Abelian Sandpile Models in Statistical Mechanics : Dissipative Abelian Sandpile Models

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Abelian Sandpile Models in Statistical Mechanics – Dissipative Abelian Sandpile Models –

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Abstract

We introduce a family of abelian sandpile models with two parameters $n, m \in \mathbb{N}$ defined on finite lattices on *d*-dimensional torus. Sites with 2dn + m or more grains of sand are unstable and topple, and in each toppling *m* grains dissipate from the system. Because of dissipation in bulk, the models are well-defined on the shift-invariant lattices and the infinitevolume limit of systems can be taken. From the determinantal expressions, we obtain the asymptotic forms of the avalanche propagators and the height-(0, 0) correlations of sandpiles for large distances in the infinite-volume limit in any dimensions $d \ge 2$. We show that both of them decay exponentially with the correlation length

$$\xi(d,a) = (\sqrt{d}\sinh^{-1}\sqrt{a(a+2)})^{-1},$$

if the dissipation rate $a = \frac{m}{2dn}$ is positive. Considering a series of models with increasing n, we discuss the limit $a \downarrow 0$ and the critical exponent defined by $\nu_a = -\lim_{a\downarrow 0} \frac{\log \xi(d,a)}{\log a}$ is determined as 1

$$\nu_a = \frac{1}{2}$$

for all $d \ge 2$. Comparison with the $q \downarrow 0$ limit of q-state Potts model in external magnetic field is discussed.

Key words. Abelian sandpile models, Dissipation, Avalanches, Height correlations, Determinantal expressions, Correlation length exponent.

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1 Introduction

Let $d \in \{2, 3, ...\}$ and $L \in \mathbb{N} \equiv \{1, 2, 3, ...\}$. Consider a box in the *d*-dimensional hypercubic lattice $B_L = \{-L, -L + 1, ..., L\}^d \subset \mathbb{Z}^d$, where \mathbb{Z} denotes the collection of all integers. We impose *periodic boundary conditions* for all *d* directions and obtain a lattice on a torus (toroidal), which is denoted by Λ_L . The number of sites in Λ_L is given by $|\Lambda_L| = (2L+1)^d$. In the present paper we study a family of Markov processes on Λ_L , $h_t = \{h_t(\mathbf{z})\}_{\mathbf{z}\in\Lambda_L}$, with discrete-time $t \in \mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$.

Assume $n, m \in \mathbb{N}$ and let

$$a = \frac{m}{2dn}$$
 and $h_c = 2d(1+a).$

Define a real symmetric matrix with size $(2L+1)^d$,

$$\Delta_L(\mathbf{x}, \mathbf{y}) = \begin{cases} h_c, & \text{if } \mathbf{x} = \mathbf{y}, \\ -1, & \text{if } |\mathbf{x} - \mathbf{y}| = 1, \\ 0, & \text{otherwise}, \end{cases}$$
(1.1)

where $\mathbf{x} = (x_1, \ldots, x_d), \mathbf{y} = (y_1, \ldots, y_d) \in \Lambda_L$ and $|\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{i=1}^d (x_i - y_i)^2}$. Let $\mathbf{1}(\omega)$ be the indicator function of an event ω ; $\mathbf{1}(\omega) = 1$, if ω occurs and $\mathbf{1}(\omega) = 0$, otherwise. The configuration space is

$$\mathcal{S}_L = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, h_c - \frac{1}{n}\right\}^{\Lambda_L}.$$

Given a configuration $h_t \in S_L, t \in \mathbb{N}_0, h_{t+1} \in S_L$ is determined by the following algorithm.

(i) Choose one site in Λ_L at random. Let **x** be the chosen site and define

$$\eta_{(1)}^{\mathbf{x}}(\mathbf{z}) = h_t(\mathbf{z}) + \frac{1}{n}\mathbf{1}(\mathbf{z} = \mathbf{x}), \quad \mathbf{z} \in \Lambda_L$$

If $\eta_{(1)}^{\mathbf{x}}(\mathbf{x}) < h_{c}$, then $\eta_{(1)}^{\mathbf{x}} \equiv \{\eta_{(1)}^{\mathbf{x}}(\mathbf{z})\}_{\mathbf{z} \in \Lambda_{L}} \in \mathcal{S}_{L}$. In this case, we set $h_{t+1} = \eta_{(1)}^{\mathbf{x}}$.



Figure 1: A toppling for the DASM with the parameters d = 2, n = 2 and m = 1. In this case $h_c = 2dn + m = 9$, and thus the site **x** with height $h(\mathbf{x}) = 10$ is unstable. In a toppling, $h_c = 9$ grains of sand drop from the site **x**, in which n = 2 grains land on each nearest-neighbor site, m = 1 grain is dissipated from the system, while $h(\mathbf{x}) - h_c = 1$ grain remains on the site **x**.

(ii) If $\eta_{(1)}^{\mathbf{x}}(\mathbf{x}) = h_c$, then $\eta_{(1)}^{\mathbf{x}} \notin S_L$. In this case, we consider a finite series of configurations $\{\eta_{(1)}^{\mathbf{x}}, \cdots, \eta_{(\tau)}^{\mathbf{x}}\}$ with $\exists \tau \in \mathbb{N}$ recursively as follows. Assume that $\eta_{(\ell)}^{\mathbf{x}} \notin S_L$ with $\ell \geq 1$, then $A_{(\ell)}^{\mathbf{x}}(h_t) \equiv \{\mathbf{z} \in \Lambda_L : \eta_{(\ell)}^{\mathbf{x}}(\mathbf{z}) \geq h_c\} \neq \emptyset$ and define

$$\eta_{(\ell+1)}^{\mathbf{x}}(\mathbf{z}) = \eta_{(\ell)}^{\mathbf{x}}(\mathbf{z}) - \sum_{\mathbf{y}:\mathbf{y}\in A_{(\ell)}^{\mathbf{x}}(h_t)} \Delta_L(\mathbf{y},\mathbf{z}), \quad \mathbf{z}\in \Lambda_L.$$

If $\eta_{(\ell+1)}^{\mathbf{x}} \in \mathcal{S}_L$, then $\tau = \ell + 1$ and $h_{t+1} = \eta_{(\tau)}^{\mathbf{x}}$. Remark that $\tau = \tau(\mathbf{x}, h_t)$ and $\tau < \infty$ by $\sum_{\mathbf{z}:\mathbf{z}\in\Lambda_L} \Delta_L(\mathbf{y}, \mathbf{z}) > 0, \forall \mathbf{y} \in \Lambda_L$ as explained below.

We think that 1/n is a unit of grain of sand and $h_t(\mathbf{z})n$ represents the height of sandpile at site \mathbf{z} measured in this unit. The step (i) simulates a random deposit of a grain of sand. In the step (ii), for each $1 \le \ell \le \tau$, the sites $\mathbf{y} \in A^{\mathbf{x}}_{(\ell)}(h_t)$ are regarded as unstable sites and the process

$$\{\eta_{(\ell)}^{\mathbf{x}}(\mathbf{z})\}_{\mathbf{z}\in\Lambda_L} \to \{\eta_{(\ell)}^{\mathbf{x}}(\mathbf{z}) - \Delta_L(\mathbf{y},\mathbf{z})\}_{\mathbf{z}\in\Lambda_L},\$$

is called a *toppling* of the site \mathbf{y} such that

 $\Delta_L(\mathbf{y}, \mathbf{y})n = h_c n = 2dn + m$ grains of sand drop from the unstable site \mathbf{y}

and

 $|\Delta_L(\mathbf{y}, \mathbf{z})| n = n$ grains of sand land on each nearest-neighbor site $\mathbf{z}, |\mathbf{x} - \mathbf{z}| = 1$.

Since there are 2*d* nearest-neighbor sites of each site, *m* grains are annihilated in a toppling. (See Fig.1.) The total number of grains on Λ_L decreases in each toppling and it guarantees $\tau < \infty$. The configuration space S_L is a set of all stable configurations of sandpiles in which height of sandpile is less than the threshold value h_c at every site; $h(\mathbf{z}) < h_c, \forall \mathbf{z} \in \Lambda_L$. From a stable configuration h_t to another stable configuration $h_{t+1}, \sum_{\ell=1}^{\tau-1} |A_{\ell\ell}^{\mathbf{x}}(h_\ell)|$ topplings occur. Such a series of toppling is called an *avalanche*. (Note that, if $\tau = 1$, toppling does not occur. Even in such a case, we call the transition from h_t to h_{t+1} an avalanche, which is just a random deposit of a grain of sand.) Define

$$T(\mathbf{x}, \mathbf{y}, h) = \sum_{\ell=1}^{\tau(\mathbf{x}, h) - 1} \mathbf{1}(\mathbf{y} \in A_{(\ell)}^{\mathbf{x}}(h)), \quad \mathbf{x}, \mathbf{y} \in \Lambda_L, \quad h \in \mathcal{S}_L.$$
(1.2)

This is the number of topplings at site $\mathbf{y} \in \Lambda_L$ in an avalanche caused by a deposit of a grain of sand at a site $\mathbf{x} \in \Lambda_L$ in the configuration $h \in \mathcal{S}_L$.

We have assumed that $n, m \in \mathbb{N}$ in the above definition of processes. If we set n = 1, m = 0, however, we have a = 0 and $\Delta_L|_{a=0}$ gives the 'rule matrix' of the sandpile model introduced by Bak, Tang and Wiesenfeld (BTW) [2, 3]. The BTW model have been studied on finite lattices with *open boundary conditions* in order to make τ be finite. For example, the BTW model is considered on a box B_L . The boundary of box B_L is given by $\partial B_L = \{\mathbf{y} = (y_1, \dots, y_d) \in B_L :$ $1 \leq \exists i \leq d \text{ s.t. } y_i = -L \text{ or } L\}$. In the BTW model defined on $B_L, \sum_{\mathbf{z}:\mathbf{z}\in\Lambda_L} \Delta_L|_{a=0}(\mathbf{y}, \mathbf{z}) = 0$ if $\mathbf{y} \in B_L \setminus \partial B_L$; that is, the number of grains of sand is conserved in any toppling in the bulk of system. By imposing the open boundary condition, we have $\sum_{\mathbf{z}:\mathbf{z}\in\Lambda_L} \Delta_L|_{a=0}(\mathbf{y}, \mathbf{z}) > 0$ for $\mathbf{y} \in \partial B_L$ and dissipation of grains of sand can occur in topplings at the boundary sites. In the present model, in every toppling at any site $\mathbf{y} \in \Lambda_L$, $\sum_{\mathbf{z}:\mathbf{z}\in\Lambda_L} \Delta_L(\mathbf{y}, \mathbf{z})n = m$ grains of sand dissipate from the system and hence $\tau < \infty$ is guaranteed in the shift-invariant system. The quantity *a* indicates the rate of dissipation in a toppling.

The present process belongs to the class of *abelian sandpile models* (ASM) studied by Dhar [6]. We define the operators $\{a(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L}$ following Dhar by

$$h_{t+1} = \mathsf{a}(\mathbf{x})h_t, \quad \mathbf{x} \in \Lambda_L,$$

where $h_t, h_{t+1} \in S_L$ and the site **x** is the chosen site in the first step (i) of the algorithm at time t. That is, $\mathbf{a}(\mathbf{x})$ represents an avalanche caused by a deposit of a grain of sand at **x**. Then the above algorithm guarantees the *abelian property* of avalanches (see Lemma 2.1 in Section 2.1)

$$[\mathbf{a}(\mathbf{x}), \mathbf{a}(\mathbf{y})] \equiv \mathbf{a}(\mathbf{x})\mathbf{a}(\mathbf{y}) - \mathbf{a}(\mathbf{y})\mathbf{a}(\mathbf{x}) = 0, \quad \forall \mathbf{x}, \mathbf{y} \in \Lambda_L.$$
(1.3)

We call the present Markov process the d-dimensional dissipative abelian sandpile model (DASM for short). The two-dimensional case was studied numerically [10] and analytically [30, 28, 18]. In the present paper, we will discuss the models in general dimensions $d \ge 2$ in finite and infinite lattices. See also [29]. As shown in [17, 26, 16] the DASM is useful to construct the infinite-volume limit of avalanche models. Importance of the abelian sandpile models in the extensive study of *self-organized criticality* in the statistical mechanics and related fields is discussed in [25].

2 Basic Properties of Dissipative Abelian Sandpile Model

2.1 Abelian property

First we prove the abelian property of avalanches (1.3).

Lemma 2.1 (Dhar [6]) Assume that the avalanche operators $\{a(x)\}_{x \in \Lambda_L}$ act on S_L . Then

$$[\mathsf{a}(\mathbf{x}),\mathsf{a}(\mathbf{y})] = 0, \quad \forall \mathbf{x}, \mathbf{y} \in \Lambda_L,$$

Proof. Let $\mathcal{X}_L = \mathbb{Z}^{\Lambda_L}$. Define three sets of maps from \mathcal{X}_L to \mathcal{X}_L ; $\{\tilde{\mathbf{t}}(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L}$, $\{\mathbf{t}(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L}$ and $\{\mathbf{d}(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L}$ as follows. For $\mathbf{x}\in\Lambda_L$ and $\eta = \{\eta(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L} \in \mathcal{X}_L$ define

$$\begin{split} \tilde{\mathfrak{t}}(\mathbf{x})\eta(\mathbf{z}) &= \eta(\mathbf{z}) - \Delta_L(\mathbf{x}, \mathbf{z}), \\ \mathfrak{t}(\mathbf{x})\eta(\mathbf{z}) &= \begin{cases} \eta(\mathbf{z}) - \Delta_L(\mathbf{x}, \mathbf{z}), & \text{ if } \eta(\mathbf{x}) \geq h_c, \\ \eta(\mathbf{z}), & \text{ otherwise,} \end{cases} \\ \mathfrak{d}(\mathbf{x})\eta(\mathbf{z}) &= \eta(\mathbf{z}) + \frac{1}{n}\mathbf{1}(\mathbf{z} = \mathbf{x}), \qquad \mathbf{z} \in \Lambda_L. \end{split}$$

By definition of \tilde{t} ,

$$\tilde{\mathfrak{t}}(\mathbf{y})\tilde{\mathfrak{t}}(\mathbf{x})\eta(\mathbf{z}) = \eta(\mathbf{z}) - \Delta_L(\mathbf{x},\mathbf{z}) - \Delta_L(\mathbf{y},\mathbf{z}), \quad \mathbf{z} \in \Lambda_L.$$

Similarly we have

$$\tilde{\mathfrak{t}}(\mathbf{x})\tilde{\mathfrak{t}}(\mathbf{y})\eta(\mathbf{z}) = \eta(\mathbf{z}) - \Delta_L(\mathbf{y},\mathbf{z}) - \Delta_L(\mathbf{x},\mathbf{z}), \quad \mathbf{z} \in \Lambda_L.$$

Therefore $\tilde{t}(\mathbf{y})\tilde{t}(\mathbf{x})\eta = \tilde{t}(\mathbf{x})\tilde{t}(\mathbf{y})\eta, \forall \eta \in \mathcal{X}_L$, that is

$$[\tilde{\mathbf{t}}(\mathbf{x}), \tilde{\mathbf{t}}(\mathbf{y})] = 0, \quad \forall \mathbf{x}, \mathbf{y} \in \Lambda_L.$$
(2.1)

