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Yamato, Hajime
Kagoshima University : Professor Emeritus

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Hajime YAMATO

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THE ERDŐS-TURÁN LAW FOR MIXTURES OF DIRICHLET PROCESSES

By

Hajime Yamato*

Abstract

For a sample of size n from a random discrete distribution $\mathcal{P}$ on the real line $\mathbb{R}$, S_1 denotes the number of observations which occur only once, S_2 the number of observations which occur exactly twice, ... , and so on. Let $O_n(S^{(n)})$ be the order of the random partition $S^{(n)} = (S_1, \cdots, S_n)$ of the positive integer n. In case $\mathcal{P}$ has the Dirichlet process, that is, $S^{(n)}$ has the Ewens sampling formula, Aratia and Tavaré (1992) shows the asymptotic normality of $\log O_n(S^{(n)})$, which is an extension of Erdős and Turán (1967). Barbour and Tavaré (1994) gives the rate of convergence. In case $\mathcal{P}$ has the mixture of Dirichlet processes, we give the asymptotic distribution of $\log O_n(S^{(n)})$ and the rate of its convergence.

Key Words and Phrases: Erdős-Turán law, mixture of Dirichlet processes, order of partition, random partition, smoothing lemma.

1. Introduction

Let G_0 be a continuous distribution on the real line $\mathbb{R}$ and θ be a positive constant. Let $\mathcal{B}$ be the σ-field which consists of the subsets of $\mathbb{R}$. Let the random distribution $\mathcal{P}$ have the Dirichlet process $\mathcal{D}(\theta G_0)$ on $(\mathbb{R}, \mathcal{B})$ with parameter θG_0. Let V_j $(j = 1, 2, \ldots)$ be a sequence of independent and identically distributed (i.i.d.) random variables with the distribution G_0, and $W_j (j = 1, 2, \ldots)$ be a sequence of i.i.d. random variables with the beta distribution $Be(1, \theta)$. We assume that $V_1, V_2, \cdots$ and $W_1, W_2, \cdots$ are independent. We put $p_1 = W_1$ and $p_j = W_j(1-W_1)\cdots(1-W_{j-1})$ $(j = 2, 3, \cdots)$. Then, we can write $\mathcal{P}(B) = \sum_{j=1}^{\infty} p_j \delta_{V_j}(B)$ for any $B \in \mathcal{B}$, where $\delta_V(B) = 1$ if $V \in B$ and 0 otherwise (Sethuraman (1994)). Thus $\mathcal{P}$ $(\in \mathcal{D}(\theta G_0))$ is discrete almost surely (a.s.).

For a sample of size n from $\mathcal{P}$ having $\mathcal{D}(\theta G_0)$, S_1 denotes the number of observations which occur only once, S_2 the number of observations which occur exactly twice, ..., and S_n the number of observations which occur exactly n times. $S^{(n)} = (S_1, \cdots, S_n)$, which satisfies $\sum_{j=1}^{n} jS_j = n$, gives the random partition of the integer n. $S^{(n)}$ has the well-known Ewens sampling formula, whose distribution depends on θ and does not depend on G_0 (see, for example, Antoniak (1974; Prop. 3) and Johnson et al. (1997; Sec. 2 of Chap. 41)).

In this paper, we consider the order $O_n(S^{(n)})$ of the random partition $S^{(n)} = (S_1, \cdots, S_n)$, which is given by

$$O_n(S^{(n)}) = \text{l.c.m.}\{\, j \;:\; S_j > 0 \;(\, j = 1, 2, \cdots, n) \,\},$$

* Emeritus Kagoshima University, Take 3–32–1–708 Kagoshima 890-0045 Japan.

where l.c.m. represents the least common multiple. For the Dirichlet process, that is, for the Ewens sampling formula, Arratia and Tavaré (1992) shows

$$\frac{\log O_n(S^{(n)}) - \frac{\theta}{2}\log^2 n}{\sqrt{\frac{\theta}{3}\log^3 n}} \xrightarrow{d} N(0,1) \quad \text{as} \quad n \to \infty,$$

where $\xrightarrow{d}$ means the convergence in distribution. Especially, the case of $\theta = 1$ (random permutation) is well-known as Erdős-Turán law (Erdős and Turán (1967)). Gnedin et al. (2012) extends the asymptotic normality of $\log O_n$ for sampling from stick-breaking partition of the interval $[0, 1]$, which includes the case of the Ewens sampling formula.

