(i)}}{c_n^{(0)}}\right).
\end{aligned}
$$

Let us now introduce the canonical variables $q_n^{(i)}$ and $p_n^{(i)}$ $(1 \le i \le N; 1 \le n \le L-1)$ defined by

$$
q_n^{(i)} = \frac{c_n^{(i)}}{c_n^{(0)}}, \quad p_n^{(i)} = -b_n^{(i)} c_n^{(0)}, \tag{7.6}
$$

whose number, $2N(L-1)$, is just enough for the Hamiltonian system under consideration; see Remark 6.2. In addition we take the change of independent variables

$$
s_i = \frac{1}{u_i} = {t_i}^L
$$

so that the resulting Hamiltonian function

$$
H_i = -\frac{K_i}{{s_i}^2} = -\frac{\operatorname{tr}(A_i A_{N+1})}{s_i} + \sum_{\substack{j=0 \\ j \neq i}}^{N} \frac{s_j \operatorname{tr}(A_i A_j)}{s_i (s_i - s_j)}
$$

becomes identical with the standard one of P_{VI} when $(L, N) = (2, 1)$; see Example 7.3. The fundamental 2-form is then rewritten as

$$
\Gamma = \sum_{i=1}^{N} \left(\sum_{n=1}^{L-1} \mathrm{d}p_n^{(i)} \wedge \mathrm{d}q_n^{(i)} - \mathrm{d}H_i \wedge \mathrm{d}s_i \right).
$$

For convenience we extendedly use the symbols $q_n^{(i)}$ and $p_n^{(i)}$ also for $i = 0$ or $n = 0$; namely, we put

$$
\begin{aligned}
q_n^{(0)} &= 1, \quad p_n^{(0)} \left(= -b_n^{(0)} c_n^{(0)}\right) = \kappa_n - \sum_{i=1}^{N} q_n^{(i)} p_n^{(i)}, \\
q_0^{(i)} &= 1, \quad p_0^{(i)} \left(= -b_0^{(i)} c_0^{(i)}\right) = \theta_i - \sum_{n=1}^{L-1} q_n^{(i)} p_n^{(i)},
\end{aligned} \tag{7.7}
$$

by taking (6.9) and (7.5) into account. We have then the

Lemma 7.1. *It holds that*

$$\mathrm{tr}(A_iA_j) = \sum_{m,n=0}^{L-1} q_m^{(i)} p_m^{(j)} q_n^{(j)} p_n^{(i)}, \tag{7.8}$$

$$\mathrm{tr}(A_iA_{N+1}) = -\sum_{n=0}^{L-1} e_n q_n^{(i)} p_n^{(i)} - \sum_{j=0}^{N} \sum_{0\le m<n\le L-1} q_m^{(i)} p_m^{(j)} q_n^{(j)} p_n^{(i)} \tag{7.9}$$

for $i, j = 0, 1, \dots, N$.

Proof. It follows from $(A_i)_{m,n} = b_m^{(i)} c_n^{(i)}$ that $\mathrm{tr}(A_iA_j) = \sum_{m,n=0}^{L-1} (A_i)_{n,m}(A_j)_{m,n} = \sum_{m,n=0}^{L-1} b_n^{(i)} c_m^{(i)} b_m^{(j)} c_n^{(j)}$, which thus yields (7.8) via $b_n^{(i)} c_n^{(j)} = -p_n^{(i)} q_n^{(j)}$. The diagonal entries of A_iA_{N+1} read

$$\begin{aligned}
(0,0):&\quad \left(e_0 b_0^{(i)} + w_{0,1} b_1^{(i)} + w_{0,2} b_2^{(i)} + \cdots + w_{0,L-1} b_{L-1}^{(i)}\right) c_0^{(i)},\\
(1,1):&\quad \left(e_1 b_1^{(i)} + w_{1,2} b_2^{(i)} + \cdots + w_{1,L-1} b_{L-1}^{(i)}\right) c_1^{(i)},\\
&\quad\vdots\\
(L-2,L-2):&\quad \left(e_{L-2} b_{L-2}^{(i)} + w_{L-2,L-1} b_{L-1}^{(i)}\right) c_{L-2}^{(i)},\\
(L-1,L-1):&\quad e_{L-1} b_{L-1}^{(i)} c_{L-1}^{(i)}.
\end{aligned}$$

Therefore $\mathrm{tr}(A_iA_{N+1}) = \sum_{n=0}^{L-1} e_n b_n^{(i)} c_n^{(i)} + \sum_{0\le m<n\le L-1} w_{m,n} b_n^{(i)} c_m^{(i)}$. If we remember $w_{m,n} = \sum_{j=0}^{N} v_{m,n}^{(j)} = -\sum_{j=0}^{N} b_m^{(j)} c_n^{(j)}$, then we find $\mathrm{tr}(A_iA_{N+1}) = \sum_{n=0}^{L-1} e_n b_n^{(i)} c_n^{(i)} - \sum_{j=0}^{N} \sum_{0\le m<n\le L-1} b_m^{(j)} c_n^{(j)} b_n^{(i)} c_m^{(i)}$; thus (7.9) is verified. □

By virtue of Lemma 7.1 together with (7.7), the Hamiltonian function H_i can be explicitly expressed as a polynomial in the $2N(L-1)$ canonical variables $q_n^{(i)}$ and $p_n^{(i)}$ ($1 \le i \le N; 1 \le n \le L-1$). Finally we arrive at the

