

# Study on the stability of stationary parallel flow of the compressible Navier-Stokes equation in a cylindrical domain

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Study on the stability of stationary parallel flow of  
the compressible Navier-Stokes equation in a  
cylindrical domain

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## Abstract

Stability of parallel flow of the compressible Navier-Stokes equation in a cylindrical domain is studied. It is shown that if the Reynolds and Mach numbers are sufficiently small, then the linearized semigroup is decomposed into two parts; one behaves like a solution of a one dimensional heat equation as time goes to infinity and the other one decays exponentially. Based on the linearized analysis, it is shown that if the Reynolds and Mach numbers are sufficiently small, then parallel flow is asymptotically stable and the asymptotic leading part of the disturbances is described by a one dimensional viscous Burgers equation.

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# 1 Introduction

This paper studies the large time behavior of solutions of the initial boundary value problem for the compressible Navier-Stokes equation

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \quad (1.1)$$

$$\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu') \nabla \operatorname{div} v + \nabla p(\rho) = \rho g, \quad (1.2)$$

$$v|_{\partial D_*} = 0, \quad (1.3)$$

$$(\rho, v)|_{t=0} = (\rho_0, v_0) \quad (1.4)$$

in a cylindrical domain  $\Omega_* = D_* \times \mathbf{R}$ :

$$\Omega_* = \{x = (x', x_3); x' = (x_1, x_2) \in D_*, x_3 \in \mathbf{R}\}.$$

Here  $D_*$  is a bounded and connected domain in  $\mathbf{R}^2$  with smooth boundary  $\partial D_*$ ;  $\rho = \rho(x, t)$  and  $v = {}^T(v^1(x, t), v^2(x, t), v^3(x, t))$  denote the unknown density and velocity, respectively, at time  $t \geq 0$  and position  $x \in \Omega_*$ ;  $p(\rho)$  is the pressure that is a smooth function of  $\rho$  and satisfies

$$p'(\rho_*) > 0$$

for a given positive constant  $\rho_*$ ;  $\mu$  and  $\mu'$  are the viscosity coefficients that satisfy

$$\mu > 0, \quad \frac{2}{3}\mu + \mu' \geq 0;$$

and  $g$  is an external force of the form  $g = {}^T(g^1(x'), g^2(x'), g^3(x'))$  with  $g^1$  and  $g^2$  satisfying

$$(g^1(x'), g^2(x')) = (\partial_{x_1} \Phi(x'), \partial_{x_2} \Phi(x')),$$

where  $\Phi$  and  $g^3$  are given smooth functions of  $x'$ . Here and in what follows  ${}^T$  stands for the transposition.

Problem (1.1)-(1.3) has a stationary solution  $\bar{u}_s = {}^T(\bar{\rho}_s(x'), \bar{v}_s(x'))$  which represents parallel flow. Here  $\bar{\rho}_s$  is determined by

$$\begin{cases} \text{Const.} - \Phi(x') = \int_{\rho_*}^{\bar{\rho}_s} \frac{p'(\eta)}{\eta} d\eta, \\ \int_{D_*} \bar{\rho}_s - \rho_* dx' = 0, \end{cases}$$

while  $\bar{v}_s$  takes the form

$$\bar{v}_s = {}^T(0, 0, \bar{v}_s^3(x')),$$

where  $\bar{v}_s^3(x')$  is the solution of

$$\begin{cases} -\mu \Delta' \bar{v}_s^3 = \bar{\rho}_s g^3, \\ \bar{v}_s^3|_{\partial D_*} = 0. \end{cases}$$

Here

$$\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2.$$

The purpose of this paper is to investigate the large time behavior of solutions to problem (1.1)-(1.4) when the initial value  $(\rho, v)|_{t=0} = (\rho_0, v_0)$  is sufficiently close to the stationary solution  $\bar{u}_s = {}^T(\bar{\rho}_s, \bar{v}_s)$ .

As for the asymptotic behavior of multi-dimensional compressible Navier-Stokes equations on unbounded domains, a lot of results have been obtained through the studies on the problems about global existence, stability, convergence rates and so on, see, e.g., [6, 8, 17, 19, 20, 21, 22, 23, 25] and references therein. Concerning the stability of parallel flows, in [16], the stability of a plane Poiseuille type flow in an infinite layer of  $\mathbf{R}^n$  was considered under the disturbances in some  $L^2$ -Sobolev space on the infinite layer. It was shown in [16] that the low frequency part of the linearized semigroup behaves like  $n - 1$  dimensional heat kernel and the high frequency part decays exponentially as  $t \rightarrow \infty$ , provided that the Reynolds and Mach numbers are sufficiently small and the density of the parallel flow is sufficiently close to the given constant  $\rho_*$ . The nonlinear problem was studied by Kagei [12]; and it was proved that the stationary parallel flow is asymptotically stable under sufficiently small initial disturbances in some  $L^2$ -Sobolev space. Furthermore, the asymptotic behavior of the disturbance is described by an  $n - 1$  dimensional heat equation when  $n \geq 3$ . When  $n = 2$ , the asymptotic behavior of the disturbance is no longer described by a linear equation but by a one dimensional viscous Burgers equation. (See also [3, 4, 5] for the stability of time periodic parallel flow.)

As for the case of the cylindrical domain  $\Omega_*$ , Iooss and Padula [9] studied the linearized stability of a stationary parallel flow in  $\Omega_*$  under the disturbances periodic in  $x_3$ . It was shown in [9] that the linearized operator generates a  $C_0$ -semigroup in  $L^2$  on the basic periodicity cell under vanishing average condition for the density-component. In particular, if the Reynolds number is suitably small, then the semigroup decays exponentially as time goes to infinity. Furthermore, the essential spectrum of the linearized operator lies in the left-half plane strictly away from the imaginary axis and the part of the spectrum lying in the right-half to the line  $\text{Re}\lambda = -c$  for some number  $c > 0$  consists of finite number of eigenvalues with finite multiplicities. As for the stability under local disturbances on  $\Omega_*$ , i.e., disturbances which are non-periodic but decay at spatial infinity, the stability of the motionless state  $\tilde{u}_s = {}^T(\rho_*, 0)$  was studied in [18]; and it was shown in [18] that the disturbance decays in  $L^2(\Omega_*)$  in the order  $t^{-\frac{1}{4}}$  if the initial disturbance is sufficiently small in  $H^3(\Omega_*) \cap L^1(\Omega_*)$ , where  $H^3(\Omega_*)$  denotes the  $L^2$ -Sobolev space on  $\Omega_*$  of order 3. Furthermore, the asymptotic behavior of the disturbance is described by a solution of a one dimensional linear heat equation. (See also [10] for the analysis in  $L^p(\Omega_*)$ .)

In this paper we will consider the stability of parallel flow  $\bar{u}_s$  under local disturbances on  $\Omega_*$ . After introducing suitable non-dimensional variables, the equations

for the disturbance  $u = {}^T(\phi, w) = {}^T(\gamma^2(\rho - \rho_s), v - v_s)$  takes the following form:

$$\partial_t \phi + v_s^3 \partial_{x_3} \phi + \gamma^2 \operatorname{div}(\rho_s w) = f^0(\phi, w), \quad (1.5)$$

$$\begin{aligned} \partial_t w - \frac{\nu}{\rho_s} \Delta w - \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} w + \nabla \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) \\ + \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \phi e_3 + v_s^3 \partial_{x_3} w + (w' \cdot \nabla' v_s^3) e_3 = f(\phi, w), \end{aligned} \quad (1.6)$$

$$w|_{\partial\Omega} = 0, \quad (1.7)$$

$$(\phi, w)|_{t=0} = (\phi_0, w_0). \quad (1.8)$$

Here  $\Omega_*$  is transformed into  $\Omega = D \times \mathbf{R}$  with  $|D| = 1$ ;  $u_s = {}^T(\rho_s, v_s)$  and  $P(\rho)$  denote the dimensionless parallel flow and pressure, respectively;  $\nu$ ,  $\tilde{\nu}$  and  $\gamma$  are the dimensionless parameters defined by

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \tilde{\nu} = \frac{\mu + \mu'}{\rho_* \ell V}, \quad \gamma = \frac{\sqrt{p'(\rho_*)}}{V}$$

with the reference velocity  $V$  which measures the strength of  $\overline{v_s}$ ;  $e_3 = {}^T(0, 0, 1) \in \mathbf{R}^3$  and  $\nabla' = {}^T(\partial_{x_1}, \partial_{x_2})$ ;  $f^0(\phi, w)$  and  $f(\phi, w)$  are the nonlinearities given by

$$f^0(\phi, w) = -\operatorname{div}(\phi w),$$

$$\begin{aligned} f(\phi, w) = -w \cdot \nabla w + \frac{\nu \phi}{(\phi + \gamma^2 \rho_s) \rho_s} \left( -\Delta w + \frac{\Delta' v_s}{\gamma^2 \rho_s} \phi \right) - \frac{\tilde{\nu} \phi}{(\phi + \gamma^2 \rho_s) \rho_s} \nabla \operatorname{div} w \\ + \frac{\phi}{\gamma^2 \rho_s} \nabla \left( \frac{P'(\rho_s) \phi}{\gamma^2 \rho_s} \right) - \frac{1}{2\gamma^4 \rho_s} \nabla \left( P''(\rho_s) \phi^2 \right) + \tilde{P}_3(\rho_s, \phi, \partial_{x'} \phi), \end{aligned}$$

where

$$\begin{aligned} \tilde{P}_3(\rho_s, \phi, \partial_{x'} \phi) = \frac{\phi^3}{\gamma^4 (\phi + \gamma^2 \rho_s) \rho_s^3} \nabla P(\rho_s) - \frac{1}{2\gamma^6 \rho_s} \nabla \left( \phi^3 P_3(\rho_s, \phi) \right) \\ + \frac{\phi}{2\gamma^6 \rho_s^2} \nabla \left( P''(\rho_s) \phi^2 + \frac{1}{\gamma^2} \phi^3 P_3(\rho_s, \phi) \right) \\ - \frac{\phi^2}{\gamma^2 (\phi + \gamma^2 \rho_s) \rho_s^2} \nabla \left( \frac{1}{\gamma^2} P'(\rho_s) \phi + \frac{1}{2\gamma^4} P''(\rho_s) \phi^2 + \frac{1}{2\gamma^6} \phi^3 P_3(\rho_s, \phi) \right), \end{aligned}$$

with

$$P_3(\rho_s, \phi) = \int_0^1 (1 - \theta)^2 P'''(\rho_s + \theta \gamma^{-2} \phi) d\theta.$$

See Section 2.2 below for the definition of non-dimensional variables. This problem is written as

$$\partial_t u + Lu = \mathbf{F}(u), \quad u = {}^T(\phi, w), \quad w|_{\partial D} = 0, \quad u|_{t=0} = u_0, \quad (1.9)$$

where  $\mathbf{F}(u) = {}^T(f^0(\phi, w), f(\phi, w))$ ; and  $L$  is the operator on  $L^2(\Omega)$  defined by

$$L = \begin{pmatrix} v_s \cdot \nabla & \gamma^2 \operatorname{div}(\rho_s \cdot) \\ \nabla \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \cdot \right) & -\frac{\nu}{\rho_s} \Delta I_3 - \frac{\nu + \nu'}{\rho_s} \nabla \operatorname{div} + v_s \cdot \nabla \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{\nu \Delta' v_s}{\gamma^2 \rho_s^2} & e_3 \otimes (\nabla v_s^3) \end{pmatrix}$$

with domain

$$D(L) = \{u = {}^T(\phi, w) \in L^2(\Omega); w \in H_0^1(\Omega), Lu \in L^2(\Omega)\}.$$

Here, for  $\mathbf{a} = {}^T(a_1, a_2, a_3)$  and  $\mathbf{b} = {}^T(b_1, b_2, b_3)$ , we denote the  $3 \times 3$  matrix  $(a_i b_j)$  by  $\mathbf{a} \otimes \mathbf{b}$ .

To investigate the nonlinear problem (1.9), we study spectral properties of the linearized semigroup  $e^{-tL}$ . We prove that there exists a bounded projection  $P_0$  satisfying  $P_0 e^{-tL} = e^{-tL} P_0$  such that if Reynolds and Mach numbers are sufficiently small, then, for the initial value  $u_0 = {}^T(\phi_0, w_0)$ , it holds that

$$\|e^{-tL} P_0 u_0 - [\mathcal{H}(t) \langle \phi_0 \rangle] u^{(0)}(t)\|_{L^2(\Omega)} \leq C(1+t)^{-\frac{3}{4}} \|u_0\|_{L^1(\Omega)}. \quad (1.10)$$

Here  $u^{(0)}$  is some function of  $x'$ ;  $\langle \phi_0 \rangle$  denotes the average of  $\phi_0$  over  $D$ , (thus,  $\langle \phi_0 \rangle$  is a function of  $x_3 \in \mathbf{R}$ ); and  $\mathcal{H}(t)$  is the heat semigroup defined by

$$\mathcal{H}(t) = \mathcal{F}^{-1} e^{-(i\kappa_1 \xi + \kappa_0 \xi^2)t} \mathcal{F}$$

with some constants  $\kappa_1 \in \mathbf{R}$  and  $\kappa_0 > 0$ , where  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  denote the Fourier transform on  $\mathbf{R}$  and the inverse Fourier transform, respectively. Furthermore, the  $(I - P_0)$ -part of  $e^{-tL}$  satisfies the exponential decay estimate

$$\|e^{-tL} (I - P_0) u_0\|_{H^1(\Omega)} \leq C e^{-dt} \{\|u_0\|_{H^1(\Omega) \times \tilde{H}^1(\Omega)} + t^{-\frac{1}{2}} \|w_0\|_{L^2(\Omega)}\} \quad (1.11)$$

for a positive constant  $d$ . Here  $\tilde{H}^1(\Omega)$  is the set of all locally  $H^1$  functions in  $L^2(\Omega)$  whose tangential derivatives near  $\partial\Omega$  belong to  $L^2(\Omega)$ .

Based on the results on spectral properties of  $e^{-tL}$ , we investigate the nonlinear problem (1.9). We prove that if the initial disturbance  $u_0 = {}^T(\phi_0, w_0)$  is sufficiently small, then the disturbance  $u(t)$  exists globally in time and it satisfies

$$\|u(t)\|_{L^2(\Omega)} = O(t^{-\frac{1}{4}}) \quad (1.12)$$

$$\|u(t) - (\sigma u^{(0)})\|_{L^2(\Omega)} = O(t^{-\frac{3}{4} + \delta}) \quad (\delta > 0) \quad (1.13)$$

as  $t \rightarrow \infty$ . Here  $\sigma = \sigma(x_3, t)$  satisfies the following one dimensional viscous Burgers equation

$$\partial_t \sigma - \kappa_0 \partial_{x_3}^2 \sigma + \kappa_1 \partial_{x_3} \sigma + \kappa_2 \partial_{x_3} (\sigma^2) = 0$$

with initial value  $\langle \phi_0 \rangle$ .

To prove (1.12) and (1.13), we first investigate spectral properties of the linearized semigroup  $e^{-tL}$ . To do so, we consider the Fourier transform of the linearized equation in  $x_3 \in \mathbf{R}$  which is written as

$$\partial_t \hat{u} + \hat{L}_\xi \hat{u} = 0, \quad \hat{u}|_{t=0} = \hat{u}_0,$$

where  $\xi \in \mathbf{R}$  denotes the dual variable. The operator  $\hat{L}_\xi$  has different properties of the cases  $|\xi| \ll 1$  and  $|\xi| \gg 1$ . We thus decompose the semigroup  $e^{-tL}$  into two parts:  $e^{-tL} = \mathcal{F}^{-1}(e^{-t\hat{L}_\xi} |_{|\xi| \leq 1}) + \mathcal{F}^{-1}(e^{-t\hat{L}_\xi} |_{|\xi| > 1})$ . As for the low frequency part,

we take a new approach. A straightforward application of the arguments in [16, 18] seems to yield a more restrictive smallness conditions for the Reynolds and Mach numbers. To overcome this, we combine the arguments in [16, 18] and the energy method in [9]. As in [16, 18], we decompose the low frequency part of the semigroup according to the spectral properties of the linearized operator with zero-frequency. The decay estimate for the  $L^2$  norm is then established with the aid of the energy method in [9] applied to the decomposed system. Based on the decay estimate for  $L^2$  norm, we obtain the estimate for the  $L^2$  norm of the derivatives. We note that this approach also enables us to improve the decay estimate in [16, Theorem 3.2]. On the other hand, in the case of the high frequency part, we employ the Fourier transformed version of Matsumura-Nishida's energy method as in [16, 18].

After establishing the decay estimates for the linearized semigroup, we then investigate the spectrum of  $-\widehat{L}_\xi$  for  $|\xi| \leq r_0$  in more detail for some small  $r_0 > 0$ . The spectrum of  $-\widehat{L}_\xi$  for  $|\xi| \leq r_0$  can be regarded as a perturbation from the one with  $\xi = 0$ , and we will show that the spectrum near the origin is given by a simple eigenvalue  $\lambda_0(\xi) = -i\kappa_0\xi - \kappa_1\xi^2 + \mathcal{O}(|\xi|^3)$  as  $|\xi| \rightarrow 0$ . Furthermore, we will establish the boundedness of the eigenprojection  $\widehat{\Pi}(\xi)$  for the eigenvalue  $\lambda_0(\xi)$  in some Sobolev space by investigating the regularity of the corresponding eigenfunctions. Setting  $P_0 = \mathcal{F}^{-1}\mathbf{1}_{\{|\xi| \leq r_0\}}\widehat{\Pi}(\xi)\mathcal{F}$  with a frequency cut-off function  $\mathbf{1}_{\{|\xi| \leq r_0\}}$  such that  $\mathbf{1}_{\{|\xi| \leq r_0\}} = 1$  for  $|\xi| \leq r_0$  and  $\mathbf{1}_{\{|\xi| \leq r_0\}} = 0$  for  $|\xi| > r_0$ , we find the asymptotic behavior of  $e^{-tL}P_0$  as described in (1.10).

The proof of (1.12) and (1.13) is then given by using the factorization of  $e^{-tL}P_0$ , estimate (1.11) and the energy method. We decompose the disturbance  $u(t)$  into its  $P_0$  and  $I - P_0$  parts. We then estimate the  $P_0$ -part by representing it in the form of variation of constants formula in terms of  $e^{-tL}P_0$  and employ the factorization result of  $e^{-tL}P_0$ . For the  $(I - P_0)$ -part of  $u(t)$ , we employ the Matsumura-Nishida energy method. In contrast to [3, 12], we make use of the estimate (1.11) and combine it with the energy method. This simplifies the argument in [3, 12] where a complicated decomposition is also used in the energy method to estimate the  $(I - P_0)$ -part of  $u(t)$ . In this paper we do not need to use such a complicated decomposition of the  $(I - P_0)$ -part in the energy method due to (1.11).

This paper is organized as follows. In Section 2 we introduce notations and non-dimensional variables. We then state the existence of stationary solution which represents parallel flow. In Section 3 we state our main results of this paper. Section 4 is devoted to the study of the linearized semigroup. We derive the decay estimate of the low frequency part in Section 4.1, and the high frequency part in Section 4.2. In Section 4.3 we will investigate the spectrum of  $-\widehat{L}_\xi$  for  $|\xi| \leq r_0$ , and in Section 4.4 we will establish a factorization of  $e^{-tL}P_0$  and prove (1.10). Section 4.5 is devoted to the proof of (1.11). The nonlinear problem is then studied in Section 5. In Section 5.1 we decompose the problem into the one for a coupled system of the  $P_0$  and  $I - P_0$  parts of  $u(t)$ . Section 5.2 is devoted to estimating the  $P_0$ -part of the disturbance  $u(t)$ , while the  $(I - P_0)$ -part is estimated in Section 5.3. Section 5.4 is devoted to the estimates for the nonlinearities. The proof of (1.13) is given in Section 5.5.

## 2 Preliminaries

In this section we introduce notations throughout this paper. We then introduce non-dimensional variables and state the existence of stationary solution which represents parallel flow.

### 2.1 Notation

We first introduce some notations which will be used throughout the paper. For  $1 \leq p \leq \infty$  we denote by  $L^p(X)$  the usual Lebesgue space on a domain  $X$  and its norm is denoted by  $\|\cdot\|_{L^p(X)}$ . Let  $m$  be a nonnegative integer.  $H^m(X)$  denotes the  $m$ th order  $L^2$  Sobolev space on  $X$  with norm  $\|\cdot\|_{H^m(X)}$ . In particular, we write  $L^2(X)$  for  $H^0(X)$ .

We denote by  $C_0^m(X)$  the set of all  $C^m$  functions with compact support in  $X$ .  $H_0^m(X)$  stands for the completion of  $C_0^m(X)$  in  $H^m(X)$ . We denote by  $H^{-1}(X)$  the dual space of  $H_0^1(X)$  with norm  $\|\cdot\|_{H^{-1}(X)}$ .

We simply denote by  $L^p(X)$  (resp.,  $H^m(X)$ ) the set of all vector fields  $w = {}^T(w^1, w^2, w^3)$  on  $X$  and its norm is denoted by  $\|\cdot\|_{L^p(X)}$  (resp.,  $\|\cdot\|_{H^m(X)}$ ). For  $u = {}^T(\phi, w)$  with  $\phi \in H^k(X)$  and  $w = {}^T(w^1, w^2, w^3) \in H^m(X)$ , we define  $\|u\|_{H^k(X) \times H^m(X)}$  by  $\|u\|_{H^k(X) \times H^m(X)} = \|\phi\|_{H^k(X)} + \|w\|_{H^m(X)}$ .

When  $X = \Omega$  we abbreviate  $L^p(\Omega)$  as  $L^p$ , and likewise,  $H^m(\Omega)$  as  $H^m$ . The norm  $\|\cdot\|_{L^p(\Omega)}$  is written as  $\|\cdot\|_{L^p}$ , and likewise,  $\|\cdot\|_{H^m(\Omega)}$  as  $\|\cdot\|_{H^m}$ .

In the case  $X = D$  we denote the norm of  $L^p(D)$  by  $|\cdot|_p$ . The norm of  $H^m(D)$  is denoted by  $|\cdot|_{H^m}$ , respectively. The inner product of  $L^2(D)$  is denoted by

$$(f, g) = \int_D f(x') \overline{g(x')} dx', \quad f, g \in L^2(D).$$

Here  $\bar{g}$  denotes the complex conjugate of  $g$ . For  $u_j = {}^T(\phi_j, w_j)$  ( $j = 1, 2$ ), we also define a weighted inner product  $\langle u_1, u_2 \rangle$  by

$$\langle u_1, u_2 \rangle = \frac{1}{\gamma^2} \int_D \phi_1 \bar{\phi}_2 \frac{P'(\rho_s)}{\gamma^2 \rho_s} dx' + \int_D w_1 \cdot \bar{w}_2 \rho_s dx',$$

where  $\rho_s = \rho_s(x')$  is the density of the parallel flow  $u_s$ . As will be seen in Proposition 2.1 below,  $\rho_s(x')$  and  $\frac{P'(\rho_s(x'))}{\rho_s(x')}$  are strictly positive in  $D$ .

For  $f \in L^1(D)$  we denote the mean value of  $f$  over  $D$  by  $\langle f \rangle$ :

$$\langle f \rangle = (f, 1) = \frac{1}{|D|} \int_D f dx',$$

where  $|D| = \int_D dx'$ . For  $u = {}^T(\phi, w) \in L^1(D)$  with  $w = {}^T(w^1, w^2, w^3)$  we define  $\langle u \rangle$  by

$$\langle u \rangle = \langle \phi \rangle + \langle w_1 \rangle + \langle w_2 \rangle + \langle w_3 \rangle.$$

Partial derivatives of a function  $u$  in  $x, x', x_3$  and  $t$  are denoted by  $\partial_x u, \partial_{x'} u, \partial_{x_3} u$  and  $\partial_t u$ . We also write higher order partial derivatives of  $u$  in  $x$  as  $\partial_x^k u = (\partial_x^\alpha u; |\alpha| = k)$ .

We set

$$\begin{aligned} \llbracket f(t) \rrbracket_k &= \left( \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \|\partial_t^j f(t)\|_{H^{k-2j}}^2 \right)^{\frac{1}{2}}, \\ \lll Df(t) \rrl_k &= \begin{cases} \|\partial_x f(t)\|_2, & k=0, \\ \left( \llbracket \partial_x f(t) \rrbracket_k^2 + \llbracket \partial_t f(t) \rrbracket_{k-1}^2 \right)^{\frac{1}{2}}, & k \geq 1. \end{cases} \end{aligned}$$

We define a function space  $Z(T)$  by

$$Z(T) = \{u = {}^T(\phi, w) \in C^0([0, T]; H^2 \times (H^2 \cap H_0^1)) \cap C^1([0, T]; L^2); \|u\|_{Z(T)} < \infty\}$$

where

$$\|u\|_{Z(T)} = \sup_{0 \leq t \leq T} \llbracket u(t) \rrbracket_2 + \left( \int_0^T \lll Dw(t) \rrl_2^2 dt \right)^{\frac{1}{2}}.$$

We denote the  $n \times n$  identity matrix by  $I_n$ . We define  $4 \times 4$  diagonal matrices  $Q_0$  and  $\tilde{Q}$  by

$$Q_0 = \text{diag}(1, 0, 0, 0), \quad \tilde{Q} = \text{diag}(0, 1, 1, 1).$$

It then follows that for  $u = {}^T(\phi, w)$  with  $w = {}^T(w^1, w^2, w^3)$ ,

$$Q_0 u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad \tilde{Q} u = \begin{pmatrix} 0 \\ w \end{pmatrix}.$$

We denote the Fourier transform of  $f = f(x_3)$  ( $x_3 \in \mathbf{R}$ ) by  $\hat{f}$  or  $\mathcal{F}[f]$ :

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbf{R}} f(x_3) e^{-i\xi x_3} dx_3, \quad \xi \in \mathbf{R}.$$

The inverse Fourier transform is denoted by  $\mathcal{F}^{-1}$ :

$$\mathcal{F}^{-1}[f](x_3) = (2\pi)^{-1} \int_{\mathbf{R}} f(\xi) e^{i\xi x_3} d\xi, \quad x_3 \in \mathbf{R}.$$

We denote the resolvent set of a closed operator  $A$  by  $\rho(A)$  and the spectrum by  $\sigma(A)$ .

We finally introduce a function space which consists of locally  $H^1$  functions in  $L^2(\Omega)$  whose tangential derivatives near  $\partial D$  belong to  $L^2(\Omega)$ . To do so, we first introduce a local curvilinear coordinate system. For any  $\bar{x}'_0 \in \partial D$ , there exist a neighborhood  $\tilde{\mathcal{O}}_{\bar{x}'_0}$  of  $\bar{x}'_0$  and a smooth diffeomorphism map  $\Psi = (\Psi_1, \Psi_2) : \tilde{\mathcal{O}}_{\bar{x}'_0} \rightarrow B_1(0) = \{z' = (z_1, z_2) : |z'| < 1\}$  such that

$$\begin{cases} \Psi(\tilde{\mathcal{O}}_{\bar{x}'_0} \cap D) = \{z' \in B_1(0) : z_1 > 0\}, \\ \Psi(\tilde{\mathcal{O}}_{\bar{x}'_0} \cap \partial D) = \{z' \in B_1(0) : z_1 = 0\}, \\ \det \nabla_{x'} \Psi \neq 0 \quad \text{on} \quad \overline{\tilde{\mathcal{O}}_{\bar{x}'_0} \cap D}. \end{cases}$$

By the tubular neighborhood theorem, there exist a neighborhood  $\mathcal{O}_{\bar{x}'_0}$  of  $\bar{x}'_0$  and a local curvilinear coordinate system  $y' = (y_1, y_2)$  on  $\mathcal{O}_{\bar{x}'_0}$  defined by

$$x' = y_1 a_1(y_2) + \Psi^{-1}(0, y_2) : \mathcal{R} \rightarrow \mathcal{O}_{\bar{x}'_0}, \quad (2.1)$$

where  $\mathcal{R} = \{y' = (y_1, y_2) : |y_1| \leq \tilde{\delta}_1, |y_2| \leq \tilde{\delta}_2\}$  for some  $\tilde{\delta}_1, \tilde{\delta}_2 > 0$ ;  $a_1(y_2)$  is the unit inward normal to  $\partial D$  that is given by

$$a_1(y_2) = \frac{\nabla_{x'} \Psi_1}{|\nabla_{x'} \Psi_1|}.$$

Setting  $y_3 = x_3$  we obtain

$$\nabla_x = e_1(y_2) \partial_{y_1} + J(y') e_2(y_2) \partial_{y_2} + e_3 \partial_{y_3},$$

$$\nabla_y = \begin{pmatrix} {}^T e_1(y_2) \\ \frac{1}{J(y')} {}^T e_2(y_2) \\ {}^T e_3 \end{pmatrix} \nabla_x,$$

where

$$e_1(y_2) = \begin{pmatrix} a_1(y_2) \\ 0 \end{pmatrix}, \quad e_2(y_2) = \begin{pmatrix} a_2(y_2) \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad (2.2)$$

$$J(y') = |\det \nabla_{x'} \Psi|, \quad a_2(y_2) = \frac{-\nabla_{x'}^\perp \Psi_1}{|\nabla_{x'}^\perp \Psi_1|}$$

with  $\nabla_{x'}^\perp \Psi_1 = {}^T(-\partial_{x_2} \Psi_1, \partial_{x_1} \Psi_1)$ . Note that  $\partial_{y_1}$  and  $\partial_{y_2}$  are the inward normal derivative and tangential derivative at  $x' = \Psi^{-1}(0, y_2) \in \partial D \cap \mathcal{O}_{\bar{x}'_0}$ , respectively. Let us denote the normal and tangential derivatives by  $\partial_n$  and  $\partial$ , i.e.,

$$\partial_n = \partial_{y_1}, \quad \partial = \partial_{y_2}.$$

Since  $\partial D$  is compact, there are bounded open sets  $\mathcal{O}_m$  ( $m = 1, \dots, N$ ) such that  $\partial D \subset \cup_{m=1}^N \mathcal{O}_m$  and for each  $m = 1, \dots, N$ , there exists a local curvilinear coordinate system  $y' = (y_1, y_2)$  as defined in (4.68) with  $\mathcal{O}_{\bar{x}'_0}$ ,  $\Psi$  and  $\mathcal{R}$  replaced by  $\mathcal{O}_m$ ,  $\Psi^m$  and  $\mathcal{R}_m = \{y' = (y_1, y_2) : |y_1| < \tilde{\delta}_1^m, |y_2| < \tilde{\delta}_2^m\}$  for some  $\tilde{\delta}_1^m, \tilde{\delta}_2^m > 0$ . At last, we take an open set  $\mathcal{O}_0 \subset D$  such that

$$\cup_{m=0}^N \mathcal{O}_m \supset D, \quad \overline{\mathcal{O}_0} \cap \partial D = \emptyset.$$

We set a local coordinate  $y' = (y_1, y_2)$  such that  $y_1 = x_1, y_2 = x_2$  on  $\mathcal{O}_0$ . We note that if  $h \in H^2(D)$ , then  $h|_{\partial D} = 0$  implies that  $\partial^k h|_{\partial D \cap \mathcal{O}_m} = 0$  ( $k = 0, 1$ ).

Let us introduce a partition of unity  $\{\chi_m\}_{m=0}^N$  subordinate to  $\{\mathcal{O}_m\}_{m=0}^N$ , satisfying

$$\sum_{m=0}^N \chi_m = 1 \text{ on } D, \quad \chi_m \in C_0^\infty(\mathcal{O}_m) \text{ } (m = 0, 1, 2, \dots, N).$$

We denote by  $\tilde{H}^1(\Omega)$  the set of all locally  $H^1$  functions in  $L^2(\Omega)$  whose tangential derivatives near  $\partial\Omega$  belong to  $L^2(\Omega)$ , and its norm is denoted by  $\|w\|_{\tilde{H}^1(\Omega)}$ :

$$\|w\|_{\tilde{H}^1(\Omega)} = \|w\|_2 + \|\partial_{x_3} w\|_2 + \|\chi_0 \partial_{x'} w\|_2 + \sum_{m=1}^N \|\chi_m \partial w\|_2.$$

Note that  $H_0^1(\Omega)$  is dense in  $\tilde{H}^1(\Omega)$ .

## 2.2 Stationary solution

In this subsection we rewrite the problem into the one in a non-dimensional form and state the existence of stationary solution which represents parallel flow. Let  $k_0$  be an integer satisfying  $k_0 \geq 3$ . We introduce the following non-dimensional variables:

$$x = \ell \tilde{x}, \quad v = V \tilde{v}, \quad \rho = \rho_* \tilde{\rho}, \quad t = \frac{\ell}{V} \tilde{t},$$

$$p = \rho_* V^2 \tilde{P}, \quad \Phi = \frac{V^2}{\ell} \tilde{\Phi}, \quad g^3 = \frac{V^2}{\ell} \tilde{g}^3,$$

$$V = |\bar{v}_s^3|_{C_*^{k_0}(D_*)} = \sum_{k=0}^{k_0} \sup_{x' \in D_*} \ell^k |\partial_{x'}^k \bar{v}_s^3(x')|, \quad \ell = \left( \int_{D_*} dx' \right)^{\frac{1}{2}}.$$

The problem (1.1)-(1.3) is then transformed into the following non-dimensional problem on  $\tilde{\Omega} = \tilde{D} \times \mathbf{R}$ :

$$\partial_{\tilde{t}} \tilde{\rho} + \operatorname{div}_{\tilde{x}}(\tilde{\rho} \tilde{v}) = 0, \tag{2.3}$$

$$\tilde{\rho}(\partial_{\tilde{t}} \tilde{v} + \tilde{v} \cdot \nabla_{\tilde{x}} \tilde{v}) - \nu \Delta_{\tilde{x}} \tilde{v} - (\nu + \nu') \nabla_{\tilde{x}} \operatorname{div}_{\tilde{x}} \tilde{v} + \tilde{P}'(\tilde{\rho}) \nabla_{\tilde{x}} \tilde{\rho} = \tilde{\rho} \tilde{g}, \tag{2.4}$$

$$\tilde{v} \mid_{\partial \tilde{D}} = 0, \tag{2.5}$$

$$(\tilde{\rho}, \tilde{v}) \mid_{\tilde{t}=0} = (\tilde{\rho}_0, \tilde{v}_0). \tag{2.6}$$

Here  $\tilde{D}$  is a bounded and connected domain in  $\mathbf{R}^2$ ;  $\tilde{g} = {}^T(\partial_{\tilde{x}_1} \tilde{\Phi}, \partial_{\tilde{x}_2} \tilde{\Phi}, \tilde{g}^3)$ ; and  $\nu$  and  $\nu'$  are non-dimensional parameters:

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \nu' = \frac{\mu'}{\rho_* \ell V}.$$

We also introduce a parameter  $\gamma$ :

$$\gamma = \sqrt{\tilde{P}'(1)} = \frac{\sqrt{p'(\rho_*)}}{V}.$$

Note that the Reynolds and Mach numbers are given by  $1/\nu$  and  $1/\gamma$ , respectively.

In what follows, for simplicity, we omit tildes of  $\tilde{x}$ ,  $\tilde{t}$ ,  $\tilde{v}$ ,  $\tilde{\rho}$ ,  $\tilde{g}$ ,  $\tilde{P}$ ,  $\tilde{\Phi}$ ,  $\tilde{D}$  and  $\tilde{\Omega}$  and write them as  $x$ ,  $t$ ,  $v$ ,  $\rho$ ,  $g$ ,  $P$ ,  $\Phi$ ,  $D$  and  $\Omega$ . Observe that, due to the non-dimensionalization, we have

$$|D| = \int_D dx' = 1,$$

and thus,

$$\langle f \rangle = \int_D f(x') dx'.$$

Let us state the existence of a stationary solution which represents parallel flow.

**Proposition 2.1.** *If  $\Phi \in C^{k_0}(\overline{D})$ ,  $g^3 \in H^{k_0}(D)$  and  $|\Phi|_{C^{k_0}}$  is sufficiently small, then (2.3)-(2.5) has a stationary solution  $u_s = {}^T(\rho_s, v_s) \in C^{k_0}(\overline{D})$ . Here  $\rho_s$  satisfies*

$$\begin{cases} \text{Const.} - \Phi(x') = \int_1^{\rho_s(x')} \frac{P'(\eta)}{\eta} d\eta, \\ \int_D \rho_s dx' = 1, \quad \rho_1 < \rho_s(x') < \rho_2 \quad (\rho_1 < 1 < \rho_2), \end{cases}$$

for some constants  $\rho_1, \rho_2 > 0$  and  $v_s$  is a function of the form  $v_s = {}^T(0, 0, v_s^3)$  with  $v_s^3 = v_s^3(x')$  being the solution of

$$\begin{cases} -\nu \Delta' v_s^3 = \rho_s g^3, \\ v_s^3|_{\partial D} = 0. \end{cases}$$

Furthermore,  $u_s = {}^T(\rho_s, v_s)$  satisfies the estimates:

$$|\rho_s(x') - 1|_{C^k} \leq C|\Phi|_{C^k}(1 + |\Phi|_{C^k})^k,$$

$$|v_s^3|_{C^k} \leq C|v_s^3|_{H^{k+2}} \leq C|\Phi|_{C^k}(1 + |\Phi|_{C^k})^k |g^3|_{H^k}$$

for  $k = 3, 4, \dots, k_0$ .

Proposition 2.1 can be proved in a similar manner to the proof of [24, Lemma 2.1].

Setting  $\rho = \rho_s + \gamma^{-2}\phi$  and  $v = v_s + w$  in (2.3)-(2.6) (without tildes), we arrive at the initial boundary value problem for the disturbance  $u = {}^T(\phi, w)$  written in (1.5)-(1.8) in section 1.

### 3 Main result

In this section we state the main result of this paper. Hereafter we set

$$\tilde{\nu} = \nu + \nu'.$$

**Theorem 3.1.** *There exist positive constant  $\nu_0$ ,  $\gamma_0$  and  $\omega_0$  such that if  $\nu \geq \nu_0$ ,  $\frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_0^2$  and  $\|\rho_s - 1\|_{C^3} \leq \omega_0$ , then the following assertions hold. There is a positive number  $\epsilon_0$  such that if  $u_0 = {}^T(\phi_0, w_0) \in [H^2 \times (H^2 \cap H_0^1)] \cap L^1$  satisfies  $\|u_0\|_{H^2 \cap L^1} \leq \epsilon_0$ , then there exists a unique global solution  $u(t) = {}^T(\phi(t), w(t))$  of (1.5)-(1.8) in  $C^0([0, \infty); H^2 \times (H^2 \cap H_0^1)) \cap C^1([0, \infty); L^2)$ ; and the following estimates hold*

$$\|\partial_{x_3}^l u(t)\|_2 = \mathcal{O}(t^{-\frac{1}{4} - \frac{l}{2}}), \quad (l = 0, 1) \quad (3.1)$$

$$\|u(t) - (\sigma u^{(0)})(t)\|_2 = \mathcal{O}(t^{-\frac{3}{4} + \delta}) \quad (\forall \delta > 0) \quad (3.2)$$

as  $t \rightarrow \infty$ . Here  $u^{(0)} = u^{(0)}(x')$  is a function given in Proposition 4.55 (iii) below; and  $\sigma = \sigma(x_3, t)$  is a function satisfying

$$\begin{aligned} \partial_t \sigma - \kappa_0 \partial_{x_3}^2 \sigma + \kappa_1 \partial_{x_3} \sigma + \kappa_2 \partial_{x_3}(\sigma^2) &= 0, \\ \sigma|_{t=0} &= \int_D \phi_0(x', x_3) dx' \end{aligned} \quad (3.3)$$

with some constants  $\kappa_0 > 0$  and  $\kappa_1, \kappa_2 \in \mathbf{R}$ .

As in [3, 12], Theorem 3.1 is proved by combining the local solvability (Proposition 5.23 below) and the appropriate a priori estimates. We will establish the necessary a priori estimates in Section 5.

To establish the a priori estimates, we will use the results on spectral properties of the linearized semigroup  $e^{-tL}$  which will be studied in Section 4. In Section 5.1, we will decompose the problem into the one for a coupled system of the  $P_0$  and  $I - P_0$  parts of  $u(t)$ . The a priori estimates will then be derived in Section 5.2–Section 5.4. The proof of (3.2) will be given in Section 5.5.

## 4 Linear problem

In this section, we treat the linearized problem of (1.5)–(1.8)

$$\begin{aligned} \partial_t \phi + v_s^3 \partial_{x_3} \phi + \gamma^2 \operatorname{div}(\rho_s w) &= 0, \\ \partial_t w - \frac{\nu}{\rho_s} \Delta w - \frac{\tilde{\nu}}{\rho_s} \nabla \operatorname{div} w + \nabla \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) \\ &+ \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \phi e_3 + v_s^3 \partial_{x_3} w + (w' \cdot \nabla' v_s^3) e_3 = 0, \\ w|_{\partial\Omega} &= 0, \\ (\phi, w)|_{t=0} &= (\phi_0, w_0). \end{aligned}$$

This problem is written as

$$\partial_t u + Lu = 0, \quad u = {}^T(\phi, w), \quad w|_{\partial D} = 0, \quad u|_{t=0} = u_0, \quad (4.1)$$

where  $L$  is the operator on  $L^2(\Omega)$  defined by

$$\begin{aligned} L &= \begin{pmatrix} v_s \cdot \nabla & \gamma^2 \operatorname{div}(\rho_s \cdot) \\ \nabla \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \cdot \right) & -\frac{\nu}{\rho_s} \Delta I_3 - \frac{\nu + \nu'}{\rho_s} \nabla \operatorname{div} + v_s \cdot \nabla \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{\nu \Delta' v_s}{\gamma^2 \rho_s^2} & e_3 \otimes (\nabla v_s^3) \end{pmatrix} \\ &\equiv L_1 + L_2 \end{aligned}$$

with domain

$$D(L) = \{u = {}^T(\phi, w) \in L^2(\Omega); \quad w \in H_0^1(\Omega), \quad Lu \in L^2(\Omega)\}.$$

In this section, we set

$$\omega = \|\rho_s - 1\|_{C^{k_0}}.$$

In a similar manner to that in [9], one can show that  $-L_1$  generates a  $C_0$ -semigroup on  $L^2(\Omega)$ . Since  $\|L_2 u\|_2 \leq C\|u\|_2$ , it follows from the standard perturbation theory that  $-L$  generates a  $C_0$ -semigroup  $e^{-tL}$  on  $L^2(\Omega)$ . It is not difficult to prove that if  $u_0 \in H^1(\Omega) \times H_0^1(\Omega)$ , then

$$\begin{aligned} e^{-tL} u_0 &\in C([0, T]; H^1(\Omega) \times H_0^1(\Omega)), \\ Q_0 e^{-tL} u_0 &\in H^1(0, T; L^2(\Omega)), \\ \tilde{Q} e^{-tL} u_0 &\in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)) \end{aligned} \quad (4.2)$$

for all  $T > 0$ . Furthermore, since  $H_0^1(\Omega)$  is dense in  $\tilde{H}^1(\Omega)$ , one can see from (4.2), Lemma 4.50 and Lemma 4.51 below that if  $u_0 \in H^1(\Omega) \times \tilde{H}^1(\Omega)$ , then

$$\begin{aligned} e^{-tL}u_0 &\in C([0, T]; H^1(\Omega) \times \tilde{H}^1(\Omega)) \cap C((0, T]; H^1(\Omega) \times H_0^1(\Omega)), \\ \nabla \tilde{Q}e^{-tL}u_0 &\in L^2(0, T; \tilde{H}^1(\Omega)) \end{aligned} \quad (4.3)$$

for all  $T > 0$ .

Our aim in this section is to analyze the spectrum of the linearized operator for the purpose of the study of the nonlinear stability in Section 5. It is shown that if the Reynolds and Mach numbers are sufficiently small, then the linearized semigroup is decomposed into two parts; one behaves like a solution of a one dimensional heat equation as time goes to infinity and the other one decays exponentially. We first consider the decay estimate for the linearized semigroup in Section 4.1 and Section 4.2. Furthermore, we analyze the spectrum of the linearized operator in Section 4.3–Section 4.5. Some estimates for the spectral projections are established, which will also be useful for the study of the nonlinear problem.

Let us state the main results in this section. In Section 4.1 and Section 4.2, we will obtain the decay estimate for  $e^{-tL}u_0$ .

**Theorem 4.1.** *Suppose that  $u_0 = {}^T(\phi_0, w_0) \in (H^1(\Omega) \times H_0^1(\Omega)) \cap L^1(\Omega)$ . There exist positive constants  $\nu_1$ ,  $\gamma_1$  and  $\omega_1$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{2\nu+\nu'} \geq \gamma_1$  and  $\omega \leq \omega_1$ , then there holds the estimate*

$$\|\partial_{x'}^k \partial_{x_3}^l e^{-tL}u_0\|_{L^2(\Omega)} \leq C \left\{ (1+t)^{-\frac{1}{4}-\frac{l}{2}} \|u_0\|_{L^1(\mathbf{R}; L^2(D))} + e^{-dt} \|u_0\|_{H^1(\Omega)} \right\}$$

for  $t \geq 0$  and  $0 \leq k+l \leq 1$  with positive constants  $C$  and  $d$ .

In Section 4.3–Section 4.5, we will analyze  $e^{-tL}u_0$  more precisely.

**Theorem 4.2.** *There exist positive constants  $\nu_1$ ,  $\gamma_1$  and  $\omega_1$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{2\nu+\nu'} \geq \gamma_1^2$  and  $\omega \leq \omega_1$ , then  $e^{-tL}u_0$  is decomposed as*

$$e^{-tL}u_0 = e^{-tL}P_0u_0 + e^{-tL}P_\infty u_0.$$

Here  $P_0$  and  $P_\infty$  are projections satisfying

$$P_0 + P_\infty = I, \quad P^2 = P,$$

$$PL \subset LP, \quad Pe^{-tL} = e^{-tL}P$$

for  $P \in \{P_0, P_\infty\}$ ; and  $e^{-tL}P_0$  and  $e^{-tL}P_\infty$  have the following properties.

(i) If  $u_0 \in L^1(\Omega) \cap L^2(\Omega)$ , then  $e^{-tL}P_0u_0$  satisfies the following estimates

$$\|\partial_{x'}^k \partial_{x_3}^l e^{-tL}P_0u_0\|_2 \leq C_{k,l}(1+t)^{-\frac{1}{4}-\frac{l}{2}} \|u_0\|_1 \quad (4.4)$$

uniformly for  $t \geq 0$  and for  $k = 0, 1, \dots, k_0$  and  $l = 0, 1, \dots$ ;

$$\|e^{-tL}P_0u_0 - [\mathcal{H}(t)\langle \phi_0 \rangle]u^{(0)}\|_2 \leq Ct^{-\frac{3}{4}} \|u_0\|_1 \quad (4.5)$$

uniformly for  $t > 0$ . Here

$$\mathcal{H}(t)\langle\phi_0\rangle = \mathcal{F}^{-1}[e^{-(i\kappa_1\xi + \kappa_0\xi^2)t}\langle\widehat{\phi}_0\rangle],$$

where  $u^{(0)} = u^{(0)}(x')$  is the function given in Lemma 4.6 below; and  $\kappa_1 \in \mathbf{R}$  and  $\kappa_0 > 0$  are some constants satisfying

$$\kappa_1 = \mathcal{O}(1),$$

$$\kappa_0 = C\frac{\gamma^2}{\nu}\left\{1 + \mathcal{O}\left(\frac{1}{\gamma^2}\right) + \left(\frac{\nu}{\gamma^2} + \frac{1}{\nu^2}\right) \times \mathcal{O}\left(\frac{2\nu + \nu'}{\gamma^2}\right)\right\},$$

where  $C$  is a positive constant.

(ii) If  $u_0 \in H^1(\Omega) \times \widetilde{H}^1(\Omega)$ , then there exists a constant  $d > 0$  such that  $e^{-tL}P_\infty u_0$  satisfies

$$\|e^{-tL}P_\infty u_0\|_{H^1} \leq Ce^{-dt}(\|u_0\|_{H^1 \times \widetilde{H}^1} + t^{-\frac{1}{2}}\|w_0\|_2) \quad (4.6)$$

uniformly for  $t > 0$ .

**Remark 4.3.** It is well-known that if  $u_0 = {}^T(\phi_0, w_0) \in L^1(\Omega)$ , then  $\|\mathcal{H}(t)\langle\phi_0\rangle\|_2 = \mathcal{O}(t^{-\frac{1}{4}})$ , and  $\sigma = \sigma(x_3, t) = \mathcal{H}(t)\langle\phi_0\rangle$  satisfies

$$\begin{cases} \partial_t \sigma - \kappa_0 \partial_{x_3}^2 \sigma + \kappa_1 \partial_{x_3} \sigma = 0, \\ \sigma|_{t=0} = \int_D \phi_0(x', x_3) dx'. \end{cases}$$

To prove Theorem 4.1 and Theorem 4.2, we consider the Fourier transform of (4.1) in  $x_3$  variable which is written as

$$\partial_t \widehat{\phi} + i\xi v_s^3 \widehat{\phi} + \gamma^2 \nabla' \cdot (\rho_s \widehat{w}') + \gamma^2 i\xi \rho_s \widehat{w}^3 = 0, \quad (4.7)$$

$$\partial_t \widehat{w}' - \frac{\nu}{\rho_s} (\Delta' - \xi^2) \widehat{w}' - \frac{\widetilde{\nu}}{\rho_s} \nabla' (\nabla' \cdot \widehat{w}' + i\xi \widehat{w}^3) + \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \widehat{\phi} \right) + i\xi v_s^3 \widehat{w}' = 0, \quad (4.8)$$

$$\begin{aligned} \partial_t \widehat{w}^3 - \frac{\nu}{\rho_s} (\Delta' - \xi^2) \widehat{w}^3 - \frac{\widetilde{\nu}}{\rho_s} i\xi (\nabla' \cdot \widehat{w}' + i\xi \widehat{w}^3) + i\xi \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \widehat{\phi} \right) + i\xi v_s^3 \widehat{w}^3 \\ + \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \widehat{\phi} + \widehat{w}' \cdot \nabla' v_s^3 = 0, \end{aligned} \quad (4.9)$$

$$\widehat{w}|_{\partial D} = 0 \quad (4.10)$$

for  $t > 0$ , and

$${}^T(\widehat{\phi}, \widehat{w})|_{t=0} = {}^T(\widehat{\phi}_0, \widehat{w}_0) = \widehat{u}_0. \quad (4.11)$$

We thus arrive at the following problem

$$\partial_t \widehat{u} + \widehat{L}_\xi \widehat{u} = 0, \quad \widehat{u}|_{t=0} = \widehat{u}_0 \quad (4.12)$$

with a parameter  $\xi \in \mathbf{R}$ . Here  $\widehat{u} = {}^T(\widehat{\phi}(x', t), \widehat{w}(x', t)) \in D(\widehat{L}_\xi)$  ( $x' \in D, t > 0$ ),  $\widehat{u}_0 \in H^1(D) \times H_0^1(D)$ , and  $\widehat{L}_\xi$  is the operator on  $L^2(D)$  of the form

$$\widehat{L}_\xi = \widehat{A}_\xi + \widehat{B}_\xi + \widehat{C}_0,$$

where

$$\widehat{A}_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\nu}{\rho_s} (\Delta' - |\xi|^2) I_2 - \frac{\widetilde{\nu}}{\rho_s} \nabla' \nabla' & -i \frac{\widetilde{\nu}}{\rho_s} \xi \nabla' \\ 0 & -i \frac{\widetilde{\nu}}{\rho_s} \xi \nabla' & -\frac{\nu}{\rho_s} (\Delta' - |\xi|^2) + \frac{\widetilde{\nu}}{\rho_s} |\xi|^2 \end{pmatrix},$$

$$\widehat{B}_\xi = \begin{pmatrix} i\xi v_s^3 & \gamma^2 \nabla' \cdot (\rho_s \cdot) & i\gamma^2 \rho_s \xi \\ \nabla' \left( \frac{P(\rho_s)}{\gamma^2 \rho_s} \cdot \right) & i\xi v_s^3 I_2 & 0 \\ i\xi \frac{P(\rho_s)}{\gamma^2 \rho_s} & 0 & i\xi v_s^3 \end{pmatrix}, \quad \widehat{C}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} & T(\nabla' v_s^3) & 0 \end{pmatrix}$$

with domain of definition

$$D(\widehat{L}_\xi) = \{\widehat{u} = {}^T(\widehat{\phi}, \widehat{w}) \in L^2(D); \widehat{w} \in H_0^1(D), \widehat{L}_\xi \widehat{u} \in L^2(D)\}.$$

Note that  $D(\widehat{L}_\xi) = D(\widehat{L}_0)$  for all  $\xi \in \mathbf{R}$ , where  $\widehat{L}_0 = \widehat{L}_\xi|_{\xi=0}$ .

As in the case of  $L$ , following [9], one can show that  $-\widehat{L}_\xi$  generates a  $C_0$ -semigroup on  $L^2(D)$ . Furthermore if  $\widehat{u}_0 \in H^1(D) \times H_0^1(D)$ , then

$$\begin{aligned} e^{-t\widehat{L}_\xi} \widehat{u}_0 &\in C([0, T]; H^1(D) \times H_0^1(D)), \\ Q_0 e^{-t\widehat{L}_\xi} \widehat{u}_0 &\in H^1(0, T; H^1(D)), \\ \widetilde{Q} e^{-t\widehat{L}_\xi} \widehat{u}_0 &\in L^2(0, T; H^2(D)) \cap H^1(0, T; L^2(D)) \end{aligned} \tag{4.13}$$

for any  $T > 0$ . Furthermore, we can see that if  $u_0 \in H^1(D) \times \widetilde{H}^1(D)$ , then

$$\begin{aligned} e^{-tL_\xi} u_0 &\in C([0, T]; H^1(D) \times \widetilde{H}^1(D)) \cap C((0, T]; H^1(D) \times H_0^1(D)), \\ \partial_{x'} \widetilde{Q} e^{-tL_\xi} u_0 &\in L^2(0, T; \widetilde{H}^1(D)) \end{aligned} \tag{4.14}$$

for any  $T > 0$ .

To prove Theorem 4.1 we decompose  $e^{-tL} u_0$  in the following way. Fix a positive number  $r_0$ . We define  $\mathbf{1}_{\{|\eta| \leq r_0\}}$  and  $\mathbf{1}_{\{|\eta| > r_0\}}$  by  $\mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) = 1$  if  $|\xi| \leq r_0$ ,  $\mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) = 0$  if  $|\xi| > r_0$ , and  $\mathbf{1}_{\{|\eta| > r_0\}}(\xi) = 1 - \mathbf{1}_{\{|\eta| \leq r_0\}}(\xi)$ .

We decompose  $e^{-tL} u_0$  as

$$e^{-tL} u_0 = U_1(t) u_0 + U_\infty(t) u_0,$$

where

$$U_1(t) u_0 = \mathcal{F}^{-1} [\mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) e^{-t\widehat{L}_\xi} \widehat{u}_0], \quad U_\infty(t) u_0 = \mathcal{F}^{-1} [\mathbf{1}_{\{|\eta| > r_0\}}(\xi) e^{-t\widehat{L}_\xi} \widehat{u}_0].$$

We can then obtain the following decay estimates for  $U_1(t) u_0$  and  $U_\infty(t) u_0$ .

**Theorem 4.4.** *There exist positive constants  $\nu_1$ ,  $\gamma_1$ ,  $\omega_1$  and  $d$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{\nu + \bar{\nu}} \geq \gamma_1^2$  and  $\omega \leq \omega_1$ , then for any  $l = 0, 1, \dots$ , there exists a positive constant  $C = C(l)$  such that the estimates*

$$\begin{aligned} \|\partial_{x_3}^l U_1(t) u_0\|_{L^2} &\leq C(1+t)^{-1/4-l/2} \|u_0\|_{L^1(\mathbf{R}; L^2(D))}, \\ \|\partial_{x'} \partial_{x_3}^l U_1(t) u_0\|_{L^2} &\leq C \left\{ (1+t)^{-1/4-l/2} \|u_0\|_{L^1(\mathbf{R}; L^2(D))} + e^{-dt} (\|u_0\|_{L^2} + \|\partial_{x'} u_0\|_{L^2}) \right\} \end{aligned}$$

hold for  $t \geq 0$ .

**Theorem 4.5.** *There exist positive constants  $\nu_1$ ,  $\gamma_1$ ,  $\omega_1$  and  $d$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{\nu+\bar{\nu}} \geq \gamma_1^2$  and  $\omega \leq \omega_1$ , then the estimate*

$$\|U_\infty(t)u_0\|_{H^1} \leq Ce^{-dt}\|u_0\|_{H^1}$$

*holds for  $t \geq 0$  with a positive constant  $C$ .*

Theorem 4.1 follows from Theorem 4.4 and Theorem 4.5. In Section 4.1 we will prove Theorem 4.4 and in Section 4.2 we will give an outline of the proof of Theorem 4.5. In Section 4.3 we will investigate the spectrum of  $-\widehat{L}_\xi$  for  $|\xi| \ll 1$ . The proof of Theorem 4.2 (i) will be given in Section 4.4; and more detailed properties of  $P_0$  and  $e^{-tL}P_0$  will be given, where we will establish a factorization of  $e^{-tL}P_0$  which will be useful in the nonlinear analysis. Theorem 4.2 (ii) is proved in Section 4.5.

## 4.1 Decay estimate of the low frequency part

In this section we give a proof of Theorem 4.4. Theorem 4.4 is a consequence of Proposition 4.17 and Proposition 4.25 below.

For simplicity we omit  $\widehat{\cdot}$  of  $\widehat{u}$ ,  $\widehat{\phi}$  and  $\widehat{w}$  in (4.7)-(4.12).

To prove Theorem 4.4 we decompose  $u(t)$  based on a spectral property of  $\widehat{L}_\xi$  with  $\xi = 0$ , namely,  $\widehat{L}_0$ .

We introduce the adjoint operator  $\widehat{L}_\xi^*$  of  $\widehat{L}_\xi$  with the weighted inner product  $\langle \cdot, \cdot \rangle$ . We define  $\widehat{L}_\xi^*$  by

$$\widehat{L}_\xi^* = \widehat{A}_\xi^* + \widehat{B}_\xi^* + \widehat{C}_0^*$$

with domain of definition

$$D(\widehat{L}_\xi^*) = \{u = {}^T(\phi, w) \in L^2(D); w \in H_0^1(D), \widehat{L}_\xi^*u \in L^2(D)\},$$

where

$$\widehat{A}_\xi^* = \widehat{A}_\xi, \quad \widehat{B}_\xi^* = -\widehat{B}_\xi$$

and

$$\widehat{C}_0^* = \begin{pmatrix} 0 & 0 & \frac{\gamma^2 \nu \Delta' v_s^3}{P'(\rho_s)} \\ 0 & 0 & \nabla' v_s^3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that  $D(\widehat{L}_\xi) = D(\widehat{L}_\xi^*)$  for any  $\xi \in \mathbf{R}$ . It follows that

$$\langle \widehat{A}_\xi u, v \rangle = \langle u, \widehat{A}_\xi^* v \rangle = \langle u, \widehat{A}_\xi v \rangle,$$

$$\langle \widehat{B}_\xi u, v \rangle = \langle u, \widehat{B}_\xi^* v \rangle = -\langle u, \widehat{B}_\xi v \rangle,$$

$$\langle \widehat{C}_0 u, v \rangle = \langle u, \widehat{C}_0^* v \rangle$$

and

$$\langle \widehat{L}_\xi u, v \rangle = \langle u, \widehat{L}_\xi^* v \rangle$$

for  $u, v \in D(\widehat{L}_\xi)$ .