Assume that $\mathbf{y} \neq \mathbf{x}$. Then

$$\tilde{\mathsf{t}}(\mathbf{y})\eta(\mathbf{x}) = \eta(\mathbf{x}) - \Delta(\mathbf{y}, \mathbf{x}) = \begin{cases} \eta(\mathbf{x}) + 1, & \text{if } |\mathbf{x} - \mathbf{y}| = 1, \\ \eta(\mathbf{x}), & \text{if } |\mathbf{x} - \mathbf{y}| > 1. \end{cases}$$

It implies that if $\eta(\mathbf{x}) \ge h_c$ then $\tilde{\mathbf{t}}(\mathbf{y})\eta(\mathbf{x}) \ge h_c$, $\forall \mathbf{y} \ne \mathbf{x}$, that is, any site cannot be stabilized by topplings which occur at other sites. Therefore, the definition of $\mathbf{t}(\mathbf{x})$ and (2.1) give

$$[\mathbf{t}(\mathbf{x}), \mathbf{t}(\mathbf{y})] = 0, \quad \forall \mathbf{x}, \mathbf{y} \in \Lambda_L.$$
(2.2)

It is obvious that

$$[\mathbf{t}(\mathbf{x}), \mathbf{d}(\mathbf{y})] = 0, \quad \forall \mathbf{x}, \mathbf{y} \in \Lambda_L.$$
(2.3)

Consider the situation that $h \in S_L$ and $A_{(\ell)}^{\mathbf{x}}(h) \neq \emptyset$, $1 \leq \ell \leq \tau$. By (2.2), $\prod_{\mathbf{z}:\mathbf{z}\in A_{(\ell)}^{\mathbf{x}}(h)} \mathbf{t}(\mathbf{z})$ is independent of the order of the products of $\mathbf{t}(\mathbf{z})$'s. Then we can write

$$\mathsf{a}(\mathbf{x})h = \left[\prod_{\ell=1}^{\tau-1} \left(\prod_{\mathbf{z}:\mathbf{z}\in A_{(\ell)}^{\mathbf{x}}(h)} \mathsf{t}(\mathbf{z})\right)\right] \mathsf{d}(\mathbf{x})h, \quad \mathbf{x}\in \Lambda_L, \quad h\in \mathcal{S}_L.$$

By (2.2) and (2.3), the lemma is proved. \blacksquare



Figure 2: The set of recurrent configurations \mathcal{R}_L is closed under avalanches.

2.2 Recurrent configurations

Consider a subset of \mathcal{S}_L defined by

$$\mathcal{R}_L = \{ h \in \mathcal{S}_L : \ \forall \mathbf{x} \in \Lambda_L, \exists k(\mathbf{x}) \in \mathbb{N}, \ \text{s.t.} \ (\mathbf{a}(\mathbf{x}))^{k(\mathbf{x})} h = h \},\$$

which is called the set of *recurrent configurations*.

Lemma 2.2 (Dhar [6]) If $h \in \mathcal{R}_L$, then $a(\mathbf{x})h \in \mathcal{R}_L$ for any $\mathbf{x} \in \Lambda_L$. That is, \mathcal{R}_L is closed under avalanches (see Fig.2).

Proof. By definition, if $h \in \mathcal{R}_L$, then for any $\mathbf{y} \in \Lambda_L$, $\exists k(\mathbf{y}) \in \mathbb{N}$, s.t. $(\mathbf{a}(\mathbf{y}))^{k(\mathbf{y})}h = h$. If we operate $\mathbf{a}(\mathbf{x}), \mathbf{x} \in \Lambda_L$ on the both sides of this equation, then we have $\mathbf{a}(\mathbf{x})(\mathbf{a}(\mathbf{y}))^{k(\mathbf{y})}h = \mathbf{a}(\mathbf{x})h$. By Lemma 2.1, LHS= $(\mathbf{a}(\mathbf{y}))^{k(\mathbf{y})}\mathbf{a}(\mathbf{x})h$. This equality implies that $\mathbf{a}(\mathbf{x})h \in \mathcal{R}_L$. Since it is valid for any $\mathbf{x} \in \Lambda_L$, the proof is completed.

Consider a $(2L+1)^d$ -dimensional vector space \mathcal{V}_L , in which the orthonormal basis is given by $\{\mathbf{e}(\mathbf{z})\}_{\mathbf{z}\in\Lambda_L}$. For each configuration $\eta \in \mathcal{X}_L$, we assign a vector

$$\boldsymbol{\eta} = \sum_{\mathbf{z}:\mathbf{z}\in\Lambda_L} \eta(\mathbf{z})\mathbf{e}(\mathbf{z}) = \sum_{\mathbf{z}:\mathbf{z}\in\Lambda_L} n\eta(\mathbf{z})\frac{\mathbf{e}(\mathbf{z})}{n},$$
(2.4)

where 1/n denotes the unit of grain of sand. Assume that $h \in \mathcal{R}_L$; for each $\mathbf{x} \in \Lambda_L$, there is $k(\mathbf{x}) \in \mathbb{N}$ such that

$$(\mathbf{a}(\mathbf{x}))^{k(\mathbf{x})}h = h. \tag{2.5}$$

Consider the vector corresponding to the configuration $(\mathbf{d}(\mathbf{x}))^{k(\mathbf{x})}h$,

$$\boldsymbol{\eta} = \left(h(\mathbf{x}) + \frac{k(\mathbf{x})}{n}\right) \mathbf{e}(\mathbf{x}) + \sum_{\mathbf{z}: \mathbf{z} \neq \mathbf{x}} h(\mathbf{z}) \mathbf{e}(\mathbf{z}) \in \mathcal{V}_L.$$
(2.6)

Then (2.5) claims that there exists a set $\{r(\mathbf{z}) \in \mathbb{N} : \mathbf{z} \in \Lambda_L\}$ such that

$$\mathbf{h} = \boldsymbol{\eta} + \sum_{\mathbf{z}:\mathbf{z}\in\Lambda_L} \left(\sum_{\mathbf{y}:\mathbf{y}\in\Lambda_L} r(\mathbf{y})\Delta_L(\mathbf{y},\mathbf{z}) \right) \mathbf{e}(\mathbf{z}).$$
(2.7)

Note that (2.7) is written as

$$\mathbf{h} = \boldsymbol{\eta} + \sum_{\mathbf{y}: \mathbf{y} \in \Lambda_L} r(\mathbf{y}) \mathbf{v}(\mathbf{y})$$

with

$$\mathbf{v}(\mathbf{x}) = \sum_{\mathbf{z}:\mathbf{z}\in\Lambda_L} \Delta_L(\mathbf{x}, \mathbf{z}) \mathbf{e}(\mathbf{z}), \quad \mathbf{x} \in \Lambda_L.$$
(2.8)

Figure 3: Hypercubic lattice Ω with the basis $\{\mathbf{v}(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L}$ in \mathcal{V}_L . Every avalanche from an unstable configuration $\boldsymbol{\eta}$ given by (2.6) to a recurrent configuration $h \in \mathcal{R}_L$ is represented by a lattice path $\boldsymbol{\eta} \rightsquigarrow \mathbf{h}$ on Ω .

We can say that, given $h \in \mathcal{R}_L$, all points $\{\eta\}$ given by (2.6) are identified with sites of a hypercubic lattice Ω with the basis $\{\mathbf{v}(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L}$ in \mathcal{V}_L . (See Fig.3.) Consider a primitive cell (fundamental domain) of the lattice defined by

$$\mathcal{U}_{L} = \left\{ \sum_{\mathbf{x}:\mathbf{x}\in\Lambda_{L}} \mathbf{c}(\mathbf{x})\mathbf{v}(\mathbf{x}): 0 \le \mathbf{c}(\mathbf{x}) < 1, \mathbf{x}\in\Lambda_{L} \right\} \subset \mathcal{V}_{L}.$$
(2.9)

By definition, the intersection of the lattice Ω and \mathcal{U}_L is a singleton, say \mathbf{p} . We assume that the origin of this lattice is given by \mathbf{p} and express the lattice by $\Omega^{\mathbf{p}}$. We consider a collection of all lattices with the same basis (2.8) having distinct origin in \mathcal{U}_L , $\{\Omega^{\mathbf{p}} : \mathbf{p} \in \mathcal{U}_L\}$. Then there establishes a bijection between $\mathcal{R}_L = \{h\}$ and $\{\Omega^{\mathbf{p}} : \mathbf{p} \in \mathcal{U}_L\}$.

Lemma 2.3 (Dhar [6]) The number of recurrent configuration is given by

$$\mathcal{R}_L| = n^{(2L+1)^d} \det \Delta_L.$$

Proof. The above bijection implies $|\mathcal{R}_L| = |\{\Omega^{\mathbf{p}} : \mathbf{p} \in \mathcal{U}_L\}|$. Since the unit of grain of sand is 1/n, the origins $\{\mathbf{p}\}$ of lattices $\{\Omega^{\mathbf{p}}\}$ should be in $(\mathbb{Z}/n)^{\Lambda_L}$, and hence (see Fig.4)

$$\left|\{\Omega^{\mathbf{p}}:\mathbf{p}\in\mathcal{U}_{L}\}\right|=\left|\mathcal{U}_{L}\cap(\mathbb{Z}/n)^{\Lambda_{L}}\right|=n^{(2L+1)^{d}}\times(\text{the volume of }\mathcal{U}_{L}).$$

The volume of \mathcal{U}_L given by (2.9) with (2.8) is det Δ_L and the proof is completed.



Figure 4: A primitive cell of Ω on the lattice $(\mathbb{Z}/n)^{\Lambda_L}$. Since the unit of grain of sand is 1/n, the origin **p** of lattice Ω should be at a site of $(\mathbb{Z}/n)^{\Lambda_L}$.

2.3 Stationary distribution

For $h \in \mathcal{R}_L$, let \mathbb{P}^h_L be the probability law of the DASM starting from the configuration $h_0 = h$.

Definition 2.4 If we restrict $\{a(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L}$ to \mathcal{R}_L , inverse of the avalanche operator can be defined by

$$\mathsf{a}(\mathbf{x})^{-1} = \mathsf{a}(\mathbf{x})^{k(\mathbf{x})-1}, \quad \mathbf{x} \in \Lambda_L.$$

Assume that $h \in \mathcal{R}_L$ is given. Define

$$\begin{split} \mu_t(X) &= \mathbb{P}^h(h_t = X), \\ W(X \to Y) &= \mathbb{P}^h(h_{t+1} = Y | h_t = X), \quad t \in \mathbb{N}_0, \quad X, Y \in \mathcal{R}_L \end{split}$$

Consider the Master equation

$$\mu_{t+1}(X) = \mu_t(X) - \sum_{Y:Y \in \mathcal{R}_L} \mu_t(X) W(X \to Y) + \sum_{Y:Y \in \mathcal{R}_L} \mu_t(Y) W(Y \to X),$$

where we have used the assumption that $h_0 = h \in \mathcal{R}_L$ and Lemma 2.2. By definition of the DASM, we can find that, for $X, Y \in \mathcal{R}_L$,

$$\begin{split} W(X \to Y) &= \sum_{\mathbf{x}:\mathbf{x} \in \Lambda_L} \operatorname{Prob}(\mathbf{x} \text{ is chosen}) \mathbf{1}(\mathsf{a}(\mathbf{x})X = Y) \\ &= \frac{1}{|\Lambda_L|} \sum_{\mathbf{x}:\mathbf{x} \in \Lambda_L} \mathbf{1}(\mathsf{a}(\mathbf{x})X = Y) \\ &= \frac{1}{(2L+1)^d} \sum_{\mathbf{x}:\mathbf{x} \in \Lambda_L} \mathbf{1}(X = \mathsf{a}^{-1}(\mathbf{x})Y). \end{split}$$

Then we have

$$\mu_{t+1}(X) - \mu_t(X) = \frac{1}{(2L+1)^d} \sum_{\mathbf{x}:\mathbf{x}\in\Lambda_L} \{\mu_t(\mathbf{a}(\mathbf{x})^{-1}X) - \mu_t(X)\}, \quad \forall X \in \mathcal{R}_L.$$

It implies that the uniform measure on \mathcal{R}_L ,

$$\mu(X) = \frac{1}{|\mathcal{R}_L|} \mathbf{1}(X \in \mathcal{R}_L) = \frac{1}{n^{(2L+1)^d} \det \Delta_L} \mathbf{1}(X \in \mathcal{R}_L), \quad X \in \mathcal{X}_L$$

is a stationary distribution of the process.

Lemma 2.5 The DASM on Λ_L is irreducible on \mathcal{R}_L .

Proof. Consider the configuration $\overline{h} \in S_L$, such that $\overline{h}(\mathbf{x}) = h_c - 1/n, \forall \mathbf{x} \in \Lambda_L$. Now we take two arbitrary configurations X and Y from \mathcal{R}_L . We have

$$\overline{h} = \prod_{\mathbf{x}: X(\mathbf{x}) < h_c - 1/n} (\mathbf{a}(\mathbf{x}))^{h_c - 1/n - X(\mathbf{x})} X = \prod_{\mathbf{x}: Y(\mathbf{x}) < h_c - 1/n} (\mathbf{a}(\mathbf{x}))^{h_c - 1/n - Y(\mathbf{x})} Y.$$
(2.10)

Since this means that the configuration \overline{h} is reachable form X and Y by avalanches, Lemma 2.2 guarantees that $\overline{h} \in \mathcal{R}_L$. Since we have assumed that $Y \in \mathcal{R}_L$, $(\mathfrak{a}(\mathbf{x}))^{k(\mathbf{x})}Y = Y$ with some $k(\mathbf{x}) \in \mathbb{N}$ for any $\mathbf{x} \in \Lambda_L$. Therefore, the second equality of (2.10) gives (see Definition 2.4)

$$Y = \prod_{\mathbf{x}:Y(\mathbf{x}) < h_c - 1/n} (\mathbf{a}(\mathbf{x}))^{k(\mathbf{x}) - (h_c - 1/n - Y(\mathbf{x}))} \overline{h}.$$
 (2.11)

Combining (2.10) and (2.11) gives

$$Y = \prod_{\mathbf{x}:Y(\mathbf{x}) < h_{c}-1/n} (\mathsf{a}(\mathbf{x}))^{k(\mathbf{x})-(h_{c}-1/n-Y(\mathbf{x}))} \prod_{\mathbf{y}:X(\mathbf{y}) < h_{c}-1/n} (\mathsf{a}(\mathbf{y}))^{h_{c}-1/n-X(\mathbf{y})} X.$$

Let $\sigma = \sum_{\mathbf{x}:Y(\mathbf{x}) < h_c - 1/n} \{k(\mathbf{x}) - (h_c - 1/n - Y(\mathbf{x}))\} + \sum_{\mathbf{x}:X(\mathbf{x}) < h_c - 1/n} \{h_c - 1/n - X(\mathbf{x})\}$. Then we see

$$\mathbb{P}^{h_0}(h_{t+s} = Y | h_t = X) \ge \left(\frac{1}{|\Lambda_L|}\right)^{\sigma} \quad \text{for } s \ge \sigma.$$

Since RHS is strictly positive for finite L, this completes the proof.

Then the following is concluded by the general theory of Markov chains (see, for example, Chapter 6.4 of [12]).

Proposition 2.6 The stationary distribution of the DASM is uniquely given by the uniform measure on \mathcal{R}_L .