Theorem 7.2. *The system* $\mathcal{G}_{L,N}$ *is equivalent to the canonical Hamiltonian system*

$$\frac{\partial q_n^{(i)}}{\partial s_j} = \frac{\partial H_j}{\partial p_n^{(i)}}, \quad \frac{\partial p_n^{(i)}}{\partial s_j} = -\frac{\partial H_j}{\partial q_n^{(i)}} \tag{7.10}$$

for $i, j = 1, 2, \dots, N$ *and* $n = 1, 2, \dots, L-1$, *where the Hamiltonian function* H_i *is defined by*

$$s_iH_i = \sum_{n=0}^{L-1} e_n q_n^{(i)} p_n^{(i)} + \sum_{j=0}^{N} \sum_{0\le m<n\le L-1} q_m^{(i)} p_m^{(j)} q_n^{(j)} p_n^{(i)} + \sum_{\substack{j=0\\ j\ne i}}^{N} \frac{s_j}{s_i - s_j} \sum_{m,n=0}^{L-1} q_m^{(i)} p_m^{(j)} q_n^{(j)} p_n^{(i)} \tag{7.11}$$

with (7.7).

We write the constant parameters contained in (7.10) as

$$\vec{\kappa} = (e_0, e_1, \dots, e_{L-1}, \kappa_0, \kappa_1, \dots, \kappa_{L-1}, \theta_0, \theta_1, \dots, \theta_N), \tag{7.12}$$

whose number is essentially $2L + N - 1$ according to (6.12). Let $\mathcal{H}_{L,N} = \mathcal{H}_{L,N}(\vec{\kappa})$ denote the Hamiltonian system (7.10). Since all the differential equations originate from a single equation (1.3), the system $\mathcal{H}_{L,N}$ is *a priori* completely integrable (in the Frobenius sense). Or it can be shown directly by noticing the following facts: (i) the 1-form $\omega = \sum_{i=1}^{N} H_i \mathrm{d}s_i$ is closed for an arbitrary solution of (7.10); (ii) the relation

$$\left(\frac{\partial}{\partial s_j}\right) H_i = \left(\frac{\partial}{\partial s_i}\right) H_j = \frac{1}{(s_i - s_j)^2} \sum_{m,n=0}^{L-1} q_m^{(i)} p_m^{(j)} q_n^{(j)} p_n^{(i)} \quad (i \neq j)$$

holds, where the symbol $(\partial/\partial s_i)$ denotes the differentiation such that $q_n^{(i)}$ and $p_n^{(i)}$ are viewed to be independent of s_i. These facts imply the commutativity of the flows induced by $H_1, H_2, \ldots, H_N$.

The correspondence between the canonical variables $q_n^{(i)}$ and $p_n^{(i)}$ and the dependent variables given in Sect. 4 is summarized as

$$q_n^{(i)} = \frac{c_n^{(i)}}{c_n^{(0)}} = \left(\frac{t_i}{t_0}\right)^n \prod_{m=1}^{n} \frac{f_{m,-m+1}^{(i)}}{f_{m,-m+1}^{(0)}} = \left(\frac{t_i}{t_0}\right)^n \frac{\sigma_{n,-n}(\theta_i + 1)\sigma_{0,0}(\theta_0 + 1)}{\sigma_{0,0}(\theta_i + 1)\sigma_{n,-n}(\theta_0 + 1)}, \tag{7.13a}$$

$$q_n^{(i)} p_n^{(i)} = -b_n^{(i)} c_n^{(i)} = \frac{g_{n,-n}^{(i)}}{L} = \frac{\theta_i}{L} \frac{\sigma_{n-1,-n-1}(\theta_i - 1)\sigma_{n,-n}(\theta_i + 1)}{\sigma_{n,-n-1}\sigma_{n-1,-n}}. \tag{7.13b}$$

Example 7.3 (Case $N = 1$). Let us restrict ourselves to the case $N = 1$; thus $\mathcal{H}_{L,1}$ becomes a system of ordinary differential equations. We begin with the case $L = 2$, which is the first nontrivial one. Write $(q, p, H, s) = (q_1^{(1)}, p_1^{(1)}, H_1, s_1)$ and $\theta = \theta_1$. Then the Hamiltonian function can be expressed as

$$H = H_{\mathrm{VI}}(a_0, a_1, a_2, a_3, a_4; q, p) + \frac{\theta(e_0(s-1) + \kappa_0 - \theta)}{s(s-1)}$$

under the substitution

$$a_0 = e_0 - e_1 + \kappa_1 + 1, \quad a_1 = -\kappa_1 + \theta, \quad a_2 = -\theta, \quad a_3 = -e_0 + e_1 + \kappa_0, \quad a_4 = -\kappa_0 + \theta.$$

Here $H_{\mathrm{VI}} = H_{\mathrm{VI}}(a_0, a_1, a_2, a_3, a_4; q, p)$ denotes the Hamiltonian function of P_{VI} and is defined by

$$\begin{aligned} s(s-1)H_{\mathrm{VI}} = {} & q(q-1)(q-s)p^2 \\ & - ((a_0 - 1)q(q-1) + a_3 q(q-s) + a_4(q-1)(q-s))\, p \\ & + a_2(a_1 + a_2)q \end{aligned}$$

with a_i being constant parameters such that $a_0 + a_1 + 2a_2 + a_3 + a_4 = 1$; see [Malm22, Oka87].

