Study on decay properties of solutions of the compressible Navier-Stokes equation in critical spaces

沖田, 匡聡

https://doi.org/10.15017/1500514

出版情報：九州大学, 2014, 博士（数理学）, 課程博士
バージョン：

Study on decay properties of solutions of the compressible Navier-Stokes equation in critical spaces

Masatoshi Okita

Graduate School of Mathematics,
Kyushu University,
Fukuoka 819-0395, Japan
Email: m-okita@math.kyushu-u.ac.jp

Acknowledgement

I would like to express my sincere gratitude to Professor Yoshiyuki Kagei for his valuable advice, continuous encouragement and for leading me to the study of nonlinear PDE's. In addition, I am grateful to Professors Shuichi Kawashima and Takayuki Kobayashi for their important suggestions and helpful comments.

Abstract

This thesis studies the convergence rates of strong solutions of the compressible Navier-Stokes equations on the whole space $\mathbb{R}^n$. The main part of this thesis is divided into two part.

In Part I, we study the optimal convergence rates for the compressible Navier-Stokes equation with a potential external force $\nabla\Phi$ for space dimension $n \geq 3$. It is proved that the perturbation and its first order derivatives decay in L^2 norm in $O(t^{-\frac{n}{4}})$ and $O(t^{-\frac{n}{4}-\frac{1}{2}})$ respectively, if the initial perturbation is small in $H^{s_0}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $s_0 = [\frac{n}{2}] + 1$ and the potential force Φ is small in some Sobolev space.

In Part II, we consider the optimal decay estimates in critical Besov spaces. The optimal decay estimates in critical spaces are established if the initial perturbations of density and velocity are small in $\dot{B}^{\frac{n}{2}}_{2,1}(\mathbb{R}^n) \cap \dot{B}^0_{p,\infty}(\mathbb{R}^n)$ and $\dot{B}^{\frac{n}{2}-1}_{2,1}(\mathbb{R}^n) \cap \dot{B}^0_{p,\infty}(\mathbb{R}^n)$ with $1 \leq p < \frac{2n}{n+1}$, respectively, for $n \geq 2$.

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1 Introduction

This thesis studies the initial value problem for the compressible Navier-Stokes equation (with potential force) on $\mathbb{R}^n$:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t u + (u \cdot \nabla) u + \frac{\nabla P(\rho)}{\rho} = \frac{\mu}{\rho} \Delta u + \frac{\mu + \mu'}{\rho} \nabla (\nabla \cdot u) - \nabla \Phi(x), \\ (\rho, u)(0, x) = (\rho_0, u_0)(x) \longrightarrow (\bar{\rho}, 0) \qquad |x| \to \infty. \end{cases} \tag{1}$$

Here $t > 0$ and $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$; the unknown functions $\rho = \rho(t, x) > 0$ and $u = u(t, x) = (u_1(t, x), u_2(t, x), \cdots, u_n(t, x))$ denote the density and velocity, respectively; $P = P(\rho)$ is the pressure that is assumed to be a function of the density ρ; μ and μ' are the viscosity coefficients satisfying the conditions $\mu > 0$ and $\mu' + \frac{2}{n}\mu > 0$; and $\nabla\cdot$, ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to x, respectively.

We assume that $P(\rho)$ is smooth and $P'(\rho)$ is positive for ρ in a neighborhood of $\bar{\rho}$, where $\bar{\rho}$ is a given positive constant .

When $\Phi = 0$, $(1)_1 - (1)_2$ has a (constant) stationary solution $(\rho_*(x), u_*) = (\bar{\rho}, 0)$. On the other hand, when Φ is small but $\Phi \neq 0$, the Navier-Stokes equation $(1)_1 - (1)_2$ with potential force has the stationary solution $(\rho_*, u_*) = (\rho_*(x), 0)$, where ρ_* satisfies, cf. [20]

$$\int_{\bar{\rho}}^{\rho_*(x)} \frac{P'(s)}{s} ds + \Phi(x) = 0. \tag{2}$$

In this thesis we derive the convergence rate of solutions of problem (1) to the stationary solution $(\rho_*, 0)$ as $t \to \infty$ when the initial perturbation is sufficiently small.

We first state our result on the convergence rate when $\Phi \neq 0$.