We begin with a lemma on the zero-eigenvalue of  $\widehat{L}_0$  and  $\widehat{L}_0^*$ .

**Lemma 4.6.** (i) *There exists a constant  $\omega_1 > 0$  such that if  $\frac{\nu+\tilde{\nu}}{\nu}\omega \leq \omega_1$ , then  $\lambda = 0$  is a simple eigenvalue of  $\widehat{L}_0$  and  $\widehat{L}_0^*$ .*

(ii) *The eigenspaces for  $\lambda = 0$  of  $\widehat{L}_0$  and  $\widehat{L}_0^*$  are spanned by  $u^{(0)}$  and  $u^{(0)*}$ , respectively, where*

$$u^{(0)} = T(\phi^{(0)}, w^{(0)}), \quad w^{(0)} = T(0, 0, w^{(0),3})$$

and

$$u^{(0)*} = T(\phi^{(0)*}, 0).$$

Here

$$\phi^{(0)}(x') = \alpha_0 \frac{\gamma^2 \rho_s(x')}{P'(\rho_s(x'))}, \quad \alpha_0 = \left( \int_D \frac{\gamma^2 \rho_s(x')}{P'(\rho_s(x'))} dx' \right)^{-1};$$

and  $w^{(0),3}$  is the solution of the following problem

$$\begin{cases} -\Delta' w^{(0),3} = -\frac{1}{\gamma^2 \rho_s} \Delta' v_s^3 \phi^{(0)}, \\ w^{(0),3} |_{\partial D} = 0; \end{cases}$$

and

$$\phi^{(0)*}(x') = \frac{\gamma^2}{\alpha_0} \phi^{(0)}(x').$$

(iii) *The eigenprojections  $\widehat{\Pi}^{(0)}$  and  $\widehat{\Pi}^{(0)*}$  for  $\lambda = 0$  of  $\widehat{L}_0$  and  $\widehat{L}_0^*$  are given by*

$$\begin{aligned} \widehat{\Pi}^{(0)} u &= \langle u, u^{(0)*} \rangle u^{(0)} = \langle Q_0 u \rangle u^{(0)}, \\ \widehat{\Pi}^{(0)*} u &= \langle u, u^{(0)} \rangle u^{(0)*} \end{aligned}$$

for  $u = T(\phi, w)$ , respectively.

(iv) *Let  $u^{(0)}$  be written as  $u^{(0)} = u_0^{(0)} + u_1^{(0)}$ , where*

$$u_0^{(0)} = T(\phi^{(0)}, 0), \quad u_1^{(0)} = T(0, w^{(0)}).$$

Then

$$u^{(0)*} = \frac{\gamma^2}{\alpha_0} u_0^{(0)}$$

and

$$\langle u, u^{(0)} \rangle = \frac{\alpha_0}{\gamma^2} \langle \phi \rangle + (w^3, w^{(0),3} \rho_s)$$

for  $u = T(\phi, w) = T(\phi, w', w^3)$ .

**Remark 4.7.**  $\phi^{(0)} = O(1)$ ,  $\alpha_0 = O(1)$  and  $w^{(0),3} = O(\frac{1}{\gamma^2})$  as  $\gamma^2 \rightarrow \infty$ .

**Proof.** Let  $\widehat{L}_0 u = 0$  for  $u = {}^T(\phi, w', w^3) \in D(\widehat{L}_0)$ . Then

$$\begin{cases} \gamma^2 \nabla' \cdot (\rho_s w') = 0, \\ -\frac{\nu}{\rho_s} \Delta' w' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot w' + \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) = 0, \\ -\frac{\nu}{\rho_s} \Delta' w^3 + \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3 \phi + w' \cdot \nabla' v_s^3 = 0, \\ w|_{\partial D} = 0. \end{cases} \quad (4.15)$$

We take the weighted inner product of (4.15) with  ${}^T(\phi, w', 0)$  to get

$$\nu |\nabla' w'|_2^2 + \tilde{\nu} |\nabla' \cdot w'|_2^2 = 0.$$

Since  $w' \in H_0^1(D)$ , we have  $w' = 0$ . It then follows that

$$\begin{cases} \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) = 0, \\ -\Delta' w^3 = -\frac{1}{\gamma^2 \rho_s} \Delta' v_s^3 \phi, \\ w^3|_{\partial D} = 0. \end{cases}$$

This implies that  $\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi$  is a constant since  $D$  is connected, and we conclude that  $\text{Ker}(\widehat{L}_0) = \text{span}\{u^{(0)}\}$ . Note that  $\int_D \phi^{(0)} dx' = 1$ .

Let  $\widehat{L}_0^* u = 0$  for  $u = {}^T(\phi, w', w^3)$ . Then

$$\begin{cases} -\gamma^2 \nabla' \cdot (\rho_s w') + \frac{\gamma^2 \nu}{P'(\rho_s)} \Delta' v_s^3 w^3 = 0, \\ -\frac{\nu}{\rho_s} \Delta' w' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot w' - \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) + w^3 \nabla' v_s^3 = 0, \\ -\frac{\nu}{\rho_s} \Delta' w^3 = 0, \\ w|_{\partial D} = 0. \end{cases}$$

The third equation, together with  $w^3|_{\partial D} = 0$ , implies that  $w^3 = 0$ , and hence,

$$\begin{cases} -\gamma^2 \nabla' \cdot (\rho_s w') = 0, \\ -\frac{\nu}{\rho_s} \Delta' w' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot w' - \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \right) = 0, \\ w'|_{\partial D} = 0. \end{cases}$$

Similarly to the case of  $\widehat{L}_0$ , one can show that  $w' = 0$  and  $\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi$  is a constant. We set  $\phi^{(0)*} = \frac{\gamma^2}{\alpha_0} \phi^{(0)}(x')$ . Note that  $\int_D \phi^{(0)} \overline{\phi^{(0)*}} \frac{P'(\rho_s)}{\gamma^4 \rho_s} dx' = 1$ . We thus proved (i), (ii) and (iii) except the simplicity of  $\lambda = 0$ . The assertion (iv) can be verified by direct computations.

It remains to prove the simplicity of  $\lambda = 0$ . Since we have already proved that  $\text{Ker}(\widehat{L}_0) = \text{span}\{u^{(0)}\}$  and  $\text{Ker}(\widehat{L}_0^*) = \text{span}\{u^{(0)*}\}$ , if we would prove the following lemma, then the proof of the simplicity of  $\lambda = 0$  would be complete.

**Lemma 4.8.** *There exists a constant  $\omega_1 > 0$  such that if  $\frac{\nu + \tilde{\nu}}{\nu} \omega \leq \omega_1$ , then  $R(\widehat{L}_0)$  and  $R(\widehat{L}_0^*)$  are closed; and there hold that*

$$L^2(D) = \text{Ker}(\widehat{L}_0) \oplus R(\widehat{L}_0), \quad L^2(D) = \text{Ker}(\widehat{L}_0^*) \oplus R(\widehat{L}_0^*).$$

To prove Lemma 4.8, we show the following proposition.

**Proposition 4.9.** *There exists a constant  $\omega_1 > 0$  such that if  $\frac{\nu+\tilde{\nu}}{\nu}\omega \leq \omega_1$ , then for any  $f = {}^T(f^0, g) = {}^T(f^0, g', g^3) \in L^2(D)$  with  $\langle f^0 \rangle = 0$ , there is a unique solution  $u = {}^T(\phi, w) \in D(\widehat{L}_0)$  of the following problem:*

$$\begin{cases} \widehat{L}_0 u = f, \\ \langle \phi \rangle = 0. \end{cases} \quad (4.16)$$

**Proof.** Let us prove that if  $\langle f^0 \rangle = 0$ , then (4.16) has a unique solution  $u = {}^T(\phi, w) \in L^2(D) \times H_0^1(D)$  with  $\langle \phi \rangle = 0$ . The problem (4.16) is rewritten as the following system:

$$\begin{cases} \nabla' \cdot w' = F[w'; f^0], \\ -\nu \Delta' w' + \nabla' \phi = G'[\phi, w'; f^0, g'], \\ w' |_{\partial D} = 0, \\ \langle \phi \rangle = 0 \end{cases} \quad (4.17)$$

and

$$\begin{cases} -\nu \Delta' w^3 = G^3[\phi, w'; g^3], \\ w^3 |_{\partial D} = 0, \end{cases} \quad (4.18)$$

where

$$F[w'; f^0] = \nabla' \cdot ((1 - \rho_s)w') + \frac{1}{\gamma^2} f^0,$$

$$\begin{aligned} G'[\phi, w'; f^0, g'] &= \tilde{\nu} \nabla' F[w'; f^0] + \nabla'((1 - \rho_s)\phi) + \nabla' \rho_s \phi \\ &\quad + \rho_s \nabla' \left( \left(1 - \frac{P'(\rho_s)}{\gamma^2 \rho_s}\right) \phi \right) + \rho_s g', \end{aligned}$$

$$G^3[\phi, w'; g^3] = -\frac{\nu}{\gamma^2 \rho_s} \Delta' v_s^3 \phi - \rho_s w' \cdot \nabla' v_s^3 + \rho_s g^3.$$

We define a set  $\dot{X}$  by

$$\dot{X} = \{(p, v'); p \in L^2(D), v' = {}^T(v^1, v^2) \in H_0^1(D), \langle p \rangle = 0\}$$

with norm

$$|(p, v')|_{\dot{X}} = |p|_2 + \nu |\nabla' v'|_2.$$

We assume that  $(\tilde{\phi}, \tilde{w}') \in \dot{X}$ . Let us consider the following problem

$$\begin{cases} \nabla' \cdot w' = F[\tilde{w}'; f^0], \\ -\nu \Delta' w' + \nabla' \phi = G'[\tilde{\phi}, \tilde{w}'; f^0, g'], \\ w' |_{\partial D} = 0. \end{cases} \quad (4.19)$$

It holds that

$$F[\tilde{w}'; f^0] \in L^2(D), \quad \langle F[\tilde{w}'; f^0] \rangle = 0,$$

$$G'[\tilde{\phi}, \tilde{w}'; f^0, g'] \in H^{-1}(D),$$

where  $H^{-1}(D)$  denotes the dual space to  $H_0^1(D)$  with norm  $|\cdot|_{H^{-1}}$ . In fact, we have from the Poincaré inequality that

$$\begin{aligned} |F[\tilde{w}'; f^0]|_2 &\leq |\nabla' \cdot ((1 - \rho_s)\tilde{w}')|_2 + \frac{1}{\gamma^2}|f^0|_2 \\ &\leq C\{\omega|\tilde{w}'|_{H^1} + \frac{1}{\gamma^2}|f^0|_2\} \\ &\leq C\{\omega|\nabla'\tilde{w}'|_2 + \frac{1}{\gamma^2}|f^0|_2\}, \end{aligned}$$

$$\begin{aligned} |G'[\tilde{\phi}, \tilde{w}'; f^0, g']|_{H^{-1}} &\leq C\{|\nabla' F[\tilde{w}'; f^0]|_{H^{-1}} + |\nabla'((1 - \rho_s)\tilde{\phi})|_{H^{-1}} \\ &\quad + |\nabla' \rho_s \tilde{\phi}|_{H^{-1}} + |\rho_s \nabla'((1 - \frac{P'(\rho_s)}{\gamma^2 \rho_s})\tilde{\phi})|_{H^{-1}} + |g'|_{H^{-1}}\} \\ &\leq C\{\tilde{\nu}|F[\tilde{w}'; f^0]|_2 + \omega|\tilde{\phi}|_2 + |g'|_2\} \\ &\leq C\{\omega(|\tilde{\phi}|_2 + \tilde{\nu}|\nabla'\tilde{w}'|_2) + \frac{\tilde{\nu}}{\gamma^2}|f^0|_2 + |g'|_2\}. \end{aligned}$$

From [26, III.1.4, Theorem 1.4.1], we see that there is a unique solution  $(\phi, w') \in \dot{X}$  of (4.19) and there holds the following estimate

$$\begin{aligned} |\phi|_2 + \nu|\nabla' w'|_2 &\leq C\{\nu|F[\tilde{w}'; f^0]|_2 + |G'[\tilde{\phi}, \tilde{w}'; f^0, g']|_{H^{-1}}\} \\ &\leq C_1\{\omega(|\tilde{\phi}|_2 + (\nu + \tilde{\nu})|\nabla'\tilde{w}'|_2) + \frac{\nu + \tilde{\nu}}{\gamma^2}|f^0|_2 + |g'|_2\}, \end{aligned} \tag{4.20}$$

where  $C_1$  is a positive constant. Let us define a map  $\Gamma : \dot{X} \rightarrow \dot{X}$  by

$$\Gamma(\tilde{\phi}, \tilde{w}') = (\phi, w') \quad \text{for } (\tilde{\phi}, \tilde{w}') \in \dot{X},$$

where  $(\phi, w') \in \dot{X}$  is a solution of (4.19). We see from (4.20) that

$$|\Gamma(\tilde{\phi}, \tilde{w}')|_{\dot{X}} \leq C_1\{\omega(|\tilde{\phi}|_2 + (\nu + \tilde{\nu})|\nabla'\tilde{w}'|_2) + \frac{\nu + \tilde{\nu}}{\gamma^2}|f^0|_2 + |g'|_2\}.$$

Since we have the estimate

$$\left| \Gamma(\tilde{\phi}_1, \tilde{w}'_1) - \Gamma(\tilde{\phi}_2, \tilde{w}'_2) \right|_{\dot{X}} \leq C_1\omega\{|\tilde{\phi}|_2 + (\nu + \tilde{\nu})|\nabla'\tilde{w}'|_2\}$$

for  $(\tilde{\phi}_1, \tilde{w}'_1), (\tilde{\phi}_2, \tilde{w}'_2) \in \dot{X}$ , if we take  $\omega$  sufficiently small satisfying  $\omega < \frac{1}{2C_1} \min\{1, \frac{\nu}{\nu + \tilde{\nu}}\}$ , then we see that  $\Gamma : \dot{X} \rightarrow \dot{X}$  is a contraction map. This implies that there is a unique  $(\phi, w') \in \dot{X}$  such that  $\Gamma(\phi, w') = (\phi, w')$ , i.e., there is a unique solution  $(\phi, w') \in \dot{X}$  of (4.17).

Furthermore, for a solution  $(\phi, w') \in \dot{X}$  of (4.17), since

$$G^3[\phi, w'; g^3] \in L^2(D),$$

there is a unique solution  $w^3 \in H_0^1(D)$  of (4.18). Consequently, we have

$$\widehat{L}_0 u = f \quad \text{in the sense of distribution,}$$

where  $f = {}^T(f^0, g', g^3) \in L^2(D)$  with  $\langle f^0 \rangle = 0$ . Since  $f \in L^2(D)$ , it holds that  $\widehat{L}_0 u \in L^2(D)$ . It then follows that

$$u \in D(\widehat{L}_0).$$

This completes the proof.  $\square$

**Proof of Lemma 4.8** We have already proved that

$$\text{Ker}(\widehat{L}_0) = \widehat{\Pi}^{(0)} L^2(D).$$

To prove  $R(\widehat{L}_0) = (I - \widehat{\Pi}^{(0)}) L^2(D)$ , we first show that

$$u = {}^T(\phi, w) \in (I - \widehat{\Pi}^{(0)}) L^2(D) \quad \text{if and only if} \quad \langle \phi \rangle = 0. \quad (4.21)$$

Let us prove (4.21). We can decompose  $u = {}^T(\phi, w)$  as

$$u = \langle \phi \rangle u^{(0)} + u_1.$$

Here

$$\langle \phi \rangle u^{(0)} \in \Pi^{(0)} L^2(D), \quad u_1 = {}^T(\phi_1, w_1) \in (I - \Pi^{(0)}) L^2(D).$$

This implies that if  $\langle \phi \rangle = 0$ , then

$$u = \langle \phi \rangle u^{(0)} + u_1 = u_1 \in (I - \widehat{\Pi}^{(0)}) L^2(D).$$

On the other hand, if  $u = {}^T(\phi, w) \in (I - \widehat{\Pi}^{(0)}) L^2(D)$ , then there exists  $\widetilde{u} = {}^T(\widetilde{\phi}, \widetilde{w}) \in L^2(D)$  such that

$$u = \widetilde{u} - \langle \widetilde{\phi} \rangle u^{(0)}.$$

It then follows that

$$\langle \phi \rangle = \langle \widetilde{\phi} \rangle - \langle \widetilde{\phi} \rangle = 0.$$

We thus conclude that (4.21) holds true.

We next show that  $R(\widehat{L}_0) = (I - \widehat{\Pi}^{(0)}) L^2(D)$ . Since  $\langle Q_0 \widehat{L}_0 u \rangle = \langle \nabla' \cdot (\rho_s w') \rangle = 0$ , we see from (4.21) that  $\widehat{L}_0 u \in (I - \widehat{\Pi}^{(0)}) L^2(D)$ , and, therefore,

$$R(\widehat{L}_0) \subset (I - \Pi^{(0)}) L^2(D).$$

On the other hand, if  $f = {}^T(f^0, g', g^3) \in (I - \widehat{\Pi}^{(0)}) L^2(D)$ , then it follows from (4.21) that  $\langle f^0 \rangle = 0$ . By Proposition 4.9, there exists a unique solution  $u = {}^T(\phi, w) \in D(\widehat{L}_0)$  such that  $\widehat{L}_0 u = f$  with  $\langle \phi \rangle = 0$ . This implies that  $f \in R(\widehat{L}_0)$ , and, thus,

$$(I - \Pi^{(0)}) L^2(D) \subset R(\widehat{L}_0).$$

Therefore we see that  $R(\widehat{L}_0) = (I - \widehat{\Pi}^{(0)}) L^2(D)$ . Consequently, we have  $R(\widehat{L}_0)$  is closed and

$$L^2(D) = \text{Ker}(\widehat{L}_0) \oplus R(\widehat{L}_0).$$

Similarly, one can prove that  $\text{Ker}(\widehat{L}_0^*) = \widehat{\Pi}^{(0)*} L^2(D)$  and  $R(\widehat{L}_0^*) = (I - \widehat{\Pi}^{(0)*}) L^2(D)$ . We thus see that  $R(\widehat{L}_0^*)$  is closed and

$$L^2(D) = \text{Ker}(\widehat{L}_0^*) \oplus R(\widehat{L}_0^*).$$

This completes the proof of Lemma 4.6.  $\square$

We are now ready to prove Theorem 4.4. We decompose  $u(t)$  as follows

$$\begin{aligned} u(t) &= \sigma(t)u^{(0)} + u_1(t), \\ \sigma(t) &= \langle Q_0 u(t) \rangle = \langle u(t), u^{(0)*} \rangle, \\ u_1(t) &= (I - \widehat{\Pi}^{(0)})u(t). \end{aligned}$$

The density component of  $u_1$  is denoted by  $\phi_1$  and the velocity component is denoted by  $w_1$ , namely,

$$u_1 = {}^T(\phi_1, w_1).$$

Note that  $\langle \phi_1 \rangle = 0$  and  $w_1|_{\partial D} = 0$ ; the latter follows from  $u^{(0)} \in D(\widehat{L}_0)$  which implies that  $w^{(0),3}|_{\partial D} = 0$ .

**Remark 4.10.** (i) The boundary condition  $w_1|_{\partial D} = 0$  implies that the Poincaré inequality holds for  $w_1$ :  $|w_1|_2 \leq C|\partial_{x'} w_1|_2$ .  
(ii) The vanishing mean value condition  $\langle \phi_1 \rangle = 0$  implies that the Poincaré inequality holds for  $\phi_1$ :  $|\phi_1|_2 \leq C|\partial_{x'} \phi_1|_2$ .

We define  $\widetilde{M}_\xi$  by

$$\widetilde{M}_\xi = \widehat{L}_\xi - \widehat{L}_0 = \widetilde{A}_\xi + \widetilde{B}_\xi,$$

where

$$\widetilde{A}_\xi = \widehat{A}_\xi - \widehat{A}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu}{\rho_s} \xi^2 I_2 & -\frac{\widetilde{\nu}}{\rho_s} i \xi \nabla' \\ 0 & -\frac{\widetilde{\nu}}{\rho_s} i \xi \nabla' & \frac{\rho_s + \widetilde{\nu}}{\rho_s} \xi^2 \end{pmatrix},$$

$$\widetilde{B}_\xi = \widehat{B}_\xi - \widehat{B}_0 = \begin{pmatrix} i \xi v_s^3 & 0 & \gamma^2 i \xi \rho_s \\ 0 & i \xi v_s^3 I_2 & 0 \\ i \xi \frac{P'(\rho_s)}{\gamma^2 \rho_s} & 0 & i \xi v_s^3 \end{pmatrix}.$$

Decomposing  $u(t)$  in (4.12) as  $u(t) = \sigma(t)u^{(0)} + u_1(t)$ , we obtain

$$\partial_t(\sigma u^{(0)} + u_1) + \widehat{L}_0 u_1 + \widetilde{M}_\xi(\sigma u^{(0)} + u_1) = 0.$$

Applying  $\Pi^{(0)}$  and  $I - \Pi^{(0)}$  to this equation, we have

$$\begin{cases} \partial_t \sigma + \langle Q_0 \widetilde{M}_\xi(\sigma u^{(0)} + u_1) \rangle = 0, \\ \partial_t u_1 + \widehat{L}_0 u_1 + (I - \widehat{\Pi}^{(0)}) \widetilde{M}_\xi(\sigma u^{(0)} + u_1) = 0. \end{cases}$$

Since  $\widehat{\Pi}^{(0)} \widetilde{M}_\xi u = \langle Q_0 \widetilde{M}_\xi u \rangle u^{(0)}$  and  $Q_0 \widetilde{M}_\xi = Q_0 \widetilde{B}_\xi$ , we get

$$\partial_t \sigma + \langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle = 0, \tag{4.22}$$

$$\partial_t u_1 + \widehat{L}_\xi u_1 + \widetilde{M}_\xi(\sigma u^{(0)}) - \langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle u^{(0)} = 0, \tag{4.23}$$

$$w_1|_{\partial D} = 0, \quad \sigma(0) = \sigma_0, \quad u_1(0) = u_{1,0}, \tag{4.24}$$

where  $\sigma_0$  and  $u_{1,0}$  are given by

$$\sigma_0 = \langle u_0, u^{(0)*} \rangle, \quad u_{1,0} = (I - \widehat{\Pi}^{(0)})u_0.$$

We see from (4.14) that if  $u_0 \in H^1(D) \times H_0^1(D)$ , then

$$\begin{aligned} \sigma &\in H^1(0, T), \\ u_1 &\in C([0, T]; H^1(D) \times H_0^1(D)), \\ \phi_1 &\in H^1(0, T; H^1(D)), \\ w_1 &\in L^2(0, T; H^2(D)) \cap H^1(0, T; L^2(D)) \end{aligned}$$

for all  $T > 0$ .

**Lemma 4.11.** *For  $u_1 = {}^T(\phi_1, w'_1, w_1^3) \in R(I - \widehat{\Pi}^{(0)})$ , there hold the estimates:*

- (i)  $|\langle Q_0 \widetilde{B}_\xi u^{(0)} \rangle| \leq C|\xi|.$
- (ii)  $|\langle Q_0 \widetilde{B}_\xi u_1 \rangle| \leq C|\xi|(|\phi_1|_2 + \gamma^2|w_1^3|_2).$
- (iii)  $|\langle Q_0 \widetilde{B}_\xi u_1 \rangle| \leq C(|\xi||\phi_1|_2 + \gamma^2|\nabla' \cdot w'_1 + i\xi w_1^3|_2 + \gamma^2\omega|w'_1|_2).$

Lemma 4.11 can be proved by direct computations. We omit the proof.

We will employ an energy method to obtain the decay estimate on solutions of (4.22)-(4.24). We write (4.23) as:

$$\left\{ \begin{aligned} &\partial_t \phi_1 + i\xi v_s^3 \phi_1 + \gamma^2 \nabla' \cdot (\rho_s w'_1) + \gamma^2 i\xi \rho_s w_1^3 \\ &\quad + i\xi v_s^3 \sigma \phi^{(0)} + \gamma^2 i\xi \rho_s \sigma w^{(0),3} - \langle Q_0 \widetilde{B}_\xi (\sigma u^{(0)} + u_1) \rangle \phi^{(0)} = 0, \\ &\partial_t w'_1 - \frac{\nu}{\rho_s} (\Delta' - \xi^2) w'_1 - \frac{\tilde{\nu}}{\rho_s} \nabla' (\nabla' \cdot w'_1 + i\xi w_1^3) + \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi_1 \right) + i\xi v_s^3 w'_1 \\ &\quad - \frac{\tilde{\nu}}{\rho_s} i\xi \nabla' (\sigma w^{(0),3}) = 0, \\ &\partial_t w_1^3 - \frac{\nu}{\rho_s} (\Delta' - \xi^2) w_1^3 - \frac{\tilde{\nu}}{\rho_s} i\xi (\nabla' \cdot w'_1 + i\xi w_1^3) + i\xi \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi_1 \right) + i\xi v_s^3 w_1^3 \\ &\quad + \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3 \phi_1 + w'_1 \cdot \nabla' v_s^3 + \frac{\nu + \tilde{\nu}}{\rho_s} \xi^2 \sigma w^{(0),3} + i\xi \alpha_0 \sigma \\ &\quad + i\xi v_s^3 \sigma w^{(0),3} - \langle Q_0 \widetilde{B}_\xi (\sigma u^{(0)} + u_1) \rangle w^{(0),3} = 0. \end{aligned} \right. \quad (4.25)$$

Before proceeding further we introduce some notations. For  $u = {}^T(\phi, w)$  we define  $E_0[u]$  by

$$E_0[u] = \frac{1}{\gamma^2} \left| \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \phi \right|_2^2 + |\sqrt{\rho_s} w|_2^2.$$

For  $w = {}^T(w', w^3)$  with  $w' = {}^T(w^1, w^2)$  we define  $\widetilde{D}_\xi[w]$  by

$$\widetilde{D}_\xi[w] = \nu(|\nabla' w|_2^2 + |\xi|^2 |w|_2^2) + \tilde{\nu} |\nabla' \cdot w' + i\xi w^3|_2^2.$$

For  $\phi$  we define  $\dot{\phi}$  by

$$\dot{\phi} = \partial_t \phi + i\xi v_s^3 \phi.$$

**Proposition 4.12.** *There exist constants  $\nu_1 > 0$  and  $\omega_1 > 0$  such that if  $\nu \geq \nu_1$  and  $\frac{\nu+\tilde{\nu}}{\nu}\omega \leq \omega_1$ , then there hold the estimates:*

$$\frac{1}{2} \frac{d}{dt} \left( \frac{\alpha_0}{\gamma^2} |\sigma|^2 + E_0[u_1] \right) + \frac{1}{2} \tilde{D}_\xi[w_1] \leq C \left\{ \left( \frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \left( \frac{1}{\gamma^2} + \frac{\nu}{\gamma^4} \right) |\phi_1|_2^2 \right\}, \quad (4.26)$$

$$\frac{\nu+\tilde{\nu}}{\gamma^4} |\dot{\phi}_1|_2^2 \leq C \left\{ \frac{\nu+\tilde{\nu}}{\gamma^4} |\xi|^2 |\sigma|^2 + \frac{\nu+\tilde{\nu}}{\gamma^4} |\xi|^2 |\phi_1|_2^2 + \left( 1 + \frac{\nu+\tilde{\nu}}{\nu} \omega^2 \right) \tilde{D}_\xi[w_1] \right\}. \quad (4.27)$$

**Proof.** Multiplying (4.22) by  $\bar{\sigma}(t)$  and taking real part of the resulting equation, we have

$$\frac{1}{2} \frac{d}{dt} |\sigma|^2 + \operatorname{Re} \{ \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \bar{\sigma} \} = 0. \quad (4.28)$$

Since  $\tilde{B}_\xi^* = -\tilde{B}_\xi$  and  $u^{(0)*} = \frac{\gamma^2}{\alpha_0} u_0^{(0)}$ , we see that

$$\begin{aligned} \langle Q_0 \tilde{B}_\xi u_1 \rangle \bar{\sigma} &= \langle \tilde{B}_\xi u_1, \sigma u^{(0)*} \rangle \\ &= -\langle u_1, \tilde{B}_\xi(\sigma u^{(0)*}) \rangle \\ &= -\frac{\gamma^2}{\alpha_0} \langle u_1, \tilde{B}_\xi(\sigma u_0^{(0)}) \rangle, \end{aligned} \quad (4.29)$$

where  $u_0^{(0)} = T(\phi^{(0)}, 0)$ . On the other hand, since

$$\langle Q_0 \tilde{B}_\xi(\sigma u^{(0)}) \rangle \bar{\sigma} = i\xi |\sigma|^2 \{ \langle v_s^3 \phi_1 \rangle + \langle \gamma^2 \rho_s w_1^3 \rangle \},$$

we have

$$\operatorname{Re} \{ \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)}) \rangle \bar{\sigma} \} = 0. \quad (4.30)$$

We thus obtain from (4.28)-(4.30) that

$$\frac{1}{2} \frac{d}{dt} |\sigma|^2 - \frac{\gamma^2}{\alpha_0} \operatorname{Re} \langle u_1, \tilde{B}_\xi(\sigma u_0^{(0)}) \rangle = 0. \quad (4.31)$$

We next take the weighted inner product of (4.25) with  $u_1$ . The real part of the resulting equation then gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E_0[u_1] + \operatorname{Re} \langle \widehat{L}_0 u_1, u_1 \rangle + \operatorname{Re} \langle \widetilde{M}_\xi(\sigma u^{(0)} + u_1), u_1 \rangle \\ - \operatorname{Re} \{ \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \langle u^{(0)}, u_1 \rangle \} = 0. \end{aligned} \quad (4.32)$$

Since  $\widehat{B}_\xi^* = -\widehat{B}_\xi$ , we see that  $\operatorname{Re} \langle \widehat{B}_\xi u_1, u_1 \rangle = 0$ . It then follows that

$$\begin{aligned} \operatorname{Re} \langle \widehat{L}_0 u_1, u_1 \rangle + \operatorname{Re} \langle \widetilde{M}_\xi(\sigma u^{(0)} + u_1), u_1 \rangle \\ = \operatorname{Re} \langle \widehat{C}_0 u_1, u_1 \rangle + \operatorname{Re} \langle \widehat{A}_\xi u_1, u_1 \rangle + \operatorname{Re} \langle \widetilde{A}_\xi(\sigma u^{(0)}), u_1 \rangle + \operatorname{Re} \langle \widetilde{B}_\xi(\sigma u^{(0)}), u_1 \rangle \\ = \operatorname{Re} \langle \widehat{C}_0 u_1, u_1 \rangle + \tilde{D}_\xi[w_1] + \operatorname{Re} \langle \widetilde{A}_\xi(\sigma u^{(0)}), u_1 \rangle + \operatorname{Re} \langle \widetilde{B}_\xi(\sigma u^{(0)}), u_1 \rangle. \end{aligned}$$

This, together with (4.32), gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E_0[u_1] + \tilde{D}_\xi[w_1] + \operatorname{Re} \langle \widehat{C}_0 u_1, u_1 \rangle + \operatorname{Re} \langle \widetilde{A}_\xi(\sigma u^{(0)}), u_1 \rangle \\ + \operatorname{Re} \langle \widetilde{B}_\xi(\sigma u^{(0)}), u_1 \rangle - \operatorname{Re} \{ \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \langle u^{(0)}, u_1 \rangle \} = 0. \end{aligned} \quad (4.33)$$

We add  $\frac{\alpha_0}{\gamma^2} \times (4.31)$  to (4.33), to get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \frac{\alpha_0}{\gamma^2} |\sigma|^2 + E_0[u_1] \right) + \tilde{D}_\xi[w_1] + \operatorname{Re} \langle \widehat{C}_0 u_1, u_1 \rangle + \operatorname{Re} \langle \tilde{A}_\xi(\sigma u^{(0)}), u_1 \rangle \\ & + \operatorname{Re} \langle \tilde{B}_\xi(\sigma u_1^{(0)}), u_1 \rangle - \operatorname{Re} \{ \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \langle u^{(0)}, u_1 \rangle \} = 0, \end{aligned} \quad (4.34)$$

where  $u_1^{(0)} = T(0, w^{(0)})$ . Here we used the equation

$$-\operatorname{Re} \langle u_1, \tilde{B}_\xi(\sigma u_0^{(0)}) \rangle + \operatorname{Re} \langle \tilde{B}_\xi(\sigma u^{(0)}), u_1 \rangle = \operatorname{Re} \langle \tilde{B}_\xi(\sigma u_1^{(0)}), u_1 \rangle.$$

By the Poincaré inequality we have

$$\begin{aligned} |\operatorname{Re} \langle \tilde{A}_\xi(\sigma u^{(0)}), u_1 \rangle| & \leq C \left( \frac{\nu}{\gamma^2} |\xi|^2 |\sigma| |w_1^3|_2 + \frac{\tilde{\nu}}{\gamma^2} |\xi| |\sigma| |\nabla' \cdot w_1' + i\xi w_1^3|_2 \right) \\ & \leq \frac{1}{8} \tilde{D}_\xi[w_1] + C \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\sigma|^2, \end{aligned}$$

$$\begin{aligned} |\operatorname{Re} \langle \tilde{B}_\xi(\sigma u_1^{(0)}), u_1 \rangle| & \leq C |\xi| |\sigma| \left( \frac{1}{\gamma^2} |\phi_1|_2 + \frac{1}{\gamma^2} |w_1^3|_2 \right) \\ & \leq C \left( \frac{1}{\gamma^2} + \frac{1}{\nu \gamma^4} \right) |\xi|^2 |\sigma|^2 + \frac{1}{\gamma^2} |\phi_1|_2^2 + \frac{1}{8} \tilde{D}_\xi[w_1]. \end{aligned}$$

Since  $\langle \phi_1 \rangle = 0$ , there holds that

$$|\langle u^{(0)}, u_1 \rangle| \leq C \frac{1}{\gamma^2} |w_1^3|_2.$$

Applying Lemma 4.11 and the Poincaré and Hölder inequalities, we thus have the following estimates:

$$\begin{aligned} & |\operatorname{Re} \{ \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \langle u^{(0)}, u_1 \rangle \}| \\ & \leq C |\xi| (|\sigma| + |\phi_1|_2 + \gamma^2 |w_1^3|_2) \frac{1}{\gamma^2} |w_1^3|_2 \\ & \leq \frac{1}{8} \tilde{D}_\xi[w_1] + C \left( \frac{1}{\nu \gamma^4} |\xi|^2 |\sigma|^2 + \frac{1}{\nu \gamma^4} |\phi_1|_2^2 + \frac{1}{\nu} \tilde{D}_\xi[w_1] \right), \\ & |\operatorname{Re} \{ \langle \widehat{C}_0 u_1, u_1 \rangle \}| \leq C \left( \frac{\nu}{\gamma^4} |\phi_1|_2^2 + \frac{1}{\nu} \tilde{D}_\xi[w_1] \right). \end{aligned}$$

Therefore we find that there exists a constant  $\nu_1 > 0$  such that if  $\nu \geq \nu_1$ , then

$$\frac{1}{2} \frac{d}{dt} \left( \frac{\alpha_0}{\gamma^2} |\sigma|^2 + E_0[u_1] \right) + \frac{1}{2} \tilde{D}_\xi[w_1] \leq C \left\{ \left( \frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \left( \frac{1}{\gamma^2} + \frac{\nu}{\gamma^4} \right) |\phi_1|_2^2 \right\}.$$

We next estimate  $\dot{\phi}_1$ . By the first equation of (4.25) there holds that

$$\begin{aligned} \frac{1}{\gamma^2} \dot{\phi}_1 & = -(\nabla' \cdot (\rho_s w_1') + i\xi \rho_s w_1^3) \\ & - \frac{1}{\gamma^2} \{ i\xi v_s^3 \sigma \phi^{(0)} + \gamma^2 i\xi \rho_s \sigma w^{(0),3} - \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \}. \end{aligned}$$

We thus obtain

$$\frac{1}{\gamma^4} |\dot{\phi}_1|_2^2 \leq C \left\{ \frac{1}{\gamma^4} |\xi|^2 |\sigma|^2 + \frac{1}{\gamma^4} |\xi|^2 |\phi_1|_2^2 + \left( \frac{1}{\nu + \tilde{\nu}} + \frac{\omega^2}{\nu} \right) \tilde{D}_\xi[w_1] \right\}.$$

Multiplying by  $\nu + \tilde{\nu}$  to both sides, we have the desired estimate. This completes the proof.  $\square$

Let us estimate  $|\phi_1|_2$ . We first introduce the Bogovskii lemma.

**Proposition 4.13.** *Let  $\dot{L}^2(D)$  be defined by*

$$\dot{L}^2(D) = \{f \in L^2(D) : \langle f \rangle = 0\}.$$

*There exists a bounded operator  $\mathcal{B} : \dot{L}^2(D) \rightarrow H_0^1(D)$  such that*

$$-\nabla' \cdot \mathcal{B}f = f,$$

$$|\nabla' \mathcal{B}f|_2 \leq C|f|_2$$

*for any  $f \in \dot{L}^2(D)$ .*

**Proof.** See, e.g., [7, III.3, Theorem 3.2]. □

The proof of the following proposition is based on the argument in [9].

**Proposition 4.14.** *There exist constants  $\nu_1 > 0$  and  $\omega_1 > 0$  such that if  $\nu \geq \nu_1$ ,  $\gamma \geq 1$  and  $\frac{\nu+\tilde{\nu}}{\nu}\omega \leq \omega_1$ , then there hold the estimates:*

$$\begin{aligned} & \frac{d}{dt} J_0[u_1] + \frac{1}{2} \frac{1}{\nu+\tilde{\nu}} |\phi_1|_2^2 \\ & \leq C \left\{ \left( \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} + 1 \right) \tilde{D}_\xi[w_1] + \frac{\nu}{\nu+\tilde{\nu}} |\xi|^2 \tilde{D}_\xi[w_1] + \left( \frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 \right\}, \end{aligned} \quad (4.35)$$

$$|J_0[u_1]| \leq C \left\{ \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} |w_1|_2^2 + \frac{\nu}{\gamma^2(\nu+\tilde{\nu})} |\phi_1|_2^2 \right\},$$

where

$$J_0[u_1] = \frac{1}{\nu+\tilde{\nu}} \operatorname{Re}(w'_1, \rho_s \psi')$$

with  $\psi' = \mathcal{B}\phi_1$ .

**Proof.** Set  $\psi' = \mathcal{B}\phi_1$ . Taking the inner product of (4.25)<sub>2</sub> with  $\rho_s \psi'$ , we get

$$\operatorname{Re}(\partial_t w'_1, \rho_s \psi') + \operatorname{Re}\left(\nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi_1 \right), \rho_s \psi'\right) = \operatorname{Re} I, \quad (4.36)$$

where

$$\begin{aligned} I = & -\nu(\nabla' w'_1, \nabla' \psi') - \nu \xi^2(w'_1, \psi') + \tilde{\nu}(\nabla' \cdot w'_1, \phi_1) + \tilde{\nu} i \xi(w_1^3, \phi_1) \\ & - i \xi(\rho_s v_s^3 w'_1, \psi') + \tilde{\nu} i \xi(\sigma w^{(0),3}, \phi_1). \end{aligned}$$

Let us estimate the first term of the left-hand side of (4.36). It holds that

$$\operatorname{Re}(\partial_t w'_1, \rho_s \psi') = \frac{d}{dt} \operatorname{Re}(w'_1, \rho_s \psi') - \operatorname{Re}(w'_1, \rho_s \partial_t \psi').$$

Since

$$-\nabla' \cdot \partial_t \psi' = \partial_t \phi_1$$

and

$$\begin{aligned} \partial_t \phi_1 = & -\{i \xi v_s^3 \phi_1 + \gamma^2 \nabla' \cdot (\rho_s w'_1) + i \xi \gamma^2 \rho_s w_1^3 \\ & + i \xi v_s^3 \sigma \phi^{(0)} + \gamma^2 i \xi \rho_s \sigma w^{(0),3} - \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \phi^{(0)}\}, \end{aligned}$$

we obtain

$$\begin{aligned}
& |\operatorname{Re}(w'_1, \rho_s \partial_t \psi')| \\
& \leq C |w'_1|_2 |\partial_t \psi'|_2 \\
& \leq C |w'_1|_2 \{ |\xi| |\phi_1|_2 + \gamma^2 |\nabla' \cdot w'_1|_2 + \gamma^2 |w'_1|_2 + \gamma^2 |\xi|^2 |w_1^3|_2 + |\xi| |\sigma| \} \\
& \leq \frac{1}{8} |\phi_1|_2^2 + C \{ (\gamma^2 + \frac{\gamma^2}{\nu+\tilde{\nu}}) |w_1|_2^2 + (1 + \gamma^2) |\xi|^2 |w_1|_2^2 + \gamma^2 |\nabla' \cdot w'_1|_2^2 + \frac{\nu+\tilde{\nu}}{\gamma^2} |\xi|^2 |\sigma|^2 \} \\
& \leq \frac{1}{8} |\phi_1|_2^2 + C \{ (\frac{1}{\nu} + \frac{\gamma^2}{\nu} + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})}) \tilde{D}_\xi[w_1] + \frac{\nu+\tilde{\nu}}{\gamma^2} |\xi|^2 |\sigma|^2 \}.
\end{aligned}$$

We next estimate the second term of the left-hand side of (4.36). There exists  $\omega_1 > 0$  such that if  $\omega \leq \omega_1$ , then it holds that

$$\begin{aligned}
\operatorname{Re} \left( \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi_1 \right), \rho_s \psi' \right) &= \left| \sqrt{\frac{P'(\rho_s)}{\gamma^2}} \phi_1 \right|_2^2 - \operatorname{Re} \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi_1, (\nabla' \rho_s) \cdot \psi' \right) \\
&\geq C(1 - \omega) \left| \sqrt{\frac{P'(\rho_s)}{\gamma^2}} \phi_1 \right|_2^2 \\
&\geq \frac{3}{4} |\phi_1|_2^2.
\end{aligned}$$

As for  $I$ , we have

$$|I| \leq \frac{1}{8} |\phi_1|_2^2 + C \left\{ \left( \nu + \frac{\tilde{\nu}^2}{\nu+\tilde{\nu}} + \frac{1}{\nu} \right) \tilde{D}_\xi[w_1] + \nu |\xi|^2 \tilde{D}_\xi[w_1] + \frac{\tilde{\nu}^2}{\gamma^4} |\xi|^2 |\sigma|^2 \right\}.$$

Therefore it holds that

$$\begin{aligned}
\frac{d}{dt} \operatorname{Re}(w'_1, \rho_s \psi') + \frac{1}{2} |\phi_1|_2^2 &\leq C \left\{ \left( \frac{1}{\nu} + \frac{\gamma^2}{\nu} + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} + \nu + \frac{\tilde{\nu}^2}{\nu+\tilde{\nu}} \right) \tilde{D}_\xi[w_1] \right. \\
&\quad \left. + \nu |\xi|^2 \tilde{D}_\xi[w_1] + \left( \frac{\nu+\tilde{\nu}}{\gamma^2} + \frac{(\nu+\tilde{\nu})^2}{\gamma^4} \right) |\xi|^2 |\sigma|^2 \right\}.
\end{aligned}$$

Multiplying by  $\frac{1}{\nu+\tilde{\nu}}$  to both sides of this inequality, we have the desired estimate. This completes the proof.  $\square$

We next derive the estimate for  $\sigma$ . We introduce a notation. Let us define  $J_1[u]$  by

$$J_1[u] = \operatorname{Re} \{ i \xi \frac{1}{\nu+\tilde{\nu}} \langle \rho_s (A + |\xi|^2)^{-1} [\rho_s w_1^3] \rangle \bar{\sigma} \}$$

for  $u = \sigma u^{(0)} + u_1$  with  $u_1 = {}^T(\phi_1, w_1^1, w_1^2, w_1^3)$ . Here  $A$  is an operator on  $L^2(D)$  defined by

$$A\varphi = -\Delta' \varphi \quad \text{for} \quad \varphi \in D(A) = H^2(D) \cap H_0^1(D).$$

**Proposition 4.15.** *There exist constants  $\nu_1 > 0$ ,  $\gamma_1 > 0$ ,  $\omega_1 > 0$  and  $\tilde{\alpha}_0 > 0$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_1^2$  and  $\frac{\nu+\tilde{\nu}}{\nu} \omega \leq \omega_1$ , then there hold the estimates:*

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left( \frac{\nu}{(\nu+\tilde{\nu})\gamma^2} |\sigma|^2 + J_1[u] \right) + \frac{1}{2} \frac{\tilde{\alpha}_0}{\nu+\tilde{\nu}} |\xi|^2 |\sigma|^2 \\
& \leq C \left\{ \frac{\nu}{(\nu+\tilde{\nu})\gamma^2} |\phi_1|_2^2 + \frac{1}{\nu+\tilde{\nu}} |\xi|^2 |\phi_1|_2^2 + \frac{\nu^2}{(\nu+\tilde{\nu})\gamma^4} \max\{1, |\xi|^2\} |\xi|^2 |\phi_1|_2^2 \right. \\
& \quad \left. + \frac{\gamma^2}{(\nu+\tilde{\nu})\nu} \tilde{D}_\xi[w_1] + \left( \frac{\tilde{\nu}^2}{(\nu+\tilde{\nu})^2} + \frac{1}{\nu} \right) |\xi|^2 \tilde{D}_\xi[w_1] \right\}, \tag{4.37}
\end{aligned}$$

$$|J_1[u]| \leq \frac{1}{\gamma^2} |\sigma|^2 + C \frac{\gamma^2}{(\nu+\tilde{\nu})^2} |w_1|_2^2,$$

where  $\tilde{\alpha}_0$  is a positive constant.

**Proof.** Since

$$\langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle = \langle Q_0 \tilde{B}_\xi u^{(0)} \rangle \sigma + \gamma^2 i \xi \langle \rho_s w_1^3 \rangle + i \xi \langle v_s^3 \phi_1 \rangle,$$

(4.22) is written as

$$\partial_t \sigma + \langle Q_0 \tilde{B}_\xi u^{(0)} \rangle \sigma + \gamma^2 i \xi \langle \rho_s w_1^3 \rangle = -i \xi \langle v_s^3 \phi_1 \rangle. \quad (4.38)$$

Set

$$\tilde{B}_\xi^3 = \begin{pmatrix} i \xi \frac{P'(\rho_s)}{\gamma^2 \rho_s} & 0 & i \xi v_s^3 \end{pmatrix}.$$

Since  $\frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi^{(0)} = \alpha_0$ , we have

$$\tilde{B}_\xi^3 u_0^{(0)} = i \xi \frac{P'(\rho_s)}{\gamma^2 \rho_s} \sigma \phi^{(0)} = i \xi \alpha_0.$$

We thus obtain

$$-(\Delta' - \xi^2) w_1^3 = -\frac{\alpha_0}{\nu} i \xi \sigma \rho_s - \frac{\rho_s}{\nu} \partial_t w_1^3 + I_1^3. \quad (4.39)$$

Here

$$I_1^3 = -\frac{\rho_s}{\nu} \{ \hat{C}_0^3 u_1 - \frac{\tilde{\nu}}{\rho_s} i \xi (\nabla' \cdot w_1' + i \xi w_1^3) + \tilde{B}_\xi^3 u_1 + \sigma \tilde{M}_\xi^3 u_1^{(0)} - \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle w^{(0),3} \},$$

where  $\hat{C}_0^3$  and  $\tilde{M}_\xi^3$  are  $1 \times 4$  matrix operators defined by

$$\tilde{M}_\xi^3 = \begin{pmatrix} 0 & 0 & \frac{\nu + \tilde{\nu}}{\rho_s} \xi^2 \end{pmatrix} + \tilde{B}_\xi^3, \quad \hat{C}_0^3 = \begin{pmatrix} \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3 & {}^T(\nabla' v_s^3) & 0 \end{pmatrix}.$$

It then follows from (4.39) that

$$w_1^3 = -\frac{\alpha_0}{\nu} i \xi \sigma (A + |\xi|^2)^{-1} \rho_s - (A + |\xi|^2)^{-1} [\frac{\rho_s}{\nu} \partial_t w_1^3] + (A + |\xi|^2)^{-1} I_1^3.$$

Substituting this into (4.38) we obtain

$$\partial_t \sigma + \langle Q_0 \tilde{B}_\xi u^{(0)} \rangle \sigma + \frac{\alpha_0 \gamma^2}{\nu} \langle \rho_s (A + |\xi|^2)^{-1} \rho_s \rangle |\xi|^2 \sigma = I_1^0 - I_2^0, \quad (4.40)$$

where

$$I_1^0 = -\gamma^2 i \xi \langle \rho_s (A + |\xi|^2)^{-1} I_1^3 \rangle - i \xi \langle v_s^3 \phi_1 \rangle, \\ I_2^0 = \gamma^2 i \xi \langle \rho_s (A + |\xi|^2)^{-1} [\frac{\rho_s}{\nu} \partial_t w_1^3] \rangle.$$

Let us calculate (4.40)  $\times \bar{\sigma}$  and take its real part. Since  $\text{Re}\{\langle Q_0 \tilde{B}_\xi u^{(0)} \rangle\} = 0$ , we have

$$\frac{1}{2} \frac{d}{dt} |\sigma|^2 + \frac{\alpha_0 \gamma^2}{\nu} \langle \rho_s (A + |\xi|^2)^{-1} \rho_s \rangle |\xi|^2 |\sigma|^2 = \text{Re}(I_1^0 \bar{\sigma}) + \text{Re}(I_2^0 \bar{\sigma}).$$

Since  $\langle \rho_s (A + |\xi|^2)^{-1} \rho_s \rangle = |(A + |\xi|^2)^{-\frac{1}{2}} \rho_s|_2^2$  is continuous in  $\xi$  and is positive for all  $\xi \in \mathbf{R}$ , we see that there exists a positive constant  $\tilde{\alpha}_0 = \mathcal{O}(|\xi|^{-2})$  as  $|\xi| \rightarrow \infty$  such that

$$\frac{\alpha_0 \gamma^2}{\nu} \langle \rho_s (A + |\xi|^2)^{-1} \rho_s \rangle \geq \frac{\tilde{\alpha}_0 \gamma^2}{\nu}$$

for all  $\xi \in \mathbf{R}$  with  $|\xi| \leq R$ . We thus obtain

$$\frac{1}{2} \frac{d}{dt} |\sigma|^2 + \frac{\tilde{\alpha}_0 \gamma^2}{\nu} |\xi|^2 |\sigma|^2 \leq \operatorname{Re}(I_1^0 \bar{\sigma}) + \operatorname{Re}(I_2^0 \bar{\sigma}). \quad (4.41)$$

As for the right-hand side of (4.41), we see from

$$|(A + |\xi|^2)^{-1} p|_2 \leq \frac{C}{|\xi|^2 + 1} |p|_2$$

that

$$\begin{aligned} |\operatorname{Re}(I_1^0 \bar{\sigma})| &\leq \frac{\tilde{\alpha}_0 \gamma^2}{\nu} \left( \frac{1}{10} + C \frac{1}{\gamma^2} \right) \min\{1, |\xi|^2\} |\sigma|^2 \\ &\quad + C \left\{ |\phi_1|_2^2 + \left( \frac{\gamma^2}{\nu} + \frac{1}{\nu \gamma^2} \right) |\xi|^2 |\phi_1|_2^2 + \frac{\nu}{\gamma^2} \max\{1, |\xi|^2\} |\xi|^2 |\phi_1|_2^2 \right. \\ &\quad \left. + \frac{\gamma^2}{\nu^2} \tilde{D}_\xi[w_1] + \left( \frac{(\nu + \tilde{\nu}) \gamma^2}{\nu^2} + \frac{\gamma^2 \tilde{\nu}^2}{(\nu + \tilde{\nu}) \nu} \right) |\xi|^2 \tilde{D}_\xi[w_1] \right\}. \end{aligned} \quad (4.42)$$

We next derive the estimate for  $I_2^0 \bar{\sigma}$ . There holds that

$$\begin{aligned} \operatorname{Re}(I_2^0 \bar{\sigma}) &= \operatorname{Re} \left\{ \gamma^2 i \xi \langle \rho_s (A + |\xi|^2)^{-1} (\frac{\rho_s}{\nu} \partial_t w_1^3) \rangle \bar{\sigma} \right\} \\ &= \frac{d}{dt} \operatorname{Re} \left\{ i \xi \frac{\gamma^2}{\nu} \langle \rho_s (A + |\xi|^2)^{-1} (\rho_s w_1^3) \rangle \bar{\sigma} \right\} \\ &\quad - \operatorname{Re} \left\{ i \xi \frac{\gamma^2}{\nu} \langle \rho_s (A + |\xi|^2)^{-1} (\rho_s w_1^3) \rangle \partial_t \bar{\sigma} \right\} \\ &= \frac{d}{dt} \left( \frac{\gamma^2 (\nu + \tilde{\nu})}{\nu} J_1[u] \right) - \operatorname{Re} \left\{ i \xi \frac{\gamma^2}{\nu} \langle \rho_s (A + |\xi|^2)^{-1} (\rho_s w_1^3) \rangle \partial_t \bar{\sigma} \right\}. \end{aligned} \quad (4.43)$$

Let us estimate the second term of the right-hand side of this equation. We see from (4.22) that

$$\begin{aligned} &\left| \operatorname{Re} \left\{ i \xi \frac{\gamma^2}{\nu} \langle \rho_s (A + |\xi|^2)^{-1} [\rho_s w_1^3] \rangle \partial_t \bar{\sigma} \right\} \right| \\ &= \left| \operatorname{Re} \left\{ i \xi \frac{\gamma^2}{\nu} \langle \rho_s (A + |\xi|^2)^{-1} (\rho_s w_1^3) \rangle \left\{ -\langle Q_0 \tilde{B}_\xi u^{(0)} \rangle - \gamma^2 i \xi \langle \rho_s w_1^3 \rangle - i \xi \langle v_s^3 \phi_1 \rangle \right\} \right\} \right| \\ &\leq C \frac{\gamma^2}{\nu} \frac{|\xi|}{1 + |\xi|^2} |w_1|_2 \left\{ |\xi| |\sigma| + \gamma^2 |\xi| |w_1|_2 + |\xi| |\phi|_2 \right\} \\ &\leq \frac{1}{10} \frac{\tilde{\alpha}_0 \gamma^2}{\nu} \min\{1, |\xi|^2\} |\sigma|^2 + C \left\{ \frac{1}{\nu} |\xi|^2 |\phi_1|_2^2 + \frac{\gamma^4}{\nu^2} \tilde{D}_\xi[w_1] \right\}. \end{aligned} \quad (4.44)$$

If  $\frac{1}{\nu}$ ,  $\frac{1}{\gamma^2}$  and  $\frac{\nu + \tilde{\nu}}{\gamma^2}$  are sufficiently small, it then follows from (4.41), (4.42), (4.43) and (4.44) that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left( |\sigma|^2 + \frac{\gamma^2 (\nu + \tilde{\nu})}{\nu} J_1[u] \right) + \frac{1}{2} \frac{\tilde{\alpha}_0 \gamma^2}{\nu} |\xi|^2 |\sigma|^2 \\ &\leq C \left\{ |\phi_1|_2^2 + \frac{\gamma^2}{\nu} |\xi|^2 |\phi_1|_2^2 + \frac{\nu}{\gamma^2} \max\{1, |\xi|^2\} |\xi|^2 |\phi_1|_2^2 \right. \\ &\quad \left. + \frac{\gamma^4}{\nu^2} \tilde{D}_\xi[w_1] + \left( \frac{\gamma^2 (\nu + \tilde{\nu})}{\nu^2} + \frac{\gamma^2 \tilde{\nu}^2}{\nu (\nu + \tilde{\nu})} \right) |\xi|^2 \tilde{D}_\xi[w_1] \right\}. \end{aligned} \quad (4.45)$$

Furthermore we have the estimate

$$\begin{aligned} |J_1[u]| &= \left| \operatorname{Re} \left( i \xi \frac{1}{\nu + \tilde{\nu}} \langle \rho_s (A + |\xi|^2)^{-1} [\rho_s w_1^3] \rangle \bar{\sigma} \right) \right| \\ &\leq \frac{1}{\gamma^2} |\sigma|^2 + C \frac{\gamma^2}{(\nu + \tilde{\nu})^2} |w_1|_2^2. \end{aligned} \quad (4.46)$$

Multiplying by  $\frac{\nu}{\gamma^2(\nu+\tilde{\nu})}$  to both sides of (4.45), we obtain the desired estimates. This completes the proof.  $\square$

From Proposition 4.12, Proposition 4.14 and Proposition 4.15, we get the estimate of  $|\sigma|$ ,  $|\phi_1|_2$  and  $|w_1|_2$ .

**Proposition 4.16.** *Let  $R > 0$ . There exist positive constants  $\nu_1, \gamma_1, \omega_1$  independent of  $R$  and an energy functional  $E_1[u]$  such that if  $\nu \geq \nu_1 R^2$ ,  $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_1^2 R^2$ ,  $\frac{\nu+\tilde{\nu}}{\nu} \omega \leq \omega_1$  and  $|\xi| \leq R$ , then there hold the estimates:*

$$\frac{d}{dt} E_1[u] + \frac{1}{\nu+\tilde{\nu}} (|\xi|^2 |\sigma|^2 + |\phi_1|_2^2) + \tilde{D}_\xi[w_1] \leq 0, \quad (4.47)$$

$$\frac{1}{2} \left( \frac{1}{\gamma^2} |\sigma|^2 + E_0[u_1] \right) \leq C E_1[u] \leq \frac{3}{2} \left( \frac{1}{\gamma^2} |\sigma|^2 + E_0[u_1] \right),$$

where  $C$  is a positive constant independent of  $u$ .