Barbour and Tavaré (1994) gives the rate of the above convergence, which is

$$\sup_{-\infty<x<\infty}\left|P\left[\left\{\frac{\theta}{3}\log^3 n\right\}^{-1/2}\left(\log O_n(S^{(n)}) - \frac{\theta}{2}\log^2 n + \theta\log n\log\log n\right) \le x\right] - \Phi(x)\right| = O\left(\frac{1}{\log^{1/2} n}\right), \quad (1)$$

where $\Phi(x)$ is the standard normal distribution function.

Hereafter, in this paper, we consider θ as a positive random variable having a distribution γ. Given θ, let the random discrete distribution $\mathcal{P}$ have the Dirichlet process $\mathcal{D}(\theta G_0)$ on $(\mathbb{R}, \mathcal{B})$ with parameter θG_0. Then this random discrete (a.s.) distribution $\mathcal{P}$ has the mixture of Dirichlet processes $\mathcal{D}(\theta G_0)$ with the mixing distribution γ (Antoniak (1974)). Concerning the random partition based on the sample, the relation between the Dirichlet process and the mixture of Dirichlet processes is equivalent to (ii) of Theorem 12 of Gnedin and Pitman (2006) which is the characterization of the random partition which are consistent and exchangeable.

For the random partition $S^{(n)} = (S_1, \cdots, S_n)$ of the integer n, based on a sample of size n from the mixture of Dirichlet processes $\mathcal{D}(\theta G_0)$ with the mixing distribution γ, we consider its order $O_n(S^{(n)})$. In the next section 2, we give the asymptotic distribution of $\log O_n(S^{(n)})$ and its rate of the convergence, which are our main results. In addition, in the section 2 we give the outline of the proof for the main results. In the section 3 we give the proof in detail.

2. Asymptotic distribution of $\log O_n(S^{(n)})$

We assume that the distribution γ of the positive random variable θ has the bounded density, and E_γ denotes the expectation with respect to γ. W assume $E_\gamma(\theta e^{c_0\theta}) < \infty$, where c_0 is a positive constant such that $0 < c_0 \le 0.41$. Then, since θ is the positive random variable, we have $E_\gamma\theta^2 < \infty$. We put $\gamma^*(x) = \gamma(2x)$, which is the distribution function of $\theta/2$. The distribution $\gamma^*(x)$ has also the bounded density.

THEOREM 2.1. *Asymptotically it holds that*

$$\frac{\log O_n(S^{(n)})}{\log^2 n} \xrightarrow{d} \frac{\theta}{2} \quad \text{as} \quad n \to \infty.$$

The rate of the convergence is

$$\sup_{-\infty<x<\infty}\left|P\left(\frac{\log O_n(S^{(n)})}{\log^2 n}\le x\right)-\gamma^*(x)\right| = O\left(\frac{1}{\log^{1/3} n}\right). \tag{2}$$

The assumptions about θ (γ) are satisfied by the following distributions: (1) The distribution whose support is finite and has the bounded density. (2) The Rayleigh distribution whose density is given by $g(x) = (x/b^2)\exp(-x^2/2b^2)$ $(x>0;\ b>0)$. (3) The half-normal distribution whose density given by $g(x) = \sqrt{2}/(\sqrt{\pi}\sigma)\exp[-x^2/(2\sigma^2]$ $(x>0;\ \sigma>0)$. (4) The gamma distribution whose density is given by $g(x) = (x/b)^{c-1}e^{-x/b}/b\Gamma(c)$ $(x>0;\ 1/c_0>b>0, c>0)$.