Now we turn to the case of general $L \geq 2$. Let $(q_n, p_n, H, s) = (q_n^{(1)}, p_n^{(1)}, H_1, s_1)$ and $\theta = \theta_1$. Then the Hamiltonian function of $\mathcal{H}_{L,1}$ takes a *coupled* form of P_{VI} ones as follows:

$$\begin{aligned} H = {} & \sum_{n=1}^{L-1} H_{\mathrm{VI}}(a_{0,n}, a_{1,n}, a_{2,n}, a_{3,n}, a_{4,n}; q_n, p_n) + \frac{\theta(e_0(s-1) + \kappa_0 - \theta)}{s(s-1)} \\ & + \sum_{1 \leq m < n \leq L-1} \frac{(q_m - 1)p_m q_n((q_n - s)p_n - \kappa_n) + (q_n - s)p_n q_m((q_m - 1)p_m - \kappa_m)}{s(s-1)}, \end{aligned} \tag{7.14}$$

where the last term reflects an interaction and the correspondence of constant parameters reads

$$a_{0,n} = e_0 - e_n + \kappa_n + 1, \quad a_{1,n} = -\kappa_n + \theta, \quad a_{2,n} = -\theta, \quad a_{3,n} = -e_0 + e_n + \kappa_0, \quad a_{4,n} = -\kappa_0 + \theta.$$

Interestingly enough, as has been pointed out by Fuji and Suzuki (see [FS09]), the coupled Hamiltonian (7.14) can be derived alternatively from the deformation of a certain linear system that is not Fuchsian but has one regular and one irregular singularities; cf. Sect. 6. It is expected to exist some integral transform (like a Laplace one) between the two kinds of Lax formalism.

Remark 7.4. We cite the recent result by Dubrovin and Mazzocco [DM07]; they have studied Hamiltonian formalism of the Schlesinger system associated with the general spectral type (cf. (6.14)). Their construction is based on a scalar differential equation of higher order that is reduced from a Fuchsian system of the form (6.13); and the apparent singularities (see [KO83]) produced by the reduction procedure are adopted as the half of the canonical variables, i.e., the generalized coordinates. The resulting Hamiltonian system has movable algebraic branch points and thereby, unlike $\mathcal{H}_{L,N}$, does not enjoy the Painlevé property (see [Malg83, Miw81]). It would be an interesting problem to transform the *general* Schlesinger system into a Hamiltonian form enjoying the Painlevé property whose Hamiltonian functions are polynomials in the dependent variables.

8 Birational canonical transformations

This section is devoted to birational symmetries of the Hamiltonian system $\mathcal{H}_{L,N} = \mathcal{H}_{L,N}(\vec{\kappa})$. Here, to be precise, a birational canonical transformation of variables $(q_n^{(i)}, p_n^{(i)}, s_i)$ is said to be a *symmetry* if it keeps the system invariant except changing the constant parameters $\vec{\kappa}$.

First we translate the action of $\langle r_k \rangle$ discussed in Sect. 3 into birational canonical transformations of $\mathcal{H}_{L,N}$. Note that $\langle r_k \rangle$ is isomorphic to an affine Weyl group of type $A^{(1)}_{L-1}$, denoted by $W(A^{(1)}_{L-1})$. For each $k \in \mathbb{Z}/L\mathbb{Z}$, let $r_k(\sigma_{k,n}) = \hat{\sigma}_{k,n}$ and $r_k(\sigma_{m,n}) = \sigma_{m,n}$ $(m \neq k)$. Substitution of (4.1) and $t = t_i$ in (3.5) yields

$$D_i \hat{\sigma}_{k,n} \cdot \sigma_{k,n} = \theta_i \sigma_{k-1,n}(\theta_i - 1)\sigma_{k+1,n}(\theta_i + 1)$$

with the aid of (4.2). Therefore we have

$$\sum_{i=0}^{N} t_i D_i \hat{\sigma}_{k,n} \cdot \sigma_{k,n} = \sum_{i=0}^{N} \theta_i t_i \sigma_{k-1,n}(\theta_i - 1)\sigma_{k+1,n}(\theta_i + 1).$$

In view of the homogeneity (4.4d) we conclude that

$$\hat{\sigma}_{k,n} = \frac{1}{\alpha_k \sigma_{k,n}} \sum_{i=0}^{N} \theta_i t_i \sigma_{k-1,n}(\theta_i - 1)\sigma_{k+1,n}(\theta_i + 1); \tag{8.1}$$

recall (3.1). Similarly, we deduce from (3.6) that

$$t_i \sigma_{k-1,n+1}(\theta_i - 1)\sigma_{k+1,n} - \hat{\sigma}_{k,n+1}\sigma_{k,n}(\theta_i - 1) + \hat{\sigma}_{k,n}(\theta_i - 1)\sigma_{k,n+1} = 0. \tag{8.2}$$