We write the probability law of the DASM on Λ_L in the stationary distribution as \mathbf{P}_L and its expectation as \mathbf{E}_L .

2.4 Allowed configurations and spanning trees

Dhar also introduced a subset of S_L called a collection of allowed configurations \mathcal{A}_L [6]. He defined that for $h \in S_L$, if there is a subset $F \subset \Lambda_L$ such that $F \neq \emptyset$ and

$$h(\mathbf{y}) < \sum_{\mathbf{x}: \mathbf{x} \in F, \mathbf{x} \neq y} (-\Delta_L(\mathbf{x}, \mathbf{y})), \quad \forall \mathbf{y} \in F,$$
(2.12)

then $h \in S_L$ has a forbidden subconfiguration (FSC) on F. Then define

$$\mathcal{A}_L = \{h \in \mathcal{S}_L : h \text{ has no FSC}\}.$$

Lemma 2.7 For the DASM on Λ_L ,

 $\mathcal{R}_L \subset \mathcal{A}_L.$

Proof. In the proof of Lemma 2.5 we have shown that $\overline{h} \in \mathcal{R}_L$ and all recurrent stares are reachable from this configuration \overline{h} . We can prove that $\overline{h} \in \mathcal{A}_L$ as follows. We assume that the contrary; there exists a finite nonempty set $F \subset \Lambda_L$ satisfying (2.12). In the DASM, however, for any $\mathbf{y} \in F$, $\overline{h}(\mathbf{y}) = h_c - 1/n = 2d + (m-1)/n \ge 2d \ge \sum_{\mathbf{x}:\mathbf{x}\in F:\mathbf{x}\neq\mathbf{y}}(-\Delta_L(\mathbf{x},\mathbf{y}))$, which contradicts our assumption. Since both \mathcal{R}_L and \mathcal{A}_L include \overline{h} , it is enough to show that \mathcal{A}_L is closed under the process of avalanche to prove the lemma, since we have already proved that \mathcal{R}_L is so in Lemma 2.2. Remark that addition of particles only increases h and such procedure on an allowed configurations cannot create any FSC. Here we assume that there exists an allowed configuration h such that by a single toppling at the site \mathbf{x} it becomes to contain a FSC. Write $h' = \mathbf{t}(\mathbf{x})\mathbf{d}(\mathbf{x})h$, that is,

$$h'(\mathbf{y}) = h(\mathbf{y}) + \frac{1}{n} \mathbf{1}(\mathbf{y} = \mathbf{x}) - \Delta_L(\mathbf{x}, \mathbf{y}), \quad \forall \mathbf{y} \in \Lambda_L.$$
(2.13)

By assumption, there exists $F \neq \emptyset$ such that

$$h'(\mathbf{y}) < \sum_{\mathbf{z}:\mathbf{z}\in F:\mathbf{z}\neq\mathbf{y}} (-\Delta_L(\mathbf{z},\mathbf{y})), \quad \forall \mathbf{y}\in F.$$
 (2.14)

Combining (2.13) and (2.14) gives

$$h(\mathbf{y}) < \sum_{\mathbf{z}: \mathbf{z} \in F, \mathbf{z} \neq \mathbf{y}} (-\Delta_L(\mathbf{z}, \mathbf{y})) + \Delta_L(\mathbf{x}, \mathbf{y}), \quad \forall \mathbf{y} \in F \setminus \{\mathbf{x}\}.$$

Since $\Delta_L(\mathbf{x}, \mathbf{y}) \leq 0$ for $\mathbf{x} \neq \mathbf{y}$, this inequality means that h has a FSC on $F \setminus {\mathbf{x}}$ and this contradicts our assumption that h is allowed. Since any avalanche consists of addition of a particle and a series of topplings, the proof is completed.

Definition 2.8 Given a pair (Λ_L, Δ_L) , let $G_L^{(v)} = \Lambda_L \cup \{\mathbf{r}\}$ with an additional vertex \mathbf{r} (the 'root'), and $G_L^{(e)}$ be the collection of $|\Delta_L(\mathbf{x}, \mathbf{y})|n = n$ edges between $\mathbf{x}, \mathbf{y} \in \Lambda_L, \mathbf{x} \neq \mathbf{y}$, and $\sum_{\mathbf{y}:\mathbf{y}\in\Lambda_L}\Delta(\mathbf{x},\mathbf{y})n = m$ edges between $\mathbf{x} \in \Lambda_L$ and \mathbf{r} . (See Fig.5.) Graph G_L associated to (Λ_L, Δ_L) is defined as

$$G_L = (G_L^{(v)}, G_L^{(e)}).$$

Definition 2.9 We say a graph T on G_L is a spanning tree, if the number of vertices of T is $|G_L^{(v)}| = |\Lambda_L| + 1$, the number of connected components is one, and the number of loops is zero.

Lemma 2.10 Let $\mathcal{T}_L = \{\text{spanning tree on } G_L \text{ associated to } (\Lambda_L, \Delta_L) \}$. Then

$$|\mathcal{T}_L| = n^{(2L+1)^d} \det \Delta_L.$$

Proof. See p.133 of [20] and Theorem 6.3 in [4].

Lemma 2.11 (Majumdar and Dhar [20]) There establishes a bijection between A_L and T_L .



Figure 5: A part of the graph $G_L = (G_L^{(v)}, G_L^{(e)})$ associated to the DASM (Λ_L, Δ_L) is illustrated for the case that d = 2, n = 2 and m = 1. In this case, each pair of the nearest-neighbor vertices are connected by n = 2 edges and each vertex is connected to the 'root' **r** by m = 1 edge.

Proof. First we order all edges incident on each site $\mathbf{x} \in G_L^{(v)}$ in some order of preference. For each configuration $h \in \mathcal{A}_L$, we consider a following discrete-time growth process of graph on G_L , which is called a *burning process* on (G_L, h) . Let $\tilde{V}_0 = V_0 = \{\mathbf{r}\}, E_0 = \emptyset$ and $T_0 = (V_0, E_0)$. Assume that we have nonempty sets $T_t = (V_t, E_t)$ and \tilde{V}_t with $t \in \mathbb{N}_0$. Let

$$\tilde{V}_{t+1} = \left\{ \mathbf{y} \in G_L^{(v)} \setminus V_t : h(\mathbf{y}) \ge \sum_{\mathbf{x}: \mathbf{x} \in G_L^{(v)} \setminus V_t} (-\Delta_L(\mathbf{x}, \mathbf{y})) \right\}.$$

For each $\mathbf{y} \in \tilde{V}_{t+1}$, consider

$$\tilde{E}_{t+1}(\mathbf{y}) = \left\{ e \in G_L^{(e)} : e \text{ connects } \mathbf{y} \text{ and a site in } \tilde{V}_t \right\}.$$

We must have

$$h(\mathbf{y}) \leq \sum_{\mathbf{x}: \mathbf{x} \in G_L^{(v)} \setminus V_t} (-\Delta_L(\mathbf{x}, \mathbf{y})) + |\tilde{E}_{t+1}(\mathbf{y})|,$$

since $h \in \mathcal{S}_L$. If $|\tilde{E}_{t+1}(\mathbf{y})| = 1$, then name that edge as $e(\mathbf{y})$. If $|\tilde{E}_{t+1}(\mathbf{y})| \ge 2$, then write

$$h(\mathbf{y}) = \sum_{\mathbf{x}:\mathbf{x}\in G_L^{(v)}\setminus V_t} (-\Delta_L(\mathbf{x},\mathbf{y})) + \frac{s}{n},$$

and choose the (s+1)-th edge in $\tilde{E}_{t+1}(\mathbf{y})$ as $e(\mathbf{y})$. We define

$$V_{t+1} = V_t \cup \tilde{V}_{t+1}, \quad E_{t+1} = E_t \cup \{e(\mathbf{y}) : \mathbf{y} \in \tilde{V}_{t+1}\}, \text{ and } T_{t+1} = (V_{t+1}, E_{t+1}).$$

By the assumption $h \in \mathcal{A}_L$, there is a finite time $\sigma < \infty$ such that $V_{\sigma} = G_L^{(v)}$ and $E_{\sigma} = G_L^{(s)}$. By the construction, $T_{\sigma} = (V_{\sigma}, E_{\sigma})$ is a spanning tree on G_L . Since this growth process of $T_t, t \in \{0, 1, \dots, \sigma\}$ is deterministic for a given configuration $h \in \mathcal{A}_L$, it gives an injection from \mathcal{A}_L to \mathcal{T}_L . This fact and Lemma 2.10 give $|\mathcal{A}_L| \leq |\mathcal{T}_L| = n^{(2L+1)^d} \det \Delta_L$. On the other hand, Lemmas 2.3 and 2.7 give $n^{(2L+1)^d} \det \Delta_L \leq |\mathcal{A}_L|$. Then we can conclude $|\mathcal{A}_L| = n^{(2L+1)^d} \det \Delta_L$ and the burning process gives a bijection between \mathcal{A}_L and \mathcal{T}_L .

Combining Lemmas 2.3, 2.7, 2.10, and 2.11, we have the following proposition.

Proposition 2.12 For the DASM on Λ_L , $\mathcal{R}_L = \mathcal{A}_L$.

3 Avalanche Propagators

3.1 Integral expressions for propagators

Define

$$G_L(\mathbf{x}, \mathbf{y}) = \mathbf{E}_L[T(\mathbf{x}, \mathbf{y}, h)], \quad \mathbf{x}, \mathbf{y} \in \Lambda,$$

where $T(\mathbf{x}, \mathbf{y}, h)$ is given by (1.2) and the expectation is taken over configurations $\{h\}$ in the stationary distribution \mathbf{P}_L . $G_L(\mathbf{x}, \mathbf{y})$ is regarded as the *avalanche propagator* from \mathbf{x} to \mathbf{y} [6]. Sometime in an avalanche caused by a deposit of a grain of sand at \mathbf{x} , this site \mathbf{x} topples many times. The set of topplings between the first and the second toppling at \mathbf{x} is called the first wave of toppling. There can occur many waves in one avalanche and $G_L(\mathbf{x}, \mathbf{x})$ gives the average number of waves of topplings in an avalanche [15].

Consider the stationary distribution \mathbf{P}_L of the DASM. For addition of a particle at any site $\mathbf{x} \in \Lambda_L$, the averaged influx of grains of sand into a site $\mathbf{z} \in \Lambda_L$ is given by $\mathbf{1}(\mathbf{z} = \mathbf{x}) + \sum_{\mathbf{y}:\mathbf{y}\neq\mathbf{z}} G_L(\mathbf{x},\mathbf{y})|\Delta_L(\mathbf{y},\mathbf{z})|n$, and the averaged outflux of them out of \mathbf{z} by $G_L(\mathbf{x},\mathbf{z})\Delta_L(\mathbf{z},\mathbf{z})n$ using the avalanche propagators. In \mathbf{P}_L , equivalence between influx and outflux must hold at any site $\mathbf{z} \in \Lambda_L$. This balance equation is written as

$$\sum_{\mathbf{y}:\mathbf{y}\in\Lambda_L}G_L(\mathbf{x},\mathbf{y})\Delta_L(\mathbf{y},\mathbf{z}) = \frac{1}{n}\mathbf{1}(\mathbf{z}=\mathbf{x}) \quad \forall \mathbf{x},\mathbf{z}\in\Lambda_L$$

and thus the propagator is given using the inverse matrix of Δ_L .

Lemma 3.1 (Dhar [6])

$$G_L(\mathbf{x}, \mathbf{y}) = \frac{1}{n} [\Delta_L^{-1}](\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \Lambda_L.$$
(3.1)

The matrix Δ_L can be diagonalized by the Fourier transformation from $\mathbf{x} = (x_1, \dots, x_d)$ to $\mathbf{n} = (n_1, \dots, n_d),$

$$U_L(\mathbf{n}, \mathbf{x}) = U_L^{-1}(\mathbf{x}, \mathbf{n}) = \frac{1}{(2L+1)^{d/2}} \exp\left(\frac{2\pi}{2L+1} \mathbf{x} \cdot \mathbf{n}\right),$$

where $\mathbf{x} \cdot \mathbf{n} = \sum_{i=1}^{d} x_i n_i$, as

$$\sum_{\mathbf{x}:\mathbf{x}\in\Lambda_L} \sum_{\mathbf{y}:\mathbf{y}\in\Lambda_L} U_L(\mathbf{n},\mathbf{x})\Delta_L(\mathbf{x},\mathbf{y})U_L^{-1}(\mathbf{y},\mathbf{m})$$
$$= 2d\left\{ (1+a) - \frac{1}{d}\sum_{i=1}^d \cos\left(\frac{2\pi}{2L+1}n_i\right) \right\} \mathbf{1}(\mathbf{n}=\mathbf{m})$$
$$\equiv \Lambda_L(\mathbf{n},\mathbf{m}), \quad \mathbf{n},\mathbf{m}\in\Lambda_L.$$

Then, (3.1) is obtained as

$$G_{L}(\mathbf{x}, \mathbf{y}) = \frac{1}{n} \sum_{\mathbf{n}:\mathbf{n}\in\Lambda_{L}} \sum_{\mathbf{m}:\mathbf{m}\in\Lambda_{L}} U_{L}^{-1}(\mathbf{x}, \mathbf{n}) [\Delta_{L}^{-1}](\mathbf{n}, \mathbf{m}) U_{L}(\mathbf{m}, \mathbf{y})$$

$$= \frac{1}{2dn} \frac{1}{(2L+1)^{d}} \sum_{\mathbf{n}:\mathbf{n}\in\Lambda_{L}} \frac{e^{-2\pi\sqrt{-1}(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}/(2L+1)}}{(1+a) - (1/d)\sum_{i=1}^{d} \cos(\frac{2\pi}{2L+1}n_{i})} .nonumber \quad (3.2)$$

Lemma 3.2 There exists a limit $G(\mathbf{x} - \mathbf{y}) = \lim_{L \uparrow \infty} G_L(\mathbf{x}, \mathbf{y}), \mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ and

$$G(\mathbf{x}) = \frac{1}{2dn} \prod_{i=1}^{d} \int_{-\pi}^{\pi} \frac{d\theta_i}{2\pi} \frac{\mathrm{e}^{-\sqrt{-1}\mathbf{x}\cdot\boldsymbol{\theta}}}{(1+a) - (1/d)\sum_{i=1}^{d}\cos\theta_i}, \quad \mathbf{x} \in \mathbb{Z}^d.$$
(3.3)