Theorem 1.1. *Assume that $n \geq 3$. Let (ρ, u) be a global solution in H^{s_0} with $s_0 = [\frac{n}{2}] + 1$, to the problem (1). Then there exist $\epsilon > 0$ such that if $(\rho_0 - \rho_*, u_0) \in H^{s_0} \cap L^1$ and*

$$\|(\rho_0 - \rho_*, u_0)\|_{H^{s_0} \cap L^1} \leq \epsilon$$

$$\|\Phi\|_{H^{s_0+1}} + \|(1 + |x|)\nabla\Phi\|_{L^2} \leq \epsilon$$

then, the estimates

$$\|\nabla^k (\rho - \rho_*, u)(t)\|_{L^2} \leq C_0 (1 + t)^{-\frac{n}{4} - \frac{k}{2}}, \quad k = 0, 1, \tag{3}$$

hold for $t \geq 0$.

The proof of Theorem 1.1 will be given in Part I.

Remark 1.2. *When $\Phi = 0$, one can also obtain the decay rates for the perturbation of higher-order spatial derivatives. In fact, one can prove the following estimates. Let $\Phi = 0$ and let (ρ, u) be a global solution in H^l with $l \geq s_0 = [\frac{n}{2}] + 1$, to the* *problem* (1) *and assume that* $(\rho_0 - \bar{\rho}, u)$ *is sufficiently small in* H^l. *Then it holds that*

$$\|\nabla^k U(t)\|_2 \leq C_0(1+t)^{-\frac{n}{4}-\frac{k}{2}}, \quad k = 0, 1, \cdots, s_0,$$

$$\|\nabla^k U(t)\|_2 \leq C_0(1+t)^{-\frac{n}{4}-\frac{s_0}{2}}, \quad s_0 \leq k \leq l$$

for $t \geq 0$.

We next states our result on the convergence rate of solution in critical Besov spaces when $\Phi = 0$. When $\Phi = 0$, we have the constant stationary solution $(\rho_*, 0) = (\bar{\rho}, 0)$.

Theorem 1.3. *Assume that* $n \geq 2$ *and* $\Phi = 0$ *and* $1 \leq p < \frac{2n}{n+1}$. *Then there exists* $\epsilon > 0$ *such that if*

$$(\rho_0 - \bar{\rho}) \in \dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^0_{p,\infty}, \ \ u_0 \in \dot{B}^{\frac{n}{2}-1}_{2,1} \cap \dot{B}^0_{p,\infty}$$

and

$$\|\rho_0 - \bar{\rho}\|_{\dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^0_{p,\infty}} + \|u_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1} \cap \dot{B}^0_{p,\infty}} \leq \epsilon,$$

then problem (1) *has a unique global solution* (ρ, u) *satisfying*

$$(\rho - \bar{\rho}, u) \in C([0,\infty); B^{\frac{n}{2}}_{2,1}) \times \big(C([0,\infty); B^{\frac{n}{2}-1}_{2,1}) \cap L^1(0,\infty; \dot{B}^{\frac{n}{2}+1}_{2,1})\big).$$

Furthermore, there exists a constant $C_0 > 0$ *such that the estimates*

$$\|(\rho - \bar{\rho}, u)(t)\|_{L^2} \leq C_0(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}, \tag{4}$$

$$\|u(t)\|_{\dot{B}^{s_1}_{2,1}} \leq C_0(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{s_1}{2}}, \qquad \text{for} \ \ 0 \leq s_1 \leq \tfrac{n}{2} - 1 \tag{5}$$

$$\|(\rho - \bar{\rho})(t)\|_{\dot{B}^{s_2}_{2,1}} \leq C_0(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{s_2}{2}}, \qquad \text{for} \ \ 0 \leq s_2 \leq \tfrac{n}{2} \tag{6}$$

hold for $t \geq 0$.

The proof of Theorem 1.3 will be given in Part II

For the compressible Navier-Stokes equations, a lot of works on the large time behavior of solutions have been done. Concerning the convergence rate to the motionless stationary solution, which is the main subject of this thesis, we first mention that, when $\Phi = 0$, Matsumura-Nishida [18] showed the global in time existence of the solution of (1) for $n = 3$, provided that the initial perturbation $(\rho_0 - \bar{\rho}, u_0)$ is sufficiently small in $H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. (See also [19].) Furthermore, the following decay estimates were obtained in [18]

$$\|\nabla^k(\rho - \bar{\rho}, u)(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}} \quad k = 0, 1. \tag{7}$$

These results were proved by combining the energy method and the decay estimates of the semigroup $E(t)$ generated by the linearized operator A at the constant state $(\bar{\rho}, 0)$.

