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Construction of Spline Dyadic Wavelet Filters

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Abstract: A sufficient condition for constructing spline dyadic wavelets is derived using Fourier transform of a reconstructing wavelet. Several spline dyadic wavelets satisfying this condition are designed, and applied to a medical image to enhance it.

Keywords: Dyadic wavelet, Fourier transform, Spline function, Image enhancement

1. Introduction

Recently, wavelet transform has been successfully used for signal and image processing. There are two types of wavelet transforms, one is of down-sampling type, and the other of dyadic type. For signal and image analysis, wavelet transform of down-sampling type has been frequently used because this type of wavelet is biorthogonal transform. However, down-sampling type of wavelet has a problem that is not shift-invariant. If wavelet transform does not have shift-invariant property, low and high frequency components obtained by the wavelet transform change by odd-shift of signals. To overcome this disadvantage, Mallat[3] proposed dyadic type of wavelet transform. Though dyadic wavelet is not biorthogonal transform, components of a signal decomposed using this wavelet are shift-invariant. Therefore, dyadic wavelet is convenient for noise reduction, edge detection and enhancement of images. Mallat derived a fast decomposition and reconstruction algorithm for dyadic wavelet transform. Using this algorithm, we can carry out effectively such image processing. Box-spline functions are often used to construct spline dyadic wavelets. Mallat constructed a quadratic spline dyadic wavelet and successfully applied to edge extraction of images.

In this paper, we give a sufficient condition for constructing spline dyadic wavelets. This condition is derived using Fourier transform of reconstructing wavelet filters. We design several spline dyadic wavelets satisfying the condition. Filters of a spline dyadic wavelet not satisfying the condition are not convergent to zero. In simulations, the designed filters are used to enhance a medical image.

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2. Dyadic Wavelet Theory

In this section, we shall describe how to design dyadic wavelets and fast dyadic wavelet transform.

2.1 Dyadic Wavelet Transform

Dyadic wavelets are designed by starting from a two-scale relation in terms of scaling function $\phi(t)$:

$$\phi(t) = \sum_k h[k]\sqrt{2}\phi(2t-k). \tag{1}$$

The Fourier transform of both sides in (1) yields

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2}}\hat{h}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right), \tag{2}$$

where the symbol ω represents the frequency, $\hat{\phi}(\omega)$ is the Fourier transform of $\phi(t)$ and $\hat{h}(\omega)$ is given by

$$\hat{h}(\omega) = \sum_k h[k]e^{-i\omega k}. \tag{3}$$

The equation (1) is equivalent to (2).

A dyadic wavelet function can be obtained in the form of

$$\psi(t) = \sum_k g[k]\sqrt{2}\phi(2t-k). \tag{4}$$

The equation (4) is equivalent to its Fourier transform

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right), \tag{5}$$

where $\hat{\psi}(\omega)$ is the Fourier transform of $\psi(t)$ and $\hat{g}(\omega)$ is obtained by replacing $h[k]$ in (3) by $g[k]$.

If the filters $h[k]$ and $g[k]$ are given, then the scaling function $\phi(t)$ and the dyadic wavelet function $\psi(t)$ are determined. To determine $h[k]$ and $g[k]$, we need dual scaling and wavelet functions. The dual scaling function $\tilde{\phi}(t)$ takes the same form as (1):

$$\tilde{\phi}(t) = \sum_k \tilde{h}[k]\sqrt{2}\tilde{\phi}(2t-k) \qquad (6)$$

which is equivalent to its Fourier transform

$$\hat{\tilde{\phi}}(\omega) = \frac{1}{\sqrt{2}}\hat{\tilde{h}}\left(\frac{\omega}{2}\right)\hat{\tilde{\phi}}\left(\frac{\omega}{2}\right).$$

We also need the dual wavelet function

$$\tilde{\psi}(t) = \sum_k \tilde{g}[k]\sqrt{2}\tilde{\phi}(2t-k) \qquad (7)$$

which is equivalent to its Fourier transform

$$\hat{\tilde{\psi}}(\omega) = \frac{1}{\sqrt{2}}\hat{\tilde{g}}\left(\frac{\omega}{2}\right)\hat{\tilde{\phi}}\left(\frac{\omega}{2}\right).$$