Through (4.5) and (4.6), the action of r_k on $(f^{(i)}_{m,n}, g^{(i)}_{m,n})$ is determined by (8.1) and (8.2) as follows:

$$r_k(f^{(i)}_{k,n}) = f^{(i)}_{k,n} \left(1 + \frac{\alpha_k t_i f^{(i)}_{k+1,n-1}}{\sum_{j=0}^{N} t_j f^{(j)}_{k+1,n-1} g^{(j)}_{k,n-1}} \right),$$

$$r_k(f^{(i)}_{k+1,n-1}) = f^{(i)}_{k+1,n-1} \left(1 - \frac{\alpha_k t_i f^{(i)}_{k+1,n-1}}{\alpha_k t_i f^{(i)}_{k+1,n-1} + \sum_{j=0}^{N} t_j f^{(j)}_{k+1,n-1} g^{(j)}_{k,n-1}} \right),$$

$$r_k(g^{(i)}_{k,n}) = g^{(i)}_{k,n} \left(1 + \frac{\alpha_k t_i f^{(i)}_{k+1,n}}{\sum_{j=0}^{N} t_j f^{(j)}_{k+1,n} g^{(j)}_{k,n}} \right),$$

$$r_k(g^{(i)}_{k+1,n-1}) = g^{(i)}_{k+1,n-1} - \frac{\alpha_k t_i f^{(i)}_{k+1,n} g^{(i)}_{k,n}}{\sum_{j=0}^{N} t_j f^{(j)}_{k+1,n} g^{(j)}_{k,n}},$$

for $n \in \mathbb{Z}/L\mathbb{Z}$. It is then easy to construct the corresponding transformation of $(q_n^{(i)}, p_n^{(i)})$ by virtue of (7.13). Moreover, as has been mentioned in Sect. 3, the system $\mathcal{H}_{L,N}$ enjoys another action $\langle r_k{}' \rangle$ of $W(A_{L-1}^{(1)})$ associated with the root variables $\beta_k = -d_{m,k-1} + 2d_{m,k} - d_{m,k+1} + 1$, which commutes with the previous one $\langle r_k \rangle$.

Next we observe that a cyclic permutation of the suffixes $\pi : (\sigma_{m,n}, d_{m,n}) \mapsto (\sigma_{m+1,n-1}, d_{m+1,n-1})$ keeps the bilinear expression (4.4) of $\mathcal{H}_{L,N}$ invariant, and so does the interchange of suffixes $\rho : (\sigma_{m,n}, d_{m,n}, t_i) \mapsto (\sigma_{n,m}, -d_{n,m}, 1/t_i)$. These trivial symmetries can be lifted to birational canonical transformations of $\mathcal{H}_{L,N}$. Note that π realizes a Dynkin automorphism which rotates simultaneously the two Dynkin diagrams of type $A_{L-1}^{(1)}$ and that ρ represents an interchange of the two diagrams.

For notational simplicity we extend the suffix n of the canonical variables $(q_n^{(i)}, p_n^{(i)})$ and parameters e_n and κ_n for any $n \in \mathbb{Z}$ by the conditions (cf. (7.13))

$$q_{n+L}^{(i)} = s_i q_n^{(i)}, \quad p_{n+L}^{(i)} = \frac{p_n^{(i)}}{s_i}, \quad e_{n+L} = e_n + 1, \quad \kappa_{n+L} = \kappa_n.$$

We set

$$\mathfrak{a}_n = \frac{\alpha_n}{L} = e_{n+1} - e_n, \quad \mathfrak{b}_n = \frac{\beta_n}{L} = e_{L-n} - e_{L-n-1} - \kappa_{L-n} + \kappa_{L-n-1} \tag{8.3}$$

for $0 \le n \le L-1$. It thus holds that $\sum_{n=0}^{L-1} \mathfrak{a}_n = \sum_{n=0}^{L-1} \mathfrak{b}_n = 1$.

We now state the result.

Theorem 8.1. *The Hamiltonian system $\mathcal{H}_{L,N}(\vec{\kappa})$ is invariant under the birational canonical transformations r_n, $r_n{}'$, π, and ρ $(n = 0, 1, \ldots, L-1)$ defined as follows*:

- *Action on the parameters $\vec{\kappa}$.*

$$r_n : e_n \mapsto e_n + \mathfrak{a}_n, \quad e_{n+1} \mapsto e_{n+1} - \mathfrak{a}_n, \quad \kappa_n \mapsto \kappa_n + \mathfrak{a}_n, \quad \kappa_{n+1} \mapsto \kappa_{n+1} - \mathfrak{a}_n.$$
$$r_n{}' : \kappa_{L-n} \mapsto \kappa_{L-n} + \mathfrak{b}_n, \quad \kappa_{L-n-1} \mapsto \kappa_{L-n-1} - \mathfrak{b}_n.$$
$$\pi : e_n \mapsto e_{n+1} - \frac{1}{L}, \quad \kappa_n \mapsto \kappa_{n+1}.$$
$$\rho : e_n \mapsto \kappa_{L-n} - e_{L-n} - \frac{\sum_{i=0}^N \theta_i}{L} + 1, \quad \kappa_n \mapsto \kappa_{L-n}.$$

- *Action on the canonical variables $(q_n^{(i)}, p_n^{(i)})$.*

$$r_n \ (n \neq 0) : \begin{cases} q_n^{(i)} \mapsto q_n^{(i)} + \dfrac{\mathfrak{a}_n (q_{n+1}^{(i)} - q_n^{(i)})}{\mathfrak{a}_n + \sum_{j=0}^N q_{n+1}^{(j)} p_n^{(j)}}, \\ p_n^{(i)} \mapsto p_n^{(i)} \left(1 + \dfrac{\mathfrak{a}_n}{\sum_{j=0}^N q_{n+1}^{(j)} p_n^{(j)}} \right), \\ p_{n+1}^{(i)} \mapsto p_{n+1}^{(i)} - \dfrac{\mathfrak{a}_n p_n^{(i)}}{\sum_{j=0}^N q_{n+1}^{(j)} p_n^{(j)}}. \end{cases}$$

