Abelian Sandpile Models in Statistical Mechanics : Dissipative Abelian Sandpile Models

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Abelian Sandpile Models in Statistical Mechanics – Dissipative Abelian Sandpile Models –

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Abstract

We introduce a family of abelian sandpile models with two parameters $n, m \in \mathbb{N}$ defined on finite lattices on d-dimensional torus. Sites with $2dn + m$ or more grains of sand are unstable and topple, and in each toppling m grains dissipate from the system. Because of dissipation in bulk, the models are well-defined on the shift-invariant lattices and the infinite-volume limit of systems can be taken. From the determinantal expressions, we obtain the asymptotic forms of the avalanche propagators and the height-$(0,0)$ correlations of sandpiles for large distances in the infinite-volume limit in any dimensions $d \geq 2$. We show that both of them decay exponentially with the correlation length

$$\xi(d,a) = (\sqrt{d}\sinh^{-1}\sqrt{a(a+2)}\,)^{-1},$$

if the dissipation rate $a = \dfrac{m}{2dn}$ is positive. Considering a series of models with increasing n, we discuss the limit $a \downarrow 0$ and the critical exponent defined by $\nu_a = -\lim_{a\downarrow 0} \dfrac{\log \xi(d,a)}{\log a}$ is determined as

$$\nu_a = \frac{1}{2}$$

for all $d \geq 2$. Comparison with the $q \downarrow 0$ limit of q-state Potts model in external magnetic field is discussed.

Key words. Abelian sandpile models, Dissipation, Avalanches, Height correlations, Determinantal expressions, Correlation length exponent.

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1 Introduction

Let $d \in \{2,3,\dots\}$ and $L \in \mathbb{N} \equiv \{1,2,3,\dots\}$. Consider a box in the d-dimensional hypercubic lattice $B_L = \{-L,-L+1,\dots,L\}^d \subset \mathbb{Z}^d$, where $\mathbb{Z}$ denotes the collection of all integers. We impose *periodic boundary conditions* for all d directions and obtain a lattice on a torus (toroidal), which is denoted by Λ_L. The number of sites in Λ_L is given by $|\Lambda_L| = (2L+1)^d$. In the present paper we study a family of Markov processes on Λ_L, $h_t = \{h_t(\mathbf{z})\}_{\mathbf{z}\in\Lambda_L}$, with discrete-time $t \in \mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$.

Assume $n, m \in \mathbb{N}$ and let

$$a = \frac{m}{2dn} \qquad \text{and} \qquad h_{\mathrm{c}} = 2d(1+a).$$

Define a real symmetric matrix with size $(2L+1)^d$,

$$\Delta_L(\mathbf{x},\mathbf{y}) = \begin{cases} h_{\mathrm{c}}, & \text{if} \quad \mathbf{x}=\mathbf{y}, \\ -1, & \text{if} \quad |\mathbf{x}-\mathbf{y}|=1, \\ 0, & \text{otherwise}, \end{cases} \tag{1.1}$$

where $\mathbf{x} = (x_1,\dots,x_d), \mathbf{y} = (y_1,\dots,y_d) \in \Lambda_L$ and $|\mathbf{x}-\mathbf{y}| = \sqrt{\sum_{i=1}^d (x_i-y_i)^2}$. Let $\mathbf{1}(\omega)$ be the indicator function of an event ω; $\mathbf{1}(\omega) = 1$, if ω occurs and $\mathbf{1}(\omega) = 0$, otherwise. The configuration space is

$$\mathcal{S}_L = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, h_{\mathrm{c}} - \frac{1}{n}\right\}^{\Lambda_L}.$$

Given a configuration $h_t \in \mathcal{S}_L, t \in \mathbb{N}_0$, $h_{t+1} \in \mathcal{S}_L$ is determined by the following algorithm.

(i) Choose one site in Λ_L at random. Let $\mathbf{x}$ be the chosen site and define

$$\eta^{\mathbf{x}}_{(1)}(\mathbf{z}) = h_t(\mathbf{z}) + \frac{1}{n}\mathbf{1}(\mathbf{z}=\mathbf{x}), \quad \mathbf{z} \in \Lambda_L.$$

If $\eta^{\mathbf{x}}_{(1)}(\mathbf{x}) < h_{\mathrm{c}}$, then $\eta^{\mathbf{x}}_{(1)} \equiv \{\eta^{\mathbf{x}}_{(1)}(\mathbf{z})\}_{\mathbf{z}\in\Lambda_L} \in \mathcal{S}_L$. In this case, we set $h_{t+1} = \eta^{\mathbf{x}}_{(1)}$.

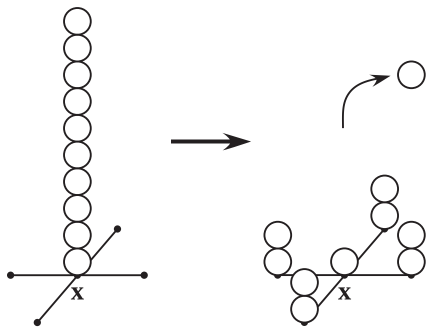

Figure 1: A toppling for the DASM with the parameters $d = 2, n = 2$ and $m = 1$. In this case $h_c = 2dn + m = 9$, and thus the site $\mathbf{x}$ with height $h(\mathbf{x}) = 10$ is unstable. In a toppling, $h_c = 9$ grains of sand drop from the site $\mathbf{x}$, in which $n = 2$ grains land on each nearest-neighbor site, $m = 1$ grain is dissipated from the system, while $h(\mathbf{x}) - h_c = 1$ grain remains on the site $\mathbf{x}$.

(ii) If $\eta^{\mathbf{x}}_{(1)}(\mathbf{x}) = h_c$, then $\eta^{\mathbf{x}}_{(1)} \notin \mathcal{S}_L$. In this case, we consider a finite series of configurations $\{\eta^{\mathbf{x}}_{(1)}, \cdots, \eta^{\mathbf{x}}_{(\tau)}\}$ with $\exists \tau \in \mathbb{N}$ recursively as follows. Assume that $\eta^{\mathbf{x}}_{(\ell)} \notin \mathcal{S}_L$ with $\ell \geq 1$, then $A^{\mathbf{x}}_{(\ell)}(h_t) \equiv \{\mathbf{z} \in \Lambda_L : \eta^{\mathbf{x}}_{(\ell)}(\mathbf{z}) \geq h_c\} \neq \emptyset$ and define

$$\eta^{\mathbf{x}}_{(\ell+1)}(\mathbf{z}) = \eta^{\mathbf{x}}_{(\ell)}(\mathbf{z}) - \sum_{\mathbf{y}:\mathbf{y} \in A^{\mathbf{x}}_{(\ell)}(h_t)} \Delta_L(\mathbf{y}, \mathbf{z}), \quad \mathbf{z} \in \Lambda_L.$$

If $\eta^{\mathbf{x}}_{(\ell+1)} \in \mathcal{S}_L$, then $\tau = \ell + 1$ and $h_{t+1} = \eta^{\mathbf{x}}_{(\tau)}$. Remark that $\tau = \tau(\mathbf{x}, h_t)$ and $\tau < \infty$ by $\sum_{\mathbf{z}:\mathbf{z} \in \Lambda_L} \Delta_L(\mathbf{y}, \mathbf{z}) > 0, \forall \mathbf{y} \in \Lambda_L$ as explained below.

We think that $1/n$ is a unit of grain of sand and $h_t(\mathbf{z})n$ represents the height of sandpile at site $\mathbf{z}$ measured in this unit. The step (i) simulates a random deposit of a grain of sand. In the step (ii), for each $1 \leq \ell \leq \tau$, the sites $\mathbf{y} \in A^{\mathbf{x}}_{(\ell)}(h_t)$ are regarded as unstable sites and the process

$$\{\eta^{\mathbf{x}}_{(\ell)}(\mathbf{z})\}_{\mathbf{z} \in \Lambda_L} \to \{\eta^{\mathbf{x}}_{(\ell)}(\mathbf{z}) - \Delta_L(\mathbf{y}, \mathbf{z})\}_{\mathbf{z} \in \Lambda_L},$$

is called a *toppling* of the site $\mathbf{y}$ such that

$$\Delta_L(\mathbf{y}, \mathbf{y})n = h_c n = 2dn + m \text{ grains of sand drop from the unstable site } \mathbf{y}$$

and

$$|\Delta_L(\mathbf{y}, \mathbf{z})|n = n \text{ grains of sand land on each nearest-neighbor site } \mathbf{z}, |\mathbf{x} - \mathbf{z}| = 1.$$

Since there are $2d$ nearest-neighbor sites of each site, m grains are annihilated in a toppling. (See Fig.1.) The total number of grains on Λ_L decreases in each toppling and it guarantees $\tau < \infty$. The configuration space $\mathcal{S}_L$ is a set of all stable configurations of sandpiles in which height of sandpile is less than the threshold value h_c at every site; $h(\mathbf{z}) < h_c, \forall \mathbf{z} \in \Lambda_L$. From a stable configuration h_t to another stable configuration h_{t+1}, $\sum_{\ell=1}^{\tau-1} |A^{\mathbf{x}}_{(\ell)}(h_t)|$ topplings occur.

Such a series of toppling is called an *avalanche*. (Note that, if $\tau = 1$, toppling does not occur. Even in such a case, we call the transition from h_t to h_{t+1} an avalanche, which is just a random deposit of a grain of sand.) Define

$$T(\mathbf{x}, \mathbf{y}, h) = \sum_{\ell=1}^{\tau(\mathbf{x},h)-1} \mathbf{1}(\mathbf{y} \in A^{\mathbf{x}}_{(\ell)}(h)), \quad \mathbf{x}, \mathbf{y} \in \Lambda_L, \quad h \in \mathcal{S}_L. \tag{1.2}$$

This is the number of topplings at site $\mathbf{y} \in \Lambda_L$ in an avalanche caused by a deposit of a grain of sand at a site $\mathbf{x} \in \Lambda_L$ in the configuration $h \in \mathcal{S}_L$.

We have assumed that $n, m \in \mathbb{N}$ in the above definition of processes. If we set $n = 1, m = 0$, however, we have $a = 0$ and $\Delta_L|_{a=0}$ gives the 'rule matrix' of the sandpile model introduced by Bak, Tang and Wiesenfeld (BTW) [2, 3]. The BTW model have been studied on finite lattices with *open boundary conditions* in order to make τ be finite. For example, the BTW model is considered on a box B_L. The boundary of box B_L is given by $\partial B_L = \{\mathbf{y} = (y_1, \cdots, y_d) \in B_L : 1 \leq \exists i \leq d \text{ s.t. } y_i = -L \text{ or } L\}$. In the BTW model defined on B_L, $\sum_{\mathbf{z}:\mathbf{z}\in\Lambda_L} \Delta_L|_{a=0}(\mathbf{y}, \mathbf{z}) = 0$ if $\mathbf{y} \in B_L \setminus \partial B_L$; that is, the number of grains of sand is conserved in any toppling in the bulk of system. By imposing the open boundary condition, we have $\sum_{\mathbf{z}:\mathbf{z}\in\Lambda_L} \Delta_L|_{a=0}(\mathbf{y}, \mathbf{z}) > 0$ for $\mathbf{y} \in \partial B_L$ and dissipation of grains of sand can occur in topplings at the boundary sites. In the present model, in every toppling at any site $\mathbf{y} \in \Lambda_L$, $\sum_{\mathbf{z}:\mathbf{z}\in\Lambda_L} \Delta_L(\mathbf{y}, \mathbf{z})n = m$ grains of sand dissipate from the system and hence $\tau < \infty$ is guaranteed in the shift-invariant system. The quantity a indicates the rate of dissipation in a toppling.

The present process belongs to the class of *abelian sandpile models* (ASM) studied by Dhar [6]. We define the operators $\{\mathsf{a}(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L}$ following Dhar by

$$h_{t+1} = \mathsf{a}(\mathbf{x})h_t, \quad \mathbf{x} \in \Lambda_L,$$

where $h_t, h_{t+1} \in \mathcal{S}_L$ and the site $\mathbf{x}$ is the chosen site in the first step (i) of the algorithm at time t. That is, $\mathsf{a}(\mathbf{x})$ represents an avalanche caused by a deposit of a grain of sand at $\mathbf{x}$. Then the above algorithm guarantees the *abelian property* of avalanches (see Lemma 2.1 in Section 2.1)

$$[\mathsf{a}(\mathbf{x}), \mathsf{a}(\mathbf{y})] \equiv \mathsf{a}(\mathbf{x})\mathsf{a}(\mathbf{y}) - \mathsf{a}(\mathbf{y})\mathsf{a}(\mathbf{x}) = 0, \quad \forall \mathbf{x}, \mathbf{y} \in \Lambda_L. \tag{1.3}$$

We call the present Markov process the d-dimensional *dissipative abelian sandpile model* (DASM for short). The two-dimensional case was studied numerically [10] and analytically [30, 28, 18]. In the present paper, we will discuss the models in general dimensions $d \geq 2$ in finite and infinite lattices. See also [29]. As shown in [17, 26, 16] the DASM is useful to construct the infinite-volume limit of avalanche models. Importance of the abelian sandpile models in the extensive study of *self-organized criticality* in the statistical mechanics and related fields is discussed in [25].

2 Basic Properties of Dissipative Abelian Sandpile Model

2.1 Abelian property

First we prove the abelian property of avalanches (1.3).