Under some conditions on the filters $h[k]$, $g[k]$, $\tilde{h}[k]$ and $\tilde{g}[k]$, we can expand a function $f(t)$ belonging to $L^2(R)$ in terms of the dual wavelet function $\tilde{\psi}(t)$. Let us define

$$\psi_{2^j}(t) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{t}{2^j}\right).$$

Then, the dyadic wavelet transform $d_j[n]$ of $f(t)$ is defined by

$$d_j[n] = \int_{-\infty}^{\infty} f(t)\psi_{2^j}(t-n)dt = (f * \bar{\psi}_{2^j})(n)$$

with $\bar{\psi}_{2^j}(t) = \psi_{2^j}(-t)$. The dyadic wavelet transform $d_j[n]$ is often called the dyadic wavelet coefficient.

Mallat proved the following theorem.

Theorem 1 (Mallat[3]). *If the Fourier transforms of filters $h[k]$, $g[k]$, $\tilde{h}[k]$ and $\tilde{g}[k]$ satisfy*

$$\hat{\tilde{h}}(\omega)\hat{h}^*(\omega) + \hat{\tilde{g}}(\omega)\hat{g}^*(\omega) = 2, \qquad \omega \in [-\pi, \pi], \quad (8)$$

then we have

$$\sum_{j=-\infty}^{\infty} \hat{\psi}^*(2^j\omega)\hat{\tilde{\psi}}(2^j\omega) = 1, \qquad \omega \in R - \{0\} \quad (9)$$

and

$$f(t) = \sum_{j=-\infty}^{\infty} \frac{1}{2^j}(d_j[\cdot] * \tilde{\psi}_{2^j})(t). \qquad (10)$$

*The symbol * denotes complex conjugation.*

The relation (9) implies that $\tilde{\psi}_{2^j}$ are biorthogonal. Therefore, (10) is a biorthogonal expansion of $f(t)$. The relation (8) will be used to design spline dyadic wavelets.

2.2 Fast Dyadic Wavelet Transform

Mallat proposed a method for computing the dyadic wavelet coefficients $d_j[n]$ fast. To describe the method, we shall introduce the dyadic scaling coefficients $a_j[n]$ as follows:

$$a_j[n] = \int_{-\infty}^{\infty} f(t)\phi_{2^j}(t-n)dt$$

with

$$\phi_{2^j}(t) = \frac{1}{\sqrt{2^j}}\phi\left(\frac{t}{2^j}\right).$$

Mallat's fast dyadic wavelet transform and its inverse are given by the following theorem.

Theorem 2 (Mallat[3]). *The fast dyadic wavelet transform is given by the recursion formulas*

$$a_{j+1}[n] = \sum_k h[k]a_j[n+2^jk], \quad j = 0,1,\cdots, \quad (11)$$

$$d_{j+1}[n] = \sum_k g[k]a_j[n+2^jk], \quad j = 0,1,\cdots. \quad (12)$$

The inverse formula is

$$a_j[n] = \frac{1}{2}\sum_k (\tilde{h}[k]a_{j+1}[n-2^jk] + \tilde{g}[k]d_{j+1}[n-2^jk]), \quad j = 0,1,\cdots. \quad (13)$$

The dyadic wavelet coefficients $d_j[n]$ in (10) can be computed recursively by (11) and (12) starting from

$$a_0[n] = \int_{-\infty}^{\infty} f(t)\phi(t-n)dt.$$

The signal $f(t)$ is given in the discrete form practically. So we must compute $a_0[n]$ approximately by using a quadrature formula. However, since the scaling function $\phi(t)$ is a local function, we usually approximate $a_0[n]$ by the value $f[n]$ of signal at the time n.

Theorem 2 describes decomposition and reconstruction formulas for one-dimensional signals. However, images can be decomposed and reconstructed if the formulas (11), (12) and (13) are applied to the image in the horizontal direction first and in the vertical direction next.

3. Spline Dyadic Wavelets

We design spline dyadic wavelets. A box-spline function $\phi(t)$ of degree m is defined as a translation of $m+1$ convolutions of characteristic function $\chi(t)$ defined by

$$\chi(t) = \begin{cases} 1, & -\frac{1}{2} \le t < \frac{1}{2}, \\ 0, & \text{otherwise} \end{cases}$$

if m is odd. If m is even, the resulting translation is centered at $t = 1/2$. The aim of this section is to obtain the filters $h[k]$, $g[k]$, $\tilde{h}[k]$ and $\tilde{g}[k]$ in (1), (4), (6) and (7), respectively, for the box-spline function $\phi(t)$.