$$r_0:\begin{cases} q_n^{(i)} \mapsto q_n^{(i)}\left(1-\dfrac{\mathfrak{a}_0(q_1^{(i)}-1)}{\mathfrak{a}_0 q_1^{(i)}+\sum_{j=0}^N q_1^{(j)}p_0^{(j)}}\right), \\ p_n^{(i)} \mapsto p_n^{(i)}\left(1+\dfrac{\mathfrak{a}_0(q_1^{(i)}-1)}{\mathfrak{a}_0+\sum_{j=0}^N q_1^{(j)}p_0^{(j)}}\right) \quad (n\neq 1), \\ p_1^{(i)} \mapsto \left(p_1^{(i)}-\dfrac{\mathfrak{a}_0 p_0^{(i)}}{\sum_{j=0}^N q_1^{(j)}p_0^{(j)}}\right)\left(1+\dfrac{\mathfrak{a}_0(q_1^{(i)}-1)}{\mathfrak{a}_0+\sum_{j=0}^N q_1^{(j)}p_0^{(j)}}\right). \end{cases}$$

$$r_n{}'\ (n\neq 0):\begin{cases} q_{L-n}^{(i)} \mapsto q_{L-n}^{(i)}+\dfrac{\mathfrak{b}_n(q_{L-n-1}^{(i)}-q_{L-n}^{(i)})}{\mathfrak{b}_n+\sum_{j=0}^N q_{L-n-1}^{(j)}p_{L-n}^{(j)}}, \\ p_{L-n}^{(i)} \mapsto p_{L-n}^{(i)}\left(1+\dfrac{\mathfrak{b}_n}{\sum_{j=0}^N q_{L-n-1}^{(j)}p_{L-n}^{(j)}}\right), \\ p_{L-n-1}^{(i)} \mapsto p_{L-n-1}^{(i)}-\dfrac{\mathfrak{b}_n p_{L-n}^{(i)}}{\sum_{j=0}^N q_{L-n-1}^{(j)}p_{L-n}^{(j)}}. \end{cases}$$

$$r_0{}':\begin{cases} q_n^{(i)} \mapsto q_n^{(i)}\left(1-\dfrac{\mathfrak{b}_0(q_{-1}^{(i)}-1)}{\mathfrak{b}_0 q_{-1}^{(i)}+\sum_{j=0}^N q_{-1}^{(j)}p_0^{(j)}}\right), \\ p_n^{(i)} \mapsto p_n^{(i)}\left(1+\dfrac{\mathfrak{b}_0(q_{-1}^{(i)}-1)}{\mathfrak{b}_0+\sum_{j=0}^N q_{-1}^{(j)}p_0^{(j)}}\right) \quad (n\neq L-1), \\ p_{L-1}^{(i)} \mapsto \dfrac{1}{s_i}\left(p_{-1}^{(i)}-\dfrac{\mathfrak{b}_0 p_0^{(i)}}{\sum_{j=0}^N q_{-1}^{(j)}p_0^{(j)}}\right)\left(1+\dfrac{\mathfrak{b}_0(q_{-1}^{(i)}-1)}{\mathfrak{b}_0+\sum_{j=0}^N q_{-1}^{(j)}p_0^{(j)}}\right). \end{cases}$$

$$\pi : q_n^{(i)} \mapsto \frac{q_{n+1}^{(i)}}{q_1^{(i)}}, \quad p_n^{(i)} \mapsto p_{n+1}^{(i)} q_1^{(i)}.$$

$$\rho : s_i \mapsto \frac{1}{s_i}, \quad q_n^{(i)} \mapsto \frac{q_{L-n}^{(i)}}{s_i}, \quad p_n^{(i)} \mapsto s_i p_{L-n}^{(i)}.$$

(*Here we have omitted to write the action on the variables if it is trivial.*) *Moreover, these transformations satisfy the relations*: $r_n{}^2 = (r_n r_{n\pm1})^3 = (r_n{}')^2 = (r_n{}' r_{n\pm1}{}')^3 = \pi^L = \rho^2 = \mathrm{id}$, $\pi r_n = r_{n+1}\pi$, $\pi r_n{}' = r_{n-1}{}'\pi$, *and* $\rho r_n = r_n{}'\rho$.

Let us explore further symmetries of $\mathcal{H}_{L,N}$ besides those in Theorem 8.1. First we consider a symmetry shifting the parameter θ_i to $\theta_i - 1$ at the level of the variables $f_{m,n}^{(i)}$ and $g_{m,n}^{(i)}$. It readily follows from (4.5) and (4.6) that

$$f_{m,n}^{(i)}(\theta_i - 1) = \frac{g_{m,n}^{(i)}}{g_{m+1,n-1}^{(i)}} f_{m+1,n}^{(i)}. \tag{8.4}$$

Combining this with (4.11) shows that

$$f_{m,n}^{(j)}(\theta_i - 1) = \frac{t_i U_{m,n}^{(i,j)} g_{m,n}^{(i)}}{t_j V_{m,n}^{(i,j)} g_{m+1,n-1}^{(i)}} f_{m+1,n}^{(i)} \quad (i \neq j). \tag{8.5}$$