Proof. Consider the Euler-Maclaurin formula for $f \in C^2(\mathbb{R})$,

$$\sum_{n=0}^{M} f(b+nc) = \frac{1}{c} \int_{b}^{b+Mc} f(\theta) d\theta + \frac{1}{2} [f(b) + f(b+Mc)] + \frac{1}{12} c^{2} \sum_{n=0}^{M-1} f^{(2)} (b+c(n+\phi)), \quad (3.4)$$

where $M \in \mathbb{N}$, $b, c \in \mathbb{R}$, $f^{(2)}(\theta)$ is the second derivative of $f(\theta)$, and $0 < \phi < 1$ (see, for instance, Appendix D in [1]). Assume that

$$f(\theta) = \frac{\mathrm{e}^{-\sqrt{-1}\alpha_1\theta}}{(1+a) - (1/d)(\cos\theta + \alpha_2)},$$

where a, α_1, α_2 are constants. Applying the Euler-Maclaurin formula (3.4) with $b = -2\pi L/(2L+1)$, M = 2L and $c = 2\pi/(2L+1)$, we have

$$\begin{split} \sum_{n=0}^{2L} \frac{\mathrm{e}^{-2\pi\sqrt{-1}\alpha_1(n-L)/(2L_1+1)}}{(1+a) - (1/d)\{\cos(\frac{2\pi}{2L+1}(n-L)) + \alpha_2\}} \\ &= (2L+1) \int_{-2\pi L/(2L+1)}^{2\pi L/(2L+1)} \frac{d\theta}{2\pi} \frac{\mathrm{e}^{-\sqrt{-1}\alpha_1\theta}}{(1+a) - (1/d)(\cos\theta + \alpha_2)} \\ &\quad + \frac{1}{2} \left[f\left(-\frac{2\pi L}{2L+1}\right) + f\left(\frac{2\pi L}{2L+1}\right) \right] \\ &\quad + \frac{1}{12} \left(\frac{2\pi}{2L+1}\right)^2 \sum_{n=0}^{2L-1} f^{(2)} \left(\frac{2\pi}{2L+1}(n+\phi-L)\right). \end{split}$$

By dividing the both sides of the equality by 2L + 1 and take the limit $L \uparrow \infty$, we obtain

$$\lim_{L \uparrow \infty} \frac{1}{2L+1} \sum_{n=-L}^{L} \frac{e^{-2\pi\sqrt{-1}\alpha_1 n/(2L_1+1)}}{(1+a) - (1/d) \{\cos(\frac{2\pi}{2L+1}n) + \alpha_2\}}$$
$$= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{e^{-\sqrt{-1}\alpha_1 \theta}}{(1+a) - (1/d)(\cos\theta + \alpha_2)}.$$

Repeating this procedure d times, we can prove Lemma 3.2. \blacksquare

3.2 Long-distance asymptotics

Now we consider the asymptotic form in $|\mathbf{x}| \uparrow \infty$ of $G(\mathbf{x})$. Here we follow the calculation found in Section XII.4 of [21] for the asymptotic expansion of two-point spin correlation function of the two-dimensional Ising model. By using the identity

$$\int_0^\infty ds \mathrm{e}^{-\alpha s} = \frac{1}{\alpha}$$

and the definition of the modified Bessel function of the first kind

$$I_n(z) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{-\sqrt{-1}n\phi + z\cos\phi},$$

we have

$$G(\mathbf{x}) = \frac{1}{2dn} \int_0^\infty ds e^{-(1+a)s} \prod_{i=1}^d I_{x_i}(s/d).$$

The asymptotic expansion of $I_n(z)$ for large n is found on p.86 in [9],

$$I_n(z) = \frac{1}{\sqrt{2\pi}} \frac{\exp\left[(n^2 + z^2)^{1/2} - n\sinh^{-1}(n/z)\right]}{(n^2 + z^2)^{1/4}} \times (1 + \mathcal{O}(1/n)),$$

and we obtain

$$G(\mathbf{x}) = \frac{1}{2dn} \left(\frac{1}{2\pi}\right)^{d/2} \int_0^\infty ds \prod_{i=1}^d \frac{1}{[x_i^2 + (s/d)^2]^{1/4}} \exp[-g(\mathbf{x}, s)] \\ \times \left(1 + \mathcal{O}(\max_i\{1/x_i\})\right),$$
(3.5)

where

$$g(\mathbf{x},s) = (1+a)s - \sum_{i=1}^{d} \left[x_i^2 + \left(\frac{s}{d}\right)^2 \right]^{1/2} + \sum_{i=1}^{d} x_i \sinh^{-1} \left(\frac{d}{s} x_i\right).$$

We can evaluate (3.5) by the saddle-point method and obtain the following result.

Theorem 3.3 Let

$$c_1(d,a) = \frac{1}{4\pi(a+1)} \left[\frac{\sqrt{a(a+2)d}}{2\pi(a+1)} \right]^{(d-3)/2}$$
(3.6)

and

$$\xi(d,a) = \frac{1}{\sqrt{d}\sinh^{-1}\sqrt{a(a+2)}}.$$
(3.7)

Then, for the DASM with $d \ge 2, m, n \in \mathbb{N}, a = m/(2dn)$,

$$\lim_{r \uparrow \infty} -\frac{1}{r} \log \left[\frac{n r^{(d-1)/2}}{c_1(d,a)} G(\mathbf{x}(r)) \right] = \frac{1}{\xi(d,a)},\tag{3.8}$$

where

$$\mathbf{x}(r) = \left(\frac{r}{\sqrt{d}}, \cdots, \frac{r}{\sqrt{d}}\right) \in \mathbb{Z}^d, \quad r > 0.$$
(3.9)

Proof. Let $g^{(1)}(\mathbf{x}, s)$ and $g^{(2)}(\mathbf{x}, s)$ be the first and second derivatives of $g(\mathbf{x}, s)$ with respect to s,

$$g^{(1)}(\mathbf{x},s) = (1+a) - \frac{1}{d} \sum_{i=1}^{d} \left[1 + \left(\frac{d}{s}x_i\right)^2 \right]^{1/2},$$

$$g^{(2)}(\mathbf{x},s) = \frac{d}{s^3} \sum_{i=1}^{d} x_i^2 \left[1 + \left(\frac{d}{s}x_i\right)^2 \right]^{-1/2}.$$

For each \mathbf{x} , let $s_0(\mathbf{x})$ be the saddle point at which $g^{(1)}(\mathbf{x}, s)$ vanishes,

$$g^{(1)}(\mathbf{x}, s_0(\mathbf{x})) = 0.$$
 (3.10)

Then

$$G(\mathbf{x}) = \frac{1}{2dn} \left(\frac{1}{2\pi}\right)^{d/2} \prod_{i=1}^{d} \frac{1}{(x_i^2 + s_0(\mathbf{x})^2/d^2)^{1/4}} \exp[-g(x, s_0(\mathbf{x}))] \\ \times \int_{-\infty}^{\infty} du \exp\left[-\frac{1}{2}g^{(2)}(\mathbf{x}, s_0(\mathbf{x}))u^2\right] \times \left(1 + \mathcal{O}(\max_i\{1/x_i\})\right) \\ = \frac{1}{2dn} \left(\frac{1}{2\pi}\right)^{d/2} \prod_{i=1}^{d} \frac{1}{(x_i^2 + s_0(x)^2/d^2)^{1/4}} \exp[-g(x, s_0(x))] \\ \times \left(\frac{2\pi}{g^{(2)}(\mathbf{x}, s_0(\mathbf{x}))}\right)^{1/2} \times \left(1 + \mathcal{O}(\max_i\{1/x_i\})\right).$$

Here we can prove that the higher derivatives of $g(\mathbf{x}, s)$ only give the contributions of order $\mathcal{O}(\max_i\{1/x_i\})$. See p.304 in [21]. Now we consider the case

$$x_i = \frac{r}{\sqrt{d}} + \varepsilon_i,$$

in which ε_i 's are finite and fixed and $r \gg 1$. The equation (3.10) for the saddle point is now

$$\sum_{i=1}^{d} \left(1 + \frac{d^2}{s_0(\mathbf{x})^2} \left(\frac{r}{\sqrt{d}} + \varepsilon_i \right)^2 \right)^{1/2} = (1+a)d,$$

and it is solved as

$$s_0(\mathbf{x}) = \sqrt{\frac{d}{a(a+2)}} \left(r + \frac{1}{\sqrt{d}} \sum_{i=1}^d \varepsilon_i + \mathcal{O}(1/r) \right).$$

This gives

$$g(\mathbf{x}, s_0(\mathbf{x})) = \sum_{i=1}^d \left(\frac{r}{\sqrt{d}} + \varepsilon_i\right) \sinh^{-1} \left[\frac{d}{s_0(x)} \left(\frac{r}{\sqrt{d}} + \varepsilon_i\right)\right]$$
$$= \sqrt{dr} \sinh^{-1} \sqrt{a(a+2)} + \sinh^{-1} \sqrt{a(a+2)} \times \sum_{i=1}^d \varepsilon_i + \mathcal{O}(1/r)$$

and

$$g^{(2)}(\mathbf{x}, s_0(x)) = \frac{1}{\sqrt{d}} \frac{(a(a+2))^{3/2}}{a+1} \frac{1}{r} + \mathcal{O}(1/r^2).$$

Then we have the estimation

$$G(\mathbf{x}) = \frac{c_1(d,a)}{n} \frac{1}{r^{(d-1)/2}} \exp\left[-\frac{r}{\xi(d,a)} - \lambda(a) \sum_{i=1}^d \varepsilon_i\right] \times \left(1 + \mathcal{O}(1/r)\right), \quad \text{as } r \uparrow \infty$$

for $\mathbf{x} = (r/\sqrt{d} + \varepsilon_1, \cdots, r/\sqrt{d} + \varepsilon_d)$, where $c_1(d, a)$ and $\xi(d, a)$ are given by (3.6) and (3.7), respectively, and

$$\lambda(a) \equiv \frac{\sqrt{d}}{\xi(d,a)}$$

= $\sinh^{-1}\sqrt{a(a+2)}$
= $\log(1+a+\sqrt{a(a+2)}).$ (3.11)

If we put $\varepsilon_i = 0, 1 \leq i \leq d$, then $G(\mathbf{x})$ is reduced to be

$$G(\mathbf{x}(r)) = \overline{G}(r) \times (1 + \mathcal{O}(1/r)), \text{ as } r \uparrow \infty$$

with

$$\bar{G}(r) = \frac{c_1(d,a)}{n} \frac{\mathrm{e}^{-r/\xi(d,a)}}{r^{(d-1)/2}}.$$
(3.12)

It proves the theorem. \blacksquare

4 Height-0 Density and Height-(0,0) Correlations

For

$$\alpha, \beta \in \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, h_{\rm c} - \frac{1}{n}\right\},$$

define

$$P_{\alpha,L}(\mathbf{x}) = \mathbf{E}_L[\mathbf{1}(h(\mathbf{x}) = \alpha)],$$

$$P_{\alpha\beta,L}(\mathbf{x}, \mathbf{y}) = \mathbf{E}_L[\mathbf{1}(h(\mathbf{x}) = \alpha)\mathbf{1}(h(\mathbf{y}) = \beta)], \quad \mathbf{x}, \mathbf{y} \in \Lambda_L.$$
(4.1)

 $P_{\alpha,L}(\mathbf{x})$ is the probability that the site \mathbf{x} has the height αn measured in the unit of grain of sand, 1/n, and $P_{\alpha\beta,L}(\mathbf{x}, \mathbf{y})$ is the (α, β) -height correlation function [19, 5, 23].

For the two-dimensional BTW model on B_L with open boundary condition, Majumdar and Dhar [19] proved the existence of the infinite-volume limits

$$P_0 = \lim_{L\uparrow\infty} P_{0,L}(\mathbf{x}),$$

$$P_{00}(\mathbf{x}(r)) = \lim_{L\uparrow\infty} P_{00,L}(0,\mathbf{x}(r)),$$

where $\mathbf{x}(r) = (r/\sqrt{2}, r/\sqrt{2})$. They gave an 8×8 matrix $M_L(r)$, whose elements depend on L and r, such that

$$P_{00,L}(0,\mathbf{x}(r)) = \det M_L(r), \quad \forall L > \frac{r}{\sqrt{2}},$$

and showed that every elements converge in the infinite-volume limit $L \uparrow \infty$ with a finite r. Then the matrix $M(r) = \lim_{L \uparrow \infty} M_L(r)$ is well-defined and we have the determinantal expression

$$P_{00}(\mathbf{x}(r)) = \det M(r).$$

Moreover, they showed that

$$\lim_{r \uparrow \infty} P_{00}(\mathbf{x}(r)) = P_0^2$$

and

$$C_{00}(\mathbf{x}(r)) \equiv \frac{P_{00}(\mathbf{x}(r)) - P_0^2}{P_0^2} \simeq -\frac{1}{2}r^{-4}, \quad \text{as } r \uparrow \infty.$$
(4.2)

Majumdar and Dhar claimed [19] that the result (4.2) is generalized for the *d*-dimensional BTW model with $d \ge 2$ as

$$C_{00}(\mathbf{x}(r)) \sim r^{-2d}, \quad \text{as } r \uparrow \infty.$$
 (4.3)

In an earlier paper [28], all these facts also hold for the two-dimensional DASM, if we prepare 10×10 matrix $M_L(r)$. (See also [5] and [23] for other generalizations of [19].) Here we show the result for the height-0 density and the height-(0,0) correlations of the DASM with general $d \ge 2$.

4.1 Nearest-neighbor correlations

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First we prove the following Lemma.

Lemma 4.1 Any configuration $h \in S_L$, in which there are two adjacent sites $\mathbf{z}_1, \mathbf{z}_2 \in \Lambda_L$, $|\mathbf{z}_1 - \mathbf{z}_2| = 1$, such that $h(\mathbf{z}_1) < 1$ and $h(\mathbf{z}_2) < 1$, is not allowed.

Proof. Let $F = {\mathbf{z}_1, \mathbf{z}_2} \subset \Lambda_L$. Then

$$\sum_{\mathbf{x}:\mathbf{x}\in F, \mathbf{x}\neq\mathbf{z}_1} (-\Delta_L(\mathbf{x},\mathbf{z}_1)) = -\Delta_L(\mathbf{z}_2,\mathbf{z}_1) = 1,$$

and

$$\sum_{\mathbf{x}\in F, \mathbf{x}\neq\mathbf{z}_2} (-\Delta_L(\mathbf{x}, \mathbf{z}_2)) = -\Delta_L(\mathbf{z}_1, \mathbf{z}_2) = 1,$$

by (1.1). Then if $h(\mathbf{z}_1) < 1$ and $h(\mathbf{z}_2) < 1$, the condition of FSC (2.12) is satisfied.

By Propositions 2.6 and 2.12, the above lemma implies the following.

Proposition 4.2 For any $L \geq 2$,

$$P_{\alpha\beta,L}(0, \pm \mathbf{e}_i) = 0, \quad 1 \le i \le d, \quad \alpha, \beta \in \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, 1 - \frac{1}{n}\right\}.$$

Then,

$$P_{\alpha\beta}(0,\pm\mathbf{e}_i) = \lim_{L\uparrow\infty} P_{\alpha\beta,L}(0,\pm\mathbf{e}_i) = 0, \quad 1 \le i \le d, \quad \alpha,\beta \in \left\{0,\frac{1}{n},\frac{2}{n},\dots,1-\frac{1}{n}\right\}.$$

4.2 Determinatal expressions of $P_{0,L}(0)$ and $P_{00,L}(0, \mathbf{x})$

Let $\mathbf{e}_i, 1 \leq i \leq d$ be the *i*-th unit vector in \mathbb{Z}^d . Define a real symmetric matrix with size $(2L+1)^d$ as

$$B_{L}^{(0)}(\mathbf{v}, \mathbf{w}) = \begin{cases} -h_{c} + 1/n, & \text{if } \mathbf{v} = \mathbf{w} = 0, \\ -1, & \text{if } \mathbf{v} = \mathbf{w}, |\mathbf{v}| = 1, \mathbf{v} \neq -\mathbf{e}_{d}, \\ -1 + 1/n, & \text{if } \mathbf{v} = \mathbf{w} = -\mathbf{e}_{d}, \\ 1, & \text{if } \mathbf{v} = 0, |\mathbf{w}| = 1, \mathbf{w} \neq -\mathbf{e}_{d}, \\ 1 - 1/n, & \text{if } \mathbf{v} = 0, \mathbf{w} = -\mathbf{e}_{d}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathbf{v}, \mathbf{w} \in \Lambda_L$.