It is well known that the function $\phi(t)$ satisfies a two-scale relation (1) and its Fourier transform is

$$\hat{\phi}(\omega) = e^{-i\varepsilon\omega/2}\left(\frac{\sin(\omega/2)}{\omega/2}\right)^{m+1} \tag{14}$$

with

$$\varepsilon = \begin{cases} 1 & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

Therefore, we can use the Fourier transform (2) to compute $\hat{h}(\omega)$ as

$$\hat{h}(\omega) = \sqrt{2}\frac{\hat{\phi}(2\omega)}{\hat{\phi}(\omega)} = \sqrt{2}e^{-i\varepsilon\omega/2}(\cos\frac{\omega}{2})^{m+1}. \tag{15}$$

For latter use, we write $\hat{h}_{m+1}(\omega,\varepsilon) = \hat{h}(\omega)$. The computation of the coefficients $h[k]$ is carried out by using the relation (3). It is convenient to write (15) in the form of z-transform

$$\hat{h}(\omega) = \sum_k h[k]z^{-k},$$

where $z = e^{i\omega}$. Using this z, (15) can be written as

$$\hat{h}(\omega) = \sqrt{2}z^{-\varepsilon/2}\left(\frac{z^{1/2}+z^{-1/2}}{2}\right)^{m+1}.$$

The coefficients $h[k]$ can be found by expanding the right hand side in terms of z.

Next, we compute the coefficients $g[k]$. Following the idea of Daubechies[1)], we choose

$$\hat{g}(\omega) = e^{-i\omega}\hat{h}_r^*(\omega+\pi, s).$$

The degree r is independent of m. Therefore,

$$\hat{g}(\omega) = (-1)^r\sqrt{2}e^{-i(\frac{2-s}{2}\omega - \frac{\pi}{2}s)}(\sin\frac{\omega}{2})^r, \tag{16}$$

where

$$s = \begin{cases} 1 & \text{if } r \text{ is odd,} \\ 0 & \text{if } r \text{ is even.} \end{cases}$$

The filters $g[k]$ can be computed by putting $z = e^{i\omega}$ in (16) and expanding the resulting equation in terms of z.

The dual scaling filters $\tilde{h}[k]$ are chosen as $\tilde{h}[k] = h[k]$.

The dual wavelet filters $\tilde{g}[k]$ are obtained from (8) as follows:

$$\hat{\tilde{g}}(\omega) = \frac{2-|\hat{h}(\omega)|^2}{\hat{g}^*(\omega)} \tag{17}$$

because we have $\hat{\tilde{h}}(\omega) = \hat{h}(\omega)$.

Theorem 3. *For $r = 1, 2$, the filters of $\hat{\tilde{g}}(\omega)$ are convergent to zero, and have finite length, that is, compact support. For $r \ge 3$, they are not convergent to zero.*

Proof. We have

$$2 - |\hat{h}(\omega)|^2 = 2\left(1 - \left(\cos\frac{\omega}{2}\right)^{2(m+1)}\right).$$

Using a well-known formula

$$\sum_{l=0}^{m} b^{2l} = \frac{1-b^{2(m+1)}}{1-b^2},$$

we get

$$1 - \left(\cos\frac{\omega}{2}\right)^{2(m+1)} = \left(\sin\frac{\omega}{2}\right)^2\sum_{l=0}^{m}\left(\cos\frac{\omega}{2}\right)^{2l}.$$

Therefore,

$$\hat{\tilde{g}}(\omega) = (-1)^r\sqrt{2}e^{-i(\frac{2-s}{2}\omega-\frac{\pi}{2}s)}\left(\sin\frac{\omega}{2}\right)^{2-r} \times \sum_{l=0}^{m}\left(\cos\frac{\omega}{2}\right)^{2l}. \tag{18}$$

From (18), we see that the power $2-r$ is negative for $r \ge 3$ and then the filters $\tilde{g}[n]$ included in (18) are not convergent to zero.

The filters $\tilde{g}[k]$ are obtainable by expanding (18) in terms of $z = e^{i\omega}$.