We observe for $i \neq j$ that

$$g_{m,n}^{(j)}(\theta_i - 1) = \theta_j \frac{\sigma_{m-1,n-1}(\theta_i - 1, \theta_j - 1)\sigma_{m,n}(\theta_i - 1, \theta_j + 1)}{\sigma_{m,n-1}(\theta_i - 1)\sigma_{m-1,n}(\theta_i - 1)}$$

$$= \frac{t_i - t_j}{t_j} \frac{\sigma_{m-1,n-1}(\theta_i - 1, \theta_j - 1)\sigma_{m,n}}{\sigma_{m,n-1}(\theta_i - 1)\sigma_{m-1,n}(\theta_j - 1)} \times \frac{g^{(j)}_{m,n+1}}{g^{(i)}_{m,n+1}} U^{(i,j)}_{m,n+1}$$

$$= \left(\frac{t_i}{t_j} \frac{\sigma_{m,n-1}(\theta_j - 1)\sigma_{m-1,n}(\theta_i - 1)}{\sigma_{m,n-1}(\theta_i - 1)\sigma_{m-1,n}(\theta_j - 1)} - 1 \right) \frac{g^{(j)}_{m,n+1}}{g^{(i)}_{m,n+1}} U^{(i,j)}_{m,n+1}, \quad \text{using (4.4a)},$$

$$= \left(\frac{t_i f^{(i)}_{m,n+1}(\theta_i - 1)}{t_j f^{(j)}_{m,n+1}(\theta_j - 1)} - 1 \right) \frac{g^{(j)}_{m,n+1}}{g^{(i)}_{m,n+1}} U^{(i,j)}_{m,n+1}, \quad \text{using (4.5)},$$

$$= \left(\frac{t_i f^{(i)}_{m+1,n+1} g^{(j)}_{m+1,n}}{t_j f^{(j)}_{m+1,n+1} g^{(i)}_{m+1,n}} - \frac{g^{(j)}_{m,n+1}}{g^{(i)}_{m,n+1}} \right) U^{(i,j)}_{m,n+1}, \quad \text{using (8.4)}. \tag{8.6}$$

By (4.13) we have

$$g^{(i)}_{m,n}(\theta_i - 1) = \kappa_{m,n} - 1 - \sum_{j \neq i} g^{(j)}_{m,n}(\theta_i - 1). \tag{8.7}$$

The transformations (8.4)–(8.7) provide a symmetry of the system $\mathcal{G}_{L,N}$, (4.12), shifting the parameter θ_i to $\theta_i - 1$; however, they do not naively give a symmetry of $\mathcal{H}_{L,N}$. To reach a birational canonical transformation of $\mathcal{H}_{L,N}$, we need to combine a trivial symmetry of (4.12) shifting the suffixes: $(f^{(i)}_{m,n}, g^{(i)}_{m,n}, d_{m,n}) \mapsto (f^{(i)}_{m-1,n}, g^{(i)}_{m-1,n}, d_{m-1,n})$. As a result we obtain a symmetry η_i of $\mathcal{H}_{L,N}$ which acts on the parameters as $\theta_i \mapsto \theta_i - 1$ and $\mathfrak{a}_n \mapsto \mathfrak{a}_{n-1}$; see Theorem 8.2 below. We do not go into detail of computations.

It is easy to find a group of symmetries $\langle \zeta_{ij} \rangle (\simeq \mathfrak{S}_{N+1})$, which is generated by a permutation of the singularities $z = u_i = 1/s_i$ $(0 \leq i \leq N)$ of the associated Fuchsian system; see Sect. 6.

Finally we deal with a symmetry deduced from the bilinear expression of $\mathcal{H}_{L,N}$ again. Observe that (4.4) is invariant under the transformation

$$\iota : \sigma_{m,n} = \sigma_{m,n}(\boldsymbol{\theta}, \boldsymbol{t}) \mapsto \sigma_{-m-1,-n-1}(-\boldsymbol{\theta}, \boldsymbol{t}), \quad d_{m,n} \mapsto d_{-m-1,-n-1}, \quad \theta_i \mapsto -\theta_i.$$

Hence we have

$$\iota(q^{(i)}_n) = \iota\left(\left(\frac{t_i}{t_0}\right)^n \frac{\sigma_{n,-n}(\theta_i + 1)\sigma_{0,0}(\theta_0 + 1)}{\sigma_{0,0}(\theta_i + 1)\sigma_{n,-n}(\theta_0 + 1)} \right), \quad \text{using (7.13a)},$$

$$= \left(\frac{t_i}{t_0}\right)^n \frac{\sigma_{-n-1,n-1}(\theta_i - 1)\sigma_{-1,-1}(\theta_0 - 1)}{\sigma_{-1,-1}(\theta_i - 1)\sigma_{-n-1,n-1}(\theta_0 - 1)}$$

$$= \left(\frac{t_i}{t_0}\right)^n \prod_{m=-n}^{-1} \frac{f^{(i)}_{m,-m}(\theta_i - 1)}{f^{(0)}_{m,-m}(\theta_0 - 1)}$$

$$= \left(\frac{t_i}{t_0}\right)^n \prod_{m=-n}^{-1} \frac{g^{(i)}_{m,-m} g^{(0)}_{m+1,-m-1}}{g^{(0)}_{m,-m} g^{(i)}_{m+1,-m-1}} \frac{f^{(i)}_{m+1,-m}}{f^{(0)}_{m+1,-m}}, \quad \text{using (8.4)},$$

$$= \frac{g^{(i)}_{-n,n} g^{(0)}_{0,0}}{g^{(0)}_{-n,n} g^{(i)}_{0,0}} \frac{1}{q^{(i)}_{-n}}$$

$$= \frac{s_i p^{(i)}_{L-n} p^{(0)}_0}{p^{(0)}_{L-n} p^{(i)}_0}. \tag{8.8}$$

Similarly, it follows that

$$\begin{aligned}\iota(q_n^{(i)}p_n^{(i)}) &= \iota\left(\frac{\theta_i}{L}\frac{\sigma_{n-1,-n-1}(\theta_i-1)\sigma_{n,-n}(\theta_i+1)}{\sigma_{n,-n-1}\sigma_{n-1,-n}}\right), \quad \text{using (7.13b)},\\ &= \frac{-\theta_i}{L}\frac{\sigma_{-n,n}(\theta_i+1)\sigma_{-n-1,n-1}(\theta_i-1)}{\sigma_{-n-1,n}\sigma_{-n,n-1}}\\ &= -q_{L-n}^{(i)}p_{L-n}^{(i)}. \end{aligned} \tag{8.9}$$

These formulae (8.8) and (8.9) define a birational canonical transformation of $\mathcal{H}_{L,N}$.