On the other hand, Kawashita [14] showed the global existence of solutions for initial perturbations sufficiently small in $H^{s_0}(\mathbb{R}^n)$ with $s_0 = [\frac{n}{2}] + 1$, $n \geq 2$. (Note that $s_0 = 2$ for $n = 3$). Wang-Tan [26] then considered the case $n = 3$ when the initial perturbation $(\rho_0 - \bar{\rho}, u_0)$ is sufficiently small in $H^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, and proved the decay estimates (7). Li-Zhang [17] showed that the density and momentum converge at the rates $(1+t)^{-\frac{3}{4}-\frac{s}{2}}$ in the L^2-norm, when initial perturbations are sufficiently small in $H^l(\mathbb{R}^3) \cap \dot{B}^{-s}_{1,\infty}(\mathbb{R}^3)$ with $l \geq 4$ and $s \in [0,1]$. Note that L^1 is included in $\dot{B}^0_{1,\infty}$. We also mention interesting works in [9, 16] where decay estimates in L^p norm were studied.

Danchin [2] proved the global existence in a critical homogeneous Besov space, i.e., a scaling invariant Besov space. The system $(1)_1 - (1)_2$ is invariant under the following transformation

$$\rho_\lambda(t,x) := \rho(\lambda^2 t, \lambda x), \quad u_\lambda(t,x) := \lambda u(\lambda^2 t, \lambda x).$$

More precisely, if (ρ, u) solves (1), so dose $(\rho_\lambda, u_\lambda)$ provided that the pressure law P has been changed into $\lambda^2 P$. Usually, we call that a functional space is a critical space for (1) if the associated norm is invariant under the transformation $(\rho, u) \to (\rho_\lambda, u_\lambda)$ (up to a constant independent of λ). It is easy to see that homogeneous Besov space $C\big([0,\infty); \dot{B}^{\frac{n}{p}}_{p,1} \times \dot{B}^{\frac{n}{p}-1}_{p,1}\big)$ is a critical space for (1); and Danchin [2] proved the global existence in $C\big([0,\infty); \dot{B}^{\frac{n}{p}}_{p,1}\big) \times \Big(C\big([0,\infty); \dot{B}^{\frac{n}{p}-1}_{p,1}\big) \cap L^1\big(0,\infty; \dot{B}^{\frac{n}{p}+1}_{p,1}\big)\Big)$, together with the estimate,

$$\begin{aligned} &\sup_{t\geq 0}\{\|\rho(t)-\bar{\rho}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|u(t)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\} + \int_0^\infty \|u\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} dt \\ \leq\ & M\big(\|\rho_0 - \bar{\rho}\|_{\dot{B}^{\frac{n}{2}}_{2,1}\cap \dot{B}^{\frac{n}{2}-1}_{2,1}} + \|u_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\big), \end{aligned} \tag{8}$$

if the initial perturbation is sufficiently small in $\big(\dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^{\frac{n}{2}-1}_{2,1}\big) \times \dot{B}^{\frac{n}{2}-1}_{2,1}$ for $n \geq 2$.

On the other hand, nonhomogeneous Besov space $B^{\frac{n}{p}}_{p,1} \times B^{\frac{n}{p}-1}_{p,1}$ is called a critical regularity space for (1). Haspot [8] proved the local solvability in a critical regularity space. In Theorem 1.3, we obtained decay estimate of solution if initial perturbation is sufficiently small in the critical regularity space and L^1.

Concerning the case $\Phi \neq 0$, Matsumura-Nishida [20] proved the global in time existence of solution of (1) for $n = 3$, provided that the initial perturbation $(\rho_0 - \rho_*, u_0)$ and Φ are sufficiently small. Moreover, Duan-Liu-Ukai-Yang [5] established the decay estimates :

$$\|\nabla^k(\rho - \rho_*, u)(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}} \quad k = 0, 1$$

for initial perturbation $(\rho_0 - \rho_*, u_0)$ sufficiently small in $H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. (Cf, [22].) Concerning the problem on the half-space and exterior domains we refer the reader to [4, 10, 11, 12, 15]. (See also [13].)