Lemma 4.3 Let E_L be the unit matrix with size $(2L+1)^d$. Then

$$P_{0,L}(0) = \det\left(E_L + nG_L B_L^{(0)}\right).$$

Proof. Define a set of allowed configurations conditioned h(0) = 0,

$$\mathcal{A}_{L}^{(0)} = \{ h \in \mathcal{A}_{L} : h(0) = 0 \}.$$

By definition (4.1), Proposition 2.6 with Lemma 2.3 and Proposition 2.12 gives

$$P_{0,L}(0) = \frac{|\mathcal{A}_L^{(0)}|}{n^{(2L+1)^d} \det \Delta_L}.$$
(4.4)

Assume that $h \in \mathcal{A}_L^{(0)}$. Then as shown in the proof of Lemma 2.11 we can uniquely define a burning process $T_t, t \in \{0, 1, \ldots, \exists \sigma\}$ on (G_L, h) associated that T_t becomes a spanning tree on G_L at time $t = \sigma$. Define a configuration h' as

$$h'(\mathbf{z}) = \begin{cases} h(\mathbf{z}) - 1, & \text{if } |\mathbf{z}| = 1, \mathbf{z} \neq -\mathbf{e}_d, \\ h(\mathbf{z}) - 1 + 1/n, & \text{if } \mathbf{z} = -\mathbf{e}_d, \\ h(\mathbf{z}), & \text{otherwise} \end{cases}$$

for $\mathbf{z} \in \Lambda_L$. Now we consider a new DASM which is defined by the matrix Δ'_L given by

$$\Delta'_L = \Delta_L + B_L^{(0)}, \tag{4.5}$$

and let \mathcal{A}'_L be a set of all allowed configurations of this DASM and G'_L be an associated graph to (Λ_L, Δ'_L) . Then we consider a burning process $T'_t = (V'_t, E'_t), t \in \{0, 1, \ldots, \sigma\}$ on (G'_L, h') . By definition of Δ'_L and h', we can make

$$V_t = V'_t, \quad \forall t \in \{0, 1, \dots, \sigma\},$$

and T'_{σ} gives a spanning tree on G'_L . By Lemma 2.11, this means $h' \in \mathcal{A}'_L$. Since there is a bijection between h and its associated burning process $T_t, t \in \{0, 1, \ldots, \sigma\}$, we have a bijection

between $\mathcal{A}_L^{(0)}$ and \mathcal{A}_L' . By Lemmas 2.10 and 2.11, $|\mathcal{A}_L^{(0)}| = |\mathcal{A}_L'| = n^{(2L+1)^d} \det \Delta_L'$. Combining (4.4) and (4.5) gives

$$P_{0,L}(0) = \frac{\det \Delta'_L}{\det \Delta_L}$$

= $\det(\Delta_L^{-1}\Delta'_L)$
= $\det(E_L + \Delta_L^{-1}B_L^{(0)}).$

Then we use Lemma 3.1 and the proof is completed. \blacksquare

Next we consider the two-point function $P_{00,L}(0, \mathbf{x})$, where we assume that $2 \leq |\mathbf{x}| < L$. We define a real symmetric matrix with size $(2L+1)^d$ as follows. For $\mathbf{v}, \mathbf{w} \in \Lambda_L$,

$$B_L^{(0,\mathbf{x})}(\mathbf{v},\mathbf{w}) = \begin{cases} -h_c + 1/n, & \text{if } \mathbf{v} = \mathbf{w} = 0 \text{ or if } \mathbf{v} = \mathbf{w} = \mathbf{x}, \\ -1, & \text{if } \mathbf{v} = \mathbf{w}, |\mathbf{v}| = 1, \mathbf{v} \neq -\mathbf{e}_d, \\ & \text{or if } \mathbf{v} = \mathbf{w}, |\mathbf{v} - \mathbf{x}| = 1, \mathbf{v} \neq \mathbf{x} - \mathbf{e}_d, \\ -1 + 1/n, & \text{if } \mathbf{v} = \mathbf{w} = -\mathbf{e}_d, \text{ or if } \mathbf{v} = \mathbf{w} = \mathbf{x} - \mathbf{e}_d, \\ & \text{or if } \mathbf{v} = 0, |\mathbf{w}| = 1, \mathbf{w} \neq -\mathbf{e}_d, \\ & \text{or if } \mathbf{v} = \mathbf{x}, |\mathbf{w} - \mathbf{x}| = 1, \mathbf{w} \neq \mathbf{x} - \mathbf{e}_d, \\ & 1 - 1/n, & \text{if } \mathbf{v} = 0, \mathbf{w} = -\mathbf{e}_d, \\ & \text{or if } \mathbf{v} = \mathbf{x}, \mathbf{w} = \mathbf{x} - \mathbf{e}_d, \\ & 0, & \text{otherwise.} \end{cases}$$

Following the same argument as $P_{0,L}(0)$ we can prove the next lemma. (See Fig.6.)

Lemma 4.4 For $2 \le |\mathbf{x}| < L$,

$$P_{00,L}(0,\mathbf{x}) = \det\left(E_L + nG_L B_L^{(0,\mathbf{x})}\right).$$



Figure 6: The matrix $\Delta_L'' \equiv \Delta_L + B_L^{(0,\mathbf{x})}$ is considered for $P_{00,L}(0,\mathbf{x})$ with $|\mathbf{x}| = r$. In the corresponding graph G_L'' the site 0 (resp. \mathbf{x}) is connected to $-\mathbf{e}_d$ (resp. $\mathbf{x} - \mathbf{e}_d$) by a single edge, but all other edges between 0 (resp. \mathbf{x}) and its nearest-neighbor sites are deleted.

4.3 Infinite-volume limit

Since the number of nonzero elements of $B_L^{(0)}$ (resp. $B_L^{(0,\mathbf{x})}$) is only 6d + 1 (resp. 2(6d + 1)), we can replace the matrix $E_L + nG_LB_L^{(0)}$ (resp. $E_L + nG_LB_L^{(0,\mathbf{x})}$) with size $(2L + 1)^d$ by a matrix with size (2d + 1) (resp. 2(2d + 1)) without changing the value of determinant. Explicit expressions are given as follows.

Let

$$\mathbf{q}_{i} = \begin{cases} 0, & \text{if } i = 1, \\ \mathbf{e}_{i-1}, & \text{if } 2 \le i \le d+1, \\ -\mathbf{e}_{i-d-1}, & \text{if } d+2 \le i \le 2d+1. \end{cases}$$

Define a matrix $\mathcal{G}^{(L)}(\mathbf{x}) = (\mathcal{G}^{(L)}_{ij})_{1 \le i,j \le 2d+1}$ with elements

$$\mathcal{G}_{ij}^{(L)}(\mathbf{x}) = G_L(0, \mathbf{x} + \mathbf{q}_j - \mathbf{q}_i), \quad 1 \le i, j \le 2d + 1.$$

$$(4.6)$$

We also define a real symmetric matrix $\mathcal{B} = (\mathcal{B}_{ij})_{1 \leq i,j \leq 2d+1}$ with elements

$$\mathcal{B}_{ij} = \begin{cases} -h_{\rm c} + 1/n, & \text{if} \quad i = j = 1, \\ -1, & \text{if} \quad 2 \le i = j \le 2d, \\ -1 + 1/n, & \text{if} \quad i = j = 2d + 1, \\ 1, & i = 1, 2 \le j \le 2d, \\ 1 - 1/n, & \text{if} \quad i = 1, j = 2d + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then define $2(2d+1) \times 2(2d+1)$ matrices

$$ilde{\mathcal{G}}^{(L)}(0,\mathbf{x}) = \left(egin{array}{cc} \mathcal{G}^{(L)}(0) & \mathcal{G}^{(L)}(\mathbf{x}) \ {}^t\mathcal{G}^{(L)}(\mathbf{x}) & \mathcal{G}^{(L)}(0) \end{array}
ight), \quad \mathbf{x} \in \Lambda_L,$$

where ${}^{t}\mathcal{G}^{(L)}(\mathbf{x})$ is a transpose of $\mathcal{G}^{(L)}(\mathbf{x})$, and

$$\tilde{\mathcal{B}} = \left(\begin{array}{cc} \mathcal{B} & 0\\ 0 & \mathcal{B} \end{array}\right).$$

We have

$$P_{0,L}(0) = \det\left(E + n\mathcal{G}^{(L)}(0)\mathcal{B}\right)$$
(4.7)

and

$$P_{00,L}(0,\mathbf{x}) = \det\left(E + n\tilde{\mathcal{G}}^{(L)}(0,\mathbf{x})\tilde{\mathcal{B}}\right),\tag{4.8}$$

where E denotes the unit matrix with size 2d + 1 in (4.7) and with size 2(2d + 1) in (4.8), respectively.

It should be remarked that the sizes of the matrices in the RHS's are independent of the lattice size L and determined only by the dimension d of lattice. The dependence of L is introduced only through each elements of $\mathcal{G}^{(L)}(\mathbf{x})$ given by (4.6). Lemma 3.2 guarantees the existence of infinite-volume limit $L \uparrow \infty$ of these elements and we put

$$\begin{aligned} \mathcal{G}_{ij}(\mathbf{x}) &= \lim_{L \uparrow \infty} \mathcal{G}_{ij}^{(L)}(\mathbf{x}) = G(\mathbf{x} + \mathbf{q}_j - \mathbf{q}_i), \quad 1 \le i, j \le 2d + 1 \\ \mathcal{G}(\mathbf{x}) &= (\mathcal{G}_{ij}(\mathbf{x}))_{1 \le i, j \le 2d + 1}, \\ \tilde{\mathcal{G}}(0, \mathbf{x}) &= \lim_{L \uparrow \infty} \tilde{\mathcal{G}}^{(L)}(0, \mathbf{x}) = \begin{pmatrix} \mathcal{G}(0) & \mathcal{G}(\mathbf{x}) \\ {}^t \mathcal{G}(\mathbf{x}) & \mathcal{G}(0) \end{pmatrix}, \end{aligned}$$

where $G(\mathbf{x})$ is explicitly given by (3.3). Then we have the following.

Proposition 4.5 There exist the infinite-volume limits

$$P_0 = \lim_{L \uparrow \infty} P_{0,L}(0), \quad P_{00}(\mathbf{x}) = \lim_{L \uparrow \infty} P_{00,L}(0, \mathbf{x}), \quad \mathbf{x} \in \mathbb{Z}^d,$$

and they are given by

$$P_0 = \det \left(E + n\mathcal{G}(0)\mathcal{B} \right)$$

and

$$P_{00}(\mathbf{x}) = \det\left(E + n\tilde{\mathcal{G}}(0, \mathbf{x})\tilde{\mathcal{B}}\right), \quad \mathbf{x} \in \mathbb{Z}^d.$$

4.4 Evaluations of determinantal expressions

From the determinantal expressions of P_0 and $P_{00}(\mathbf{x})$ given in Proposition 4.5, the following explicit evaluations of these quantities are obtained.

Theorem 4.6 (i) Define

$$\gamma_1 = \frac{1}{2d} \prod_{i=1}^d \int_{-\pi}^{\pi} \frac{d\theta_i}{2\pi} \frac{1}{(1+a) - (1/d) \sum_{i=1}^d \cos \theta_i}$$

and

$$\gamma_2 = \frac{1}{2d} \prod_{i=1}^d \int_{-\pi}^{\pi} \frac{d\theta_i}{2\pi} \frac{e^{-2\sqrt{-1}(\theta_1 + \theta_2)}}{(1+a) - (1/d) \sum_{i=1}^d \cos \theta_i}.$$

Then, for the DASM with $d \geq 2, m, n \in \mathbb{N}$,

$$P_{0} = \frac{1 - 2da\gamma_{1}}{2dn} \left[2\{1 - d(\gamma_{1} - \gamma_{2})\} + (1 - 4d\gamma_{1})a - 2d\gamma_{1}a^{2} \right] \\ \times \left[2(d-1)(\gamma_{1} - \gamma_{2}) - (1 - 4d\gamma_{1})a + 2d\gamma_{1}a^{2} \right]^{2} \\ \times \left[\{1 - (\gamma_{1} - \gamma_{2})\}^{2} - \{(2d(1+a)^{2} - 1)\gamma_{1} - (2d-1)\gamma_{2} - (1+a)\}^{2} \right]^{d-2}, (4.9)$$

where a = m/(2dn).

(ii) Let

$$C_{00}(\mathbf{x}) = \frac{P_{00}(\mathbf{x}) - P_0^2}{P_0^2}, \quad \mathbf{x} \in \mathbb{Z}^d.$$
(4.10)

Then, there exists a nonzero factor $c_2(d, a, n)$ such that for the DASM with $d \ge 2, m, n \in \mathbb{N}$

$$\lim_{r \uparrow \infty} -\frac{1}{r} \log \left[\frac{r^{d-1}}{c_2(d,a,n)} C_{00}(\mathbf{x}(r)) \right] = \frac{2}{\xi(d,a)},\tag{4.11}$$

where a = m/(2dn), $\xi(d, a)$ and $\mathbf{x}(r)$ are given by (3.7) and (3.9), respectively, and that

$$\lim_{a \downarrow 0} \frac{c_2(d, a, m/(2da))}{a^{(d+1)/2}} = \left(\frac{d}{2\pi^2}\right)^{(d-3)/2} \left[\frac{d\{1 + (d-1)\bar{\gamma}\}}{2\pi(d-1)\bar{\gamma}}\right]^2,\tag{4.12}$$

where

$$\bar{\gamma} = \frac{1}{2d} \prod_{i=1}^{d} \int_{-\pi}^{\pi} \frac{d\theta_i}{2\pi} \frac{1 - e^{-2\sqrt{-1}(\theta_1 + \theta_2)}}{1 - (1/d) \sum_{i=1}^{d} \cos \theta_i}.$$