By inserting (14) and (16) into (5), we obtain

$$\hat{\psi}_r(\omega) = \left(-\frac{\omega}{4}\right)^r e^{-i(\frac{2+\varepsilon-s}{4}\omega-\frac{\pi}{2}s)} \times \left(\frac{\sin(\omega/4)}{\omega/4}\right)^{m+r+1}. \quad (19)$$

The inverse Fourier transform $\psi_r(t)$ of (19) is a desirable spline dyadic wavelet. The wavelet function $\psi_r(t)$ has a support of length $(m+r+1)/2$, and is antisymmetric in case of $r=1$, and symmetric in case of $r=2$.

We restrict $m=2$. **Figure 1** shows the quadratic spline function $\phi(t)$ and the first ($r=1$) quadratic spline dyadic wavelet $\psi_1(t)$. **Figure 2** illustrates the quadratic spline function $\phi(t)$ and the second ($r=2$) quadratic spline dyadic wavelet $\psi_2(t)$.

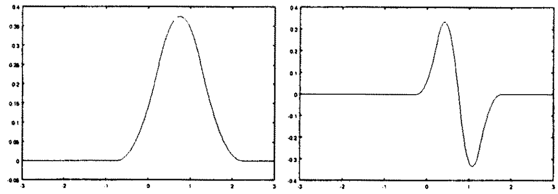

Fig.1 The quadratic spline function $\phi(t)$ (left) and the first ($r=1$) quadratic spline dyadic wavelet $\psi_1(t)$ (right).

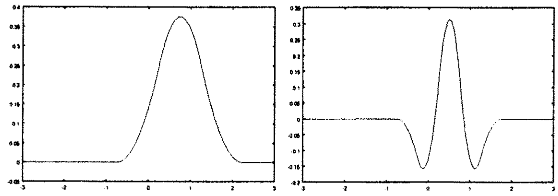

Fig.2 The quadratic spline function $\phi(t)$ (left) and the second ($r=2$) quadratic spline dyadic wavelet $\psi_2(t)$ (right).

Actual spline dyadic wavelet transform for signals and images is carried out by using the fast dyadic wavelet transform in Theorem 2. We need there only the filters $h[k]$, $g[k]$, $\tilde{h}[k]$ and $\tilde{g}[k]$. In Tables below, we list them in case of $r=1,2$, from $m=0$ until $m=4$.

4. Simulations

In simulations, we applied the fast dyadic wavelet transform algorithm with the obtained filters in Section 3 to a medical image in **Fig. 4** (left) to enhance it. First, we decompose the medical image until 3 times using the decomposition algorithm in Theorem 2. Next, we multiply the obtained high frequency components by an enhancement function

$$e(x) = \begin{cases} x-(K-1)T & \text{if } x < -T, \\ Kx & \text{if } |x| \le T, \\ x+(K-1)T & \text{if } x > T \end{cases}$$

and reconstruct using the reconstruction algorithm in Theorem 2 to get an enhanced image. We selected parameters K and T as $K=20$ and $T=1$[2]. The enhancement function $e(x)$ with these parameters is shown in **Fig. 3**.

Figure 4 (right) , **Fig. 5** (left) and **Fig. 5** (right) illustrate the enhancement of the medical image obtained by using spline dyadic wavelet filters with $r=2$ and $m=2$, where the analysis was performed up to the first, second and third level, respectively.

Table 1 Spline dyadic wavelet filters ($r=1, m=0,1,2,3,4$).

k	$\frac{g[k]}{\sqrt{2}}$	$m=0$: $\frac{h[k]}{\sqrt{2}}=\frac{\tilde{h}[k]}{\sqrt{2}}$	$m=0$: $\frac{\tilde{g}[k]}{\sqrt{2}}$	$m=1$: $\frac{h[k]}{\sqrt{2}}=\frac{\tilde{h}[k]}{\sqrt{2}}$	$m=1$: $\frac{\tilde{g}[k]}{\sqrt{2}}$
-1				0.25	-0.125
0	-0.5	0.5	-0.5	0.50	-0.625
1	0.5	0.5	0.5	0.25	0.625
2					0.125