**Proof.** For a given  $R > 0$  we assume that  $|\xi| \leq R$ . Let  $b_1 > 1$  and  $b_2 > 1$  be constants. Define  $E_1[u]$  by

$$E_1[u] = b_1 \left( 1 + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \right) \left( \frac{\alpha_0}{\gamma^2} |\sigma|^2 + E_0[u_1] \right) + b_2 J_0[u_1] + \frac{\nu}{\gamma^2(\nu+\tilde{\nu})} |\sigma|^2 + J_1[u].$$

Since we have

$$\frac{1}{2} (|\phi_1|_2^2 + |w_1|_2^2) \leq C_0 E_0[u_1] \leq \frac{3}{2} (|\phi_1|_2^2 + |w_1|_2^2),$$

$$|J_0[u_1]| \leq C_1 \left\{ \frac{\nu}{\gamma^2(\nu+\tilde{\nu})} |\phi_1|_2^2 + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} |w_1|_2^2 \right\},$$

$$|J_1[u]| \leq \frac{1}{\gamma^2} |\sigma|^2 + C_2 \frac{\gamma^2}{(\nu+\tilde{\nu})^2} |w_1|_2^2,$$

if  $\frac{1}{\nu+\tilde{\nu}} < 1$ ,  $\frac{\nu+\tilde{\nu}}{\gamma^2} < 1$  and  $b_1 > 8 \max\{C_0 C_1 b_2, C_0 C_2, \alpha_0^{-1}\}$ , then there exists a constant  $C > 0$  such that

$$\frac{1}{2} \left( \frac{1}{\gamma^2} |\sigma|^2 + E_0[u_1] \right) \leq C E_1[u] \leq \frac{3}{2} \left( \frac{1}{\gamma^2} |\sigma|^2 + E_0[u_1] \right). \quad (4.48)$$

Let us compute  $b_1 \times \left( 1 + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \right) \times (4.26) + b_2 \times (4.35) + (4.37)$  then

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} E_1[u] + \frac{b_1}{2} \left( 1 + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \right) \tilde{D}_\xi[w_1] + \frac{b_2}{2} \frac{1}{\nu+\tilde{\nu}} |\phi_1|_2^2 + \frac{\tilde{\alpha}_0}{2} \frac{1}{\nu+\tilde{\nu}} |\xi|^2 |\sigma|^2 \\ & \leq C_3 \left\{ b_1 \left( 1 + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \right) \left( \frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + b_1 \left( 1 + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \right) \left( \frac{1}{\gamma^2} + \frac{\nu}{\gamma^4} \right) |\phi_1|_2^2 \right. \\ & \quad + b_2 \left( \frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + b_2 \left( 1 + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \right) \tilde{D}_\xi[w_1] + b_2 \frac{\nu}{\nu+\tilde{\nu}} |\xi|^2 \tilde{D}_\xi[w_1] \\ & \quad + \frac{\nu}{(\nu+\tilde{\nu})\gamma^2} |\phi_1|_2^2 + \frac{1}{\nu+\tilde{\nu}} |\xi|^2 |\phi_1|_2^2 + \frac{\nu^2}{(\nu+\tilde{\nu})\gamma^4} \max\{1, |\xi|^2\} |\xi|^2 |\phi_1|_2^2 \\ & \quad \left. + \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \tilde{D}_\xi[w_1] + \left( \frac{\tilde{\nu}^2}{(\nu+\tilde{\nu})^2} + \frac{1}{\nu} \right) |\xi|^2 \tilde{D}_\xi[w_1] \right\}. \end{aligned}$$

Fix  $b_1 > 1$  and  $b_2 > 1$  so large that  $b_2 \geq 16C_3 R^2$  and  $b_1 \geq 16 \max\{C_0 C_1 b_2, C_0 C_2, \alpha_0^{-1}, C_3 b_2, C_3 R^2\}$ . We assume that  $\nu \geq \nu_1$  and  $\gamma \geq \gamma_1$  are so large that  $\nu \geq 16C_3 b_1 \max\{\tilde{\alpha}_0^{-1}, b_2^{-1}, 1\}$  and  $\gamma^2 > 16C_3 (1 + \tilde{\alpha}^{-1} + \tilde{\alpha}^{-\frac{1}{2}}) (\nu + \tilde{\nu}) \max\{b_1, b_2, b_2^{-1} (1 + R^2)\}$ . It then follows that there exists a constant  $C > 0$  such that

$$\frac{d}{dt} E_1[u] + C \left\{ \frac{1}{\nu+\tilde{\nu}} |\xi|^2 |\sigma|^2 + \frac{1}{\nu+\tilde{\nu}} |\phi_1|_2^2 + \tilde{D}_\xi[w_1] \right\} \leq 0. \quad (4.49)$$

We thus obtain the desired estimates. This completes the proof.  $\square$

We are now in a position to prove the estimate of the  $L^2$  norm of  $U_1(t)u_0$ . Before proceeding further we introduce a notation. For  $R > 0$  we define  $\mathbf{1}_{\{|\eta| \leq R\}}$  by  $\mathbf{1}_{\{|\eta| \leq R\}}(\xi) = 1$  for  $|\xi| \leq R$  and  $\mathbf{1}_{\{|\eta| \leq R\}}(\xi) = 0$  for  $|\xi| > R$ .

**Proposition 4.17.** *Let  $R > 0$ . There exist positive constants  $\nu_1$ ,  $\gamma_1$  and  $\omega_1$  such that if  $\nu \geq \nu_1 R^2$ ,  $\frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_1^2 R^2$ , and  $\frac{\nu + \tilde{\nu}}{\nu} \omega \leq \omega_1$ , then for any  $l = 0, 1, \dots$ , there exists a constant  $C = C(l) > 0$  such that the estimate*

$$\|\partial_{x_3}^l \mathcal{F}^{-1}[\mathbf{1}_{\{|\eta| \leq R\}}(\xi) e^{-t\hat{L}_\xi} \hat{u}_0]\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{l}{2}} \|u_0\|_{L^1(\mathbf{R}; L^2(D))} \quad (4.50)$$

holds for  $t \geq 0$ .

**Proof.** For a given  $R > 0$ , we assume that  $|\xi| \leq R$ . Since

$$|\xi|^2 |\sigma|^2 + |\phi_1|_2^2 + \tilde{D}_\xi[w_1] \geq \tilde{d}_0 |\xi|^2 (|\sigma|^2 + |\phi_1|_2^2 + |w_1|_2^2)$$

for some constant  $\tilde{d}_0 = \tilde{d}_0(R) > 0$ , we see from (4.47) that there exists a constant  $d_0 > 0$  such that

$$\frac{d}{dt} E_1[u](t) + d_0 |\xi|^2 |u|_2^2 \leq 0.$$

This implies that

$$|e^{-t\hat{L}_\xi} \hat{u}_0(\xi)|_{L^2} \leq C e^{-d_0 |\xi|^2 t} |\hat{u}_0(\xi)|_{L^2}. \quad (4.51)$$

We thus obtain the desired estimate. This completes the proof.  $\square$

We next estimate derivatives of  $u$ . We introduce some notations. We define  $J_2^{(0)}[u]$  by

$$J_2^{(0)}[u] = -2\text{Re}\langle \sigma u^{(0)} + u_1, \hat{B}_\xi \tilde{Q} u_1 \rangle \quad \text{for } u = \sigma u^{(0)} + u_1.$$

In addition, we set

$$E_2^{(0)}[u] = \left(1 + \frac{b_3 \gamma^2}{\nu}\right) \left(\frac{\alpha_0}{\gamma^2} |\sigma|^2 + E_0[u_1]\right) + \tilde{D}_\xi[w_1],$$

$$\tilde{E}_2^{(0)}[u] = E_2^{(0)}[u] + J_2^{(0)}[u],$$

where  $b_3$  is a positive constant to be determined later. We note that there exists a constant  $b_3^* > 0$  such that if  $b_3 \geq b_3^*$  and  $\gamma^2 \geq 1$  then

$$\frac{1}{2} E_2^{(0)}[u] \leq \tilde{E}_2^{(0)}[u] \leq \frac{3}{2} E_2^{(0)}[u].$$

Taking  $b_3$  suitably large, we have the following estimate for  $\tilde{E}_2^{(0)}[u]$ .

**Proposition 4.18.** *There exist constants  $b_3 \geq b_3^*$ ,  $\nu_1 > 0$  and  $\omega_1 > 0$  such that if  $\nu \geq \nu_1$ ,  $\gamma^2 \geq 1$  and  $\omega \leq \omega_1$ , then there holds the estimate:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \widetilde{E}_2^{(0)}[u] + \frac{1}{4} b_3 \frac{\gamma^2}{\nu} \widetilde{D}_\xi[w_1] + \frac{1}{2} |\sqrt{\rho_s} \partial_t w_1|_2^2 \\ & \leq C \left\{ \left( \frac{1}{\nu} + \frac{\nu + \bar{\nu}}{\nu \gamma^2} + \frac{\bar{\nu}^2}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \frac{(\nu + \bar{\nu})^2}{\gamma^4} |\xi|^4 |\sigma|^2 \right. \\ & \quad \left. + \left( \frac{1}{\nu} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4} \right) |\phi_1|_2^2 + \frac{1}{\gamma^2} |\xi|^2 |\phi_1|_2^2 \right\}. \end{aligned} \quad (4.52)$$

**Proof.** Since  $u$  is a solution of

$$\partial_t u + \widehat{L}_\xi u = 0,$$

it holds that

$$\langle \partial_t u, \partial_t \widetilde{Q} u_1 \rangle + \langle \widehat{L}_\xi u, \partial_t \widetilde{Q} u_1 \rangle = 0. \quad (4.53)$$

We first consider the first term on the left-hand side of (4.53). Since

$$\partial_t \sigma = -\langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle,$$

$$\langle u^{(0)}, \partial_t \widetilde{Q} u_1 \rangle = \langle u_1^{(0)}, \partial_t \widetilde{Q} u_1 \rangle,$$

applying Remark 4.10 and Lemma 4.11, we obtain

$$\begin{aligned} \operatorname{Re} \langle \partial_t u, \partial_t \widetilde{Q} u_1 \rangle &= \operatorname{Re} \{ \langle \partial_t \sigma u^{(0)}, \partial_t \widetilde{Q} u_1 \rangle + \langle \partial_t u_1, \partial_t \widetilde{Q} u_1 \rangle \} \\ &= \operatorname{Re} \{ -\langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \langle u_1^{(0)}, \partial_t \widetilde{Q} u_1 \rangle + |\sqrt{\rho_s} \partial_t w_1|_2^2 \} \\ &\geq \frac{7}{8} |\sqrt{\rho_s} \partial_t w_1|_2^2 - C \left\{ \frac{1}{\gamma^4} |\xi|^2 (|\sigma|^2 + |\phi_1|_2^2) + \frac{1}{\nu} \widetilde{D}_\xi[w_1] \right\}. \end{aligned} \quad (4.54)$$

As for the second term on the left-hand side of (4.53), we see from  $\widehat{L}_0 u^{(0)} = 0$  and  $\widehat{B}_0 u^{(0)} = 0$  that

$$\begin{aligned} \langle \widehat{L}_\xi u, \partial_t \widetilde{Q} u_1 \rangle &= \langle \widetilde{M}_\xi(\sigma u^{(0)}), \partial_t \widetilde{Q} u_1 \rangle + \langle \widehat{L}_\xi u_1, \partial_t \widetilde{Q} u_1 \rangle \\ &= \langle \widetilde{A}_\xi(\sigma u^{(0)}), \partial_t \widetilde{Q} u_1 \rangle + \langle \widehat{B}_\xi(\sigma u^{(0)} + u_1), \partial_t \widetilde{Q} u_1 \rangle \\ &\quad + \langle \widehat{A}_\xi u_1, \partial_t \widetilde{Q} u_1 \rangle + \langle \widehat{C}_0 u_1, \partial_t \widetilde{Q} u_1 \rangle. \end{aligned} \quad (4.55)$$

It follows from (4.53), (4.54) and (4.55) that

$$\begin{aligned} & \frac{7}{8} |\sqrt{\rho_s} \partial_t w_1|_2^2 + \operatorname{Re} \langle \widetilde{A}_\xi(\sigma u^{(0)}), \partial_t \widetilde{Q} u_1 \rangle + \operatorname{Re} \langle \widehat{B}_\xi(\sigma u^{(0)} + u_1), \partial_t \widetilde{Q} u_1 \rangle \\ & \quad + \operatorname{Re} \langle \widehat{A}_\xi u_1, \partial_t \widetilde{Q} u_1 \rangle + \operatorname{Re} \langle \widehat{C}_0 u_1, \partial_t \widetilde{Q} u_1 \rangle \\ & \leq C \left\{ \frac{1}{\gamma^4} |\xi|^2 |\sigma|^2 + \frac{1}{\gamma^4} |\xi|^2 |\phi_1|_2^2 + \frac{1}{\nu} \widetilde{D}_\xi[w_1] \right\}. \end{aligned} \quad (4.56)$$

Next we show the estimate

$$\begin{aligned} & \operatorname{Re} \{ \langle \widehat{B}_\xi(\sigma u^{(0)} + u_1), \partial_t \widetilde{Q} u_1 \rangle + \langle \widehat{A}_\xi u_1, \partial_t \widetilde{Q} u_1 \rangle \} \\ & \geq \frac{1}{2} \frac{d}{dt} \left( \widetilde{D}_\xi[w_1] + J_2^{(0)}[u] \right) - \epsilon |\sqrt{\rho_s} \partial_t w_1|_2^2 \\ & \quad - C \left\{ \left( \frac{1}{\gamma^2} + \frac{1}{\epsilon \gamma^4} \right) |\xi|^2 |\sigma|^2 + \frac{1}{\gamma^2} |\xi|^2 |\phi_1|_2^2 + \left( \frac{1}{\nu \gamma^2} + \frac{\gamma^2}{\nu} + \frac{1}{\epsilon \nu} \right) \widetilde{D}_\xi[w_1] \right\} \end{aligned} \quad (4.57)$$

for any  $\epsilon > 0$  with  $C$  independent of  $\epsilon$ . In fact, it holds by integrating by parts that

$$\operatorname{Re}\langle \widehat{A}_\xi u_1, \partial_t \widetilde{Q} u_1 \rangle = \frac{1}{2} \frac{d}{dt} \widetilde{D}_\xi[w_1]. \quad (4.58)$$

Since  $\widehat{B}_\xi^* = -\widehat{B}_\xi$ , we see that

$$\operatorname{Re}\langle \widehat{B}_\xi(\sigma u_0^{(0)}), \partial_t \widetilde{Q} u_1 \rangle = -\frac{d}{dt} \{ \operatorname{Re}\langle \sigma u_0^{(0)}, \widehat{B}_\xi \widetilde{Q} u_1 \rangle \} + \operatorname{Re}\langle \partial_t(\sigma u_0^{(0)}), \widehat{B}_\xi \widetilde{Q} u_1 \rangle. \quad (4.59)$$

By (4.22) we have

$$\partial_t \sigma = -\langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle.$$

It then follows from Lemma 4.11 that

$$\begin{aligned} |\operatorname{Re}\langle \partial_t(\sigma u_0^{(0)}), \widehat{B}_\xi \widetilde{Q} u_1 \rangle| &= |\operatorname{Re}\{ \langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \langle u_0^{(0)}, \widehat{B}_\xi \widetilde{Q} u_1 \rangle \}| \\ &\leq C \{ \frac{1}{\gamma^2} |\xi|^2 (|\sigma|^2 + |\phi_1|_2^2) + (\frac{1}{\nu \gamma^2} + \frac{\gamma^2}{\nu}) \widetilde{D}_\xi[w_1] \}. \end{aligned} \quad (4.60)$$

Similarly to above, there holds the following equation:

$$\begin{aligned} \operatorname{Re}\langle \widehat{B}_\xi u_1, \partial_t \widetilde{Q} u_1 \rangle &= -\frac{d}{dt} \{ \operatorname{Re}\langle u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle \} + \operatorname{Re}\langle \partial_t u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle \\ &= -\frac{d}{dt} \{ \operatorname{Re}\langle u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle \} + \operatorname{Re}\langle \partial_t Q_0 u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle + \operatorname{Re}\langle \partial_t \widetilde{Q} u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle. \end{aligned} \quad (4.61)$$

We estimate the second term on the right hand of (4.61). By (4.25) we have

$$\begin{aligned} \partial_t Q_0 u_1 &= -Q_0 \{ \widehat{L}_\xi u_1 + \widetilde{M}_\xi(\sigma u^{(0)}) - \langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle u^{(0)} \} \\ &= -Q_0 \widehat{B}_\xi u_1 - Q_0 \widetilde{B}_\xi(\sigma u^{(0)}) + \langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle u^{(0)}. \end{aligned}$$

Since  $\langle \partial_t Q_0 u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle = \langle \partial_t Q_0 u_1, Q_0 \widehat{B}_\xi \widetilde{Q} u_1 \rangle$ , we see from Lemma 4.11 that

$$\begin{aligned} |\operatorname{Re}\langle \partial_t Q_0 u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle| &\leq C \{ |Q_0 \widehat{B}_\xi u_1|_2 + |Q_0 \widetilde{B}_\xi(\sigma u^{(0)})|_2 \\ &\quad + |\langle Q_0 \widetilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle u^{(0)}|_2 \} \times \frac{1}{\gamma^2} |Q_0 \widehat{B}_\xi \widetilde{Q} u_1|_2 \\ &\leq C \{ \frac{1}{\gamma^2} |\xi|^2 (|\sigma|^2 + |\phi_1|_2^2) + (\frac{1}{\nu \gamma^2} + \frac{\gamma^2}{\nu}) \widetilde{D}_\xi[w_1] \}. \end{aligned} \quad (4.62)$$

The third term on the right-hand of (4.61) is estimated as

$$\begin{aligned} |\operatorname{Re}\langle \partial_t \widetilde{Q} u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle| &\leq C |\sqrt{\rho_s} \partial_t w_1|_2 (|\nabla' \cdot (\rho_s w_1') + i \xi \rho_s w_1^3|_2 + |\xi| |w_1|_2) \\ &\leq \epsilon |\sqrt{\rho_s} \partial_t w_1|_2^2 + C \frac{1}{\epsilon \nu} \widetilde{D}_\xi[w_1] \end{aligned}$$

for any  $\epsilon > 0$  with  $C$  independent of  $\epsilon$ . This, together with (4.61) and (4.62), leads to the inequality

$$\begin{aligned} \operatorname{Re}\langle \widehat{B}_\xi u_1, \partial_t \widetilde{Q} u_1 \rangle &\geq -\frac{d}{dt} \{ \operatorname{Re}\langle u_1, \widehat{B}_\xi \widetilde{Q} u_1 \rangle \} - \epsilon |\sqrt{\rho_s} \partial_t w_1|_2^2 \\ &\quad - C \{ \frac{1}{\gamma^2} |\xi|^2 (|\sigma|^2 + |\phi_1|_2^2) + (\frac{1}{\nu \gamma^2} + \frac{\gamma^2}{\nu} + \frac{1}{\epsilon \nu}) \widetilde{D}_\xi[w_1] \} \end{aligned} \quad (4.63)$$

for any  $\epsilon > 0$  with  $C$  independent of  $\epsilon$ . Furthermore, we have

$$\begin{aligned} |\operatorname{Re}\langle \widehat{B}_\xi(\sigma u_1^{(0)}), \partial_t \widetilde{Q}u_1 \rangle| &\leq C|\sqrt{\rho_s}\partial_t w_1|_2 |i\xi\rho_s\sigma w^{(0),3} + i\xi v_s^3\sigma w^{(0),3}|_2 \\ &\leq \epsilon|\sqrt{\rho_s}\partial_t w_1|_2^2 + C\frac{1}{\epsilon\gamma^4}|\xi|^2|\sigma|^2 \end{aligned} \quad (4.64)$$

for any  $\epsilon > 0$  with  $C$  independent of  $\epsilon$ . By (4.59), (4.60), (4.63) and (4.64), we obtain

$$\begin{aligned} &\operatorname{Re}\langle \widehat{B}_\xi(\sigma u^{(0)} + u_1), \partial_t \widetilde{Q}u_1 \rangle \\ &\geq -\frac{1}{2}\frac{d}{dt}J_2^{(0)}[u] - \epsilon|\sqrt{\rho_s}\partial_t w_1|_2^2 \\ &\quad - C\left\{\left(\frac{1}{\gamma^2} + \frac{1}{\epsilon\gamma^4}\right)|\xi|^2|\sigma|^2 + \frac{1}{\gamma^2}|\xi|^2|\phi_1|_2^2 + \left(\frac{1}{\nu\gamma^2} + \frac{\gamma^2}{\nu} + \frac{1}{\epsilon\nu}\right)\widetilde{D}_\xi[w_1]\right\}. \end{aligned}$$

This, together with (4.58), gives (4.57).

The remaining terms on the left-hand side of (4.56) are estimated as

$$\begin{aligned} &|\operatorname{Re}\langle \widetilde{A}_\xi(\sigma u^{(0)}), \partial_t \widetilde{Q}u_1 \rangle| \\ &\leq C\{\widetilde{\nu}|\xi||\nabla' w^{(0),3}|_2 + (\nu + \widetilde{\nu})|\xi|^2|w^{(0),3}|_2\}|\sigma||\sqrt{\rho_s}\partial_t w_1|_2 \\ &\leq \epsilon|\sqrt{\rho_s}\partial_t w_1|_2^2 + C\frac{1}{\epsilon}\left\{\frac{\widetilde{\nu}^2}{\gamma^4}|\xi|^2|\sigma|^2 + \frac{(\nu+\widetilde{\nu})^2}{\gamma^4}|\xi|^4|\sigma|^2\right\}, \end{aligned} \quad (4.65)$$

$$\begin{aligned} |\operatorname{Re}\langle \widehat{C}_0 u_1, \partial_t \widetilde{Q}u_1 \rangle| &\leq C\left(\frac{\nu}{\gamma^2}|\phi_1|_2 + |w'_1|_2\right)|\sqrt{\rho_s}\partial_t w_1|_2 \\ &\leq \epsilon|\sqrt{\rho_s}\partial_t w_1|_2^2 + C\frac{1}{\epsilon}\left\{\frac{\nu^2}{\gamma^4}|\phi_1|_2^2 + \frac{1}{\nu}\widetilde{D}_\xi[w_1]\right\}. \end{aligned} \quad (4.66)$$

Here  $\epsilon$  is an arbitrary positive number and  $C$  is a constant independent of  $\epsilon$ . Taking  $\epsilon > 0$  suitably small, we see from (4.56) with (4.57), (4.65) and (4.66) that if  $\nu \geq 1$  and  $\gamma^2 \geq 1$ , then

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\widetilde{D}_\xi[w_1] + J_2^{(0)}[u]\right) + \frac{3}{4}|\sqrt{\rho_s}\partial_t w_1|_2^2 \\ &\leq C_0\left\{\left(\frac{1}{\gamma^2} + \frac{\widetilde{\nu}^2}{\gamma^4}\right)|\xi|^2|\sigma|^2 + \frac{(\nu+\widetilde{\nu})^2}{\gamma^4}|\xi|^4|\sigma|^2\right. \\ &\quad \left.+ \frac{\nu^2}{\gamma^4}|\phi_1|_2^2 + \frac{1}{\gamma^2}|\xi|^2|\phi_1|_2^2 + \frac{\gamma^2}{\nu}\widetilde{D}_\xi[w_1]\right\}. \end{aligned} \quad (4.67)$$

We take  $b_3$  as  $b_3 \geq \max\{b_3^*, 4C_0\}$  and then add (4.67) to  $b_3\frac{\gamma^2}{\nu} \times (4.26)$ , to get (4.52). This completes the proof.  $\square$

We next establish the estimate for higher order derivatives near the boundary  $\partial D$ . We introduce the local curvilinear coordinate system.

For any  $\bar{x}'_0 \in \partial D$ , there exist a neighborhood  $\widetilde{\mathcal{O}}_{\bar{x}'_0}$  of  $\bar{x}'_0$  and a smooth diffeomorphism map  $\Psi = (\Psi_1, \Psi_2) : \widetilde{\mathcal{O}}_{\bar{x}'_0} \rightarrow B_1(0) = \{z' = (z_1, z_2) : |z'| < 1\}$  such that

$$\begin{cases} \Psi(\widetilde{\mathcal{O}}_{\bar{x}'_0} \cap D) = \{z' \in B_1(0) : z_1 > 0\}, \\ \Psi(\widetilde{\mathcal{O}}_{\bar{x}'_0} \cap \partial D) = \{z' \in B_1(0) : z_1 = 0\}, \\ \det \nabla_{x'} \Psi \neq 0 \quad \text{on} \quad \overline{\widetilde{\mathcal{O}}_{\bar{x}'_0} \cap D}. \end{cases}$$

By the tubular neighborhood theorem, there exist a neighborhood  $\mathcal{O}_{\bar{x}'_0}$  of  $\bar{x}'_0$  and a local curvilinear coordinate system  $y' = (y_1, y_2)$  on  $\mathcal{O}_{\bar{x}'_0}$  defined by

$$x' = y_1 a_1(y_2) + \Psi^{-1}(0, y_2) : \mathcal{R} \rightarrow \mathcal{O}_{\bar{x}'_0}, \quad (4.68)$$

where  $\mathcal{R} = \{y' = (y_1, y_2) : |y_1| \leq \tilde{\delta}_1, |y_2| \leq \tilde{\delta}_2\}$  for some  $\tilde{\delta}_1, \tilde{\delta}_2 > 0$ ;  $a_1(y_2)$  is the unit inward normal to  $\partial D$  that is given by

$$a_1(y_2) = \frac{\nabla_{x'} \Psi_1}{|\nabla_{x'} \Psi_1|}.$$

We set  $y_3 = x_3$ . It then follows that

$$\nabla_x = e_1(y_2) \partial_{y_1} + J(y') e_2(y_2) \partial_{y_2} + e_3 \partial_{y_3},$$

$$\nabla_y = \begin{pmatrix} {}^T e_1(y_2) \\ \frac{1}{J(y')} {}^T e_2(y_2) \\ {}^T e_3 \end{pmatrix} \nabla_x,$$

where

$$e_1(y_2) = \begin{pmatrix} a_1(y_2) \\ 0 \end{pmatrix}, \quad e_2(y_2) = \begin{pmatrix} a_2(y_2) \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad (4.69)$$

$$J(y') = |\det \nabla_{x'} \Psi|, \quad a_2(y_2) = \frac{-\nabla_{x'}^\perp \Psi_1}{|\nabla_{x'}^\perp \Psi_1|}$$

with  $\nabla_{x'}^\perp \Psi_1 = {}^T(-\partial_{x_2} \Psi_1, \partial_{x_1} \Psi_1)$ . Note that  $\partial_{y_1}$  and  $\partial_{y_2}$  are the inward normal derivative and tangential derivative at  $x' = \Psi^{-1}(0, y_2) \in \partial D \cap \mathcal{O}_{\bar{x}'_0}$ , respectively. We denote the normal and tangential derivatives by  $\partial_n$  and  $\partial$ , i.e.,

$$\partial_n = \partial_{y_1}, \quad \partial = \partial_{y_2}.$$

Since  $\partial D$  is compact, there are bounded open sets  $\mathcal{O}_m$  ( $m = 1, \dots, N$ ) such that  $\partial D \subset \cup_{m=1}^N \mathcal{O}_m$  and for each  $m = 1, \dots, N$ , there exists a local curvilinear coordinate system  $y' = (y_1, y_2)$  as defined in (4.68) with  $\mathcal{O}_{\bar{x}'_0}$ ,  $\Psi$  and  $\mathcal{R}$  replaced by  $\mathcal{O}_m$ ,  $\Psi^m$  and  $\mathcal{R}_m = \{y' = (y_1, y_2) : |y_1| < \tilde{\delta}_1^m, |y_2| < \tilde{\delta}_2^m\}$  for some  $\tilde{\delta}_1^m, \tilde{\delta}_2^m > 0$ . At last, we take an open set  $\mathcal{O}_0 \subset D$  such that

$$\cup_{m=0}^N \mathcal{O}_m \supset D, \quad \overline{\mathcal{O}_0} \cap \partial D = \emptyset.$$

We set a local coordinate  $y' = (y_1, y_2)$  such that  $y_1 = x_1, y_2 = x_2$  on  $\mathcal{O}_0$ .

Note that if  $h \in H^2(D)$ , then  $h|_{\partial D} = 0$  implies that  $\partial^k h|_{\partial D \cap \mathcal{O}_m} = 0$  ( $k = 0, 1$ ).

Let us introduce a partition of unity  $\{\chi_m\}_{m=0}^N$  subordinate to  $\{\mathcal{O}_m\}_{m=0}^N$ , satisfying

$$\sum_{m=0}^N \chi_m = 1 \text{ on } D, \quad \chi_m \in C_0^\infty(\mathcal{O}_m) \quad (m = 0, 1, \dots, N).$$

In the following we will denote by  $[A, B]$  the commutator of  $A$  and  $B$ , i.e.,

$$[A, B] = AB - BA.$$

**Lemma 4.19.** For  $1 \leq m \leq N$  there hold the following estimates.

- (i)  $|[\partial, \partial_{x_j}]h| \leq C|\partial_{x'}h|$  for  $h \in H^2(D)$  and  $j = 1, 2$ .
- (ii)  $|(\chi_m[\partial, \partial_{x_j}]h, \chi_m \partial h)| \leq C|\chi_m \partial_{x'}h|_2^2$  for  $h \in H^2(D)$  and  $j = 1, 2$ .
- (iii)  $|(\chi_m[\partial, \partial_{x_k} \partial_{x_l}]h, \chi_m \partial h)| \leq \eta|\chi_m \partial_{x'} \partial h|_2^2 + C(1 + \frac{1}{\eta})|\partial_{x'}h|_{L^2(D \cap \mathcal{O}_m)}^2$  for all  $\eta > 0$ ,  $h \in H^2(D)$  with  $\partial h|_{\partial D \cap \mathcal{O}_m} = 0$  and  $k, l = 1, 2$ .

**Proof.** (i) For  $x' \in D \cap \mathcal{O}_m$ , we set  $y' = \Psi^m(x')$ ,  $h(x') = \tilde{h}(y')$ . Then there exists a smooth matrix valued function  $A_1(y')$  such that  $\nabla_{x'} = A_1(y')\nabla_{y'}$ . We thus find that

$$[\partial, \partial_{x_j}]h = \partial \partial_{x_j} h - \partial_{x_j} \partial h = \sum_{0 \leq l_1, 0 \leq l_2, l_1 + l_2 = 1} h_{l_1 l_2} \partial^{l_1} \partial_n^{l_2} \tilde{h},$$

where  $h_{l_1 l_2} = h_{l_1 l_2}(y')$  are smooth functions depending only on  $D \cap \mathcal{O}_m$ . Since

$$\frac{1}{C}|\partial_{y'} \tilde{h}| \leq |\partial_{x'} h| \leq C|\partial_{y'} \tilde{h}|$$

for some constant  $C > 0$ , we have the desired inequality. This completes the proof of (i).

(ii) The estimate in (ii) immediately follows from (i).

(iii) We have  $\nabla_{y'} = A_1(y')^{-1}\nabla_{x'}$ . We set  $A_1(y')^{-1} = (c^{ij}(x'))_{ij}$ . There holds that

$$[\partial, \partial_{x_k x_l}]h = - \sum_{j=1}^2 \{ \partial_{x_k} \partial_{x_l} c^{2j} \partial_{x_j} h + \partial_{x_l} c^{2j} \partial_{x_k} \partial_{x_j} h + \partial_{x_k} c^{2j} \partial_{x_l} \partial_{x_j} h \}.$$

It follows from integration by parts that

$$\begin{aligned} & |(\chi_m \partial_{x_l} c^{2j} \partial_{x_k} \partial_{x_j} h, \chi_m \partial h)| \\ &= |(\chi_m \partial_{x_l} c^{2j} \partial_{x_j} h, \chi_m \partial_{x_k} \partial h) + (\chi_m \partial_{x_k} \partial_{x_l} c^{2j} \partial_{x_j} h, \chi_m \partial h) + (\partial_{x_k} \chi_m^2 \partial_{x_l} c^{2j} \partial_{x_j} h, \partial h)| \\ &\leq C \{ |\chi_m \partial_{x_j} h|_2 |\chi_m \partial_{x_k} \partial h|_2 + |\chi_m \partial_{x_j} h|_2 |\chi_m \partial h|_2 + |\partial_{x_j} h|_{L^2(D \cap \mathcal{O}_m)} |\chi_m \partial h|_2 \} \\ &\leq \eta |\chi_m \partial_{x'} \partial h|_2^2 + C(1 + \frac{1}{\eta}) |\partial_{x'} h|_{L^2(D \cap \mathcal{O}_m)}^2. \end{aligned}$$

This complete the proof of (iii).  $\square$

We are in a position to estimate higher order derivatives. We first derive the estimate for  $\partial \phi_1$ .

**Proposition 4.20.** For  $1 \leq m \leq N$ , there exist constants  $\nu_1 > 0$ ,  $\omega_1 > 0$  and  $b > 0$  such that if  $\nu \geq \nu_1$ ,  $\gamma^2 \geq 1$  and  $\frac{\nu + \tilde{\nu}}{\nu} \omega \leq \omega_1$ , then there holds the estimate:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial \phi_1 \right|_2^2 + |\chi_m \sqrt{\rho_s} \partial w_1|_2^2 \right) + b \frac{\nu + \tilde{\nu}}{\gamma^4} |\chi_m \partial \phi_1|_2^2 \\ & + \frac{1}{2} \nu (|\chi_m \nabla' \partial w_1|_2^2 + |\xi|^2 |\chi_m \partial w_1|_2^2) + \frac{1}{2} \tilde{\nu} |\chi_m (\nabla' \cdot \partial w_1' + i \xi \partial w_1^3)|_2^2 \\ & \leq C \left\{ \left( \frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \left( \eta + \frac{1}{\gamma^2} \right) |\phi_1|_2^2 + \left( \eta + \frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\phi_1|_2^2 \right. \\ & \quad \left. + \left( \eta + \frac{1}{\gamma^2} \right) |\partial_{x'} \phi_1|_2^2 + \left( \frac{1}{\eta \nu} + \frac{\nu}{\gamma^2} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w_1] + \left( \frac{\tilde{\nu}}{\nu} + 1 \right) |\xi|^2 \tilde{D}_\xi[w_1] \right\} \end{aligned} \tag{4.70}$$

for any  $\eta > 0$  with  $C$  independent of  $\eta$ .

**Proof.** Applying  $\partial$  to (4.25), we have

$$\begin{cases} \partial_t \partial \phi_1 + i\xi v_s^3 \partial \phi_1 + \gamma^2 \nabla' \cdot (\rho_s \partial w_1') + \gamma^2 i\xi \rho_s \partial w_1^3 = \tilde{F}^0, \\ \partial_t \partial w_1' - \frac{\nu}{\rho_s} (\Delta' - |\xi|^2) \partial w_1' - \frac{\tilde{\nu}}{\rho_s} \nabla' (\nabla' \cdot \partial w_1' + i\xi \partial w_1^3) \\ \quad + \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \partial \phi_1 \right) + i\xi v_s^3 \partial w_1' = \tilde{G}', \\ \partial_t \partial w_1^3 - \frac{\nu}{\rho_s} (\Delta' - |\xi|^2) \partial w_1^3 - \frac{\tilde{\nu}}{\rho_s} i\xi (\nabla' \cdot \partial w_1' + i\xi \partial w_1^3) \\ \quad + i\xi \frac{P'(\rho_s)}{\gamma^2 \rho_s} \partial \phi_1 + i\xi v_s^3 \partial w_1^3 + \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3 \partial \phi_1 + \partial w_1' \cdot \nabla' v_s^3 = \tilde{G}^3 \end{cases} \quad (4.71)$$

on  $D \cap \mathcal{O}_m$  and

$$\partial w_1|_{\partial D \cap \mathcal{O}_m} = 0.$$

Here  $\tilde{F}^0 = F_1^0 + F_2^0$ ,  $\tilde{G}' = G_1' + G_2'$  and  $\tilde{G}^3 = G_1^3 + G_2^3$ , with

$$\begin{aligned} F_1^0 &= -[\partial, i\xi v_s^3] \phi_1 - \gamma^2 [\partial, \nabla' \cdot \rho_s] w_1' - \gamma^2 [\partial, i\xi \rho_s] w_1^3, \\ G_1' &= \nu [\partial, \frac{1}{\rho_s} \Delta'] w_1' - \nu [\partial, \frac{1}{\rho_s} |\xi|^2] w_1' + \tilde{\nu} [\partial, \frac{1}{\rho_s} \nabla' \nabla' \cdot] w_1' + \tilde{\nu} [\partial, \frac{1}{\rho_s} \nabla' (i\xi)] w_1^3 \\ &\quad - [\partial, \nabla' \frac{P'(\rho_s)}{\gamma^2 \rho_s}] \phi_1 - [\partial, i\xi v_s^3] w_1', \\ G_1^3 &= \nu [\partial, \frac{1}{\rho_s} \Delta'] w_1^3 - \nu [\partial, \frac{1}{\rho_s} |\xi|^2] w_1^3 + \tilde{\nu} [\partial, \frac{1}{\rho_s} i\xi \nabla' \cdot] w_1' - \tilde{\nu} [\partial, \frac{1}{\rho_s} |\xi|^2] w_1^3 \\ &\quad - [\partial, \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3] \phi_1 - [\partial, i\xi \frac{P'(\rho_s)}{\gamma^2 \rho_s}] \phi_1 - [\partial, i\xi v_s^3] w_1^3 - [\partial, {}^T(\nabla' v_s^3)] w_1', \\ F_2^0 &= -\{i\xi \sigma \partial (v_s^3 \phi^{(0)}) + \gamma^2 i\xi \sigma \partial (\rho_s w^{(0),3}) - \langle Q_0 \tilde{B}_\xi (\sigma u^{(0)} + u_1) \rangle \partial \phi^{(0)}\}, \\ G_2' &= -\{-\tilde{\nu} i\xi \sigma \partial (\frac{1}{\rho_s} \nabla' w^{(0),3})\}, \\ G_2^3 &= -\{(\nu + \tilde{\nu}) \xi^2 \sigma \partial (\frac{1}{\rho_s} w^{(0),3}) + i\xi \sigma \partial (v_s^3 w^{(0),3}) - \langle Q_0 \tilde{B}_\xi (\sigma u^{(0)} + u_1) \rangle \partial w^{(0),3}\}. \end{aligned}$$

We set  $\tilde{F} = {}^T(\tilde{F}^0, \tilde{G}', \tilde{G}^3)$ ,  $F_1 = {}^T(F_1^0, G_1', G_1^3)$  and  $F_2 = {}^T(F_2^0, G_2', G_2^3)$ . Taking the weighted inner product of (4.71) with  $\chi_m^2 \partial u_1$ , we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left( \frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial \phi_1 \right|_2^2 + |\chi_m \sqrt{\rho_s} \partial w_1|_2^2 \right) \\ &\quad + \nu \{ |\chi_m \nabla' \partial w_1|_2^2 + |\xi|^2 |\chi_m \partial w_1|_2^2 \} + \tilde{\nu} |\chi_m (\nabla' \cdot \partial w_1' + i\xi \partial w_1^3)|_2^2 \\ &= \text{Re} \{ \langle F, \chi_m^2 \partial u_1 \rangle - I \}, \end{aligned} \quad (4.72)$$

where

$$\begin{aligned} I &= \nu (\nabla' \partial w_1, \nabla' (\chi_m^2) \partial w_1) + \tilde{\nu} (\nabla' \cdot \partial w_1' + i\xi \partial w_1^3, \nabla' (\chi_m^2) \cdot \partial w_1') \\ &\quad - \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \partial \phi_1, \nabla' (\chi_m^2) \cdot \rho_s \partial w_1' \right) + (i\xi v_s^3 \partial w_1, \chi_m^2 \rho_s \partial w_1) \\ &\quad + \left( \frac{\nu}{\gamma^2 \rho_s} \Delta' v_s^3 \partial \phi_1, \chi_m^2 \partial w_1^3 \right) + (\partial w_1' \cdot \nabla' v_s^3, \chi_m^2 \rho_s \partial w_1^3). \end{aligned}$$

Let us estimate the right-hand side of (4.72). By Lemma 4.19 and the Poincaré inequality we have

$$\begin{aligned} &|\text{Re} \langle F_1, \chi_m^2 \partial u_1 \rangle| \\ &\leq \left( \eta + \frac{C}{\gamma^2} \right) |\phi_1|_2^2 + \left( \eta + \frac{C}{\gamma^2} \right) |\xi|^2 |\phi_1|_2^2 + \left( \eta + \frac{C}{\gamma^2} \right) |\partial_{x'} \phi_1|_2^2 \\ &\quad + C \left( \frac{1}{\nu \eta} + \frac{\nu}{\gamma^2} + \frac{\tilde{\nu}}{\nu} + \frac{1}{\nu} + 1 \right) \tilde{D}_\xi [w_1] + \frac{1}{8} \nu (|\chi_m \nabla' \partial w_1|_2^2 + |\xi|^2 |\chi_m \partial w_1|_2^2) \\ &\quad + \frac{1}{8} \tilde{\nu} |\chi_m (\nabla' \cdot \partial w_1' + i\xi \partial w_1^3)|_2^2, \end{aligned}$$

$$|\operatorname{Re} I| \leq \left(\eta + \frac{C}{\gamma^2}\right) |\partial_{x'} \phi_1|_2^2 + C \left(\frac{\nu}{\gamma^2} + \frac{1}{\nu\eta} + \frac{1}{\nu} + 1\right) \tilde{D}_\xi[w_1] \\ + \frac{1}{8} \nu \left(|\chi_m \nabla' \partial w_1|_2^2 + |\xi|^2 |\chi_m \partial w_1|_2^2\right) + \frac{1}{8} \tilde{\nu} |\chi_m (\nabla' \cdot \partial w_1' + i\xi \partial w_1^3)|_2^2$$

for any  $\eta > 0$  with  $C$  independent of  $\eta > 0$ . By Lemma 4.11 and the Hölder inequality we deduce that

$$|\operatorname{Re} \langle F_2, \chi_m^2 \partial u_1 \rangle| \leq C \left\{ \left(\frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4}\right) |\xi|^2 |\sigma|^2 + \left(\frac{1}{\gamma^2} + \frac{1}{\gamma^4}\right) |\xi|^2 |\phi_1|_2^2 + \left(\eta + \frac{1}{\gamma^2}\right) |\partial_{x'} \phi_1|_2^2 \right. \\ \left. + \left(\frac{1}{\nu} + \frac{1}{\nu\eta}\right) \tilde{D}_\xi[w_1] + \left(\frac{\tilde{\nu}}{\nu} + 1\right) |\xi|^2 \tilde{D}_\xi[w_1] \right\}$$

for any  $\eta > 0$  with  $C$  independent of  $\eta > 0$ . Therefore we see from (4.72) that if  $\nu \geq 1$ ,  $\gamma^2 \geq 1$  and  $\omega \leq 1$ , then

$$\frac{1}{2} \frac{d}{dt} \left( \frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial \phi_1 \right|_2^2 + |\chi_m \sqrt{\rho_s} \partial w_1|_2^2 \right) \\ + \frac{3}{4} \nu \left(|\chi_m \nabla' \partial w_1|_2^2 + |\xi|^2 |\chi_m \partial w_1|_2^2\right) + \frac{3}{4} \tilde{\nu} |\chi_m (\nabla' \cdot \partial w_1' + i\xi \partial w_1^3)|_2^2 \\ \leq C \left\{ \left(\frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4}\right) |\xi|^2 |\sigma|^2 + \left(\eta + \frac{1}{\gamma^2}\right) |\phi_1|_2^2 + \left(\eta + \frac{1}{\gamma^2} + \frac{1}{\gamma^4}\right) |\xi|^2 |\phi_1|_2^2 \right. \\ \left. + \left(\eta + \frac{1}{\gamma^2}\right) |\partial_{x'} \phi_1|_2^2 + \left(\frac{1}{\nu\eta} + \frac{\nu}{\gamma^2} + \frac{\tilde{\nu}}{\nu} + 1\right) \tilde{D}_\xi[w_1] + \left(\frac{\tilde{\nu}}{\nu} + 1\right) |\xi|^2 \tilde{D}_\xi[w_1] \right\}. \quad (4.73)$$

We next estimate  $\partial \dot{\phi}_1$ . The first equation of (4.71) leads to

$$\frac{1}{\gamma^2} \partial \dot{\phi}_1 = \frac{1}{\gamma^2} (\partial_t \partial \phi_1 + i\xi \partial (v_s^3 \phi_1)) \\ = \frac{1}{\gamma^2} \tilde{F}^0 - \left\{ \frac{1}{\gamma^2} i\xi \partial v_s^3 \phi_1 + \nabla' \cdot (\rho_s \partial w_1') + i\xi \rho_s \partial w_1^3 \right\}.$$

We thus have

$$\frac{1}{\gamma^2} |\chi_m \partial \dot{\phi}_1|_2^2 \leq C \left\{ \frac{1}{\gamma^4} |\xi|^2 |\sigma|^2 + \frac{1}{\gamma^4} |\xi|^2 |\phi_1|_2^2 + \frac{1}{\nu} \tilde{D}_\xi[w_1] + |\chi_m (\nabla' \cdot \partial w_1' + i\xi \partial w_1^3)|_2^2 \right\}.$$

Take  $b > 0$  suitably small and add  $b \frac{\nu + \tilde{\nu}}{\gamma^4} |\chi_m \partial \dot{\phi}_1|_2^2$  to (4.73). We thus obtain the desired estimate. This completes the proof.  $\square$

We next derive the estimate for  $\partial_n \phi_1$ .

**Proposition 4.21.** *For  $1 \leq m \leq N$ , there exist constants  $\nu_1 > 0$ ,  $\omega_1 > 0$  and  $b > 0$  such that if  $\nu \geq \nu_1$ ,  $\gamma^2 \geq 1$  and  $\frac{\nu + \tilde{\nu}}{\nu} \omega \leq \omega_1$ , then there holds the estimate:*

$$\frac{1}{2} \frac{d}{dt} \left( \frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_n \phi_1 \right|_2^2 \right) + \frac{1}{2} \frac{1}{\nu + \tilde{\nu}} \left| \chi_m \frac{P'(\rho_s)}{\gamma^2} \partial_n \phi_1 \right|_2^2 + b \frac{\nu + \tilde{\nu}}{\gamma^4} |\chi_m \partial_n \dot{\phi}_1|_2^2 \\ \leq C \left\{ \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\sigma|^2 + \left(\frac{\omega^2}{\nu + \tilde{\nu}} + \frac{\nu^2}{\gamma^4 (\nu + \tilde{\nu})}\right) |\phi_1|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\phi_1|_2^2 + \left(\frac{\tilde{\nu}}{\nu} + 1\right) \tilde{D}_\xi[w_1] \right. \\ \left. + \frac{\nu}{\nu + \tilde{\nu}} |\xi|^2 \tilde{D}_\xi[w_1] + \frac{\nu^2}{\nu + \tilde{\nu}} \left(|\chi_m \partial_n \partial w_1|_2^2 + |\chi_m \partial^2 w_1|_2^2\right) + \frac{1}{\nu + \tilde{\nu}} |\sqrt{\rho_s} \partial_t w_1|_2^2 \right\}. \quad (4.74)$$

**Proof.** For a scalar field  $p(x')$  on  $D \cap \mathcal{O}_m$ , we set

$$\tilde{p}(y') = p(x') \quad (y' = \Psi^m(x'), \quad x' \in D \cap \mathcal{O}_m).$$

Similarly we transform a vector field  $h(x') = {}^T(h^1(x'), h^2(x'), h^3(x'))$  into  $\tilde{h}(y') = {}^T(\tilde{h}^1(y'), \tilde{h}^2(y'), \tilde{h}^3(y'))$  as

$$h(x') = E(y')\tilde{h}(y'),$$

where  $E(y') = (e_1(y_2'), e_2(y_2'), e_3)$  with  $e_1(y_2')$ ,  $e_2(y_2')$  and  $e_3$  given in (4.69). Note that, since  $e_3 = {}^T(0, 0, 1)$ , the Fourier transform in  $x_3 = y_3$  commutes with these transformations. It then follows that  $\tilde{\phi}_1(y')$  and  $\tilde{w}_1(y') = {}^T(\tilde{w}_1^1(y'), \tilde{w}_1^2(y'), \tilde{w}_1^3(y'))$  are governed by the following system of equations

$$\left\{ \begin{array}{l} \partial_t \tilde{\phi}_1 + i\xi \tilde{v}_s^3 \tilde{\phi}_1 + \gamma^2 \widehat{\text{div}}_y(\tilde{\rho}_s \tilde{w}_1) + i\xi \tilde{v}_s^3 \sigma \tilde{\phi}^{(0)} + \gamma^2 i\xi \tilde{\rho}_s \sigma \tilde{w}^{(0),3} \\ \quad - \langle Q_0 \tilde{B}_\xi(\sigma \tilde{u}^{(0)} + \tilde{u}_1) \rangle \tilde{\phi}^{(0)} = 0, \\ \partial_t \tilde{w}_1^1 + \frac{\nu}{\tilde{\rho}_s} (\widehat{\text{rot}}_y \widehat{\text{rot}}_y \tilde{w}_1)^1 - \frac{\nu + \tilde{\nu}}{\tilde{\rho}_s} (\widehat{\nabla}_y \widehat{\text{div}}_y \tilde{w}_1)^1 + \partial_{y_1} \left( \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s} \tilde{\phi}_1 \right) \\ \quad + \frac{\nu}{\gamma^2 \tilde{\rho}_s^2} (\Delta_{y'} \tilde{v}_s)^1 \tilde{\phi}_1 + i\xi \tilde{v}_s^3 \tilde{w}_1^1 - \frac{\tilde{\nu}}{\tilde{\rho}_s} i\xi \sigma \partial_{y_1} \tilde{w}^{(0),3} = 0, \\ \partial_t \tilde{w}_1^2 + \frac{\nu}{\tilde{\rho}_s} (\widehat{\text{rot}}_y \widehat{\text{rot}}_y \tilde{w}_1)^2 - \frac{\nu + \tilde{\nu}}{\tilde{\rho}_s} (\widehat{\nabla}_y \widehat{\text{div}}_y \tilde{w}_1)^2 + \frac{1}{J} \partial_{y_2} \left( \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s} \tilde{\phi}_1 \right) \\ \quad + \frac{\nu}{\gamma^2 \tilde{\rho}_s^2} (\Delta_{y'} \tilde{v}_s)^2 \tilde{\phi}_1 + i\xi \tilde{v}_s^3 \tilde{w}_1^2 - \frac{\tilde{\nu}}{\tilde{\rho}_s} i\xi \sigma \frac{1}{J} \partial_{y_2} \tilde{w}^{(0),3} = 0, \\ \partial_t \tilde{w}_1^3 + \frac{\nu}{\tilde{\rho}_s} (\widehat{\text{rot}}_y \widehat{\text{rot}}_y \tilde{w}_1)^3 - \frac{\nu + \tilde{\nu}}{\tilde{\rho}_s} (\widehat{\nabla}_y \widehat{\text{div}}_y \tilde{w}_1)^3 + i\xi \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s} \tilde{\phi}_1 \\ \quad + \frac{\nu}{\gamma^2 \tilde{\rho}_s^2} (\Delta_{y'} \tilde{v}_s)^3 \tilde{\phi}_1 + i\xi \tilde{v}_s^3 \tilde{w}_1^3 + \tilde{w}_1^1 \partial_{y_1} \tilde{v}_s^3 + \frac{1}{J} \tilde{w}_1^2 \partial_{y_2} \tilde{v}_s^3 + \frac{\nu + \tilde{\nu}}{\tilde{\rho}_s} \xi^2 \sigma \tilde{w}^{(0),3} \\ \quad + i\xi \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s} \sigma \tilde{\phi}^{(0)} + i\xi \tilde{v}_s^3 \sigma \tilde{w}^{(0),3} + \langle Q_0 \tilde{B}_\xi(\sigma \tilde{u}^{(0)} + \tilde{u}_1) \rangle \tilde{w}^{(0),3} = 0 \end{array} \right. \quad (4.75)$$

with  $\tilde{\rho}_s(y') = \rho_s(x')$ ,  $\tilde{v}_s^3(y') = v_s^3(x')$  and  $\tilde{P}'(\tilde{\rho}_s(y')) = P'(\rho_s(x'))$ . Here  $\nabla_y$ ,  $\text{div}_y$  and  $\text{rot}_y$  denote the gradient, divergence and rotation in the curvilinear coordinate  $y$  which are written for  $\tilde{p} = \tilde{p}(y')$  and  $\tilde{h} = {}^T(\tilde{h}^1(y'), \tilde{h}^2(y'), \tilde{h}^3(y'))$  as

$$\begin{aligned} \nabla_y \tilde{p} &= e_1 \partial_{y_1} \tilde{p} + \frac{1}{J} e_2 \partial_{y_2} \tilde{p} + e_3 \partial_{y_3} \tilde{p}, \\ \text{div}_y \tilde{h} &= \frac{1}{J} \{ \partial_{y_1} (J \tilde{h}^1) + \partial_{y_2} \tilde{h}^2 + \partial_{y_3} (J \tilde{h}^3) \}, \\ \text{rot}_y \tilde{h} &= (\text{rot}_y \tilde{h})^1 e_1 + (\text{rot}_y \tilde{h})^2 e_2 + (\text{rot}_y \tilde{h})^3 e_3 \end{aligned}$$

with

$$\begin{aligned} (\text{rot}_y \tilde{h})^1 &= \frac{1}{J} \{ \partial_{y_2} \tilde{h}^3 - \partial_{y_3} (J \tilde{h}^2) \}, \\ (\text{rot}_y \tilde{h})^2 &= \partial_{y_3} \tilde{h}^1 - \partial_{y_1} \tilde{h}^3, \\ (\text{rot}_y \tilde{h})^3 &= \frac{1}{J} \{ \partial_{y_1} \tilde{h}^2 - \partial_{y_2} (J \tilde{h}^1) \}, \end{aligned}$$

and, therefore,

$$\begin{aligned} (\text{rot}_y \text{rot}_y \tilde{h})^1 &= \frac{1}{J} \{ \partial_{y_2} (\text{rot}_y \tilde{h})^3 - \partial_{y_3} (\text{rot}_y \tilde{h})^2 \}, \\ (\text{rot}_y \text{rot}_y \tilde{h})^2 &= \partial_{y_3} (\text{rot}_y \tilde{h})^1 - \partial_{y_1} (\text{rot}_y \tilde{h})^3, \\ (\text{rot}_y \text{rot}_y \tilde{h})^3 &= \frac{1}{J} \{ \partial_{y_1} (\text{rot}_y \tilde{h})^2 - \partial_{y_2} (\text{rot}_y \tilde{h})^1 \}; \end{aligned}$$

the Fourier transformed gradient  $\widehat{\nabla}_y$  is given by

$$\widehat{\nabla}_y \tilde{p} = e_1 \partial_{y_1} \tilde{p} + \frac{1}{J} e_2 \partial_{y_2} \tilde{p} + e_3 i\xi \tilde{p};$$

and similarly  $\widehat{\text{div}}_y$  and  $\widehat{\text{rot}}_y$  are obtained from  $\text{div}_y$  and  $\text{rot}_y$  by replacing  $\partial_{y_3}$  with  $i\xi$  respectively. Applying  $\partial_{y_1}$  to the first equation of (4.75), we have

$$\begin{aligned} & \partial_t \partial_{y_1} \tilde{\phi}_1 + i\xi \tilde{v}_s^3 \partial_{y_1} \tilde{\phi}_1 + \gamma^2 \tilde{\rho}_s \partial_{y_1} \widehat{\text{div}}_y \tilde{w}_1 \\ &= -\left\{ i\xi \partial_{y_1} \tilde{v}_s^3 \tilde{\phi}_1 + \gamma^2 \partial_{y_1} (\widehat{\text{div}}_y (\tilde{\rho}_s \tilde{w}_1)) - \gamma^2 \tilde{\rho}_s \partial_{y_1} \widehat{\text{div}}_y \tilde{w}_1 \right. \\ & \quad \left. + i\xi \partial_{y_1} (\tilde{v}_s^3 \sigma \tilde{\phi}^{(0)}) + \gamma^2 i\xi \partial_{y_1} (\tilde{\rho}_s \sigma \tilde{w}^{(0),3}) - \langle Q_0 \tilde{B}_\xi (\sigma \tilde{u}^{(0)} + \tilde{u}_1) \rangle \partial_{y_1} \tilde{\phi}^{(0)} \right\}. \end{aligned} \quad (4.76)$$

To eliminate the term  $\partial_{y_1} \partial_{y_1} \tilde{w}_1^1$  in this equation, we consider  $\frac{\gamma^2 \tilde{\rho}_s}{\nu + \tilde{\nu}} \times (4.75)_2 + \frac{1}{\tilde{\rho}_s} \times (4.76)$ . It then follows that

$$\frac{1}{\tilde{\rho}_s} \partial_t \partial_{y_1} \tilde{\phi}_1 + \frac{\tilde{P}'(\tilde{\rho}_s)}{\nu + \tilde{\nu}} \partial_{y_1} \tilde{\phi}_1 + \frac{1}{\tilde{\rho}_s} i\xi \tilde{v}_s^3 \partial_{y_1} \tilde{\phi}_1 = I, \quad (4.77)$$

where  $I = I_1 + I_2$  with

$$\begin{aligned} I_1 &= -\frac{\gamma^2}{\nu + \tilde{\nu}} \left\{ \tilde{\rho}_s \partial_t \tilde{w}_1^1 + \nu (\widehat{\text{rot}}_y \widehat{\text{rot}}_y \tilde{w}_1)^1 \right. \\ & \quad \left. + \tilde{\rho}_s \partial_{y_1} \left( \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s} \right) \tilde{\phi}_1 + \frac{\nu}{\gamma^2} \tilde{\rho}_s (\Delta_{y'} \tilde{v}_s)^1 \tilde{\phi}_1 + i\xi \tilde{\rho}_s \tilde{v}_s^3 \tilde{w}_1^1 \right\} \\ & \quad - \left\{ i\xi \frac{1}{\tilde{\rho}_s} \partial_{y_1} \tilde{v}_s^3 \tilde{\phi}_1 + \gamma^2 \frac{1}{\tilde{\rho}_s} \partial_{y_1} (\widehat{\text{div}}_y (\tilde{\rho}_s \tilde{w}_1)) - \gamma^2 \partial_{y_1} \widehat{\text{div}}_y \tilde{w}_1 \right\}, \\ I_2 &= -\frac{\gamma^2}{\nu + \tilde{\nu}} (-\tilde{\nu} i\xi \sigma \partial_{y_1} \tilde{w}^{(0),3}) - \left\{ i\xi \frac{1}{\tilde{\rho}_s} \partial_{y_1} (\tilde{v}_s^3 \sigma \tilde{\phi}^{(0)}) \right. \\ & \quad \left. + \gamma^2 i\xi \frac{1}{\tilde{\rho}_s} \partial_{y_1} (\tilde{\rho}_s \sigma \tilde{w}^{(0),3}) - \frac{1}{\tilde{\rho}_s} \langle Q_0 \tilde{B}_\xi (\sigma \tilde{u}^{(0)} + \tilde{u}_1) \rangle \partial_{y_1} \tilde{\phi}^{(0)} \right\}. \end{aligned}$$

Considering  $\int_{\Psi^m(D \cap \mathcal{O}_m)} (4.77) \times \tilde{\chi}_m^2 \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^4} \overline{\partial_{y_1} \tilde{\phi}_1} J dy'$  with  $\tilde{\chi}_m(y') = \chi_m(x')$ , we see that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \frac{1}{\gamma^2} \left| \tilde{\chi}_m \sqrt{\frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s}} \partial_{y_1} \tilde{\phi}_1 \right|_2^2 \right) + \frac{1}{\nu + \tilde{\nu}} \left| \tilde{\chi}_m \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2} \partial_{y_1} \tilde{\phi}_1 \right|_2^2 \\ &= \int_{\Psi^m(D \cap \mathcal{O}_m)} I \times \tilde{\chi}_m^2 \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^4} \overline{\partial_{y_1} \tilde{\phi}_1} J dy'. \end{aligned}$$

Since

$$(\widehat{\text{rot}}_y \widehat{\text{rot}}_y \tilde{w}_1)^1 = \frac{1}{J} \partial_{y_2} \left( \frac{1}{J} \partial_{y_1} (J \tilde{w}_1^2) - \frac{1}{J} \partial_{y_2} \tilde{w}_1^1 \right) - i\xi (i\xi \tilde{w}_1^1 - \partial_{y_1} \tilde{w}_1^3),$$

we obtain

$$\begin{aligned} \frac{\nu + \tilde{\nu}}{\gamma^4} |\tilde{\chi}_m I_1|_2^2 &\leq C \left\{ \left( \frac{\omega^2}{\nu + \tilde{\nu}} + \frac{\nu^2}{\gamma^4(\nu + \tilde{\nu})} \right) |\tilde{\chi}_m \tilde{\phi}_1|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\tilde{\chi}_m \tilde{\phi}_1|_2^2 + \frac{1}{\nu + \tilde{\nu}} |\tilde{\chi}_m \sqrt{\tilde{\rho}_s} \partial_t \tilde{w}_1|_2^2 \right. \\ & \quad \left. + (\nu + \tilde{\nu}) \omega^2 |\tilde{\chi}_m \tilde{w}_1|_2^2 + \frac{1}{\nu + \tilde{\nu}} |\xi|^2 |\tilde{\chi}_m \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\xi|^4 |\tilde{\chi}_m \tilde{w}_1|_2^2 \right. \\ & \quad \left. + (\nu + \tilde{\nu}) \omega^2 |\tilde{\chi}_m \partial_{y'} \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\xi|^2 |\tilde{\chi}_m \partial_{y'} \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\tilde{\chi}_m \partial_{y'} \partial_{y_2} \tilde{w}_1|_2^2 \right\}, \\ \frac{\nu + \tilde{\nu}}{\gamma^4} |\tilde{\chi}_m I_2|_2^2 &\leq C \left\{ \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\sigma|^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\tilde{\phi}_1|_{L^2(\Psi^m(D \cap \mathcal{O}_m))}^2 + (\nu + \tilde{\nu}) |\tilde{w}_1|_{L^2(\Psi^m(D \cap \mathcal{O}_m))}^2 \right\}. \end{aligned}$$

It then follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \frac{1}{\gamma^2} \left| \tilde{\chi}_m \sqrt{\frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s}} \partial_{y_1} \tilde{\phi}_1 \right|_2^2 \right) + \frac{3}{4} \frac{1}{\nu + \tilde{\nu}} \left| \tilde{\chi}_m \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2} \partial_{y_1} \tilde{\phi}_1 \right|_2^2 \\ &\leq C \left\{ \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\sigma|^2 + \left( \frac{\omega^2}{\nu + \tilde{\nu}} + \frac{\nu^2}{\gamma^4(\nu + \tilde{\nu})} \right) |\tilde{\chi}_m \tilde{\phi}_1|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\tilde{\phi}_1|_{L^2(\Psi^m(D \cap \mathcal{O}_m))}^2 \right. \\ & \quad \left. + (\nu + \tilde{\nu}) |\tilde{w}_1|_{L^2(\Psi^m(D \cap \mathcal{O}_m))}^2 + \frac{1}{\nu + \tilde{\nu}} |\xi|^2 |\tilde{\chi}_m \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\xi|^4 |\tilde{\chi}_m \tilde{w}_1|_2^2 \right. \\ & \quad \left. + (\nu + \tilde{\nu}) \omega^2 |\tilde{\chi}_m \partial_{y'} \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\xi|^2 |\tilde{\chi}_m \partial_{y'} \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\tilde{\chi}_m \partial_{y'} \partial_{y_2} \tilde{w}_1|_2^2 \right. \\ & \quad \left. + \frac{1}{\nu + \tilde{\nu}} |\tilde{\chi}_m \sqrt{\tilde{\rho}_s} \partial_t \tilde{w}_1|_2^2 \right\}. \end{aligned} \quad (4.78)$$

We next consider  $\partial_{y_1} \dot{\tilde{\phi}}_1$  where  $\dot{\tilde{\phi}}_1 = \partial_t \tilde{\phi}_1 + i\xi \tilde{v}_s^3 \tilde{\phi}_1$ . (4.77) gives that

$$\frac{1}{\gamma^2} \partial_{y_1} \dot{\tilde{\phi}}_1 = \frac{1}{\gamma^2 \tilde{\rho}_s} (I + i\xi \partial_{y_1} \tilde{v}_s^3 \tilde{\phi}_1 - \frac{\gamma^2}{\nu + \tilde{\nu}} \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s} \partial_{y_1} \tilde{\phi}_1).$$

This equation leads to the estimate

$$\begin{aligned} \frac{\nu + \tilde{\nu}}{\gamma^4} |\tilde{\chi}_m \partial_{y_1} \dot{\tilde{\phi}}_1|_2^2 &\leq C \left\{ \frac{\nu + \tilde{\nu}}{\gamma^4} |\tilde{\chi}_m \tilde{\rho}_s (I + i\xi \partial_{y_1} \tilde{v}_s^3 \tilde{\phi}_1)|_2^2 + \frac{1}{\nu + \tilde{\nu}} \left| \tilde{\chi}_m \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2} \partial_{y_1} \tilde{\phi}_1 \right|_2^2 \right\} \\ &\leq C \left\{ \frac{\nu + \tilde{\nu}}{\gamma^4} |\tilde{\chi}_m I|_2^2 + \frac{1}{\nu + \tilde{\nu}} \left| \tilde{\chi}_m \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2} \partial_{y_1} \tilde{\phi}_1 \right|_2^2 \right\}. \end{aligned}$$

Therefore if we take  $b > 0$  suitably small and add  $b \frac{\nu + \tilde{\nu}}{\gamma^4} |\tilde{\chi}_m \partial_{y_1} \dot{\tilde{\phi}}_1|_2^2$  to (4.78), we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left( \frac{1}{\gamma^2} \left| \tilde{\chi}_m \sqrt{\frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s}} \partial_{y_1} \tilde{\phi}_1 \right|_2^2 \right) + \frac{1}{2} \frac{1}{\nu + \tilde{\nu}} \left| \tilde{\chi}_m \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2} \partial_{y_1} \tilde{\phi}_1 \right|_2^2 + b \frac{\nu + \tilde{\nu}}{\gamma^4} |\tilde{\chi}_m \partial_{y_1} \dot{\tilde{\phi}}_1|_2^2 \\ &\leq C \left\{ \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\sigma|^2 + \left( \frac{\omega^2}{\nu + \tilde{\nu}} + \frac{\nu^2}{\gamma^4 (\nu + \tilde{\nu})} \right) |\tilde{\chi}_m \tilde{\phi}_1|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} |\xi|^2 |\tilde{\phi}_1|_{L^2(\Psi^m(D \cap \mathcal{O}_m))}^2 \right. \\ &\quad + (\nu + \tilde{\nu}) |\tilde{w}_1|_{L^2(\Psi^m(D \cap \mathcal{O}_m))}^2 + \frac{1}{\nu + \tilde{\nu}} |\xi|^2 |\tilde{\chi}_m \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\xi|^4 |\tilde{\chi}_m \tilde{w}_1|_2^2 \\ &\quad + (\nu + \tilde{\nu}) \omega^2 |\tilde{\chi}_m \partial_{y'} \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\xi|^2 |\tilde{\chi}_m \partial_{y'} \tilde{w}_1|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} |\tilde{\chi}_m \partial_{y'} \partial_{y_2} \tilde{w}_1|_2^2 \\ &\quad \left. + \frac{1}{\nu + \tilde{\nu}} |\tilde{\chi}_m \sqrt{\tilde{\rho}_s} \partial_t \tilde{w}_1|_2^2 \right\}. \end{aligned} \quad (4.79)$$

The desired estimate follows from (4.79) by inverting to the original coordinates  $x'$  and noting that  $\partial_{y_1} = \partial_n$ ,  $\partial_{y_2} = \partial$ . This completes the proof.  $\square$

We next derive the interior estimate for the derivative of  $\phi_1$ .