Theorem 1.1 is an extension of the results in [5] and [26] by an approach different to [5, 26]. To prove Theorem 1.1, as in [13], we introduce a decomposition of the perturbation $U(t) = (\rho - \rho_*, u)(t)$ associated with the spectral properties of the linearized operator A at the constant state $(\bar{\rho}, 0)$. In the case of our problem, we simply decompose the perturbation $U(t)$ into low and high frequency parts. As for the low frequency part, we apply the decay estimates for the low frequency part of $E(t)$; while the high frequency part is estimated by using the energy method. One of the points of our approach is that by restricting the use of the decay estimates for $E(t)$ to its low frequency part, one can avoid the derivative loss due to the convective term of the transport equation $(1)_1$. On the other hand, the convective term of $(1)_1$ can be controlled by the energy method which we apply to the high frequency part. Another point is that in the high frequency part we have a Poincaré type inequality: $\|\nabla U_\infty\|_{L^2} \geq C\|U_\infty\|_{L^2}$, where U_∞ is the high frequency part of the perturbation U. This yields the strict positivity inequality $(AU_\infty, U_\infty)_{L^2} + \gamma\|\nabla\sigma_\infty\|_{L^2}^2 \geq C_0\|U_\infty\|_{L^2}^2$ for some positive constants C_0 and γ, where σ_∞ denotes the density component of U_∞. Furthermore, the Poincaré type inequality makes the estimate of the nonlinearity a little bit simpler in the energy method. Using these properties we can deal with the time decay of $\|U(t)\|_{H^{s_0}}$ in contrast to the approach in [5, 26] which, roughly speaking, deals mainly with $\|\nabla U(t)\|_{H^{s_0-1}}$.

To prove Theorem 1.3, as in [13] and Theorem 1.1, we introduce a decomposition of the perturbation $U(t) = (\rho - \bar{\rho}, u)(t)$ associated with the spectral properties of the linearized operator A. In the case of our problem, we decompose the perturbation $U(t)$ into low and high frequency parts. As for the low frequency part, we apply the decay estimates for the low frequency part of the semigroup $E(t)$; while the high frequency part is estimated by using the energy method by Danchin. We note that in estimating the low frequency part, we also make use of the fact that any order of differentiation acts as a bounded operator on the low frequency part, so that we can establish the decay estimate for the norm of the velocity with critical regularity. (See Remark 7.4 below.) On the other hand, the convective term of $(1)_1$ can be controlled by the energy method and commutator estimate which we apply to the high frequency part. In the estimates of nonlinearities we carefully compute nonlinear interactions between low-low, low-high and high-high frequency parts. We also use the estimate $\int_0^\infty \|u\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} dt < M\epsilon$, that follows from (8) established by Danchin [2].

The thesis is organized as follows. In Section 2 we introduce the notations, some properties of Besov spaces, and auxiliary lemmas used in this thesis. The main part of this thesis is divided into two parts. In Part I, which consists of Sections 3, 4 and 5, we study the compressible Navier-Stokes equation with potential force. In Section 3 we state the existence and spacial decay property of stationary solution for the compressible Navier-Stokes equation with potential force. We then rewrite the problem to perturbation equations. In Section 4 we introduce a decomposition of the solution into low and high frequency parts, and we state properties of functions of low and high frequency parts. In Section 5 we give the proof of Theorem 1.1. In Part II, which consists of Sections 6 and 7, we study the compressible Navier-Stokes equation in critical spaces. In Section 6, we rewrite the system into the one for the perturbation and introduce auxiliary lemmas used in the proof of Theorem 1.3. In Section 6, we give a proof of Theorem 1.3.

2 Preliminaries

In this section we first introduce the notation which will be used throughout this thesis. Some useful lemmas will be given subsequently.

2.1 Notation

Let $L^p(1 \le p \le \infty)$ denote the usual L^p-Lebesgue space on $\mathbb{R}^n$ with norm $\|\cdot\|_p$. For nonnegative integer m, we denote by $W^{m,p}(1 \le p \le \infty)$ the usual L^p-Sobolev space of order m whose norm is denoted by $\|\cdot\|_{W^{m,p}}$. When $p = 2$, we define $H^m = W^{m,2}$. $\mathcal{S}'$ denotes dual space of the Schwartz space. The inner-product of L^2 is denoted by $(\cdot,\cdot)$. If S is any nonempty subset of $\mathbb{Z}$, sequence space $l^p(S)$ denote the usual l^p sequence space on S.

We introduce the following notation for spatial derivatives. For a multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, we denote

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}, \quad |\alpha| = \sum_{i=1}^n \alpha_i,$$

and for any integer $l \ge 0, \nabla^l f$ denotes all of l-th derivatives of f.