In the following, we will explain how to prove this theorem. Let

$$M^{(1)}(r) = E + n\tilde{\mathcal{G}}(0, \mathbf{x}(r))\tilde{\mathcal{B}}, \quad r > 0, \quad \mathbf{x}(r) \in \mathbb{Z}^d,$$

where E is a unit matrix with size 2(2d+1). That is,

$$M^{(1)}(r) = \begin{pmatrix} m^{(1)} & \tilde{m}^{(1)}(r) \\ \hat{m}^{(1)}(r) & m^{(1)} \end{pmatrix},$$

where for $1 \le i \le 2d + 1$

$$m_{ij}^{(1)} = \begin{cases} \mathbf{1}(i=1) + \sum_{k=1}^{2d+1} n \mathcal{G}_{ik}(0) \\ -\{(1-1/n) + h_c\} n \mathcal{G}_{i1}(0) - \mathcal{G}_{i\,2d+1}(0), & \text{if } j = 1, \\ \mathbf{1}(i=j) + n[\mathcal{G}_{i1}(0) - \mathcal{G}_{ij}(0)], & \text{if } 2 \le j \le 2d, \\ \mathbf{1}(i=2d+1) + (1-1/n)n[\mathcal{G}_{i1}(0) - \mathcal{G}_{i\,2d+1}(0)], & \text{if } j = 2d+1, \end{cases}$$

$$\tilde{m}_{ij}^{(1)}(r) = n \times \begin{cases} \sum_{k=1}^{2d+1} \mathcal{G}_{ik}(\mathbf{x}(r)) \\ -\{(1-1/n) + h_c\} \mathcal{G}_{i1}(\mathbf{x}(r)) - (1/n) \mathcal{G}_{i\,2d+1}(\mathbf{x}(r)), & \text{if } j = 1, \\ \mathcal{G}_{i1}(\mathbf{x}(r)) - \mathcal{G}_{ij}(\mathbf{x}(r)), & \text{if } 2 \le j \le 2d, \\ (1-1/n) (\mathcal{G}_{i1}(\mathbf{x}(r)) - \mathcal{G}_{i\,2d+1}(\mathbf{x}(r))), & \text{if } j = 2d+1, \end{cases}$$

$$\hat{m}_{ij}^{(1)}(r) = n \times \begin{cases} \sum_{k=1}^{2d+1} \mathcal{G}_{ki}(\mathbf{x}(r)) \\ -\{(1-1/n) + \eta_c\} \mathcal{G}_{1i}(\mathbf{x}(r)) - (1/n) \mathcal{G}_{2d+1\,i}(\mathbf{x}(r)), & \text{if } j = 1, \\ \mathcal{G}_{1i}(\mathbf{x}(r)) - \mathcal{G}_{ji}(\mathbf{x}(r)), & \text{if } 2 \le j \le 2d, \\ (1-1/n) (\mathcal{G}_{1i}(\mathbf{x}(r)) - \mathcal{G}_{2d+1\,i}(\mathbf{x}(r))), & \text{if } j = 2d+1. \end{cases}$$

We find that

$$\begin{split} m_{i1}^{(1)} + \sum_{j=2}^{2d+1} m_{ij}^{(1)} &= 1 - 2dan\mathcal{G}_{i1}(0), \\ \tilde{m}_{i1}^{(1)}(r) + \sum_{j=2}^{2d+1} \tilde{m}_{ij}^{(1)} &= -2dan\mathcal{G}_{i1}(\mathbf{x}(r)), \\ \hat{m}_{i1}^{(1)}(r) + \sum_{j=2}^{2d+1} \hat{m}_{ij}^{(1)} &= -2dan\mathcal{G}_{1i}(\mathbf{x}(r)), \quad 1 \le i \le 2d+1. \end{split}$$

For $1 \leq i \leq 2d + 1$, let

$$\begin{split} m_{ij} &= \begin{cases} 1 - 2dan\mathcal{G}_{i1}(0), & \text{if } j = 1, \\ m_{ij}^{(1)}, & \text{if } 2 \le j \le 2d + 1, \end{cases} \\ \tilde{m}_{ij}(r) &= \begin{cases} -2dan\mathcal{G}_{i1}(\mathbf{x}(r)), & \text{if } j = 1, \\ \tilde{m}_{ij}^{(1)}(r), & \text{if } 2 \le j \le 2d + 1, \end{cases} \\ \hat{m}_{ij}(r) &= \begin{cases} -2dan\mathcal{G}_{1i}(\mathbf{x}(r)), & \text{if } j = 1, \\ \hat{m}_{ij}^{(1)}(r), & \text{if } 2 \le j \le 2d + 1. \end{cases} \end{split}$$

Then

$$P_{0} = \det m^{(1)} = \det m,$$

$$P_{00}(\mathbf{x}(r)) = \det M^{(1)}(r) = \det M(r) \quad \text{with} \quad M(r) = \begin{pmatrix} m & \tilde{m}(r) \\ \hat{m}(r) & m \end{pmatrix}.$$
 (4.13)

Note that, if we introduce the the *dipole potential*

$$\phi_{(i_1,j_1),(i_2,j_2)}(\mathbf{x}(r)) = \mathcal{G}_{i_1j_1}(\mathbf{x}(r)) - \mathcal{G}_{i_2j_2}(\mathbf{x}(r)), \quad 1 \le i_1, i_2, j_1, j_2 \le 2d+1,$$

the elements of the matrix M(r) are expressed as follows; for $1 \le i \le 2d + 1$,

$$m_{ij} = \begin{cases} 1 - 2dan\mathcal{G}_{i1}(0), & \text{if } j = 1, \\ \mathbf{1}(i=j) + n\phi_{(i,1),(i,j)}(0), & \text{if } 2 \le j \le 2d, \\ \mathbf{1}(i=2d+1) + (1-1/n)n\phi_{(i,1),(i,2d+1)}(0), & \text{if } j = 2d+1, \end{cases}$$
(4.14)

$$\tilde{m}_{ij}(r) = n \times \begin{cases} -2da\mathcal{G}_{i1}(\mathbf{x}(r)), & \text{if } j = 1, \\ \phi_{(i,1),(i,j)}(\mathbf{x}(r)), & \text{if } 2 \le j \le 2d, \\ (1 - 1/n)\phi_{(i,1),(i,2d+1)}(\mathbf{x}(r)), & \text{if } j = 2d + 1, \end{cases}$$

$$\hat{m}_{ij}(r) = n \times \begin{cases} -2da\mathcal{G}_{1i}(\mathbf{x}(r)), & \text{if } j = 1, \\ \phi_{(1,i),(j,i)}(\mathbf{x}(r)), & \text{if } 2 \le j \le 2d, \\ (1 - 1/n)\phi_{(1,i),(2d+1,i)}(\mathbf{x}(r)), & \text{if } j = 2d + 1. \end{cases}$$

Now we study the asymptotics of $P_{00}(r)$ in $r \uparrow \infty$. Theorem 3.3 and its proof given in Section 3 implies that with any finite c_i 's,

$$G\left(\mathbf{x}(r) + \sum_{i=1}^{d} c_i \mathbf{e}_i\right) = \bar{G}(r) \exp\left(-\lambda(a) \sum_{i=1}^{d} c_i\right) \times \left(1 + \mathcal{O}(1/r)\right), \quad \text{as } r \uparrow \infty$$

with (3.6), (3.7), (3.11), and (3.12). Then we see

$$\begin{split} \tilde{m}(r) &= n\bar{G}(r)n(r,\lambda)(1+\mathcal{O}(1/r)), \\ \hat{m}(r) &= n\bar{G}(r)n(r,-\lambda)(1+\mathcal{O}(1/r)), \quad \text{as } r\uparrow\infty, \end{split}$$

where $n(r, \lambda) = (n_{ij}(r, \lambda))_{1 \le i,j \le 2d+1}$ with elements,

$$n_{ij}(r,\lambda) = \begin{cases} -2da, & \text{if } i = j = 1, \\ (1 - e^{-\lambda}), & \text{if } i = 1, 2 \le j \le d+1, \\ (1 - e^{\lambda}), & \text{if } i = 1, d+2 \le j \le 2d, \\ (1 - 1/n)(1 - e^{\lambda}), & \text{if } i = 1, j = 2d+1, \\ -2dae^{\lambda}, & \text{if } 2 \le i \le d+1, j = 1, \\ -2dae^{-\lambda}, & \text{if } d+2 \le i \le 2d+1, j = 1, \\ e^{\lambda}(1 - e^{-\lambda}), & \text{if } 2 \le i \le d+1, d+2 \le j \le 2d, \\ (1 - 1/n)e^{\lambda}(1 - e^{\lambda}), & \text{if } 2 \le i \le d+1, d+2 \le j \le 2d, \\ (1 - 1/n)e^{\lambda}(1 - e^{\lambda}), & \text{if } 2 \le i \le d+1, j = 2d+1, \\ e^{-\lambda}(1 - e^{-\lambda}), & \text{if } d+2 \le i \le 2d+1, 2 \le j \le d+1, \\ e^{-\lambda}(1 - e^{\lambda}), & \text{if } d+2 \le i \le 2d+1, 2 \le j \le d+1, \\ (1 - 1/n)e^{-\lambda}(1 - e^{\lambda}), & \text{if } d+2 \le i \le 2d+1, j = 2d+1. \end{cases}$$

We obtain a matrix M'(r) from M(r) by subtracting (the first row) $\times e^{\lambda}$ from the *i*-th row with $2 \leq i \leq d+1$, (the first row) $\times e^{-\lambda}$ from the *i*-th row with $d+2 \leq i \leq 2d+1$, (the (2d+2)-th row) $\times e^{-\lambda}$ from the *i*-th row with $2d+3 \leq i \leq 3d+2$, and (the (2d+2)-th row) $\times e^{\lambda}$ from the *i*-th row with $3d+3 \leq i \leq 2(2d+1)$). We have

$$M'(r) = \begin{pmatrix} m'(\lambda) & \tilde{m}'(r,\lambda) \\ \tilde{m}'(r,-\lambda) & m'(-\lambda) \end{pmatrix}$$

with

$$m'_{ij}(\lambda) = \begin{cases} 1 - 2dan\mathcal{G}_{11}(0), & \text{if } i = j = 1, \\ n\phi_{(1,1),(1,j)}(0), & \text{if } i = 1, 2 \le j \le 2d, \\ (1 - 1/n)n\phi_{(1,1),(1,2d+1)}(0), & \text{if } i = 1, j = 2d + 1, \\ (1 - e^{\lambda}) - 2dan(\mathcal{G}_{i1}(0) - e^{\lambda}\mathcal{G}_{11}(0)), & \text{if } 2 \le i \le d + 1, j = 1, \\ (1 - e^{-\lambda}) - 2dan(\mathcal{G}_{i1}(0) - e^{-\lambda}\mathcal{G}_{11}(0)), & \text{if } d + 2 \le i \le 2d + 1, j = 1, \\ 1(i = j) + n[\phi_{(i,1),(i,j)}(0) - e^{\lambda}\phi_{(1,1),(1,j)}(0)], & \text{if } 2 \le i \le d + 1, 2 \le j \le 2d, \\ 1(i = j) + n[\phi_{(i,1),(i,j)}(0) - e^{-\lambda}\phi_{(1,1),(1,j)}(0)], & \text{if } d + 2 \le i \le 2d + 1, \\ 2 \le j \le 2d, \\ (1 - 1/n) \\ \times n[\phi_{(i,1),(i,2d+1)}(0) - e^{\lambda}\phi_{(1,1),(1,2d+1)}(0)], & \text{if } 2 \le i \le d + 1, j = 2d + 1, \\ 1(i = 2d + 1) + (1 - 1/n) \\ \times n[\phi_{(i,1),(i,2d+1)}(0) - e^{-\lambda}\phi_{(1,1),(1,2d+1)}(0)], & \text{if } d + 2 \le i \le 2d + 1, \\ j = 2d + 1, \end{cases}$$

$$(4.15)$$

and with

$$\tilde{m}'_{ij}(r,\lambda) = n\bar{G}(r) \times \begin{cases} -2da(1+\mathcal{O}(1/r)), & \text{if } i=j=1, \\ (1-\mathrm{e}^{-\lambda})(1+\mathcal{O}(1/r)), & \text{if } i=1, 2\leq j\leq d+1, \\ (1-\mathrm{e}^{\lambda})(1+\mathcal{O}(1/r)), & \text{if } i=1, d+2\leq j\leq 2d, \\ (1-1/n)(1-\mathrm{e}^{\lambda})(1+\mathcal{O}(1/r)), & \text{if } i=1, j=2d+1, \\ \mathcal{O}(1/r), & \text{otherwise,} \end{cases}$$
(4.16)

so that

$$P_0(\mathbf{x}(r)) = \det M(r) = \det M'(r), \quad r > 0, \quad \mathbf{x}(r) \in \mathbb{Z}^d.$$

Now we expand det M'(r) along the first and the (2d + 2)-th rows. Let |M'(j,k)| be the determinant of M'(r) with the first and the (2d + 2)-th rows and the *j*-th and the *k*-th columns removed and multiplied by $-(-1)^{1+j} \times (-1)^{2d+2+k} = (-1)^{j+k}$. Then we have

$$\det M'(r) = \sum_{j=1}^{2(2d+1)} \sum_{k=1, k \neq j}^{2(2d+1)} M'(r)_{1j} M'(r)_{2d+2,k} |M'(j,k)|.$$

Remark that, by (4.15) and (4.16),

$$|M'(j,k)| = \mathcal{O}(1/r), \text{ as } r \to \infty,$$

if $1\leq j,k\leq 2d+1$ or $2d+2\leq j,k\leq 2(2d+1),$ and

$$|M'(j,k)| = |m'^{(j)}(\lambda)| \times |m'^{(k)}(\lambda)| \times (1 + \mathcal{O}(1/r)), \quad \text{as } r \to \infty,$$

if $1 \le j \le 2d + 1 < k \le 2(2d + 1)$ or $1 \le k \le 2d + 1 < j \le 2(2d + 1)$, where $|m'^{(j)}(\lambda)|$ is the (1, j)-cofactor of $m'(\lambda)$. Then

$$\det M'(r) = \left(\sum_{j=1}^{2d+1} m'_{1j}(\lambda) |m'^{(j)}(\lambda)|\right) \left(\sum_{j=1}^{2d+1} m'_{1j}(-\lambda) |m'^{(j)}(-\lambda)|\right) + \left(\sum_{j=1}^{2d+1} \tilde{m}'_{1j}(r,\lambda) |m'^{(j)}(-\lambda)|\right) \left(\sum_{j=1}^{2d+1} \tilde{m}'_{1j}(r,-\lambda) |m'^{(j)}(-\lambda)|\right) = \det m'(\lambda) \times \det m'(-\lambda) + \det \bar{m}(\lambda) \times \det \bar{m}(-\lambda) \times \left(n\bar{G}(r)\right)^2 (1 + \mathcal{O}(1/r)), (4.17)$$

where $\bar{m}(\lambda) = (\bar{m}_{ij}(\lambda))_{1 \le i,j \le 2d+1}$ with elements

$$\bar{m}_{ij}(\lambda) = \begin{cases} -2da, & \text{if } i = j = 1, \\ 1 - e^{\lambda}, & \text{if } i = 1, 2 \le j \le d+1, \\ 1 - e^{-\lambda}, & \text{if } i = 1, d+2 \le j \le 2d, \\ (1 - 1/n)(1 - e^{-\lambda}), & \text{if } i = 1, j = 2d+1, \\ m'_{ij}(\lambda), & \text{otherwise.} \end{cases}$$

We find that

$$\det m'(\lambda) = \det m'(-\lambda) = \det m. \tag{4.18}$$

The determinantal expressions (4.13) with (3.12), (4.17), and (4.18) give

$$\lim_{r \uparrow \infty} P_{00}(\mathbf{x}(r)) = \lim_{r \uparrow \infty} \{ (\det m)^2 + \det \bar{m}(\lambda) \det \tilde{m}(-\lambda) (n\bar{G}(r))^2 \}$$
$$= (\det m)^2 = P_0^2.$$