**Proposition 4.22.** *There exist constants  $\nu_1 > 0$ ,  $\omega_1 > 0$  and  $b > 0$  such that if  $\nu \geq \nu_1$ ,  $\gamma^2 \geq 1$  and  $\frac{\nu + \tilde{\nu}}{\nu} \omega \leq \omega_1$ , then there holds the estimate:*

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left( \frac{1}{\gamma^2} \left| \chi_0 \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_{x'} \phi_1 \right|_2^2 + |\chi_0 \sqrt{\rho_s} \partial_{x'} w_1|_2^2 \right) + b \frac{\nu + \tilde{\nu}}{\gamma^4} |\chi_0 \partial_{x'} \dot{\phi}_1|_2^2 \\ &\quad + \frac{1}{2} \nu (|\chi_0 \nabla' \partial_{x'} w_1|_2^2 + |\xi|^2 |\chi_0 \partial_{x'} w_1|_2^2) + \frac{1}{2} \tilde{\nu} |\chi_0 (\nabla' \cdot \partial_{x'} w_1' + i\xi \partial_{x'} w_1^3)|_2^2 \\ &\leq C \left\{ \left( \frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \frac{1}{\gamma^2} |\phi_1|_2^2 + \left( \frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{\omega^2}{\nu + \tilde{\nu}} \right) |\xi|^2 |\phi_1|_2^2 \right. \\ &\quad \left. + \left( \eta + \frac{1}{\gamma^2} \right) |\partial_{x'} \phi_1|_2^2 + \left( \frac{1}{\eta \nu} + \frac{\nu}{\gamma^2} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w_1] + \left( \frac{\tilde{\nu}}{\nu} + 1 \right) |\xi|^2 \tilde{D}_\xi[w_1] \right\} \end{aligned} \quad (4.80)$$

for any  $\eta > 0$  with  $C$  independent of  $\eta$ .

Since  $\text{supp}(\chi_0 w_1) \subset D$  we have  $\partial_{x'} w_1|_{\partial D \cap \mathcal{O}_0} = 0$ . Therefore we can prove this proposition similarly to the proof of Proposition 4.20. We omit the details.

Before proceeding further we introduce an energy functional. We define  $E_3^{(0)}[u_1]$  by

$$\begin{aligned} E_3^{(0)}[u_1] &= \frac{1}{\gamma^2} \left| \chi_0 \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_{x'} \phi_1 \right|_2^2 + |\chi_0 \sqrt{\rho_s} \partial_{x'} w_1|_2^2 \\ &\quad + b_4 \sum_{m=1}^N \left( \frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial \phi_1 \right|_2^2 + |\chi_m \sqrt{\rho_s} \partial w_1|_2^2 \right) + \sum_{m=1}^N \frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_n \phi_1 \right|_2^2, \end{aligned}$$

where  $b_4$  is a positive constant. Taking  $b_4$  suitably large, we have the following estimate for  $E_3^{(0)}[u_1]$ .

**Proposition 4.23.** *There exist constants  $\nu_1 > 0$ ,  $\omega_1 > 0$ ,  $b > 0$  and  $b_4 > 0$  such that if  $\nu \geq \nu_1$ ,  $\gamma^2 \geq 1$  and  $\frac{\nu+\tilde{\nu}}{\nu}\omega \leq \omega_1$ , then there holds the estimate:*

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} E_3^{(0)}[u_1] + b \frac{\nu+\tilde{\nu}}{\gamma^4} |\partial_{x'} \dot{\phi}_1|_2^2 \\
& + \frac{1}{2} \left\{ \nu (|\chi_0 \nabla' \partial_{x'} w_1|_2^2 + |\xi|^2 |\chi_0 \partial_{x'} w_1|_2^2) + \tilde{\nu} |\chi_0 (\nabla' \cdot \partial_{x'} w_1' + i \xi \partial_{x'} w_1^3)|_2^2 \right\} \\
& + \frac{1}{2} \sum_{m=1}^N \left\{ \nu (|\chi_m \nabla' \partial w_1|_2^2 + |\xi|^2 |\chi_m \partial w_1|_2^2) + \tilde{\nu} |\chi_m (\nabla' \cdot \partial w_1' + i \xi \partial w_1^3)|_2^2 \right\} \\
& \leq C \left\{ \left( \frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \left( \eta + \frac{\omega^2}{\nu+\tilde{\nu}} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4(\nu+\tilde{\nu})} \right) |\phi_1|_2^2 \right. \\
& \quad + \left( \eta + \frac{\omega^2}{\nu+\tilde{\nu}} + \frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\phi_1|_2^2 + \left( \eta + \frac{1}{\gamma^2} \right) |\partial_{x'} \phi_1|_2^2 \\
& \quad \left. + \left( \frac{1}{\nu\eta} + \frac{\nu}{\gamma^2} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w_1] + \left( \frac{\tilde{\nu}}{\nu} + 1 \right) |\xi|^2 \tilde{D}_\xi[w_1] + \frac{1}{\nu+\tilde{\nu}} |\sqrt{\rho_s} \partial_t w_1|_2^2 \right\}
\end{aligned} \tag{4.81}$$

for any  $\eta > 0$  with  $C$  independent of  $\eta$ .

Using Proposition 4.20, Proposition 4.21 and Proposition 4.22, we obtain the estimate of Proposition 4.23.

We next derive a dissipative estimate for  $|\partial_{x'}^2 w_1|_2$  and  $|\partial_{x'} \phi_1|_2$ .

**Proposition 4.24.** *There exist constants  $\nu_1 > 0$  and  $\omega_1 > 0$  such that if  $\nu \geq \nu_1$ ,  $\frac{\nu+\tilde{\nu}}{\nu}\omega \leq \omega_1$  and  $\gamma^2 \geq 1$ , then there holds the estimate:*

$$\begin{aligned}
& \frac{\nu^2}{\nu+\tilde{\nu}} |\partial_{x'}^2 w_1'|_2^2 + \frac{1}{\nu+\tilde{\nu}} |\partial_{x'} \phi_1|_2^2 \\
& \leq C \left\{ \left( \frac{1}{\nu+\tilde{\nu}} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \frac{\nu^2}{\gamma^4(\nu+\tilde{\nu})} |\phi_1|_2^2 + \left( \frac{1}{\nu+\tilde{\nu}} + \frac{\nu^2+\tilde{\nu}^2}{\gamma^4(\nu+\tilde{\nu})} \right) |\xi|^2 |\phi_1|_2^2 \right. \\
& \quad \left. + \left( \frac{\tilde{\nu}}{\nu} + 1 \right) (1 + |\xi|^2) \tilde{D}_\xi[w_1] + \frac{1}{\nu+\tilde{\nu}} |\sqrt{\rho_s} \partial_t w_1|_2^2 + \frac{\nu^2+\tilde{\nu}^2}{\gamma^4(\nu+\tilde{\nu})} |\dot{\phi}_1|_{H^1}^2 \right\}.
\end{aligned} \tag{4.82}$$

**Proof.** We first derive the estimate for  $\partial_{x'}^2 w_1'$  and  $\partial_{x'} \phi_1$ . We will employ the following estimate for solutions of Stokes equation. If  $(p, h')$  is the solution of

$$\begin{cases} \nabla' \cdot h' = F^0, \\ -\Delta' h' + \frac{1}{\nu} \nabla' p = \frac{1}{\nu} G', \\ h' |_{\partial D} = 0, \end{cases}$$

then there holds

$$|\partial_{x'}^2 h'|_2^2 + \frac{1}{\nu^2} |\partial_{x'} p|_2^2 \leq C \{ |F^0|_{H^1}^2 + \frac{1}{\nu^2} |G'|_2^2 \}. \tag{4.83}$$

(See, e.g., [7, IV.6], [26, III.1.5].) By the first and second equations of (4.25), with the boundary condition of  $w_1'$ , we see that  $(\phi_1, w_1')$  satisfies the following Stokes

equation

$$\begin{cases} \nabla' \cdot w'_1 = F_1^0, \\ -\Delta' w'_1 + \frac{1}{\nu} \nabla' \left( \frac{P'(\rho_s)}{\gamma^2} \phi_1 \right) = \frac{1}{\nu} G'_1, \\ w'_1|_{\partial D} = 0, \end{cases}$$

where

$$\begin{aligned} F_1^0 &= -\frac{1}{\gamma^2 \rho_s} \{ \partial_t \phi_1 + i\xi v_s^3 \phi_1 + \gamma^2 (\nabla' \rho_s) \cdot w'_1 + \gamma^2 i\xi w_1^3 \\ &\quad + i\xi v_s^3 \sigma \phi^{(0)} + \gamma^2 i\xi \rho_s \sigma w^{(0),3} - \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle \phi^{(0)} \}, \\ G'_1 &= -\rho_s \{ \partial_t w'_1 + \frac{\nu}{\rho_s} \xi^2 w'_1 - \frac{\tilde{\nu}}{\rho_s} (\nabla' \cdot w'_1 + i\xi w_1^3) + i\xi v_s^3 w'_1 \\ &\quad + \nabla' \left( \frac{1}{\rho_s} \right) \frac{P'(\rho_s)}{\gamma^2} \phi_1 - \frac{\tilde{\nu}}{\rho_s} i\xi \nabla'(\sigma w^{(0),3}) \}. \end{aligned}$$

By Lemma 4.11 and the Poincaré inequality, we have

$$\begin{aligned} |F_1^0|_2^2 &\leq C \{ \frac{1}{\gamma^4} |\xi|^2 |\sigma|_2^2 + \frac{1}{\gamma^4} |\phi_1|_2^2 + \frac{1}{\nu} \tilde{D}_\xi[w_1] + \frac{1}{\gamma^4} |\dot{\phi}_1|_2^2 \}, \\ |\partial_{x'} F_1^0|_2^2 &\leq C \{ \frac{1}{\gamma^4} |\xi|^2 |\sigma|^2 + \frac{1}{\gamma^4} |\xi|^2 |\phi_1|_2^2 + \frac{1}{\nu} (1 + |\xi|^2) \tilde{D}_\xi[w_1] + \frac{1}{\gamma^4} |\dot{\phi}_1|_{H^1}^2 \}, \\ |G'_1|_2^2 &\leq C \{ \frac{\tilde{\nu}^2}{\gamma^4} |\xi|^2 |\sigma|^2 + (\omega^2 + \frac{\tilde{\nu}^2}{\gamma^4}) |\xi|^2 |\phi_1|_2^2 + (\frac{1}{\nu} + \frac{\tilde{\nu}^2}{\nu}) \tilde{D}_\xi[w_1] \\ &\quad + (\nu + \frac{\tilde{\nu}^2}{\nu}) |\xi|^2 \tilde{D}_\xi[w_1] + \frac{\tilde{\nu}^2}{\gamma^4} |\dot{\phi}_1|_{H^1}^2 + |\sqrt{\rho_s} \partial_t w_1|_2^2 \}. \end{aligned}$$

Since

$$\begin{aligned} \partial_{x'} \left( \frac{P'(\rho_s)}{\gamma^2} \phi_1 \right) &= \frac{P'(\rho_s)}{\gamma^2} \partial_{x'} \phi_1 + \frac{P''(\rho_s) \partial_{x'} \rho_s}{\gamma^2} \phi_1, \\ \frac{P'(\rho_s)}{\gamma^2} &\geq \frac{1}{2}, \end{aligned}$$

and

$$|\phi_1|_2 \leq C |\partial_{x'} \phi_1|_2$$

by the Poincaré inequality, we see that

$$\begin{aligned} |\partial_{x'} \left( \frac{P'(\rho_s)}{\gamma^2} \phi_1 \right)|_2^2 &\geq C \{ |\partial_{x'} \phi_1|_2^2 - \omega^2 |\phi_1|_2^2 \} \\ &\geq C (1 - \omega^2) |\partial_{x'} \phi_1|_2^2 \\ &\geq C |\partial_{x'} \phi_1|_2^2 \end{aligned}$$

for  $\omega^2 < \frac{1}{2}$ . We thus find the estimate

$$\begin{aligned} &|\partial_{x'}^2 w'_1|_2^2 + \frac{1}{\nu^2} |\partial_{x'} \phi_1|_2^2 \\ &\leq C \frac{1}{\nu^2} \{ \frac{\nu^2 + \tilde{\nu}^2}{\gamma^4} |\xi|^2 |\sigma|^2 + \frac{\nu^2}{\gamma^4} |\phi_1|_2^2 + (\omega^2 + \frac{\nu^2}{\gamma^4} + \frac{\tilde{\nu}^2}{\gamma^4}) |\xi|^2 |\phi_1|_2^2 \\ &\quad + (\nu + \frac{1}{\nu} + \frac{\tilde{\nu}^2}{\nu}) \tilde{D}_\xi[w_1] + (\nu + \frac{\tilde{\nu}^2}{\nu}) |\xi|^2 \tilde{D}_\xi[w_1] + |\sqrt{\rho_s} \partial_t w_1|_2^2 + \frac{\nu^2 + \tilde{\nu}^2}{\gamma^4} |\dot{\phi}_1|_{H^1}^2 \}. \end{aligned} \tag{4.84}$$

We next derive the estimate for  $\partial_{x'}^2 w_1^3$ . The third equation of (4.25), with the boundary condition of  $w_1^3$ , is written as

$$\begin{cases} -\Delta' w_1^3 = G_1^3, \\ w_1^3|_{\partial D} = 0, \end{cases}$$

where

$$\begin{aligned} G_1^3 = & -\frac{\rho_s}{\nu} \left\{ \partial_t w_1^3 + \frac{\nu}{\rho_s} \xi^2 w_1^3 - \frac{\tilde{\nu}}{\rho_s} i \xi (\nabla' \cdot w_1' + i \xi w_1^3) \right. \\ & + i \xi \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi_1 \right) + i \xi v_s^3 w_1^3 + \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3 \phi_1 + w_1' \cdot \nabla' v_s^3 \\ & \left. + \frac{\nu + \tilde{\nu}}{\rho_s} \xi^2 \sigma w^{(0),3} + i \xi \alpha_0 \sigma + i \xi v_s^3 \sigma w^{(0),3} - \langle Q_0 \tilde{B}_\xi(\sigma u^{(0)} + u_1) \rangle w^{(0),3} \right\}. \end{aligned}$$

We thus obtain

$$|w_1^3|_{H^2}^2 \leq C |G_1^3|_2^2.$$

It then follows that

$$\begin{aligned} |\partial_{x'}^2 w_1^3|_2^2 \leq & C \frac{1}{\nu^2} \left\{ \left( 1 + \frac{1}{\gamma^4} + \frac{(\nu + \tilde{\nu})^2}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \frac{\nu^2}{\gamma^4} |\phi_1|_2^2 + \left( 1 + \frac{1}{\gamma^4} \right) |\xi|^2 |\phi_1|_2^2 \right. \\ & \left. + \left( \nu + \tilde{\nu} + \frac{1}{\nu} \right) \tilde{D}_\xi[w_1] + \frac{\tilde{\nu}^2}{\nu + \tilde{\nu}} |\xi|^2 \tilde{D}_\xi[w_1] + |\sqrt{\rho_s} \partial_t w_1|_2^2 \right\}. \end{aligned} \quad (4.85)$$

Multiplying  $\frac{\nu^2}{\nu + \tilde{\nu}}$  to (4.84) + (4.85), we have the desired estimate. This completes the proof.  $\square$

We are now in a position to prove Theorem 4.4.

**Proposition 4.25.** *Let  $R > 0$ . There exist positive constants  $\nu_1, \gamma_1, \omega_1$  and  $d$  such that if  $\nu \geq \nu_1 R^2$ ,  $\frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_1^2 R^2$  and  $\frac{\nu + \tilde{\nu}}{\nu} \omega \leq \omega_1$ , then for any  $l = 0, 1, \dots$ , there exists a constant  $C = C(l) > 0$  such that the estimate*

$$\begin{aligned} & \|\partial_{x'} \partial_{x_3}^l \mathcal{F}^{-1}[\mathbf{1}_{\{|\eta| \leq R\}}(\xi) e^{-t \hat{L}_\xi} \hat{u}_0]\|_{L^2} \\ & \leq C \left\{ (1+t)^{-\frac{1}{4} - \frac{l}{2}} \|u_0\|_{L^1(\mathbf{R}; L^2(D))} + e^{-dt} (\|u_0\|_{L^2} + \|\partial_{x'} u_0\|_{L^2}) \right\} \end{aligned}$$

holds for  $t \geq 0$ .

**Proof.** Let  $b_5$  and  $b_6$  be constants satisfying  $b_5, b_6 > 1$ . Define  $E_4^{(0)}[u]$  by

$$E_4^{(0)}[u] = b_5 \frac{\nu}{\nu + \tilde{\nu}} \tilde{E}_2^{(0)}[u] + b_6 E_3^{(0)}[u_1].$$

If  $\gamma^2 \geq 1$ , then there exists a constant  $C > 0$  such that

$$\begin{aligned} & \frac{1}{2} \left\{ \frac{1}{\gamma^2} |\sigma|^2 + E_0[u_1] + \frac{1}{\gamma^2} |\partial_{x'} \phi_1|_2^2 + \tilde{D}_\xi[w_1] \right\} \\ & \leq C E_4^{(0)} \leq \frac{3}{2} \left\{ \frac{1}{\gamma^2} |\sigma|^2 + E_0[u_1] + \frac{1}{\gamma^2} |\partial_{x'} \phi_1|_2^2 + \tilde{D}_\xi[w_1] \right\}. \end{aligned}$$

We compute  $b_5 \frac{\nu}{\nu + \tilde{\nu}} \times (4.52) + b_6 \times (4.81) + b b_6 \times (4.27) + (4.82)$ . It holds that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} E_4^{(0)}[u] + \frac{\nu^2}{\nu+\tilde{\nu}} |\partial_{x'}^2 w_1|_2^2 + \frac{1}{\nu+\tilde{\nu}} |\partial_{x'} \phi_1|_2^2 \\
& + \frac{b_3 b_5}{4} \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \tilde{D}_\xi[w_1] + \frac{b_5}{2} \frac{1}{\nu+\tilde{\nu}} |\sqrt{\rho_s} \partial_t w_1|_2^2 + b b_6 \frac{\nu+\tilde{\nu}}{\gamma^4} |\dot{\phi}_1|_{H^1}^2 \\
& + \frac{b_6}{2} \left\{ \nu (|\chi_0 \nabla' \partial_{x'} w_1|_2^2 + |\xi|^2 |\chi_0 \partial_{x'} w_1|_2^2) + \tilde{\nu} |\chi_0 (\nabla' \cdot \partial_{x'} w_1' + i \xi \partial_{x'} w_1^3)|_2^2 \right\} \\
& + \frac{b_6}{2} \sum_{m=1}^N \left\{ \nu (|\chi_m \nabla' \partial_{x'} w_1|_2^2 + |\xi|^2 |\chi_m \partial_{x'} w_1|_2^2) + \tilde{\nu} |\chi_m (\nabla' \cdot \partial_{x'} w_1' + i \xi \partial_{x'} w_1^3)|_2^2 \right\} \\
& \leq C_4 \left\{ b_5 \frac{\nu}{\nu+\tilde{\nu}} \left( \frac{1}{\nu} + \frac{\nu+\tilde{\nu}}{\nu \gamma^2} + \frac{\tilde{\nu}^2}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + b_5 \frac{\nu}{\nu+\tilde{\nu}} \frac{(\nu+\tilde{\nu})^2}{\gamma^4} |\xi|^4 |\sigma|^2 \right. \\
& + b_5 \frac{\nu}{\nu+\tilde{\nu}} \left( \frac{1}{\nu} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4} \right) |\phi_1|_2^2 + b_5 \frac{\nu}{\nu+\tilde{\nu}} \frac{1}{\gamma^2} |\xi|^2 |\phi_1|_2^2 + b_6 \left( \frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 \\
& + b_6 \left( \eta + \frac{\omega^2}{\nu+\tilde{\nu}} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4(\nu+\tilde{\nu})} \right) |\phi_1|_2^2 + b_6 \left( \eta + \frac{\omega^2}{\nu+\tilde{\nu}} + \frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\phi_1|_2^2 \\
& + b_6 \left( \eta + \frac{1}{\gamma^2} \right) |\partial_{x'} \phi_1|_2^2 + b_6 \left( \frac{1}{\nu \eta} + \frac{\nu}{\gamma^2} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w_1] + b_6 \left( \frac{\tilde{\nu}}{\nu} + 1 \right) |\xi|^2 \tilde{D}_\xi[w_1] \\
& + b_6 \frac{1}{\nu+\tilde{\nu}} |\sqrt{\rho_s} \partial_t w_1|_2^2 + b b_6 \frac{\nu+\tilde{\nu}}{\gamma^4} |\xi|^2 |\sigma|^2 + b b_6 \frac{\nu+\tilde{\nu}}{\gamma^4} |\xi|^2 |\phi_1|_2^2 + b b_6 \left( 1 + \frac{\nu+\tilde{\nu}}{\nu} \omega^2 \right) \tilde{D}_\xi[w_1] \\
& + \left( \frac{1}{\nu+\tilde{\nu}} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\sigma|^2 + \frac{\nu^2}{\gamma^4(\nu+\tilde{\nu})} |\phi_1|_2^2 + \left( \frac{1}{\nu+\tilde{\nu}} + \frac{\nu^2+\tilde{\nu}^2}{\gamma^4(\nu+\tilde{\nu})} \right) |\xi|^2 |\phi_1|_2^2 \\
& + \left( \frac{\tilde{\nu}}{\nu} + 1 \right) (1 + |\xi|^2) \tilde{D}_\xi[w_1] + \frac{1}{\nu+\tilde{\nu}} |\sqrt{\rho_s} \partial_t w_1|_2^2 + \frac{\nu^2+\tilde{\nu}^2}{\gamma^4(\nu+\tilde{\nu})} |\dot{\phi}_1|_{H^1}^2 \left. \right\}.
\end{aligned}$$

Fix  $b_5 > 1$  and  $b_6 > 1$  sufficiently large such that  $b_6 \geq \frac{2C_4}{b}$  and  $b_5 \geq 8b_6 C_4$ , respectively. Let us take  $\eta > 0$  so small satisfying  $\eta \leq \min\{1, \frac{1}{8b_6 C_4}\}$ . We assume that  $\nu \geq \nu_1$  and  $\gamma \geq \gamma_1$  are so large that  $\nu \geq \nu_1 > 1$  and  $\gamma^2 \geq 8b_6 C_4(\nu + \tilde{\nu})$ . Since we have that

$$\begin{aligned}
\tilde{D}_\xi[w_1] & \leq C(1+R)|w_1|_2 |\partial_{x'}^2 w_1|_2 \\
& \leq \epsilon |\partial_{x'}^2 w_1|_2^2 + C_\epsilon^1 (1+R)^2 |w_1|_2^2
\end{aligned}$$

for any  $\epsilon > 0$ , if we take  $\epsilon$  sufficiently small such that  $\epsilon < \frac{1}{2} \frac{\nu^2}{\nu+\tilde{\nu}}$ , then we get

$$\frac{d}{dt} E_4^{(0)}[u] + d(|\nabla' \phi_1|_2^2 + |\nabla' w_1|_{H^1}^2) \leq C|u|_2^2.$$

Now we decompose  $E_4^{(0)}[u]$  as

$$E_4^{(0)}[u] = E_{4,0}^{(0)}[u] + E_{4,1}^{(0)}[u],$$

where

$$\begin{aligned}
\frac{1}{2}|u|_2^2 & \leq C E_{4,0}^{(0)}[u] \leq \frac{3}{2}|u|_2^2, \\
\frac{1}{2}(|\nabla' \phi_1|_2^2 + |\nabla' w_1|_{H^1}^2) & \leq C E_{4,1}^{(0)}[u] \leq \frac{3}{2}(|\nabla' \phi_1|_2^2 + |\nabla' w_1|_{H^1}^2).
\end{aligned}$$

It then follows that

$$\frac{d}{dt} E_{4,1}^{(0)}[u](t) + d_1 E_{4,1}^{(0)}[u] + \frac{d}{2}(|\nabla' \phi_1|_2^2 + |\nabla' w_1|_{H^1}^2) \leq C|u|_2^2 - \frac{d}{dt} E_{4,0}^{(0)}[u](t).$$

We thus obtain

$$\begin{aligned}
& E_{4,1}^{(0)}[u](t) + \frac{d}{2} \int_0^t e^{-d_1(t-\tau)} (|\nabla' \phi_1|_2^2 + |\nabla' w_1|_{H^1}^2) d\tau \\
& \leq e^{-d_1 t} E_{4,1}^{(0)}[u_0] + C \int_0^t e^{-d_1(t-\tau)} |u|_2^2 d\tau - \int_0^t e^{-d_1(t-\tau)} \frac{d}{d\tau} E_{4,0}^{(0)}[u](\tau) d\tau.
\end{aligned}$$

Since

$$e^{-d_1(t-\tau)} \frac{d}{d\tau} E_{4,0}^{(0)}[u](\tau) = \frac{d}{d\tau} \{e^{-d_1(t-\tau)} E_{4,0}^{(0)}[u](\tau)\} + d_1 e^{-d_1(t-\tau)} E_{4,0}^{(0)}[u](\tau)$$

and

$$E_{4,0}^{(0)}[u] \leq C|u|_2^2,$$

we see that

$$E_{4,1}^{(0)}[u](t) \leq e^{-d_1 t} E_4^{(0)}[u_0] + C \int_0^t e^{-d_1(t-\tau)} |u(\tau)|_2^2 d\tau.$$

From (4.51), we obtain

$$E_{4,1}^{(0)}[u](t) \leq e^{-d_1 t} E_4^{(0)}[u_0] + C|u_0|_2^2 \int_0^t e^{-d_1(t-\tau)} e^{-d_0|\xi|^2\tau} d\tau.$$

Let us estimate the second term on the right-hand side of this inequality. We have

$$\begin{aligned} \int_0^{t/2} \exp\{-d_1(t-\tau) - d_0|\xi|^2\tau\} d\tau &\leq \int_0^{t/2} \exp\{-d_1(t-\tau)\} d\tau \\ &\leq \frac{1}{d_1} \exp\left\{-\frac{d_1}{2}t\right\} \\ &\leq \frac{1}{d_1} \exp\left\{-\frac{d_1}{2} \frac{|\xi|^2}{R^2}t\right\}, \\ \int_{t/2}^t \exp\{-d_1(t-\tau) - d_0|\xi|^2\tau\} d\tau &\leq \exp\left\{-\frac{d_0}{2}|\xi|^2t\right\} \int_{t/2}^t \exp\{-d_1(t-\tau)\} d\tau \\ &\leq \frac{1}{d_1} \exp\left\{-\frac{d_0}{2}|\xi|^2t\right\}. \end{aligned}$$

We set  $d_2 = \min\{d_0, \frac{d_1}{R^2}\}$ . It then follows that there exist positive constants  $\nu_1, \gamma_1, \omega_1, d_1$  and  $d_2$  such that if  $\nu \geq \nu_1 R^2$ ,  $\frac{\gamma^2}{\nu+\bar{\nu}} \geq \gamma_1^2 R^2$  and  $\frac{\nu+\bar{\nu}}{\nu} \omega \leq \omega_1$ , then

$$E_{4,1}^{(0)}[u](t) \leq C \{e^{-\frac{d_2}{2}|\xi|^2t} |u_0|_2^2 + e^{-d_1 t} E_4^{(0)}[u_0]\}. \quad (4.86)$$

□

Combining Proposition 4.17 and Proposition 4.25 with  $R = 1$  we obtain the desired estimates in Theorem 4.4.

## 4.2 Decay estimate of the high frequency part

In this section we will give a proof of Theorem 4.5. To prove Theorem 4.5, we will employ an energy method to obtain the estimate on solutions of

$$\partial_t u + \widehat{L}_\xi u = 0, \quad w|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0$$

similarly to Section 4.1. The following Propositions 4.26-4.31 can be proved in a similar manner in Section 4.1. So we give the statements only and omit the proofs.

**Proposition 4.26.** *There exists a constant  $\nu_1 > 0$  such that if  $\nu \geq \nu_1$ , then there hold the estimates:*

$$\frac{1}{2} \frac{d}{dt} E_0[u] + \frac{1}{2} \tilde{D}_\xi[w] \leq C \frac{\nu}{\gamma^4} |\phi|_2^2, \quad (4.87)$$

$$\frac{\nu+\tilde{\nu}}{\gamma^4} |\dot{\phi}|_2^2 \leq C(1 + \frac{\nu+\tilde{\nu}}{\nu} \omega^2) \tilde{D}_\xi[w]. \quad (4.88)$$

We proceed to estimate derivatives of  $u$ . We introduce some notations. We define  $J_2^{(\infty)}[u]$  by

$$J_2^{(\infty)}[u] = -2\text{Re}\langle u, \widehat{B}_\xi \tilde{Q}u \rangle.$$

In addition, we set

$$E_2^{(\infty)}[u] = (1 + \frac{\tilde{b}_3 \gamma^2}{\nu}) E_0[u] + \tilde{D}_\xi[w],$$

$$\tilde{E}_2^{(\infty)}[u] = E_2^{(\infty)}[u] + J_2^{(\infty)}[u],$$

where  $\tilde{b}_3$  is a positive constant to be determined later. We note that there exists a constant  $\tilde{b}_3^* > 0$  such that if  $\tilde{b}_3 \geq \tilde{b}_3^*$  and  $\gamma^2 \geq 1$ , then

$$\frac{1}{2} E_2^{(\infty)}[u] \leq \tilde{E}_2^{(\infty)}[u] \leq \frac{3}{2} E_2^{(\infty)}[u].$$

Taking  $\tilde{b}_3$  suitably large, we have the following estimate for  $\tilde{E}_2^{(\infty)}[u]$ .

**Proposition 4.27.** *There exist constants  $\tilde{b}_3 \geq \tilde{b}_3^*$  and  $\nu_1 > 0$  such that if  $\nu \geq \nu_1$  and  $\gamma^2 \geq 1$ , then there holds the estimate:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \tilde{E}_2^{(\infty)}[u] + \frac{1}{4} \tilde{b}_3 \frac{\gamma^2}{\nu} \tilde{D}_\xi[w] + \frac{1}{2} |\sqrt{\rho_s} \partial_t w|_2^2 \\ & \leq C \left\{ \left( \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4} \right) |\phi|_2^2 + \frac{1}{\gamma^2} |\xi|^2 |\phi|_2^2 \right\}. \end{aligned} \quad (4.89)$$

**Proposition 4.28.** *For  $1 \leq m \leq N$ , there exist constants  $\nu_1 > 0$  and  $b > 0$  such that if  $\nu \geq \nu_1$ ,  $\gamma^2 \geq 1$  and  $\omega \leq 1$ , then there holds the estimate:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial \phi \right|_2^2 + \left| \chi_m \sqrt{\rho_s} \partial w \right|_2^2 \right) + b \frac{\nu+\tilde{\nu}}{\gamma^4} \left| \chi_m \partial \dot{\phi} \right|_2^2 \\ & \quad + \frac{1}{2} \nu \left( \left| \chi_m \nabla' \partial w \right|_2^2 + |\xi|^2 \left| \chi_m \partial w \right|_2^2 \right) + \frac{1}{2} \tilde{\nu} \left| \chi_m (\nabla' \cdot \partial w' + i \xi \partial w^3) \right|_2^2 \\ & \leq C \left\{ \left( \eta + \frac{1}{\gamma^2} \right) |\phi|_2^2 + \left( \eta + \frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\phi|_2^2 + \left( \eta + \frac{1}{\gamma^2} \right) |\partial_{x'} \phi|_2^2 \right. \\ & \quad \left. + \left( \frac{1}{\eta \nu} + \frac{\nu}{\gamma^2} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w] \right\} \end{aligned} \quad (4.90)$$

for any  $\eta > 0$  with  $C$  independent of  $\eta$ .

**Proposition 4.29.** *For  $1 \leq m \leq N$ , there exist constants  $\nu_1 > 0$  and  $b > 0$  such that if  $\nu \geq \nu_1$ ,  $\gamma^2 \geq 1$  and  $\omega \leq 1$ , then there holds the estimate:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_n \phi \right|_2^2 \right) + \frac{1}{2} \frac{1}{\nu+\tilde{\nu}} \left| \chi_m \frac{P'(\rho_s)}{\gamma^2} \partial_n \phi \right|_2^2 + b \frac{\nu+\tilde{\nu}}{\gamma^4} \left| \chi_m \partial_n \dot{\phi} \right|_2^2 \\ & \leq C \left\{ \left( \frac{\omega^2}{\nu+\tilde{\nu}} + \frac{\nu^2}{\gamma^4(\nu+\tilde{\nu})} \right) |\phi|_2^2 + \frac{\nu+\tilde{\nu}}{\gamma^4} |\xi|^2 |\phi|_2^2 + \left( \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w] + \frac{\nu}{\nu+\tilde{\nu}} |\xi|^2 \tilde{D}_\xi[w] \right. \\ & \quad \left. + \frac{\nu^2}{\nu+\tilde{\nu}} \left( \left| \chi_m \partial_n \partial w \right|_2^2 + \left| \chi_m \partial^2 w \right|_2^2 \right) + \frac{1}{\nu+\tilde{\nu}} |\sqrt{\rho_s} \partial_t w|_2^2 \right\}. \end{aligned} \quad (4.91)$$

**Proposition 4.30.** *There exist constants  $\nu_1 > 0$  and  $b > 0$  such that if  $\nu \geq \nu_1$ ,  $\gamma^2 \geq 1$  and  $\omega \leq 1$ , then there holds the estimate:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \frac{1}{\gamma^2} \left| \chi_0 \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_{x'} \phi \right|_2^2 + \left| \chi_0 \sqrt{\rho_s} \partial_{x'} w \right|_2^2 \right) + b \frac{\nu + \tilde{\nu}}{\gamma^4} \left| \chi_0 \partial_{x'} \dot{\phi} \right|_2^2 \\ & + \frac{1}{2} \nu \left( \left| \chi_0 \nabla' \partial_{x'} w \right|_2^2 + \left| \xi \right|^2 \left| \chi_0 \partial_{x'} w \right|_2^2 \right) + \frac{1}{2} \tilde{\nu} \left| \chi_0 (\nabla' \cdot \partial_{x'} w' + i \xi \partial_{x'} w^3) \right|_2^2 \\ & \leq C \left\{ \frac{1}{\gamma^2} |\phi|_2^2 + \left( \frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} + \frac{\omega^2}{\nu + \tilde{\nu}} \right) |\xi|^2 |\phi|_2^2 + \left( \eta + \frac{1}{\gamma^2} \right) |\partial_{x'} \phi|_2^2 \right. \\ & \quad \left. + \left( \frac{1}{\eta \nu} + \frac{\nu}{\gamma^2} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w] \right\} \end{aligned} \quad (4.92)$$

for any  $\eta > 0$  with  $C$  independent of  $\eta$ .

Before proceeding further we introduce an energy functional. We define  $E_3^{(\infty)}[u]$  by

$$\begin{aligned} E_3^{(\infty)}[u] &= \frac{1}{\gamma^2} \left| \chi_0 \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_{x'} \phi \right|_2^2 + \left| \chi_0 \sqrt{\rho_s} \partial_{x'} w \right|_2^2 \\ &+ \tilde{b}_4 \sum_{m=1}^N \left( \frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial \phi \right|_2^2 + \left| \chi_m \sqrt{\rho_s} \partial w \right|_2^2 \right) + \sum_{m=1}^N \frac{1}{\gamma^2} \left| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_n \phi \right|_2^2, \end{aligned}$$

where  $\tilde{b}_4$  is a positive constant. Taking  $\tilde{b}_4$  suitably large, we have the following estimate for  $E_3^{(\infty)}[u]$ .

**Proposition 4.31.** *There exist constants  $\nu_1 > 0$ ,  $b > 0$  and  $\tilde{b}_4 > 0$  such that if  $\nu \geq \nu_1$ ,  $\gamma^2 \geq 1$  and  $\omega \leq 1$ , then there holds the estimate:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} E_3^{(\infty)}[u] + b \frac{\nu + \tilde{\nu}}{\gamma^4} \left| \partial_{x'} \dot{\phi} \right|_2^2 \\ & + \frac{1}{2} \left\{ \nu \left( \left| \chi_0 \nabla' \partial_{x'} w \right|_2^2 + \left| \xi \right|^2 \left| \chi_0 \partial_{x'} w \right|_2^2 \right) + \tilde{\nu} \left| \chi_0 (\nabla' \cdot \partial_{x'} w' + i \xi \partial_{x'} w^3) \right|_2^2 \right\} \\ & + \frac{1}{2} \sum_{m=1}^N \left\{ \nu \left( \left| \chi_m \nabla' \partial w \right|_2^2 + \left| \xi \right|^2 \left| \chi_m \partial w \right|_2^2 \right) + \tilde{\nu} \left| \chi_m (\nabla' \cdot \partial w' + i \xi \partial w^3) \right|_2^2 \right\} \\ & \leq C \left\{ \left( \eta + \frac{\omega^2}{\nu + \tilde{\nu}} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4(\nu + \tilde{\nu})} \right) |\phi|_2^2 + \left( \eta + \frac{\omega^2}{\nu + \tilde{\nu}} + \frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\phi|_2^2 \right. \\ & \quad + \left( \eta + \frac{1}{\gamma^2} \right) |\partial_{x'} \phi|_2^2 + \left( \frac{1}{\eta \nu} + \frac{\nu}{\gamma^2} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w] \\ & \quad \left. + \frac{\nu}{\nu + \tilde{\nu}} |\xi|^2 \tilde{D}_\xi[w] + \frac{1}{\nu + \tilde{\nu}} \left| \sqrt{\rho_s} \partial_t w \right|_2^2 \right\} \end{aligned} \quad (4.93)$$

for any  $\eta > 0$  with  $C$  independent of  $\eta$ .

We do not have the estimate for  $\phi$  such as  $|\phi|_2 \leq C |\partial_{x'} \phi|_2$  similar to that for  $\phi_1$  in Section 4.1. We thus use the estimate for a solution of the Fourier transformed Stokes equation of the case  $|\xi|^2 \gg 1$ .

**Proposition 4.32.** *Assume that  $(p, h) \in H^1(D) \times H^2(D)$  is a solution of the following Stokes equation*

$$\begin{cases} \nabla' \cdot h' + i \xi h^3 = F^0, \\ (|\xi|^2 - \Delta') h' + \frac{1}{\nu} \partial_{x'} p = \frac{1}{\nu} G', \\ (|\xi|^2 - \Delta') h^3 + \frac{1}{\nu} i \xi p = \frac{1}{\nu} G^3, \\ h|_{\partial D} = 0. \end{cases}$$

There exists a constant  $R_0 = R_0(D) > 0$  such that if  $|\xi| \geq R_0$ , then there holds the following estimate:

$$\begin{aligned} & \frac{1}{\nu^2} |p|_2^2 + \frac{1}{\nu^2} |\xi|^2 |p|_2^2 + \frac{1}{\nu^2} |\partial_{x'} p|_2^2 \\ & + |h|_2^2 + |\xi|^2 |h|_2^2 + |\partial_{x'} h|_2^2 + \sum_{j=0}^2 |\xi|^{2j} |\partial_{x'}^{2-j} h|_2^2 \\ & \leq CR_0^2 \{ |F^0|_2^2 + |\xi|^2 |F^0|_2^2 + |\partial_{x'} F^0|_2^2 + \frac{1}{\nu^2} |G|_2^2 + |\partial_{x'} h|_2^2 \}, \end{aligned}$$

where  $C$  is a positive constant independent of  $|\xi|$ .

Proposition 4.32 can be proved similarly to the proof of [12, Lemma6.6] and we omit the proof. Applying Proposition 4.32, we have the following estimate.

**Proposition 4.33.** *There exist constant  $\nu_1 > 0$  such that if  $\nu \geq \nu_1$ ,  $\gamma^2 \geq 1$  and  $\omega \leq 1$ , then there holds the estimate:*

$$\begin{aligned} & \frac{1}{\nu+\tilde{\nu}} (|\phi|_2^2 + |\xi|^2 |\phi|_2^2 + |\partial_{x'} \phi|_2^2) \\ & + \frac{\nu^2}{\nu+\tilde{\nu}} \left( |w|_2^2 + |\xi|^2 |w|_2^2 + |\partial_{x'} w|_2^2 + \sum_{j=0}^2 |\xi|^{2j} |\partial_{x'}^{2-j} w|_2^2 \right) \\ & \leq CR_0^2 \left\{ \left( \frac{\omega^2}{\nu+\tilde{\nu}} + \frac{\nu^2}{\gamma^4(\nu+\tilde{\nu})} \right) |\phi|_2^2 + \frac{\nu}{\nu+\tilde{\nu}} \tilde{D}_\xi[w] \right. \\ & \quad \left. + \frac{\nu^2+\tilde{\nu}^2}{\gamma^4(\nu+\tilde{\nu})} (|\dot{\phi}|_2^2 + |\xi|^2 |\dot{\phi}|_2^2 + |\partial_{x'} \dot{\phi}|_2^2) + \frac{1}{\nu+\tilde{\nu}} |\sqrt{\rho_s} \partial_t w|_2^2 \right\} \end{aligned} \quad (4.94)$$

for  $|\xi| \geq R_0$ , where  $R_0$  is the constant given in Proposition 4.32 and  $C$  is a positive constant independent of  $|\xi|$ .

**Proof.** We observe that  $(\phi, w)$  satisfies the following Stokes equation

$$\begin{cases} \nabla' \cdot w' + i\xi w^3 = F^0, \\ (\xi^2 - \Delta') w' + \frac{1}{\nu} \nabla' \left( \frac{P'(\rho_s)}{\gamma^2} \phi \right) = \frac{1}{\nu} G', \\ (\xi^2 - \Delta') w^3 + \frac{1}{\nu} i\xi \frac{P'(\rho_s)}{\gamma^2} \phi = \frac{1}{\nu} G^3, \\ w|_{\partial D} = 0, \end{cases}$$

where

$$\begin{aligned} F^0 &= -\frac{1}{\rho_s} \{ \partial_t \phi + i\xi v_s^3 \phi + (\nabla' \rho_s) \cdot w' \}, \\ G' &= -\rho_s \{ \partial_t w' - \frac{\tilde{\nu}}{\rho_s} \nabla' (\nabla' \cdot w' + i\xi w^3) - \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi \nabla' \rho_s + i\xi v_s^3 w' \}, \\ G^3 &= -\rho_s \{ \partial_t w^3 - \frac{\tilde{\nu}}{\rho_s} i\xi (\nabla' \cdot w' + i\xi w^3) + i\xi v_s^3 w^3 + \frac{\nu}{\gamma^2 \rho_s} \Delta' v_s^3 \phi + w' \cdot \nabla' v_s^3 \}. \end{aligned}$$

Therefore we get the desired estimate from Proposition 4.32. This completes the proof.  $\square$

We finally prove Theorem 4.5.

**Proof of Theorem 4.5** Let  $\tilde{b}_5, \tilde{b}_6$  and  $\tilde{b}_7$  be constants satisfying  $\tilde{b}_5, \tilde{b}_6, \tilde{b}_7 > 1$ . Define  $\tilde{E}_4^{(\infty)}[u]$  by

$$\tilde{E}_4^{(\infty)}[u] = \tilde{b}_5 E_3^{(\infty)}[u] + \frac{\tilde{b}_6}{\nu+\tilde{\nu}} \tilde{E}_2^{(\infty)}[u] + \tilde{b}_7 \left(1 + \frac{\tilde{\nu}}{\nu}\right) (1 + |\xi|^2) E_0[u].$$

We compute (4.94) +  $\tilde{b}_5 \times \{(4.93) + b \frac{\nu+\tilde{\nu}}{\gamma^4} (1 + |\xi|^2) |\dot{\phi}|_2^2\} + \frac{\tilde{b}_6}{\nu+\tilde{\nu}} \times (4.89) + \tilde{b}_7 (1 + \frac{\tilde{\nu}}{\nu}) (1 + |\xi|^2) \times (4.87)$  then

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \tilde{E}_4^{(\infty)}[u] + \frac{\nu^2}{\nu+\tilde{\nu}} \left( |w|_2^2 + |\xi|^2 |w|_2^2 + |\partial_{x'} w|_2^2 + \sum_{j=0}^2 |\xi|^{2j} |\partial_{x'}^j w|_2^2 \right) \\ & + \frac{1}{\nu+\tilde{\nu}} (|\phi|_2^2 + |\xi|^2 |\phi|_2^2 + |\partial_{x'} \phi|_2^2) + \tilde{b} \tilde{b}_5 \frac{\nu+\tilde{\nu}}{\gamma^4} (|\dot{\phi}|_2^2 + |\xi|^2 |\dot{\phi}|_2^2 + |\partial_{x'} \dot{\phi}|_2^2) \\ & + \frac{\tilde{b}_5}{2} \left\{ \nu (|\chi_0 \nabla' \partial_{x'} w|_2^2 + |\xi|^2 |\chi_0 \partial_{x'} w|_2^2) + \tilde{\nu} |\chi_0 (\nabla' \cdot \partial_{x'} w' + i \xi \partial_{x'} w^3)|_2^2 \right\} \\ & + \frac{\tilde{b}_5}{2} \sum_{m=1}^N \left\{ \nu (|\chi_m \nabla' \partial w|_2^2 + |\xi|^2 |\chi_m \partial w|_2^2) + \tilde{\nu} |\chi_m (\nabla' \cdot \partial w' + i \xi \partial w^3)|_2^2 \right\} \\ & + \frac{\tilde{b}_3 \tilde{b}_6}{4} \frac{\gamma^2}{\nu(\nu+\tilde{\nu})} \tilde{D}_\xi[w] + \frac{\tilde{b}_6}{2} \frac{1}{\nu+\tilde{\nu}} |\sqrt{\rho_s} \partial_t w|_2^2 + \frac{\tilde{b}_7}{2} \left(1 + \frac{\tilde{\nu}}{\nu}\right) (1 + |\xi|^2) \tilde{D}_\xi[w] \\ & \leq \tilde{C}_4 \left\{ R_0^2 \left( \frac{\omega^2}{\nu+\tilde{\nu}} + \frac{\nu^2}{\gamma^4(\nu+\tilde{\nu})} \right) |\phi|_2^2 + R_0^2 \frac{\nu}{\nu+\tilde{\nu}} \tilde{D}_\xi[w] + R_0^2 \frac{\nu^2+\tilde{\nu}^2}{\gamma^4(\nu+\tilde{\nu})} (|\dot{\phi}|_2^2 + |\xi|^2 |\dot{\phi}|_2^2 + |\partial_{x'} \dot{\phi}|_2^2) \right. \\ & + R_0^2 \frac{1}{\nu+\tilde{\nu}} |\sqrt{\rho_s} \partial_t w|_2^2 + \tilde{b}_5 \left( \eta + \frac{\omega^2}{\nu+\tilde{\nu}} + \frac{1}{\gamma^2} + \frac{\nu^2}{\gamma^4(\nu+\tilde{\nu})} \right) |\phi|_2^2 + \tilde{b}_5 \left( \eta + \frac{\omega^2}{\nu+\tilde{\nu}} + \frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) |\xi|^2 |\phi|_2^2 \\ & + \tilde{b}_5 \left( \eta + \frac{1}{\gamma^2} \right) |\partial_{x'} \phi|_2^2 + \tilde{b}_5 \left( \frac{1}{\nu\eta} + \frac{\nu}{\gamma^2} + \frac{\tilde{\nu}}{\nu} + 1 \right) \tilde{D}_\xi[w] + \tilde{b}_5 \frac{\nu}{\nu+\tilde{\nu}} |\xi|^2 \tilde{D}_\xi[w] \\ & + \tilde{b}_5 \frac{1}{\nu+\tilde{\nu}} |\sqrt{\rho_s} \partial_t w|_2^2 + \tilde{b} \tilde{b}_5 \left( \frac{\nu+\tilde{\nu}}{\gamma^4} + \frac{(\nu+\tilde{\nu})^2}{\gamma^4 \nu} \right) (1 + |\xi|^2) \tilde{D}_\xi[w] \\ & \left. + \tilde{b}_6 \left( \frac{1}{\gamma^2(\nu+\tilde{\nu})} + \frac{\nu^2}{\gamma^4(\nu+\tilde{\nu})} \right) |\phi|_2^2 + \tilde{b}_6 \frac{1}{\gamma^2(\nu+\tilde{\nu})} |\xi|^2 |\phi|_2^2 + \tilde{b}_7 \frac{\nu+\tilde{\nu}}{\gamma^4} (1 + |\xi|^2) |\phi|_2^2 \right\}. \end{aligned}$$

Fix  $\tilde{b}_5 > 1$ ,  $\tilde{b}_6 > 1$  and  $\tilde{b}_7 > 1$  so large that  $\tilde{b}_5 \geq \frac{2\tilde{C}_4}{b} R_0^2$ ,  $\tilde{b}_6 \geq 8\tilde{C}_4 \max\{R_0^2, \tilde{b}_5\}$  and  $\tilde{b}_7 > 20\tilde{C}_4 \max\{R_0^2, \tilde{b}_5 \frac{1}{\eta(\nu+\tilde{\nu})}, \tilde{b}_5, \tilde{b} \tilde{b}_5\}$ , respectively. We take  $\eta > 0$  and  $\omega > 0$  sufficiently small such that  $\eta < \frac{1}{20\tilde{C}_4 \tilde{b}_5} \frac{1}{\nu+\tilde{\nu}}$  and  $\omega^2 < \frac{1}{20\tilde{C}_4} \min\{\frac{1}{R_0^2}, \frac{1}{\tilde{b}_5}\}$ , respectively. We assume that  $\nu \geq \nu_1$  and  $\gamma \geq \gamma_1$  are large enough such that  $\nu \geq \nu_1 > 1$  and  $\gamma^2 > 20\tilde{C}_4 \max\{\tilde{b}_6(\nu+\tilde{\nu}), \frac{\tilde{b}_5}{\tilde{b}_7} \frac{\nu^2}{\nu+\tilde{\nu}}, \sqrt{\tilde{b}_7}(\nu+\tilde{\nu})\}$ . We then arrive at the estimate

$$\begin{aligned} & \frac{d}{dt} \tilde{E}_4^{(\infty)}[u] + \frac{\nu^2}{\nu+\tilde{\nu}} \left( |w|_2^2 + |\xi|^2 |w|_2^2 + |\partial_{x'} w|_2^2 + \sum_{j=1}^2 |\xi|^{2j} |\partial_{x'}^j w|_2^2 \right) \\ & + \frac{1}{\nu+\tilde{\nu}} (|\phi|_2^2 + |\xi|^2 |\phi|_2^2 + |\partial_{x'} \phi|_2^2) + \frac{\nu+\tilde{\nu}}{\gamma^4} (|\dot{\phi}|_2^2 + |\xi|^2 |\dot{\phi}|_2^2 + |\dot{\phi}|_{H^1}^2) \\ & + \nu (|\chi_0 \nabla' \partial_{x'} w|_2^2 + |\xi|^2 |\chi_0 \partial_{x'} w|_2^2) + \tilde{\nu} |\chi_0 (\nabla' \cdot \partial_{x'} w' + i \xi \partial_{x'} w^3)|_2^2 \\ & + \sum_{m=1}^N \left\{ \nu (|\chi_m \nabla' \partial w|_2^2 + |\xi|^2 |\chi_m \partial w|_2^2) + \tilde{\nu} |\chi_m (\nabla' \cdot \partial w' + i \xi \partial w^3)|_2^2 \right\} \\ & + \frac{1}{\nu+\tilde{\nu}} |\sqrt{\rho_s} \partial_t w|_2^2 + \frac{\nu+\tilde{\nu}}{\nu} (1 + |\xi|^2) \tilde{D}_\xi[w] \\ & \leq 0 \end{aligned}$$

for all  $\xi \in \mathbf{R}$  with  $|\xi| \geq R_0$ . We define  $E_4^{(\infty)}[u]$  by

$$E_4^{(\infty)}[u] = |\phi|_2^2 + |\xi|^2 |\phi|_2^2 + |\partial_{x'} \phi|_2^2 + |w|_2^2 + |\xi|^2 |w|_2^2 + |\partial_{x'} w|_2^2.$$

Since

$$\frac{1}{2} \left\{ \left(1 + \frac{\tilde{b}_3 \gamma^2}{\nu}\right) E_0[u] + \tilde{D}_\xi[w] \right\} \leq \tilde{E}_2^{(\infty)}[u] \leq \frac{3}{2} \left\{ \left(1 + \frac{\tilde{b}_3 \gamma^2}{\nu}\right) E_0[u] + \tilde{D}_\xi[w] \right\},$$

$$\frac{1}{2} \frac{1}{\gamma^2} |\partial_{x'} \phi|_2^2 \leq \tilde{C}_5 E_3^{(\infty)}[u] \leq \frac{3}{2} \left( \frac{1}{\gamma^2} |\partial_{x'} \phi|_2^2 + |\partial_{x'} w|_2^2 \right)$$

for a positive constant  $\tilde{C}_5$ , we see that

$$\frac{1}{2} E_4^{(\infty)}[u] \leq \tilde{C}_6 \tilde{E}_4^{(\infty)}[u] \leq \frac{3}{2} E_4^{(\infty)}[u]$$

for a positive constant  $\tilde{C}_6$ . We thus see that there exist positive constants  $\nu_1$ ,  $\gamma_1$ ,  $\omega_1$  and  $d$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_1^2 R_0^2$  and  $\omega \leq \omega_1 R_0^{-2}$ , then

$$E_4^{(\infty)}[u](t) \leq C e^{-dt} E_4^{(\infty)}[u_0]$$

for  $|\xi| \geq R_0$ . On the other hand, for  $1 \leq |\xi| \leq R_0$ , we obtain the desired estimate from (4.86) with  $R = R_0$ . This completes the proof.  $\square$

### 4.3 Spectrum of $-\hat{L}_\xi$ for $|\xi| \ll 1$

In this section, we consider the spectrum of  $-\hat{L}_\xi$  for  $|\xi| \ll 1$ .