For a function f, we denote its Fourier transform by $\mathfrak{F}[f] = \hat{f}$:

$$\mathfrak{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx \quad (\xi \in \mathbb{R}).$$

The inverse Fourier transform is denoted by $\mathfrak{F}^{-1}[f] = \check{f}$,

$$\mathfrak{F}^{-1}[f](x) = \check{f}(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} f(\xi)e^{i\xi\cdot x}d\xi \quad (x \in \mathbb{R}).$$

For operators A, B, we denote the commutator of A and B by $[A, B]$:

$$[A, B]f = A(Bf) - B(Af).$$

BC^k denotes the set of all functions such that $\nabla^l f$ is a bounded function for $l \le k$.

Let us next define the homogeneous and nonhomogeneous Besov spaces. First we introduce the dyadic partition of unity. We can use for instance any $\{\phi, \chi\} \in C^\infty$, such that

$$\operatorname{Supp}\phi \subset \{\xi \in \mathbb{R}^n | \frac{3}{4} \le |\xi| \le \frac{8}{3}\},$$

$$\operatorname{Supp}\chi \subset \{\xi \in \mathbb{R}^n | |\xi| \le \frac{4}{3}\},$$

$$\chi(\xi) + \sum_{j\ge 0}\phi(2^{-j}\xi) = 1 \ \ for\ \xi \in \mathbb{R}^n,$$

$$\sum_{j\in\mathbb{Z}}\phi(2^{-j}\xi) = 1 \quad for\ \xi \in \mathbb{R}^n\backslash\{0\},$$

$$\operatorname{Supp}\phi(2^{-j}\cdot)\cap\operatorname{Supp}\phi(2^{-j'}\cdot)=\emptyset\quad for\ |j-j'|\geq 2,$$
$$\operatorname{Supp}\chi\cap\operatorname{Supp}\phi(2^{-j}\cdot)=\emptyset\quad for\ j\geq 1.$$

Denoting $h=\mathfrak{F}^{-1}\phi$ and $\tilde{h}=\mathfrak{F}^{-1}\chi$, we then define the dyadic blocks by

$$\begin{aligned}\triangle_{-1}u&=\tilde{h}*u,\\ \triangle_j u&=2^{jn}\int_{\mathbb{R}^n}h(2^jy)u(x-y)dy\quad \text{if } j\geq 0,\\ \dot{\triangle}_j u&=2^{jn}\int_{\mathbb{R}^n}h(2^jy)u(x-y)dy\quad \text{if } j\in\mathbb{Z}.\end{aligned}$$

The low-frequency cut-off operators are defined by

$$S_ju=\sum_{-1\leq k\leq j-1}\triangle_k u,\quad \dot{S}_ju=\sum_{k\leq j-1}\dot{\triangle}_k u.$$

Obviously we can write that: $Id=\sum_j\triangle_j$. The high-frequency cut-off operators $\tilde{S}_j$ are defined by

$$\tilde{S}_ju=\sum_{k\geq j}\dot{\triangle}_k u.$$

We define ϕ_j by $\phi_j(\xi)=\phi(2^{-j}\xi)$.

To begin with, we define Besov spaces.

Definition 1. For $s\in\mathbb{R}$ and $1\leq p,r\leq\infty$, and $u\in\mathcal{S}'$ we set

$$\|u\|_{B^s_{p,r}}:=\big\|2^{js}\|\triangle_j u\|_{L^p}\big\|_{l^r(\{j\geq -1\})},$$

$$\|u\|_{\dot{B}^s_{p,r}}:=\big\|2^{js}\|\dot{\triangle}_j u\|_{L^p}\big\|_{l^r(\mathbb{Z})}.$$

The nonhomogeneous Besov space $B^s_{p,r}$ and the homogeneous Besov space $\dot{B}^s_{p,r}$ are the sets of functions $u\in\mathcal{S}'$ such that $\|u\|_{B^s_{p,r}}$ and $\|u\|_{\dot{B}^s_{p,r}}<\infty$ respectively.

2.2 Useful lemmas

The following lemmas will be used frequently.

Lemma 2.1 (Hardy's inequality). *Assume that $n\geq 3$. Then there holds the inequality*

$$\Big\|\frac{u}{|x|}\Big\|_2\leq C\|\nabla u\|_2$$

for $u\in H^1$.

See, e.g., [6], for the proof.

Lemma 2.2. *Assume that $n\geq 3$. Then there holds the inequality*

$$\|f\|_\infty\leq C\|\nabla f\|_{H^{s_0-1}}$$

for $f\in H^{s_0}$.