Lemma 2.1 (Dhar [6]) *Assume that the avalanche operators* $\{\mathsf{a}(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L}$ *act on* $\mathcal{S}_L$. *Then*

$$[\mathsf{a}(\mathbf{x}),\mathsf{a}(\mathbf{y})]=0,\quad \forall \mathbf{x},\mathbf{y}\in\Lambda_L.$$

Proof. Let $\mathcal{X}_L=\mathbb{Z}^{\Lambda_L}$. Define three sets of maps from $\mathcal{X}_L$ to $\mathcal{X}_L$; $\{\tilde{\mathsf{t}}(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L}$, $\{\mathsf{t}(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L}$ and $\{\mathsf{d}(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L}$ as follows. For $\mathbf{x}\in\Lambda_L$ and $\eta=\{\eta(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L}\in\mathcal{X}_L$ define

$$\begin{aligned}
\tilde{\mathsf{t}}(\mathbf{x})\eta(\mathbf{z}) &= \eta(\mathbf{z})-\Delta_L(\mathbf{x},\mathbf{z}),\\
\mathsf{t}(\mathbf{x})\eta(\mathbf{z}) &= \begin{cases}\eta(\mathbf{z})-\Delta_L(\mathbf{x},\mathbf{z}), & \text{if}\quad \eta(\mathbf{x})\ge h_{\mathrm{c}},\\ \eta(\mathbf{z}), & \text{otherwise},\end{cases}\\
\mathsf{d}(\mathbf{x})\eta(\mathbf{z}) &= \eta(\mathbf{z})+\frac{1}{n}\mathbf{1}(\mathbf{z}=\mathbf{x}),\qquad \mathbf{z}\in\Lambda_L.
\end{aligned}$$

By definition of $\tilde{\mathsf{t}}$,

$$\tilde{\mathsf{t}}(\mathbf{y})\tilde{\mathsf{t}}(\mathbf{x})\eta(\mathbf{z})=\eta(\mathbf{z})-\Delta_L(\mathbf{x},\mathbf{z})-\Delta_L(\mathbf{y},\mathbf{z}),\quad \mathbf{z}\in\Lambda_L.$$

Similarly we have

$$\tilde{\mathsf{t}}(\mathbf{x})\tilde{\mathsf{t}}(\mathbf{y})\eta(\mathbf{z})=\eta(\mathbf{z})-\Delta_L(\mathbf{y},\mathbf{z})-\Delta_L(\mathbf{x},\mathbf{z}),\quad \mathbf{z}\in\Lambda_L.$$

Therefore $\tilde{\mathsf{t}}(\mathbf{y})\tilde{\mathsf{t}}(\mathbf{x})\eta=\tilde{\mathsf{t}}(\mathbf{x})\tilde{\mathsf{t}}(\mathbf{y})\eta, \forall\eta\in\mathcal{X}_L$, that is

$$[\tilde{\mathsf{t}}(\mathbf{x}),\tilde{\mathsf{t}}(\mathbf{y})]=0,\quad \forall \mathbf{x},\mathbf{y}\in\Lambda_L. \tag{2.1}$$

Assume that $\mathbf{y}\neq\mathbf{x}$. Then

$$\tilde{\mathsf{t}}(\mathbf{y})\eta(\mathbf{x})=\eta(\mathbf{x})-\Delta(\mathbf{y},\mathbf{x})=\begin{cases}\eta(\mathbf{x})+1, & \text{if}\quad |\mathbf{x}-\mathbf{y}|=1,\\ \eta(\mathbf{x}), & \text{if}\quad |\mathbf{x}-\mathbf{y}|>1.\end{cases}$$

It implies that if $\eta(\mathbf{x})\ge h_{\mathrm{c}}$ then $\tilde{\mathsf{t}}(\mathbf{y})\eta(\mathbf{x})\ge h_{\mathrm{c}}, \forall\mathbf{y}\neq\mathbf{x}$, that is, any site cannot be stabilized by topplings which occur at other sites. Therefore, the definition of $\mathsf{t}(\mathbf{x})$ and (2.1) give

$$[\mathsf{t}(\mathbf{x}),\mathsf{t}(\mathbf{y})]=0,\quad \forall \mathbf{x},\mathbf{y}\in\Lambda_L. \tag{2.2}$$

It is obvious that

$$[\mathsf{t}(\mathbf{x}),\mathsf{d}(\mathbf{y})]=0,\quad \forall \mathbf{x},\mathbf{y}\in\Lambda_L. \tag{2.3}$$

Consider the situation that $h\in\mathcal{S}_L$ and $A^{\mathbf{x}}_{(\ell)}(h)\neq\emptyset$, $1\le\ell\le\tau$. By (2.2), $\prod_{\mathbf{z}:\mathbf{z}\in A^{\mathbf{x}}_{(\ell)}(h)}\mathsf{t}(\mathbf{z})$ is independent of the order of the products of $\mathsf{t}(\mathbf{z})$'s. Then we can write

$$\mathsf{a}(\mathbf{x})h=\left[\prod_{\ell=1}^{\tau-1}\left(\prod_{\mathbf{z}:\mathbf{z}\in A^{\mathbf{x}}_{(\ell)}(h)}\mathsf{t}(\mathbf{z})\right)\right]\mathsf{d}(\mathbf{x})h,\quad \mathbf{x}\in\Lambda_L,\quad h\in\mathcal{S}_L.$$

By (2.2) and (2.3), the lemma is proved. ∎

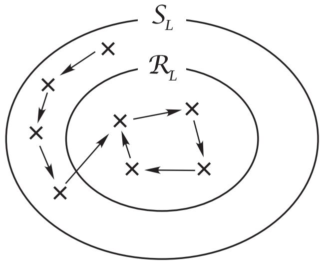

Figure 2: The set of recurrent configurations $\mathcal{R}_L$ is closed under avalanches.

2.2 Recurrent configurations

Consider a subset of $\mathcal{S}_L$ defined by

$$\mathcal{R}_L = \{h \in \mathcal{S}_L : \ \forall \mathbf{x} \in \Lambda_L, \exists k(\mathbf{x}) \in \mathbb{N}, \text{ s.t. } (\mathsf{a}(\mathbf{x}))^{k(\mathbf{x})} h = h\},$$

which is called the set of *recurrent configurations*.

Lemma 2.2 (Dhar [6]) *If $h \in \mathcal{R}_L$, then $\mathsf{a}(\mathbf{x})h \in \mathcal{R}_L$ for any $\mathbf{x} \in \Lambda_L$. That is, $\mathcal{R}_L$ is closed under avalanches (see Fig.2).*

Proof. By definition, if $h \in \mathcal{R}_L$, then for any $\mathbf{y} \in \Lambda_L$, $\exists k(\mathbf{y}) \in \mathbb{N}$, s.t. $(\mathsf{a}(\mathbf{y}))^{k(\mathbf{y})} h = h$. If we operate $\mathsf{a}(\mathbf{x}), \mathbf{x} \in \Lambda_L$ on the both sides of this equation, then we have $\mathsf{a}(\mathbf{x})(\mathsf{a}(\mathbf{y}))^{k(\mathbf{y})} h = \mathsf{a}(\mathbf{x})h$. By Lemma 2.1, LHS= $(\mathsf{a}(\mathbf{y}))^{k(\mathbf{y})}\mathsf{a}(\mathbf{x})h$. This equality implies that $\mathsf{a}(\mathbf{x})h \in \mathcal{R}_L$. Since it is valid for any $\mathbf{x} \in \Lambda_L$, the proof is completed. ∎

Consider a $(2L+1)^d$-dimensional vector space $\mathcal{V}_L$, in which the orthonormal basis is given by $\{\mathbf{e}(\mathbf{z})\}_{\mathbf{z} \in \Lambda_L}$. For each configuration $\eta \in \mathcal{X}_L$, we assign a vector

$$\boldsymbol{\eta} = \sum_{\mathbf{z}:\mathbf{z} \in \Lambda_L} \eta(\mathbf{z})\mathbf{e}(\mathbf{z}) = \sum_{\mathbf{z}:\mathbf{z} \in \Lambda_L} n\eta(\mathbf{z})\frac{\mathbf{e}(\mathbf{z})}{n}, \tag{2.4}$$

where $1/n$ denotes the unit of grain of sand. Assume that $h \in \mathcal{R}_L$; for each $\mathbf{x} \in \Lambda_L$, there is $k(\mathbf{x}) \in \mathbb{N}$ such that

$$(\mathsf{a}(\mathbf{x}))^{k(\mathbf{x})} h = h. \tag{2.5}$$

Consider the vector corresponding to the configuration $(\mathsf{d}(\mathbf{x}))^{k(\mathbf{x})} h$,

$$\boldsymbol{\eta} = \left(h(\mathbf{x}) + \frac{k(\mathbf{x})}{n}\right)\mathbf{e}(\mathbf{x}) + \sum_{\mathbf{z}:\mathbf{z} \neq \mathbf{x}} h(\mathbf{z})\mathbf{e}(\mathbf{z}) \in \mathcal{V}_L. \tag{2.6}$$

Then (2.5) claims that there exists a set $\{r(\mathbf{z}) \in \mathbb{N} : \mathbf{z} \in \Lambda_L\}$ such that

$$\mathbf{h} = \boldsymbol{\eta} + \sum_{\mathbf{z}:\mathbf{z} \in \Lambda_L} \left(\sum_{\mathbf{y}:\mathbf{y} \in \Lambda_L} r(\mathbf{y})\Delta_L(\mathbf{y}, \mathbf{z})\right)\mathbf{e}(\mathbf{z}). \tag{2.7}$$

Note that (2.7) is written as

$$\mathbf{h} = \boldsymbol{\eta} + \sum_{\mathbf{y}:\mathbf{y}\in\Lambda_L} r(\mathbf{y})\mathbf{v}(\mathbf{y})$$

with

$$\mathbf{v}(\mathbf{x}) = \sum_{\mathbf{z}:\mathbf{z}\in\Lambda_L} \Delta_L(\mathbf{x},\mathbf{z})\mathbf{e}(\mathbf{z}), \quad \mathbf{x}\in\Lambda_L. \tag{2.8}$$

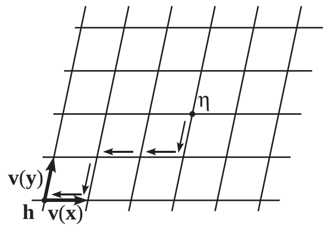

Figure 3: Hypercubic lattice Ω with the basis $\{\mathbf{v}(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L}$ in $\mathcal{V}_L$. Every avalanche from an unstable configuration $\boldsymbol{\eta}$ given by (2.6) to a recurrent configuration $h\in\mathcal{R}_L$ is represented by a lattice path $\boldsymbol{\eta}\rightsquigarrow\mathbf{h}$ on Ω.

We can say that, given $h\in\mathcal{R}_L$, all points $\{\boldsymbol{\eta}\}$ given by (2.6) are identified with sites of a hypercubic lattice Ω with the basis $\{\mathbf{v}(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L}$ in $\mathcal{V}_L$. (See Fig.3.) Consider a primitive cell (fundamental domain) of the lattice defined by

$$\mathcal{U}_L = \left\{ \sum_{\mathbf{x}:\mathbf{x}\in\Lambda_L} \mathbf{c}(\mathbf{x})\mathbf{v}(\mathbf{x}) : 0\le\mathbf{c}(\mathbf{x})<1, \mathbf{x}\in\Lambda_L \right\} \subset \mathcal{V}_L. \tag{2.9}$$

By definition, the intersection of the lattice Ω and $\mathcal{U}_L$ is a singleton, say $\mathbf{p}$. We assume that the origin of this lattice is given by $\mathbf{p}$ and express the lattice by $\Omega^{\mathbf{p}}$. We consider a collection of all lattices with the same basis (2.8) having distinct origin in $\mathcal{U}_L$, $\{\Omega^{\mathbf{p}}:\mathbf{p}\in\mathcal{U}_L\}$. Then there establishes a bijection between $\mathcal{R}_L=\{h\}$ and $\{\Omega^{\mathbf{p}}:\mathbf{p}\in\mathcal{U}_L\}$.

Lemma 2.3 (Dhar [6]) *The number of recurrent configuration is given by*

$$|\mathcal{R}_L| = n^{(2L+1)^d}\det\Delta_L.$$

Proof. The above bijection implies $|\mathcal{R}_L| = |\{\Omega^{\mathbf{p}}:\mathbf{p}\in\mathcal{U}_L\}|$. Since the unit of grain of sand is $1/n$, the origins $\{\mathbf{p}\}$ of lattices $\{\Omega^{\mathbf{p}}\}$ should be in $(\mathbb{Z}/n)^{\Lambda_L}$, and hence (see Fig.4)

$$|\{\Omega^{\mathbf{p}}:\mathbf{p}\in\mathcal{U}_L\}| = \left|\mathcal{U}_L\cap(\mathbb{Z}/n)^{\Lambda_L}\right| = n^{(2L+1)^d}\times(\text{the volume of }\mathcal{U}_L).$$

The volume of $\mathcal{U}_L$ given by (2.9) with (2.8) is $\det\Delta_L$ and the proof is completed. ∎

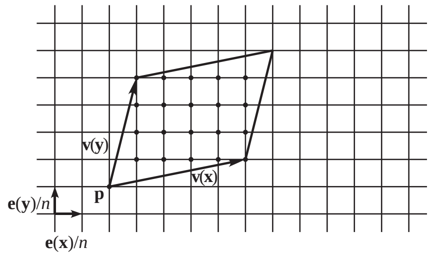

Figure 4: A primitive cell of Ω on the lattice $(\mathbb{Z}/n)^{\Lambda_L}$. Since the unit of grain of sand is $1/n$, the origin $\mathbf{p}$ of lattice Ω should be at a site of $(\mathbb{Z}/n)^{\Lambda_L}$.

2.3 Stationary distribution

For $h \in \mathcal{R}_L$, let $\mathbb{P}_L^h$ be the probability law of the DASM starting from the configuration $h_0 = h$.

Definition 2.4 *If we restrict* $\{\mathsf{a}(\mathbf{x})\}_{\mathbf{x}\in\Lambda_L}$ *to* $\mathcal{R}_L$, *inverse of the avalanche operator can be defined by*

$$\mathsf{a}(\mathbf{x})^{-1} = \mathsf{a}(\mathbf{x})^{k(\mathbf{x})-1}, \quad \mathbf{x} \in \Lambda_L.$$

Assume that $h \in \mathcal{R}_L$ is given. Define

$$\begin{aligned} &\mu_t(X) = \mathbb{P}^h(h_t = X), \\ &W(X \to Y) = \mathbb{P}^h(h_{t+1} = Y | h_t = X), \quad t \in \mathbb{N}_0, \quad X, Y \in \mathcal{R}_L. \end{aligned}$$

Consider the Master equation

$$\mu_{t+1}(X) = \mu_t(X) - \sum_{Y:Y\in\mathcal{R}_L} \mu_t(X)W(X \to Y) + \sum_{Y:Y\in\mathcal{R}_L} \mu_t(Y)W(Y \to X),$$

where we have used the assumption that $h_0 = h \in \mathcal{R}_L$ and Lemma 2.2. By definition of the DASM, we can find that, for $X, Y \in \mathcal{R}_L$,

$$\begin{aligned} W(X \to Y) &= \sum_{\mathbf{x}:\mathbf{x}\in\Lambda_L} \text{Prob}(\mathbf{x} \text{ is chosen}) \mathbf{1}(\mathsf{a}(\mathbf{x})X = Y) \\ &= \frac{1}{|\Lambda_L|} \sum_{\mathbf{x}:\mathbf{x}\in\Lambda_L} \mathbf{1}(\mathsf{a}(\mathbf{x})X = Y) \\ &= \frac{1}{(2L+1)^d} \sum_{\mathbf{x}:\mathbf{x}\in\Lambda_L} \mathbf{1}(X = \mathsf{a}^{-1}(\mathbf{x})Y). \end{aligned}$$

Then we have

$$\mu_{t+1}(X) - \mu_t(X) = \frac{1}{(2L+1)^d} \sum_{\mathbf{x}:\mathbf{x}\in\Lambda_L} \{\mu_t(\mathsf{a}(\mathbf{x})^{-1}X) - \mu_t(X)\}, \quad \forall X \in \mathcal{R}_L.$$

It implies that the uniform measure on $\mathcal{R}_L$,

$$\mu(X) = \frac{1}{|\mathcal{R}_L|}\mathbf{1}(X \in \mathcal{R}_L) = \frac{1}{n^{(2L+1)^d} \det \Delta_L}\mathbf{1}(X \in \mathcal{R}_L), \quad X \in \mathcal{X}_L$$

is a stationary distribution of the process.