Let us consider the resolvent problem

$$(\lambda + \hat{L}_\xi)u = f$$

with  $|\xi| \ll 1$ , where  $u = {}^T(\phi, w) \in D(\hat{L}_\xi) = D(\hat{L}_0)$  and  $f = {}^T(f^0, g) \in L^2(D)$ .

We first establish the resolvent estimate for  $|\xi| \ll 1$ . To do so, let us consider the resolvent problem for  $\xi = 0$

$$(\lambda + L_0)u = f, \tag{4.95}$$

where  $u = {}^T(\phi, w) \in D(L_0)$  and  $f = {}^T(f^0, g) \in L^2(D)$ . Decomposing  $u$  in (4.134) as

$$u = \langle \phi \rangle u^{(0)} + u_1$$

with

$$u_1 = (I - \Pi^{(0)})u,$$

we obtain

$$\lambda(\langle \phi \rangle u^{(0)} + u_1) + L_0 u_1 = f.$$

Applying  $\Pi^{(0)}$  and  $I - \Pi^{(0)}$  to this equation, we have

$$\begin{cases} \lambda \langle \phi \rangle = \langle f^0 \rangle, \\ \lambda u_1 + L_0 u_1 = f_1, \end{cases} \tag{4.96}$$

where  $f_1 = (I - \Pi^{(0)})f$ . We see from the first equation of (4.135) that if  $\lambda \neq 0$ , then

$$\langle \phi \rangle = \frac{1}{\lambda} \langle f^0 \rangle.$$

This implies that

$$|\langle \phi \rangle| \leq \frac{1}{|\lambda|} |f^0|_2. \quad (4.97)$$

On the other hand, the  $u_1$ -part has the following properties. The second equation of (4.135) is written as

$$\begin{cases} \lambda \phi_1 + \gamma^2 \nabla' \cdot (\rho_s w_1') = f_1^0, \\ \lambda w_1' - \frac{\nu}{\rho_s} \Delta' w_1' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot w_1' + \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi_1 \right) = g_1', \\ \lambda w_1^3 - \frac{\nu}{\rho_s} \Delta' w_1^3 + \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3 \phi_1 + w_1' \cdot \nabla' v_s^3 = g_1^3, \end{cases} \quad (4.98)$$

where  $u_1 = {}^T(\phi_1, w_1) = {}^T(\phi_1, w_1', w_1^3)$  and  $f_1 = {}^T(f_1^0, g_1) = {}^T(f_1^0, g_1', g_1^3)$ .

To state the estimates for the  $u_1$ -part, we introduce a quantity  $\tilde{D}_0[w_1]$  defined by

$$\tilde{D}_0[w_1] = |\nabla' w_1|_2^2 + |\nabla' \cdot w_1'|_2^2$$

for  $w_1 = {}^T(w_1', w_1^3)$ .

**Proposition 4.34.** *There exist constants  $\nu_1 > 0$ ,  $\gamma_1 > 0$  and  $\omega_1 > 0$  and an energy functional  $E_0[u_1]$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_1^2$  and  $\omega \leq \omega_1$ , then there holds the estimate*

$$(\operatorname{Re} \lambda) E_0[u_1] + c(|\phi_1|_2^2 + \tilde{D}_0[w_1]) \leq C|f_1|_2|u_1|_2,$$

where  $c$  and  $C$  are positive constants independent of  $u_1$  and  $\lambda$ ; and  $E_0[u_1]$  is equivalent to  $|u_1|_2^2$ .

Proposition 4.52 can be proved in a similar manner to the proof of [1, Proposition 4.7] by replacing  $\frac{d}{dt}$  with  $\operatorname{Re} \lambda$  and taking  $\xi = 0$  there.

The Poincaré inequality yields  $\tilde{D}_0[w_1] \geq C|w_1|_2^2$  with a positive constant  $C$ . Therefore, the resolvent estimates for  $-L_0$  now follow from (4.136) and Proposition 4.52.

**Proposition 4.35.** *There exist constants  $\nu_1 > 0$ ,  $\gamma_1 > 0$  and  $\omega_1 > 0$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_1^2$  and  $\omega \leq \omega_1$ , then there is a positive constant  $c_0 > 0$  such that*

$$\Sigma_0 \equiv \{\lambda \neq 0 : \operatorname{Re} \lambda > -c_0\} \subset \rho(-L_0).$$

Furthermore, the following estimates

$$|(\lambda + L_0)^{-1} f|_2 \leq C \left\{ \frac{1}{|\lambda|} |f^0|_2 + \frac{1}{(\operatorname{Re} \lambda + c_0)} |f_1|_2 \right\},$$

$$|\partial_{x'} \{ \tilde{Q}(\lambda + L_0)^{-1} f \}|_2 \leq C \left\{ \frac{1}{|\lambda|} |f^0|_2 + \frac{1}{(\operatorname{Re} \lambda + c_0)^{1/2}} |f_1|_2 \right\}$$

hold uniformly for  $\lambda \in \Sigma_0$ . The same assertions also hold for  $-L_0^*$ .

Based on Proposition 4.53, we have the resolvent estimates for  $-L_\xi$  with  $|\xi| \ll 1$ .

**Theorem 4.36.** *There exist constants  $\nu_1 > 0$ ,  $\gamma_1 > 0$  and  $\omega_1 > 0$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_1^2$  and  $\omega \leq \omega_1$ , then the following assertions hold. For any  $\eta$  satisfying  $0 < \eta \leq \frac{c_0}{2}$  there is a number  $r_0 = r_0(\eta)$  such that*

$$\Sigma_1 \equiv \{\lambda \neq 0 : |\lambda| \geq \eta, \operatorname{Re} \lambda \geq -\frac{c_0}{2}\} \subset \rho(-L_\xi)$$

for  $|\xi| \leq r_0$ . Furthermore, the following estimates

$$|(\lambda + L_\xi)^{-1}f|_2 \leq C|f|_2,$$

$$|\partial_{x'}\{\tilde{Q}(\lambda + L_\xi)^{-1}f\}|_2 \leq C|f|_2$$

hold uniformly for  $\lambda \in \Sigma_1$  and  $\xi$  with  $|\xi| \leq r_0$ . The same assertions also hold for  $-L_\xi^*$ .

**Proof.** Let us decompose  $L_\xi$  as

$$L_\xi = L_0 + \xi L^{(1)} + \xi^2 L^{(2)},$$

where

$$L^{(1)} = i \begin{pmatrix} v_s^3 & 0 & \gamma^2 \rho_s \\ 0 & v_s^3 I_2 & -\frac{\tilde{\nu}}{\rho_s} \nabla' \\ \frac{P'(\rho_s)}{\gamma^2 \rho_s} & -\frac{\tilde{\nu}}{\rho_s} \nabla' & v_s^3 \end{pmatrix}, \quad L^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu}{\rho_s} I_2 & 0 \\ 0 & 0 & \frac{\nu+\tilde{\nu}}{\rho_s} \end{pmatrix}.$$

For  $u = {}^T(\phi, w) \in L^2(D) \times H_0^1(D)$  we have

$$|L^{(1)}u|_2 \leq C|u|_{L^2 \times H^1}, \quad |L^{(2)}u|_2 \leq C|u|_2. \quad (4.99)$$

Therefore, we see from Proposition 4.53 that for any  $0 < \eta \leq \frac{c_0}{2}$  there exists  $r_0 > 0$  such that if  $|\xi| \leq r_0$ , then

$$|(\xi L^{(1)} + \xi^2 L^{(2)})(\lambda + L_0)^{-1}f|_2 \leq \frac{1}{2}|f|_2. \quad (4.100)$$

It then follows that

$$\Sigma_1 \equiv \{\lambda : |\lambda| > \eta, \operatorname{Re} \lambda \geq -\frac{c_0}{2}\} \subset \rho(-L_\xi),$$

and that, if  $\lambda \in \Sigma_1$ , then  $(\lambda + L_\xi)^{-1}$  is given by the Neumann series expansion

$$(\lambda + L_\xi)^{-1} = (\lambda + L_0)^{-1} + \sum_{N=0}^{\infty} (-1)^N [(\xi L^{(1)} + \xi^2 L^{(2)})(\lambda + L_0)^{-1}]$$

for  $|\xi| \leq r_0$ , and it holds that

$$|(\lambda + L_\xi)^{-1}f|_2 \leq C|f|_2 \quad (4.101)$$

for  $\lambda \in \Sigma_1$  and  $|\xi| \leq r_0$ . We thus obtain the desired estimates. This completes the proof.  $\square$

As for the spectrum of  $-L_\xi$  near  $\lambda = 0$ , we have the following result.

**Theorem 4.37.** *There exist positive constants  $\nu_1$ ,  $\gamma_1$ ,  $\omega_1$  and  $r_0$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_1^2$  and  $\omega \leq \omega_1$ , then it holds that*

$$\sigma(-L_\xi) \cap \{\lambda : |\lambda| \leq \frac{c_0}{2}\} = \{\lambda_0(\xi)\}$$

for  $\xi$  with  $|\xi| \leq r_0$ , where  $\lambda_0(\xi)$  is a simple eigenvalue of  $-L_\xi$  that has the form

$$\lambda_0(\xi) = -i\kappa_0\xi - \kappa_1\xi^2 + \mathcal{O}(|\xi|^3)$$

as  $\xi \rightarrow 0$ . Here  $\kappa_0 \in \mathbf{R}$  and  $\kappa_1 > 0$  are the numbers given by

$$\kappa_0 = \langle v_s^3 \phi^{(0)} + \gamma^2 \rho_s w^{(0),3} \rangle = \mathcal{O}(1),$$

$$\kappa_1 = \frac{\gamma^2}{\nu} \left\{ \alpha_0 |(-\Delta')^{-\frac{1}{2}} \rho_s|_2^2 + \mathcal{O}\left(\frac{1}{\gamma^2}\right) + \left(\frac{\nu}{\gamma^2} + \frac{1}{\nu^2}\right) \times \mathcal{O}\left(\frac{\nu+\tilde{\nu}}{\gamma^2}\right) \right\},$$

where  $-\Delta'$  denotes the Laplace operator on  $L^2(D)$  under the zero Dirichlet boundary condition with domain

$$D(-\Delta') = H^2(D) \cap H_0^1(D).$$

**Proof.** For  $u \in L^2(D) \times H_0^1(D)$  we see from Theorem 4.54 and (4.138) that

$$|L^{(1)}u|_2 \leq C(|L_0u|_2 + |u|_2), \quad |L^{(2)}u|_2 \leq C|u|_2.$$

Therefore, since 0 is a simple eigenvalue of  $-L_0$ , we see from the analytic perturbation theory that there exists a positive constant  $r_0$  such that

$$\sigma(-L_\xi) \cap \{\lambda : |\lambda| \leq \frac{c_0}{2}\} = \{\lambda_0(\xi)\}$$

for all  $\xi$  with  $|\xi| \leq r_0$ . Here  $\lambda_0(\xi)$  is a simple eigenvalue of  $-L_\xi$ . Furthermore,  $\lambda_0(\xi)$  and the eigenprojection  $\Pi(\xi)$  for  $\lambda_0(\xi)$  are expanded as

$$\lambda_0(\xi) = \lambda^{(0)} + \xi\lambda^{(1)} + \xi^2\lambda^{(2)} + \mathcal{O}(|\xi|^3),$$

$$\Pi(\xi) = \Pi^{(0)} + \xi\Pi^{(1)} + \mathcal{O}(|\xi|^2) \tag{4.102}$$

with

$$\begin{aligned} \lambda^{(0)} &= 0, \\ \lambda^{(1)} &= -\langle L^{(1)}u^{(0)}, u^{(0)*} \rangle, \\ \lambda^{(2)} &= -\langle L^{(2)}u^{(0)}, u^{(0)*} \rangle + \langle L^{(1)}SL^{(1)}u^{(0)}, u^{(0)*} \rangle, \\ \Pi^{(1)} &= -\Pi^{(0)}L^{(1)}S - SL^{(1)}\Pi^{(0)}, \end{aligned}$$

where

$$S = \left\{ (I - \Pi^{(0)})L_0(I - \Pi^{(0)}) \right\}^{-1}.$$

We first consider  $\lambda^{(1)}$ . Since

$$L^{(1)}u^{(0)} = i \begin{pmatrix} v_s^3 \phi^{(0)} + \gamma^2 \rho_s w^{(0),3} \\ -\frac{\tilde{\nu}}{\rho_s} \nabla' w^{(0),3} \\ \alpha_0 + v_s^3 w^{(0),3} \end{pmatrix},$$

we obtain

$$\lambda^{(1)} = -\langle L^{(1)}u^{(0)}, u^{(0)*} \rangle = -\langle Q_0 L^{(1)}u^{(0)} \rangle = -i\langle v_s^3 \phi^{(0)} + \gamma^2 \rho_s w^{(0),3} \rangle = i\mathcal{O}(1)$$

as  $\gamma^2 \rightarrow \infty$ .

We next consider  $\lambda^{(2)}$ . Since  $Q_0 L^{(2)}u^{(0)} = 0$ , we have

$$\langle L^{(2)}u^{(0)}, u^{(0)*} \rangle = \langle Q_0 L^{(2)}u^{(0)} \rangle = 0.$$

It then follows that

$$\lambda^{(2)} = \langle L^{(1)}SL^{(1)}u^{(0)}, u^{(0)*} \rangle = \langle Q_0 L^{(1)}SL^{(1)}u^{(0)} \rangle.$$

We define  $\tilde{u}$  by

$$\tilde{u} = SL^{(1)}u^{(0)},$$

which satisfies

$$\begin{cases} L_0 \tilde{u} = (I - \Pi^{(0)})L^{(1)}u^{(0)} = L^{(1)}u^{(0)} + \lambda^{(1)}u^{(0)}, \\ \tilde{w} |_{\partial D} = 0, \\ \langle \tilde{\phi} \rangle = 0. \end{cases} \quad (4.103)$$

Note that  $\tilde{u} = {}^T(\tilde{\phi}, \tilde{w}) \in i\mathbf{R}^4$  and  $\lambda^{(1)} \in i\mathbf{R}$ . We rewrite  $\lambda^{(2)}$  as

$$\lambda^{(2)} = \langle Q_0 L^{(1)}\tilde{u} \rangle = \langle iv_s^3 \tilde{\phi} + i\gamma^2 \rho_s \tilde{w}^3 \rangle,$$

where  $\tilde{u} = {}^T(\tilde{\phi}, \tilde{w}) = {}^T(\tilde{\phi}, \tilde{w}', \tilde{w}^3)$ . To show the strict negativity of  $\lambda^{(2)}$ , we estimate  $\tilde{u}$ . The problem (4.142) is written as

$$\begin{cases} \gamma^2 \nabla' \cdot (\rho_s \tilde{w}') = i\xi v_s^3 \phi^{(0)} + i\gamma^2 \rho_s w^{(0),3} + \lambda^{(1)} \phi^{(0)}, \\ -\frac{\nu}{\rho_s} \Delta' \tilde{w}' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot \tilde{w}' + \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \tilde{\phi} \right) = -i \frac{\tilde{\nu}}{\rho_s} \nabla' w^{(0),3}, \\ -\frac{\nu}{\rho_s} \Delta' \tilde{w}^3 + \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \tilde{\phi} + \tilde{w}' \cdot \nabla' v_s^3 = i \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi^{(0)} + iv_s^3 w^{(0),3} + \lambda^{(1)} w^{(0),3}, \\ \tilde{w} |_{\partial D} = 0, \\ \langle \tilde{\phi} \rangle = 0, \end{cases}$$

i.e.,  $\tilde{u} = {}^T(\tilde{\phi}, \tilde{w}) = {}^T(\tilde{\phi}, \tilde{w}', \tilde{w}^3)$  is a solution of

$$\begin{cases} \nabla' \cdot \tilde{w}' = F^0[\tilde{w}'], \\ -\nu \Delta' \tilde{w}' + \nabla' \tilde{\phi} = G'[\tilde{\phi}, \tilde{w}'], \\ \tilde{w}' |_{\partial D} = 0, \\ \langle \tilde{\phi} \rangle = 0 \end{cases} \quad (4.104)$$

and

$$\begin{cases} -\nu \Delta' \tilde{w}^3 = G^3[\tilde{\phi}, \tilde{w}'], \\ \tilde{w}^3 |_{\partial D} = 0, \end{cases} \quad (4.105)$$

where  $F^0[\tilde{w}']$ ,  $G'[\tilde{\phi}, \tilde{w}']$  and  $G^3[\tilde{\phi}, \tilde{w}']$  are defined as

$$\begin{aligned} F^0[\tilde{w}'] &= \frac{1}{\gamma^2} \{ i v_s^3 \phi^{(0)} + i \gamma^2 \rho_s w^{(0),3} + \lambda^{(1)} \phi^{(0)} \} - \nabla' \cdot ((1 - \rho_s) \tilde{w}'), \\ G'[\tilde{\phi}, \tilde{w}'] &= -i \tilde{\nu} \nabla' w^{(0),3} + \tilde{\nu} \nabla' F^0[\tilde{w}'] + \nabla' ((1 - \rho_s) \tilde{\phi}) \\ &\quad + (\nabla' \rho_s) \tilde{\phi} + \rho_s \nabla' \left\{ \left( 1 - \frac{P'(\rho_s)}{\gamma^2 \rho_s} \right) \right\} \tilde{\phi}, \\ G^3[\tilde{\phi}, \tilde{w}'] &= \rho_s \left\{ i \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi^{(0)} + i v_s^3 w^{(0),3} + \lambda^{(1)} w^{(0),3} \right\} - \rho_s \left\{ \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \tilde{\phi} + \tilde{w}' \cdot \nabla' v_s^3 \right\}. \end{aligned}$$

As for the problem (4.143), since  $\lambda^{(1)} = -i \langle v_s^3 \phi^{(0)} + \gamma^2 \rho_s w^{(0),3} \rangle$ , it holds that  $\langle F^0[\tilde{w}'] \rangle = 0$ . Furthermore, we have

$$\begin{aligned} |F^0[\tilde{w}']|_2 &\leq C \left\{ \frac{1}{\gamma^2} (|\lambda^{(1)}| |\phi^{(0)}|_2 + |\phi^{(0)}|_2 + \gamma^2 |w^{(0),3}|_2) + \omega |\nabla' \tilde{w}'|_2 \right\} \\ &\leq C \omega |\nabla' \tilde{w}'|_2 + \mathcal{O}\left(\frac{1}{\gamma^2}\right), \end{aligned}$$

$$\begin{aligned} |G'[\tilde{\phi}, \tilde{w}']|_{H^{-1}} &\leq C \left\{ \tilde{\nu} |\nabla' w^{(0),3}|_{H^{-1}} + \tilde{\nu} |\nabla' F^0[\tilde{w}']|_{H^{-1}} + |\nabla' ((1 - \rho_s) \tilde{\phi})|_{H^{-1}} \right. \\ &\quad \left. + |\nabla' \rho_s \tilde{\phi}|_{H^{-1}} + |\rho_s ((1 - \frac{P'(\rho_s)}{\gamma^2 \rho_s}) \tilde{\phi})|_{H^{-1}} \right\} \\ &\leq C \omega \{ |\tilde{\phi}|_2 + \tilde{\nu} |\nabla' \tilde{w}'|_2 \} + \mathcal{O}\left(\frac{\tilde{\nu}}{\gamma^2}\right). \end{aligned}$$

Since  $(\tilde{\phi}, \tilde{w}') \in \dot{X} \equiv \{(p, v') \in L^2(D) \times H_0^1(D) : \langle p \rangle = 0\}$  and it is a solution of the Stokes system (4.143), we see from estimate for the Stokes system (see, e.g., [26]) that there holds the estimate

$$\begin{aligned} |\tilde{\phi}|_2^2 + \nu^2 |\nabla' \tilde{w}'|_2^2 &\leq \nu^2 \{ C \omega^2 |\tilde{w}'|_2^2 + \mathcal{O}\left(\frac{1}{\gamma^4}\right) \} + \{ C \omega^2 (|\tilde{\phi}|_2^2 + \tilde{\nu}^2 |\nabla' \tilde{w}'|_2^2) + \mathcal{O}\left(\frac{\tilde{\nu}^2}{\gamma^4}\right) \} \\ &\leq C_1 \omega^2 \{ |\tilde{\phi}|_2^2 + (\nu + \tilde{\nu})^2 |\nabla' \tilde{w}'|_2^2 \} + \mathcal{O}\left(\frac{(\nu + \tilde{\nu})^2}{\gamma^4}\right). \end{aligned}$$

Therefore, if  $\omega$  is so small that  $\omega^2 < \frac{1}{2C_1} \min\{1, (\frac{\nu}{\nu + \tilde{\nu}})^2\}$ , then

$$|\tilde{\phi}|_2^2 + \nu^2 |\nabla' \tilde{w}'|_2^2 = \mathcal{O}\left(\frac{(\nu + \tilde{\nu})^2}{\gamma^4}\right). \quad (4.106)$$

As for the problem (4.144), since

$$\begin{aligned} |G^3[\tilde{\phi}, \tilde{w}']|_2 &\leq C \left\{ |\lambda^{(1)}| |w^{(0),3}|_2 + \frac{1}{\gamma^2} |\phi^{(0)}|_2 + |w^{(0),3}|_2 + \frac{\nu}{\gamma^2} |\tilde{\phi}|_2 + |\tilde{w}'|_2 \right\} \\ &\leq C \left\{ \frac{\nu}{\gamma^2} |\tilde{\phi}|_2 + |\tilde{w}'|_2 \right\} + \mathcal{O}\left(\frac{1}{\gamma^2}\right), \end{aligned}$$

we have  $G^3[\tilde{\phi}, \tilde{w}'] \in L^2(D)$ . It then follows that

$$\tilde{w}^3 = \frac{1}{\nu} (-\Delta')^{-1} G^3[\tilde{\phi}, \tilde{w}'].$$

Since  $\phi^{(0)} = \alpha_0 \frac{\gamma^2 \rho_s}{P'(\rho_s)}$  (see Lemma 4.6 (ii)), we find that

$$\begin{aligned} \langle \rho_s \tilde{w}^3 \rangle &= \frac{1}{\nu} \langle \rho_s (-\Delta')^{-1} G^3[\tilde{\phi}, \tilde{w}'] \rangle \\ &= \frac{1}{\nu} \langle \rho_s (-\Delta')^{-1} (i \alpha_0 \rho_s) \rangle \\ &\quad + \frac{1}{\nu} \langle \rho_s (-\Delta')^{-1} \{ i \rho_s v_s^3 w^{(0),3} + \rho_s \lambda^{(1)} w^{(0),3} - \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s} \tilde{\phi} - \rho_s \tilde{w}' \cdot \nabla' v_s^3 \} \rangle \\ &= i \frac{\alpha_0}{\nu} |(-\Delta')^{-\frac{1}{2}} \rho_s|_2^2 \\ &\quad + \frac{1}{\nu} \langle \rho_s (-\Delta')^{-1} \{ i \rho_s v_s^3 w^{(0),3} + \rho_s \lambda^{(1)} w^{(0),3} - \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s} \tilde{\phi} - \rho_s \tilde{w}' \cdot \nabla' v_s^3 \} \rangle. \end{aligned}$$

Furthermore, since  $\tilde{u} = {}^T(\tilde{\phi}, \tilde{w}') \in i\mathbf{R}^4$  and  $\lambda^{(1)} \in i\mathbf{R}$ , we see from (4.145) that

$$\begin{aligned} & \langle \rho_s (-\Delta')^{-1} \{ i \rho_s v_s^3 w^{(0),3} + \rho_s \lambda^{(1)} w^{(0),3} - \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s} \tilde{\phi} - \rho_s \tilde{w}' \cdot \nabla' v_s^3 \} \rangle \\ &= i \mathcal{O}\left(\frac{1}{\gamma^2}\right) + i \left(\frac{\nu}{\gamma^2} + \frac{1}{\nu^2}\right) \times \mathcal{O}\left(\frac{\nu+\tilde{\nu}}{\gamma^2}\right). \end{aligned}$$

It then follows that

$$\langle \rho_s \tilde{w}^3 \rangle = i \frac{\alpha_0}{\nu} |(-\Delta')^{-\frac{1}{2}} \rho_s|_2^2 + i \frac{1}{\nu} \left\{ \mathcal{O}\left(\frac{1}{\gamma^2}\right) + \left(\frac{\nu}{\gamma^2} + \frac{1}{\nu^2}\right) \times \mathcal{O}\left(\frac{\nu+\tilde{\nu}}{\gamma^2}\right) \right\}.$$

By (4.145) we also have

$$\langle v_s^3 \tilde{\phi} \rangle = i \mathcal{O}\left(\frac{\nu+\tilde{\nu}}{\gamma^2}\right).$$

We conclude that

$$\begin{aligned} \lambda^{(2)} &= \langle i v_s^3 \tilde{\phi} + i \gamma^2 \rho_s \tilde{w}^3 \rangle \\ &= i \gamma^2 \left[ i \frac{\alpha_0}{\nu} |(-\Delta')^{-\frac{1}{2}} \rho_s|_2^2 + i \frac{1}{\nu} \left\{ \mathcal{O}\left(\frac{1}{\gamma^2}\right) + \left(\frac{\nu}{\gamma^2} + \frac{1}{\nu^2}\right) \times \mathcal{O}\left(\frac{\nu+\tilde{\nu}}{\gamma^2}\right) \right\} \right] + i \cdot i \mathcal{O}\left(\frac{\nu+\tilde{\nu}}{\gamma^2}\right) \\ &= -\frac{\gamma^2}{\nu} \left[ \alpha_0 |(-\Delta')^{-\frac{1}{2}} \rho_s|_2^2 + \left\{ \mathcal{O}\left(\frac{1}{\gamma^2}\right) + \left(\frac{1}{\nu^2} + \frac{\nu}{\gamma^2}\right) \times \mathcal{O}\left(\frac{\nu+\tilde{\nu}}{\gamma^2}\right) \right\} \right] \\ &< 0 \end{aligned}$$

for sufficiently small  $\frac{1}{\nu}$  and  $\frac{\nu+\tilde{\nu}}{\gamma^2}$ . We thus obtain the desired estimates. This completes the proof.  $\square$

We next establish some estimates related to  $\Pi(\xi)$  in  $H^k(D)$ . We first consider estimates for higher order derivatives of  $(\lambda + L_0)^{-1} f$ .

**Proposition 4.38.** *For any  $f = {}^T(f^0, g) \in H^k(D) \times H^{k-1}(D)$ . There exist positive constants  $\nu_1, \gamma_1, \omega_1$  and  $c_1$  such that if  $\nu \geq \nu_1, \frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_1^2, \omega \leq \omega_1$  and  $\lambda \in \Sigma_2 \equiv \{\lambda \neq 0 : |\lambda| \leq c_1\}$ , then  $(\lambda + L_0)^{-1} f \in H^k(D) \times (H^{k+1}(D) \cap H_0^1(D))$  for  $k = 0, 1, \dots, k_0$ . Furthermore, the following estimate holds:*

$$|(\lambda + L_0)^{-1} f|_{H^k \times H^{k+1}} \leq C \left(1 + \frac{1}{|\lambda|}\right) |f|_{H^k \times H^{k-1}},$$

where  $C$  is a positive constant independent of  $\lambda \in \Sigma_2$ . The same assertions also hold for  $-L_0^*$ .

**Proof.** For a given  $f = {}^T(f^0, g) \in H^k(D) \times H^{k-1}(D)$ , we consider the problem

$$\begin{cases} (\lambda + \mathcal{L}_0)U = f, \\ W|_{\partial D} = 0 \end{cases} \quad (4.107)$$

for  $U = {}^T(\Phi, W)$ . Here  $\mathcal{L}_0$  is differential operator given by

$$\mathcal{L}_0 U = \begin{pmatrix} \gamma^2 \nabla' \cdot (\rho_s W') \\ -\frac{\nu}{\rho_s} \Delta' W' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot W' + \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \Phi \right) \\ -\frac{\nu}{\rho_s} \Delta' W^3 + \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \Phi + W' \cdot \nabla' v_s^3 \end{pmatrix}$$

for  $U = {}^T(\Phi, W)$ . To solve the problem (4.146), we decompose  $\Phi$  and  $f^0$  as

$$\Phi = \Phi_1 + \sigma, \quad f^0 = f_1^0 + \langle f^0 \rangle,$$

where  $\sigma = \langle \Phi \rangle$ ,  $\Phi_1 = \Phi - \sigma$  and  $f_1^0 = f^0 - \langle f^0 \rangle$ . Note that

$$\langle \Phi_1 \rangle = 0, \quad \langle f_1^0 \rangle = 0.$$

Then (4.146) is equivalent to the problem

$$\lambda \sigma = \langle f^0 \rangle, \quad (4.108)$$

$$\lambda \Phi_1 + \gamma^2 \nabla' \cdot (\rho_s W') = f_1^0, \quad (4.109)$$

$$\lambda W' - \frac{\nu}{\rho_s} \Delta' W' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot W' + \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} (\sigma + \Phi_1) \right) = g', \quad (4.110)$$

$$\lambda W^3 - \frac{\nu}{\rho_s} \Delta' W^3 + \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} (\sigma + \Phi_1) - W' \cdot \nabla' v_s^3 = g^3 \quad (4.111)$$

with  $W|_{\partial D} = 0$ . If  $\lambda \neq 0$ , then we find from (4.147) that

$$\sigma = \frac{1}{\lambda} \langle f^0 \rangle. \quad (4.112)$$

Substituting  $\sigma = \frac{1}{\lambda} \langle f^0 \rangle$  into (4.149) and (4.150), we obtain

$$\begin{cases} \lambda \Phi_1 + \gamma^2 \nabla' \cdot (\rho_s W') = f_1^0, \\ \lambda W' - \frac{\nu}{\rho_s} \Delta' W' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot W' + \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \Phi_1 \right) = g' - \frac{1}{\lambda} \langle f^0 \rangle \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \right), \\ \lambda W^3 - \frac{\nu}{\rho_s} \Delta' W^3 + \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \Phi_1 - W' \cdot \nabla' v_s^3 = g^3 - \frac{1}{\lambda} \langle f^0 \rangle \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \end{cases} \quad (4.113)$$

with  $W|_{\partial D} = 0$ . Let us write the problem (4.152) as

$$\begin{cases} \nabla' \cdot W' = F^0[\Phi_1, W' : f_1^0], \\ -\nu \Delta' W' + \nabla' \Phi_1 = G'[\Phi_1, W' : f^0, g'], \\ W'|_{\partial D} = 0 \end{cases} \quad (4.114)$$

and

$$\begin{cases} -\nu \Delta' W^3 = G^3[\Phi_1, W', W^3 : f^0, g^3], \\ W^3|_{\partial D} = 0. \end{cases} \quad (4.115)$$

Here

$$\begin{aligned} F^0[\Phi_1, W' : f_1^0] &= -\frac{1}{\gamma^2} \lambda \Phi_1 + \nabla' \cdot ((1 - \rho_s) W') + \frac{1}{\gamma^2} f_1^0, \\ G'[\Phi_1, W' : f^0, g'] &= -\lambda \rho_s W' + \tilde{\nu} \nabla' F^0[\Phi_1, W' : f_1^0] + \nabla' ((1 - \rho_s) \Phi_1) + \nabla' \rho_s \Phi_1 \\ &\quad - \frac{1}{\lambda} \langle f^0 \rangle \rho_s \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \right) + \rho_s \nabla' \left( \left( 1 - \frac{P'(\rho_s)}{\gamma^2 \rho_s} \right) \Phi_1 \right) + \rho_s g', \\ G^3[\Phi_1, W', W^3 : f^0, g^3] &= -\lambda \rho_s W^3 - \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \frac{1}{\lambda} \langle f^0 \rangle - \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \Phi_1 - \rho_s W' \cdot \nabla' v_s^3 + \rho_s g^3. \end{aligned}$$

We now define a set  $\dot{X}_k$  by

$$\dot{X}_k = \{(p, v') \in H^k(D) \times (H^{k+1}(D) \cap H_0^1(D)) : \langle p \rangle = 0\}$$

with norm

$$|(p, v')|_{\dot{X}_k} = |p|_{H^k} + \nu|v'|_{H^{k+1}}.$$

For a given  $(\tilde{\Phi}_1, \tilde{W}') \in \dot{X}_k$ , we consider the problem

$$\begin{cases} \nabla' \cdot W' = F^0[\tilde{\Phi}_1, \tilde{W}' : f_1^0], \\ -\nu\Delta' W' + \nabla' \Phi_1 = G'[\tilde{\Phi}_1, \tilde{W}' : f^0, g'], \\ W'|_{\partial D} = 0. \end{cases} \quad (4.116)$$

It holds that

$$\begin{aligned} \langle F^0[\tilde{\Phi}_1, \tilde{W}' : f_1^0] \rangle &= 0, \quad F^0[\tilde{\Phi}_1, \tilde{W}' : f_1^0] \in H^k(D), \\ G'[\tilde{\Phi}_1, \tilde{W}' : f^0, g'] &\in H^{k-1}(D). \end{aligned}$$

In fact, we see that

$$\langle F^0[\tilde{\Phi}_1, \tilde{W}' : f_1^0] \rangle = -\frac{1}{\gamma^2} \lambda \langle \tilde{\Phi}_1 \rangle + \langle \nabla' \cdot ((1 - \rho_s) \tilde{W}') \rangle + \frac{1}{\gamma^2} \langle f_1^0 \rangle = 0,$$

$$|F^0[\tilde{\Phi}_1, \tilde{W}' : f_1^0]|_{H^k} \leq C \left\{ \frac{1}{\gamma^2} |\lambda| |\tilde{\Phi}_1|_{H^k} + \omega |\tilde{W}'|_{H^{k+1}} + \frac{1}{\gamma^2} |f_1^0|_{H^k} \right\}$$

and

$$\begin{aligned} &|G'[\tilde{\Phi}_1, \tilde{W}' : f^0, g']|_{H^{k-1}} \\ &\leq C \left\{ |\lambda| |\tilde{W}'|_{H^{k-1}} + \tilde{\nu} |F^0[\tilde{\Phi}_1, \tilde{W}' : f_1^0]|_{H^k} + \omega |\tilde{\Phi}_1|_{H^k} + \frac{1}{|\lambda|} |\langle f^0 \rangle| + |g'|_{H^{k-1}} \right\} \\ &\leq C \left\{ \left( \frac{\tilde{\nu}}{\gamma^2} |\lambda| + \omega \right) |\tilde{\Phi}_1|_{H^k} + \nu \left( \frac{1}{\nu} |\lambda| + \frac{\tilde{\nu}}{\nu} \omega \right) |\tilde{W}'|_{H^{k+1}} + \left( \frac{\tilde{\nu}}{\gamma^2} + \frac{1}{|\lambda|} \right) |f^0|_{H^k} + |g'|_{H^{k-1}} \right\} \end{aligned}$$

for a positive constant  $C$  independent of  $\lambda$ . From [26], we see that there is a unique solution  $(\Phi_1, W') \in \dot{X}_k$  of (4.155) and there holds the estimate

$$\begin{aligned} &|\Phi|_{H^k} + \nu|W'|_{H^{k+1}} \\ &\leq C \left\{ \nu |F^0[\tilde{\Phi}_1, \tilde{W}' : f_1^0]|_{H^k} + |G'[\tilde{\Phi}_1, \tilde{W}' : f^0, g']|_{H^{k-1}} \right\} \\ &\leq C \left\{ \left( \frac{\nu + \tilde{\nu}}{\gamma^2} |\lambda| + \omega \right) |\tilde{\Phi}_1|_{H^k} + \nu \left( \frac{1}{\nu} |\lambda| + \frac{\nu + \tilde{\nu}}{\nu} \omega \right) |\tilde{W}'|_{H^{k+1}} \right. \\ &\quad \left. + \left( \frac{\nu + \tilde{\nu}}{\gamma^2} + \frac{1}{|\lambda|} \right) |f^0|_{H^k} + |g'|_{H^{k-1}} \right\} \end{aligned} \quad (4.117)$$

for a positive constant  $C$  independent of  $\lambda$ . Let us define a map  $\Gamma_1 : \dot{X}_k \rightarrow \dot{X}_k$  such that

$$\Gamma_1(\tilde{\Phi}_1, \tilde{W}') = (\Phi_1, W'),$$

where  $(\Phi_1, W') \in \dot{X}_k$  is a solution of (4.155). From (4.156), for  $(\tilde{\Phi}_{1,1}, \tilde{W}'_1), (\tilde{\Phi}_{1,2}, \tilde{W}'_2) \in \dot{X}_k$ , the estimate

$$\begin{aligned} &|\Gamma_1(\tilde{\Phi}_{1,1}, \tilde{W}'_1) - \Gamma_1(\tilde{\Phi}_{1,2}, \tilde{W}'_2)|_{H^k \times H^{k+1}} \\ &\leq C_1 \left\{ \left( \frac{\nu + \tilde{\nu}}{\gamma^2} + \frac{1}{\nu} \right) |\lambda| + \left( \frac{\nu + \tilde{\nu}}{\nu} + 1 \right) \omega \right\} |(\tilde{\Phi}_{1,1} - \tilde{\Phi}_{1,2}, \tilde{W}'_1 - \tilde{W}'_2)|_{\dot{X}_k} \end{aligned}$$

holds for a positive constant  $C_1$  independent of  $\lambda$ . If  $\omega$  and  $|\lambda|$  are so small that  $\omega < \frac{1}{2C_1} \frac{\nu}{\nu+\bar{\nu}}$  and  $|\lambda| < \frac{1}{2C_1}$ , then  $\Gamma_1 : \dot{X}_k \rightarrow \dot{X}_k$  is a contraction map. This implies that there is a unique  $(\Phi_1, W') \in \dot{X}_k$  such that  $\Gamma_1(\Phi_1, W') = (\Phi_1, W')$ , i.e., there is a unique solution  $(\Phi_1, W') \in \dot{X}_k$  of (4.153). Furthermore, from (4.156),  $(\Phi_1, W')$  satisfies the estimate

$$|\Phi_1|_{H^k} + |W'|_{H^{k+1}} \leq C \left\{ \left(1 + \frac{1}{|\lambda|}\right) |f^0|_{H^k} + |g'|_{H^{k-1}} \right\}, \quad (4.118)$$

where  $C$  is a positive constant independent of  $\lambda$ .

As for (4.154), for a given  $\widetilde{W}^3 \in H^{k+1}(D) \cap H_0^1(D)$ , we consider the problem

$$\begin{cases} -\nu \Delta' W^3 = G^3[\Phi_1, W', \widetilde{W}^3 : f^0, g^3], \\ W^3|_{\partial D} = 0, \end{cases} \quad (4.119)$$

where  $(\Phi_1, W') \in \dot{X}_k$  is a solution of (4.153). It holds that

$$G^3[\Phi_1, W', \widetilde{W}^3 : f^0, g^3] \in H^{k-1}(D).$$

In fact, we have

$$\begin{aligned} & |G^3[\Phi_1, W', \widetilde{W}^3 : f^0, g^3]|_{H^{k-1}} \\ & \leq C \left\{ |\lambda| |\widetilde{W}^3|_{H^{k-1}} + |\Phi_1|_{H^{k-1}} + |W'|_{H^{k-1}} + |g^3|_{H^{k-1}} + \frac{1}{|\lambda|} |\langle f^0 \rangle| \right\} \\ & \leq C_2 \left\{ |\lambda| |\widetilde{W}^3|_{H^{k-1}} + \left(1 + \frac{1}{|\lambda|}\right) |f^0|_{H^k} + |g|_{H^{k-1}} \right\} \end{aligned} \quad (4.120)$$

for a positive constant  $C_2$  independent of  $\lambda$ . If  $|\lambda|$  is sufficiently small satisfying  $|\lambda| < \min\{\frac{1}{2C_1}, \frac{1}{C_2}\}$ , then there is a unique solution  $W^3 \in H^{k+1}(D) \cap H_0^1(D)$  of (4.154). Furthermore, from (4.159),  $W^3$  satisfies the estimate

$$|W^3|_{H^{k+1}} \leq C \left\{ \left(1 + \frac{1}{|\lambda|}\right) |f^0|_{H^k} + |g|_{H^{k-1}} \right\}, \quad (4.121)$$

where  $C$  is a positive constant independent of  $\lambda$ .

Now we set

$$\Sigma_2 \equiv \left\{ \lambda \neq 0 : |\lambda| < \min\left\{ \frac{1}{2C_1}, \frac{1}{C_2} \right\} \right\}.$$

Since  $\Phi = \sigma + \Phi_1$ , we see that if  $\omega < \frac{1}{2C_1} \frac{\nu}{\nu+\bar{\nu}}$  and  $\lambda \in \Sigma_2$ , then there is a unique solution  $(\Phi, W) \in H^k(D) \times (H^{k+1}(D) \cap H_0^1(D))$  of (4.146). Moreover, from (4.151), (4.157) and (4.160),  $\Phi$  and  $W$  satisfies the estimate

$$\begin{aligned} |\Phi|_{H^k} + |W|_{H^{k+1}} & \leq |\sigma| + |\Phi_1|_{H^k} + |W'|_{H^{k+1}} + |W^3|_{H^{k+1}} \\ & \leq C \left\{ \left(1 + \frac{1}{|\lambda|}\right) |f^0|_{H^k} + |g|_{H^{k-1}} \right\} \end{aligned}$$

for a positive constant  $C$  independent of  $\lambda \in \Sigma_2$ .

Since  $D(L_0) \supset H^k(D) \times (H^{k+1}(D) \cap H_0^1(D))$ , we can replace  $\mathcal{L}_0$  with  $L_0$ ; and we find that if  $\omega < \frac{1}{2C_1} \frac{\nu}{\nu+\bar{\nu}}$  and  $\lambda \in \Sigma_2$ , then  $(\lambda + L_0)^{-1} f \in H^k(D) \times (H^{k+1}(D) \cap H_0^1(D))$ . Furthermore,  $(\lambda + L_0)^{-1} f$  satisfies the estimate

$$|(\lambda + L_0)^{-1} f|_{H^k \times H^{k+1}} \leq C \left\{ \left(1 + \frac{1}{|\lambda|}\right) |f^0|_{H^k} + |g|_{H^{k-1}} \right\},$$

where  $C$  is a positive constant independent of  $\lambda \in \Sigma_2$ . We thus obtain the desired estimates. The assertions for  $L_0^*$  can be proved in a similar manner. This completes the proof.  $\square$

We finally obtain the following estimates for the eigenfunctions  $u_\xi$  and  $u_\xi^*$  associated with  $\lambda_0(\xi)$  and  $\bar{\lambda}_0(\xi)$ , respectively, which yields the boundedness of  $\Pi(\xi)$  on  $H^k(D)$ .

**Theorem 4.39.** *There exist positive constants  $\nu_1$ ,  $\gamma_1$  and  $\omega_1$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{\nu+\bar{\nu}} \geq \gamma_1^2$  and  $\omega \leq \omega_1$ , then there exists a positive constant  $r_0$  such that for any  $\xi \in \mathbf{R}$  with  $|\xi| \leq r_0$  the following assertions hold. There exist  $u_\xi$  and  $u_\xi^*$  eigenfunctions associated with  $\lambda_0(\xi)$  and  $\bar{\lambda}_0(\xi)$ , respectively, that satisfy*

$$\langle u_\xi, u_\xi^* \rangle = 1,$$

and the eigenprojection  $\Pi(\xi)$  for  $\lambda_0(\xi)$  is given by

$$\Pi(\xi)u = \langle u, u_\xi^* \rangle u_\xi.$$

Furthermore,  $u_\xi$  and  $u_\xi^*$  are written in the form

$$\begin{aligned} u_\xi(x') &= u^{(0)}(x') + i\xi u^{(1)}(x') + |\xi|^2 u^{(2)}(x', \xi), \\ u_\xi^*(x') &= u^{*(0)}(x') + i\xi u^{*(1)}(x') + |\xi|^2 u^{*(2)}(x', \xi), \end{aligned}$$

and the following estimates hold

$$|u|_{H^{k+2}} \leq C_{k,r_0}$$

for  $u \in \{u_\xi, u_\xi^*, u^{(1)}, u^{*(1)}, u^{(2)}, u^{*(2)}\}$  and  $k = 0, 1, \dots, k_0$ : and a positive constant  $C_{k,r_0}$  depending on  $k$  and  $r_0$ .

We can prove Theorem 4.57 by using Proposition 4.56, similarly to the proof of [12, Lemma 4.3]. We thus omit the proof.

## 4.4 Spectral properties of $e^{-tL}P_0$

In this section we give a factorization of  $e^{-tL}P_0$  and prove Theorem 4.2 (i).

We denote the characteristic function of a set  $\{\xi \in \mathbf{R} : |\xi| \leq r_0\}$  by  $\mathbf{1}_{\{|\xi| \leq r_0\}}$ , i.e.,

$$\mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) = \begin{cases} 1, & |\xi| \leq r_0, \\ 0, & |\xi| > r_0. \end{cases}$$

We define the projection  $P_0$  by

$$P_0 = \mathcal{F}^{-1} \mathbf{1}_{\{|\xi| \leq r_0\}} \Pi(\xi) \mathcal{F}.$$

$P_0$  is a bounded projection on  $L^2(\Omega)$  satisfying

$$P_0 L \subset L P_0, \quad P_0 e^{-tL} = e^{-tL} P_0.$$

As in [3, 5], to investigate  $e^{-tL} P_0$ , we introduce operators related to  $u_\xi$  and  $u_\xi^*$ . We define the operators  $\mathcal{T} : L^2(\mathbf{R}) \rightarrow L^2(\Omega)$ ,  $\mathcal{P} : L^2(\Omega) \rightarrow L^2(\mathbf{R})$  and  $\Lambda : L^2(\mathbf{R}) \rightarrow L^2(\mathbf{R})$  by

$$\begin{aligned} \mathcal{T}\sigma &= \mathcal{F}^{-1}[\mathcal{T}_\xi \sigma], & \mathcal{T}_\xi \sigma &= \mathbf{1}_{\{|\xi| \leq r_0\}} u_\xi \sigma; \\ \mathcal{P}u &= \mathcal{F}^{-1}[\mathcal{P}_\xi u], & \mathcal{P}_\xi u &= \mathbf{1}_{\{|\xi| \leq r_0\}} \langle u, u_\xi^* \rangle; \\ \Lambda\sigma &= \mathcal{F}^{-1}[\mathbf{1}_{\{|\xi| \leq r_0\}} \lambda_0(\xi) \sigma] \end{aligned}$$

for  $u \in L^2(\Omega)$  and  $\sigma \in L^2(\mathbf{R})$ . It then follows that

$$P_0 = \mathcal{T}\mathcal{P}, \quad e^{-tL} P_0 = \mathcal{T} e^{t\Lambda} \mathcal{P}.$$

We investigate boundedness properties of  $\mathcal{T}$ ,  $\mathcal{P}$  and  $e^{t\Lambda}$ .

As for  $\mathcal{T}$ , we have the following

**Proposition 4.40.** *The operator  $\mathcal{T}$  has the following properties:*

- (i)  $\partial_{x_3}^l \mathcal{T} = \mathcal{T} \partial_{x_3}^l$  for  $l = 0, 1, \dots$ .
- (ii)  $\|\partial_{x_3}^k \partial_{x_3}^l \mathcal{T} \sigma\|_2 \leq C \|\sigma\|_{L^2(\mathbf{R})}$  for  $k = 0, 1, \dots, k_0$ ,  $l = 0, 1, \dots$  and  $\sigma \in L^2(\mathbf{R})$ .
- (iii)  $\mathcal{T}$  is decomposed as

$$\mathcal{T} = \mathcal{T}^{(0)} + \partial_{x_3} \mathcal{T}^{(1)} + \partial_{x_3}^2 \mathcal{T}^{(2)}.$$

Here  $\mathcal{T}^{(j)} \sigma = \mathcal{F}^{-1}[\mathcal{T}^{(j)} \sigma]$  ( $j = 0, 1, 2$ ) with

$$\begin{aligned} \mathcal{T}^{(0)} \sigma &= \mathbf{1}_{\{|\xi| \leq r_0\}} \sigma u^{(0)}, \\ \mathcal{T}^{(1)} \sigma &= \mathbf{1}_{\{|\xi| \leq r_0\}} \sigma u^{(1)}(\cdot), \\ \mathcal{T}^{(2)} \sigma &= -\mathbf{1}_{\{|\xi| \leq r_0\}} \sigma u^{(2)}(\cdot, \xi), \end{aligned}$$

where  $u^{(j)}$  ( $j = 0, 1, 2$ ) are the functions given in Theorem 4.57. The assertions (i) and (ii) hold with  $\mathcal{T}$  replaced by  $\mathcal{T}^{(j)}$  ( $j = 0, 1, 2$ ).

**Proof.** It is clear that (i) is true. As for (ii), we can prove the estimates by using the properties of  $u_\xi$  in Theorem 4.57 and the Sobolev inequality. From the expansion of  $u_\xi$  given in Theorem 4.57, we can expand  $\mathcal{T}$  as in (iii).  $\square$

As for  $\mathcal{P}$ , we have the following properties.

**Proposition 4.41.** *The operator  $\mathcal{P}$  has the following properties:*

- (i)  $\partial_{x_3}^l \mathcal{P} = \mathcal{P} \partial_{x_3}^l$  for  $l = 0, 1, \dots$ .
- (ii)  $\|\partial_{x_3}^k \mathcal{P} u\|_{L^2(\mathbf{R})} \leq C \|u\|_2$  for  $k = 0, 1, \dots, k_0$ ,  $l = 0, 1, \dots$  and  $u \in L^2(\Omega)$ . Furthermore,  $\|\mathcal{P} u\|_{L^2(\mathbf{R})} \leq C \|u\|_1$  for  $u \in L^1(\Omega)$ .
- (iii)  $\mathcal{P}$  is decomposed as

$$\mathcal{P} = \mathcal{P}^{(0)} + \partial_{x_3} \mathcal{P}^{(1)} + \partial_{x_3}^2 \mathcal{P}^{(2)}.$$

Here  $\mathcal{P}^{(j)}u = \mathcal{F}^{-1}[\mathcal{P}^{(j)}u]$  ( $j = 0, 1, 2$ ) with

$$\begin{aligned}\mathcal{P}^{(0)}u &= \mathbf{1}_{\{|\xi| \leq r_0\}} \langle u, u^{*(0)} \rangle = \mathbf{1}_{\{|\xi| \leq r_0\}} \langle Q_0 u \rangle, \\ \mathcal{P}^{(1)}u &= \mathbf{1}_{\{|\xi| \leq r_0\}} \langle u, u^{*(1)} \rangle, \\ \mathcal{P}^{(2)}u &= -\mathbf{1}_{\{|\xi| \leq r_0\}} \langle u, u^{*(2)}(\xi) \rangle,\end{aligned}$$

where  $u^{(j)*}$  ( $j = 0, 1, 2$ ) are the functions given in Theorem 4.57. The assertions (i) and (ii) hold with  $\mathcal{P}$  replaced by  $\mathcal{P}^{(j)}$  ( $j = 0, 1, 2$ ).

**Proof.** It is clear that (i) holds true. As for (ii), we can prove the estimates by using the properties of  $u_\xi^*$  in Theorem 4.57 and the Sobolev inequality. From the expansion of  $u_\xi^*$  given in Theorem 4.57, we can expand  $\mathcal{P}$  as in (iii).  $\square$

As for  $\Lambda$ , we have the following decay estimates for  $e^{t\Lambda}$ .

**Proposition 4.42.** *The operator  $e^{t\Lambda}$  satisfies the following decay estimates.*

- (i)  $\|\partial_{x_3}^l e^{t\Lambda} \mathcal{P}u\|_{L^2(\mathbf{R})} \leq C(1+t)^{-\frac{1}{4}-\frac{l}{2}} \|u\|_1,$
  - (ii)  $\|\partial_{x_3}^l e^{t\Lambda} \mathcal{P}^{(j)}u\|_{L^2(\mathbf{R})} \leq C(1+t)^{-\frac{1}{4}-\frac{l}{2}} \|u\|_1, \quad j = 0, 1, 2,$
  - (iii)  $\|\partial_{x_3}^l (\mathcal{T} - \mathcal{T}^{(0)}) e^{t\Lambda} \mathcal{P}u\|_2 \leq C(1+t)^{-\frac{3}{4}-\frac{l}{2}} \|u\|_1,$
- for  $u \in L^1(\Omega)$  and  $l = 0, 1, 2, \dots$ .

**Proof.** Since  $\lambda_0(\xi) = -i\kappa_0\xi - \kappa_1\xi^2 + \mathcal{O}(|\xi|^3)$ , we see from Theorem 4.57 that

$$\begin{aligned}\|\partial_{x_3}^l e^{t\Lambda} \mathcal{P}^{(j)}u\|_{L^2(\mathbf{R})} &\leq C \int_{\mathbf{R}} \mathbf{1}_{\{|\xi| \leq r_0\}} |\xi|^{2l} e^{-t(i\kappa_0\xi + \kappa_1\xi^2)} |\langle u(\xi), u^{*(j)} \rangle|^2 d\xi \\ &\leq C \int_{\mathbf{R}} \mathbf{1}_{\{|\xi| \leq r_0\}} |\xi|^{2l} e^{-t(i\kappa_0\xi + \kappa_1\xi^2)} |u(\xi)|_1^2 d\xi \\ &\leq C \begin{cases} \|u\|_1^2, \\ t^{-\frac{1}{2}-l} \|u\|_1^2. \end{cases}\end{aligned}\tag{4.122}$$

This implies (i) and (ii). As for (iii), since  $\mathcal{T} - \mathcal{T}^{(0)} = \partial_{x_3} \mathcal{T}^{(1)} + \partial_{x_3}^2 \mathcal{T}^{(2)}$ , we obtain the desired estimate from (i) and Proposition 4.40.  $\square$

The estimate (4.4) in Theorem 4.2 follows from Propositions 4.40 and 4.42.

We next investigate the asymptotic behavior of  $e^{-tL}$ . Recall that  $\mathcal{H}(t)$  is defined by

$$\mathcal{H}(t)\sigma = \mathcal{F}^{-1}[e^{-(i\kappa_0\xi + \kappa_1\xi^2)t}\sigma]$$

for  $\sigma \in L^2(\mathbf{R})$ , where  $\kappa_0 \in \mathbf{R}$  and  $\kappa_1 > 0$  are given in Theorem 4.55. We first introduce the well-known decay estimate for  $\mathcal{H}(t)$ .

**Proposition 4.43.** *There holds the estimate*

$$\|\partial_{x_3}^l (\mathcal{H}(t)\sigma)\|_{L^2(\mathbf{R})} \leq Ct^{-\frac{1}{4}-\frac{l}{2}} \|\sigma\|_{L^1(\mathbf{R})} \quad (l = 0, 1, \dots)$$

for  $\sigma \in L^1(\mathbf{R})$ .

We next consider the asymptotic behavior of  $e^{t\Lambda}$ . The asymptotic leading part of  $e^{t\Lambda} \mathcal{P}$  is given by  $\mathcal{H}(t)$ . In fact, we have the following

**Proposition 4.44.** For  $u \in L^2(\Omega)$ , we set  $\sigma = \langle Q_0 u \rangle$ . If  $u \in L^1(\Omega)$ , then there holds the estimate

$$\|\partial_{x_3}^l (e^{t\Lambda} \mathcal{P}u - \mathcal{H}(t)\sigma)\|_{L^2(\mathbf{R})} \leq Ct^{-\frac{3}{4}-\frac{l}{2}} \|u\|_1 \quad (l = 0, 1, \dots).$$

**Proof.** By Proposition 4.41 we have

$$e^{t\Lambda} \mathcal{P} = e^{t\Lambda} \mathcal{P}^{(0)} + \partial_{x_3} e^{t\Lambda} \mathcal{P}^{(1)} + \partial_{x_3}^2 e^{t\Lambda} \mathcal{P}^{(2)}.$$

Set  $\sigma = \langle Q_0 u \rangle$ . Since  $e^{t\Lambda} \mathcal{P}^{(0)} u = \mathcal{F}^{-1}[\mathbf{1}_{\{|\xi| \leq r_0\}} e^{\lambda_0(\xi)t} \sigma]$ , we see that

$$\mathcal{F}[e^{t\Lambda} \mathcal{P}^{(0)} u - \mathcal{H}(t)\sigma] = (\mathbf{1}_{\{|\xi| \leq r_0\}} - 1) e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t} \sigma + \mathbf{1}_{\{|\xi| \leq r_0\}} (e^{\lambda_0(\xi)t} - e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t}) \sigma.$$

By using the relation

$$\lambda_0(\xi) + (i\kappa_0 \xi + \kappa_1 \xi^2) = \mathcal{O}(|\xi|^3)$$

we obtain

$$\begin{aligned} e^{\lambda_0(\xi)t} - e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t} &= e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t} (e^{(\lambda_0(\xi) + i\kappa_0 \xi + \kappa_1 \xi^2)t} - 1) \\ &= e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t} \mathcal{O}(|\xi|^3)t. \end{aligned}$$

It then follows that

$$\begin{aligned} \int_{|\xi| \leq r_0} |\xi|^{2l} |e^{\lambda_0(\xi)t} - e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t}|^2 d\xi &\leq C \int_{|\xi| \leq r_0} |\xi|^{2(l+3)} t^2 e^{-2\kappa_1 \xi^2 t} d\xi \|\sigma\|_{L^1(\mathbf{R})}^2 \\ &\leq C \int_{|\xi| \leq r_0} (|\xi|^2 t)^2 e^{-\kappa_1 \xi^2 t} |\xi|^{2(l+1)} e^{-\kappa_1 \xi^2 t} d\xi \|\sigma\|_{L^1(\mathbf{R})}^2 \\ &\leq C \int_{|\xi| \leq r_0} |\xi|^{2(l+1)} e^{-\kappa_1 \xi^2 t} d\xi \|\sigma\|_{L^1(\mathbf{R})}^2 \\ &\leq Ct^{-\frac{3}{2}-l} \|\sigma\|_{L^1(\mathbf{R})}^2. \end{aligned}$$

On the other hand, we also have

$$\int_{|\xi| \leq r_0} |\xi|^{2l} |e^{\lambda_0(\xi)t} - e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t}|^2 d\xi \leq C \|\sigma\|_{L^1(\mathbf{R})}^2.$$

We thus obtain

$$\int_{|\xi| \leq r_0} |\xi|^{2l} |e^{\lambda_0(\xi)t} - e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t}|^2 d\xi \leq C(1+t)^{-\frac{3}{2}-l} \|\sigma\|_{L^1(\mathbf{R})}^2.$$

Similarly, we have

$$\|(\mathbf{1}_{\{|\xi| \leq r_0\}} - 1) e^{-(i\kappa_0 \xi + \kappa_1 \xi^2)t} \sigma\|_2^2 \leq Ct^{-\frac{1}{2}-l} e^{-\kappa_1 r_0^2 t} \|\sigma\|_{L^1(\mathbf{R})}^2.$$

We thus see that

$$\|e^{t\Lambda} \mathcal{P}^{(0)} u - \mathcal{H}(t)\sigma\|_{L^2(\mathbf{R})} \leq Ct^{-\frac{3}{4}-\frac{l}{2}} \|u\|_1.$$

This estimate and Proposition 4.42 (ii) give the desired estimate. This completes the proof.  $\square$

We are now in a position to prove estimate (4.5) in Theorem 4.2 (i). In fact, we have

$$e^{-tL}P_0u - [\mathcal{H}(t)\sigma]u^{(0)} = (\mathcal{T} - \mathcal{T}^{(0)})e^{t\Lambda}\mathcal{P}u + [e^{t\Lambda}\mathcal{P}u - \mathcal{H}(t)\sigma]u^{(0)}.$$

This, together with Proposition 4.42 (iii) and Proposition 4.44, yields the desired estimate (4.5).