The above results are summed up in the

Theorem 8.2. *The Hamiltonian system $\mathcal{H}_{L,N}(\vec{\kappa})$ is invariant under the birational canonical transformations η_i, ζ_{ij}, and ι $(i,j=0,1,\dots,N; i\neq j)$ defined as follows*:

- *Action on the parameters $\vec{\kappa}$.*

$$\begin{aligned}\eta_i &: e_n \mapsto e_{n-1}+\frac{1}{L}, \quad \kappa_n \mapsto \kappa_n - e_n + e_{n-1}, \quad \theta_i \mapsto \theta_i - 1.\\ \zeta_{ij} &: \theta_i \leftrightarrow \theta_j.\\ \iota &: e_n \mapsto -e_{L-n}+1, \quad \kappa_n \mapsto -\kappa_{L-n}, \quad \theta_i \mapsto -\theta_i.\end{aligned}$$

- *Action on the canonical variables $(q_n^{(i)}, p_n^{(i)})$.*

$$\eta_i : \begin{cases} q_n^{(j)} \mapsto \eta_i(q_n^{(j)}) = \dfrac{\left(\sum_{m=1}^L p_{-m}^{(i)}\right)\left(\sum_{m=1}^L p_{n-m}^{(i)}q_{n-m}^{(j)}\right)}{\left(\sum_{m=1}^L p_{n-m}^{(i)}\right)\left(\sum_{m=1}^L p_{-m}^{(i)}q_{-m}^{(j)}\right)} \quad (\textit{for } \forall j),\\ p_n^{(j)} \mapsto \eta_i(p_n^{(j)}) = \dfrac{1}{\eta_i(q_n^{(j)})}\dfrac{s_j}{s_i-s_j}\left(\dfrac{p_n^{(j)}}{p_n^{(i)}}-\dfrac{p_{n-1}^{(j)}}{p_{n-1}^{(i)}}\right)\displaystyle\sum_{m=1}^L p_{n-m}^{(i)}q_{n-m}^{(j)} \quad (j\neq i),\\ p_n^{(i)} \mapsto \dfrac{1}{\eta_i(q_n^{(i)})}\left(\kappa_n - e_n + e_{n-1} - \displaystyle\sum_{j\neq i}\eta_i(q_n^{(j)}p_n^{(j)})\right).\end{cases}$$

$$\zeta_{ij}\ (i,j\neq 0) : s_i \leftrightarrow s_j, \quad q_n^{(i)} \leftrightarrow q_n^{(j)}, \quad p_n^{(i)} \leftrightarrow p_n^{(j)}.$$

$$\zeta_{i0}=\zeta_{0i} : \begin{cases} s_i \mapsto \dfrac{1}{s_i}, \quad s_j \mapsto \dfrac{s_j}{s_i}, \quad q_n^{(i)} \mapsto \dfrac{1}{q_n^{(i)}}, \quad q_n^{(j)} \mapsto \dfrac{q_n^{(j)}}{q_n^{(i)}},\\ p_n^{(i)} \mapsto q_n^{(i)}p_n^{(0)}, \quad p_n^{(j)} \mapsto q_n^{(i)}p_n^{(j)} \quad (j\neq i).\end{cases}$$

$$\iota : q_n^{(i)} \mapsto \frac{s_i p_{L-n}^{(i)}p_0^{(0)}}{p_{L-n}^{(0)}p_0^{(i)}}, \quad p_n^{(i)} \mapsto -\frac{q_{L-n}^{(i)}p_0^{(i)}p_{L-n}^{(0)}}{s_i p_0^{(0)}}.$$

Remark 8.3. We may regard $\mathfrak{a}_n$, $\mathfrak{b}_n$ $(1\le n\le L-1)$ and θ_i $(0\le i\le N)$ as the $2L+N-1$ constant parameters of $\mathcal{H}_{L,N}$ instead of $\vec{\kappa}$; see (7.12) and (8.3). For reference we summarize how the birational symmetries in Theorems 8.1 and 8.2 act on $\mathfrak{a}_n$, $\mathfrak{b}_n$, and θ_i below.

$$\begin{aligned} r_n &: \mathfrak{a}_n \mapsto -\mathfrak{a}_n, \quad \mathfrak{a}_{n\pm1} \mapsto \mathfrak{a}_{n\pm1}+\mathfrak{a}_n.\\ r_n{}' &: \mathfrak{b}_n \mapsto -\mathfrak{b}_n, \quad \mathfrak{b}_{n\pm1} \mapsto \mathfrak{b}_{n\pm1}+\mathfrak{b}_n.\end{aligned}$$

$$\begin{aligned}
\pi &: \mathfrak{a}_n \mapsto \mathfrak{a}_{n+1}, \quad \mathfrak{b}_n \mapsto \mathfrak{b}_{n-1}.\\
\rho &: \mathfrak{a}_n \leftrightarrow \mathfrak{b}_n.\\
\eta_i &: \mathfrak{a}_n \mapsto \mathfrak{a}_{n-1}, \quad \theta_i \mapsto \theta_i - 1.\\
\zeta_{ij} &: \theta_i \leftrightarrow \theta_j.\\
\iota &: \mathfrak{a}_n \mapsto \mathfrak{a}_{L-n}, \quad \mathfrak{b}_n \mapsto \mathfrak{b}_{L-n}, \quad \theta_i \mapsto -\theta_i.
\end{aligned}$$

Recall that the groups of canonical transformations $\langle r_n \rangle$ and $\langle r_n' \rangle$ mutually commute and each of them gives a birational realization of $W(A_{L-1}^{(1)})$.