Here we set

$$\det \bar{m}(\lambda) = a \det m^*(\lambda),$$

with a matrix $m^*(\lambda) = (m^*_{ij}(\lambda))_{1 \leq i,j \leq 2d+1}$ with elements

$$m_{ij}^{*}(\lambda) = \begin{cases} -2d, & \text{if } i = j = 1, \\ (1 - e^{\lambda})/a^{1/2}, & \text{if } i = 1, 2 \le j \le d + 1, \\ (1 - e^{-\lambda})/a^{1/2}, & \text{if } i = 1, d + 2 \le j \le 2d, \\ (1 - 1/n)(1 - e^{-\lambda})/a^{1/2}, & \text{if } i = 1, j = 2d + 1, \\ (1 - e^{\lambda})/a^{1/2} & & \\ -2da^{1/2}n(\mathcal{G}_{i1}(0) - e^{\lambda}\mathcal{G}_{11}(0)), & \text{if } 2 \le i \le d + 1, j = 1, \\ (1 - e^{-\lambda})/a^{1/2} & & \\ -2da^{1/2}n(\mathcal{G}_{i1}(0) - e^{-\lambda}\mathcal{G}_{11}(0)), & \text{if } d + 2 \le i \le 2d + 1, j = 1, \\ (1 - e^{-\lambda})/a^{1/2} & & \\ 1(i = j) + n[\phi_{(i,1),(i,j)}(0) - e^{\lambda}\phi_{(1,1),(1,j)}(0)], & \text{if } 2 \le i \le d + 1, 2 \le j \le 2d, \\ 1(i = j) + n[\phi_{(i,1),(i,j)}(0) - e^{-\lambda}\phi_{(1,1),(1,j)}(0)], & \text{if } d + 2 \le i \le 2d + 1, \\ & 2 \le j \le 2d, \\ (1 - 1/n) & & \\ \times n[\phi_{(i,1),(i,2d+1)}(0) - e^{-\lambda}\phi_{(1,1),(1,2d+1)}(0)], & \text{if } 2 \le i \le d + 1, j = 2d + 1, \\ 1(i = 2d + 1) + (1 - 1/n), & & \\ \times n[\phi_{(i,1),(i,2d+1)}(0) - e^{-\lambda}\phi_{(1,1),(1,2d+1)}(0)], & \text{if } d + 2 \le i \le 2d + 1, \\ & j = 2d + 1. \end{cases}$$

By the definition (4.10), we see

$$C_{00}(\mathbf{x}(r)) = a^2 \frac{\det m^*(\lambda) \det m^*(-\lambda)}{(\det m)^2} (n\bar{G}(r))^2 \times (1 + \mathcal{O}(1/r)), \quad \text{as } r \uparrow \infty.$$

Since $\overline{G}(r)$ is given by (3.12), (4.11) of Theorem 4.6 (ii) is proved with

$$c_2(d, a, n) = (ac_1(d, a))^2 \frac{\det m^*(\lambda) \times \det m^*(-\lambda)}{(\det m)^2}.$$

Now the problem is reduced to the calculation of det m and det $m^*(\lambda)$. Consider a matrix $R = (R_{ij})_{1 \le i,j \le N}$ with elements

$$R_{ij} = \begin{cases} u, & \text{if } i = j = 1, \\ b, & \text{if } i = 1, 2 \le j \le d + 1, \\ c, & \text{if } i = 1, d + 2 \le j \le 2d, \\ (1 - 1/n)c, & \text{if } i = 1, j = 2d + 1, \\ q, & \text{if } 2 \le i \le d + 1, j = 1, \\ e, & \text{if } d + 2 \le i \le 2d + 1, j = 1, \\ f, & \text{if } 2 \le i \le d + 1, 2 \le j \le 2d, j \ne i, j \ne i + d, \\ 1 + v, & \text{if } 2 \le i \le d + 1, 2 \le j \le 2d, j \ne i, j \ne i + d, \\ h, & \text{if } 2 \le i \le d, j = i + d, \\ (1 - 1/n)f, & \text{if } 2 \le i \le d, j = 2d + 1, \\ (1 - 1/n)h, & \text{if } i = d + 1, j = 2d + 1, \\ s, & \text{if } d + 2 \le i \le 2d + 1, 2 \le j \le 2d, j \ne i, j \ne i - d, \\ t, & \text{if } d + 2 \le i \le 2d + 1, j = i - d, \\ 1 + k, & \text{if } d + 2 \le i \le 2d, j = 2d + 1, \\ 1 + (1 - 1/n)k, & \text{if } i = j = 2d + 1. \end{cases}$$
(4.20)

We perform the following procedure on R.

- (i) Subtract (the first row) $\times q/u$ from the *i*-th row with $2 \le i \le d+1$.
- (ii) Subtract (the first row) $\times e/u$ from the *i*-th row with $d+2 \le i \le 2d+1$.
- (iii) Subtract the second row from the *i*-th row with $3 \le i \le d+1$.
- (iv) Subtract the (d+2)-th row from the *i*-th row with $d+3 \le i \le 2d+1$.
- (v) Add the *j*-th column to the second column with $3 \le j \le d+1$.
- (vi) Add the *j*-th column to the (d+2)-th column with $d+3 \le j \le 2d$.
- (vii) Add (the (2d + 1)-th column) $\times 1/(1 1/n)$ to the (d + 2)-th column.
- (viii) Subtract (the (d+j)-th column) $\times (t-s)/(1+k-s)$ from the j-th column with $3\leq j\leq d.$

After these procedures, by changing the orders of rows and columns appropriately, we obtain the following identity.

$$\det R = u \times \left[1 + v - f - \frac{t - s}{1 + k - s} (h - f) \right]^{d-2} \times (1 + k - s)^{d-2} \times \det S,$$
(4.21)

where $S = (S_i j)_{1 \le i,j \le 4}$ with elements

$$\begin{array}{ll} S_{11}=1+v+(d-1)f-dbq/u, & S_{12}=h+(d-1)f-dcq/u, \\ S_{13}=(1-1/n)(f-cq/u), & S_{14}=f-bq/u, \\ S_{21}=t+(d-1)s-dbe/u, & S_{22}=1+k+(d-1)s-dce/u, \\ S_{23}=(1-1/n)(s-ce/u), & S_{24}=s-be/u, \\ S_{31}=0, & S_{32}=1/(n-1), \\ S_{33}=1+(1-1/n)(k-s), & S_{34}=t-s, \\ S_{41}=0, & S_{42}=0, \\ S_{43}=(1-1/n)(h-f), & S_{44}=1+v-f. \end{array}$$

Define

$$g_0 = nG(0), \quad g_1 = nG(\mathbf{e}_1), \quad g_2 = nG(2\mathbf{e}_1), \quad g_3 = nG(\mathbf{e}_1 + \mathbf{e}_2),$$

where $G(\mathbf{x})$ is given by (3.3) and $\mathbf{e}_1, \mathbf{e}_2$ are the unit vectors in the first and second directions in \mathbb{Z}^d . Since the system is isotropic, we can find that the matrix m defined by (4.14) is in the form (4.20) with

$$u = 1 - 2dag_0, \qquad b = c = g_0 - g_1, q = e = 1 - 2dag_1, \qquad f = s = g_1 - g_3, v = k = g_1 - g_0, \qquad h = t = g_1 - g_2.$$
(4.22)

By Lemma 3.2 and the isotropy of the system gives

$$2d(1+a)g_0 - 2dg_1 = 1,$$

$$2d(1+a)g_1 - (g_0 + g_2 + 2(d-1)g_3) = 0,$$

which are written as

$$g_1 = (1+a)g_0 - \frac{1}{2d},$$

$$g_2 = [2d(1+a)^2 - 1]g_0 - 2(d-1)g_3 - (1+a).$$
(4.23)

The formula (4.21) with (4.22) and (4.23) gives

$$P_{0} = \det m = \frac{1 - 2dag_{0}}{2dn} [2\{1 - d(g_{0} - g_{3})\} + (1 - 4dg_{0})a - 2dg_{0}a] \\ \times [2(d - 1)(g_{0} - g_{3}) - (1 - 4dg_{0})a + 2dg_{0}a^{2}]^{2} \\ \times [\{1 - (g_{0} - g_{3})\}^{2} - (g_{2} - g_{3})^{2}]^{d-2}.$$

$$(4.24)$$

It proves (4.9) of Theorem 4.6 (i).

It should be noted that, if we put n = 1 and take $a \downarrow 0$ limit in (4.24), we have the formula

$$P_0 = \frac{4(d-1)^2}{d} (1 - d\bar{g}_{03})\bar{g}_{03}^2 [(1 - \bar{g}_{03})^2 - \bar{g}_{23}^2]^{d-2},$$

where

$$\bar{g}_{03} = \lim_{a \downarrow 0} (g_0 - g_3), \qquad \bar{g}_{23} = \lim_{a \downarrow 0} (g_2 - g_3).$$

In particular, $\bar{g}_{03} = 1/\pi$ and $\bar{g}_{23} = 1 - 1/\pi$ for d = 2 [27], and thus we have

$$P_0 = \frac{2}{\pi^2} \left(1 - \frac{2}{\pi} \right), \quad d = 2.$$

This coincides with the value of P_0 obtained by Majumdar and Dhar [19] for the two-dimensional BTW model.

We can also find that the matrix $m^*(\lambda)$ defined by (4.19) is in the form (4.20) with

$$\begin{split} & u = -2d, & b = (1 - e^{\lambda})/a^{1/2}, \\ & c = (1 - e^{-\lambda})/a^{1/2}, & q = (1 - e^{\lambda})/a^{1/2} - 2da^{1/2}(g_1 - e^{\lambda}g_0), \\ & e = (1 - e^{-\lambda})/a^{1/2} - 2da^{1/2}(g_1 - e^{-\lambda}g_0), & f = (g_1 - g_3) - e^{\lambda}(g_0 - g_1), \\ & s = (g_1 - g_3) - e^{-\lambda}(g_0 - g_1), & v = (g_1 - g_0) - e^{\lambda}(g_0 - g_1), \\ & k = (g_1 - g_2) - e^{-\lambda}(g_0 - g_1), & h = (g_1 - g_2) - e^{\lambda}(g_0 - g_1), \\ & t = (g_1 - g_2) - e^{-\lambda}(g_0 - g_1). \end{split}$$

The formula (4.21) gives

$$\det m^*(\lambda) = -2d \left[\{1 - (g_0 - g_3)\}^2 - (g_2 - g_3)^2 \right]^{d-2} \times \det S,$$

where

$$\det S = b_1(d, a, \lambda) + b_2(d, a, \lambda) \frac{1}{n}$$

with some functions b_1 and b_2 of d, a, λ . Since (3.11) gives

$$e^{\lambda(a)} = 1 + a + \sqrt{a(a+2)} = 1 + \sqrt{2}a^{1/2} + \mathcal{O}(a), \text{ as } a \downarrow 0,$$

we found that

$$b_1(d, a, \lambda) = \mathcal{O}(a^2),$$

$$b_2(d, a, \lambda) = \frac{4(d-1)}{d}(g_0 - g_3)\{1 - d(g_0 - g_3)\}\{1 + (d-1)(g_0 - g_3)\} + \mathcal{O}(a^{1/2}), \text{ as } a \downarrow 0.$$

Thus we obtain

$$\lim_{a \downarrow 0} \frac{\det m^*(\lambda) \det m^*(-\lambda)}{(\det m)^2} = \left[\frac{2d\{1 + (d-1)\bar{g}_{03}\}}{(d-1)\bar{g}_{03}}\right]^2.$$

Since $\lim_{a\downarrow 0} c_1(d,a)/a^{(d-3)/4} = (d/(2\pi^2))^{(d-3)/4}/(4\pi)$, (4.12) of Theorem 4.6 is proved.

5 Discussions

5.1 Critical exponent ν_a

The results (3.8) of Theorem 3.3 and (4.11) of Theorem 4.6 mean that both of $G(\mathbf{x}(r))$ and $C_{00}(\mathbf{x}(r))$ decay exponentially as increasing r with a correlation length $\xi(d, a)$. Since $\xi(d, a) < \infty$ for any a > 0, the stationary state of the DASM is non-critical [28]. Moreover the theorems imply that, if we make the parameter n be large with a fixed m, then the value of a = m/(2dn) can be small and

$$nG(\mathbf{x}(r)) \simeq c_1(d)a^{(d-3)/4} \frac{e^{-r/\xi(d,a)}}{r^{(d-1)/2}},$$
(5.1)

$$C_{00}(\mathbf{x}(r)) \simeq c_2(d) a^{(d+1)/2} \frac{\mathrm{e}^{-2r/\xi(d,a)}}{r^{d-1}}, \quad \text{as } r \uparrow \infty,$$
 (5.2)

where $c_1(d) = (d/(2\pi^2))^{(d-3)/4}/(4\pi)$ and $c_2(d)$ is given by (4.12).

Consider a series of DASMs with increasing n with a fixed m. Then we will have an increasing series of correlation lengths $\{\xi(d, a)\}$ and we will see the asymptotic divergence,

$$\xi(d,a) \simeq \frac{1}{\sqrt{2d}} a^{-\nu_a} \quad \text{as} \quad a \to 0$$
(5.3)

with

$$\nu_a = \frac{1}{2} \quad \text{for all} \quad d \ge 2. \tag{5.4}$$

We notice that, if we identify a with a reduced temperature

$$t = \frac{|T - T_{\rm c}|}{T_{\rm c}} \tag{5.5}$$

around a critical temperature T_c in the equilibrium spin system, (5.1) with (5.3) and (5.4) is exactly in the Ornstein-Zernike form of correlations in the mean-field theory of equilibrium phase transitions (see, for instance, Eq.(61) in Section 3.1 of [14]). This implies that we can regard (5.3) as a critical phenomenon with a parameter *a* approaching to its critical value $a_c = 0$ and we can say that the associated *critical exponent* ν_a is exactly determined as (5.4). Vanderzande and Daerden discussed the exponent ν_a for the DASM on more general lattices [29]. This exponent may be identified with the critical exponent $\nu = 1/2$ obtained by Vespignani and Zapperi by the generalized mean-field theory [30]. They claimed that they made only use of conservation laws to evaluate $\nu = 1/2$ and thus at least on this result their mean-field theory is exact for any $d \ge 2$. The present work justifies their conjecture. We can conclude that with respect to the avalanche propagators and height-(0,0) correlation functions the upper critical dimension of the ASM is two. This result does not contradict to the result by Priezzhev [24], since he studied the intersection phenomena of avalanches and for them the upper critical dimension is four.

The results (5.1) and (5.2) suggest that there exists a scaling limit such that

$$\lim_{\substack{r\uparrow\infty,a\downarrow0:\\a^{1/2}r=\kappa/\sqrt{2d}}} r^{d-2}nG(\mathbf{x}(r)) = \mathcal{F}_G(\kappa),$$
$$\lim_{\substack{r\uparrow\infty,a\downarrow0:\\a^{1/2}r=\kappa/\sqrt{2d}}} r^{2d}C_{00}(\mathbf{x}(r)) = \mathcal{F}_C(\kappa), \quad 0 < \kappa < \infty$$

with

$$\mathcal{F}_{G}(\kappa) = 2^{-(d+1)/2} \pi^{-(d-1)/2} \kappa^{(d-3)/2} e^{-\kappa},$$

$$\mathcal{F}_{C}(\kappa) = 2^{-(d+1)} \pi^{-(d-1)} \left[\frac{1 + (d-1)\bar{\gamma}}{(d-1)\bar{\gamma}} \right]^{2} \kappa^{d+1} e^{-\kappa}$$

This observation is consistent with the statement

$$G(\mathbf{x}(r)) \sim r^{-(d-2)}, \quad \text{as } r \uparrow \infty$$
 (5.6)

and (4.3) claimed by Majumdar and Dhar [19] for the self-organized criticality realized in the *d*dimensional BTW model with $d \ge 2$. (Note that for the two-dimensional BTW model, $G(\mathbf{x}(r)) - G(0) \simeq -(1/2\pi) \log r$, as $r \uparrow \infty$.)