We finally state the estimates for the projection  $P_0$ .

**Theorem 4.45.** *The projection  $P_0$  has the following properties:*

- (i)  $\partial_{x_3}^l P_0 = P_0 \partial_{x_3}^l$  for  $l = 0, 1, \dots$ .
- (ii)  $\|\partial_x^k \partial_{x_3}^l P_0 u\|_2 \leq C_k \|u\|_1$  for  $k = 0, 1, \dots, k_0$ ,  $l = 0, 1, \dots$  and  $u \in L^1(\Omega)$ .
- (iii)  $P_0$  is decomposed as

$$P_0 = P_0^{(0)} + \partial_{x_3} P_0^{(1)} + \partial_{x_3}^2 P_0^{(2)},$$

where  $P_0^{(j)}u = \mathcal{F}^{-1}[P_0^{(j)}u]$  ( $j = 0, 1, 2$ ) with

$$P_0^{(0)} = \mathcal{T}^{(0)}\mathcal{P}^{(0)} = \mathbf{1}_{\{|\xi| \leq r_0\}}\Pi^{(0)}, \quad (4.123)$$

$$P_0^{(1)} = \mathcal{T}^{(0)}\mathcal{P}^{(1)} + \mathcal{T}^{(1)}\mathcal{P}^{(0)} = -i\mathbf{1}_{\{|\xi| \leq r_0\}}\Pi^{(1)}, \quad (4.124)$$

$$P_0^{(2)} = \mathcal{T}^{(0)}\mathcal{P}^{(2)} + \mathcal{T}^{(1)}\{\mathcal{P}^{(1)} + \partial_{x_3}\mathcal{P}^{(2)}\} + \mathcal{T}^{(2)}\{\mathcal{P}^{(0)} + \partial_{x_3}\mathcal{P}^{(1)} + \partial_{x_3}^2\mathcal{P}^{(2)}\}. \quad (4.125)$$

Furthermore,  $P_0^{(j)}$  ( $j = 0, 1, 2$ ) satisfy assertions (i) and (ii) by replacing  $P_0$  with  $P_0^{(j)}$ .

**Proof.** It is clear that (i) is true. Estimates in (ii) are given by Propositions 4.40, 4.41. As for (iii), it is easy to see that  $\partial_{x_3}^l P_0^{(j)} = P_0^{(j)} \partial_{x_3}^l$ . The estimates

$$\|\partial_x^k \partial_{x_3}^l P_0^{(j)}u\|_2 \leq C_k \|u\|_1$$

can also be obtained by Propositions 4.40, 4.41. The relations (4.124) and (4.125) can be verified by equating the coefficients of each power of  $\xi$  in the expansions of  $\Pi(\xi)$  in (4.141) and  $\langle u, u_\xi^* \rangle u_\xi$ . This completes the proof.  $\square$

## 4.5 Decay estimate for $e^{-tL}(I - P_0)$

In this section we prove Theorem 4.2 (ii). We set

$$P_\infty = I - P_0.$$

To prove Theorem 4.2 (ii), we first introduce the decay estimate of  $e^{-tL}P_\infty u_0$  for  $u_0 \in H^1(\Omega) \times H_0^1(\Omega)$ .

**Proposition 4.46.** *There exist constants  $\nu_1$ ,  $\gamma_1$  and  $\omega_1$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{\nu+\bar{\nu}} \geq \gamma_1^2$  and  $\omega \leq \omega_1$ , then  $e^{-tL}P_\infty u_0$  have the following properties. If  $u_0 \in H^1(\Omega) \times H_0^1(\Omega)$ , then there exists a constant  $d > 0$  such that  $e^{-tL}P_\infty u_0$  satisfies*

$$\|e^{-tL}P_\infty u_0\|_{H^1} \leq Ce^{-dt}\|u_0\|_{H^1} \quad (4.126)$$

for  $t \geq 0$ .

**Proof.**  $P_\infty$  is written as

$$P_\infty = P_{\infty,0} + \tilde{P}_\infty,$$

where

$$\begin{aligned} P_{\infty,0}u &= \mathcal{F}^{-1}[P_{\infty,0}u], & P_{\infty,0}u &= \mathbf{1}_{\{|\xi| \leq r_0\}}(I - P_0)u, \\ \tilde{P}_\infty u &= \mathcal{F}^{-1}[\tilde{P}_\infty u], & \tilde{P}_\infty u &= (1 - \mathbf{1}_{\{|\xi| \leq r_0\}})u. \end{aligned}$$

The estimate  $\|e^{-tL}\tilde{P}_\infty u_0\|_{H^1} \leq Ce^{-dt}\|u_0\|_{H^1}$  was proved in [1, Theorem 3.3]. As for  $P_{\infty,0}$  part, since  $\rho(-L_\xi|_{(I-\Pi_0(\xi))L^2}) \subset \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \geq -\frac{c_0}{2}\}$  by Theorem 4.54, we have

$$|e^{-tL_\xi}P_{\infty,0}u_0|_2 \leq Ce^{-\frac{c_0}{4}t}|u_0|_2. \quad (4.127)$$

We now apply the argument of the proof of [1, Proposition 4.20] to  $u(t) = e^{-tL}P_{\infty,0}u_0$ . Due to (4.127), one can replace  $e^{-\frac{d_2}{2}|\xi|^2 t}|u_0|_2^2$  in the inequality (4.72) of [1] by  $e^{-\frac{c_0}{2}t}|u_0|_2^2$  to obtain  $E_{4,1}^{(0)}[u](t) \leq Ce^{-2\tilde{d}_1 t}|u_0|_{H^1}^2$  for a positive constant  $\tilde{d}_1$ . Integrating this over  $|\xi| \leq r_0$  and using the Plancherel Theorem, we have

$$\|e^{-tL}P_{\infty,0}u_0\|_{H^1} \leq Ce^{-\tilde{d}t}\|u_0\|_{H^1}$$

for a positive constant  $\tilde{d}$ . Combining the estimates for  $e^{-tL}\tilde{P}_\infty u_0$  and  $e^{-tL}P_{\infty,0}u_0$  we obtain the desired estimate. This completes the proof.  $\square$

We next consider the estimate for  $e^{-tL}u$  for  $0 < t \leq 1$ .

**Proposition 4.47.** *Let  $T > 0$ . If  $u_0 \in H^1(\Omega) \times \tilde{H}^1(\Omega)$ , then  $e^{-tL}u_0$  satisfies  $e^{-tL}u_0 \in H^1(\Omega) \times H_0^1(\Omega)$  for  $t > 0$  and*

$$\|e^{-tL}u_0\|_{H^1} \leq C_T\{\|u_0\|_{H^1 \times \tilde{H}^1} + t^{-\frac{1}{2}}\|w_0\|_2\} \quad (4.128)$$

for  $0 < t \leq T$ .

Let  $u_0 \in H^1(\Omega) \times \tilde{H}^1(\Omega)$ . Applying Proposition 4.47 with  $t = 1$ , we have  $u_1 = e^{-tL}u_0|_{t=1} \in H^1(\Omega) \times H_0^1(\Omega)$  and

$$\|u_1\|_{H^1} \leq C\|u_0\|_{H^1 \times \tilde{H}^1}.$$

This, together with Proposition 4.46 and Proposition 4.47, implies Theorem 4.2 (ii).

It remains to prove Proposition 4.47.

**Lemma 4.48.** *Let  $T > 0$ . There hold the following estimates for  $0 \leq t \leq T$ :*

(i) for  $\ell = 0, 1$ ,

$$\|\partial_{x_3}^\ell u(t)\|_2^2 + c \int_0^t \|\nabla \partial_{x_3}^\ell w\|_2^2 + \|\operatorname{div} \partial_{x_3}^\ell w\|_2^2 d\tau \leq C_T \|\partial_{x_3}^\ell u_0\|_2^2,$$

(ii)

$$\begin{aligned} & \|\chi_0 \partial_{x'} u(t)\|_2^2 + c \int_0^t \|\chi_0 \nabla \partial_{x'} w(\tau)\|_2^2 + \|\chi_0 \operatorname{div} \partial_{x'} w\|_2^2 d\tau \\ & \leq C_T \left\{ \|u_0\|_2^2 + \|\partial_{x_3} u_0\|_2^2 + \|\chi_0 \partial_{x'} u_0\|_2^2 + \int_0^t \|\partial_{x'} \phi(\tau)\|_2^2 d\tau \right\}, \end{aligned}$$

(iii) for  $1 \leq m \leq N$ ,

$$\begin{aligned} & \|\chi_m \partial u(t)\|_2^2 + c \int_0^t \|\chi_m \nabla \partial w\|_2^2 + \|\chi_m \operatorname{div} \partial w\|_2^2 d\tau \\ & \leq C_T \left\{ \|u_0\|_2^2 + \|\partial_{x_3} u_0\|_2^2 + \|\chi_m \partial u_0\|_2^2 + \int_0^t \|\partial_{x'} \phi\|_2^2 d\tau \right\}. \end{aligned}$$

Lemma 4.48 can be proved by the energy method as those in the proof of [1, Propositions 4.7, 4.15, 4.17]. Note that here are no restrictions on  $\nu$ ,  $\tilde{\nu}$  and  $\gamma$  but  $C_T$  depends on  $T$ .

We next consider the  $L^2$  estimate of the normal derivative for  $\phi$ .

**Lemma 4.49.** *Let  $T > 0$ . For  $1 \leq m \leq N$ , there holds the estimate for  $0 \leq t \leq T$ :*

$$\begin{aligned} & \|\chi_m \partial_n \phi(t)\|_2^2 \\ & \leq C_T \left\{ \|u_0\|_2^2 + \|\partial_{x_3} u_0\|_2^2 + \|\chi_m \partial u_0\|_2^2 + \|\chi_m \partial_n \phi_0\|_2^2 + \int_0^t \|\partial_{x'} \phi\|_2^2 d\tau \right\}. \end{aligned}$$

**Proof.** Let us transform a scalar field  $p(x')$  on  $D \cap \mathcal{O}_m$  as

$$\tilde{p}(y') = p(x') \quad (y' = \Psi^m(x'), \quad x' \in D \cap \mathcal{O}_m),$$

where  $\Psi^m(x')$  is a function given in Section 2. Similarly we transform a vector field  $h(x') = {}^T(h^1(x'), h^2(x'), h^3(x'))$  into  $\tilde{h}(y') = {}^T(\tilde{h}^1(y'), \tilde{h}^2(y'), \tilde{h}^3(y'))$  as

$$h(x') = E(y') \tilde{h}(y')$$

where  $E(y') = (e_1(y'), e_2(y'), e_3)$  with  $e_1(y')$ ,  $e_2(y')$  and  $e_3$  given in Section 2. From the proof of [1, Proposition 4.16], we have

$$\partial_\tau \partial_{y_1} \tilde{\phi} + (a + b \partial_{y_3}) \partial_{y_1} \tilde{\phi} = \tilde{\rho}_s I - \frac{\gamma^2 \tilde{\rho}_s^2}{\nu + \tilde{\nu}} \partial_\tau \tilde{w}^1, \quad (4.129)$$

where

$$a(y') = \frac{\tilde{\rho}_s \tilde{P}'(\tilde{\rho}_s)}{\nu + \tilde{\nu}}, \quad b(y') = \tilde{v}_s^3(y'),$$

$$I = -\frac{\gamma^2}{\nu+\tilde{\nu}}\left\{\nu(\operatorname{rot}_y \operatorname{rot}_y \tilde{w})^1 + \tilde{\rho}_s \partial_{y_1} \left( \frac{\tilde{P}'(\tilde{\rho}_s)}{\gamma^2 \tilde{\rho}_s} \right) \tilde{\phi} + \frac{\nu}{\gamma^2} \tilde{\rho}_s (\Delta_{y'} \tilde{v}_s)^1 \tilde{\phi} + \tilde{\rho}_s \tilde{v}_s^3 \partial_{y_3} \tilde{w}^1 \right\} \\ - \left\{ \frac{1}{\tilde{\rho}_s} \partial_{y_1} \tilde{v}_s^3 \partial_{y_3} \tilde{\phi} + \gamma^2 \frac{1}{\tilde{\rho}_s} \partial_{y_1} (\operatorname{div}_y (\tilde{\rho}_s \tilde{w})) - \gamma^2 \partial_{y_1} \operatorname{div}_y \tilde{w} \right\}.$$

Here  $(\operatorname{rot}_y \tilde{w})^1$  denotes the  $e_1(y')$  component of  $\operatorname{rot}_y \tilde{w}$ , and so on. We note that  $(\operatorname{rot}_y \operatorname{rot}_y \tilde{w})^1$  does not contain  $\partial_{y_1}^2$ . See the proof of [1, Proposition 4.16]. We also note that there is a positive constant  $a_0$  such that

$$a(y') \geq a_0 > 0$$

for any  $y' \in \Psi^m(D)$ .

We denote by  $e^{-t(a+b\partial_{y_3})}$  the semigroup generated by  $-(a+b\partial_{y_3})$ , i.e.,

$$e^{-t(a+b\partial_{y_3})} \tilde{\phi}_0 = \mathcal{F}^{-1} [e^{-(a(y')+i\xi b(y'))t} \widehat{\tilde{\phi}_0}].$$

Then it is easy to see that

$$\|\tilde{\chi}_m e^{-t(a+b\partial_{y_3})} \tilde{\phi}_0\|_2 \leq e^{-a_0 t} \|\tilde{\chi}_m \tilde{\phi}_0\|_2.$$

In terms of  $e^{-t(a+b\partial_{y_3})}$ ,  $\partial_{y_1} \tilde{\phi}$  is written as

$$\begin{aligned} \partial_{y_1} \tilde{\phi}(t) &= e^{-t(a+b\partial_{y_3})} \partial_{y_1} \tilde{\phi}_0 + \int_0^t e^{-(t-\tau)(a+b\partial_{y_3})} \tilde{\rho}_s \tilde{I}(\tau) d\tau \\ &\quad - \frac{\gamma^2 \tilde{\rho}_s^2}{\nu + \tilde{\nu}} \int_0^t e^{-(t-\tau)(a+b\partial_{y_3})} \partial_\tau \tilde{w}^1 d\tau \\ &\equiv J_1 + J_2 + J_3. \end{aligned}$$

As for  $J_1$  and  $J_2$ , we have

$$\begin{aligned} \|\tilde{\chi}_m J_1\|_2 &\leq e^{-a_0 t} \|\tilde{\chi}_m \partial_{y_1} \tilde{\phi}_0\|_2, \\ \|\tilde{\chi}_m J_2\|_2 &\leq C \int_0^t e^{-a_0(t-\tau)} \|\tilde{\chi}_m \tilde{I}(\tau)\|_2 d\tau. \end{aligned}$$

As for  $J_3$ , integrating by parts, we have

$$J_3 = \frac{\gamma^2 \tilde{\rho}_s^2}{\nu + \tilde{\nu}} \left[ e^{-t(a+b\partial_{y_3})} \tilde{w}_0^1 - \tilde{w}^1(t) + (a+b\partial_{y_3}) \int_0^t e^{-(t-\tau)(a+b\partial_{y_3})} \tilde{w}^1(\tau) d\tau \right].$$

We thus obtain

$$\|\tilde{\chi}_m J_3\|_2 \leq C \left\{ e^{-a_0 t} \|\tilde{\chi}_m \tilde{w}_0^1\|_2 + \|\tilde{\chi}_m \tilde{w}^1(t)\|_2 + \int_0^t e^{-a_0(t-\tau)} \|\tilde{\chi}_m \partial_{y_3} \tilde{w}^1(\tau)\|_2 d\tau \right\}.$$

Furthermore, we have

$$\begin{aligned} \|\tilde{\chi}_m I(\tau)\|_2 &\leq C \left\{ \|\tilde{\chi}_m \tilde{\phi}(\tau)\|_2 + \|\tilde{\chi}_m \partial_{y_3} \tilde{\phi}(\tau)\|_2 + \|\tilde{\chi}_m \tilde{w}(\tau)\|_2 \right. \\ &\quad \left. + \|\tilde{\chi}_m \nabla_y \tilde{w}(\tau)\|_2 + \|\tilde{\chi}_m \nabla_y \partial_{y_2} \tilde{w}(\tau)\|_2 + \|\tilde{\chi}_m \nabla_y \partial_{y_3} \tilde{w}(\tau)\|_2 \right\}. \end{aligned}$$

It then follows that

$$\begin{aligned} \|\tilde{\chi}_m \partial_{y_1} \tilde{\phi}(t)\|_2 &\leq C \left[ e^{-a_0 t} (\|\tilde{\chi}_m \partial_{y_1} \tilde{\phi}_0\|_2 + \|\tilde{\chi}_m \tilde{w}_0^1\|_2) + \|\tilde{\chi}_m \tilde{w}^1(t)\|_2 \right. \\ &\quad + \int_0^t e^{-a_0(t-\tau)} \left\{ \|\tilde{\chi}_m \tilde{\phi}(\tau)\|_2 + \|\tilde{\chi}_m \partial_{y_3} \tilde{\phi}(\tau)\|_2 + \|\tilde{\chi}_m \tilde{w}(\tau)\|_2 \right. \\ &\quad \left. \left. + \|\tilde{\chi}_m \nabla_y \tilde{w}(\tau)\|_2 + \|\tilde{\chi}_m \nabla_y \partial_{y_2} \tilde{w}(\tau)\|_2 + \|\tilde{\chi}_m \nabla_y \partial_{y_3} \tilde{w}(\tau)\|_2 \right\} d\tau \right]. \end{aligned}$$

Inverting to the original coordinates  $x'$  and noting that  $\partial_{y_1} = \partial_n$ ,  $\partial_{y_2} = \partial$ , we see that

$$\begin{aligned} \|\chi_m \partial_n \phi(t)\|_2 &\leq C \left\{ e^{-a_0 t} (\|\chi_m \partial_n \phi_0\|_2 + \|\chi_m w_0^1\|_2) + \|\chi_m w^1(t)\|_2 \right. \\ &\quad + \int_0^t \|\chi_m \phi(\tau)\|_2 + \|\chi_m \partial_{x_3} \phi(\tau)\|_2 + \|\chi_m w(\tau)\|_2 \\ &\quad \left. + \|\chi_m \nabla w(\tau)\|_2 + \|\chi_m \nabla \partial w(\tau)\|_2 + \|\chi_m \nabla \partial_{x_3} w(\tau)\|_2 d\tau \right\}. \end{aligned}$$

This, together with Lemma 4.48, yields the desired estimate. This completes the proof.  $\square$

By Lemma 4.48 and Lemma 4.49, we have the following estimate.

**Lemma 4.50.** *Let  $T > 0$ . There exists a positive constant  $c$  such that the estimate*

$$\begin{aligned} &\|u(t)\|_{H^1 \times \tilde{H}^1}^2 + c \int_0^t \|\nabla w(\tau)\|_2^2 + \|\operatorname{div} w(\tau)\|_2^2 + \|\nabla \partial_{x_3} w(\tau)\|_2^2 + \|\operatorname{div} \partial_{x_3} w(\tau)\|_2^2 \\ &+ \|\chi_0 \nabla \partial_{x'} w(\tau)\|_2^2 + \|\chi_0 \operatorname{div} \partial_{x'} w(\tau)\|_2^2 + \sum_{m=1}^N \left\{ \|\chi_m \nabla \partial w(\tau)\|_2^2 + \|\chi_m \operatorname{div} \partial w(\tau)\|_2^2 \right\} d\tau \\ &\leq C_T \|u_0\|_{H^1 \times \tilde{H}^1}^2 \end{aligned}$$

holds for  $0 \leq t \leq T$ .

We finally consider the  $L^2$  estimate for  $\partial_{x'} w$ .

**Lemma 4.51.** *Let  $T > 0$ . There holds the estimate*

$$\|\partial_{x'} w(t)\|_2 \leq C_T \{ \|u_0\|_{H^1 \times \tilde{H}^1} + t^{-\frac{1}{2}} \|w_0\|_2 \}$$

for  $0 < t \leq T$ .

**Proof.** We see that  $w$  satisfies the equation

$$\partial_t w + \bar{A} w + \bar{B} u = 0,$$

where  $\bar{A}$  is the  $3 \times 3$  operator defined by

$$\bar{A} = -\frac{\nu}{\rho_s} \Delta - \frac{\nu + \tilde{\nu}}{\rho_s} \nabla \operatorname{div},$$

$\bar{B}$  is the  $3 \times 4$  operator defined by

$$\bar{B} = \begin{pmatrix} \nabla' \left( \frac{P(\rho_s)}{\gamma^2 \rho_s} \cdot \right) & v_s^3 \partial_{x_3} I_2 & 0 \\ \partial_{x_3} \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \cdot \right) + \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} & T(\nabla' v_s^3) & v_s^3 \partial_{x_3} \cdot \end{pmatrix}$$

We write  $w(t)$  as

$$w(t) = e^{-t\bar{A}} w_0 + \int_0^t e^{-(t-\tau)\bar{A}} \bar{B} u(\tau) d\tau.$$

Then

$$\nabla' w(t) = \nabla' e^{-t\bar{A}} w_0 + \int_0^t \nabla' e^{-(t-\tau)\bar{A}} \bar{B} u(\tau) d\tau. \quad (4.130)$$

Since  $\bar{A}$  is strongly elliptic, we have

$$\|\nabla' e^{-t\bar{A}} w_0\|_2 \leq C t^{-\frac{1}{2}} \|w_0\|_2$$

for  $0 < t \leq T$ . Furthermore, we see from Lemma 4.48 and Lemma 4.50 that

$$\begin{aligned} \left\| \int_0^t \nabla' e^{-(t-\tau)\bar{A}} \bar{B} u(\tau) d\tau \right\|_2 &\leq C \int_0^t (t-\tau)^{-\frac{1}{2}} \|\bar{B} u(\tau)\|_2 d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{1}{2}} \|u(\tau)\|_{H^1 \times \tilde{H}^1} d\tau \\ &\leq C \|u_0\|_{H^1 \times \tilde{H}^1} \int_0^t (t-\tau)^{-\frac{1}{2}} d\tau \\ &\leq C T^{\frac{1}{2}} \|u_0\|_{H^1 \times \tilde{H}^1} \end{aligned} \quad (4.131)$$

for  $0 < t \leq T$ . It then follows from (4.130) and (4.131) that

$$\|\partial_{x'} w(t)\|_2 \leq C_T \{ \|u_0\|_{H^1 \times \tilde{H}^1} + t^{-\frac{1}{2}} \|w_0\|_2 \} \quad (4.132)$$

for  $0 < t \leq T$ . This completes the proof.  $\square$

**Proof of Proposition 4.47.** Let  $u(t) = e^{-tL} u_0$ . It is not difficult to see that if  $u_0 \in H^1(\Omega) \times H_0^1(\Omega)$ , then  $u(t)$  satisfies

$$u \in C([0, T]; H^1(\Omega) \times H_0^1(\Omega)), \quad \tilde{Q}u \in L^2(0, T; H^2(\Omega)). \quad (4.133)$$

Using Lemma 4.50 and Lemma 4.51, we obtain the estimate

$$\|u(t)\|_{H^1}^2 + c \int_0^t \bar{D}_1[w](\tau) d\tau \leq C_T \{ \|u_0\|_{H^1 \times \tilde{H}^1}^2 + t^{-1} \|w_0\|_2^2 \}$$

for  $0 < t \leq T$ . Here

$$\begin{aligned} \bar{D}_1[w] &= (\|\nabla w\|_2^2 + \|\operatorname{div} w\|_2^2) + (\|\nabla \partial_{x_3} w\|_2^2 + \|\operatorname{div} \partial_{x_3} w\|_2^2) \\ &\quad + (\|\chi_0 \nabla \partial_{x'} w\|_2^2 + \|\chi_0 \operatorname{div} \partial_{x'} w\|_2^2) + \sum_{m=1}^N (\|\chi_m \nabla \partial w\|_2^2 + \|\chi_m \operatorname{div} \partial w\|_2^2). \end{aligned}$$

We thus obtain estimate (4.128) if  $u_0 \in H^1(\Omega) \times H_0^1(\Omega)$ . Since  $H_0^1(\Omega)$  is dense in  $\tilde{H}^1(\Omega)$ , one can see from Lemma 4.50, (4.128) and (4.133) that if  $u_0 \in H^1(\Omega) \times \tilde{H}^1(\Omega)$ , then  $u(t)$  satisfies

$$u \in C([0, T]; H^1(\Omega) \times \tilde{H}^1(\Omega)) \cap C((0, T]; H^1(\Omega) \times H_0^1(\Omega))$$

and estimate (4.128). This completes the proof.  $\square$

$$(\lambda + \hat{L}_0)u = f, \quad (4.134)$$

where  $u = {}^T(\phi, w) \in D(\hat{L}_0)$  and  $f = {}^T(f^0, g) \in L^2(D)$ . Decomposing  $u$  in (4.134) as

$$u = \langle \phi \rangle u^{(0)} + u_1$$

with

$$u_1 = (I - \hat{\Pi}^{(0)})u,$$

we obtain

$$\lambda(\langle \phi \rangle u^{(0)} + u_1) + \hat{L}_0 u_1 = f.$$

Applying  $\hat{\Pi}^{(0)}$  and  $I - \hat{\Pi}^{(0)}$  to this equation, we have

$$\begin{cases} \lambda \langle \phi \rangle = \langle f^0 \rangle, \\ \lambda u_1 + \hat{L}_0 u_1 = f_1, \end{cases} \quad (4.135)$$

where  $f_1 = (I - \hat{\Pi}^{(0)})f$ . We see from the first equation of (4.135) that if  $\lambda \neq 0$ , then

$$\langle \phi \rangle = \frac{1}{\lambda} \langle f^0 \rangle.$$

This implies that

$$|\langle \phi \rangle| \leq \frac{1}{|\lambda|} |f^0|_2. \quad (4.136)$$

On the other hand, the  $u_1$ -part has the following properties. The second equation of (4.135) is written as

$$\begin{cases} \lambda \phi_1 + \gamma^2 \nabla' \cdot (\rho_s w_1') = f_1^0, \\ \lambda w_1' - \frac{\nu}{\rho_s} \Delta' w_1' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot w_1' + \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi_1 \right) = g_1', \\ \lambda w_1^3 - \frac{\nu}{\rho_s} \Delta' w_1^3 + \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3 \phi_1 + w_1' \cdot \nabla' v_s^3 = g_1^3, \end{cases} \quad (4.137)$$

where  $u_1 = {}^T(\phi_1, w_1) = {}^T(\phi_1, w_1', w_1^3)$  and  $f_1 = {}^T(f_1^0, g_1) = {}^T(f_1^0, g_1', g_1^3)$ . We can obtain the  $L^2$  estimate for  $u_1$  in a similar manner to the proof of Proposition 4.12 by replacing  $\frac{d}{dt}$  with  $\text{Re}\lambda$  and taking  $\xi = 0$ , which is stated as follows. We introduce a quantity  $\tilde{D}_0[w_1]$  defined by

$$\tilde{D}_0[w_1] = |\nabla' w_1|_2^2 + |\nabla' \cdot w_1'|_2^2$$

for  $w_1 = {}^T(w_1', w_1^3)$ .

**Proposition 4.52.** *There exist constants  $\nu_1 > 0$ ,  $\gamma_1 > 0$  and  $\omega_1 > 0$  and an energy functional  $E_0[u_1]$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{\nu+\bar{\nu}} \geq \gamma_1^2$  and  $\omega \leq \omega_1$ , then there hold the estimate*

$$(\operatorname{Re}\lambda)E_0[u_1] + c(|\phi_1|_2^2 + \tilde{D}_0[w_1]) \leq C|f_1|_2|u_1|_2,$$

where  $c$  and  $C$  are positive constants independent of  $u_1$  and  $\lambda$ ; and  $E_0[u_1]$  is equivalent to  $|u_1|_2^2$ .

The Poincaré inequality yields  $\tilde{D}_0[w_1] \geq C|w_1|_2^2$  with a positive constant  $C$ . Therefore, the resolvent estimates for  $-\hat{L}_0$  now follow from (4.136) and Proposition 4.52.

**Proposition 4.53.** *There exist constants  $\nu_1 > 0$ ,  $\gamma_1 > 0$  and  $\omega_1 > 0$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{\nu+\bar{\nu}} \geq \gamma_1^2$  and  $\omega \leq \omega_1$ , then there is a positive constant  $c_0 > 0$  such that*

$$\Sigma_0 \equiv \{\lambda \neq 0 : \operatorname{Re}\lambda > -c_0\} \subset \rho(-\hat{L}_0).$$

Furthermore, the following estimates

$$|(\lambda + \hat{L}_0)^{-1}f|_2 \leq C\left\{\frac{1}{|\lambda|}|f^0|_2 + \frac{1}{(\operatorname{Re}\lambda + c_0)}|f_1|_2\right\},$$

$$|\partial_{x'}\{\tilde{Q}(\lambda + \hat{L}_0)^{-1}f\}|_2 \leq C\left\{\frac{1}{|\lambda|}|f^0|_2 + \frac{1}{(\operatorname{Re}\lambda + c_0)^{1/2}}|f_1|_2\right\}$$

hold uniformly for  $\lambda \in \Sigma_0$ . The same assertions also hold for  $-\hat{L}_0^*$ .

Based on Proposition 4.53, we have the resolvent estimates for  $-\hat{L}_\xi$  with  $|\xi| \ll 1$ .

**Theorem 4.54.** *There exist constants  $\nu_1 > 0$ ,  $\gamma_1 > 0$  and  $\omega_1 > 0$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{\nu+\bar{\nu}} \geq \gamma_1^2$  and  $\omega \leq \omega_1$ , then the following assertions hold. For any  $\eta$  satisfying  $0 < \eta \leq \frac{c_0}{2}$  there is a number  $r_0 = r_0(\eta)$  such that*

$$\Sigma_1 \equiv \{\lambda \neq 0 : |\lambda| \geq \eta, \operatorname{Re}\lambda \geq -\frac{c_0}{2}\} \subset \rho(-\hat{L}_\xi)$$

for  $|\xi| \leq r_0$ . Furthermore, the following estimates

$$|(\lambda + \hat{L}_\xi)^{-1}f|_2 \leq C|f|_2,$$

$$|\partial_{x'}\{\tilde{Q}(\lambda + \hat{L}_\xi)^{-1}f\}|_2 \leq C|f|_2$$

hold uniformly for  $\lambda \in \Sigma_1$  and  $\xi$  with  $|\xi| \leq r_0$ . The same assertions also hold for  $-\hat{L}_\xi^*$ .

**Proof.** Let us decompose  $\hat{L}_\xi$  as

$$\hat{L}_\xi = \hat{L}_0 + \xi\hat{L}^{(1)} + \xi^2\hat{L}^{(2)},$$

where

$$\widehat{L}^{(1)} = i \begin{pmatrix} v_s^3 & 0 & \gamma^2 \rho_s \\ 0 & v_s^3 I_2 & -\frac{\tilde{\nu}}{\rho_s} \nabla' \\ \frac{P'(\rho_s)}{\gamma^2 \rho_s} & -\frac{\tilde{\nu}}{\rho_s} \nabla' & v_s^3 \end{pmatrix}, \quad \widehat{L}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu}{\rho_s} I_2 & 0 \\ 0 & 0 & \frac{\nu + \tilde{\nu}}{\rho_s} \end{pmatrix}.$$

For  $u = {}^T(\phi, w) \in L^2(D) \times H_0^1(D)$  we have

$$|\widehat{L}^{(1)}u|_2 \leq C|u|_{L^2 \times H^1}, \quad |\widehat{L}^{(2)}u|_2 \leq C|u|_2. \quad (4.138)$$

Therefore, we see from Proposition 4.53 that for any  $0 < \eta \leq \frac{c_0}{2}$  there exists  $r_0 > 0$  such that if  $|\xi| \leq r_0$ , then

$$|(\xi \widehat{L}^{(1)} + \xi^2 \widehat{L}^{(2)})(\lambda + \widehat{L}_0)^{-1}f|_2 \leq \frac{1}{2}|f|_2. \quad (4.139)$$

It then follows that

$$\Sigma_1 \equiv \{\lambda : |\lambda| > \eta, \operatorname{Re} \lambda \geq -\frac{c_0}{2}\} \subset \rho(-\widehat{L}_\xi),$$

and that, if  $\lambda \in \Sigma_1$ , then  $(\lambda + \widehat{L}_\xi)^{-1}$  is given by the Neumann series expansion

$$(\lambda + \widehat{L}_\xi)^{-1} = (\lambda + \widehat{L}_0)^{-1} + \sum_{N=0}^{\infty} (-1)^N [(\xi \widehat{L}^{(1)} + \xi^2 \widehat{L}^{(2)})(\lambda + \widehat{L}_0)^{-1}]$$

for  $|\xi| \leq r_0$ , and it holds that

$$|(\lambda + \widehat{L}_\xi)^{-1}f|_2 \leq C|f|_2 \quad (4.140)$$

for  $\lambda \in \Sigma_1$  and  $|\xi| \leq r_0$ . We thus obtain the desired estimates. This completes the proof.  $\square$

As for the spectrum of  $-\widehat{L}_\xi$  near  $\lambda = 0$ , we have the following result.

**Theorem 4.55.** *There exist positive constants  $\nu_1$ ,  $\gamma_1$ ,  $\omega_1$  and  $r_0$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_1^2$  and  $\omega \leq \omega_1$ , then it holds that*

$$\sigma(-\widehat{L}_\xi) \cap \{\lambda : |\lambda| \leq \frac{c_0}{2}\} = \{\lambda_0(\xi)\}$$

for  $\xi$  with  $|\xi| \leq r_0$ , where  $\lambda_0(\xi)$  is a simple eigenvalue of  $-\widehat{L}_\xi$  that has the form

$$\lambda_0(\xi) = -i\kappa_1\xi - \kappa_0\xi^2 + \mathcal{O}(|\xi|^3)$$

as  $\xi \rightarrow 0$ . Here  $\kappa_1 \in \mathbf{R}$  and  $\kappa_0 > 0$  are the numbers given by

$$\kappa_1 = \langle v_s^3 \phi^{(0)} + \gamma^2 \rho_s w^{(0),3} \rangle = \mathcal{O}(1),$$

$$\kappa_0 = \frac{\gamma^2}{\nu} \left\{ \alpha_0 \left| (-\Delta')^{-\frac{1}{2}} \rho_s \right|_2^2 + \mathcal{O}\left(\frac{1}{\gamma^2}\right) + \left(\frac{\nu}{\gamma^2} + \frac{1}{\nu^2}\right) \times \mathcal{O}\left(\frac{\nu + \tilde{\nu}}{\gamma^2}\right) \right\},$$

where  $-\Delta'$  denotes the Laplace operator on  $L^2(D)$  under the zero Dirichlet boundary condition with domain

$$D(-\Delta') = H^2(D) \cap H_0^1(D).$$

**Proof.** For  $u \in L^2(D) \times H_0^1(D)$  we see from Theorem 4.54 and (4.138) that

$$|\widehat{L}^{(1)}u|_2 \leq C(|\widehat{L}_0u|_2 + |u|_2), \quad |\widehat{L}^{(2)}u|_2 \leq C|u|_2.$$

Therefore, since 0 is a simple eigenvalue of  $-\widehat{L}_0$ , we see from the analytic perturbation theory that there exists a positive constant  $r_0$  such that

$$\sigma(-\widehat{L}_\xi) \cap \{\lambda : |\lambda| \leq \frac{c_0}{2}\} = \{\lambda_0(\xi)\}$$

for all  $\xi$  with  $|\xi| \leq r_0$ . Here  $\lambda_0(\xi)$  is a simple eigenvalue of  $-\widehat{L}_\xi$ . Furthermore,  $\lambda_0(\xi)$  and the eigenprojection  $\widehat{\Pi}(\xi)$  for  $\lambda_0(\xi)$  are expanded as

$$\begin{aligned} \lambda_0(\xi) &= \lambda^{(0)} + \xi\lambda^{(1)} + \xi^2\lambda^{(2)} + \mathcal{O}(|\xi|^3), \\ \widehat{\Pi}(\xi) &= \widehat{\Pi}^{(0)} + \xi\widehat{\Pi}^{(1)} + \mathcal{O}(|\xi|^2) \end{aligned} \tag{4.141}$$

with

$$\begin{aligned} \lambda^{(0)} &= 0, \\ \lambda^{(1)} &= \langle \widehat{L}^{(1)}u^{(0)}, u^{(0)*} \rangle, \\ \lambda^{(2)} &= \langle \widehat{L}^{(2)}u^{(0)}, u^{(0)*} \rangle - \langle \widehat{L}^{(1)}\widehat{S}\widehat{L}^{(1)}u^{(0)}, u^{(0)*} \rangle, \\ \widehat{\Pi}^{(1)} &= -\widehat{\Pi}^{(0)}\widehat{L}^{(1)}\widehat{S} - \widehat{S}\widehat{L}^{(1)}\widehat{\Pi}^{(0)}, \end{aligned}$$

where

$$\widehat{S} = \left\{ (I - \widehat{\Pi}^{(0)})\widehat{L}_0(I - \widehat{\Pi}^{(0)}) \right\}^{-1}.$$

We first consider  $\lambda^{(1)}$ . Since

$$\widehat{L}^{(1)}u^{(0)} = i \begin{pmatrix} v_s^3\phi^{(0)} + \gamma^2\rho_s w^{(0),3} \\ -\frac{\tilde{\nu}}{\rho_s}\nabla' w^{(0),3} \\ \alpha_0 + v_s^3 w^{(0),3} \end{pmatrix},$$

we obtain

$$\lambda^{(1)} = \langle \widehat{L}^{(1)}u^{(0)}, u^{(0)*} \rangle = \langle Q_0\widehat{L}^{(1)}u^{(0)} \rangle = i\langle v_s^3\phi^{(0)} + \gamma^2\rho_s w^{(0),3} \rangle = i\mathcal{O}(1)$$

as  $\gamma^2 \rightarrow \infty$ .

We next consider  $\lambda^{(2)}$ . Since  $Q_0\widehat{L}^{(2)}u^{(0)} = 0$ , we have

$$\langle \widehat{L}^{(2)}u^{(0)}, u^{(0)*} \rangle = \langle Q_0\widehat{L}^{(2)}u^{(0)} \rangle = 0.$$

It then follows that

$$\lambda^{(2)} = -\langle \widehat{L}^{(1)}\widehat{S}\widehat{L}^{(1)}u^{(0)}, u^{(0)*} \rangle = -\langle Q_0\widehat{L}^{(1)}\widehat{S}\widehat{L}^{(1)}u^{(0)} \rangle.$$

We define  $\tilde{u}$  by

$$\tilde{u} = \widehat{S}\widehat{L}^{(1)}u^{(0)},$$

which satisfies

$$\begin{cases} \widehat{L}_0 \widetilde{u} = (I - \widehat{\Pi}^{(0)}) \widehat{L}^{(1)} u^{(0)}, \\ \widetilde{w} |_{\partial D} = 0, \\ \langle \widetilde{\phi} \rangle = 0. \end{cases} \quad (4.142)$$

Note that  $\widetilde{u} = T(\widetilde{\phi}, \widetilde{w}) \in i\mathbf{R}^4$  and  $\lambda^{(1)} \in i\mathbf{R}$ . We rewrite  $\lambda^{(2)}$  as

$$\begin{aligned} \lambda^{(2)} &= -\{\langle Q_0 \widehat{L}^{(1)} \widetilde{u} \rangle + \langle Q_0 \widehat{L}^{(1)} \langle \widetilde{\phi} \rangle u^{(0)} \rangle\} \\ &= -\{\langle i v_s^3 \widetilde{\phi} + i \gamma^2 \rho_s \widetilde{w}^3 \rangle + \lambda^{(1)} \langle \widetilde{\phi} \rangle\}, \end{aligned}$$

where  $\widetilde{u} = T(\widetilde{\phi}, \widetilde{w}) = T(\widetilde{\phi}, \widetilde{w}', \widetilde{w}^3)$ . To show the strict negativity of  $\lambda^{(2)}$ , we estimate  $\widetilde{u}$ . The problem (4.142) is written as

$$\begin{cases} \gamma^2 \nabla' \cdot (\rho_s \widetilde{w}') = \lambda^{(1)} \phi^{(0)} - i \xi v_s^3 \phi^{(0)} - i \gamma^2 \rho_s w^{(0),3}, \\ -\frac{\nu}{\rho_s} \Delta' \widetilde{w}' - \frac{\widetilde{\nu}}{\rho_s} \nabla' \nabla' \cdot \widetilde{w}' + \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \widetilde{\phi} \right) = -i \frac{\widetilde{\nu}}{\rho_s} \nabla' w^{(0),3}, \\ -\frac{\nu}{\rho_s} \Delta' \widetilde{w}^3 + \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \widetilde{\phi} + \widetilde{w}' \cdot \nabla' v_s^3 = \lambda^{(1)} w^{(0),3} - i \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi^{(0)} - i v_s^3 w^{(0),3}, \\ \widetilde{w} |_{\partial D} = 0, \\ \langle \widetilde{\phi} \rangle = 0, \end{cases}$$

i.e.,  $\widetilde{u} = T(\widetilde{\phi}, \widetilde{w}) = T(\widetilde{\phi}, \widetilde{w}', \widetilde{w}^3)$  is a solution of

$$\begin{cases} \nabla' \cdot \widetilde{w}' = F^0[\widetilde{w}'], \\ -\nu \Delta' \widetilde{w}' + \nabla' \widetilde{\phi} = G'[\widetilde{\phi}, \widetilde{w}'], \\ \widetilde{w}' |_{\partial D} = 0, \\ \langle \widetilde{\phi} \rangle = 0 \end{cases} \quad (4.143)$$

and

$$\begin{cases} -\nu \Delta' \widetilde{w}^3 = G^3[\widetilde{\phi}, \widetilde{w}'], \\ \widetilde{w}^3 |_{\partial D} = 0, \end{cases} \quad (4.144)$$

where  $F^0[\widetilde{w}']$ ,  $G'[\widetilde{\phi}, \widetilde{w}']$  and  $G^3[\widetilde{\phi}, \widetilde{w}']$  are defined as

$$\begin{aligned} F^0[\widetilde{w}'] &= \frac{1}{\gamma^2} \{ \lambda^{(1)} \phi^{(0)} - i v_s^3 \phi^{(0)} - i \gamma^2 \rho_s w^{(0),3} \} - \nabla' \cdot ((1 - \rho_s) \widetilde{w}'), \\ G'[\widetilde{\phi}, \widetilde{w}'] &= -i \widetilde{\nu} \nabla' w^{(0),3} + \widetilde{\nu} \nabla' F^0[\widetilde{w}'] + \nabla' ((1 - \rho_s) \widetilde{\phi}) \\ &\quad + (\nabla' \rho_s) \widetilde{\phi} + \rho_s \nabla' \left\{ \left( 1 - \frac{P'(\rho_s)}{\gamma^2 \rho_s} \right) \right\} \widetilde{\phi}, \\ G^3[\widetilde{\phi}, \widetilde{w}'] &= \rho_s \left\{ \lambda^{(1)} w^{(0),3} - i \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi^{(0)} - i v_s^3 w^{(0),3} \right\} - \rho_s \left\{ \frac{\nu}{\gamma^2 \rho_s^2} \Delta' v_s^3 \widetilde{\phi} + \widetilde{w}' \cdot \nabla' v_s^3 \right\}. \end{aligned}$$

As for the problem (4.143), since  $\lambda^{(1)} = -i \langle v_s^3 \phi^{(0)} + \gamma^2 \rho_s w^{(0),3} \rangle$ , it holds that  $\langle F^0[\widetilde{w}'] \rangle = 0$ . Furthermore, we have

$$\begin{aligned} |F^0[\widetilde{w}']|_2 &\leq C \left\{ \frac{1}{\gamma^2} (|\lambda^{(1)}| |\phi^{(0)}|_2 + |\phi^{(0)}|_2 + \gamma^2 |w^{(0),3}|_2) + \omega |\nabla' \widetilde{w}'|_2 \right\} \\ &\leq C \omega |\nabla' \widetilde{w}'|_2 + \mathcal{O}\left(\frac{1}{\gamma^2}\right), \end{aligned}$$

$$\begin{aligned}
|G'[\tilde{\phi}, \tilde{w}']|_{H^{-1}} &\leq C\{\tilde{\nu}|\nabla' w^{(0),3}|_{H^{-1}} + \tilde{\nu}|\nabla' F^0[\tilde{w}']|_{H^{-1}} + |\nabla'((1 - \rho_s)\tilde{\phi})|_{H^{-1}} \\
&\quad + |\nabla' \rho_s \tilde{\phi}|_{H^{-1}} + |\rho_s((1 - \frac{P'(\rho_s)}{\gamma^2 \rho_s})\tilde{\phi})|_{H^{-1}}\} \\
&\leq C\omega\{|\tilde{\phi}|_2 + \tilde{\nu}|\nabla' \tilde{w}'|_2\} + \mathcal{O}(\frac{\tilde{\nu}}{\gamma^2}).
\end{aligned}$$

Since  $(\tilde{\phi}, \tilde{w}') \in \dot{X} \equiv \{(p, v') \in L^2(D) \times H_0^1(D) : \langle p \rangle = 0\}$  and it is a solution of the Stokes system (4.143), we see from estimate for the Stokes system (see, e.g., [26]) that there holds the estimate

$$\begin{aligned}
|\tilde{\phi}|_2^2 + \nu^2 |\nabla' \tilde{w}'|_2^2 &\leq \nu^2 \{C\omega^2 |\tilde{w}'|_2^2 + \mathcal{O}(\frac{1}{\gamma^4})\} + \{C\omega^2 (|\tilde{\phi}|_2^2 + \tilde{\nu}^2 |\nabla' \tilde{w}'|_2^2) + \mathcal{O}(\frac{\tilde{\nu}^2}{\gamma^4})\} \\
&\leq C_1 \omega^2 \{|\tilde{\phi}|_2^2 + (\nu + \tilde{\nu})^2 |\nabla' \tilde{w}'|_2^2\} + \mathcal{O}(\frac{(\nu + \tilde{\nu})^2}{\gamma^4}).
\end{aligned}$$

Therefore, if  $\omega$  is so small that  $\omega^2 < \frac{1}{2C_1} \min\{1, (\frac{\nu}{\nu + \tilde{\nu}})^2\}$ , then

$$|\tilde{\phi}|_2^2 + \nu^2 |\nabla' \tilde{w}'|_2^2 = \mathcal{O}(\frac{(\nu + \tilde{\nu})^2}{\gamma^4}). \quad (4.145)$$

As for the problem (4.144), since

$$\begin{aligned}
|G^3[\tilde{\phi}, \tilde{w}']|_2 &\leq C\{\lambda^{(1)}|w^{(0),3}|_2 + \frac{1}{\gamma^2}|\phi^{(0)}|_2 + |w^{(0),3}|_2 + \frac{\nu}{\gamma^2}|\tilde{\phi}|_2 + |\tilde{w}'|_2\} \\
&\leq C\{\frac{\nu}{\gamma^2}|\tilde{\phi}|_2 + |\tilde{w}'|_2\} + \mathcal{O}(\frac{1}{\gamma^2}),
\end{aligned}$$

we have  $G^3[\tilde{\phi}, \tilde{w}'] \in L^2(D)$ . It then follows that

$$\tilde{w}^3 = \frac{1}{\nu}(-\Delta')^{-1}G^3[\tilde{\phi}, \tilde{w}'].$$

Since  $\phi^{(0)} = \alpha_0 \frac{\gamma^2 \rho_s}{P'(\rho_s)}$ , we see that

$$\begin{aligned}
\langle \rho_s \tilde{w}^3 \rangle &= \frac{1}{\nu} \langle \rho_s (-\Delta')^{-1} G^3[\tilde{\phi}, \tilde{w}'] \rangle \\
&= \frac{1}{\nu} \langle \rho_s (-\Delta')^{-1} (-i\alpha_0 \rho_s) \rangle \\
&\quad + \frac{1}{\nu} \langle \rho_s (-\Delta')^{-1} \{ \rho_s \lambda^{(1)} w^{(0),3} - \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s} \tilde{\phi} - \rho_s \tilde{w}' \cdot \nabla' v_s^3 - i \rho_s v_s^3 w^{(0),3} \} \rangle \\
&= -i \frac{\alpha_0}{\nu} |(-\Delta')^{-\frac{1}{2}} \rho_s|_2^2 \\
&\quad + \frac{1}{\nu} \langle \rho_s (-\Delta')^{-1} \{ \rho_s \lambda^{(1)} w^{(0),3} - \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s} \tilde{\phi} - \rho_s \tilde{w}' \cdot \nabla' v_s^3 - i \rho_s v_s^3 w^{(0),3} \} \rangle.
\end{aligned}$$

Furthermore, since  $\tilde{u} = {}^T(\tilde{\phi}, \tilde{w}') \in i\mathbf{R}^4$  and  $\lambda^{(1)} \in i\mathbf{R}$ , we see from (4.145) that

$$\begin{aligned}
&\langle \rho_s (-\Delta')^{-1} \{ \rho_s \lambda^{(1)} w^{(0),3} - \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s} \tilde{\phi} - \rho_s \tilde{w}' \cdot \nabla' v_s^3 - i \rho_s v_s^3 w^{(0),3} \} \rangle \\
&= i\mathcal{O}(\frac{1}{\gamma^2}) + i(\frac{\nu}{\gamma^2} + \frac{1}{\nu^2}) \times \mathcal{O}(\frac{\nu + \tilde{\nu}}{\gamma^2}).
\end{aligned}$$

It then follows that

$$\langle \rho_s \tilde{w}^3 \rangle = -i \frac{\alpha_0}{\nu} |(-\Delta')^{-\frac{1}{2}} \rho_s|_2^2 + i \frac{1}{\nu} \left\{ \mathcal{O}(\frac{1}{\gamma^2}) + (\frac{\nu}{\gamma^2} + \frac{1}{\nu^2}) \times \mathcal{O}(\frac{\nu + \tilde{\nu}}{\gamma^2}) \right\}.$$

By (4.145) we also have

$$\langle v_s^3 \tilde{\phi} \rangle + \lambda^{(1)} \langle \tilde{\phi} \rangle = i\mathcal{O}(\frac{\nu + \tilde{\nu}}{\gamma^2}).$$

We conclude that

$$\begin{aligned}
-\lambda^{(2)} &= \langle iv_s^3 \tilde{\phi} + i\gamma^2 \rho_s \tilde{w}^3 \rangle + \lambda^{(1)} \langle \tilde{\phi} \rangle \\
&= i\gamma^2 \left[ -i \frac{\alpha_0}{\nu} |(-\Delta')^{-\frac{1}{2}} \rho_s|_2^2 + i \frac{1}{\nu} \left\{ \mathcal{O}\left(\frac{1}{\gamma^2}\right) + \left(\frac{\nu}{\gamma^2} + \frac{1}{\nu^2}\right) \times \mathcal{O}\left(\frac{\nu+\tilde{\nu}}{\gamma^2}\right) \right\} \right] + i \cdot i \mathcal{O}\left(\frac{\nu+\tilde{\nu}}{\gamma^2}\right) \\
&= \frac{\gamma^2}{\nu} \left[ \alpha_0 |(-\Delta')^{-\frac{1}{2}} \rho_s|_2^2 + \left\{ \mathcal{O}\left(\frac{1}{\gamma^2}\right) + \left(\frac{1}{\nu^2} + \frac{\nu}{\gamma^2}\right) \times \mathcal{O}\left(\frac{\nu+\tilde{\nu}}{\gamma^2}\right) \right\} \right] \\
&> 0
\end{aligned}$$

for sufficiently small  $\frac{1}{\nu}$  and  $\frac{\nu+\tilde{\nu}}{\gamma^2}$ . We thus obtain the desired estimates. This completes the proof.  $\square$

We next establish some estimates related to  $\widehat{\Pi}(\xi)$  in  $H^k(D)$ . We first consider estimates for higher order derivatives of  $(\lambda + \widehat{L}_0)^{-1}f$ .

**Proposition 4.56.** *For any  $f = {}^T(f^0, g) \in H^k(D) \times H^{k-1}(D)$ . There exist positive constants  $\nu_1, \gamma_1, \omega_1$  and  $c_1$  such that if  $\nu \geq \nu_1, \frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_1^2, \omega \leq \omega_1$  and  $\lambda \in \Sigma_2 \equiv \{\lambda \neq 0 : |\lambda| \leq c_1\}$ , then  $(\lambda + \widehat{L}_0)^{-1}f \in H^k(D) \times (H^{k+1}(D) \cap H_0^1(D))$  for  $k = 0, 1, \dots, k_0$ . Furthermore, the following estimate holds:*

$$|(\lambda + \widehat{L}_0)^{-1}f|_{H^k \times H^{k+1}} \leq C(1 + \frac{1}{|\lambda|})|f|_{H^k \times H^{k-1}},$$

where  $C$  is a positive constant independent of  $\lambda \in \Sigma_2$ . The same assertions also hold for  $-\widehat{L}_0^*$ .