Remark 8.4 (Additional symmetry valid for only $N = 1$). Let us consider the case $N = 1$. Write $(q_n, p_n, s) = (q_n^{(1)}, p_n^{(1)}, s_1)$ and $\theta = \theta_1$. We then find another symmetry given as follows (see also the appendix):

$$\varphi : \begin{cases} e_0 \mapsto \kappa_0 - e_0 - 1, \quad e_n \mapsto -e_{L-n} \quad (n \neq 0), \\ \kappa_0 \mapsto \kappa_0, \quad \kappa_n \mapsto -\kappa_{L-n} \quad (n \neq 0), \quad \theta \mapsto \kappa_0 - \theta, \\ q_n \mapsto \dfrac{s p_{L-n}}{q_{L-n} p_{L-n} - \kappa_{L-n}}, \quad p_n \mapsto \dfrac{q_{L-n}(\kappa_{L-n} - q_{L-n} p_{L-n})}{s}. \end{cases}$$

A Case $L = 2$: the Garnier system

If $L = 2$ then the canonical Hamiltonian system $\mathcal{H}_{L,N}$ is equivalent to the Garnier system in N variables (see [Gar12]). This fact is guaranteed by the Lax formalism given in Sect. 6. But, however, our polynomial Hamiltonian function (7.11) is different from that given in [KO84] (see also [IKSY91]). In this appendix we describe explicitly the canonical transformation between the two Hamiltonian systems.

First we concerns the general (L, N) case. Define the canonical transformation $(q_n^{(i)}, p_n^{(i)}, H_i, s_i) \mapsto (Q_n^{(i)}, P_n^{(i)}, \widetilde{H}_i, s_i)$ by

$$\begin{aligned}
Q_n^{(i)} &= -s_i \frac{p_n^{(i)}}{p_n^{(0)}} \left(= -s_i \frac{b_n^{(i)}}{b_n^{(0)}} \right),\\
Q_n^{(i)} P_n^{(i)} &= -q_n^{(i)} p_n^{(i)} \left(= b_n^{(i)} c_n^{(i)} \right),\\
\widetilde{H}_i &= H_i - \sum_{n=1}^{L-1} \frac{q_n^{(i)} p_n^{(i)}}{s_i}.
\end{aligned}$$

Clearly the new Hamiltonian function $\widetilde{H}_i$ becomes again a polynomial in $Q_n^{(i)}$ and $P_n^{(i)}$. This canonical transformation is, in short, derived from an interchange of the roles of $b_n^{(i)}$ and $c_n^{(i)}$ in the definition (7.6) of the canonical variables. Note that only if $N = 1$ it keeps the form of the Hamiltonian function unchanged, thereby giving rise to a birational symmetry; see Remark 8.4.

Next we let $L = 2$ and write the variables as $(Q_1^{(i)}, P_1^{(i)}) = (q_i, p_i)$ for $i = 1, 2, \ldots, N$. The

Hamiltonian function $\widetilde{H}_i$ thus takes the following expression:

$$\begin{aligned} s_i(s_i-1)\widetilde{H}_i \equiv q_i\left(\kappa_1+\sum_j q_jp_j\right)\left(\kappa_1-\theta_0+\sum_j q_jp_j\right)+s_ip_i(q_ip_i+\theta_i) \\ -\sum_{j(\neq i)} R_{ji}(q_jp_j+\theta_j)q_ip_j-\sum_{j(\neq i)} S_{ij}(q_ip_i+\theta_i)q_jp_i \\ -\sum_{j(\neq i)} R_{ij}q_jp_j(q_ip_i+\theta_i)-\sum_{j(\neq i)} R_{ij}q_ip_i(q_jp_j+\theta_j) \\ -(s_i+1)(q_ip_i+\theta_i)q_ip_i-(\theta_{N+2}s_i+\theta_{N+1}+1)q_ip_i \end{aligned}$$

modulo some function in only $\boldsymbol{s}=(s_1,\ldots,s_N)$. Here we put $R_{ij}=s_i(s_j-1)/(s_j-s_i)$, $S_{ij}=s_i(s_i-1)/(s_i-s_j)$, $\theta_{N+1}=d_{1,0}-d_{1,1}-1/2$, $\theta_{N+2}=d_{1,1}-d_{0,1}-1/2$, and $\kappa_1=\left(\sum_{i=0}^{N+2}\theta_i+1\right)/2$. The symbols $\sum_j$ and $\sum_{j(\neq i)}$ stand for the summation over $j=1,2,\ldots,N$ and over $j=1,\ldots,i-1,i+1,\ldots,N$, respectively. This is exactly the (usual) polynomial Hamiltonian function for the Garnier system; cf. [KO84, Tsu06].

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