Given θ, let random variables B_j $(j=1,2,\cdots)$ be independent and take the value 0,1 with the probabilities given by

$$P(B_j=0)=\frac{j-1}{\theta+j-1},\quad P(B_j=1)=\frac{\theta}{\theta+j-1}\quad (j=1,2,\cdots).$$

For $j=1,2,\cdots$ and $m=0,1,2,\cdots$, we let

$$Z_{jm}=\sum_{i=m+1}^{\infty}B_i(1-B_{i+1})\cdots(1-B_{i+j-1})B_{i+j}.$$

We put $Y_n = Z_{1n}+Z_{2n}+\cdots+Z_{nn}$, and $Z^{(n)}=(Z_1,\cdots,Z_n)$, where $Z_j = Z_{j0}$ $(j=1,2,\cdots)$. Then, the following lemma holds.

LEMMA 2.2. (Arratia and Tavaré (1992), Theorem 1) *Given θ, the random partition $S^{(n)}=(S_1,\cdots,S_n)$ of n based on a mixture of Dirichlet processes is expressed by the following equivalent form such that*

$$S_j=\sum_{i=1}^{n-j}B_i(1-B_{i+1})\cdots(1-B_{i+j-1})B_{i+j}+B_{n-j+1}(1-B_{n-j+2})\cdots(1-B_n)$$

for each $j=1,2,\cdots,n$. Given θ, $Z_1,\cdots,Z_n$ are independent and Z_j has Poisson distribution with mean θ/j $(j=1,\cdots,n)$.

For S_j of Lemma 2.2, for example,

$$S_1=B_1B_2+B_2B_3+\cdots+B_{n-1}B_n+B_n$$

is the number of components of size 1 for the partition of n.

$$S_2=B_1(1-B_2)B_3+B_2(1-B_3)B_4+\cdots+B_{n-2}(1-B_{n-1})B_n+B_{n-1}(1-B_n)$$

is the number of components of size 2 for the partition of n.

For $Z^{(n)}=(Z_1,\cdots,Z_n)$, we put

$$O_n(Z^{(n)})=\text{l.c.m.}\{\,j\ :\ Z_j>0\ (j=1,2,\cdots,n)\,\},\quad T_n(Z^{(n)})=\prod_{j=1}^{n}j^{Z_j}$$

and

$$\mu_n(\theta)=E\big[\log T_n\big(Z^{(n)}\big)-\log O_n\big(Z^{(n)}\big)\ \big|\ \theta\,\big].$$

We put

$$S_{1n}^* = \frac{\log T_n(Z^{(n)})}{\log^2 n}, \quad S_{2n}^* = \frac{\log O_n(Z^{(n)}) + \mu_n(\theta)}{\log^2 n} \quad \text{and} \quad S_{3n}^* = \frac{\log O_n(S^{(n)})}{\log^2 n}.$$

For S_{1n}^*, we have the following proposition whose proof is given in the next section.

PROPOSITION 2.3. *S_{1n}^* converges in distribution to γ^* and its rate is given by*

$$\sup_{-\infty<x<\infty} \Big| P(S_{1n}^* \le x) - \gamma^*(x) \Big| = O\bigg(\frac{1}{\log^{1/3} n}\bigg). \tag{3}$$

In order to evaluate the probabilities $P(S_{2n}^* \le x)$ and $P(S_{3n}^* \le x)$ based on this proposition, we need the folling lemma.

LEMMA 2.4. *Let H be a distribution function, which has the density and $H'(x) \le \xi$ for a positive constant ξ. Let U be a random variable satisfying $\sup_{-\infty<x<\infty} |P(U \le x) - H(x)| \le \eta$. Then for any random variable X and any $\epsilon > 0$*

$$\sup_{-\infty<x<\infty} |P(U + X \le x) - H(x)| \le \eta + \epsilon\xi + P(|X| > \epsilon). \tag{4}$$

The lemma 2.4 is easily proved by the equation (1.20) of Petrov (1995). Using the relation (4) to $S_{2n}^* = S_{1n}^* + (S_{2n}^* - S_{1n}^*)$ with (3), we obtain the following lemma 2.5. The detail is given in the next section.