k	$m=2$: $\frac{h[k]}{\sqrt{2}}=\frac{\tilde{h}[k]}{\sqrt{2}}$	$m=2$: $\frac{\tilde{g}[k]}{\sqrt{2}}$	$m=3$: $\frac{h[k]}{\sqrt{2}}=\frac{\tilde{h}[k]}{\sqrt{2}}$	$m=3$: $\frac{\tilde{g}[k]}{\sqrt{2}}$
-3				-0.008125
-2		-0.03125	0.0625	-0.057125
-1	0.125	-0.21875	0.2500	-0.300625
0	0.375	-0.68750	0.3750	-0.755635
1	0.375	0.68750	0.2500	0.755635
2	0.125	0.21895	0.0625	0.300625
3		0.03125		0.057125
4				0.008125

k	$m=4$: $\frac{h[k]}{\sqrt{2}}=\frac{\tilde{h}[k]}{\sqrt{2}}$	$m=4$: $\frac{\tilde{g}[k]}{\sqrt{2}}$
-4		-0.001953125
-3		-0.021484375
-2	0.03121875	-0.109375000
-1	0.15609375	-0.343750000
0	0.31218750	-0.691406250
1	0.31218750	0.691406250
2	0.15609375	0.343750000
3	0.03121875	0.109375000
4		0.021484375
5		0.001953125

Table 2 Spline dyadic wavelet filters ($r = 2, m = 0, 1, 2, 3, 4$)

k	$\frac{g[k]}{\sqrt{2}}$	$m=0$: $\frac{h[k]}{\sqrt{2}} = \frac{\tilde{h}[k]}{\sqrt{2}}$	$m=0$: $\frac{\tilde{g}[k]}{\sqrt{2}}$	$m=1$: $\frac{h[k]}{\sqrt{2}} = \frac{\tilde{h}[k]}{\sqrt{2}}$	$m=1$: $\frac{\tilde{g}[k]}{\sqrt{2}}$
-1				0.25	
0	-0.25	0.5		0.50	0.25
1	0.50	0.5	1	0.25	1.25
2	-0.25				0.25

k	$m=2$: $\frac{h[k]}{\sqrt{2}} = \frac{\tilde{h}[k]}{\sqrt{2}}$	$m=2$: $\frac{\tilde{g}[k]}{\sqrt{2}}$	$m=3$: $\frac{h[k]}{\sqrt{2}} = \frac{\tilde{h}[k]}{\sqrt{2}}$	$m=3$: $\frac{\tilde{g}[k]}{\sqrt{2}}$
-2			0.0625	0.01563125
-1	0.125	0.06250	0.2500	0.15631250
0	0.375	0.50000	0.3750	0.73466875
1	0.375	1.87500	0.2500	2.18837500
2	0.125	0.50000	0.0625	0.73466875
3		0.06250		0.15631250
4				0.01563125

k	$m=4$: $\frac{h[k]}{\sqrt{2}} = \frac{\tilde{h}[k]}{\sqrt{2}}$	$m=4$: $\frac{\tilde{g}[k]}{\sqrt{2}}$
-3		0.00390625
-2	0.03121875	0.04687500
-1	0.15609375	0.26562500
0	0.31218750	0.95312500
1	0.31218750	2.33593750
2	0.15609375	0.95312500
3	0.03121875	0.26562400
4		0.04687500
5		0.00390625

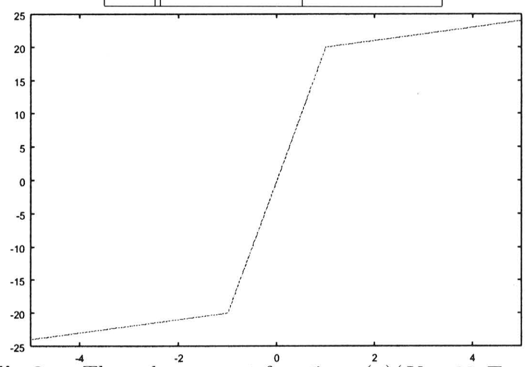

Fig.3 The enhancement function $e(x)(K = 20, T = 1)$.

We also carried out the enhancement of the medical image shown in **Fig. 4** in case of $r = 1$ and $m = 2$. This case has often been treated by Mallat. The experimental results are shown in **Fig. 6** and **Fig. 7**.

Comparing the results shown in **Fig. 4** and **Fig. 5** with those shown in **Fig. 6** and **Fig. 7**, we see that the former results are clear in comparison with the latter ones.