5.2 The $q \rightarrow 0$ limit of the Potts model

Majumdar and Dhar [20] discussed the relationship between the ASM and the $q \downarrow 0$ limit of the q-state Potts model. For $q \in \{2, 3, ...\}$, the q-state Potts model on the lattice $G_L = (G_L^{(v)}, G_L^{(e)})$ given by Definition 2.8 is defined as follows. At each vertex $\mathbf{v} \in G_L^{(v)} = \Lambda_L \cup \{\mathbf{r}\}$, put a spin variable $s(\mathbf{x}) \in \{1, 2, ..., q\}$. The Hamiltonian for the configuration $s = \{s(\mathbf{v})\}_{\mathbf{v} \in G_L^{(v)}}$ is given by

$$\mathcal{H}(s) = -\sum_{e = \{\mathbf{v}, \mathbf{w}\} \in G_L^{(e)}} \mathbf{1}(s(\mathbf{v}) = s(\mathbf{w})).$$

The partition function of the Potts model in the Gibbs ensemble with a temperature T > 0 is defined by

$$Z(q,T) = \sum_{s \in \{1,2,...,q\}^{G_L^{(v)}}} e^{-\mathcal{H}(s)/T}$$

=
$$\sum_{s \in \{1,2,...,q\}^{G_L^{(v)}}} \prod_{e=\{\mathbf{v},\mathbf{w}\}\in G_L^{(e)}} \left[1 + \chi \mathbf{1}(s(\mathbf{v}) = s(\mathbf{w}))\right]$$
(5.7)

with $\chi = e^{1/T} - 1$. We consider a subset of $G_L^{(e)}$ denoted by $E \subset G_L^{(e)}$. Each connected component in E is called a cluster. Let c(E) be the number of disconnected clusters of E; $E = \bigcup_{i=1}^{c(E)} E_i$, where $E_i \cap E_j = \emptyset, i \neq j$. If a vertex $\mathbf{v} \in G_L^{(v)}$ is not connected by any edge in E, we write $\mathbf{v} \notin E$. By performing binomial expansions and taking the summation over spin configurations in (5.7), we obtain the Fortuin-Kasteleyn representation of partition function,

$$Z(q,T) = \sum_{E \subset G_L^{(e)}} q^{|\{\mathbf{v} \in G_L^{(v)}; \mathbf{v} \notin E\}|} q^{c(E)} \chi^{|E|},$$
(5.8)

where |E| denotes the number of edges in E. Note that we can regard (5.8) as a function of $q \in \mathbb{R}$ and T > 0. We consider the asymptotics of (5.8) in the limit $q \downarrow 0$. The dominant terms in this limit should be with E such that c(E) = 1 and $\{\mathbf{v} \in G_L^{(v)} : \mathbf{v} \notin E\} = \emptyset \iff E$ contains all vertices in $G_L^{(v)} \iff E$ is a spanning subgraph of G_L . If we further take the high-temperature limit $T \uparrow \infty \iff \chi \downarrow 0$, we have only spanning subgraphs with a minimal number of edges, which are just the spanning trees. Then we have

$$\lim_{T\uparrow\infty}\lim_{q\downarrow 0}T^{(2L+1)^d}q^{-1}Z(q,T)=|\mathcal{T}_L|,$$

where \mathcal{T}_L is the collection of all spanning trees on G_L . As shown in Section 2.4, there establishes a bijection between \mathcal{T}_L and \mathcal{A}_L (Lemma 2.11) and $\mathcal{A}_L = \mathcal{R}_L$ (Proposition 2.12). (The relation between the $q \downarrow 0$ limit of the q-state Potts model with finite temperatures and the ASM is discussed in Section 7.2 in [7].) The two-dimensional q-state Potts model shows a continuous phase transition associated with critical phenomena at a finite temperature $0 < T_c < \infty$ without external magnetic field B = 0, when q = 2, 3 and 4 [31].

Usual critical phenomena of spin models are specified by the behavior of two-point correlation functions for the energy density $G_{\epsilon}(r, t, b, L)$ and for the order-parameter density $G_{\sigma}(r, t, b, L)$. Here r denotes the distance of two points, t the reduced temperature (5.5), b the reduced external field

$$b = \frac{|B|}{T_{\rm c}},$$

and L the size of the lattice on which the model is defined. It is conjectured in the scaling theory that, if L is sufficiently large and we observe the system in the very vicinity of the critical point; $t \ll 1, b \ll 1$, the correlation functions behave as

$$G_{\epsilon}(r,t,b,L) = L^{2x_{\epsilon}} \mathcal{F}_{\epsilon} \left(\frac{r}{L}, tL^{y_{t}}, bL^{y_{b}}\right),$$

$$G_{\sigma}(r,t,b,L) = L^{2x_{\sigma}} \mathcal{F}_{\sigma} \left(\frac{r}{L}, tL^{y_{t}}, bL^{y_{b}}\right),$$
(5.9)

with the scaling exponents $x_{\epsilon}, x_{\sigma}, y_{\epsilon}, y_{\sigma}$, and the scaling functions $\mathcal{F}_{\epsilon}, \mathcal{F}_{\sigma}$. If the system is of *d*-dimensional, the hyperscaling relations $x_{\epsilon} + y_t = d, x_{\sigma} + y_b = d$ hold (see, for instance, [13, 14]). From the scaling forms (5.9), we expect the power-law behavior of correlation functions at the critical point $(t = b = 0, L \uparrow \infty)$ such that

$$G_{\epsilon}(r) \sim r^{-2x_{\epsilon}}, \quad G_{\sigma}(t) \sim r^{-2x_{\sigma}}, \quad \text{as } r \uparrow \infty,$$

and in the off-critical regions with $L \uparrow \infty$, the correlation length $\xi = \xi(t, b)$ behaves as

$$\begin{split} \xi(t,0) &\sim t^{-\nu_t} \quad \text{with} \quad \nu_t = \frac{1}{y_t}, \\ \xi(0,b) &\sim b^{-\nu_b} \quad \text{with} \quad \nu_b = \frac{1}{y_b}, \quad \text{as } t \downarrow 0, b \downarrow 0. \end{split}$$

For the two-dimensional q-state Potts model, the critical exponents are determined as functions of q through the parameter

$$u = u(q) = \frac{2}{\pi} \cos^{-1}\left(\frac{\sqrt{q}}{2}\right)$$

as [31]

$$\begin{aligned} x_{\epsilon} &= \frac{1+u}{2-u}, \qquad \qquad y_t = 2 - x_{\epsilon} = \frac{3(1-u)}{2-u}, \\ x_{\sigma} &= \frac{1-u^2}{4(2-u)}, \qquad \qquad y_b = 2 - x_{\sigma} = \frac{(3-u)(5-u)}{4(2-u)} \end{aligned}$$

They give the limits

$$x_{\epsilon} \to 2, \quad y_t \to 0, \quad x_{\sigma} \to 0, \quad y_b \to 2, \quad \text{as } q \downarrow 0 \iff u \uparrow 1.$$

Majumdar and Dhar [20] noted by their results (4.3) and (5.6) for the BTW models that the avalanche propagator $G(\mathbf{x}(r))$ and the height-(0,0) correlation function $C_{00}(\mathbf{x}(r))$ in ASM play the roles of the order-parameter density correlation function $G_{\sigma}(r)$ and the energy density correlation function $G_{\epsilon}(r)$ in the critical phenomena, respectively. In particular, in the two-dimensional case, the power-law exponents are respectively given as

$$2x_{\sigma}\Big|_{q\downarrow 0} = 0 = (d-2)\Big|_{d=2}, \qquad 2x_{\epsilon}\Big|_{q\downarrow 0} = 4 = 2d\Big|_{d=2}.$$

Our interpretation of the present result (5.4) is that introduction of dissipation to the ASM may correspond to imposing an external magnetic field B to the Potts models and hence $\nu_a = 1/2$ is identified with

$$\nu_b\Big|_{q\downarrow 0} = \left.\frac{1}{y_b}\right|_{q\downarrow 0} = \frac{1}{2}.$$

We remark that the critical exponents for the specific heat α , for the order parameter β , and for the magnetic-field susceptibility γ of the

$$\alpha = \frac{2(1-2u)}{3(1-u)} \to -\infty, \quad \beta = \frac{1+u}{12} \to \frac{1}{6}, \quad \gamma = \frac{7-4u+u^2}{6(1-u)} \to \infty, \quad \text{as } q \downarrow 0 \Longleftrightarrow u \uparrow 1.$$

We suspect some interpretation of the value $\beta|_{q\downarrow 0} = 1/6$ in the DASM.

5.3 Recent topics on height correlations

In Section 4 the one-point and the two-point correlations of height-0 sites were calculated for the DASM with general $d \ge 2$. In the two-dimensional case, the three-point and the four-point correlations were also calculated for height-0 sites and general property of 'the height-0 field of ASMs' have been extensively studied from the view point of a c = -2 conformal field theory [18, 8].

For the two-dimensional BTW model, in which the values of stable height of sandpile are h = 0, 1, 2, and 3, the height correlations have been calculated also for $h \ge 1$. Priezzhev determined P_{α} for $\alpha \in \{0, 1, 2, 3\}$, where the results with $\alpha \ge 1$ are expressed using multivariate integrals of determinantal integrands [23]. Poghosyan et al. [22] claimed that the height-0 state is the only one showing pure power-law-correlations and that general form of height correlations for $h \ge 1$ contains logarithmic functions. They showed that for $\alpha \ge 1$

$$C_{0\alpha}(\mathbf{x}(r)) = \frac{P_{0\alpha}(\mathbf{x}(r)) - P_0 P_\alpha}{P_0 P_\alpha} \simeq \frac{1}{r^4} (c_1 \log r + c_2), \quad \text{as } r \uparrow \infty$$

with some constants c_1, c_2 . Moreover, they predicted that $C_{\alpha\beta}(\mathbf{x}(r)) \sim \log^2 r/r^4$ if $\alpha \geq 1$ and $\beta \geq 1$. These results are discussed with the logarithmic conformal field theory. See also [11]. We will see a lot of interesting open problems concerning height correlations for the BTW models and the DASMs in higher dimensions.

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References

- Andrews, G. E., Askey, R., Roy, R.: Special Functions. Cambridge: Cambridge University Press, 1999
- Bak, P., Tang, C., Wiesenfeld, K.: Self-organized criticality: an explanation of 1/f noise. Phys. Rev. Lett. 59, 381-384 (1987)
- [3] Bak, P., Tang, C., Wiesenfeld, K.: Self-organized criticality. Phys. Rev. A 38, 364-374 (1988)
- [4] Biggs, N.: Algebraic Graph Theory, second edition. Cambridge: Cambridge Univ. Press, 1993
- [5] Brankov, J. G., Ivashkevich, E. V., Priezzhev, V. B.: Boundary effects in a two-dimensional Abelian sandpile. J. Phys. I France 3, 1729-1740 (1993)
- [6] Dhar, D.: Self-organized critical state of sandpile automaton models. Phys. Rev. Lett. 64, 1613-1616 (1990)
- [7] Dhar, D.: Theoretical studies of self-organized criticality. Physica A 369, 29-70 (2006)
- [8] Dürre M.: Conformal covariance of the Abelian sandpile height one field. Stochastic Process. Appl. 119, 2725-2743 (2009)
- [9] Erdélyi, A. et al: Higher Transcendental Functions. vol.II, New York: McGraw-Hill, 1953

- [10] Ghaffari, P., Lise, S., Jensen, H. J.: Nonconservative sandpile models, Phys. Rev. E 56, 6702-6709 (1997)
- [11] Gorsky, A., Nechaev, S., Poghosyan, V. S., Priezzhev, V. B.: From elongated spanning trees to vicious random walks. Nucl. Phys. B 870 [FS], 55-77 (2013)
- [12] Grimmett, G. R., Stirzaker, D. R.: Probability and Random Processes. 2nd edition, Oxford: Clarendon Press, 1992
- [13] Henkel, M.: Conformal Invariance and Critical Phenomena. Berlin: Springer, 1999
- [14] Itzykson, C., Drouffe, J.-M.: Statistical Field Theory I, II. Cambridge: Cambridge University Press, 1989
- [15] Ivashkevich, E. V., Ktitarev, D. V., Priezzhev, V. B.: Waves of topplings in an Abelian sandpile. Physica A 209, 347-360 (1994)
- [16] Járai, A., Redig, F., Saada, E.: Approaching criticality via the zero dissipation limit in the abelian avalanche model. to appear in J. Stat. Phys. ; arXiv:math.PR/0906.3128v4
- [17] Maes, C., Redig, F., Saada, E.: The infinite volume limit of dissipative abelian sandpiles. Commun. Math. Phys. 244, 395-417 (2004)
- [18] Mahieu, S., Ruelle, P.: Scaling fields in the two-dimensional abelian sandpile model. Phys. Rev. E 64, 066130/1-10 (2001)
- [19] Majumdar, S. N., Dhar, D.: Height correlations in the Abelian sandpile model. J. Phys. A: Math. Gen. 24, L357-L362 (1991)
- [20] Majumdar, S. N., Dhar, D.: Equivalence between the Abelian sandpile model and the $q \rightarrow 0$ limit of the Potts model. Physica A **185**, 129-145 (1992)
- [21] McCoy, B. M., Wu, T. T.: The Two-Dimensional Ising Model. Cambridge, Massachusetts: Harvard University Press, 1973
- [22] Poghosyan, V. S., Grigorov, S. Y., Priezzhev, V. B., Ruelle, P.: Logarithmic two-point correlations in the Abelian sandpile model. J. Stat. Mech. P07025/1-27 (2010)
- [23] Priezzhev, V. B.: Structure of two-dimensional sandpile. I. Height probability. J. Stat. Phys. 74, 955-979 (1994)
- [24] Priezzhev, V. B.: The upper critical dimension of the Abelian sandpile model. J. Stat. Phys. 98, 667-684 (2000)
- [25] Prussner, G.: Self-Organised Criticality Theory, Models and Characterisation. Cambridge: Cambridge University Press, 2012
- [26] Schmidt, K., Verbitskiy, E.: Abelian sandpiles and the Harmonic model. Commun. Math. Phys. 292, 721-750 (2009)
- [27] Spitzer, F.: Principle of Random Walk. New York: Van Nostrand, 1964, p.148
- [28] Tsuchiya, T., Katori, M.: Proof of breaking of self-organized criticality in a nonconservative Abelian sandpile model. Phys. Rev. E 61, 1183-1188 (2000)
- [29] Vanderzande, C., Daerden, F.: Dissipative abelian sandpiles and random walks. Phys. Rev. E 63, 030301-30304 (R) (2001)
- [30] Vespignani, A., Zapperi, S.: How self-organized criticality works: A unified mean-field picture. Phys. Rev. E57, 6345-6362 (1998)
- [31] Wu, F. Y.: The Potts model. Rev. Mod. Phys. 54, 235-268 (1982)