Lemma 2.5 *The DASM on Λ_L is irreducible on $\mathcal{R}_L$.*

Proof. Consider the configuration $\overline{h} \in \mathcal{S}_L$, such that $\overline{h}(\mathbf{x}) = h_c - 1/n, \forall \mathbf{x} \in \Lambda_L$. Now we take two arbitrary configurations X and Y from $\mathcal{R}_L$. We have

$$\overline{h} = \prod_{\mathbf{x}:X(\mathbf{x})<h_c-1/n} (\mathsf{a}(\mathbf{x}))^{h_c-1/n-X(\mathbf{x})} X = \prod_{\mathbf{x}:Y(\mathbf{x})<h_c-1/n} (\mathsf{a}(\mathbf{x}))^{h_c-1/n-Y(\mathbf{x})} Y. \tag{2.10}$$

Since this means that the configuration $\overline{h}$ is reachable form X and Y by avalanches, Lemma 2.2 guarantees that $\overline{h} \in \mathcal{R}_L$. Since we have assumed that $Y \in \mathcal{R}_L$, $(\mathsf{a}(\mathbf{x}))^{k(\mathbf{x})} Y = Y$ with some $k(\mathbf{x}) \in \mathbb{N}$ for any $\mathbf{x} \in \Lambda_L$. Therefore, the second equality of (2.10) gives (see Definition 2.4)

$$Y = \prod_{\mathbf{x}:Y(\mathbf{x})<h_c-1/n} (\mathsf{a}(\mathbf{x}))^{k(\mathbf{x})-(h_c-1/n-Y(\mathbf{x}))} \overline{h}. \tag{2.11}$$

Combining (2.10) and (2.11) gives

$$Y = \prod_{\mathbf{x}:Y(\mathbf{x})<h_c-1/n} (\mathsf{a}(\mathbf{x}))^{k(\mathbf{x})-(h_c-1/n-Y(\mathbf{x}))} \prod_{\mathbf{y}:X(\mathbf{y})<h_c-1/n} (\mathsf{a}(\mathbf{y}))^{h_c-1/n-X(\mathbf{y})} X.$$

Let $\sigma = \sum_{\mathbf{x}:Y(\mathbf{x})<h_c-1/n} \{k(\mathbf{x}) - (h_c - 1/n - Y(\mathbf{x}))\} + \sum_{\mathbf{x}:X(\mathbf{x})<h_c-1/n} \{h_c - 1/n - X(\mathbf{x})\}$. Then we see

$$\mathbb{P}^{h_0}(h_{t+s} = Y | h_t = X) \geq \left(\frac{1}{|\Lambda_L|}\right)^{\sigma} \qquad \text{for } s \geq \sigma.$$

Since RHS is strictly positive for finite L, this completes the proof. ∎

Then the following is concluded by the general theory of Markov chains (see, for example, Chapter 6.4 of [12]).

Proposition 2.6 *The stationary distribution of the DASM is uniquely given by the uniform measure on $\mathcal{R}_L$.*

We write the probability law of the DASM on Λ_L in the stationary distribution as $\mathbf{P}_L$ and its expectation as $\mathbf{E}_L$.

2.4 Allowed configurations and spanning trees

Dhar also introduced a subset of $\mathcal{S}_L$ called a collection of *allowed configurations* $\mathcal{A}_L$ [6]. He defined that for $h \in \mathcal{S}_L$, if there is a subset $F \subset \Lambda_L$ such that $F \neq \emptyset$ and

$$h(\mathbf{y}) < \sum_{\mathbf{x}:\mathbf{x}\in F, \mathbf{x}\neq y} (-\Delta_L(\mathbf{x}, \mathbf{y})), \quad \forall \mathbf{y} \in F, \tag{2.12}$$

then $h \in \mathcal{S}_L$ has a *forbidden subconfiguration* (FSC) on F. Then define

$$\mathcal{A}_L = \{h \in \mathcal{S}_L : h \text{ has no FSC}\}.$$

Lemma 2.7 *For the DASM on Λ_L,*

$$\mathcal{R}_L \subset \mathcal{A}_L.$$

Proof. In the proof of Lemma 2.5 we have shown that $\overline{h} \in \mathcal{R}_L$ and all recurrent stares are reachable from this configuration $\overline{h}$. We can prove that $\overline{h} \in \mathcal{A}_L$ as follows. We assume that the contrary; there exists a finite nonempty set $F \subset \Lambda_L$ satisfying (2.12). In the DASM, however, for any $\mathbf{y} \in F$, $\overline{h}(\mathbf{y}) = h_{\rm c} - 1/n = 2d + (m-1)/n \geq 2d \geq \sum_{\mathbf{x}:\mathbf{x}\in F:\mathbf{x}\neq\mathbf{y}}(-\Delta_L(\mathbf{x},\mathbf{y}))$, which contradicts our assumption. Since both $\mathcal{R}_L$ and $\mathcal{A}_L$ include $\overline{h}$, it is enough to show that $\mathcal{A}_L$ is closed under the process of avalanche to prove the lemma, since we have already proved that $\mathcal{R}_L$ is so in Lemma 2.2. Remark that addition of particles only increases h and such procedure on an allowed configurations cannot create any FSC. Here we assume that there exists an allowed configuration h such that by a single toppling at the site $\mathbf{x}$ it becomes to contain a FSC. Write $h' = \mathsf{t}(\mathbf{x})\mathsf{d}(\mathbf{x})h$, that is,

$$h'(\mathbf{y}) = h(\mathbf{y}) + \frac{1}{n}\mathbf{1}(\mathbf{y}=\mathbf{x}) - \Delta_L(\mathbf{x},\mathbf{y}), \quad \forall \mathbf{y} \in \Lambda_L. \tag{2.13}$$

By assumption, there exists $F \neq \emptyset$ such that

$$h'(\mathbf{y}) < \sum_{\mathbf{z}:\mathbf{z}\in F:\mathbf{z}\neq\mathbf{y}} (-\Delta_L(\mathbf{z},\mathbf{y})), \quad \forall \mathbf{y} \in F. \tag{2.14}$$

Combining (2.13) and (2.14) gives

$$h(\mathbf{y}) < \sum_{\mathbf{z}:\mathbf{z}\in F,\mathbf{z}\neq\mathbf{y}} (-\Delta_L(\mathbf{z},\mathbf{y})) + \Delta_L(\mathbf{x},\mathbf{y}), \quad \forall \mathbf{y} \in F \setminus \{\mathbf{x}\}.$$

Since $\Delta_L(\mathbf{x},\mathbf{y}) \leq 0$ for $\mathbf{x} \neq \mathbf{y}$, this inequality means that h has a FSC on $F \setminus \{\mathbf{x}\}$ and this contradicts our assumption that h is allowed. Since any avalanche consists of addition of a particle and a series of topplings, the proof is completed. ∎

Definition 2.8 *Given a pair (Λ_L, Δ_L), let $G_L^{(v)} = \Lambda_L \cup \{\mathbf{r}\}$ with an additional vertex $\mathbf{r}$ (the 'root'), and $G_L^{(e)}$ be the collection of $|\Delta_L(\mathbf{x},\mathbf{y})|n = n$ edges between $\mathbf{x},\mathbf{y} \in \Lambda_L, \mathbf{x} \neq \mathbf{y}$, and $\sum_{\mathbf{y}:\mathbf{y}\in\Lambda_L} \Delta(\mathbf{x},\mathbf{y})n = m$ edges between $\mathbf{x} \in \Lambda_L$ and $\mathbf{r}$. (See Fig.5.) Graph G_L associated to (Λ_L, Δ_L) is defined as*

$$G_L = (G_L^{(v)}, G_L^{(e)}).$$

Definition 2.9 *We say a graph T on G_L is a spanning tree, if the number of vertices of T is $|G_L^{(v)}| = |\Lambda_L| + 1$, the number of connected components is one, and the number of loops is zero.*

Lemma 2.10 *Let $\mathcal{T}_L = \{$spanning tree on G_L associated to $(\Lambda_L, \Delta_L)\}$. Then*

$$|\mathcal{T}_L| = n^{(2L+1)^d} \det \Delta_L.$$

Proof. See p.133 of [20] and Theorem 6.3 in [4].

Lemma 2.11 (Majumdar and Dhar [20]) *There establishes a bijection between $\mathcal{A}_L$ and $\mathcal{T}_L$.*

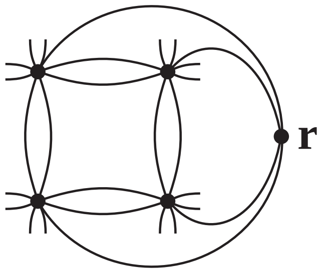

Figure 5: A part of the graph $G_L = (G_L^{(v)}, G_L^{(e)})$ associated to the DASM (Λ_L, Δ_L) is illustrated for the case that $d = 2, n = 2$ and $m = 1$. In this case, each pair of the nearest-neighbor vertices are connected by $n = 2$ edges and each vertex is connected to the 'root' $\mathbf{r}$ by $m = 1$ edge.

Proof. First we order all edges incident on each site $\mathbf{x} \in G_L^{(v)}$ in some order of preference. For each configuration $h \in \mathcal{A}_L$, we consider a following discrete-time growth process of graph on G_L, which is called a *burning process* on (G_L, h). Let $\tilde{V}_0 = V_0 = \{\mathbf{r}\}$, $E_0 = \emptyset$ and $T_0 = (V_0, E_0)$. Assume that we have nonempty sets $T_t = (V_t, E_t)$ and $\tilde{V}_t$ with $t \in \mathbb{N}_0$. Let

$$\tilde{V}_{t+1} = \left\{ \mathbf{y} \in G_L^{(v)} \setminus V_t : h(\mathbf{y}) \geq \sum_{\mathbf{x}:\mathbf{x} \in G_L^{(v)} \setminus V_t} (-\Delta_L(\mathbf{x}, \mathbf{y})) \right\}.$$

For each $\mathbf{y} \in \tilde{V}_{t+1}$, consider

$$\tilde{E}_{t+1}(\mathbf{y}) = \left\{ e \in G_L^{(e)} : e \text{ connects } \mathbf{y} \text{ and a site in } \tilde{V}_t \right\}.$$

We must have

$$h(\mathbf{y}) \leq \sum_{\mathbf{x}:\mathbf{x} \in G_L^{(v)} \setminus V_t} (-\Delta_L(\mathbf{x}, \mathbf{y})) + |\tilde{E}_{t+1}(\mathbf{y})|,$$

since $h \in \mathcal{S}_L$. If $|\tilde{E}_{t+1}(\mathbf{y})| = 1$, then name that edge as $e(\mathbf{y})$. If $|\tilde{E}_{t+1}(\mathbf{y})| \geq 2$, then write

$$h(\mathbf{y}) = \sum_{\mathbf{x}:\mathbf{x} \in G_L^{(v)} \setminus V_t} (-\Delta_L(\mathbf{x}, \mathbf{y})) + \frac{s}{n},$$

and choose the $(s+1)$-th edge in $\tilde{E}_{t+1}(\mathbf{y})$ as $e(\mathbf{y})$. We define

$$V_{t+1} = V_t \cup \tilde{V}_{t+1}, \quad E_{t+1} = E_t \cup \{e(\mathbf{y}) : \mathbf{y} \in \tilde{V}_{t+1}\}, \quad \text{and} \quad T_{t+1} = (V_{t+1}, E_{t+1}).$$

By the assumption $h \in \mathcal{A}_L$, there is a finite time $\sigma < \infty$ such that $V_\sigma = G_L^{(v)}$ and $E_\sigma = G_L^{(s)}$. By the construction, $T_\sigma = (V_\sigma, E_\sigma)$ is a spanning tree on G_L. Since this growth process of $T_t, t \in \{0, 1, \cdots, \sigma\}$ is deterministic for a given configuration $h \in \mathcal{A}_L$, it gives an injection from $\mathcal{A}_L$ to $\mathcal{T}_L$. This fact and Lemma 2.10 give $|\mathcal{A}_L| \leq |\mathcal{T}_L| = n^{(2L+1)^d} \det \Delta_L$. On the other hand, Lemmas 2.3 and 2.7 give $n^{(2L+1)^d} \det \Delta_L \leq |\mathcal{A}_L|$. Then we can conclude $|\mathcal{A}_L| = n^{(2L+1)^d} \det \Delta_L$ and the burning process gives a bijection between $\mathcal{A}_L$ and $\mathcal{T}_L$. ∎

Combining Lemmas 2.3, 2.7, 2.10, and 2.11, we have the following proposition.

Proposition 2.12 *For the DASM on Λ_L, $\mathcal{R}_L = \mathcal{A}_L$.*

3 Avalanche Propagators

3.1 Integral expressions for propagators

Define

$$G_L(\mathbf{x}, \mathbf{y}) = \mathbf{E}_L[T(\mathbf{x}, \mathbf{y}, h)], \quad \mathbf{x}, \mathbf{y} \in \Lambda,$$

where $T(\mathbf{x}, \mathbf{y}, h)$ is given by (1.2) and the expectation is taken over configurations $\{h\}$ in the stationary distribution $\mathbf{P}_L$. $G_L(\mathbf{x}, \mathbf{y})$ is regarded as the *avalanche propagator* from $\mathbf{x}$ to $\mathbf{y}$ [6]. Sometime in an avalanche caused by a deposit of a grain of sand at $\mathbf{x}$, this site $\mathbf{x}$ topples many times. The set of topplings between the first and the second toppling at $\mathbf{x}$ is called the first *wave* of toppling. There can occur many waves in one avalanche and $G_L(\mathbf{x}, \mathbf{x})$ gives the average number of waves of topplings in an avalanche [15].

Consider the stationary distribution $\mathbf{P}_L$ of the DASM. For addition of a particle at any site $\mathbf{x} \in \Lambda_L$, the averaged influx of grains of sand into a site $\mathbf{z} \in \Lambda_L$ is given by $\mathbf{1}(\mathbf{z} = \mathbf{x}) + \sum_{\mathbf{y}:\mathbf{y} \neq \mathbf{z}} G_L(\mathbf{x}, \mathbf{y})|\Delta_L(\mathbf{y}, \mathbf{z})|n$, and the averaged outflux of them out of $\mathbf{z}$ by $G_L(\mathbf{x}, \mathbf{z})\Delta_L(\mathbf{z}, \mathbf{z})n$ using the avalanche propagators. In $\mathbf{P}_L$, equivalence between influx and outflux must hold at any site $\mathbf{z} \in \Lambda_L$. This balance equation is written as

$$\sum_{\mathbf{y}:\mathbf{y} \in \Lambda_L} G_L(\mathbf{x}, \mathbf{y})\Delta_L(\mathbf{y}, \mathbf{z}) = \frac{1}{n}\mathbf{1}(\mathbf{z} = \mathbf{x}) \quad \forall \mathbf{x}, \mathbf{z} \in \Lambda_L$$

and thus the propagator is given using the inverse matrix of Δ_L.