**Proof.** For a given  $f = {}^T(f^0, g) \in H^k(D) \times H^{k-1}(D)$ , we consider the problem

$$\begin{cases} (\lambda + \widehat{\mathcal{L}}_0)U = f, \\ W|_{\partial D} = 0 \end{cases} \quad (4.146)$$

for  $U = {}^T(\Phi, W)$ . Here  $\widehat{\mathcal{L}}_0$  is differential operator given by

$$\widehat{\mathcal{L}}_0 U = \begin{pmatrix} \gamma^2 \nabla' \cdot (\rho_s W') \\ -\frac{\nu}{\rho_s} \Delta' W' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot W' + \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \Phi \right) \\ -\frac{\nu}{\rho_s} \Delta' W^3 + \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \Phi + W' \cdot \nabla' v_s^3 \end{pmatrix}$$

for  $U = {}^T(\Phi, W)$ . To solve the problem (4.146), we decompose  $\Phi$  and  $f^0$  as

$$\Phi = \Phi_1 + \sigma, \quad f^0 = f_1^0 + \langle f^0 \rangle,$$

where  $\sigma = \langle \Phi \rangle$ ,  $\Phi_1 = \Phi - \sigma$  and  $f_1^0 = f^0 - \langle f^0 \rangle$ . Note that

$$\langle \Phi_1 \rangle = 0, \quad \langle f_1^0 \rangle = 0.$$

Then (4.146) is equivalent to the problem

$$\lambda \sigma = \langle f^0 \rangle, \quad (4.147)$$

$$\lambda\Phi_1 + \gamma^2\nabla' \cdot (\rho_s W') = f_1^0, \quad (4.148)$$

$$\lambda W' - \frac{\nu}{\rho_s} \Delta' W' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot W' + \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} (\sigma + \Phi_1) \right) = g', \quad (4.149)$$

$$\lambda W^3 - \frac{\nu}{\rho_s} \Delta' W^3 + \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} (\sigma + \Phi_1) - W' \cdot \nabla' v_s^3 = g^3 \quad (4.150)$$

with  $W|_{\partial D} = 0$ . If  $\lambda \neq 0$ , then we find from (4.147) that

$$\sigma = \frac{1}{\lambda} \langle f^0 \rangle. \quad (4.151)$$

Substituting  $\sigma = \frac{1}{\lambda} \langle f^0 \rangle$  into (4.149) and (4.150), we obtain

$$\begin{cases} \lambda\Phi_1 + \gamma^2\nabla' \cdot (\rho_s W') = f_1^0, \\ \lambda W' - \frac{\nu}{\rho_s} \Delta' W' - \frac{\tilde{\nu}}{\rho_s} \nabla' \nabla' \cdot W' + \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \Phi_1 \right) = g' - \frac{1}{\lambda} \langle f^0 \rangle \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \right), \\ \lambda W^3 - \frac{\nu}{\rho_s} \Delta' W^3 + \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \Phi_1 - W' \cdot \nabla' v_s^3 = g^3 - \frac{1}{\lambda} \langle f^0 \rangle \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \end{cases} \quad (4.152)$$

with  $W|_{\partial D} = 0$ . Let us write the problem (4.152) as

$$\begin{cases} \nabla' \cdot W' = F^0[\Phi_1, W' : f_1^0], \\ -\nu \Delta' W' + \nabla' \Phi_1 = G'[\Phi_1, W' : f^0, g'], \\ W' |_{\partial D} = 0 \end{cases} \quad (4.153)$$

and

$$\begin{cases} -\nu \Delta' W^3 = G^3[\Phi_1, W', W^3 : f^0, g^3], \\ W^3 |_{\partial D} = 0. \end{cases} \quad (4.154)$$

Here

$$\begin{aligned} F^0[\Phi_1, W' : f_1^0] &= -\frac{1}{\gamma^2} \lambda \Phi_1 + \nabla' \cdot ((1 - \rho_s) W') + \frac{1}{\gamma^2} f_1^0, \\ G'[\Phi_1, W' : f^0, g'] &= -\lambda \rho_s W' + \tilde{\nu} \nabla' F^0[\Phi_1, W' : f_1^0] + \nabla' ((1 - \rho_s) \Phi_1) + \nabla' \rho_s \Phi_1 \\ &\quad - \frac{1}{\lambda} \langle f^0 \rangle \rho_s \nabla' \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \right) + \rho_s \nabla' \left( \left( 1 - \frac{P'(\rho_s)}{\gamma^2 \rho_s} \right) \Phi_1 \right) + \rho_s g', \\ G^3[\Phi_1, W', W^3 : f^0, g^3] &= -\lambda \rho_s W^3 - \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \frac{1}{\lambda} \langle f^0 \rangle - \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s^2} \Phi_1 - \rho_s W' \cdot \nabla' v_s^3 + \rho_s g^3. \end{aligned}$$

We now define a set  $\dot{X}_k$  by

$$\dot{X}_k = \{(p, v') \in H^k(D) \times (H^{k+1}(D) \cap H_0^1(D)) : \langle p \rangle = 0\}$$

with norm

$$|(p, v')|_{\dot{X}_k} = |p|_{H^k} + \nu |v'|_{H^{k+1}}.$$

For a given  $(\tilde{\Phi}_1, \tilde{W}') \in \dot{X}_k$ , we consider the problem

$$\begin{cases} \nabla' \cdot W' = F^0[\tilde{\Phi}_1, \tilde{W}' : f_1^0], \\ -\nu \Delta' W' + \nabla' \Phi_1 = G'[\tilde{\Phi}_1, \tilde{W}' : f^0, g'], \\ W' |_{\partial D} = 0. \end{cases} \quad (4.155)$$

It holds that

$$\langle F^0[\tilde{\Phi}_1, \tilde{W}' : f_1^0] \rangle = 0, \quad F^0[\tilde{\Phi}_1, \tilde{W}' : f_1^0] \in H^k(D),$$

$$G'[\tilde{\Phi}_1, \tilde{W}' : f^0, g'] \in H^{k-1}(D).$$

In fact, we see that

$$\langle F^0[\tilde{\Phi}_1, \tilde{W}' : f_1^0] \rangle = -\frac{1}{\gamma^2} \lambda \langle \tilde{\Phi}_1 \rangle + \langle \nabla' \cdot ((1 - \rho_s) \tilde{W}') \rangle + \frac{1}{\gamma^2} \langle f_1^0 \rangle = 0,$$

$$|F^0[\tilde{\Phi}_1, \tilde{W}' : f_1^0]|_{H^k} \leq C \left\{ \frac{1}{\gamma^2} |\lambda| |\tilde{\Phi}_1|_{H^k} + \omega |\tilde{W}'|_{H^{k+1}} + \frac{1}{\gamma^2} |f_1^0|_{H^k} \right\}$$

and

$$\begin{aligned} & |G'[\tilde{\Phi}_1, \tilde{W}' : f^0, g']|_{H^{k-1}} \\ & \leq C \left\{ |\lambda| |\tilde{W}'|_{H^{k-1}} + \tilde{\nu} |F^0[\tilde{\Phi}_1, \tilde{W}' : f_1^0]|_{H^k} + \omega |\tilde{\Phi}_1|_{H^k} + \frac{1}{|\lambda|} |\langle f^0 \rangle| + |g'|_{H^{k-1}} \right\} \\ & \leq C \left\{ \left( \frac{\tilde{\nu}}{\gamma^2} |\lambda| + \omega \right) |\tilde{\Phi}_1|_{H^k} + \nu \left( \frac{1}{\nu} |\lambda| + \frac{\tilde{\nu}}{\nu} \omega \right) |\tilde{W}'|_{H^{k+1}} + \left( \frac{\tilde{\nu}}{\gamma^2} + \frac{1}{|\lambda|} \right) |f^0|_{H^k} + |g'|_{H^{k-1}} \right\} \end{aligned}$$

for a positive constant  $C$  independent of  $\lambda$ . From [26], we see that there is a unique solution  $(\Phi_1, W') \in \dot{X}_k$  of (4.155) and there holds the estimate

$$\begin{aligned} & |\Phi|_{H^k} + \nu |W'|_{H^{k+1}} \\ & \leq C \left\{ \nu |F^0[\tilde{\Phi}_1, \tilde{W}' : f_1^0]|_{H^k} + |G'[\tilde{\Phi}_1, \tilde{W}' : f^0, g']|_{H^{k-1}} \right\} \\ & \leq C \left\{ \left( \frac{\nu + \tilde{\nu}}{\gamma^2} |\lambda| + \omega \right) |\tilde{\Phi}_1|_{H^k} + \nu \left( \frac{1}{\nu} |\lambda| + \frac{\nu + \tilde{\nu}}{\nu} \omega \right) |\tilde{W}'|_{H^{k+1}} \right. \\ & \quad \left. + \left( \frac{\nu + \tilde{\nu}}{\gamma^2} + \frac{1}{|\lambda|} \right) |f^0|_{H^k} + |g'|_{H^{k-1}} \right\} \end{aligned} \tag{4.156}$$

for a positive constant  $C$  independent of  $\lambda$ . Let us define a map  $\Gamma_1 : \dot{X}_k \rightarrow \dot{X}_k$  such that

$$\Gamma_1(\tilde{\Phi}_1, \tilde{W}') = (\Phi_1, W'),$$

where  $(\Phi_1, W') \in \dot{X}_k$  is a solution of (4.155). From (4.156), for  $(\tilde{\Phi}_{1,1}, \tilde{W}'_1), (\tilde{\Phi}_{1,2}, \tilde{W}'_2) \in \dot{X}_k$ , the estimate

$$\begin{aligned} & |\Gamma_1(\tilde{\Phi}_{1,1}, \tilde{W}'_1) - \Gamma_1(\tilde{\Phi}_{1,2}, \tilde{W}'_2)|_{H^k \times H^{k+1}} \\ & \leq C_1 \left\{ \left( \frac{\nu + \tilde{\nu}}{\gamma^2} + \frac{1}{\nu} \right) |\lambda| + \left( \frac{\nu + \tilde{\nu}}{\nu} + 1 \right) \omega \right\} |(\tilde{\Phi}_{1,1} - \tilde{\Phi}_{1,2}, \tilde{W}'_1 - \tilde{W}'_2)|_{\dot{X}_k} \end{aligned}$$

holds for a positive constant  $C_1$  independent of  $\lambda$ . If  $\omega$  and  $|\lambda|$  are so small that  $\omega < \frac{1}{2C_1} \frac{\nu}{\nu + \tilde{\nu}}$  and  $|\lambda| < \frac{1}{2C_1}$ , then  $\Gamma_1 : \dot{X}_k \rightarrow \dot{X}_k$  is a contraction map. This implies that there is a unique  $(\Phi_1, W') \in \dot{X}_k$  such that  $\Gamma_1(\Phi_1, W') = (\Phi_1, W')$ , i.e., there is a unique solution  $(\Phi_1, W') \in \dot{X}_k$  of (4.153). Furthermore, from (4.156),  $(\Phi_1, W')$  satisfies the estimate

$$|\Phi_1|_{H^k} + |W'|_{H^{k+1}} \leq C \left\{ \left( 1 + \frac{1}{|\lambda|} \right) |f^0|_{H^k} + |g'|_{H^{k-1}} \right\}, \tag{4.157}$$

where  $C$  is a positive constant independent of  $\lambda$ .

As for (4.154), for a given  $\widetilde{W}^3 \in H^{k+1}(D) \cap H_0^1(D)$ , we consider the problem

$$\begin{cases} -\nu \Delta' W^3 = G^3[\Phi_1, W', \widetilde{W}^3 : f^0, g^3], \\ W^3|_{\partial D} = 0, \end{cases} \quad (4.158)$$

where  $(\Phi_1, W') \in \dot{X}_k$  is a solution of (4.153). It holds that

$$G^3[\Phi_1, W', \widetilde{W}^3 : f^0, g^3] \in H^{k-1}(D).$$

In fact, we have

$$\begin{aligned} & |G^3[\Phi_1, W', \widetilde{W}^3 : f^0, g^3]|_{H^{k-1}} \\ & \leq C \{ |\lambda| |\widetilde{W}^3|_{H^{k-1}} + |\Phi_1|_{H^{k-1}} + |W'|_{H^{k-1}} + |g^3|_{H^{k-1}} + \frac{1}{|\lambda|} |\langle f^0 \rangle| \} \\ & \leq C_2 \{ |\lambda| |\widetilde{W}^3|_{H^{k-1}} + (1 + \frac{1}{|\lambda|}) |f^0|_{H^k} + |g|_{H^{k-1}} \} \end{aligned} \quad (4.159)$$

for a positive constant  $C_2$  independent of  $\lambda$ . If  $|\lambda|$  is sufficiently small satisfying  $|\lambda| < \min\{\frac{1}{2C_1}, \frac{1}{C_2}\}$ , then there is a unique solution  $W^3 \in H^{k+1}(D) \cap H_0^1(D)$  of (4.154). Furthermore, from (4.159),  $W^3$  satisfies the estimate

$$|W^3|_{H^{k+1}} \leq C \{ (1 + \frac{1}{|\lambda|}) |f^0|_{H^k} + |g|_{H^{k-1}} \}, \quad (4.160)$$

where  $C$  is a positive constant independent of  $\lambda$ .

Now we set

$$\Sigma_2 \equiv \{ \lambda \neq 0 : |\lambda| < \min\{ \frac{1}{2C_1}, \frac{1}{C_2} \} \}.$$

Since  $\Phi = \sigma + \Phi_1$ , we see that if  $\omega < \frac{1}{2C_1} \frac{\nu}{\nu+\bar{\nu}}$  and  $\lambda \in \Sigma_2$ , then there is a unique solution  $(\Phi, W) \in H^k(D) \times (H^{k+1}(D) \cap H_0^1(D))$  of (4.146). Moreover, from (4.151), (4.157) and (4.160),  $\Phi$  and  $W$  satisfies the estimate

$$\begin{aligned} |\Phi|_{H^k} + |W|_{H^{k+1}} & \leq |\sigma| + |\Phi_1|_{H^k} + |W'|_{H^{k+1}} + |W^3|_{H^{k+1}} \\ & \leq C \{ (1 + \frac{1}{|\lambda|}) |f^0|_{H^k} + |g|_{H^{k-1}} \} \end{aligned}$$

for a positive constant  $C$  independent of  $\lambda \in \Sigma_2$ .

Since  $D(\widehat{L}_0) \supset H^k(D) \times (H^{k+1}(D) \cap H_0^1(D))$ , we can replace  $\widehat{\mathcal{L}}_0$  with  $\widehat{L}_0$ ; and we find that if  $\omega < \frac{1}{2C_1} \frac{\nu}{\nu+\bar{\nu}}$  and  $\lambda \in \Sigma_2$ , then  $(\lambda + \widehat{L}_0)^{-1} f \in H^k(D) \times (H^{k+1}(D) \cap H_0^1(D))$ . Furthermore,  $(\lambda + \widehat{L}_0)^{-1} f$  satisfies the estimate

$$|(\lambda + \widehat{L}_0)^{-1} f|_{H^k \times H^{k+1}} \leq C \{ (1 + \frac{1}{|\lambda|}) |f^0|_{H^k} + |g|_{H^{k-1}} \},$$

where  $C$  is a positive constant independent of  $\lambda \in \Sigma_2$ . We thus obtain the desired estimates. The assertions for  $\widehat{L}_0^*$  can be proved in a similar manner. This completes the proof.  $\square$

We finally obtain the following estimates for the eigenfunctions  $u_\xi$  and  $u_\xi^*$  associated with  $\lambda_0(\xi)$  and  $\bar{\lambda}_0(\xi)$ , respectively, which yields the boundedness of  $\widehat{\Pi}(\xi)$  on  $H^k(D)$ .

**Theorem 4.57.** *There exist positive constants  $\nu_1$ ,  $\gamma_1$  and  $\omega_1$  such that if  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{\nu+\bar{\nu}} \geq \gamma_1^2$  and  $\omega \leq \omega_1$ , then there exists a positive constant  $r_0$  such that for any  $\xi \in \mathbf{R}$  with  $|\xi| \leq r_0$  the following assertions hold. There exist  $u_\xi$  and  $u_\xi^*$  eigenfunctions associated with  $\lambda_0(\xi)$  and  $\bar{\lambda}_0(\xi)$ , respectively, that satisfy*

$$\langle u_\xi, u_\xi^* \rangle = 1,$$

*and the eigenprojection  $\widehat{\Pi}(\xi)$  for  $\lambda_0(\xi)$  is given by*

$$\widehat{\Pi}(\xi)u = \langle u, u_\xi^* \rangle u_\xi.$$

*Furthermore,  $u_\xi$  and  $u_\xi^*$  are written in the form*

$$\begin{aligned} u_\xi(x') &= u^{(0)}(x') + i\xi u^{(1)}(x') + |\xi|^2 u^{(2)}(x', \xi), \\ u_\xi^*(x') &= u^{*(0)}(x') + i\xi u^{*(1)}(x') + |\xi|^2 u^{*(2)}(x', \xi), \end{aligned}$$

*and the following estimates hold*

$$|u|_{H^{k+2}} \leq C_{k,r_0}$$

*for  $u \in \{u_\xi, u_\xi^*, u^{(1)}, u^{*(1)}, u^{(2)}, u^{*(2)}\}$  and  $k = 0, 1, \dots, k_0$ : and a positive constant  $C_{k,r_0}$  depending on  $k$  and  $r_0$ .*

We can prove Theorem 4.57 by using Proposition 4.56, similarly to the proof of [12, Lemma 4.3]. We thus omit the proof.

## 5 Nonlinear problem

In this section we treat the nonlinear problem (1.5)-(1.8). This problem is written as

$$\frac{du}{dt} + Lu = \mathbf{F}, \quad w|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0. \quad (5.1)$$

Here  $u = {}^T(\phi, w)$ ;  $\mathbf{F} = \mathbf{F}(u)$  denotes the nonlinearity:

$$\mathbf{F} = {}^T(f^0(\phi, w), f(\phi, w)).$$

Our aim in this section is to establish the a priori estimates of  $u(t)$  for the proof of Theorem 3.1.

In what follows we set

$$\omega = \|\rho_s - 1\|_{C^3}.$$

## 5.1 Decomposition of Problem

In this section we formulate the problem.

The local solvability in  $Z(T)$  for (5.28) follows from [13].

**Proposition 5.1.** *If  $u_0 = {}^T(\phi_0, w_0)$  satisfies the following conditions;*

- (i)  $u_0 \in H^2 \times (H^2 \cap H_0^1)$ ,
- (ii)  $-\frac{\gamma^2}{4}\rho_1 \leq \phi_0$ ,

*then there exists a number  $T_0 > 0$  depending on  $\|u_0\|_{H^2}$  and  $\rho_1$  such that the following assertions hold. Problem (5.28) has a unique solution  $u(t) \in Z(T)$  satisfying*

$$\phi(x, t) \geq -\frac{\gamma^2}{2}\rho_1 \quad \text{for } \forall (x, t) \in \Omega \times [0, T_0];$$

*and the following estimate holds*

$$\|u\|_{Z(T)}^2 \leq C_0 \{1 + \|u_0\|_{H^2}^2\}^\alpha \|u_0\|_{H^2}^2$$

*for some positive constants  $C_0$  and  $\alpha$ .*

Theorem 3.1 would follow if we would establish the a priori estimates of  $u(t)$  in  $Z(T)$  uniformly for  $T$ .

To obtain the appropriate a priori estimates, we decompose the solution  $u$  into its  $P_0$  and  $P_\infty$  parts. Let us decompose the solution  $u(t)$  of (5.28) as

$$u(t) = (\sigma_1 u^{(0)})(t) + u_1(t) + u_\infty(t),$$

where

$$\sigma_1(t) = \mathcal{P}u(t), \quad u_1(t) = (\mathcal{T} - \mathcal{T}^{(0)})\mathcal{P}u(t), \quad u_\infty(t) = P_\infty u(t).$$

Note that  $P_0 u(t) = (\sigma_1 u^{(0)})(t) + u_1(t)$ .

Since  $u_1(t)$  is written as

$$u_1(t) = (\mathcal{T} - \mathcal{T}^{(0)})\mathcal{P}u(t) = (\partial_{x_3} \mathcal{T}^{(1)} + \partial_{x_3}^2 \mathcal{T}^{(2)})\sigma_1(t),$$

we see from Proposition 4.40 and Proposition 4.41 the following estimates for  $\sigma_1(t)$  and  $u_1(t)$ .

**Proposition 5.2.** *Let  $u(t)$  be a solution of (5.28) in  $Z(T)$ . Then there hold the estimates*

$$\|\partial_{x_3}^l \sigma_1(t)\|_2 \leq C \|\partial_{x_3} \sigma_1(t)\|_2$$

*for  $1 \leq l \leq 3$ ; and*

$$\|\partial_x^k \partial_{x_3}^l \partial_t^m u_1(t)\|_2 \leq C \{\|\partial_{x_3} \sigma_1(t)\|_2 + \|\partial_t \sigma_1(t)\|_2\}$$

*for  $1 \leq k + l + 2m \leq 3$ .*

We derive the equations for  $\sigma_1(t)$  and  $u_\infty(t)$ .

**Proposition 5.3.** *Let  $T > 0$  and assume that  $u(t)$  is a solution of (5.28) in  $Z(T)$ . Then the following assertions hold.*

$$\sigma_1 \in \bigcap_{j=0}^1 C^j([0, T] : H^2(\mathbf{R})), \quad u_\infty \in Z(T), \quad \phi_\infty \in C^1([0, T]; H^1).$$

Furthermore,  $\sigma_1$  and  $u_\infty$  satisfy

$$\sigma_1(t) = e^{t\Lambda} \mathcal{P}u_0 + \int_0^T e^{(t-\tau)\Lambda} \mathcal{P}\mathbf{F}(\tau) d\tau; \quad (5.2)$$

and

$$\partial_t u_\infty + Lu_\infty = \mathbf{F}_\infty, \quad w_\infty|_{\partial\Omega} = 0, \quad u_\infty|_{t=0} = u_{\infty,0}, \quad (5.3)$$

where  $\mathbf{F}_\infty = P_\infty \mathbf{F}$  and  $u_{\infty,0} = P_\infty u_0$ .

Let  $u(t)$  be a solution of (5.28) in  $Z(T)$ . From Proposition 5.24, we obtain

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{4}} \{ \|u_1(\tau)\|_2 + \|\partial_x u_1(\tau)\|_2 \} \\ & \leq C \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{4}} \{ \|\partial_{x_3} \sigma_1(\tau)\|_2 + \|\partial_\tau \sigma_1(\tau)\|_2 \}, \end{aligned}$$

and thus, the estimates for  $u_1(t)$  follows from the ones for  $\sigma_1(t)$ . Therefore, as in [3], we introduce the quantity  $M_1(t)$  defined by

$$M_1(t) = \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{1}{4}} \|\sigma_1(\tau)\|_2 + \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{4}} \{ \|\partial_{x_3} \sigma_1(\tau)\|_2 + \|\partial_\tau \sigma_1(\tau)\|_2 \};$$

and we define the quantity  $M(t) \geq 0$  by

$$M(t)^2 = M_1(t)^2 + \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{2}} E_\infty(\tau) \quad (t \in [0, T])$$

with

$$E_\infty(t) = \|u_\infty(t)\|_2^2.$$

We define a quantity  $D_\infty(t)$  for  $u_\infty = {}^T(\phi_\infty, w_\infty)$  by

$$D_\infty(t) = \|D\phi_\infty(t)\|_1^2 + \|Dw_\infty(t)\|_2^2.$$

If we could show  $M(t) \leq C$  uniformly for  $t \geq 0$ , then Theorem 3.1 would follow. The uniform estimate for  $M(t)$  is given by using the following estimates for  $M_1(t)$  and  $E_\infty(t)$ .

**Proposition 5.4.** *There exist positive constants  $\nu_0$ ,  $\gamma_0$  and  $\omega_0$  such that if  $\nu \geq \nu_0$ ,  $\frac{\gamma^2}{\nu + \bar{\nu}} \geq \gamma_0^2$  and  $\omega \leq \omega_0$ , then the following assertions hold. There is a positive number  $\epsilon_1$  such that if a solution  $u(t)$  of (5.28) in  $Z(T)$  satisfies  $\sup_{0 \leq \tau \leq t} \|u(\tau)\|_2 \leq \epsilon_1$  and*

*$M(t) \leq 1$  for  $t \in [0, T]$ , then the estimates*

$$M_1(t) \leq C \{ \|u_0\|_{L^1} + M(t)^2 \} \quad (5.4)$$

and

$$\begin{aligned} E_\infty(t) + \int_0^\infty e^{-a(t-\tau)} D_\infty(\tau) d\tau \\ \leq C\{e^{-at} E_\infty(0) + (1+t)^{-\frac{3}{2}} M(t)^4 + \int_0^t e^{-a(t-\tau)} \mathcal{R}(\tau) d\tau\} \end{aligned} \quad (5.5)$$

hold uniformly for  $t \in [0, T]$  with  $C > 0$  independent of  $T$ . Here  $a = a(\nu, \tilde{\nu}, \gamma)$  is a positive constant; and  $\mathcal{R}(t)$  is a function satisfying the estimate

$$\mathcal{R}(t) \leq C\{(1+t)^{-\frac{3}{2}} M(t)^3 + (1+t)^{-\frac{1}{4}} M(t) D_\infty(t)\} \quad (5.6)$$

provided that  $\sup_{0 \leq \tau \leq t} \|u(\tau)\|_2 \leq \epsilon_2$  and  $M(t) \leq 1$ .

Proposition 5.26 follows from Propositions 5.5, 5.8 and 5.14 below.

As in [3, 12], one can see from Propositions 5.23 and 5.26 that if  $\|u_0\|_{H^2 \cap L^1}$  is sufficiently small, then

$$M(t) \leq C\|u_0\|_{H^2 \cap L^1}$$

uniformly for  $t \geq 0$ , which proves Theorem 3.1.

## 5.2 Estimates for $P_0$ -part of $u(t)$

In this section, we estimate the  $P_0$ -part of  $u(t)$

$$P_0 u(t) = (\sigma_1 u^{(0)})(t) + u_1(t),$$

where  $\sigma_1(t) = \mathcal{P}u(t)$  and  $u_1(t) = (\mathcal{T} - \mathcal{T}^{(0)})\mathcal{P}u(t)$ . We will prove the following estimate for  $M_1(t)$ .

**Proposition 5.5.** *Let  $T > 0$  and assume that  $\nu \geq \nu_1$ ,  $\frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_1^2$  and  $\omega \leq \omega_1$ . Then there exists a positive constant  $\epsilon$  independent of  $T$  such that if a solution  $u(t)$  of (5.28) in  $Z(T)$  satisfies  $\sup_{0 \leq \tau \leq t} \|u(\tau)\|_2 \leq \epsilon$  and  $M(t) \leq 1$  for all  $t \in [0, T]$ , then the estimate*

$$M_1(t) \leq C\{\|u_0\|_1 + M(t)^2\}$$

*holds uniformly for  $t \in [0, T]$ , where  $C$  is a positive constant independent of  $T$ .*

Let us prove Proposition 5.5. We decompose the nonlinearity  $\mathbf{F}$  into

$$\mathbf{F} = \sigma_1^2 \mathbf{F}_1 + \mathbf{F}_2,$$

where

$$\begin{aligned} \mathbf{F}_1 &= \mathbf{F}_1(x') = -^T \left( 0, \frac{1}{2\gamma^4 \rho_s(x')} \nabla' \{P''(\rho_s(x')) (\phi^{(0)}(x'))^2\}, 0 \right), \\ \mathbf{F}_2 &= \mathbf{F} - \sigma_1^2 \mathbf{F}_1. \end{aligned}$$

Here  $\sigma_1^2 \mathbf{F}_1(x')$  is the part of  $\mathbf{F}$  containing only  $\sigma_1^2(t)$  but not  $\partial_{x_3} \sigma_1(t)$ ,  $u_1(t)$ ,  $u_\infty(t)$ ,  $\sigma_1^3(t)$  and so on.

Before going further, we introduce a notation. For a function  $g$  we define  $\langle g \rangle_0$  by

$$\langle g \rangle_0 = \mathcal{F}^{-1}[\mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) \langle \widehat{g} \rangle].$$

The nonlinearity  $\mathbf{F}$  has the following properties.

**Lemma 5.6.** *There hold the following assertions.*

- (i)  $\langle Q_0 \mathbf{F} \rangle = -\partial_{x_3} \langle \phi w^3 \rangle.$
- (ii)  $\mathcal{P} \mathbf{F} = -\partial_{x_3} \langle \phi w^3 \rangle_0 + \partial_{x_3} \mathcal{P}^{(1)} \mathbf{F} + \partial_{x_3}^2 \mathcal{P}^{(2)} \mathbf{F}.$

**Proof.** As for (i), we see from integration by parts that  $\langle \nabla' \cdot (\phi w') \rangle = 0$ . It then follows that

$$\langle Q_0 \mathbf{F} \rangle = -\langle \operatorname{div}(\phi w) \rangle = -\langle \partial_{x_3}(\phi w^3) \rangle = -\partial_{x_3} \langle \phi w^3 \rangle.$$

We next prove (ii). From the definition of  $\mathcal{P}^{(0)}$  and (i), there holds that

$$\mathcal{P}^{(0)} \mathbf{F} = \mathcal{F}^{-1}[\mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) \langle Q_0 \widehat{\mathbf{F}} \rangle] = \langle Q_0 \mathbf{F} \rangle_0 = -\partial_{x_3} \langle \phi w^3 \rangle_0.$$

We thus obtain (ii). This completes the proof.  $\square$

Noting that  $\|\sigma_1\|_\infty \leq C \|\sigma_1\|_2^{\frac{1}{2}} \|\partial_{x_3} \sigma_1\|_2^{\frac{1}{2}}$ , one can obtain the following estimates by straightforward computations.

**Lemma 5.7.** *There exists a positive constant  $\epsilon$  such that if a solution  $u(t)$  of (5.28) in  $Z(T)$  satisfies  $\sup_{0 \leq \tau \leq t} \|u(\tau)\|_2 \leq \epsilon$  and  $M(t) \leq 1$  for all  $t \in [0, T]$ , then the following estimates hold for  $t \in [0, T]$  with a positive constant  $C$  independent of  $T$ .*

- (i)  $\|\partial_{x_3}(\sigma_1^2(t))\|_1 \leq C(1+t)^{-1} M(t)^2.$
- (ii)  $\|\partial_{x_3} \langle \phi w^3 \rangle(t)\|_1 \leq C(1+t)^{-1} M(t)^2.$
- (iii)  $\|\langle \phi w^3 \rangle(t)\|_1 \leq C(1+t)^{-\frac{1}{2}} M(t)^2.$
- (iv)  $\|\mathbf{F}(t)\|_1 \leq C(1+t)^{-\frac{1}{2}} M(t)^2.$
- (v)  $\|\mathbf{F}_2(t)\|_1 \leq C(1+t)^{-1} M(t)^2.$
- (vi)  $\|\mathbf{F}(t)\|_2 \leq C(1+t)^{-\frac{3}{4}} M(t)^2.$

**Proof of Proposition 5.5** We see from Proposition 4.42 that

$$\|\partial_{x_3}^l e^{t\Lambda} \mathcal{P} u_0\|_2 \leq C(1+t)^{-\frac{1}{4}-\frac{l}{2}} \|u_0\|_1 \quad (l = 0, 1).$$

We next consider  $\int_0^t e^{(t-\tau)\Lambda} \mathcal{P} \mathbf{F}(\tau) d\tau$ . We write it as

$$\int_0^t e^{(t-\tau)\Lambda} \mathcal{P} \mathbf{F}(\tau) d\tau = \left( \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) e^{(t-\tau)\Lambda} \mathcal{P} \mathbf{F}(\tau) d\tau =: I_1(t) + I_2(t).$$

We see from Lemma 5.6 (ii) that

$$\begin{aligned} e^{(t-\tau)\Lambda}\mathcal{P}\mathbf{F}(\tau) &= e^{(t-\tau)\Lambda}\{-\partial_{x_3}\langle\phi w^3\rangle_0 + \partial_{x_3}\mathcal{P}^{(1)}\mathbf{F} + \partial_{x_3}^2\mathcal{P}^{(2)}\mathbf{F}\} \\ &= \partial_{x_3}e^{(t-\tau)\Lambda}\{-\langle\phi w^3\rangle_0 + \mathcal{P}^{(1)}\mathbf{F} + \partial_{x_3}\mathcal{P}^{(2)}\mathbf{F}\}. \end{aligned}$$

By Proposition 4.42 and Lemma 5.7 we then have

$$\begin{aligned} \|\partial_{x_3}^l I_1(t)\|_2 &\leq C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-\frac{l}{2}} (\|\langle\phi w^3\rangle_0(\tau)\|_1 + \|\mathbf{F}(\tau)\|_1) d\tau \\ &\leq C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-\frac{l}{2}} (1+\tau)^{-\frac{1}{2}} d\tau M(t)^2 \\ &\leq C(1+t)^{-\frac{1}{4}-\frac{l}{2}} M(t)^2 \end{aligned}$$

for  $l = 0, 1$ . Applying Lemma 5.7 (ii) and (v) we have

$$\begin{aligned} \|\partial_{x_3}^l I_2(t)\|_2 &\leq C \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{1}{4}-\frac{l}{2}} (1+\tau)^{-1} d\tau M(t)^2 \\ &\leq C(1+t)^{-\frac{1}{4}-\frac{l}{2}} M(t)^2 \end{aligned}$$

for  $l = 0, 1$ . We thus obtain

$$\|\partial_{x_3}^l \sigma_1(t)\|_2 \leq C(1+t)^{-\frac{1}{4}-\frac{l}{2}} \{\|u_0\|_1 + M(t)^2\} \quad (5.7)$$

for  $l = 0, 1$ .

Let us estimate the time derivative. Since  $\lambda_0(\xi) = -(i\kappa_1\xi + \kappa_0\xi^2 + \mathcal{O}(\xi^3)) = \mathcal{O}(\xi)$ , we obtain

$$\|\Lambda\sigma_1(t)\|_2 = \|\mathcal{F}^{-1}[\mathbf{1}_{\{|\eta|\leq r_0\}}(\xi)\lambda_0(\xi)\widehat{\sigma}_1(t)]\|_2 \leq C\|\partial_{x_3}\sigma_1(t)\|_2.$$

This, together with (5.29), (5.7) and Lemma 5.7, implies that

$$\|\partial_t\sigma_1(t)\|_2 \leq C\{\|\partial_{x_3}\sigma_1(t)\|_2 + \|\mathbf{F}(t)\|_2\} \leq C(1+t)^{-\frac{3}{4}}\{\|u_0\|_1 + M(t)^2\}. \quad (5.8)$$

By (5.7) and (5.8) we deduce the desired estimate. This completes the proof.  $\square$

### 5.3 Estimates for $P_\infty$ -part of $u(t)$

In this section we derive the estimates for the  $P_\infty$ -part of  $u(t)$ .

Throughout this section, we assume that  $u(t)$  is a solution of (5.28) in  $Z(T)$  for a given  $T > 0$ . We show the following estimate.

**Proposition 5.8.** *There exist positive constants  $\nu_0 (\geq \nu_1)$ ,  $\gamma_0 (\geq \gamma_1)$  and  $\omega_0 (\leq \omega_1)$  such that if  $\nu \geq \nu_0$ ,  $\frac{\gamma^2}{\nu+\widehat{\nu}} \geq \gamma_0^2$  and  $\omega \leq \omega_0$ , then*

$$\begin{aligned} E_\infty(t) + \int_0^t e^{-a(t-\tau)} D_\infty(\tau) d\tau \\ \leq C\{e^{-at}E_\infty(0) + (1+t)^{-\frac{3}{2}}M(t)^4 + \int_0^t e^{-a(t-\tau)}\mathcal{R}(\tau)d\tau\}. \end{aligned}$$

uniformly for  $t \in [0, T]$  with  $C > 0$  independent of  $T$ .

Proposition 5.8 is proved by the estimate (4.6) for  $e^{-tL}P_\infty$  and the Matsumura-Nishida energy method.

We introduce notations. In what follows  $C$  and  $C_j$  ( $j = 1, 2, \dots$ ) denote various constants independent of  $T$ ,  $\nu$ ,  $\tilde{\nu}$  and  $\gamma$ , whereas,  $C_{\nu\tilde{\nu}\gamma\dots}$  denotes various constants which depends on  $\nu, \tilde{\nu}, \gamma, \dots$  but not on  $T$ .

We first establish the  $H^1$ -estimate for  $u_\infty$  which follows from the estimate (4.6) for  $e^{-tL}P_\infty$ .

**Proposition 5.9.** *There exist positive constants  $\nu_0$  ( $\geq \nu_1$ ),  $\gamma_0$  ( $\geq \gamma_1$ ) and  $\omega_0$  ( $\leq \omega_1$ ) such that if*

$$\nu \geq \nu_0, \quad \frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_0^2, \quad \omega \leq \omega_0; \quad (5.9)$$

then, for any  $0 < a < 2a_0$ ,

$$\begin{aligned} & \|u_\infty(t)\|_{H^1}^2 + \int_0^t e^{-a(t-\tau)} \|u_\infty(\tau)\|_{H^1}^2 d\tau \\ & \leq C_{\nu\tilde{\nu}\gamma} \left\{ e^{-at} \|u_{\infty,0}\|_{H^1}^2 + \sup_{0 \leq \tau \leq t} \|\mathbf{F}_\infty(\tau)\|_2^2 + \int_0^t e^{-a(t-\tau)} \|\mathbf{F}_\infty(\tau)\|_{H^1}^2 d\tau \right\}. \end{aligned}$$

**Proof.** We write  $u_\infty(t)$  as

$$u_\infty(t) = e^{-tL}u_{\infty,0} + \int_0^t e^{-(t-\tau)L} \mathbf{F}_\infty(\tau) d\tau.$$

Since  $u_{\infty,0} \in H^1 \times H_0^1$ , we see from (4.6) that

$$\begin{aligned} \|u_\infty(t)\|_{H^1} & \leq C \left\{ e^{-a_0 t} \|u_{\infty,0}\|_{H^1}^2 + \int_0^t e^{-a_0(t-\tau)} \|\mathbf{F}_\infty(\tau)\|_{H^1 \times \widehat{H}^1} d\tau \right. \\ & \quad \left. + \int_0^t e^{-a_0(t-\tau)} (t-\tau)^{-\frac{1}{2}} \|\mathbf{F}_\infty(\tau)\|_2 d\tau \right\} \\ & \leq C \left\{ e^{-a_0 t} \|u_{\infty,0}\|_{H^1}^2 + \sup_{0 \leq \tau \leq t} \|\mathbf{F}_\infty(\tau)\|_2 \right. \\ & \quad \left. + \int_0^t e^{-a_0(t-\tau)} \|\mathbf{F}_\infty(\tau)\|_{H^1 \times \widehat{H}^1} d\tau \right\}, \end{aligned}$$

from which we have

$$\|u_\infty(t)\|_{H^1}^2 \leq C \left\{ e^{-2a_0 t} \|u_{\infty,0}\|_{H^1}^2 + \sup_{0 \leq \tau \leq t} \|\mathbf{F}_\infty(\tau)\|_2^2 + \int_0^t e^{-a(t-\tau)} \|\mathbf{F}_\infty(\tau)\|_{H^1}^2 d\tau \right\} \quad (5.10)$$

for any  $0 < a < 2a_0$ . Set  $V(t) = \int_0^t e^{-\tilde{a}(t-\tau)} \|\mathbf{F}_\infty(\tau)\|_{H^1}^2 d\tau$ . Then  $V(t)$  satisfies  $dV/dt + \tilde{a}V = \|\mathbf{F}_\infty\|_{H^1}^2$  and  $V(0) = 0$ . It follows that  $\int_0^t e^{-a(t-\tau)} V(t) d\tau \leq \int_0^t e^{-a(t-\tau)} \|\mathbf{F}_\infty(\tau)\|_{H^1}^2 d\tau$  for any  $0 < a < \tilde{a}$ . This, together with (5.10), yields the desired inequality. This completes the proof.  $\square$

We next derive the  $H^2$  estimate for  $u_\infty(t)$ .

In what follows we set

$$f_\infty^0 = Q_0 \mathbf{F}_\infty, \quad f_\infty = \tilde{Q} \mathbf{F}_\infty$$

and

$$\dot{\phi}_\infty = \partial_t \phi_\infty + v_s^3 \partial_{x_3} \phi_\infty + w \cdot \nabla \phi_\infty,$$

where

$$\tilde{f}_\infty^0 = f_\infty^0 - w \cdot \nabla \phi_\infty.$$

Note that

$$\|\dot{\phi}_\infty\|_{H^1} \leq C_{\nu\tilde{\nu}\gamma} (\|u_\infty\|_{H^1 \times H^2}^2 + \|\tilde{f}_\infty^0\|_{H^1}^2).$$

The following Propositions 5.10 – 5.13 can be proved in a similar manner in [1, Section 4]. So we give the statements only and omit the proof.

We first state the  $L^2$  energy estimates for  $\partial_t u_\infty$  and  $\partial_{x_3}^2 u_\infty$ .

**Proposition 5.10.** *Under the assumption (5.9) (with  $\nu_0$ ,  $\gamma_0$  and  $\omega_0^{-1}$  replaced by suitably larger ones), the following assertions hold.*

(i) *There exists positive constant  $c$  such that the following inequality holds:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ \frac{1}{\gamma^2} \left\| \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_t \phi_\infty \right\|_2^2 + \left\| \sqrt{\rho_s} \partial_t w_\infty \right\|_2^2 \right\} \\ & + \frac{1}{2} \nu \|\nabla \partial_t w_\infty\|_2^2 + \frac{1}{2} \tilde{\nu} \|\operatorname{div} \partial_t w_\infty\|_2^2 + c \frac{\nu + \tilde{\nu}}{\gamma^4} \|\partial_t \dot{\phi}_\infty\|_2^2 \\ & \leq C_{\nu\tilde{\nu}\gamma} \|u_\infty\|_{H^1 \times H^2} + |A_1|. \end{aligned} \quad (5.11)$$

Here

$$\begin{aligned} A_1 = & \frac{1}{2} \left( |\partial_t \phi_\infty|^2, \operatorname{div} \left( \frac{P'(\rho_s)}{\gamma^4 \rho_s} w \right) \right) + \left( [\partial_t, w \cdot \nabla] \phi_\infty, \frac{P'(\rho_s)}{\gamma^4 \rho_s} \partial_t \phi_\infty \right) \\ & + \left( \partial_t \tilde{f}_\infty^0, \frac{P'(\rho_s)}{\gamma^4 \rho_s} \partial_t \phi_\infty \right) + \left( \partial_t f_\infty, \rho_s \partial_t w_\infty \right). \end{aligned}$$

(ii) *There exists positive constant  $b$  such that the following inequality holds:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ \frac{1}{\gamma^2} \left\| \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_{x_3}^2 \phi_\infty \right\|_2^2 + \left\| \sqrt{\rho_s} \partial_{x_3}^2 w_\infty \right\|_2^2 \right\} \\ & + \frac{1}{2} \nu \|\nabla \partial_{x_3}^2 w_\infty\|_2^2 + \frac{1}{2} \tilde{\nu} \|\operatorname{div} \partial_{x_3}^2 w_\infty\|_2^2 + b \frac{\nu + \tilde{\nu}}{\gamma^4} \|\partial_{x_3}^2 \dot{\phi}_\infty\|_2^2 \\ & \leq C \frac{\nu + \tilde{\nu}}{\gamma^4} \|\partial_{x_3}^2 \phi_\infty\|_2^2 + C_{\nu\tilde{\nu}\gamma} \|u_\infty\|_{H^1 \times H^2} + |A_{0,0,2}|. \end{aligned} \quad (5.12)$$

Here

$$\begin{aligned} A_{0,0,2} = & \frac{1}{2} \left( |\partial_{x_3}^2 \phi_\infty|^2, \operatorname{div} \left( \frac{P'(\rho_s)}{\gamma^4 \rho_s} w \right) \right) + \left( [\partial_{x_3}^2, w \cdot \nabla] \phi_\infty, \frac{P'(\rho_s)}{\gamma^4 \rho_s} \partial_{x_3}^2 \phi_\infty \right) \\ & + \left( \partial_{x_3}^2 \tilde{f}_\infty^0, \frac{P'(\rho_s)}{\gamma^4 \rho_s} \partial_{x_3}^2 \phi_\infty \right) + \left( \partial_{x_3} f_\infty, \partial_{x_3} (\rho_s \partial_{x_3}^2 w_\infty) \right) + C b \frac{\nu + \tilde{\nu}}{\gamma^4} \|\partial_{x_3}^2 \tilde{f}_\infty^0\|_2^2. \end{aligned}$$

We next state the interior estimate and the boundary estimates of the tangential derivatives.

**Proposition 5.11.** *Under the assumption (5.9) (with  $\nu_0$ ,  $\gamma_0$  and  $\omega_0^{-1}$  replaced by suitably larger ones), the following assertions hold.*

(i) *There exists positive constant  $b$  such that the estimate*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ \frac{1}{\gamma^2} \left\| \chi_0 \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_{x'}^2 \phi_\infty \right\|_2^2 + \left\| \chi_0 \sqrt{\rho_s} \partial_{x'}^2 w_\infty \right\|_2^2 \right\} \\ & + \frac{1}{2} \nu \left\| \chi_0 \nabla \partial_{x'}^2 w_\infty \right\|_2^2 + \frac{1}{2} \tilde{\nu} \left\| \chi_0 \operatorname{div} \partial_{x'}^2 w_\infty \right\|_2^2 + b \frac{\nu + \tilde{\nu}}{\gamma^4} \left\| \chi_0 \partial_{x'}^2 \dot{\phi}_\infty \right\|_2^2 \\ & \leq (\epsilon + C \frac{\nu + \tilde{\nu}}{\gamma^4}) \left\| \partial_{x'}^2 \phi_\infty \right\|_2^2 + C_{\epsilon \nu \tilde{\nu} \gamma} \|u_\infty\|_{H^1 \times H^2}^2 + |A^{(0)}| \end{aligned} \quad (5.13)$$

holds for any  $\epsilon > 0$ . Here

$$\begin{aligned} A^{(0)} = & \frac{1}{2} \left( \left| \partial_{x'}^2 \phi_\infty \right|^2, \operatorname{div} \left( \chi_0^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} w \right) \right) + \left( [\partial_{x'}^2, w \cdot \nabla] \phi_\infty, \chi_0^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} \partial_{x'}^2 \phi_\infty \right) \\ & + \left( \partial_{x'}^2 \tilde{f}_\infty^0, \chi_0^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} \partial_{x'}^2 \phi_\infty \right) + \left( \partial_{x'} f_\infty, \partial_{x'} (\chi_0^2 \rho_s \partial_{x'}^2 w_\infty) \right) \\ & + C b \frac{\nu + \tilde{\nu}}{\gamma^4} \left\| \chi_0 \partial_{x'}^2 \tilde{f}_\infty^0 \right\|_2^2. \end{aligned}$$

(ii) *Let  $1 \leq m \leq N$ . There exists positive constant  $b$  such that the estimate*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ \frac{1}{\gamma^2} \left\| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial^k \partial_{x_3}^j \phi_\infty \right\|_2^2 + \left\| \chi_m \sqrt{\rho_s} \partial^k \partial_{x_3}^j w_\infty \right\|_2^2 \right\} \\ & + \frac{1}{2} \nu \left\| \chi_m \nabla \partial^k \partial_{x_3}^j w_\infty \right\|_2^2 + \frac{1}{2} \tilde{\nu} \left\| \chi_m \operatorname{div} \partial^k \partial_{x_3}^j w_\infty \right\|_2^2 + b \frac{\nu + \tilde{\nu}}{\gamma^4} \left\| \chi_m \partial^k \partial_{x_3}^j \dot{\phi}_\infty \right\|_2^2 \\ & \leq (\epsilon + C \frac{1}{\gamma^2}) \left\| \partial_x^2 \phi_\infty \right\|_2^2 + C_{\epsilon \nu \gamma} \|u_\infty\|_{H^1 \times H^2}^2 + |A_{0,k,j}^{(m)}| \end{aligned} \quad (5.14)$$

holds for  $(k, j) = (2, 0), (1, 1)$  and any  $\epsilon > 0$ . Here

$$\begin{aligned} A_{0,k,j}^{(m)} = & \frac{1}{2} \left( \left| \partial^k \partial_{x_3}^j \phi_\infty \right|^2, \operatorname{div} \left( \chi_m^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} w \right) \right) \\ & + \left( [\partial^k \partial_{x_3}^j, w \cdot \nabla] \phi_\infty, \chi_m^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} \partial^k \partial_{x_3}^j \phi_\infty \right) \\ & + \left( \partial^k \partial_{x_3}^j \tilde{f}_\infty^0, \chi_m^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} \partial^k \partial_{x_3}^j \phi_\infty \right) + \left( \partial^{k-1} \partial_{x_3}^j f_\infty, \partial (\chi_m^2 \rho_s \partial^k \partial_{x_3}^j w_\infty) \right) \\ & + C b \frac{\nu + \tilde{\nu}}{\gamma^4} \left\| \chi_m \partial^k \partial_{x_3}^j \tilde{f}_\infty^0 \right\|_2^2. \end{aligned}$$

The normal derivatives of  $\phi_\infty$  is estimated as follows.

**Proposition 5.12.** *Let  $1 \leq m \leq N$ . Under the assumption (5.9) (with  $\nu_0$ ,  $\gamma_0$  and  $\omega_0^{-1}$  replaced by suitably larger ones), there exists positive constant  $b$  such that the estimate*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \frac{1}{\gamma^2} \left\| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_n^{l+1} \partial^k \partial_{x_3}^j \phi_\infty \right\|_2^2 \right) + \frac{1}{2} \frac{1}{\nu + \tilde{\nu}} \left\| \chi_m \frac{P'(\rho_s)}{\gamma^2} \partial_n^{l+1} \partial^k \partial_{x_3}^j \phi_\infty \right\|_2^2 \\ & + b \frac{\nu + \tilde{\nu}}{\gamma^4} \left\| \chi_m \partial_n^{l+1} \partial^k \partial_{x_3}^j \dot{\phi}_\infty \right\|_2^2 \\ & \leq C \left\{ \frac{\nu + \tilde{\nu}}{\gamma^4} \left\| \partial_x^2 \phi_\infty \right\|_2^2 + \frac{1}{\nu + \tilde{\nu}} \left\| \partial_t \partial_x w_\infty \right\|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} \left( \left\| \chi_m \partial_n^l \partial^k \partial_{x_3}^{j+2} w_\infty \right\|_2^2 \right. \right. \\ & \quad \left. \left. + \left\| \chi_m \nabla \partial_n^l \partial^k \partial_{x_3}^{j+1} w_\infty \right\|_2^2 + \left\| \chi_m \nabla \partial_n^l \partial^{k+1} \partial_{x_3}^j w_\infty \right\|_2^2 \right) \right\} \\ & + C_{\nu \tilde{\nu} \gamma} \|u_\infty\|_{H^1 \times H^2}^2 + |A_{l+1,k,j}^{(m)}| \end{aligned} \quad (5.15)$$

holds for  $j, k, l \geq 0$  satisfying  $j + k + l = 1$ . Here

$$\begin{aligned} A_{l+1,k,j}^{(m)} = & \frac{1}{2} \sum_{j+k+l=1} \left| \left( |\partial_n^{l+1} \partial^k \partial_{x_3}^j \phi_\infty|^2, \operatorname{div} \left( \chi_m^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} w \right) \right) \right| \\ & + C \sum_{j+k+l=1} \left\| \chi_m [\partial_n^{l+1} \partial^k \partial_{x_3}^j, w \cdot \nabla] \phi_\infty \right\|_2^2 \\ & + C(b+1) \left( \frac{\nu+\tilde{\nu}}{\gamma^4} \|\chi_m \partial_n^{l+1} \partial^k \partial_{x_3}^j \tilde{f}_\infty^0\|_2^2 + \|f_\infty\|_{H^1}^2 \right). \end{aligned}$$

Using the estimate for the Stokes system we have the following estimates.

**Proposition 5.13.** *Under the assumption (5.9) (with  $\nu_0$ ,  $\gamma_0$  and  $\omega_0^{-1}$  replaced by suitably larger ones), the following assertions hold.*

(i) *There holds the estimate*

$$\begin{aligned} & \frac{\nu^2}{\nu+\tilde{\nu}} \|\partial_x^3 w_\infty\|_2^2 + \frac{1}{\nu+\tilde{\nu}} \|\partial_x^2 \phi_\infty\|_2^2 \\ & \leq C \left\{ \frac{\nu+\tilde{\nu}}{\gamma^4} \|\partial_x^2 \dot{\phi}_\infty\|_2^2 + \frac{1}{\nu+\tilde{\nu}} \|\partial_t \partial_x w_\infty\|_2^2 + \frac{\nu+\tilde{\nu}}{\gamma^4} \|\tilde{f}_\infty^0\|_{H^2}^2 + \frac{1}{\nu+\tilde{\nu}} \|f_\infty\|_{H^1}^2 \right\} \quad (5.16) \\ & + C_{\nu\tilde{\nu}\gamma} \|u_\infty\|_{H^1 \times H^2}^2. \end{aligned}$$

(ii) *Let  $1 \leq m \leq N$ . There holds the estimate*

$$\begin{aligned} & \frac{\nu^2}{\nu+\tilde{\nu}} \|\chi_m \partial_x^2 \partial w_\infty\|_2^2 + \frac{1}{\nu+\tilde{\nu}} \|\chi_m \partial_x \partial \phi_\infty\|_2^2 \\ & \leq C \left\{ \frac{\nu+\tilde{\nu}}{\gamma^4} \|\chi_m \partial \partial_{x_3} \phi_\infty\|_2^2 + \frac{\nu+\tilde{\nu}}{\gamma^4} \|\chi_m \partial_x \partial \dot{\phi}_\infty\|_2^2 + \frac{1}{\nu+\tilde{\nu}} \|\partial_t \partial_x w_\infty\|_2^2 \right. \quad (5.17) \\ & \left. + \frac{\nu+\tilde{\nu}}{\gamma^4} \|\tilde{f}_\infty^0\|_{H^2}^2 + \frac{1}{\nu+\tilde{\nu}} \|f_\infty\|_{H^1}^2 \right\} + C_{\nu\tilde{\nu}\gamma} \|u_\infty\|_{H^1 \times H^2}^2. \end{aligned}$$

(iii) *There holds the estimate*

$$\begin{aligned} & \frac{\nu^2}{\nu+\tilde{\nu}} \|\partial_x^2 \partial_{x_3} w_\infty\|_2^2 + \frac{1}{\nu+\tilde{\nu}} \|\partial_x \partial_{x_3} \phi_\infty\|_2^2 \\ & \leq C \left\{ \frac{\nu+\tilde{\nu}}{\gamma^4} \|\partial_x \partial_{x_3} \dot{\phi}_\infty\|_2^2 + \frac{1}{\nu+\tilde{\nu}} \|\partial_t \partial_x w_\infty\|_2^2 + \frac{\nu+\tilde{\nu}}{\gamma^4} \|\tilde{f}_\infty^0\|_{H^2}^2 + \frac{1}{\nu+\tilde{\nu}} \|f_\infty\|_{H^1}^2 \right\} \quad (5.18) \\ & + C_{\nu\tilde{\nu}\gamma} \|u_\infty\|_{H^1 \times H^2}^2. \end{aligned}$$

We are now in a position to prove Proposition 5.8.