LEMMA 2.5. *S_{2n}^* converges in distribution to γ^* and its rate is given by*

$$\sup_{-\infty<x<\infty} |P(S_{2n}^* \le x) - \gamma^*(x)| = O\bigg(\frac{1}{\log^{1/3} n}\bigg). \tag{5}$$

Using the relation (4) to $S_{3n}^* = S_{2n}^* + (S_{2n}^* - S_{3n}^*)$ with (5), we obtain the following lemma 2.6. The detail is given in the next section.

LEMMA 2.6. *S_{3n}^* converges in distribution to γ^* and its rate is given by*

$$\sup_{-\infty<x<\infty} |P(S_{3n}^* \le x) - \gamma^*(x)| = O\bigg(\frac{1}{\log^{1/3} n}\bigg). \tag{6}$$

Thus the lemma 2.6 yields the theorem 2.1.

3. Appendix

To prove the proposition 2.3, we use the smoothing lemma (see, for example, Petrov (1995; Theorem 5.1)).

LEMMA 3.1. (smoothing lemma) *Let $F(x)$ and $G(x)$ be distribution functions with the characteristic functions $f(t)$ and $g(t)$, respectively. Suppose that $G(x)$ has a bounded derivative on the real line, so that $\sup_x G'(x) \le K$. Then for every $T > 0$ and every $b(> 1/2\pi)$ we have*

$$\sup_{-\infty<x<\infty} |F(x) - G(x)| \le b \int_{-T}^{T} \Big| \frac{f(t) - g(t)}{t} \Big| dt + c(b)\frac{K}{T},$$

where $c(b)$ is a positive constant depending only on b.

Proof of Proposition 2.3 We put

$$F_n(x) = P(S^*_{1n} \le x) = P\biggl(\frac{\log T_n(Z^{(n)})}{\log^2 n} \le x\biggr)$$

and let the characteristic functions of the distribution functions F_n and γ^* be $f_n(t)$ and $g(t)$, respectively. We have

$$\log T_n(Z^{(n)}) = \sum_{j=1}^{n} Z_j \log j$$

and given θ, $Z_1, Z_2, \cdots$ are independent, and Z_j has Poisson distribution $P(\theta/j)$ for $j = 1, 2, \cdots$. Therefore we obtain

$$E[e^{it \log T_n(Z^{(n)})}|\theta] = \exp\bigl\{\theta \sum_{j=1}^{n} \frac{1}{j}(e^{it\log j} - 1)\bigr\}$$

and therefore,

$$f_n(t) = E_\gamma\bigl[E[e^{i\frac{t}{\log^2 n}\log T_n(Z^{(n)})}|\theta]\bigr] = E_\gamma \exp\left\{\theta \sum_{j=1}^{n} \frac{1}{j}(e^{it\frac{\log j}{\log^2 n}} - 1)\right\}.$$

Since $g(t) = E_\gamma \exp\{i\theta t/2\}$, we have

$$f_n(t) - g(t) = E_\gamma e^{i\frac{\theta t}{2}}\left[\exp\left\{\theta \sum_{j=1}^{n} \frac{1}{j}(e^{it\frac{\log j}{\log^2 n}} - 1) - i\frac{\theta t}{2}\right\} - 1\right].$$

Thus we obtain

$$|f_n(t) - g(t)| \le E_\gamma\left|\exp\theta\left\{\sum_{j=1}^{n} \frac{1}{j}(e^{it\frac{\log j}{\log^2 n}} - 1) - \frac{it}{2}\right\} - 1\right| \le I_1 + I_2, \tag{7}$$

where I_1 and I_2 are given by

$$I_1 = E_\gamma\left|\exp\theta\left\{\sum_{j=1}^{n} \frac{1}{j}(e^{it\frac{\log j}{\log^2 n}} - 1) - \frac{it}{2}\right\} - \exp\theta\left\{i\frac{t}{\log^2 n}\sum_{j=1}^{n}\frac{\log j}{j} - \frac{it}{2}\right\}\right|,$$

$$I_2 = E_\gamma\left|\exp i\theta t\left\{\frac{1}{\log^2 n}\sum_{j=1}^{n}\frac{\log j}{j} - \frac{1}{2}\right\} - 1\right|.$$