Lemma 3.1 (Dhar [6])

$$G_L(\mathbf{x}, \mathbf{y}) = \frac{1}{n}[\Delta_L^{-1}](\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \Lambda_L. \tag{3.1}$$

The matrix Δ_L can be diagonalized by the Fourier transformation from $\mathbf{x} = (x_1, \cdots, x_d)$ to $\mathbf{n} = (n_1, \cdots, n_d)$,

$$U_L(\mathbf{n}, \mathbf{x}) = U_L^{-1}(\mathbf{x}, \mathbf{n}) = \frac{1}{(2L+1)^{d/2}} \exp\left(\frac{2\pi}{2L+1}\mathbf{x} \cdot \mathbf{n}\right),$$

where $\mathbf{x} \cdot \mathbf{n} = \sum_{i=1}^d x_i n_i$, as

$$\begin{aligned}
&\sum_{\mathbf{x}:\mathbf{x} \in \Lambda_L} \sum_{\mathbf{y}:\mathbf{y} \in \Lambda_L} U_L(\mathbf{n}, \mathbf{x})\Delta_L(\mathbf{x}, \mathbf{y})U_L^{-1}(\mathbf{y}, \mathbf{m}) \\
&\quad = 2d\left\{(1+a) - \frac{1}{d}\sum_{i=1}^d \cos\left(\frac{2\pi}{2L+1}n_i\right)\right\}\mathbf{1}(\mathbf{n} = \mathbf{m}) \\
&\quad \equiv \Lambda_L(\mathbf{n}, \mathbf{m}), \quad \mathbf{n}, \mathbf{m} \in \Lambda_L.
\end{aligned}$$

Then, (3.1) is obtained as

$$\begin{aligned} G_L(\mathbf{x},\mathbf{y}) &= \frac{1}{n}\sum_{\mathbf{n}:\mathbf{n}\in\Lambda_L}\sum_{\mathbf{m}:\mathbf{m}\in\Lambda_L} U_L^{-1}(\mathbf{x},\mathbf{n})[\Delta_L^{-1}](\mathbf{n},\mathbf{m})U_L(\mathbf{m},\mathbf{y}) \\ &= \frac{1}{2dn}\frac{1}{(2L+1)^d}\sum_{\mathbf{n}:\mathbf{n}\in\Lambda_L}\frac{\mathrm{e}^{-2\pi\sqrt{-1}(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}/(2L+1)}}{(1+a)-(1/d)\sum_{i=1}^d\cos(\frac{2\pi}{2L+1}n_i)}.nonumber \qquad (3.2) \end{aligned}$$

Lemma 3.2 *There exists a limit* $G(\mathbf{x}-\mathbf{y}) = \lim_{L\uparrow\infty} G_L(\mathbf{x},\mathbf{y}), \mathbf{x},\mathbf{y}\in\mathbb{Z}^d$ *and*

$$G(\mathbf{x}) = \frac{1}{2dn}\prod_{i=1}^d\int_{-\pi}^{\pi}\frac{d\theta_i}{2\pi}\frac{\mathrm{e}^{-\sqrt{-1}\mathbf{x}\cdot\boldsymbol{\theta}}}{(1+a)-(1/d)\sum_{i=1}^d\cos\theta_i}, \quad \mathbf{x}\in\mathbb{Z}^d. \tag{3.3}$$

Proof. Consider the Euler-Maclaurin formula for $f\in \mathrm{C}^2(\mathbb{R})$,

$$\sum_{n=0}^M f(b+nc) = \frac{1}{c}\int_b^{b+Mc} f(\theta)d\theta + \frac{1}{2}[f(b)+f(b+Mc)] + \frac{1}{12}c^2\sum_{n=0}^{M-1} f^{(2)}(b+c(n+\phi)), \tag{3.4}$$

where $M\in\mathbb{N}$, $b,c\in\mathbb{R}$, $f^{(2)}(\theta)$ is the second derivative of $f(\theta)$, and $0<\phi<1$ (see, for instance, Appendix D in [1]). Assume that

$$f(\theta) = \frac{\mathrm{e}^{-\sqrt{-1}\alpha_1\theta}}{(1+a)-(1/d)(\cos\theta+\alpha_2)},$$

where a, α_1, α_2 are constants. Applying the Euler-Maclaurin formula (3.4) with $b = -2\pi L/(2L+1)$, $M = 2L$ and $c = 2\pi/(2L+1)$, we have

$$\begin{aligned} &\sum_{n=0}^{2L}\frac{\mathrm{e}^{-2\pi\sqrt{-1}\alpha_1(n-L)/(2L_1+1)}}{(1+a)-(1/d)\{\cos(\frac{2\pi}{2L+1}(n-L))+\alpha_2\}} \\ &\quad = (2L+1)\int_{-2\pi L/(2L+1)}^{2\pi L/(2L+1)}\frac{d\theta}{2\pi}\frac{\mathrm{e}^{-\sqrt{-1}\alpha_1\theta}}{(1+a)-(1/d)(\cos\theta+\alpha_2)} \\ &\quad\quad +\frac{1}{2}\left[f\left(-\frac{2\pi L}{2L+1}\right)+f\left(\frac{2\pi L}{2L+1}\right)\right] \\ &\quad\quad +\frac{1}{12}\left(\frac{2\pi}{2L+1}\right)^2\sum_{n=0}^{2L-1} f^{(2)}\left(\frac{2\pi}{2L+1}(n+\phi-L)\right). \end{aligned}$$

By dividing the both sides of the equality by $2L+1$ and take the limit $L\uparrow\infty$, we obtain

$$\begin{aligned} &\lim_{L\uparrow\infty}\frac{1}{2L+1}\sum_{n=-L}^{L}\frac{\mathrm{e}^{-2\pi\sqrt{-1}\alpha_1 n/(2L_1+1)}}{(1+a)-(1/d)\{\cos(\frac{2\pi}{2L+1}n)+\alpha_2\}} \\ &\quad = \int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\frac{\mathrm{e}^{-\sqrt{-1}\alpha_1\theta}}{(1+a)-(1/d)(\cos\theta+\alpha_2)}. \end{aligned}$$

Repeating this procedure d times, we can prove Lemma 3.2. ∎

3.2 Long-distance asymptotics

Now we consider the asymptotic form in $|\mathbf{x}| \uparrow \infty$ of $G(\mathbf{x})$. Here we follow the calculation found in Section XII.4 of [21] for the asymptotic expansion of two-point spin correlation function of the two-dimensional Ising model. By using the identity

$$\int_0^\infty ds \mathrm{e}^{-\alpha s} = \frac{1}{\alpha}$$

and the definition of the modified Bessel function of the first kind

$$I_n(z) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \mathrm{e}^{-\sqrt{-1}n\phi + z\cos\phi},$$

we have

$$G(\mathbf{x}) = \frac{1}{2dn} \int_0^\infty ds \mathrm{e}^{-(1+a)s} \prod_{i=1}^d I_{x_i}(s/d).$$

The asymptotic expansion of $I_n(z)$ for large n is found on p.86 in [9],

$$I_n(z) = \frac{1}{\sqrt{2\pi}} \frac{\exp\left[(n^2+z^2)^{1/2} - n\sinh^{-1}(n/z)\right]}{(n^2+z^2)^{1/4}} \times (1 + \mathcal{O}(1/n)),$$

and we obtain

$$\begin{aligned} G(\mathbf{x}) = \frac{1}{2dn} \left(\frac{1}{2\pi}\right)^{d/2} \int_0^\infty ds \prod_{i=1}^d \frac{1}{[x_i^2 + (s/d)^2]^{1/4}} \exp[-g(\mathbf{x}, s)] \\ \times \left(1 + \mathcal{O}(\max_i \{1/x_i\})\right), \end{aligned} \tag{3.5}$$

where

$$g(\mathbf{x}, s) = (1+a)s - \sum_{i=1}^d \left[x_i^2 + \left(\frac{s}{d}\right)^2\right]^{1/2} + \sum_{i=1}^d x_i \sinh^{-1}\left(\frac{d}{s}x_i\right).$$

We can evaluate (3.5) by the saddle-point method and obtain the following result.

Theorem 3.3 *Let*

$$c_1(d, a) = \frac{1}{4\pi(a+1)} \left[\frac{\sqrt{a(a+2)d}}{2\pi(a+1)}\right]^{(d-3)/2} \tag{3.6}$$

and

$$\xi(d, a) = \frac{1}{\sqrt{d}\sinh^{-1}\sqrt{a(a+2)}}. \tag{3.7}$$

Then, for the DASM with $d \geq 2, m, n \in \mathbb{N}, a = m/(2dn)$,

$$\lim_{r\uparrow\infty} -\frac{1}{r} \log\left[\frac{nr^{(d-1)/2}}{c_1(d, a)} G(\mathbf{x}(r))\right] = \frac{1}{\xi(d, a)}, \tag{3.8}$$

where

$$\mathbf{x}(r) = \left(\frac{r}{\sqrt{d}}, \cdots, \frac{r}{\sqrt{d}}\right) \in \mathbb{Z}^d, \quad r > 0. \tag{3.9}$$

Proof. Let $g^{(1)}(\mathbf{x}, s)$ and $g^{(2)}(\mathbf{x}, s)$ be the first and second derivatives of $g(\mathbf{x}, s)$ with respect to s,

$$\begin{aligned} g^{(1)}(\mathbf{x}, s) &= (1+a) - \frac{1}{d}\sum_{i=1}^{d}\left[1+\left(\frac{d}{s}x_i\right)^2\right]^{1/2}, \\ g^{(2)}(\mathbf{x}, s) &= \frac{d}{s^3}\sum_{i=1}^{d} x_i^2\left[1+\left(\frac{d}{s}x_i\right)^2\right]^{-1/2}. \end{aligned}$$

For each $\mathbf{x}$, let $s_0(\mathbf{x})$ be the saddle point at which $g^{(1)}(\mathbf{x}, s)$ vanishes,

$$g^{(1)}(\mathbf{x}, s_0(\mathbf{x})) = 0. \tag{3.10}$$

Then

$$\begin{aligned} G(\mathbf{x}) &= \frac{1}{2dn}\left(\frac{1}{2\pi}\right)^{d/2}\prod_{i=1}^{d}\frac{1}{(x_i^2+s_0(\mathbf{x})^2/d^2)^{1/4}}\exp[-g(x, s_0(\mathbf{x}))] \\ &\quad \times \int_{-\infty}^{\infty} du \exp\left[-\frac{1}{2}g^{(2)}(\mathbf{x}, s_0(\mathbf{x}))u^2\right] \times \left(1+\mathcal{O}(\max_i\{1/x_i\})\right) \\ &= \frac{1}{2dn}\left(\frac{1}{2\pi}\right)^{d/2}\prod_{i=1}^{d}\frac{1}{(x_i^2+s_0(x)^2/d^2)^{1/4}}\exp[-g(x, s_0(x))] \\ &\quad \times \left(\frac{2\pi}{g^{(2)}(\mathbf{x}, s_0(\mathbf{x}))}\right)^{1/2} \times \left(1+\mathcal{O}(\max_i\{1/x_i\})\right). \end{aligned}$$

Here we can prove that the higher derivatives of $g(\mathbf{x}, s)$ only give the contributions of order $\mathcal{O}(\max_i\{1/x_i\})$. See p.304 in [21]. Now we consider the case

$$x_i = \frac{r}{\sqrt{d}} + \varepsilon_i,$$

in which ε_i's are finite and fixed and $r \gg 1$. The equation (3.10) for the saddle point is now

$$\sum_{i=1}^{d}\left(1+\frac{d^2}{s_0(\mathbf{x})^2}\left(\frac{r}{\sqrt{d}}+\varepsilon_i\right)^2\right)^{1/2} = (1+a)d,$$

and it is solved as

$$s_0(\mathbf{x}) = \sqrt{\frac{d}{a(a+2)}}\left(r+\frac{1}{\sqrt{d}}\sum_{i=1}^{d}\varepsilon_i+\mathcal{O}(1/r)\right).$$

This gives

$$\begin{aligned} g(\mathbf{x}, s_0(\mathbf{x})) &= \sum_{i=1}^{d}\left(\frac{r}{\sqrt{d}}+\varepsilon_i\right)\sinh^{-1}\left[\frac{d}{s_0(x)}\left(\frac{r}{\sqrt{d}}+\varepsilon_i\right)\right] \\ &= \sqrt{d}r\sinh^{-1}\sqrt{a(a+2)}+\sinh^{-1}\sqrt{a(a+2)}\times\sum_{i=1}^{d}\varepsilon_i+\mathcal{O}(1/r) \end{aligned}$$

and

$$g^{(2)}(\mathbf{x}, s_0(x)) = \frac{1}{\sqrt{d}} \frac{(a(a+2))^{3/2}}{a+1} \frac{1}{r} + \mathcal{O}(1/r^2).$$

Then we have the estimation

$$G(\mathbf{x}) = \frac{c_1(d,a)}{n} \frac{1}{r^{(d-1)/2}} \exp\left[-\frac{r}{\xi(d,a)} - \lambda(a)\sum_{i=1}^{d} \varepsilon_i\right] \times (1 + \mathcal{O}(1/r)), \quad \text{as } r \uparrow \infty$$

for $\mathbf{x} = (r/\sqrt{d} + \varepsilon_1, \cdots, r/\sqrt{d} + \varepsilon_d)$, where $c_1(d,a)$ and $\xi(d,a)$ are given by (3.6) and (3.7), respectively, and

$$\begin{aligned} \lambda(a) &\equiv \frac{\sqrt{d}}{\xi(d,a)} \\ &= \sinh^{-1}\sqrt{a(a+2)} \\ &= \log(1 + a + \sqrt{a(a+2)}). \end{aligned} \tag{3.11}$$

If we put $\varepsilon_i = 0, 1 \leq i \leq d$, then $G(\mathbf{x})$ is reduced to be

$$G(\mathbf{x}(r)) = \bar{G}(r) \times (1 + \mathcal{O}(1/r)), \quad \text{as } r \uparrow \infty$$

with

$$\bar{G}(r) = \frac{c_1(d,a)}{n} \frac{e^{-r/\xi(d,a)}}{r^{(d-1)/2}}. \tag{3.12}$$

It proves the theorem. ▮

4 Height-0 Density and Height-$(0,0)$ Correlations

For

$$\alpha, \beta \in \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, h_c - \frac{1}{n}\right\},$$

define

$$\begin{aligned} P_{\alpha,L}(\mathbf{x}) &= \mathbf{E}_L[\mathbf{1}(h(\mathbf{x}) = \alpha)], \\ P_{\alpha\beta,L}(\mathbf{x},\mathbf{y}) &= \mathbf{E}_L[\mathbf{1}(h(\mathbf{x}) = \alpha)\mathbf{1}(h(\mathbf{y}) = \beta)], \quad \mathbf{x},\mathbf{y} \in \Lambda_L. \end{aligned} \tag{4.1}$$

$P_{\alpha,L}(\mathbf{x})$ is the probability that the site $\mathbf{x}$ has the height αn measured in the unit of grain of sand, $1/n$, and $P_{\alpha\beta,L}(\mathbf{x},\mathbf{y})$ is the (α,β)-*height correlation function* [19, 5, 23].

For the two-dimensional BTW model on B_L with open boundary condition, Majumdar and Dhar [19] proved the existence of the infinite-volume limits

$$\begin{aligned} P_0 &= \lim_{L\uparrow\infty} P_{0,L}(\mathbf{x}), \\ P_{00}(\mathbf{x}(r)) &= \lim_{L\uparrow\infty} P_{00,L}(0,\mathbf{x}(r)), \end{aligned}$$

where $\mathbf{x}(r) = (r/\sqrt{2}, r/\sqrt{2})$. They gave an 8×8 matrix $M_L(r)$, whose elements depend on L and r, such that

$$P_{00,L}(0, \mathbf{x}(r)) = \det M_L(r), \quad \forall L > \frac{r}{\sqrt{2}},$$

and showed that every elements converge in the infinite-volume limit $L \uparrow \infty$ with a finite r. Then the matrix $M(r) = \lim_{L\uparrow\infty} M_L(r)$ is well-defined and we have the determinantal expression

$$P_{00}(\mathbf{x}(r)) = \det M(r).$$

Moreover, they showed that

$$\lim_{r\uparrow\infty} P_{00}(\mathbf{x}(r)) = P_0^2,$$

and

$$C_{00}(\mathbf{x}(r)) \equiv \frac{P_{00}(\mathbf{x}(r)) - P_0^2}{P_0^2} \simeq -\frac{1}{2} r^{-4}, \quad \text{as } r \uparrow \infty. \tag{4.2}$$

Majumdar and Dhar claimed [19] that the result (4.2) is generalized for the d-dimensional BTW model with $d \geq 2$ as

$$C_{00}(\mathbf{x}(r)) \sim r^{-2d}, \quad \text{as } r \uparrow \infty. \tag{4.3}$$

In an earlier paper [28], all these facts also hold for the two-dimensional DASM, if we prepare 10×10 matrix $M_L(r)$. (See also [5] and [23] for other generalizations of [19].) Here we show the result for the height-0 density and the height-$(0,0)$ correlations of the DASM with general $d \geq 2$.