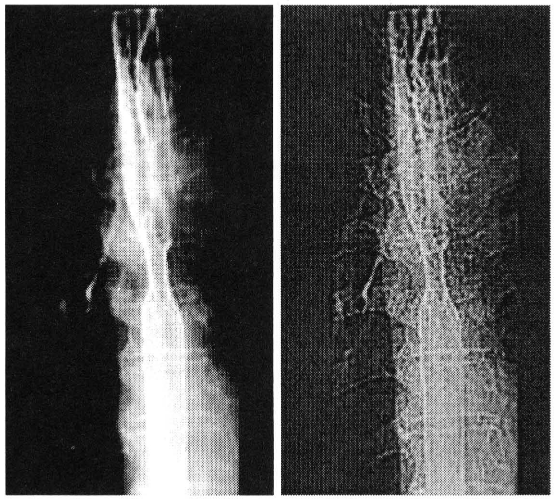

Fig.4 The medical image (left) and the first (right) enhanced image in case of $r = 2$ and $m = 2$.

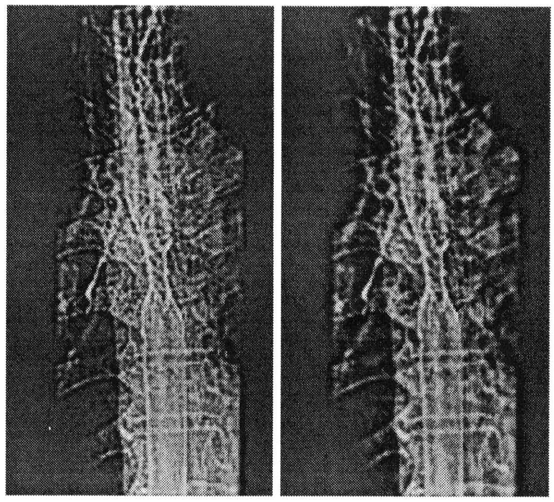

Fig.5 The second (left) and the third (right) enhanced image in case of $r = 2$ and $m = 2$.

5. Conclusion

We found a sufficient condition under which spline dyadic wavelet filters have finite length, and designed some spline dyadic wavelet filters. They contain the filter constructed by Mallat. Using these filters, we enhanced a medical image, in which an enhancement function was used. There are some problems to be resolved in the future. One problem is an application of spline dyadic wavelets to edge detection of images. Second one is to apply these wavelets to remove noise from corrupted images.

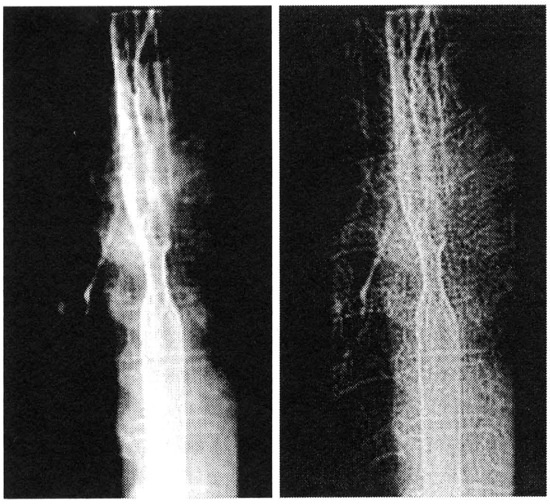

Fig.6 The medical image (left) and the first (right) enhanced image in case of $r = 1$ and $m = 2$.

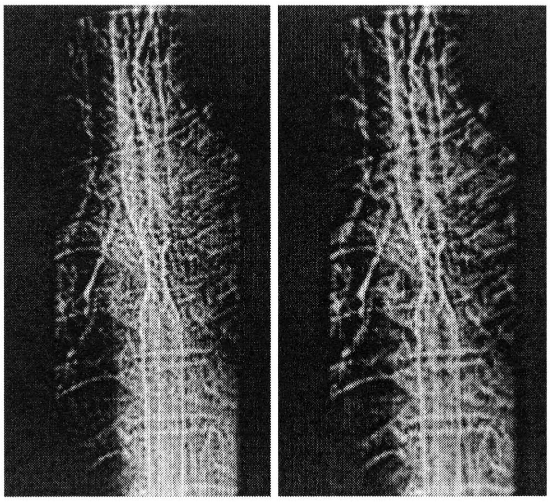

Fig.7 The second (left) and the (right) third enhanced image in case of $r = 1$ and $m = 2$.

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