At first, we evaluate I_2. Using the inequality $|e^{i\eta} - 1| \le |\eta|$, we have

$$I_2 \le E_\gamma\left\{\theta|t|\left|\frac{1}{\log^2 n}\sum_{j=1}^{n}\frac{\log j}{j} - \frac{1}{2}\right|\right\} \le E_\gamma(\theta)|t|\frac{c_3}{\log^2 n}, \tag{8}$$

where c_3 is a positive constant. Next, we evaluate I_1. We have

$$\begin{aligned} I_1 = E_\gamma\Bigg|&\exp i\theta\left\{\frac{t}{\log^2 n}\sum_{j=1}^{n}\frac{\log j}{j} - \frac{t}{2}\right\} \\ &\times\left[\exp\theta\left\{\sum_{j=1}^{n}\frac{1}{j}(e^{it\frac{\log j}{\log^2 n}} - 1) - i\frac{t}{\log^2 n}\sum_{j=1}^{n}\frac{\log j}{j}\right\} - 1\right]\Bigg| \le E_\gamma\bigl|\exp(\theta J) - 1\bigr|, \end{aligned} \tag{9}$$

where

$$J=\sum_{j=1}^{n}\frac{1}{j}(e^{it\frac{\log j}{\log^2 n}}-1)-i\frac{t}{\log^2 n}\sum_{j=1}^{n}\frac{\log j}{j}=\sum_{j=1}^{n}\frac{1}{j}\Big(e^{it\frac{\log j}{\log^2 n}}-\frac{it\log j}{\log^2 n}-1\Big).$$

Applying the inequality $|\exp(i\eta)-1-i\eta|\le\frac{1}{2}\eta^2$ $(-\infty<\eta<\infty)$ to the parentheses of the right-hand side of the above, we have

$$\big|J\big|\le\frac{t^2}{2\log n}\times\frac{1}{\log^3 n}\sum_{j=1}^{n}\frac{\log^2 j}{j}.$$

$\sum_{j=1}^{n}[\log^2 j/j]\big/\log^3 n$ is monotone decreasing and converges to $1/3$ as $n\to\infty$. It takes the maximum at $n=2$, which is smaller than 0.722. Thus we obtain

$$\big|J\big|\le 0.361\frac{t^2}{\log n}.$$

Using the above inequality and the inequality $|e^w-1|\le|w|e^{|w|}$ for any complex number w to (9), we obtain

$$I_1\le E_\gamma\bigg\{0.361\theta\frac{t^2}{\log n}\exp\bigg[\theta\frac{0.361}{\log^{1/3}n}\Big(\frac{t}{\log^{1/3}n}\Big)^2\bigg]\bigg\}.$$

We note $0.361/\log^{1/3}n<c_0$ with $c_0=0.41$. For $|t|\le\log^{1/3}n$, under the assumption of $E(\theta e^{c_0\theta})<\infty$, we have

$$\left|\frac{I_1}{t}\right|=O\Big(\frac{1}{\log^{2/3}}\Big). \tag{10}$$

With respect to I_2, by (8) we have

$$\left|\frac{I_2}{t}\right|=O\Big(\frac{1}{\log^2 n}\Big), \tag{11}$$

Using (10) and (11) to (7), for $0\le t\le\log^{1/3}n$, we obtain

$$\left|\frac{f_n(t)-g(t)}{t}\right|=O\Big(\frac{1}{\log^{2/3}n}\Big). \tag{12}$$