4.1 Nearest-neighbor correlations

First we prove the following Lemma.

Lemma 4.1 *Any configuration $h \in \mathcal{S}_L$, in which there are two adjacent sites $\mathbf{z}_1, \mathbf{z}_2 \in \Lambda_L$, $|\mathbf{z}_1 - \mathbf{z}_2| = 1$, such that $h(\mathbf{z}_1) < 1$ and $h(\mathbf{z}_2) < 1$, is not allowed.*

Proof. Let $F = \{\mathbf{z}_1, \mathbf{z}_2\} \subset \Lambda_L$. Then

$$\sum_{\mathbf{x}:\mathbf{x}\in F, \mathbf{x}\neq\mathbf{z}_1} (-\Delta_L(\mathbf{x}, \mathbf{z}_1)) = -\Delta_L(\mathbf{z}_2, \mathbf{z}_1) = 1,$$

and

$$\sum_{\mathbf{x}:\mathbf{x}\in F, \mathbf{x}\neq\mathbf{z}_2} (-\Delta_L(\mathbf{x}, \mathbf{z}_2)) = -\Delta_L(\mathbf{z}_1, \mathbf{z}_2) = 1,$$

by (1.1). Then if $h(\mathbf{z}_1) < 1$ and $h(\mathbf{z}_2) < 1$, the condition of FSC (2.12) is satisfied. ∎

By Propositions 2.6 and 2.12, the above lemma implies the following.

Proposition 4.2 *For any $L \geq 2$,*

$$P_{\alpha\beta,L}(0, \pm\mathbf{e}_i) = 0, \quad 1 \leq i \leq d, \quad \alpha, \beta \in \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, 1 - \frac{1}{n}\right\}.$$

Then,

$$P_{\alpha\beta}(0, \pm\mathbf{e}_i) = \lim_{L\uparrow\infty} P_{\alpha\beta,L}(0, \pm\mathbf{e}_i) = 0, \quad 1 \leq i \leq d, \quad \alpha, \beta \in \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, 1 - \frac{1}{n}\right\}.$$

4.2 Determinatal expressions of $P_{0,L}(0)$ and $P_{00,L}(0,\mathbf{x})$

Let $\mathbf{e}_i, 1 \le i \le d$ be the i-th unit vector in $\mathbb{Z}^d$. Define a real symmetric matrix with size $(2L+1)^d$ as

$$B_L^{(0)}(\mathbf{v},\mathbf{w}) = \begin{cases} -h_c + 1/n, & \text{if} \quad \mathbf{v} = \mathbf{w} = 0, \\ -1, & \text{if} \quad \mathbf{v} = \mathbf{w}, |\mathbf{v}| = 1, \mathbf{v} \neq -\mathbf{e}_d, \\ -1 + 1/n, & \text{if} \quad \mathbf{v} = \mathbf{w} = -\mathbf{e}_d, \\ 1, & \text{if} \quad \mathbf{v} = 0, |\mathbf{w}| = 1, \mathbf{w} \neq -\mathbf{e}_d, \\ 1 - 1/n, & \text{if} \quad \mathbf{v} = 0, \mathbf{w} = -\mathbf{e}_d, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathbf{v},\mathbf{w} \in \Lambda_L$.

Lemma 4.3 *Let E_L be the unit matrix with size $(2L+1)^d$. Then*

$$P_{0,L}(0) = \det\left(E_L + nG_L B_L^{(0)}\right).$$

Proof. Define a set of allowed configurations conditioned $h(0) = 0$,

$$\mathcal{A}_L^{(0)} = \{h \in \mathcal{A}_L : h(0) = 0\}.$$

By definition (4.1), Proposition 2.6 with Lemma 2.3 and Proposition 2.12 gives

$$P_{0,L}(0) = \frac{|\mathcal{A}_L^{(0)}|}{n^{(2L+1)^d} \det \Delta_L}. \tag{4.4}$$

Assume that $h \in \mathcal{A}_L^{(0)}$. Then as shown in the proof of Lemma 2.11 we can uniquely define a burning process $T_t, t \in \{0,1,\dots,{}^{\exists}\sigma\}$ on (G_L, h) associated that T_t becomes a spanning tree on G_L at time $t = \sigma$. Define a configuration h' as

$$h'(\mathbf{z}) = \begin{cases} h(\mathbf{z}) - 1, & \text{if} \quad |\mathbf{z}| = 1, \mathbf{z} \neq -\mathbf{e}_d, \\ h(\mathbf{z}) - 1 + 1/n, & \text{if} \quad \mathbf{z} = -\mathbf{e}_d, \\ h(\mathbf{z}), & \text{otherwise} \end{cases}$$

for $\mathbf{z} \in \Lambda_L$. Now we consider a new DASM which is defined by the matrix Δ'_L given by

$$\Delta'_L = \Delta_L + B_L^{(0)}, \tag{4.5}$$

and let $\mathcal{A}'_L$ be a set of all allowed configurations of this DASM and G'_L be an associated graph to (Λ_L, Δ'_L). Then we consider a burning process $T'_t = (V'_t, E'_t), t \in \{0,1,\dots,\sigma\}$ on (G'_L, h'). By definition of Δ'_L and h', we can make

$$V_t = V'_t, \quad \forall t \in \{0,1,\dots,\sigma\},$$

and T'_σ gives a spanning tree on G'_L. By Lemma 2.11, this means $h' \in \mathcal{A}'_L$. Since there is a bijection between h and its associated burning process $T_t, t \in \{0,1,\dots,\sigma\}$, we have a bijection between $\mathcal{A}_L^{(0)}$ and $\mathcal{A}'_L$. By Lemmas 2.10 and 2.11, $|\mathcal{A}_L^{(0)}| = |\mathcal{A}'_L| = n^{(2L+1)^d} \det \Delta'_L$. Combining (4.4) and (4.5) gives

$$\begin{aligned} P_{0,L}(0) &= \frac{\det \Delta'_L}{\det \Delta_L} \\ &= \det(\Delta_L^{-1}\Delta'_L) \\ &= \det(E_L + \Delta_L^{-1} B_L^{(0)}). \end{aligned}$$

Then we use Lemma 3.1 and the proof is completed. ∎

Next we consider the two-point function $P_{00,L}(0, \mathbf{x})$, where we assume that $2 \le |\mathbf{x}| < L$. We define a real symmetric matrix with size $(2L+1)^d$ as follows. For $\mathbf{v}, \mathbf{w} \in \Lambda_L$,

$$B_L^{(0,\mathbf{x})}(\mathbf{v}, \mathbf{w}) = \begin{cases} -h_c + 1/n, & \text{if } \mathbf{v} = \mathbf{w} = 0 \text{ or if } \mathbf{v} = \mathbf{w} = \mathbf{x}, \\ -1, & \text{if } \mathbf{v} = \mathbf{w}, |\mathbf{v}| = 1, \mathbf{v} \ne -\mathbf{e}_d, \\ & \text{or if } \mathbf{v} = \mathbf{w}, |\mathbf{v} - \mathbf{x}| = 1, \mathbf{v} \ne \mathbf{x} - \mathbf{e}_d, \\ -1 + 1/n, & \text{if } \mathbf{v} = \mathbf{w} = -\mathbf{e}_d, \quad \text{or if } \mathbf{v} = \mathbf{w} = \mathbf{x} - \mathbf{e}_d, \\ 1, & \text{if } \mathbf{v} = 0, |\mathbf{w}| = 1, \mathbf{w} \ne -\mathbf{e}_d, \\ & \text{or if } \mathbf{v} = \mathbf{x}, |\mathbf{w} - \mathbf{x}| = 1, \mathbf{w} \ne \mathbf{x} - \mathbf{e}_d \\ 1 - 1/n, & \text{if } \mathbf{v} = 0, \mathbf{w} = -\mathbf{e}_d, \\ & \text{or if } \mathbf{v} = \mathbf{x}, \mathbf{w} = \mathbf{x} - \mathbf{e}_d, \\ 0, & \text{otherwise.} \end{cases}$$

Following the same argument as $P_{0,L}(0)$ we can prove the next lemma. (See Fig.6.)

Lemma 4.4 *For* $2 \le |\mathbf{x}| < L$,

$$P_{00,L}(0, \mathbf{x}) = \det\left(E_L + nG_L B_L^{(0,\mathbf{x})}\right).$$

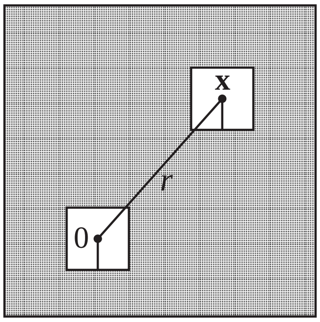

Figure 6: The matrix $\Delta''_L \equiv \Delta_L + B_L^{(0,\mathbf{x})}$ is considered for $P_{00,L}(0, \mathbf{x})$ with $|\mathbf{x}| = r$. In the corresponding graph G''_L the site 0 (resp. $\mathbf{x}$) is connected to $-\mathbf{e}_d$ (resp. $\mathbf{x} - \mathbf{e}_d$) by a single edge, but all other edges between 0 (resp. $\mathbf{x}$) and its nearest-neighbor sites are deleted.

4.3 Infinite-volume limit

Since the number of nonzero elements of $B_L^{(0)}$ (resp. $B_L^{(0,\mathbf{x})}$) is only $6d+1$ (resp. $2(6d+1)$), we can replace the matrix $E_L + nG_L B_L^{(0)}$ (resp. $E_L + nG_L B_L^{(0,\mathbf{x})}$) with size $(2L+1)^d$ by a matrix with size $(2d+1)$ (resp. $2(2d+1)$) without changing the value of determinant. Explicit expressions are given as follows.

Let

$$\mathbf{q}_i = \begin{cases} 0, & \text{if} \quad i=1, \\ \mathbf{e}_{i-1}, & \text{if} \quad 2 \le i \le d+1, \\ -\mathbf{e}_{i-d-1}, & \text{if} \quad d+2 \le i \le 2d+1. \end{cases}$$

Define a matrix $\mathcal{G}^{(L)}(\mathbf{x}) = (\mathcal{G}_{ij}^{(L)})_{1\le i,j \le 2d+1}$ with elements

$$\mathcal{G}_{ij}^{(L)}(\mathbf{x}) = G_L(0, \mathbf{x}+\mathbf{q}_j - \mathbf{q}_i), \quad 1 \le i,j \le 2d+1. \tag{4.6}$$

We also define a real symmetric matrix $\mathcal{B} = (\mathcal{B}_{ij})_{1\le i,j\le 2d+1}$ with elements

$$\mathcal{B}_{ij} = \begin{cases} -h_{\mathrm{c}} + 1/n, & \text{if} \quad i=j=1, \\ -1, & \text{if} \quad 2 \le i = j \le 2d, \\ -1+1/n, & \text{if} \quad i=j=2d+1, \\ 1, & i=1, 2\le j \le 2d, \\ 1-1/n, & \text{if} \quad i=1, j=2d+1, \\ 0, & \text{otherwise}. \end{cases}$$

Then define $2(2d+1)\times 2(2d+1)$ matrices

$$\tilde{\mathcal{G}}^{(L)}(0,\mathbf{x}) = \begin{pmatrix} \mathcal{G}^{(L)}(0) & \mathcal{G}^{(L)}(\mathbf{x}) \\ {}^t\mathcal{G}^{(L)}(\mathbf{x}) & \mathcal{G}^{(L)}(0) \end{pmatrix}, \quad \mathbf{x} \in \Lambda_L,$$

where ${}^t\mathcal{G}^{(L)}(\mathbf{x})$ is a transpose of $\mathcal{G}^{(L)}(\mathbf{x})$, and

$$\tilde{\mathcal{B}} = \begin{pmatrix} \mathcal{B} & 0 \\ 0 & \mathcal{B} \end{pmatrix}.$$

We have

$$P_{0,L}(0) = \det\left(E + n\mathcal{G}^{(L)}(0)\mathcal{B}\right) \tag{4.7}$$

and

$$P_{00,L}(0,\mathbf{x}) = \det\left(E + n\tilde{\mathcal{G}}^{(L)}(0,\mathbf{x})\tilde{\mathcal{B}}\right), \tag{4.8}$$

where E denotes the unit matrix with size $2d+1$ in (4.7) and with size $2(2d+1)$ in (4.8), respectively.

It should be remarked that the sizes of the matrices in the RHS's are independent of the lattice size L and determined only by the dimension d of lattice. The dependence of L is introduced only through each elements of $\mathcal{G}^{(L)}(\mathbf{x})$ given by (4.6). Lemma 3.2 guarantees the existence of infinite-volume limit $L \uparrow \infty$ of these elements and we put

$$\begin{aligned} \mathcal{G}_{ij}(\mathbf{x}) &= \lim_{L\uparrow\infty} \mathcal{G}_{ij}^{(L)}(\mathbf{x}) = G(\mathbf{x}+\mathbf{q}_j-\mathbf{q}_i), \quad 1\le i,j\le 2d+1, \\ \mathcal{G}(\mathbf{x}) &= (\mathcal{G}_{ij}(\mathbf{x}))_{1\le i,j\le 2d+1}, \\ \tilde{\mathcal{G}}(0,\mathbf{x}) &= \lim_{L\uparrow\infty} \tilde{\mathcal{G}}^{(L)}(0,\mathbf{x}) = \begin{pmatrix} \mathcal{G}(0) & \mathcal{G}(\mathbf{x}) \\ {}^t\mathcal{G}(\mathbf{x}) & \mathcal{G}(0) \end{pmatrix}, \end{aligned}$$

where $G(\mathbf{x})$ is explicitly given by (3.3). Then we have the following.

Proposition 4.5 *There exist the infinite-volume limits*

$$P_0 = \lim_{L\uparrow\infty} P_{0,L}(0), \quad P_{00}(\mathbf{x}) = \lim_{L\uparrow\infty} P_{00,L}(0,\mathbf{x}), \quad \mathbf{x} \in \mathbb{Z}^d,$$

and they are given by

$$P_0 = \det\left(E + n\mathcal{G}(0)\mathcal{B}\right)$$

and

$$P_{00}(\mathbf{x}) = \det\left(E + n\tilde{\mathcal{G}}(0,\mathbf{x})\tilde{\mathcal{B}}\right), \quad \mathbf{x} \in \mathbb{Z}^d.$$

4.4 Evaluations of determinantal expressions

From the determinantal expressions of P_0 and $P_{00}(\mathbf{x})$ given in Proposition 4.5, the following explicit evaluations of these quantities are obtained.