**Proof of Proposition 5.8** Let  $b_1$  and  $b_2$  be constants satisfying  $b_1, b_2 > 1$ . Define  $\tilde{\mathcal{E}}_2[u_\infty]$  by

$$\begin{aligned} \tilde{\mathcal{E}}_2[u_\infty] = & \frac{1}{\gamma^2} \sum_{m=1}^N \left\{ b_1 \left( \left\| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial^2 \phi_\infty \right\|_2^2 + \left\| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial \partial_{x_3} \phi_\infty \right\|_2^2 \right) \right. \\ & + \left\| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_n \partial \phi_\infty \right\|_2^2 + \left\| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_n \partial_{x_3} \phi_\infty \right\|_2^2 \Big\} \\ & + \frac{1}{\gamma^2} \left( \left\| \chi_0 \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_{x'}^2 \phi_\infty \right\|_2^2 + b_1 \left\| \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_{x_3}^2 \phi_\infty \right\|_2^2 \right) \\ & + b_1 \sum_{m=1}^N \left( \left\| \chi_m \sqrt{\rho_s} \partial^2 w_\infty \right\|_2^2 + \left\| \chi_m \sqrt{\rho_s} \partial \partial_{x_3} w_\infty \right\|_2^2 \right) \\ & + \left\| \chi_0 \sqrt{\rho_s} \partial_{x'}^2 w_\infty \right\|_2^2 + b_1 \left\| \sqrt{\rho_s} \partial_{x_3}^2 w_\infty \right\|_2^2 \end{aligned}$$

for  $u_\infty = T(\phi_\infty, w_\infty)$ . We compute

$$\begin{aligned} & b_2 \left[ \sum_{m=1}^N \left\{ b_1 \{ (5.14) \mid_{(k,j)=(2,0)} + (5.14) \mid_{(k,j)=(1,1)} \} \right. \right. \\ & \quad \left. \left. + (5.15) \mid_{(l,k,j)=(0,1,0)} + (5.15) \mid_{(l,k,j)=(0,0,1)} \right\} + (5.13) + b_1(5.12) \right] \\ & + \sum_{m=1}^N (5.17) + (5.18). \end{aligned}$$

Then

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} b_2 \tilde{\mathcal{E}}_2[u_\infty] + b b_2 \frac{\nu + \tilde{\nu}}{\gamma^4} \left( \sum_{m=1}^N \|\chi_m \partial_x \partial \dot{\phi}_\infty\|_2^2 + \|\partial_x \partial_{x_3} \dot{\phi}_\infty\|_2^2 \right) \\ & + \frac{\nu^2}{\nu + \tilde{\nu}} \left( \sum_{m=1}^N \|\chi_m \partial_x^2 \partial w_\infty\|_2^2 + \|\partial_x^2 \partial_{x_3} w_\infty\|_2^2 \right) \\ & + \frac{1}{\nu + \tilde{\nu}} \left( \sum_{m=1}^N \|\chi_m \partial_x \partial \phi_\infty\|_2^2 + \|\chi_m \partial_x \partial_{x_3} \phi_\infty\|_2^2 \right) \\ & + \frac{b_2}{2} \nu \left\{ b_1 \sum_{m=1}^N (\|\chi_m \nabla \partial^2 w_\infty\|_2^2 + \|\chi_m \nabla \partial \partial_{x_3} w_\infty\|_2^2) + \|\chi_0 \nabla \partial_{x'}^2 w_\infty\|_2^2 + b_1 \|\nabla \partial_{x_3}^2 w_\infty\|_2^2 \right\} \\ & + \frac{b_2}{2} \tilde{\nu} \left\{ b_1 \sum_{m=1}^N (\|\chi_m \operatorname{div} \partial^2 w_\infty\|_2^2 + \|\chi_m \operatorname{div} \partial \partial_{x_3} w_\infty\|_2^2) + \|\chi_0 \operatorname{div} \partial_{x'}^2 w_\infty\|_2^2 \right. \\ & \quad \left. + b_1 \|\operatorname{div} \partial_{x_3}^2 w_\infty\|_2^2 \right\} + \frac{b_2}{2} \frac{1}{\nu + \tilde{\nu}} \sum_{m=1}^N \left( \left\| \chi_m \frac{P'(\rho_s)}{\gamma^2} \partial_n \partial \phi_\infty \right\|_2^2 + \left\| \chi_m \frac{P'(\rho_s)}{\gamma^2} \partial_n \partial_{x_3} \phi_\infty \right\|_2^2 \right) \\ & \leq C_{b_1 b_2} \left\{ \left( \epsilon + \frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) \|\partial_x^2 \phi_\infty\|_2^2 + C_{\epsilon \nu \gamma \omega} \|u_\infty\|_{H^1 \times H^2}^2 + \frac{1}{\nu + \tilde{\nu}} \|\partial_t \partial_x w_\infty\|_2^2 + \mathcal{R}_0 \right\} \\ & + C_1 \left\{ b_2 \frac{\nu^2}{\nu + \tilde{\nu}} \sum_{m=1}^N (\|\chi_m \nabla \partial^2 w_\infty\|_2^2 + \|\chi_m \nabla \partial \partial_{x_3} w_\infty\|_2^2 + \|\chi_m \nabla \partial_{x_3}^2 w_\infty\|_2^2) \right. \\ & \quad \left. + \frac{\nu + \tilde{\nu}}{\gamma^4} \left( \sum_{m=1}^N \|\chi_m \partial_x \partial \dot{\phi}_\infty\|_2^2 + \|\partial_x \partial_{x_3} \dot{\phi}_\infty\|_2^2 \right) \right\} \end{aligned}$$

for any  $\epsilon > 0$ . Here

$$\mathcal{R}_0 = \sum_{m=1}^N (|A_{0,2,0}^{(m)}| + |A_{0,1,1}^{(m)}|) + |A^{(0)}| + |A_{0,0,2}| + \sum_{m=1}^N (|A_{1,1,0}^{(m)}| + |A_{1,0,1}^{(m)}|).$$

Fix  $b_1 > 8C_1$  and  $b_2 > 8\frac{C_1}{b}$ . It then follows that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} b_2 \tilde{\mathcal{E}}_2[u_\infty] + b \frac{\nu+\tilde{\nu}}{\gamma^4} \left( \sum_{m=1}^N \|\chi_m \partial_x \partial \dot{\phi}_\infty\|_2^2 + \|\partial_x \partial_{x_3} \dot{\phi}_\infty\|_2^2 \right) \\
& + \frac{\nu^2}{\nu+\tilde{\nu}} \left( \sum_{m=1}^N \|\chi_m \partial_x^2 \partial w_\infty\|_2^2 + \|\partial_x^2 \partial_{x_3} w_\infty\|_2^2 \right) \\
& + \frac{1}{\nu+\tilde{\nu}} \left( \sum_{m=1}^N \|\chi_m \partial_x \partial \phi_\infty\|_2^2 + \|\chi_m \partial_x \partial_{x_3} \phi_\infty\|_2^2 \right) \\
& + \frac{b_2}{2} I_1[w_\infty] + \frac{b_2}{2} \frac{1}{\nu+\tilde{\nu}} \sum_{m=1}^N \left( \left\| \chi_m \frac{P'(\rho_s)}{\gamma^2} \partial_n \partial \phi_\infty \right\|_2^2 + \left\| \chi_m \frac{P'(\rho_s)}{\gamma^2} \partial_n \partial_{x_3} \phi_\infty \right\|_2^2 \right) \\
& \leq C_{b_1 b_2} \left\{ \left( \epsilon + \frac{1}{\gamma^2} + \frac{\nu+\tilde{\nu}}{\gamma^4} \right) \|\partial_x^2 \phi_\infty\|_2^2 + C_{\epsilon \nu \gamma \omega} \|u_\infty\|_{H^1 \times H^2}^2 \right. \\
& \quad \left. + \frac{1}{\nu+\tilde{\nu}} \|\partial_t \partial_x w_\infty\|_2^2 + \mathcal{R}_0 \right\}
\end{aligned} \tag{5.19}$$

for any  $\epsilon > 0$ . Here  $I_1[w_\infty]$  is given by

$$\begin{aligned}
I_1[w_\infty] = & \nu \left\{ \sum_{m=1}^N \left( \|\chi_m \nabla \partial^2 w_\infty\|_2^2 + \|\chi_m \nabla \partial \partial_{x_3} w_\infty\|_2^2 \right) \right. \\
& + \|\chi_0 \nabla \partial_{x'}^2 w_\infty\|_2^2 + \|\nabla \partial_{x_3}^2 w_\infty\|_2^2 \Big\} \\
& + \tilde{\nu} \left\{ \sum_{m=1}^N \left( \|\chi_m \operatorname{div} \partial^2 w_\infty\|_2^2 + \|\chi_m \operatorname{div} \partial \partial_{x_3} w_\infty\|_2^2 \right) \right. \\
& + \|\chi_0 \operatorname{div} \partial_{x'}^2 w_\infty\|_2^2 + \|\operatorname{div} \partial_{x_3}^2 w_\infty\|_2^2 \Big\}.
\end{aligned}$$

Let  $b_3$  and  $b_4$  be constants satisfying  $b_3, b_4 > 1$ . Define  $\mathcal{E}_2[u_\infty]$  by

$$\begin{aligned}
\mathcal{E}_2[u_\infty] = & b_2 b_3 b_4 \tilde{\mathcal{E}}_2[u_\infty] + b_4 \frac{1}{\gamma^2} \sum_{m=1}^N \left\| \chi_m \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_n^2 \phi_\infty \right\|_2^2 \\
& + \left( \frac{1}{\gamma^2} \left\| \sqrt{\frac{P'(\rho_s)}{\gamma^2 \rho_s}} \partial_t \phi_\infty \right\|_2^2 + \left\| \sqrt{\rho_s} \partial_t w_\infty \right\|_2^2 \right)
\end{aligned}$$

for  $u_\infty = {}^T(\phi_\infty, w_\infty)$ . We compute

$$b_4 \left\{ b_3 (5.19) + \sum_{m=1}^N (5.15) \mid_{(l,k,j)=(1,0,0)} \right\} + (5.16) + b_5 (5.11).$$

It follows that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \mathcal{E}_2[u_\infty] + b b_4 \frac{\nu + \tilde{\nu}}{\gamma^4} \|\partial_x^2 \dot{\phi}_\infty\|_2^2 + \frac{\nu^2}{\nu + \tilde{\nu}} \|\partial_x^3 w_\infty\|_2^2 + \frac{1}{\nu + \tilde{\nu}} \|\partial_x^2 \phi_\infty\|_2^2 \\
& + b_3 b_4 \left\{ \frac{\nu^2}{\nu + \tilde{\nu}} \left( \sum_{m=1}^N \|\chi_m \partial_x^2 \partial w_\infty\|_2^2 + \|\partial_x^2 \partial_{x_3} w_\infty\|_2^2 \right) \right. \\
& + \frac{1}{\nu + \tilde{\nu}} \left( \sum_{m=1}^N \|\chi_m \partial_x \partial \phi_\infty\|_2^2 + \|\partial_x \partial_{x_3} \phi_\infty\|_2^2 \right) \Big\} + \frac{1}{2} b_2 b_3 b_4 I_1[w_\infty] \\
& + \frac{b_4}{2} \frac{1}{\nu + \tilde{\nu}} \sum_{m=1}^N \|\chi_m \frac{P'(\rho_s)}{\gamma^2} \partial_n \nabla \phi_\infty\|_2^2 + \frac{1}{2} \{ \nu \|\nabla \partial_t w_\infty\|_2^2 + \tilde{\nu} \|\operatorname{div} \partial_t w_\infty\|_2^2 + c \frac{\nu + \tilde{\nu}}{\gamma^4} \|\partial_t \phi_\infty\|_2^2 \} \\
& \leq C_{b_1 \dots b_4} \left\{ \left( \epsilon + \frac{1}{\gamma^2} + \frac{\nu + \tilde{\nu}}{\gamma^4} \right) \|\partial_x^2 \phi_\infty\|_2^2 + \frac{1}{\nu + \tilde{\nu}} \|\partial_t \partial_x w_\infty\|_2^2 + C_{\epsilon \nu \gamma \omega} \|u_\infty\|_{H^1 \times H^2}^2 \right. \\
& + \mathcal{R} \Big\} + C_2 \left\{ b_4 \frac{\nu^2}{\nu + \tilde{\nu}} \sum_{m=1}^N \left( \|\chi_m \partial_n \partial_{x_3}^2 w_\infty\|_2^2 + \|\chi_m \nabla \partial_n \partial_{x_3} w_\infty\|_2^2 \right) \right. \\
& + \left. \|\chi_m \nabla \partial_n \partial w_\infty\|_2^2 + \frac{\nu + \tilde{\nu}}{\gamma^4} \|\partial_x^2 \dot{\phi}_\infty\|_2^2 \right\}
\end{aligned}$$

for any  $\epsilon > 0$ . Here

$$\mathcal{R} = \mathcal{R}_0 + \sum_{m=1}^N |A_{2,0,0}^{(m)}| + |A_1|.$$

Fix  $b_3$  and  $b_4$  so large that  $b_3 > 8C_2$  and  $b_4 > 2\frac{C_2}{b}$ . We assume that  $\nu$ ,  $\tilde{\nu}$  and  $\gamma$  also satisfy  $\gamma^2 > 8C_{b_1 \dots b_4}$  and  $\gamma^2 > 8C_{b_1 \dots b_4}(\nu + \tilde{\nu})$ . We take  $\epsilon > 0$  sufficiently small such that  $\epsilon < \frac{1}{8C_{b_1 \dots b_4}} \frac{1}{\nu + \tilde{\nu}}$ . It then follows that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \mathcal{E}_2[u_\infty] + b \frac{\nu + \tilde{\nu}}{\gamma^4} \|\partial_x^2 \dot{\phi}_\infty\|_2^2 + \frac{1}{2} \frac{\nu^2}{\nu + \tilde{\nu}} \|\partial_x^3 w_\infty\|_2^2 + \frac{1}{2} \frac{1}{\nu + \tilde{\nu}} \|\partial_x^2 \phi_\infty\|_2^2 \\
& + \frac{\nu^2}{\nu + \tilde{\nu}} \left( \sum_{m=1}^N \|\chi_m \partial_x^2 \partial w_\infty\|_2^2 + \|\partial_x^2 \partial_{x_3} w_\infty\|_2^2 \right) \\
& + \frac{1}{\nu + \tilde{\nu}} \left( \sum_{m=1}^N \|\chi_m \partial_x \partial \phi_\infty\|_2^2 + \|\partial_x \partial_{x_3} \phi_\infty\|_2^2 \right) \\
& + I_1[w_\infty] + \frac{1}{\nu + \tilde{\nu}} \sum_{m=1}^N \|\chi_m \frac{P'(\rho_s)}{\gamma^2} \partial_n \nabla \phi_\infty\|_2^2 \\
& + \frac{1}{2} \{ \nu \|\nabla \partial_t w_\infty\|_2^2 + \tilde{\nu} \|\operatorname{div} \partial_t w_\infty\|_2^2 + c \frac{\nu + \tilde{\nu}}{\gamma^4} \|\partial_t \phi_\infty\|_2^2 \} \\
& \leq \{ C_{\epsilon \nu \gamma \omega} \|u_\infty\|_{H^1 \times H^2}^2 + \mathcal{R} \}.
\end{aligned} \tag{5.20}$$

We thus see that there are positive constants  $c_1$ ,  $c_2$  and  $C_3$  such that

$$\begin{aligned}
& \frac{d}{dt} \mathcal{E}_2[u_\infty] + c_1 \mathcal{E}_2[u_\infty] \\
& + c_2 \left( \|\partial_x^3 w_\infty\|_2^2 + \|\partial_x^2 \phi_\infty\|_2^2 + \|\dot{\phi}_\infty\|_{H^2}^2 + \|\partial_t w_\infty\|_{H^1}^2 + \|\partial_t \phi_\infty\|_2^2 \right) \\
& \leq C_{\nu \tilde{\nu} \gamma} (\|u_\infty\|_{H^1 \times H^2}^2 + \mathcal{R}).
\end{aligned}$$

Since

$$\|\partial_x^2 w_\infty\|_2^2 \leq \eta \|\partial_x^3 w_\infty\|_2^2 + C_\eta \|w_\infty\|_2^2$$

holds for any  $\eta > 0$ , taking  $\eta$  so small that  $\eta < \frac{1}{2} \min\{\frac{c_2}{c_2 + C_{\nu\tilde{\nu}\gamma}}, 1\}$ , we obtain

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}_2[u_\infty] + c_1 \mathcal{E}_2[u_\infty] \\ & + \frac{c_2}{2} (\|\partial_x^3 w_\infty\|_2^2 + \|\partial_x^2 \phi_\infty\|_2^2 + \|\dot{\phi}_\infty\|_{H^1}^2 + \|\partial_t w_\infty\|_{H^1}^2 + \|\partial_t \phi_\infty\|_2^2) \\ & \leq 2C_{\nu\tilde{\nu}\gamma} (\|u_\infty\|_{H^1 \times L^2}^2 + \mathcal{R}). \end{aligned} \quad (5.21)$$

We see from (5.21) and Proposition 5.9 that there exist positive constants  $\tilde{c}_1$ ,  $\tilde{c}_2$  and  $C_{\nu\tilde{\nu}\gamma}$  such that

$$\begin{aligned} & \mathcal{E}_2[u_\infty(t)] + \|u_\infty(t)\|_{H^1}^2 + \tilde{c}_2 \int_0^t e^{-\tilde{c}_1(t-\tau)} (\|\partial_x^3 w_\infty\|_2^2 + \|\partial_x^2 \phi_\infty\|_2^2 + \|u_\infty\|_{H^1}^2 \\ & + \|\dot{\phi}_\infty\|_{H^1}^2 + \|\partial_t u_\infty\|_{H^1 \times L^2}^2) d\tau \\ & \leq C_{\nu\tilde{\nu}\gamma} \left\{ e^{-\tilde{c}_1 t} (\mathcal{E}_2[u_{\infty,0}] + \|u_{\infty,0}\|_{H^1}^2) + \sup_{0 \leq \tau \leq t} \|\mathbf{F}_\infty(\tau)\|_2^2 + \int_0^t e^{-\tilde{c}_1(t-\tau)} \mathcal{R} d\tau \right\}. \end{aligned} \quad (5.22)$$

It remains to estimate the term  $\|\partial_x^2 w_\infty(t)\|_2$ . We write the second equation of (5.28) as

$$-\nu \Delta w_\infty - \tilde{\nu} \nabla \operatorname{div} w_\infty = J, \quad w_\infty|_{\partial\Omega} = 0,$$

where

$$J = -\rho_s \left\{ \partial_t w_\infty + \nabla \left( \frac{P'(\rho_s)}{\gamma^2 \rho_s} \phi_\infty \right) + \frac{\nu \Delta' v_s^3}{\gamma^2 \rho_s} \phi_\infty e_3 + v_s^3 \partial_{x_3} w_\infty + (w'_\infty \cdot \nabla' v_s^3) e_3 - f_\infty \right\}.$$

Since  $J \in L^2(\Omega)$ , we obtain, by elliptic estimate,

$$\|\partial_x^2 w_\infty\|_2^2 \leq C(\|w_\infty\|_2^2 + \|J\|_2^2) \leq C_{\nu\tilde{\nu}\gamma} (\mathcal{E}_2[u_\infty] + \|u_\infty\|_{H^1}^2 + \|f_\infty\|_2^2).$$

From this, with (5.22), we see that

$$\begin{aligned} & \mathcal{E}_2[u_\infty(t)] + \|u_\infty(t)\|_{H^1}^2 + \|\partial_x^2 w_\infty(t)\|_2^2 + \tilde{c}_2 \int_0^t e^{-\tilde{c}_1(t-\tau)} (\|\partial_x^3 w_\infty\|_2^2 + \|\partial_x^2 \phi_\infty\|_2^2 \\ & + \|u_\infty\|_{H^1}^2 + \|\dot{\phi}_\infty\|_{H^1}^2 + \|\partial_t u_\infty\|_{H^1 \times L^2}^2) d\tau \\ & \leq C_{\nu\tilde{\nu}\gamma} \left\{ e^{-\tilde{c}_1 t} (\mathcal{E}_2[u_{\infty,0}] + \|u_{\infty,0}\|_{H^1}^2) + \sup_{0 \leq \tau \leq t} \|\mathbf{F}_\infty(\tau)\|_2^2 + \int_0^t e^{-\tilde{c}_1(t-\tau)} \mathcal{R} d\tau \right\}. \end{aligned} \quad (5.23)$$

As we will see in section 8 below, it holds that

$$\sup_{0 \leq \tau \leq t} \|\mathbf{F}_\infty(\tau)\|_2^2 \leq C(1+t)^{-\frac{3}{2}} M(t)^4. \quad (5.24)$$

Proposition 5.8 follows from (5.23) and (5.24). This completes the proof.  $\square$

## 5.4 Estimates of nonlinear terms

In this section we prove the estimate (5.24) and (5.33).

We first make an observation. By the Sobolev inequality we have

$$\|\phi(t)\|_\infty \leq C\llbracket\phi(t)\rrbracket_{H^2} \leq C_1\llbracket u(t)\rrbracket_2$$

for a positive constant  $C_1$ . It then follows that

$$\rho(x, t) = \rho_s(x') + \gamma^{-2}\phi(x, t) \geq \rho_1 - \gamma^{-2}\|\phi(t)\|_\infty \geq \rho_1 - C_1\gamma^{-2}\llbracket u(t)\rrbracket_2.$$

Fix a positive constant  $\epsilon_s$  satisfying  $\epsilon_s \leq \frac{1}{4}\frac{\gamma^2\rho_1}{C_1}$ . If  $\llbracket u(t)\rrbracket_2 \leq \epsilon_s$ , then it holds that

$$\|\phi(t)\|_\infty \leq \frac{1}{4}\gamma^2\rho_1, \quad \rho(x, t) \geq \frac{3}{4}\rho_1 > 0.$$

This implies that  $\tilde{Q}\mathbf{F}(t)$  is smooth whenever  $\llbracket u(t)\rrbracket_2 \leq \epsilon_s$ .

We will show the following

**Proposition 5.14.** *If  $\llbracket u(t)\rrbracket_2 \leq \epsilon_s$  and  $M(t) \leq 1$ , then*

$$\sup_{0 \leq \tau \leq t} \|\mathbf{F}_\infty(\tau)\|_2^2 \leq C(1+t)^{-\frac{3}{2}}M(t)^4, \quad (5.25)$$

$$\mathcal{R}(t) \leq C\{(1+t)^{-\frac{3}{2}}M(t)^3 + (1+t)^{-\frac{1}{4}}M(t)D_\infty(t)\} \quad (5.26)$$

To prove Proposition 5.14, we prepare several lemmas.

**Lemma 5.15.** (i) *For  $2 \leq p \leq \infty$ . If  $j$  and  $k$  are nonnegative integers satisfying*

$$0 \leq j < k, \quad k > j + n(\frac{1}{2} - \frac{1}{p}),$$

*then there exists a positive constant  $C$  such that*

$$\|\partial_x^j f\|_{L^p(\mathbf{R}^n)} \leq \|f\|_{L^2(\mathbf{R}^2)}^{1-a} \|f\|_{H^k(\mathbf{R}^n)}^a.$$

*Here  $a = \frac{1}{k}(j + \frac{n}{2} - \frac{n}{p})$ .*

(ii) *For  $2 \leq p \leq \infty$ . If  $j$  and  $k$  are nonnegative integers satisfying*

$$0 \leq j < k, \quad k > j + 3(\frac{1}{2} - \frac{1}{p}),$$

*then there exists a positive constant  $C$  such that*

$$\|\partial_x^j f\|_{L^p(\Omega)} \leq C\|f\|_{H^k(\Omega)}.$$

(iii) *If  $f \in H^1(\Omega)$  and  $f = f(x_3)$  is independent of  $x' \in D$ , then it holds that*

$$\|f\|_{L^\infty(\Omega)} \leq C\|f\|_{L^2(\Omega)}^{\frac{1}{2}} \|\partial_{x_3} f\|_{L^2(\Omega)}^{\frac{1}{2}}$$

*for a positive constant  $C$ .*

The proof of Lemma 5.15 can be found, e.g., in [12, 17].

**Lemma 5.16.** (i) For nonnegative integers  $m$  and  $m_k$  ( $k = 1, \dots, l$ ) and multi index  $\alpha_k$  ( $k = 1, \dots, l$ ), we assume that

$$m \geq [\frac{n}{2}], \quad 0 \leq |\alpha_k| \leq m_k \leq 2 + |\alpha_k| \quad (k = 1, \dots, l)$$

and

$$\sum_{k=1}^l m_k \geq 2(l-1) + \sum_{k=1}^l |\alpha_k|,$$

then the estimate holds

$$\left\| \prod_{k=1}^l \partial_x^{\alpha_k} f_k \right\|_2 \leq C \prod_{k=1}^l \|f_k\|_{H^{m_k}}$$

for a positive constant  $C$ .

(ii) For  $1 \leq k \leq m$ . We assume that  $F(x, t, y)$  is a smooth function on  $\Omega \times [0, \infty) \times I$  with a compact interval  $I \subset \mathbf{R}$ . For  $|\alpha| + 2j = k$  the estimates hold

$$\begin{aligned} & \left\| [\partial_x^\alpha \partial_t^j, F(x, t, f_1)] f_2 \right\|_2 \\ & \leq \begin{cases} C_0(t, f_1(t)) \llbracket f_2 \rrbracket_{k-1} + C_1(t, f_1(t)) \{1 + \left\| \|Df_1\| \right\|_{m-1}^{|\alpha|+j-1}\} \left\| \|Df_1\| \right\|_{m-1} \llbracket f_2 \rrbracket_k, \\ C_0(t, f_1(t)) \llbracket f_2 \rrbracket_{k-1} + C_1(t, f_1(t)) \{1 + \left\| \|Df_1\| \right\|_{m-1}^{|\alpha|+j-1}\} \left\| \|Df_1\| \right\|_m \llbracket f_2 \rrbracket_{k-1}, \end{cases} \end{aligned}$$

where

$$\begin{aligned} C_0(t, f_1(t)) &= \sum_{(\beta, l) \leq (\alpha, j), (\beta, l) \neq (0, 0)} \sup_x |(\partial_x^\beta \partial_t^l F)(x, t, f_1(x, t))|, \\ C_1(t, f_1(t)) &= \sum_{(\beta, l) \leq (\alpha, j), 1 \leq p \leq j + |\alpha|} \sup_x |(\partial_x^\beta \partial_t^l \partial_y^p F)(x, t, f_1(x, t))|. \end{aligned}$$

(iii) For  $m \geq 2$  the estimates hold

$$\|f_1 \cdot f_2\|_{H^m} \leq C_1 \|f_1\|_{H^m} \|f_2\|_{H^m}, \quad \llbracket f_1 \cdot f_2 \rrbracket_m \leq C_2 \llbracket f_1 \rrbracket_m \llbracket f_2 \rrbracket_m$$

for a positive constants  $C_1$  and  $C_2$ .

See, e.g., [14, 17] for the proof of Lemma 5.16.

We begin with the following preliminary estimates for  $\sigma_1$  and  $u_j$  ( $j = 1, \infty$ ).

**Lemma 5.17.** We assume that  $u(t) = {}^T(\phi(t), w(t)) = (\sigma_1 u^{(0)})(t) + u_1(t) + u_\infty(t)$  be a solution of (5.28) in  $Z(T)$ . The following estimates hold for all  $t \in [0, T]$  with a positive constant  $C$  independent of  $T$ .

- (i)  $\|\sigma_1(t)\|_2 \leq C(1+t)^{-\frac{1}{4}} M(t),$
- (ii)  $\llbracket u(t) \rrbracket_2 \leq C(1+t)^{-\frac{1}{4}} M(t),$
- (iii)  $\left\| \|D\sigma_1(t)\| \right\| \leq C(1+t)^{-\frac{3}{4}} M(t),$

- (iv)  $\|u_j(t)\|_2 \leq C(1+t)^{-\frac{3}{4}}M(t), \quad (j = 1, \infty).$
- (v)  $\|\sigma_1(t)\|_\infty \leq C(1+t)^{-\frac{1}{2}}M(t),$
- (vi)  $\|u_j(t)\|_\infty \leq C(1+t)^{-\frac{3}{4}}M(t), \quad (j = 1, \infty).$
- (vii)  $\|u(t)\|_\infty \leq C(1+t)^{-\frac{1}{2}}M(t).$

Lemma 5.17 easily follows from Lemma 5.15 and the definition of  $M(t)$ .

Let us estimate the nonlinearities. For  $Q_0 \mathbf{F} = -\operatorname{div}(\phi w)$ , we have the following estimates.

**Proposition 5.18.** *We assume that  $u(t) = {}^T(\phi(t), w(t)) = (\sigma_1 u^{(0)})(t) + u_1(t) + u_\infty(t)$  be a solution of (5.28) in  $Z(T)$ . If  $M(t) \leq 1$  for all  $t \in [0, T]$ , then the estimates hold with a positive constant  $C$  independent of  $T$ .*

- (i)  $\|\phi \operatorname{div} w\|_l \leq \begin{cases} C(1+t)^{-\frac{5}{4}}M(t)^2 & (l = 1), \\ C(1+t)^{-\frac{5}{4}}M(t)^2 + (1+t)^{-\frac{1}{2}}M(t)\|Dw_\infty(t)\|_2 & (l = 2). \end{cases}$
- (ii)  $\|w \cdot \nabla \phi_\infty\|_{H^1} \leq C(1+t)^{-\frac{5}{4}}M(t)^2.$
- (iii)  $\|w \cdot \nabla(\sigma_1 \phi^{(0)} + \phi_1)\|_2 \leq C(1+t)^{-\frac{5}{4}}M(t)^2.$
- (iv)  $\begin{aligned} & \left| \left( |\partial_{x_3}^2 \phi_\infty|^2, \operatorname{div}\left(\frac{P'(\rho_s)}{\gamma^4 \rho_s} w\right) \right) \right| + \left| \left( |\partial_{x'}^2 \phi_\infty|^2, \operatorname{div}\left(\chi_0^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} w\right) \right) \right| \\ & + \left| \left( |\partial_t \phi_\infty|^2, \operatorname{div}\left(\frac{P'(\rho_s)}{\gamma^4 \rho_s} w\right) \right) \right| + \sum_{m=1}^N \left\{ \sum_{j+k=1} \left| \left( |\partial^{k+1} \partial_{x_3}^j \phi_\infty|^2, \operatorname{div}\left(\chi_m^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} w\right) \right) \right| \right. \\ & \left. + \sum_{j+k+l=1} \left| \left( |\partial_n^{l+1} \partial^k \partial_{x_3}^j \phi_\infty|^2, \operatorname{div}\left(\chi_m^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} w\right) \right) \right| \right\} \\ & \leq C(1+t)^{-\frac{1}{2}}M(t)D_\infty(t). \end{aligned}$
- (v)  $\begin{aligned} & \|[\partial_{x_3}, w \cdot \nabla] \phi_\infty\|_2 + \|\chi_0[\partial_{x'}^2, w \cdot \nabla] \phi_\infty\|_2 + \|[\partial_t, w \cdot \nabla] \phi_\infty\|_2 \\ & + \sum_{m=1}^N \left\{ \sum_{j+k=1} \left\| \chi_m[\partial^{k+1} \partial_{x_3}^j, w \cdot \nabla] \phi_\infty \right\|_2 + \sum_{j+k+l=1} \left\| \chi_m[\partial_n^{l+1} \partial^k \partial_{x_3}^j, w \cdot \nabla] \phi_\infty \right\|_2 \right\} \\ & \leq C\{(1+t)^{-1}M(t)^2 + (1+t)^{-\frac{1}{4}}M(t)\sqrt{D_\infty(t)}\}. \end{aligned}$
- (vi)  $\|\partial_t(\phi w)\|_2 \leq C(1+t)^{-1}M(t)^2.$

**Proof.** The estimates (i)-(iii) and (vi) can be proved by applying Lemmas 5.15 and 5.16 similarly to the proof of [12, Proposition 8.5]. As for (iv), we have

$$\begin{aligned} & \left| \left( |\partial_{x_3}^2 \phi_\infty|^2, \operatorname{div}\left(\frac{P'(\rho_s)}{\gamma^4 \rho_s} w\right) \right) \right| + \left| \left( |\partial_{x'}^2 \phi_\infty|^2, \operatorname{div}\left(\chi_0^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} w\right) \right) \right| + \left| \left( |\partial_t \phi_\infty|^2, \operatorname{div}\left(\frac{P'(\rho_s)}{\gamma^4 \rho_s} w\right) \right) \right| \\ & + \sum_{m=1}^N \left\{ \sum_{j+k=1} \left| \left( |\partial^{k+1} \partial_{x_3}^j \phi_\infty|^2, \operatorname{div}\left(\chi_m^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} w\right) \right) \right| \right. \\ & \left. + \sum_{j+k+l=1} \left| \left( |\partial_n^{l+1} \partial^k \partial_{x_3}^j \phi_\infty|^2, \operatorname{div}\left(\chi_m^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} w\right) \right) \right| \right\} \\ & \leq C\|D\phi_\infty\|_1^2(\|w\|_\infty + \|\nabla w\|_\infty) \\ & \leq C(1+t)^{-\frac{1}{2}}M(t)D_\infty(t). \end{aligned}$$

We next consider (v). We observe that  $[\partial^{k+1}\partial_{x_3}^j, w \cdot \nabla]\phi_\infty$  and  $[\partial_n^{l+1}\partial^k\partial_{x_3}^j, w \cdot \nabla]\phi_\infty$  are written as a linear combination of terms of the forms  $a[\partial_x^q, w]\nabla\phi_\infty$  and  $(w \cdot \nabla a)\partial_x^q\phi_\infty$  with smooth function  $a = a(x')$  and integer  $q$  satisfying  $1 \leq q \leq 2$ . Therefore, applying Lemma 5.16, we obtain the desired estimate. This completes the proof.  $\square$

Let us consider the nonlinearity  $\tilde{Q}\mathbf{F} = {}^T(0, f)$ . We write  $\tilde{Q}\mathbf{F} = {}^T(0, f)$  in the form

$$\tilde{Q}\mathbf{F} = \tilde{\mathbf{F}}_0 + \tilde{\mathbf{F}}_1 + \tilde{\mathbf{F}}_2 + \tilde{\mathbf{F}}_3,$$

where  $\tilde{\mathbf{F}}_l = {}^T(0, h_l)$  ( $l = 0, 1, 2, 3$ ). Here

$$\begin{aligned} h_0 &= -w \cdot \nabla w + f_1(\rho_s, \phi) \left( -\partial_{x_3}^2 \sigma_1 w^{(0)} + \frac{\partial_{x'}^2 v_s}{\gamma^2 \rho_s} (\phi_1 + \phi_\infty) \right) \\ &\quad + f_2(\rho_s, \phi) \left( -\partial_{x_3}^2 \sigma_1 w^{(0)} - \partial_{x_3} \sigma_1 \partial_{x'} w^{(0)} \right) \\ &\quad + f_{0,1}(x', \phi) \phi \sigma_1 + f_{0,2}(x', \phi) \partial_{x_3} \sigma_1 + f_{0,3}(x', \phi) \phi (\phi_1 + \phi_\infty), \\ h_1 &= -\operatorname{div} (f_1(\rho_s, \phi) \nabla (w_1 + w_\infty)), \\ h_2 &= -\nabla (f_2(\rho_s, \phi) \operatorname{div} (w_1 + w_\infty)) + (\operatorname{div} (w_1 + w_\infty)) \nabla' (f_2(\rho_s, \phi)), \\ h_3 &= \nabla (f_3(x', \phi) \phi (\phi_1 + \phi_\infty)) - (\phi_1 + \phi_\infty) \nabla (f_3(x', \phi) \phi). \end{aligned}$$

Here  $f_{0,l} = f_{0,l}(x', \phi)$  ( $l = 1, 2, 3$ ) and  $f_3(x', \phi)$  are smooth functions of  $x'$  and  $\phi$ .

**Proposition 5.19.** *We assume that  $u(t)$  is a solution of (5.28) in  $Z(T)$ . If  $\sup_{0 \leq \tau \leq t} \|u(\tau)\|_2 \leq \epsilon_s$  and  $M(t) \leq 1$  for all  $t \in [0, T]$ , then the following estimates hold with a positive constant  $C$  independent of  $T$ .*

- (i)  $\|\tilde{Q}\mathbf{F}(t)\|_2 \leq C(1+t)^{-\frac{3}{4}} M(t)^2$ .
- (ii)  $\|h_0(t)\|_2 \leq C\{(1+t)^{-\frac{3}{4}} M(t)^2 + (1+t)^{-\frac{1}{4}} M(t) \|Dw_\infty(t)\|_2\}$ .
- (iii)  $\|h_l(t)\|_{H^1} \leq C\{(1+t)^{-1} M(t)^2 + (1+t)^{-\frac{1}{4}} M(t) \|Dw_\infty(t)\|_2\}, \quad (l = 1, 2, 3)$ .
- (iv)  $\|\partial_t h_l(t)\|_2 \leq C\{(1+t)^{-1} M(t)^2 + (1+t)^{-\frac{1}{4}} M(t) \|Dw_\infty(t)\|_2\}, \quad (l = 1, 2, 3)$ .

Proposition 5.19 can be proved in a similar manner to the proof of [12, Proposition 8.6] and [3, Proposition 8.6].

**Proof of Proposition 5.14** We first prove (5.25). We see from Proposition 5.18 and Proposition 5.19 that

$$\begin{aligned} \|Q_0\mathbf{F}\|_2 &\leq C(1+t)^{-\frac{5}{4}} M(t)^2, \\ \|\tilde{Q}\mathbf{F}\|_2 &\leq C(1+t)^{-\frac{3}{4}} M(t)^2, \end{aligned}$$

and hence,

$$\|\tilde{Q}\mathbf{F}_\infty\|_2^2 \leq C\|\mathbf{F}\|_2^2 \leq C(1+t)^{-\frac{3}{2}} M(t)^4.$$

This shows (5.25).

We next prove (5.26). We write

$$\mathcal{R} = \sum_{i=1}^4 I_i,$$

where

$$\begin{aligned} I_1 &= \left| \left( |\partial_{x_3}^2 \phi_\infty|^2, \operatorname{div} \left( \frac{P'(\rho_s)}{\gamma^4 \rho_s} w \right) \right) \right| + \left| \left( |\partial_{x'}^2 \phi_\infty|^2, \operatorname{div} \left( \chi_0^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} w \right) \right) \right| \\ &\quad + \left| \left( |\partial_t \phi_\infty|^2, \operatorname{div} \left( \frac{P'(\rho_s)}{\gamma^4 \rho_s} w \right) \right) \right| + \sum_{m=1}^N \left\{ \sum_{j+k=1} \left| \left( |\partial^{k+1} \partial_{x_3}^j \phi_\infty|^2, \operatorname{div} \left( \chi_m^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} w \right) \right) \right| \right. \\ &\quad \left. + \sum_{j+k+l=1} \left| \left( |\partial_n^{l+1} \partial^k \partial_{x_3}^j \phi_\infty|^2, \operatorname{div} \left( \chi_m^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} w \right) \right) \right| \right\}, \\ I_2 &= \left| \left( [\partial_{x_3}, w \cdot \nabla] \phi_\infty, \frac{P'(\rho_s)}{\gamma^4 \rho_s} \partial_{x_3} \phi_\infty \right) \right| + \left| \left( \chi_0^2 [\partial_{x'}, w \cdot \nabla] \phi_\infty, \frac{P'(\rho_s)}{\gamma^4 \rho_s} \partial_{x'} \phi_\infty \right) \right| \\ &\quad + \left| \left( [\partial_t, w \cdot \nabla] \phi_\infty, \frac{P'(\rho_s)}{\gamma^4 \rho_s} \partial_t \phi_\infty \right) \right| \\ &\quad + \sum_{m=1}^N \left\{ \sum_{j+k=1} \left| \left( \chi_m^2 [\partial^{k+1} \partial_{x_3}^j, w \cdot \nabla] \phi_\infty, \frac{P'(\rho_s)}{\gamma^4 \rho_s} \partial^{k+1} \partial_{x_3}^j \phi_\infty \right) \right| \right. \\ &\quad \left. + \sum_{j+k+l=1} \left\| \chi_m^2 [\partial_n^{l+1} \partial^k \partial_{x_3}^j, w \cdot \nabla] \phi_\infty \frac{P'(\rho_s)}{\gamma^4 \rho_s} \right\|_2^2 \right\}, \\ I_3 &= \left| \left( \partial_{x_3}^2 \tilde{f}_\infty^0, \frac{P'(\rho_s)}{\gamma^4 \rho_s} \partial_{x_3}^2 \phi_\infty \right) \right| + \left| \left( \partial_{x'}^2 \tilde{f}_\infty^0, \chi_0^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} \partial_{x'}^2 \phi_\infty \right) \right| + \left| \left( \partial_t \tilde{f}_\infty^0, \frac{P'(\rho_s)}{\gamma^4 \rho_s} \partial_t \phi_\infty \right) \right| \\ &\quad + \sum_{m=1}^N \sum_{j+k=1} \left| \left( \partial^k \partial_{x_3}^j \tilde{f}_\infty^0, \chi_m^2 \frac{P'(\rho_s)}{\gamma^4 \rho_s} \partial^{k+1} \partial_{x_3}^j \phi_\infty \right) \right| + \|\tilde{f}_\infty^0\|_{H^1}^2, \\ I_4 &= \left| \left( \partial_{x_3} f_\infty, \partial_{x_3} (\rho_s \partial_{x_3}^2 w_\infty) \right) \right| + \left| \left( \partial_{x'} f_\infty, \partial_{x'} (\chi_0^2 \rho_s \partial_{x'}^2 w_\infty) \right) \right| + \left| \left( \partial_t f_\infty, \rho_s \partial_t w_\infty \right) \right| \\ &\quad + \sum_{m=1}^N \sum_{j+k=1} \left| \left( \partial^k \partial_{x_3}^j f_\infty, \partial (\chi_m^2 \rho_s \partial^{k+1} \partial_{x_3}^j w_\infty) \right) \right| + \|f_\infty\|_2^2. \end{aligned}$$

From Proposition 5.18 (iv), (v) and Lemma 5.17 we see that

$$\begin{aligned} I_1 &\leq C(1+t)^{-\frac{1}{2}} M(t) D_\infty(t), \\ I_2 &\leq C(1+t)^{-\frac{1}{4}} M(t) \sqrt{D_\infty(t)} \|\phi_\infty\|_2 \\ &\leq C(1+t)^{-1} M(t)^2 \sqrt{D_\infty(t)} \\ &\leq C \left\{ (1+t)^{-\frac{7}{4}} M(t)^3 + (1+t)^{-\frac{1}{4}} M(t) D_\infty(t) \right\}. \end{aligned}$$

As for  $I_3$  and  $I_4$ , we have

$$\begin{aligned} I_3 + I_4 &\leq C \{ \|\tilde{f}_\infty^0\|_{H^2} \|\phi_\infty\|_{H^2} + \|f_\infty\|_{H^1} \|w_\infty\|_{H^2} \\ &\quad + \|\partial_t \tilde{f}_\infty^0\|_2 \|\partial_t \phi_\infty\|_2 + \|\partial_t f_\infty\|_2 \|\partial_t w_\infty\|_2 \}. \end{aligned}$$

Since  $\|Q_0 P_\infty \mathbf{F}\|_1 + \|\tilde{Q} P_\infty \mathbf{F}\|_2 \leq C \|\mathbf{F}\|_1$ , we find from Proposition 5.18 and Proposition 5.19 that

$$\|\tilde{f}_\infty^0\|_{H^2} + \|f_\infty\|_{H^1} + \|\partial_t f_\infty\|_2 \leq C \left\{ (1+t)^{-\frac{3}{4}} M(t)^2 + (1+t)^{-\frac{1}{4}} M(t) \|Dw_\infty(t)\|_2 \right\}.$$

It then follows from Lemma 5.17 that

$$\begin{aligned} & \|\tilde{f}_\infty^0\|_{H^2}\|\phi_\infty\|_{H^2} + \|f_\infty\|_{H^1}\|w_\infty\|_{H^2} + \|\partial_t f_\infty\|_2\|\partial_t w_\infty\|_2 \\ & \leq C\{(1+t)^{-\frac{3}{2}}M(t)^3 + (1+t)^{-1}M(t)^2\sqrt{D_\infty(t)}\}. \end{aligned}$$

It remains to estimate  $\|\partial_t \tilde{f}_\infty^0\|_2\|\partial_t \phi_\infty\|_2$ . Since

$$\partial_t \phi_\infty = -Q_0 L P_\infty u + Q_0 P_\infty \mathbf{F}.$$

we see from Lemma 5.16 and Proposition 5.18 (i) – (iii) that

$$\|\partial_t \phi_\infty\|_{H^1} \leq C\{\|v_s^3 \partial_{x_3} \phi_\infty\|_{H^1} + \|\partial_x w_\infty\|_{H^1} + \|Q_0 \mathbf{F}_\infty\|_{H^1}\} \leq C(1+t)^{-\frac{3}{4}}M(t).$$

This, together with Lemma 5.16 and Proposition 5.18 (i) – (iii), then yields

$$\|\partial_t \tilde{f}_\infty^0\|_2\|\partial_t \phi_\infty\|_2 \leq C\{(1+t)^{-2}M(t)^3 + (1+t)^{-\frac{5}{4}}M(t)^2\sqrt{D_\infty(t)}\},$$

and therefore, we have

$$I_3 + I_4 \leq C\{(1+t)^{-\frac{3}{2}}M(t)^2 + (1+t)^{-\frac{1}{2}}M(t)D_\infty(t)\}.$$

We thus conclude that

$$\mathcal{R}(t) \leq C\{(1+t)^{-\frac{3}{2}}M(t)^2 + (1+t)^{-\frac{1}{4}}M(t)D_\infty(t)\}.$$

This completes the proof. □

## 5.5 Asymptotic behavior

In this section we prove the asymptotic behavior (3.2).

Since  $M(t) \leq C\|u_0\|_{H^2 \cap L^1}$  for all  $t \geq 0$ , we see that

$$\|u(t) - (\sigma_1 u^{(0)})(t)\|_2 \leq C(1+t)^{-\frac{3}{4}}\|u_0\|_{H^2 \cap L^1}.$$

Therefore, to prove (3.2), it suffices to show the following

**Proposition 5.20.** *Let  $\sigma = \sigma(x_3, t)$  be the solution of (3.3) with initial value  $\sigma|_{t=0} = \langle \phi_0 \rangle$ . Assume that  $\nu \geq \nu_0$ ,  $\frac{\gamma^2}{\nu + \bar{\nu}} \geq \gamma_0^2$  and  $\omega \leq \omega_0$ . Then there exists  $\epsilon > 0$  such that if  $\|u_0\|_{H^2 \cap L^1} \leq \epsilon$ , then*

$$\|\sigma_1(t) - \sigma(t)\|_2 \leq C(1+t)^{-\frac{3}{4}+\delta}\|u_0\|_{H^2 \cap L^1} \quad (\delta > 0).$$

To prove Proposition (5.20). We prepare two lemmas.

In what follows we denote by  $\sigma = \sigma(x_3, t)$  the solution of (3.3) with initial value  $\sigma|_{t=0} = \sigma_0$ .

It is well-known that  $\sigma(t)$  has the following decay properties.

**Lemma 5.21.** *Assume that  $\sigma(t)$  is a solution of (3.3) with  $\sigma|_{t=0} = \sigma_0 \in H^1 \cap L^1$ . Then*

$$\begin{aligned}\|\partial_{x_3}^l \sigma(t)\|_2 &\leq C(1+t)^{-\frac{1}{4}-\frac{l}{2}} \|\sigma_0\|_{H^1 \cap L^1} \quad (l = 0, 1), \\ \|\sigma(t)\|_\infty &\leq C(1+t)^{-\frac{1}{2}} \|\sigma_0\|_{H^1 \cap L^1}.\end{aligned}$$

We decompose  $\mathcal{H}(t)$  into two parts. We define  $\mathcal{H}_0(t)$  and  $\mathcal{H}_\infty(t)$  by

$$\mathcal{H}_0(t) = \mathcal{F}^{-1} \mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) e^{-(i\kappa_1 \xi + \kappa_0 \xi^2)t} \mathcal{F}, \quad \mathcal{H}_\infty(t) = \mathcal{H}(t) - \mathcal{H}_0(t).$$

Then  $\mathcal{H}(t) = \mathcal{H}_0(t) + \mathcal{H}_\infty(t)$  and  $\mathcal{H}_0(t)$  and  $\mathcal{H}_\infty(t)$  have the following properties.

**Lemma 5.22.** *There hold the following estimates.*

$$\begin{aligned}\|\partial_{x_3}^l \mathcal{H}_0(t) \sigma_0\|_2 &\leq C(1+t)^{-\frac{1}{4}-\frac{l}{2}} \|\sigma_0\|_1, \\ \|\partial_{x_3}^l \mathcal{H}_\infty(t) \sigma_0\|_2 &\leq C t^{-\frac{l}{2}} e^{-\frac{\kappa_0}{2} r_0^2 t} \|\sigma_0\|_2, \\ \|\partial_{x_3}^l (e^{t\Lambda} \sigma_0 - \mathcal{H}_0(t) \sigma_0)\|_2 &\leq C(1+t)^{-\frac{3}{4}-\frac{l}{2}} \|\sigma_0\|_1.\end{aligned}$$

Lemma 5.22 can be proved in a similar manner to the proof of [2, Proposition 5.8]; and we omit the proof.

We now prove Proposition 5.20.

**Proof of Proposition 5.20.** Let  $\sigma_0 = \langle \phi_0 \rangle$ . We define  $N(t)$  by

$$N(t) = \sup_{0 \leq \tau \leq t} (1+\tau)^{\frac{3}{4}+\delta} \|\sigma_1(t) - \sigma(t)\|_{H^1}.$$

We write  $\sigma$  as

$$\sigma(t) = \mathcal{H}(t) \sigma_0 - \kappa_2 \int_0^t \mathcal{H}(t-\tau) \partial_{x_3}(\sigma^2)(\tau) d\tau. \quad (5.27)$$

As for  $\sigma_1(t)$ , by Lemma 5.6 (ii), we have

$$\begin{aligned}\mathcal{F}[\mathcal{P}\mathbf{F}] &= -i\xi \mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) \langle \widehat{\phi w^3} \rangle + \partial_{x_3} \mathcal{F}[\mathcal{P}^{(1)} \mathbf{F}] + \partial_{x_3}^2 \mathcal{F}[\mathcal{P}^{(2)} \mathbf{F}] \\ &= -i\xi \kappa_{21} \mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) \langle \widehat{\sigma_1^2} \rangle - i\xi \mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) (\langle \widehat{\phi w^3} \rangle - \langle \phi^{(0)} w^{(0),3} \rangle \langle \widehat{\sigma_1^2} \rangle) \\ &\quad + \partial_{x_3} \mathcal{F}[\mathcal{P}^{(1)}(\sigma_1^2 \mathbf{F}_1 + \mathbf{F}_2)] + \partial_{x_3}^2 \mathcal{F}[\mathcal{P}^{(2)} \mathbf{F}],\end{aligned}$$

where  $\kappa_{21} = \langle \phi^{(0)} w^{(0),3} \rangle$ . Furthermore,

$$\begin{aligned}\mathcal{F}[\mathcal{P}^{(1)}(\sigma_1^2 \mathbf{F}_1)] &= \mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) \langle \widehat{\sigma_1^2 \mathbf{F}_1}, u^{*(1)} \rangle = \mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) \langle \mathbf{F}_1, u^{*(1)} \rangle \langle \widehat{\sigma_1^2} \rangle \\ &= -\kappa_{22} \mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) \langle \widehat{\sigma_1^2} \rangle,\end{aligned}$$

where  $\kappa_{22} = -\langle \mathbf{F}_1, u^{*(1)} \rangle$ . We thus obtain

$$\begin{aligned}e^{(t-\tau)\Lambda} \mathcal{P}\mathbf{F} &= -\kappa_2 e^{(t-\tau)\Lambda} \partial_{x_3}(\sigma_1^2) - e^{(t-\tau)\Lambda} \partial_{x_3} \{ \langle \phi w^3 \rangle - \langle \phi^{(0)} w^{(0),3} \rangle \sigma_1^2 \} \\ &\quad + e^{(t-\tau)\Lambda} J_4 + e^{(t-\tau)\Lambda} J_5.\end{aligned}$$

Here we set  $\kappa_2 = \kappa_{21} + \kappa_{22}$ ,

$$\begin{aligned} J_4 &= \partial_{x_3} \mathcal{P}^{(1)} \mathbf{F}_2 + \partial_{x_3}^2 \mathcal{P}^{(2)} \mathbf{F}_2, \\ J_5 &= \partial_{x_3}^2 \mathcal{P}^{(2)} (\sigma_1^2 \mathbf{F}_1). \end{aligned}$$

It then follows from (5.29) and (5.27) that  $\sigma_1(t) - \sigma(t)$  is written as

$$\sigma_1(t) - \sigma(t) = \sum_{j=0}^5 I_j(t),$$

where

$$\begin{aligned} I_0(t) &= e^{t\Lambda} \mathcal{P} u_0 - \mathcal{H}(t) \sigma_0 + \kappa_2 \int_0^t \mathcal{H}_\infty(t-\tau) \partial_{x_3}(\sigma^2)(\tau) d\tau, \\ I_1(t) &= -\kappa_2 \int_0^t \mathcal{H}_0(t-\tau) \partial_{x_3}(\sigma_1^2 - \sigma^2) d\tau, \\ I_2(t) &= -\kappa_2 \int_0^t (e^{(t-\tau)\Lambda} - \mathcal{H}_0(t-\tau)) \partial_{x_3}(\sigma_1^2)(\tau) d\tau, \\ I_3(t) &= - \int_0^t \partial_{x_3} e^{(t-\tau)\Lambda} (\langle \phi w^3 \rangle - \langle \phi^{(0)} w^{(0),3} \rangle \sigma_1^2) d\tau, \\ I_j(t) &= \int_0^t e^{(t-\tau)\Lambda} J_j(\tau) d\tau, \quad (j = 4, 5). \end{aligned}$$

We see from Proposition 4.44 and Lemmas 5.21, 5.22 that

$$\begin{aligned} &\|I_0(t)\|_{H^1} \\ &\leq C \left\{ (1+t)^{-\frac{3}{4}} \|u_0\|_{H^1 \cap L^1} + \int_0^t (t-\tau)^{-\frac{l}{2}} e^{-\frac{\kappa_0}{2} r_0^2(t-\tau)} \|\sigma\|_\infty \|\partial_{x_3} \sigma\|_2(\tau) d\tau \right\} \\ &\leq C \left\{ (1+t)^{-\frac{3}{4}} \|u_0\|_{H^1 \cap L^1} + \int_0^t (t-\tau)^{-\frac{l}{2}} e^{-\frac{\kappa_0}{2} r_0^2(t-\tau)} (1+\tau)^{-\frac{5}{4}} d\tau \|u_0\|_{H^1 \cap L^1}^2 \right\} \\ &\leq C (1+t)^{-\frac{3}{4}} \|u_0\|_{H^1 \cap L^1} \{1 + \|u_0\|_{H^1 \cap L^1}\}. \end{aligned}$$

As for  $I_1(t)$ , we first observe

$$\|(\sigma_1^2 - \sigma^2)(t)\|_1 \leq \|(\sigma_1 + \sigma)(t)\|_2 \|(\sigma_1 - \sigma)(t)\|_2 \leq C(1+t)^{-1+\delta} N(t) \|u_0\|_{H^2 \cap L^1}.$$

Since  $\partial_{x_3}^k \mathcal{H}_0(t) = \mathcal{H}_0(t) \partial_{x_3}^k$  ( $k = 0, 1$ ), we see from Lemma 5.22 that

$$\begin{aligned} \|\partial_{x_3}^k I_1(t)\|_2 &\leq C \int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} (1+\tau)^{-1+\delta} d\tau \|u_0\|_{H^2 \cap L^1} N(t) \\ &\leq C (1+t)^{-\frac{3}{4}+\delta} \|u_0\|_{H^2 \cap L^1} N(t) \end{aligned}$$

for  $k = 0, 1$ .

As for  $I_2(t)$ , we see from Lemma 5.22 that

$$\|\partial_{x_3}^k I_2(t)\|_2 \leq C \int_0^t (1+t-\tau)^{-\frac{5}{4}-\frac{k}{2}} (1+\tau)^{-1} d\tau M(t)^2 \leq C(1+t)^{-1} \|u_0\|_{H^2 \cap L^1}^2$$

for  $k = 0, 1$ .

As for  $I_3(t)$ , since

$$\begin{aligned} & \|\langle \phi w^3 \rangle(\tau) - \langle \phi^{(0)} w^{(0),3}(\tau) \rangle \sigma_1^2(\tau)\|_1 \\ & \leq C \{ \|\sigma_1(\tau)\|_2 \|u(\tau) - \sigma_1(\tau) u^{(0)}(\tau)\|_2 + \|u(\tau) - \sigma_1(\tau) u^{(0)}(\tau)\|_2^2 \} \\ & \leq C(1 + \tau)^{-1} M(t)^2, \end{aligned}$$

we have

$$\begin{aligned} \|\partial_{x_3}^k I_3(t)\|_2 & \leq C M(t)^2 \int_0^t (1 + \tau)^{-\frac{3}{4} - \frac{k}{2}} (1 + \tau)^{-1} d\tau \\ & \leq C(1 + t)^{-\frac{3}{4}} \log(1 + t) \|u_0\|_{H^2 \cap L^1}^2 \end{aligned}$$

for  $k = 0, 1$ .

By Proposition 4.42 and Lemma 5.7,  $I_4(t)$  is estimated as

$$\begin{aligned} \|\partial_{x_3}^k I_4(t)\|_2 & = \left\| \int_0^t e^{(t-\tau)\Lambda} \partial_{x_3} (\mathcal{P}^{(1)} \mathbf{F}_2(\tau) + \partial_{x_3} \mathcal{P}^{(2)} \mathbf{F}_2(\tau)) d\tau \right\|_2 M(t)^2 \\ & \leq C \int_0^t (1 + t - \tau)^{-\frac{3}{4} - \frac{k}{2}} (1 + \tau)^{-1} d\tau \|u_0\|_{H^2 \cap L^1}^2 \\ & \leq C(1 + t)^{-\frac{3}{4}} \log(1 + t) \|u_0\|_{H^2 \cap L^1}^2 \end{aligned}$$

for  $k = 0, 1$ .

As for  $I_5(t)$ , since  $\partial_{x_3} \mathcal{P}^{(2)}(\tau) = \mathcal{P}^{(2)} \partial_{x_3}$ , we see from Lemma 5.7 that

$$\begin{aligned} \|\partial_{x_3}^k I_5(t)\|_2 & \leq \left\| \int_0^t e^{(t-\tau)\Lambda} \partial_{x_3}^{k+1} \mathcal{P}^{(2)} (\partial_{x_3} (\sigma_1^2) \mathbf{F}_1)(\tau) d\tau \right\|_2 \\ & \leq C \left\{ \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{3}{4} - \frac{k}{2}} (1 + \tau)^{-1} d\tau M(t)^2 \right. \\ & \quad \left. \leq C(1 + t)^{-\frac{3}{4}} \log(1 + t) \|u_0\|_{H^2 \cap L^1}^2 \right\} \end{aligned}$$

for  $k = 0, 1$ .

Therefore, we obtain

$$\|(\sigma_1 - \sigma)(t)\|_{H^1} \leq C(1 + t)^{-\frac{3}{4} + \delta} \|u_0\|_{H^2 \cap L^1} \{1 + \|u_0\|_{H^2 \cap L^1} + \|u_0\|_{H^2 \cap L^1}^2 + N(t)\}.$$

It then follows that if  $\|u_0\|_{H^2 \cap L^1}$  is sufficiently small, then

$$N(t) \leq C \|u_0\|_{H^2 \cap L^1}.$$

We thus see that if  $\|u_0\|_{H^2 \cap L^1} \ll 1$ , then

$$\|\sigma_1(t) - \sigma(t)\|_2 \leq C(1 + t)^{-\frac{3}{4} + \delta} \|u_0\|_{H^2 \cap L^1}$$

This completes the proof. □

In this section we formulate the problem. The problem (1.5)-(1.8) is written as

$$\frac{du}{dt} + Lu = \mathbf{F}, \quad w|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0. \quad (5.28)$$

Here  $u = {}^T(\phi, w)$ ;  $\mathbf{F} = \mathbf{F}(u)$  denotes the nonlinearity:

$$\mathbf{F} = {}^T(f^0(\phi, w), \mathbf{f}(\phi, w)).$$

The local solvability in  $Z(T)$  for (5.28) follows from [13].

**Proposition 5.23.** *If  $u_0 = {}^T(\phi_0, w_0)$  satisfies the following conditions;*

(i)  $u_0 \in H^2 \times (H^2 \cap H_0^1)$ ,

(ii)  $-\frac{\gamma^2}{4}\rho_1 \leq \phi_0$ ,

*then there exists a number  $T_0 > 0$  depending on  $\|u_0\|_{H^2}$  and  $\rho_1$  such that the following assertions hold. Problem (5.28) has a unique solution  $u(t) \in Z(T)$  satisfying*

$$\phi(x, t) \geq -\frac{\gamma^2}{2}\rho_1 \quad \text{for } \forall (x, t) \in \Omega \times [0, T_0];$$

*and the following estimate holds*

$$\|u\|_{Z(T)}^2 \leq C_0 \{1 + \|u_0\|_{H^2}^2\}^\alpha \|u_0\|_{H^2}^2$$

*for some positive constants  $C_0$  and  $\alpha$ .*

Theorem 3.1 would follow if we would establish the a priori estimates of  $u(t)$  in  $Z(T)$  uniformly for  $T$ .

To obtain the appropriate a priori estimates, we decompose the solution  $u$  into its  $P_0$  and  $P_\infty$  parts. Let us decompose the solution  $u(t)$  of (5.28) as

$$u(t) = (\sigma_1 u^{(0)})(t) + u_1(t) + u_\infty(t),$$

where

$$\sigma_1(t) = \mathcal{P}u(t), \quad u_1(t) = (\mathcal{T} - \mathcal{T}^{(0)})\mathcal{P}u(t), \quad u_\infty(t) = P_\infty u(t).$$

Note that  $P_0 u(t) = (\sigma_1 u^{(0)})(t) + u_1(t)$ .

Since  $u_1(t)$  is written as

$$u_1(t) = (\mathcal{T} - \mathcal{T}^{(0)})\mathcal{P}u(t) = (\partial_{x_3} \mathcal{T}^{(1)} + \partial_{x_3}^2 \mathcal{T}^{(2)})\sigma_1(t),$$

we see from Proposition 4.40 and Proposition 4.41 the following estimates for  $\sigma_1(t)$  and  $u_1(t)$ .

**Proposition 5.24.** *Let  $u(t)$  be a solution of (5.28) in  $Z(T)$ . Then there hold the estimates*

$$\|\partial_{x_3}^l \sigma_1(t)\|_2 \leq C \|\partial_{x_3} \sigma_1(t)\|_2$$

*for  $1 \leq l \leq 3$ ; and*

$$\|\partial_{x'}^k \partial_{x_3}^l \partial_t^m u_1(t)\|_2 \leq C \|\partial_{x_3} \sigma_1(t)\|_2 + \|\partial_t \sigma_1(t)\|_2$$

*for  $0 \leq k + l + 2m \leq 3$ .*

We derive the equations for  $\sigma_1(t)$  and  $u_\infty(t)$ .

**Proposition 5.25.** *Let  $T > 0$  and assume that  $u(t)$  is a solution of (5.28) in  $Z(T)$ . Then the following assertions hold.*

$$\sigma_1 \in \bigcap_{j=0}^1 C^j([0, T] : H^2(\mathbf{R})), \quad u_\infty \in Z(T), \quad \phi_\infty \in C^1([0, T]; H^1).$$

Furthermore,  $\sigma_1$  and  $u_\infty$  satisfy

$$\sigma_1(t) = e^{t\Lambda} \mathcal{P}u_0 + \int_0^t e^{(t-\tau)\Lambda} \mathcal{P}\mathbf{F}(\tau) d\tau; \quad (5.29)$$

and

$$\partial_t u_\infty + Lu_\infty = \mathbf{F}_\infty, \quad w_\infty|_{\partial\Omega} = 0, \quad u_\infty|_{t=0} = u_{\infty,0}, \quad (5.30)$$

where  $\mathbf{F}_\infty = P_\infty \mathbf{F}$  and  $u_{\infty,0} = P_\infty u_0$ .

Let  $u(t)$  be a solution of (5.28) in  $Z(T)$ . From Proposition 5.24, we obtain

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{4}} \{ \|u_1(\tau)\|_2 + \|\partial_x u_1(\tau)\|_2 \} \\ & \leq C \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{4}} \{ \|\partial_{x_3} \sigma_1(\tau)\|_2 + \|\partial_\tau \sigma_1(\tau)\|_2 \}, \end{aligned}$$

and thus, the estimates for  $u_1(t)$  follows from the ones for  $\sigma_1(t)$ . Therefore, as in [3], we introduce the quantity  $M_1(t)$  defined by

$$M_1(t) = \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{1}{4}} \|\sigma_1(\tau)\|_2 + \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{4}} \{ \|\partial_{x_3} \sigma_1(\tau)\|_2 + \|\partial_\tau \sigma_1(\tau)\|_2 \};$$

and we define the quantity  $M(t) \geq 0$  by

$$M(t)^2 = M_1(t)^2 + \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{2}} E_\infty(\tau) \quad (t \in [0, T])$$

with

$$E_\infty(t) = \|u_\infty(t)\|_2^2.$$

We define a quantity  $D_\infty(t)$  for  $u_\infty = {}^T(\phi_\infty, w_\infty)$  by

$$D_\infty(t) = \|D\phi_\infty(t)\|_1^2 + \|Dw_\infty(t)\|_2^2.$$

If we could show  $M(t) \leq C$  uniformly for  $t \geq 0$ , then Theorem 3.1 would follow. The uniform estimate for  $M(t)$  is given by using the following estimates for  $M_1(t)$  and  $E_\infty(t)$ .

**Proposition 5.26.** *There exist positive constants  $\nu_0$ ,  $\gamma_0$  and  $\omega_0$  such that if  $\nu \geq \nu_0$ ,  $\frac{\gamma^2}{\nu+\bar{\nu}} \geq \gamma_0^2$  and  $\omega \leq \omega_0$ , then the following assertions hold. There is a positive number*

$\epsilon_1$  such that if a solution  $u(t)$  of (5.28) in  $Z(T)$  satisfies  $\sup_{0 \leq \tau \leq t} \|u(\tau)\|_2 \leq \epsilon_1$  and  $M(t) \leq 1$  for  $t \in [0, T]$ , then the estimates

$$M_1(t) \leq C\{\|u_0\|_{L^1} + M(t)^2\} \quad (5.31)$$

and

$$\begin{aligned} E_\infty(t) + \int_0^\infty e^{-a(t-\tau)} D_\infty(\tau) d\tau \\ \leq C\{e^{-at} E_\infty(0) + (1+t)^{-\frac{3}{2}} M(t)^4 + \int_0^t e^{-a(t-\tau)} \mathcal{R}(\tau) d\tau\} \end{aligned} \quad (5.32)$$

hold uniformly for  $t \in [0, T]$  with  $C > 0$  independent of  $T$ . Here  $a = a(\nu, \tilde{\nu}, \gamma)$  is a positive constant; and  $\mathcal{R}(t)$  is a function satisfying the estimate

$$\mathcal{R}(t) \leq C\{(1+t)^{-\frac{3}{2}} M(t)^3 + (1+t)^{-\frac{1}{4}} M(t) D_\infty(t)\} \quad (5.33)$$

provided that  $\sup_{0 \leq \tau \leq t} \|u(\tau)\|_2 \leq \epsilon_2$  and  $M(t) \leq 1$ .

Proposition 5.26 follows from Propositions 5.5, 5.8 and 5.14 below.

As in [3, 12], one can see from Propositions 5.23 and 5.26 that if  $\|u_0\|_{H^2 \cap L^1}$  is sufficiently small, then

$$M(t) \leq C\|u_0\|_{H^2 \cap L^1}$$

uniformly for  $t \geq 0$ , which proves Theorem 3.1.

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