Using the lemma 3.1 with $T=\log^{1/3}n$, and F_n and γ^* instead of F and G, respectively, by (7) we obtain

$$\sup_{-\infty<x<\infty}|F_n(x)-\gamma^*(x)|=O\Big(\frac{1}{\log^{1/3}n}\Big),$$

which yields (3). □

We have $\sum_{j=1}^{n}[\log^2 j/j]\big/\log^3 n<0.38$ for $n\ge 6$. If we take n greater than 5, we have (10) under the the assumption of $E(\theta e^{c_0\theta})<\infty$ with $c_0=0.16$. We have $\sum_{j=1}^{n}[\log^2 j/j]\big/\log^3 n<0.35$ for $n\ge 12$. If we take n greater than 11, we have (10) under the the assumption of $E(\theta e^{c_0\theta})<\infty$ with $c_0=0.13$. As concerns the condition

$E(\theta e^{c_0\theta}) < \infty$, the larger n we neglect, and the smaller positive constant c_0 is sufficient.

Proof of Lemma 2.5 With respect to the difference between S^*_{1n} and S^*_{2n}, for any $\epsilon > 0$, we have

$$P(|S^*_{1n} - S^*_{2n}| > \epsilon \mid \theta) = P\left(\left|\frac{\log O_n(Z^{(n)}) + \mu_n(\theta)}{\log^2 n} - \frac{\log T_n(Z^{(n)})}{\log^2 n}\right| > \epsilon \,\middle|\, \theta\right)$$

$$= P\left(\left|\log T_n(Z^{(n)}) - \log O_n(Z^{(n)}) - \mu_n(\theta))\right| > \epsilon \log^2 n \,\middle|\, \theta\right). \tag{13}$$

By the proposition 2.3 and its proof of Barbour and Tavaré (1994), it holds that

$$P\left(\left|\log T_n(Z^{(n)}) - \log O_n(Z^{(n)}) - \mu_n(\theta))\right| > \epsilon \log^2 n \,\middle|\, \theta\right) = \theta c_{1n} + \theta^2 c_{2n} \text{ for } \forall \epsilon > 0 \tag{14}$$

where $c_{1n} = O\big((\log\log n)^2)/\log n\big)$ and $c_{2n} = O(1/\log n)$. Therefore, under the condition $E_\gamma \theta^2 < \infty$, by (13) and (14) we have

$$P(|S^*_{1n} - S^*_{2n}| > \epsilon) = O\left(\frac{(\log\log n)^2}{\log n}\right) \tag{15}$$

for any $\epsilon > 0$. We use the relation (4) by taking $U = S^*_{1n}$, $X = S^*_{2n} - S^*_{1n}$, $H = \gamma^*$, $\eta = O(1/\log^{1/3} n)$, and $\epsilon = O(1/\log^{1/3} n)$. By the relation (3) and (15), we obtain

$$\sup_{-\infty<x<\infty} |P(S^*_{2n} \le x) - \gamma^*(x)| = O\left(\frac{1}{\log^{1/3} n}\right). \tag{16}$$

Proof of Lemma 2.6 By the relation (2.1) and (2.2) of Barbour and Tavaré (1994), we have

$$|S^*_{2n} - S^*_{3n}| = \left|\frac{\log O_n(Z^{(n)}) - \log O_n(S^{(n)})}{\log^2 n}\right| \le Y, \text{ given } \theta \tag{17}$$

where $Y = (Y_n + 1)/\log n$ and $E(Y_n) = E_\gamma(E(Y_n|\theta)) \le E_\gamma \theta^2$. Thus, by (17) we have

$$P(|S^*_{2n} - S^*_{3n}| > \epsilon) \le P(|Y| > \epsilon) \le \frac{1 + E_\gamma \theta^2}{\epsilon \log n} \quad \text{for } \forall \epsilon > 0. \tag{18}$$

We use the relation (4) by taking $U = S^*_{2n}$, $X = S^*_{3n} - S^*_{2n}$, $H = \gamma^*$, $\eta = O(1/\log^{1/3} n)$, and $\epsilon = O(1/\log^{2/3} n)$. By the relation (16) and (18), we obtain

$$\sup_{-\infty<x<\infty} |P(S^*_{3n} \le x) - \gamma^*(x)| = O\left(\frac{1}{\log^{1/3} n}\right). \tag{19}$$

The rate of convergence given by (2) is less than (1). Further work is desirable for the better rate of convergence of (2).

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