Theorem 4.6 (i) *Define*

$$\gamma_1 = \frac{1}{2d}\prod_{i=1}^{d}\int_{-\pi}^{\pi}\frac{d\theta_i}{2\pi}\frac{1}{(1+a)-(1/d)\sum_{i=1}^{d}\cos\theta_i}$$

and

$$\gamma_2 = \frac{1}{2d}\prod_{i=1}^{d}\int_{-\pi}^{\pi}\frac{d\theta_i}{2\pi}\frac{\mathrm{e}^{-2\sqrt{-1}(\theta_1+\theta_2)}}{(1+a)-(1/d)\sum_{i=1}^{d}\cos\theta_i}.$$

Then, for the DASM with $d \geq 2, m, n \in \mathbb{N}$,

$$\begin{aligned} P_0 &= \frac{1-2da\gamma_1}{2dn}\left[2\{1-d(\gamma_1-\gamma_2)\}+(1-4d\gamma_1)a-2d\gamma_1 a^2\right] \\ &\times \left[2(d-1)(\gamma_1-\gamma_2)-(1-4d\gamma_1)a+2d\gamma_1 a^2\right]^2 \\ &\times \left[\{1-(\gamma_1-\gamma_2)\}^2-\{(2d(1+a)^2-1)\gamma_1-(2d-1)\gamma_2-(1+a)\}^2\right]^{d-2}, \end{aligned} \tag{4.9}$$

where $a = m/(2dn)$.

(ii) *Let*

$$C_{00}(\mathbf{x}) = \frac{P_{00}(\mathbf{x})-P_0^2}{P_0^2}, \quad \mathbf{x} \in \mathbb{Z}^d. \tag{4.10}$$

Then, there exists a nonzero factor $c_2(d,a,n)$ such that for the DASM with $d \geq 2, m, n \in \mathbb{N}$

$$\lim_{r\uparrow\infty} -\frac{1}{r}\log\left[\frac{r^{d-1}}{c_2(d,a,n)}C_{00}(\mathbf{x}(r))\right] = \frac{2}{\xi(d,a)}, \tag{4.11}$$

where $a = m/(2dn)$, $\xi(d,a)$ and $\mathbf{x}(r)$ are given by (3.7) and (3.9), respectively, and that

$$\lim_{a\downarrow 0}\frac{c_2(d,a,m/(2da))}{a^{(d+1)/2}} = \left(\frac{d}{2\pi^2}\right)^{(d-3)/2}\left[\frac{d\{1+(d-1)\bar{\gamma}\}}{2\pi(d-1)\bar{\gamma}}\right]^2, \tag{4.12}$$

where

$$\bar{\gamma} = \frac{1}{2d}\prod_{i=1}^{d}\int_{-\pi}^{\pi}\frac{d\theta_i}{2\pi}\frac{1-\mathrm{e}^{-2\sqrt{-1}(\theta_1+\theta_2)}}{1-(1/d)\sum_{i=1}^{d}\cos\theta_i}.$$

In the following, we will explain how to prove this theorem. Let

$$M^{(1)}(r) = E + n\tilde{\mathcal{G}}(0,\mathbf{x}(r))\tilde{\mathcal{B}}, \quad r>0, \quad \mathbf{x}(r)\in\mathbb{Z}^d,$$

where E is a unit matrix with size $2(2d+1)$. That is,

$$M^{(1)}(r) = \begin{pmatrix} m^{(1)} & \tilde{m}^{(1)}(r) \\ \hat{m}^{(1)}(r) & m^{(1)} \end{pmatrix},$$

where for $1 \le i \le 2d+1$

$$m_{ij}^{(1)} = \begin{cases} \mathbf{1}(i=1)+\sum_{k=1}^{2d+1} n\mathcal{G}_{ik}(0) \\ \quad -\{(1-1/n)+h_{\mathrm{c}}\}n\mathcal{G}_{i1}(0)-\mathcal{G}_{i\,2d+1}(0), & \text{if} \quad j=1, \\ \mathbf{1}(i=j)+n[\mathcal{G}_{i1}(0)-\mathcal{G}_{ij}(0)], & \text{if} \quad 2\le j\le 2d, \\ \mathbf{1}(i=2d+1)+(1-1/n)n[\mathcal{G}_{i1}(0)-\mathcal{G}_{i\,2d+1}(0)], & \text{if} \quad j=2d+1, \end{cases}$$

$$\tilde{m}_{ij}^{(1)}(r) = n\times\begin{cases} \sum_{k=1}^{2d+1}\mathcal{G}_{ik}(\mathbf{x}(r)) \\ \quad -\{(1-1/n)+h_{\mathrm{c}}\}\mathcal{G}_{i1}(\mathbf{x}(r))-(1/n)\mathcal{G}_{i\,2d+1}(\mathbf{x}(r)), & \text{if} \quad j=1, \\ \mathcal{G}_{i1}(\mathbf{x}(r))-\mathcal{G}_{ij}(\mathbf{x}(r)), & \text{if} \quad 2\le j\le 2d, \\ (1-1/n)(\mathcal{G}_{i1}(\mathbf{x}(r))-\mathcal{G}_{i\,2d+1}(\mathbf{x}(r))), & \text{if} \quad j=2d+1, \end{cases}$$

$$\hat{m}_{ij}^{(1)}(r) = n\times\begin{cases} \sum_{k=1}^{2d+1}\mathcal{G}_{ki}(\mathbf{x}(r)) \\ \quad -\{(1-1/n)+\eta_{\mathrm{c}}\}\mathcal{G}_{1i}(\mathbf{x}(r))-(1/n)\mathcal{G}_{2d+1\,i}(\mathbf{x}(r)), & \text{if} \quad j=1, \\ \mathcal{G}_{1i}(\mathbf{x}(r))-\mathcal{G}_{ji}(\mathbf{x}(r)), & \text{if} \quad 2\le j\le 2d, \\ (1-1/n)(\mathcal{G}_{1i}(\mathbf{x}(r))-\mathcal{G}_{2d+1\,i}(\mathbf{x}(r))), & \text{if} \quad j=2d+1. \end{cases}$$

We find that

$$m_{i1}^{(1)}+\sum_{j=2}^{2d+1}m_{ij}^{(1)} = 1-2dan\mathcal{G}_{i1}(0),$$

$$\tilde{m}_{i1}^{(1)}(r)+\sum_{j=2}^{2d+1}\tilde{m}_{ij}^{(1)} = -2dan\mathcal{G}_{i1}(\mathbf{x}(r)),$$

$$\hat{m}_{i1}^{(1)}(r)+\sum_{j=2}^{2d+1}\hat{m}_{ij}^{(1)} = -2dan\mathcal{G}_{1i}(\mathbf{x}(r)), \quad 1\le i\le 2d+1.$$

For $1 \le i \le 2d+1$, let

$$
\begin{aligned}
m_{ij} &= \begin{cases} 1 - 2dan\mathcal{G}_{i1}(0), & \text{if} \quad j = 1, \\ m^{(1)}_{ij}, & \text{if} \quad 2 \le j \le 2d+1, \end{cases} \\
\tilde{m}_{ij}(r) &= \begin{cases} -2dan\mathcal{G}_{i1}(\mathbf{x}(r)), & \text{if} \quad j = 1, \\ \tilde{m}^{(1)}_{ij}(r), & \text{if} \quad 2 \le j \le 2d+1, \end{cases} \\
\hat{m}_{ij}(r) &= \begin{cases} -2dan\mathcal{G}_{1i}(\mathbf{x}(r)), & \text{if} \quad j = 1, \\ \hat{m}^{(1)}_{ij}(r), & \text{if} \quad 2 \le j \le 2d+1. \end{cases}
\end{aligned}
$$

Then

$$
\begin{aligned}
P_0 &= \det m^{(1)} = \det m, \\
P_{00}(\mathbf{x}(r)) &= \det M^{(1)}(r) = \det M(r) \quad \text{with} \quad M(r) = \begin{pmatrix} m & \tilde{m}(r) \\ \hat{m}(r) & m \end{pmatrix}.
\end{aligned} \tag{4.13}
$$

Note that, if we introduce the the *dipole potential*

$$
\phi_{(i_1,j_1),(i_2,j_2)}(\mathbf{x}(r)) = \mathcal{G}_{i_1 j_1}(\mathbf{x}(r)) - \mathcal{G}_{i_2 j_2}(\mathbf{x}(r)), \quad 1 \le i_1, i_2, j_1, j_2 \le 2d+1,
$$

the elements of the matrix $M(r)$ are expressed as follows; for $1 \le i \le 2d+1$,

$$
m_{ij} = \begin{cases} 1 - 2dan\mathcal{G}_{i1}(0), & \text{if} \quad j = 1, \\ \mathbf{1}(i = j) + n\phi_{(i,1),(i,j)}(0), & \text{if} \quad 2 \le j \le 2d, \\ \mathbf{1}(i = 2d+1) + (1 - 1/n)n\phi_{(i,1),(i,2d+1)}(0), & \text{if} \quad j = 2d+1, \end{cases} \tag{4.14}
$$

$$
\tilde{m}_{ij}(r) = n \times \begin{cases} -2da\mathcal{G}_{i1}(\mathbf{x}(r)), & \text{if} \quad j = 1, \\ \phi_{(i,1),(i,j)}(\mathbf{x}(r)), & \text{if} \quad 2 \le j \le 2d, \\ (1 - 1/n)\phi_{(i,1),(i,2d+1)}(\mathbf{x}(r)), & \text{if} \quad j = 2d+1, \end{cases}
$$

$$
\hat{m}_{ij}(r) = n \times \begin{cases} -2da\mathcal{G}_{1i}(\mathbf{x}(r)), & \text{if} \quad j = 1, \\ \phi_{(1,i),(j,i)}(\mathbf{x}(r)), & \text{if} \quad 2 \le j \le 2d, \\ (1 - 1/n)\phi_{(1,i),(2d+1,i)}(\mathbf{x}(r)), & \text{if} \quad j = 2d+1. \end{cases}
$$

Now we study the asymptotics of $P_{00}(r)$ in $r \uparrow \infty$. Theorem 3.3 and its proof given in Section 3 implies that with any finite c_i's,

$$
G\left(\mathbf{x}(r) + \sum_{i=1}^{d} c_i \mathbf{e}_i\right) = \bar{G}(r) \exp\left(-\lambda(a) \sum_{i=1}^{d} c_i\right) \times \left(1 + \mathcal{O}(1/r)\right), \quad \text{as } r \uparrow \infty
$$

with (3.6),(3.7), (3.11), and (3.12). Then we see

$$
\begin{aligned}
\tilde{m}(r) &= n\bar{G}(r)n(r, \lambda)(1 + \mathcal{O}(1/r)), \\
\hat{m}(r) &= n\bar{G}(r)n(r, -\lambda)(1 + \mathcal{O}(1/r)), \quad \text{as } r \uparrow \infty,
\end{aligned}
$$

where $n(r,\lambda) = (n_{ij}(r,\lambda))_{1\le i,j\le 2d+1}$ with elements,

$$n_{ij}(r,\lambda) = \begin{cases} -2da, & \text{if} \quad i=j=1, \\ (1-\mathrm{e}^{-\lambda}), & \text{if} \quad i=1, 2\le j\le d+1, \\ (1-\mathrm{e}^{\lambda}), & \text{if} \quad i=1, d+2\le j\le 2d, \\ (1-1/n)(1-\mathrm{e}^{\lambda}), & \text{if} \quad i=1, j=2d+1, \\ -2da\mathrm{e}^{\lambda}, & \text{if} \quad 2\le i\le d+1, j=1, \\ -2da\mathrm{e}^{-\lambda}, & \text{if} \quad d+2\le i\le 2d+1, j=1, \\ \mathrm{e}^{\lambda}(1-\mathrm{e}^{-\lambda}), & \text{if} \quad 2\le i,j\le d+1, \\ \mathrm{e}^{\lambda}(1-\mathrm{e}^{\lambda}), & \text{if} \quad 2\le i\le d+1, d+2\le j\le 2d, \\ (1-1/n)\mathrm{e}^{\lambda}(1-\mathrm{e}^{\lambda}), & \text{if} \quad 2\le i\le d+1, j=2d+1, \\ \mathrm{e}^{-\lambda}(1-\mathrm{e}^{-\lambda}), & \text{if} \quad d+2\le i\le 2d+1, 2\le j\le d+1, \\ \mathrm{e}^{-\lambda}(1-\mathrm{e}^{\lambda}), & \text{if} \quad d+2\le i,j\le 2d, \\ (1-1/n)\mathrm{e}^{-\lambda}(1-\mathrm{e}^{\lambda}), & \text{if} \quad d+2\le i\le 2d+1, j=2d+1. \end{cases}$$

We obtain a matrix $M'(r)$ from $M(r)$ by subtracting (the first row) $\times\mathrm{e}^{\lambda}$ from the i-th row with $2\le i\le d+1$, (the first row) $\times\mathrm{e}^{-\lambda}$ from the i-th row with $d+2\le i\le 2d+1$, (the $(2d+2)$-th row) $\times\mathrm{e}^{-\lambda}$ from the i-th row with $2d+3\le i\le 3d+2$, and (the $(2d+2)$-th row) $\times\mathrm{e}^{\lambda}$ from the i-th row with $3d+3\le i\le 2(2d+1)$. We have

$$M'(r) = \begin{pmatrix} m'(\lambda) & \tilde{m}'(r,\lambda) \\ \tilde{m}'(r,-\lambda) & m'(-\lambda) \end{pmatrix}$$

with

$$m'_{ij}(\lambda) = \begin{cases} 1-2dan\mathcal{G}_{11}(0), & \text{if} \quad i=j=1, \\ n\phi_{(1,1),(1,j)}(0), & \text{if} \quad i=1, 2\le j\le 2d, \\ (1-1/n)n\phi_{(1,1),(1,2d+1)}(0), & \text{if} \quad i=1, j=2d+1, \\ (1-\mathrm{e}^{\lambda})-2dan(\mathcal{G}_{i1}(0)-\mathrm{e}^{\lambda}\mathcal{G}_{11}(0)), & \text{if} \quad 2\le i\le d+1, j=1, \\ (1-\mathrm{e}^{-\lambda})-2dan(\mathcal{G}_{i1}(0)-\mathrm{e}^{-\lambda}\mathcal{G}_{11}(0)), & \text{if} \quad d+2\le i\le 2d+1, j=1, \\ \mathbf{1}(i=j)+n[\phi_{(i,1),(i,j)}(0)-\mathrm{e}^{\lambda}\phi_{(1,1),(1,j)}(0)], & \text{if} \quad 2\le i\le d+1, 2\le j\le 2d, \\ \mathbf{1}(i=j)+n[\phi_{(i,1),(i,j)}(0)-\mathrm{e}^{-\lambda}\phi_{(1,1),(1,j)}(0)], & \text{if} \quad d+2\le i\le 2d+1, \\ & \qquad 2\le j\le 2d, \\ (1-1/n) & \\ \quad \times n[\phi_{(i,1),(i,2d+1)}(0)-\mathrm{e}^{\lambda}\phi_{(1,1),(1,2d+1)}(0)], & \text{if} \quad 2\le i\le d+1, j=2d+1, \\ \mathbf{1}(i=2d+1)+(1-1/n) & \\ \quad \times n[\phi_{(i,1),(i,2d+1)}(0)-\mathrm{e}^{-\lambda}\phi_{(1,1),(1,2d+1)}(0)], & \text{if} \quad d+2\le i\le 2d+1, \\ & \qquad j=2d+1, \end{cases} \tag{4.15}$$

and with

$$\tilde{m}'_{ij}(r,\lambda) = n\bar{G}(r)\times\begin{cases} -2da(1+\mathcal{O}(1/r)), & \text{if} \quad i=j=1, \\ (1-\mathrm{e}^{-\lambda})(1+\mathcal{O}(1/r)), & \text{if} \quad i=1, 2\le j\le d+1, \\ (1-\mathrm{e}^{\lambda})(1+\mathcal{O}(1/r)), & \text{if} \quad i=1, d+2\le j\le 2d, \\ (1-1/n)(1-\mathrm{e}^{\lambda})(1+\mathcal{O}(1/r)), & \text{if} \quad i=1, j=2d+1, \\ \mathcal{O}(1/r), & \text{otherwise}, \end{cases} \tag{4.16}$$

so that

$$P_0 0(\mathbf{x}(r)) = \det M(r) = \det M'(r), \quad r > 0, \quad \mathbf{x}(r) \in \mathbb{Z}^d.$$

Now we expand $\det M'(r)$ along the first and the $(2d+2)$-th rows. Let $|M'(j,k)|$ be the determinant of $M'(r)$ with the first and the $(2d+2)$-th rows and the j-th and the k-th columns removed and multiplied by $-(-1)^{1+j} \times (-1)^{2d+2+k} = (-1)^{j+k}$. Then we have

$$\det M'(r) = \sum_{j=1}^{2(2d+1)} \sum_{k=1,k\neq j}^{2(2d+1)} M'(r)_{1j} M'(r)_{2d+2,k} |M'(j,k)|.$$

Remark that, by (4.15) and (4.16),

$$|M'(j,k)| = \mathcal{O}(1/r), \quad \text{as } r \to \infty,$$

if $1 \le j,k \le 2d+1$ or $2d+2 \le j,k \le 2(2d+1)$, and

$$|M'(j,k)| = |m'^{(j)}(\lambda)| \times |m'^{(k)}(\lambda)| \times (1 + \mathcal{O}(1/r)), \quad \text{as } r \to \infty,$$

if $1 \le j \le 2d+1 < k \le 2(2d+1)$ or $1 \le k \le 2d+1 < j \le 2(2d+1)$, where $|m'^{(j)}(\lambda)|$ is the $(1,j)$-cofactor of $m'(\lambda)$. Then

$$\begin{aligned}
\det M'(r) &= \left(\sum_{j=1}^{2d+1} m'_{1j}(\lambda) |m'^{(j)}(\lambda)| \right) \left(\sum_{j=1}^{2d+1} m'_{1j}(-\lambda) |m'^{(j)}(-\lambda)| \right) \\
&\quad + \left(\sum_{j=1}^{2d+1} \tilde{m}'_{1j}(r,\lambda) |m'^{(j)}(-\lambda)| \right) \left(\sum_{j=1}^{2d+1} \tilde{m}'_{1j}(r,-\lambda) |m'^{(j)}(-\lambda)| \right) \\
&= \det m'(\lambda) \times \det m'(-\lambda) + \det \bar{m}(\lambda) \times \det \bar{m}(-\lambda) \times \left(n\bar{G}(r) \right)^2 (1 + \mathcal{O}(1/r)), \quad (4.17)
\end{aligned}$$

where $\bar{m}(\lambda) = (\bar{m}_{ij}(\lambda))_{1 \le i,j \le 2d+1}$ with elements

$$\bar{m}_{ij}(\lambda) = \begin{cases} -2da, & \text{if} \quad i = j = 1, \\ 1 - \mathrm{e}^{\lambda}, & \text{if} \quad i = 1, 2 \le j \le d+1, \\ 1 - \mathrm{e}^{-\lambda}, & \text{if} \quad i = 1, d+2 \le j \le 2d, \\ (1 - 1/n)(1 - \mathrm{e}^{-\lambda}), & \text{if} \quad i = 1, j = 2d+1, \\ m'_{ij}(\lambda), & \text{otherwise.} \end{cases}$$

We find that

$$\det m'(\lambda) = \det m'(-\lambda) = \det m. \quad (4.18)$$

The determinantal expressions (4.13) with (3.12), (4.17), and (4.18) give

$$\begin{aligned}
\lim_{r \uparrow \infty} P_{00}(\mathbf{x}(r)) &= \lim_{r \uparrow \infty} \{ (\det m)^2 + \det \bar{m}(\lambda) \det \tilde{m}(-\lambda) (n\bar{G}(r))^2 \} \\
&= (\det m)^2 = P_0^2.
\end{aligned}$$

Here we set

$$\det \bar{m}(\lambda) = a \det m^*(\lambda),$$

with a matrix $m^*(\lambda) = (m^*_{ij}(\lambda))_{1\le i,j\le 2d+1}$ with elements

$$m^*_{ij}(\lambda) = \begin{cases} -2d, & \text{if } i=j=1, \\ (1-\mathrm{e}^{\lambda})/a^{1/2}, & \text{if } i=1, 2\le j\le d+1, \\ (1-\mathrm{e}^{-\lambda})/a^{1/2}, & \text{if } i=1, d+2\le j\le 2d, \\ (1-1/n)(1-\mathrm{e}^{-\lambda})/a^{1/2}, & \text{if } i=1, j=2d+1, \\ (1-\mathrm{e}^{\lambda})/a^{1/2} & \\ \quad -2da^{1/2}n(\mathcal{G}_{i1}(0)-\mathrm{e}^{\lambda}\mathcal{G}_{11}(0)), & \text{if } 2\le i\le d+1, j=1, \\ (1-\mathrm{e}^{-\lambda})/a^{1/2} & \\ \quad -2da^{1/2}n(\mathcal{G}_{i1}(0)-\mathrm{e}^{-\lambda}\mathcal{G}_{11}(0)), & \text{if } d+2\le i\le 2d+1, j=1, \\ \mathbf{1}(i=j)+n[\phi_{(i,1),(i,j)}(0)-\mathrm{e}^{\lambda}\phi_{(1,1),(1,j)}(0)], & \text{if } 2\le i\le d+1, 2\le j\le 2d, \\ \mathbf{1}(i=j)+n[\phi_{(i,1),(i,j)}(0)-\mathrm{e}^{-\lambda}\phi_{(1,1),(1,j)}(0)], & \text{if } d+2\le i\le 2d+1, \\ & \quad 2\le j\le 2d, \\ (1-1/n) & \\ \quad \times n[\phi_{(i,1),(i,2d+1)}(0)-\mathrm{e}^{\lambda}\phi_{(1,1),(1,2d+1)}(0)], & \text{if } 2\le i\le d+1, j=2d+1, \\ \mathbf{1}(i=2d+1)+(1-1/n), & \\ \quad \times n[\phi_{(i,1),(i,2d+1)}(0)-\mathrm{e}^{-\lambda}\phi_{(1,1),(1,2d+1)}(0)], & \text{if } d+2\le i\le 2d+1, \\ & \quad j=2d+1. \end{cases} \tag{4.19}$$

By the definition (4.10), we see

$$C_{00}(\mathbf{x}(r)) = a^2\frac{\det m^*(\lambda)\det m^*(-\lambda)}{(\det m)^2}(n\bar{G}(r))^2\times(1+\mathcal{O}(1/r)), \quad \text{as } r\uparrow\infty.$$

Since $\bar{G}(r)$ is given by (3.12), (4.11) of Theorem 4.6 (ii) is proved with

$$c_2(d,a,n) = (ac_1(d,a))^2\frac{\det m^*(\lambda)\times\det m^*(-\lambda)}{(\det m)^2}.$$

Now the problem is reduced to the calculation of $\det m$ and $\det m^*(\lambda)$. Consider a matrix $R=(R_{ij})_{1\le i,j\le N}$ with elements

$$R_{ij} = \begin{cases} u, & \text{if} \quad i=j=1, \\ b, & \text{if} \quad i=1, 2\le j\le d+1, \\ c, & \text{if} \quad i=1, d+2\le j\le 2d, \\ (1-1/n)c, & \text{if} \quad i=1, j=2d+1, \\ q, & \text{if} \quad 2\le i\le d+1, j=1, \\ e, & \text{if} \quad d+2\le i\le 2d+1, j=1, \\ f, & \text{if} \quad 2\le i\le d+1, 2\le j\le 2d, j\ne i, j\ne i+d, \\ 1+v, & \text{if} \quad 2\le i=j\le d+1, \\ h, & \text{if} \quad 2\le i\le d, j=i+d, \\ (1-1/n)f, & \text{if} \quad 2\le i\le d, j=2d+1, \\ (1-1/n)h, & \text{if} \quad i=d+1, j=2d+1, \\ s, & \text{if} \quad d+2\le i\le 2d+1, 2\le j\le 2d, j\ne i, j\ne i-d, \\ t, & \text{if} \quad d+2\le i\le 2d+1, j=i-d, \\ 1+k, & \text{if} \quad d+2\le i=j\le 2d, \\ (1-1/n)s, & \text{if} \quad d+2\le i\le 2d, j=2d+1, \\ 1+(1-1/n)k, & \text{if} \quad i=j=2d+1. \end{cases} \tag{4.20}$$

We perform the following procedure on R.

(i) Subtract (the first row) $\times q/u$ from the i-th row with $2 \leq i \leq d+1$.

(ii) Subtract (the first row) $\times e/u$ from the i-th row with $d+2 \leq i \leq 2d+1$.

(iii) Subtract the second row from the i-th row with $3 \leq i \leq d+1$.

(iv) Subtract the $(d+2)$-th row from the i-th row with $d+3 \leq i \leq 2d+1$.

(v) Add the j-th column to the second column with $3 \leq j \leq d+1$.

(vi) Add the j-th column to the $(d+2)$-th column with $d+3 \leq j \leq 2d$.

(vii) Add (the $(2d+1)$-th column) $\times 1/(1-1/n)$ to the $(d+2)$-th column.

(viii) Subtract (the $(d+j)$-th column) $\times(t-s)/(1+k-s)$ from the j-th column with $3 \leq j \leq d$.

After these procedures, by changing the orders of rows and columns appropriately, we obtain the following identity.

$$\det R = u \times \left[1+v-f-\frac{t-s}{1+k-s}(h-f)\right]^{d-2} \times (1+k-s)^{d-2} \times \det S, \tag{4.21}$$

where $S = (S_i j)_{1 \leq i,j \leq 4}$ with elements

$$\begin{aligned}
&S_{11} = 1+v+(d-1)f - dbq/u, && S_{12} = h+(d-1)f - dcq/u, \\
&S_{13} = (1-1/n)(f-cq/u), && S_{14} = f - bq/u, \\
&S_{21} = t+(d-1)s - dbe/u, && S_{22} = 1+k+(d-1)s - dce/u, \\
&S_{23} = (1-1/n)(s-ce/u), && S_{24} = s - be/u, \\
&S_{31} = 0, && S_{32} = 1/(n-1), \\
&S_{33} = 1+(1-1/n)(k-s), && S_{34} = t-s, \\
&S_{41} = 0, && S_{42} = 0, \\
&S_{43} = (1-1/n)(h-f), && S_{44} = 1+v-f.
\end{aligned}$$

Define

$$g_0 = nG(0), \quad g_1 = nG(\mathbf{e}_1), \quad g_2 = nG(2\mathbf{e}_1), \quad g_3 = nG(\mathbf{e}_1+\mathbf{e}_2),$$

where $G(\mathbf{x})$ is given by (3.3) and $\mathbf{e}_1, \mathbf{e}_2$ are the unit vectors in the first and second directions in $\mathbb{Z}^d$. Since the system is isotropic, we can find that the matrix m defined by (4.14) is in the form (4.20) with

$$\begin{aligned}
&u = 1-2dag_0, && b = c = g_0 - g_1, \\
&q = e = 1-2dag_1, && f = s = g_1 - g_3, \\
&v = k = g_1 - g_0, && h = t = g_1 - g_2.
\end{aligned} \tag{4.22}$$

By Lemma 3.2 and the isotropy of the system gives

$$\begin{aligned}
&2d(1+a)g_0 - 2dg_1 = 1, \\
&2d(1+a)g_1 - (g_0 + g_2 + 2(d-1)g_3) = 0,
\end{aligned}$$

which are written as

$$\begin{aligned} g_1 &= (1+a)g_0 - \frac{1}{2d}, \\ g_2 &= [2d(1+a)^2-1]g_0 - 2(d-1)g_3 - (1+a). \end{aligned} \tag{4.23}$$

The formula (4.21) with (4.22) and (4.23) gives

$$\begin{aligned} P_0 = \det m &= \frac{1-2dag_0}{2dn}\,[2\{1-d(g_0-g_3)\} + (1-4dg_0)a - 2dg_0a] \\ &\times \left[2(d-1)(g_0-g_3) - (1-4dg_0)a + 2dg_0a^2\right]^2 \\ &\times \left[\{1-(g_0-g_3)\}^2 - (g_2-g_3)^2\right]^{d-2}. \end{aligned} \tag{4.24}$$

It proves (4.9) of Theorem 4.6 (i).

It should be noted that, if we put $n=1$ and take $a \downarrow 0$ limit in (4.24), we have the formula

$$P_0 = \frac{4(d-1)^2}{d}(1-d\bar{g}_{03})\bar{g}_{03}^2[(1-\bar{g}_{03})^2 - \bar{g}_{23}^2]^{d-2},$$

where

$$\bar{g}_{03} = \lim_{a\downarrow 0}(g_0-g_3), \qquad \bar{g}_{23} = \lim_{a\downarrow 0}(g_2-g_3).$$

In particular, $\bar{g}_{03} = 1/\pi$ and $\bar{g}_{23} = 1-1/\pi$ for $d=2$ [27], and thus we have

$$P_0 = \frac{2}{\pi^2}\left(1-\frac{2}{\pi}\right), \quad d=2.$$

This coincides with the value of P_0 obtained by Majumdar and Dhar [19] for the two-dimensional BTW model.

We can also find that the matrix $m^*(\lambda)$ defined by (4.19) is in the form (4.20) with

$$\begin{aligned} &u = -2d, && b = (1-e^{\lambda})/a^{1/2}, \\ &c = (1-e^{-\lambda})/a^{1/2}, && q = (1-e^{\lambda})/a^{1/2} - 2da^{1/2}(g_1 - e^{\lambda}g_0), \\ &e = (1-e^{-\lambda})/a^{1/2} - 2da^{1/2}(g_1 - e^{-\lambda}g_0), && f = (g_1-g_3) - e^{\lambda}(g_0-g_1), \\ &s = (g_1-g_3) - e^{-\lambda}(g_0-g_1), && v = (g_1-g_0) - e^{\lambda}(g_0-g_1), \\ &k = (g_1-g_0) - e^{-\lambda}(g_0-g_1), && h = (g_1-g_2) - e^{\lambda}(g_0-g_1), \\ &t = (g_1-g_2) - e^{-\lambda}(g_0-g_1). && \end{aligned}$$

The formula (4.21) gives

$$\det m^*(\lambda) = -2d\left[\{1-(g_0-g_3)\}^2 - (g_2-g_3)^2\right]^{d-2} \times \det S,$$

where

$$\det S = b_1(d,a,\lambda) + b_2(d,a,\lambda)\frac{1}{n}.$$

with some functions b_1 and b_2 of d, a, λ. Since (3.11) gives

$$e^{\lambda(a)} = 1 + a + \sqrt{a(a+2)} = 1 + \sqrt{2}a^{1/2} + \mathcal{O}(a), \quad \text{as } a \downarrow 0,$$

we found that

$$\begin{aligned} b_1(d,a,\lambda) &= \mathcal{O}(a^2), \\ b_2(d,a,\lambda) &= \frac{4(d-1)}{d}(g_0-g_3)\{1-d(g_0-g_3)\}\{1+(d-1)(g_0-g_3)\}+\mathcal{O}(a^{1/2}), \text{ as } a \downarrow 0. \end{aligned}$$

Thus we obtain

$$\lim_{a\downarrow 0} \frac{\det m^*(\lambda) \det m^*(-\lambda)}{(\det m)^2} = \left[\frac{2d\{1+(d-1)\bar{g}_{03}\}}{(d-1)\bar{g}_{03}} \right]^2.$$

Since $\lim_{a\downarrow 0} c_1(d,a)/a^{(d-3)/4} = (d/(2\pi^2))^{(d-3)/4}/(4\pi)$, (4.12) of Theorem 4.6 is proved.

5 Discussions

5.1 Critical exponent ν_a

The results (3.8) of Theorem 3.3 and (4.11) of Theorem 4.6 mean that both of $G(\mathbf{x}(r))$ and $C_{00}(\mathbf{x}(r))$ decay exponentially as increasing r with a correlation length $\xi(d,a)$. Since $\xi(d,a) < \infty$ for any $a > 0$, the stationary state of the DASM is non-critical [28]. Moreover the theorems imply that, if we make the parameter n be large with a fixed m, then the value of $a = m/(2dn)$ can be small and

$$nG(\mathbf{x}(r)) \simeq c_1(d)a^{(d-3)/4}\frac{e^{-r/\xi(d,a)}}{r^{(d-1)/2}}, \tag{5.1}$$

$$C_{00}(\mathbf{x}(r)) \simeq c_2(d)a^{(d+1)/2}\frac{e^{-2r/\xi(d,a)}}{r^{d-1}}, \quad \text{as } r \uparrow \infty, \tag{5.2}$$

where $c_1(d) = (d/(2\pi^2))^{(d-3)/4}/(4\pi)$ and $c_2(d)$ is given by (4.12).

Consider a series of DASMs with increasing n with a fixed m. Then we will have an increasing series of correlation lengths $\{\xi(d,a)\}$ and we will see the asymptotic divergence,

$$\xi(d,a) \simeq \frac{1}{\sqrt{2d}}a^{-\nu_a} \quad \text{as} \quad a \to 0 \tag{5.3}$$

with

$$\nu_a = \frac{1}{2} \quad \text{for all} \quad d \geq 2. \tag{5.4}$$

We notice that, if we identify a with a reduced temperature

$$t = \frac{|T-T_c|}{T_c} \tag{5.5}$$

around a critical temperature T_c in the equilibrium spin system, (5.1) with (5.3) and (5.4) is exactly in the Ornstein-Zernike form of correlations in the mean-field theory of equilibrium phase transitions (see, for instance, Eq.(61) in Section 3.1 of [14]). This implies that we can regard (5.3) as a critical phenomenon with a parameter a approaching to its critical value $a_c = 0$ and we can say that the associated *critical exponent* ν_a is exactly determined as (5.4). Vanderzande and Daerden discussed the exponent ν_a for the DASM on more general lattices [29].

This exponent may be identified with the critical exponent $\nu = 1/2$ obtained by Vespignani and Zapperi by the generalized mean-field theory [30]. They claimed that they made only use of conservation laws to evaluate $\nu = 1/2$ and thus at least on this result their mean-field theory is exact for any $d \geq 2$. The present work justifies their conjecture. We can conclude that with respect to the avalanche propagators and height-$(0,0)$ correlation functions the upper critical dimension of the ASM is two. This result does not contradict to the result by Priezzhev [24], since he studied the intersection phenomena of avalanches and for them the upper critical dimension is four.

The results (5.1) and (5.2) suggest that there exists a scaling limit such that

$$\lim_{\substack{r\uparrow\infty,a\downarrow 0:\\ a^{1/2}r=\kappa/\sqrt{2d}}} r^{d-2}nG(\mathbf{x}(r)) = \mathcal{F}_G(\kappa),$$
$$\lim_{\substack{r\uparrow\infty,a\downarrow 0:\\ a^{1/2}r=\kappa/\sqrt{2d}}} r^{2d}C_{00}(\mathbf{x}(r)) = \mathcal{F}_C(\kappa), \quad 0 < \kappa < \infty$$

with

$$\begin{aligned}\mathcal{F}_G(\kappa) &= 2^{-(d+1)/2}\pi^{-(d-1)/2}\kappa^{(d-3)/2}e^{-\kappa},\\ \mathcal{F}_C(\kappa) &= 2^{-(d+1)}\pi^{-(d-1)}\left[\frac{1+(d-1)\overline{\gamma}}{(d-1)\overline{\gamma}}\right]^2\kappa^{d+1}\mathrm{e}^{-\kappa},\end{aligned}$$

This observation is consistent with the statement

$$G(\mathbf{x}(r)) \sim r^{-(d-2)}, \quad \text{as } r\uparrow\infty \tag{5.6}$$

and (4.3) claimed by Majumdar and Dhar [19] for the self-organized criticality realized in the d-dimensional BTW model with $d \geq 2$. (Note that for the two-dimensional BTW model, $G(\mathbf{x}(r)) - G(0) \simeq -(1/2\pi)\log r$, as $r\uparrow\infty$.)

5.2 The $q \to 0$ limit of the Potts model

Majumdar and Dhar [20] discussed the relationship between the ASM and the $q\downarrow 0$ limit of the *q-state Potts model.* For $q \in \{2,3,\dots\}$, the q-state Potts model on the lattice $G_L = (G_L^{(v)}, G_L^{(e)})$ given by Definition 2.8 is defined as follows. At each vertex $\mathbf{v} \in G_L^{(v)} = \Lambda_L \cup \{\mathbf{r}\}$, put a spin variable $s(\mathbf{x}) \in \{1,2,\dots,q\}$. The Hamiltonian for the configuration $s = \{s(\mathbf{v})\}_{\mathbf{v}\in G_L^{(v)}}$ is given by

$$\mathcal{H}(s) = -\sum_{e=\{\mathbf{v},\mathbf{w}\}\in G_L^{(e)}} \mathbf{1}(s(\mathbf{v}) = s(\mathbf{w})).$$

The partition function of the Potts model in the Gibbs ensemble with a temperature $T > 0$ is defined by

$$\begin{aligned}Z(q,T) &= \sum_{s\in\{1,2,\dots,q\}^{G_L^{(v)}}} e^{-\mathcal{H}(s)/T}\\ &= \sum_{s\in\{1,2\dots,q\}^{G_L^{(v)}}}\prod_{e=\{\mathbf{v},\mathbf{w}\}\in G_L^{(e)}}\Big[1+\chi\mathbf{1}(s(\mathbf{v})=s(\mathbf{w}))\Big]\end{aligned} \tag{5.7}$$

with $\chi = e^{1/T} - 1$. We consider a subset of $G_L^{(e)}$ denoted by $E \subset G_L^{(e)}$. Each connected component in E is called a cluster. Let $c(E)$ be the number of disconnected clusters of E; $E = \bigcup_{i=1}^{c(E)} E_i$, where $E_i \cap E_j = \emptyset, i \neq j$. If a vertex $\mathbf{v} \in G_L^{(v)}$ is not connected by any edge in E, we write $\mathbf{v} \notin E$. By performing binomial expansions and taking the summation over spin configurations in (5.7), we obtain the Fortuin-Kasteleyn representation of partition function,

$$Z(q, T) = \sum_{E \subset G_L^{(e)}} q^{|\{\mathbf{v} \in G_L^{(v)}; \mathbf{v} \notin E\}|} q^{c(E)} \chi^{|E|}, \tag{5.8}$$

where $|E|$ denotes the number of edges in E. Note that we can regard (5.8) as a function of $q \in \mathbb{R}$ and $T > 0$. We consider the asymptotics of (5.8) in the limit $q \downarrow 0$. The dominant terms in this limit should be with E such that $c(E) = 1$ and $\{\mathbf{v} \in G_L^{(v)} : \mathbf{v} \notin E\} = \emptyset \iff E$ contains all vertices in $G_L^{(v)} \iff E$ is a spanning subgraph of G_L. If we further take the high-temperature limit $T \uparrow \infty \iff \chi \downarrow 0$, we have only spanning subgraphs with a minimal number of edges, which are just the spanning trees. Then we have

$$\lim_{T \uparrow \infty} \lim_{q \downarrow 0} T^{(2L+1)^d} q^{-1} Z(q, T) = |\mathcal{T}_L|,$$

where $\mathcal{T}_L$ is the collection of all spanning trees on G_L. As shown in Section 2.4, there establishes a bijection between $\mathcal{T}_L$ and $\mathcal{A}_L$ (Lemma 2.11) and $\mathcal{A}_L = \mathcal{R}_L$ (Proposition 2.12). (The relation between the $q \downarrow 0$ limit of the q-state Potts model with finite temperatures and the ASM is discussed in Section 7.2 in [7].) The two-dimensional q-state Potts model shows a continuous phase transition associated with *critical phenomena* at a finite temperature $0 < T_{\mathrm{c}} < \infty$ without external magnetic field $B = 0$, when $q = 2, 3$ and 4 [31].

Usual critical phenomena of spin models are specified by the behavior of two-point correlation functions for the energy density $G_\epsilon(r, t, b, L)$ and for the order-parameter density $G_\sigma(r, t, b, L)$. Here r denotes the distance of two points, t the reduced temperature (5.5), b the reduced external field

$$b = \frac{|B|}{T_{\mathrm{c}}},$$

and L the size of the lattice on which the model is defined. It is conjectured in the scaling theory that, if L is sufficiently large and we observe the system in the very vicinity of the critical point; $t \ll 1, b \ll 1$, the correlation functions behave as

$$\begin{aligned} G_\epsilon(r, t, b, L) &= L^{2x_\epsilon} \mathcal{F}_\epsilon \left(\frac{r}{L}, tL^{y_t}, bL^{y_b} \right), \\ G_\sigma(r, t, b, L) &= L^{2x_\sigma} \mathcal{F}_\sigma \left(\frac{r}{L}, tL^{y_t}, bL^{y_b} \right), \end{aligned} \tag{5.9}$$

with the scaling exponents x_ϵ, x_σ, y_ϵ, y_σ, and the scaling functions $\mathcal{F}_\epsilon, \mathcal{F}_\sigma$. If the system is of d-dimensional, the hyperscaling relations $x_\epsilon + y_t = d, x_\sigma + y_b = d$ hold (see, for instance, [13, 14]). From the scaling forms (5.9), we expect the power-law behavior of correlation functions at the critical point ($t = b = 0, L \uparrow \infty$) such that

$$G_\epsilon(r) \sim r^{-2x_\epsilon}, \quad G_\sigma(t) \sim r^{-2x_\sigma}, \quad \text{as } r \uparrow \infty,$$

and in the off-critical regions with $L \uparrow \infty$, the correlation length $\xi = \xi(t, b)$ behaves as

$$\xi(t,0) \sim t^{-\nu_t} \quad \text{with} \quad \nu_t = \frac{1}{y_t},$$
$$\xi(0,b) \sim b^{-\nu_b} \quad \text{with} \quad \nu_b = \frac{1}{y_b}, \quad \text{as } t \downarrow 0, b \downarrow 0.$$

For the two-dimensional q-state Potts model, the critical exponents are determined as functions of q through the parameter

$$u = u(q) = \frac{2}{\pi} \cos^{-1}\left(\frac{\sqrt{q}}{2}\right)$$

as [31]

$$x_\epsilon = \frac{1+u}{2-u}, \qquad y_t = 2 - x_\epsilon = \frac{3(1-u)}{2-u},$$
$$x_\sigma = \frac{1-u^2}{4(2-u)}, \qquad y_b = 2 - x_\sigma = \frac{(3-u)(5-u)}{4(2-u)}.$$

They give the limits

$$x_\epsilon \to 2, \quad y_t \to 0, \quad x_\sigma \to 0, \quad y_b \to 2, \quad \text{as } q \downarrow 0 \Longleftrightarrow u \uparrow 1.$$

Majumdar and Dhar [20] noted by their results (4.3) and (5.6) for the BTW models that the avalanche propagator $G(\mathbf{x}(r))$ and the height-$(0,0)$ correlation function $C_{00}(\mathbf{x}(r))$ in ASM play the roles of the order-parameter density correlation function $G_\sigma(r)$ and the energy density correlation function $G_\epsilon(r)$ in the critical phenomena, respectively. In particular, in the two-dimensional case, the power-law exponents are respectively given as

$$2x_\sigma\Big|_{q\downarrow 0} = 0 = (d-2)\Big|_{d=2}, \qquad 2x_\epsilon\Big|_{q\downarrow 0} = 4 = 2d\Big|_{d=2}.$$

Our interpretation of the present result (5.4) is that introduction of dissipation to the ASM may correspond to imposing an external magnetic field B to the Potts models and hence $\nu_a = 1/2$ is identified with

$$\nu_b\Big|_{q\downarrow 0} = \frac{1}{y_b}\Big|_{q\downarrow 0} = \frac{1}{2}.$$

We remark that the critical exponents for the specific heat α, for the order parameter β, and for the magnetic-field susceptibility γ of the

$$\alpha = \frac{2(1-2u)}{3(1-u)} \to -\infty, \quad \beta = \frac{1+u}{12} \to \frac{1}{6}, \quad \gamma = \frac{7-4u+u^2}{6(1-u)} \to \infty, \quad \text{as } q \downarrow 0 \Longleftrightarrow u \uparrow 1.$$

We suspect some interpretation of the value $\beta|_{q\downarrow 0} = 1/6$ in the DASM.

5.3 Recent topics on height correlations

In Section 4 the one-point and the two-point correlations of height-0 sites were calculated for the DASM with general $d \geq 2$. In the two-dimensional case, the three-point and the four-point correlations were also calculated for height-0 sites and general property of 'the height-0 field of ASMs' have been extensively studied from the view point of a $c = -2$ conformal field theory [18, 8].

For the two-dimensional BTW model, in which the values of stable height of sandpile are $h = 0, 1, 2,$ and 3, the height correlations have been calculated also for $h \geq 1$. Priezzhev determined P_α for $\alpha \in \{0, 1, 2, 3\}$, where the results with $\alpha \geq 1$ are expressed using multivariate integrals of determinantal integrands [23]. Poghosyan et al. [22] claimed that the height-0 state is the only one showing pure power-law-correlations and that general form of height correlations for $h \geq 1$ contains logarithmic functions. They showed that for $\alpha \geq 1$

$$C_{0\alpha}(\mathbf{x}(r)) = \frac{P_{0\alpha}(\mathbf{x}(r)) - P_0 P_\alpha}{P_0 P_\alpha} \simeq \frac{1}{r^4}(c_1 \log r + c_2), \quad \text{as } r \uparrow \infty$$

with some constants c_1, c_2. Moreover, they predicted that $C_{\alpha\beta}(\mathbf{x}(r)) \sim \log^2 r / r^4$ if $\alpha \geq 1$ and $\beta \geq 1$. These results are discussed with the logarithmic conformal field theory. See also [11]. We will see a lot of interesting open problems concerning height correlations for the BTW models and the DASMs in higher dimensions.

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