Lemma 2.2 is proved as follows. Let $p = \frac{2n}{n-2}$. Then, since $s_0 - 1 > \frac{n}{p}$, by the Sobolev inequalities, we have

$$\|f\|_\infty \leq C\|f\|_{W^{s_0-1,p}} \leq C\|\nabla f\|_{H^{s_0-1}}.$$

This proves Lemma 2.2.

Lemma 2.3. *Suppose $a(x) \in BC^1$. For $u \in L^2$ set*

$$[a(x)\frac{\partial}{\partial x_k}, \eta_\epsilon *]u(x) = a(x)\frac{\partial}{\partial x_k}(\eta_\epsilon * u)(x) - \big(\eta_\epsilon * (a\frac{\partial u}{\partial x_k})\big)(x).$$

Here $\eta_\epsilon * u$ is standard Friedrichs mollifier. Then it holds that

$$\|[a(x)\frac{\partial}{\partial x_k}, \eta_\epsilon *]u(x)\|_2 \leq C\|\nabla a\|_\infty \|u\|_2.$$

and

$$\|[a(x)\frac{\partial}{\partial x_k}, \eta_\epsilon *]u(x)\|_2 \longrightarrow 0 \quad (\epsilon \to 0).$$

See, e.g., [21], for the proof.

Lemma 2.4. *Suppose $u \in L^2(0,T;H^1)$ and $\frac{\partial}{\partial t}u \in L^2(0,T;H^{-1})$. Then, the mapping $t \mapsto \|u(t)\|_2^2$ is absolutely continuous, with*

$$\frac{d}{dt}\|u(t)\|_2^2 = 2 < u'(t), u(t) >$$

in the sense of distribution.

See, e.g., [6], for the proof.

Lemma 2.5. *If $0 \leq s_j$ $(j = 1, 2, \cdots, l)$ satisfy $s_j \leq \frac{n}{2}$ $(j = 1, 2, \cdots, l)$ and $s_1 + s_2 + \cdots + s_l > (\frac{n}{2})(l-1)$, then there holds*

$$\|f_1 \cdot f_2 \cdots f_l\|_2 \leq C_{s_1 \cdots s_l} \prod_{j=1}^{l} \|f_j\|_{H^{s_j}}.$$

See, e.g., [14], for the proof.

By using Lemma 2.5 we have the following estimates.

Lemma 2.6. *(i) If $1 \leq |\alpha| \leq s_0$, $g \in H^{s_0}$ and $f \in H^{|\alpha|}$, then*

$$\|[\partial_x^\alpha, g]f\|_2 \leq C \left\{ \begin{array}{l} \|\nabla g\|_{H^{s_0-1}} \|f\|_{H^{|\alpha|}} \\ \|\nabla g\|_{H^{s_0}} \|f\|_{H^{|\alpha|-1}}. \end{array} \right.$$

(*ii*) *Let I be a compact interval of $\mathbb{R}$ and let $\mathcal{R}(y,x) \in C^\infty(I \times \mathbb{R}^n)$. If $1 \leq |\alpha| \leq s_0$, then there holds*

$$\begin{aligned}\|\partial_x^\alpha\big(\mathcal{R}(g(x),x)f\big)\|_2 \;\leq\; & C\{\mathcal{R}_0(g)\|f\|_2 + \mathcal{R}_1(g)\|\nabla f\|_{H^{|\alpha|-1}} \\ & +\mathcal{R}_2(g)(1+\|g\|_{H^{s_0}})^{|\alpha|-1}\|\nabla g\|_{H^{s_0-1}}\|f\|_{H^{|\alpha|}}\}\end{aligned}$$

for $g \in H^{s_0}$ such that $g(x) \in I (x \in \mathbb{R}^n)$ and $f \in H^{|\alpha|}$. Here

$$\mathcal{R}_0(g) := \sup_{x\in\mathbb{R}^n} \big|(\partial_x^\alpha \mathcal{R})(g(x),x)\big|,$$

$$\mathcal{R}_1(g) := \sup_{\beta<\alpha, x\in\mathbb{R}^n} \big|(\partial_x^\beta \mathcal{R})(g(x),x)\big|,$$

$$\mathcal{R}_2(g) := \max_{k\geq 1, k+|\beta|\leq|\alpha|} \sup_{x\in\mathbb{R}^n} |(\partial_y^k\partial_x^\beta \mathcal{R})(g(x),x)|.$$

Lemma 2.6 can be proved in a similar way to the proof of [14, Lemma 3]. (See also [12, Lemma 4.3] and [11, Lemma A.2])

Lemma 2.7. *The following inequalities hold:*

(*i*) $\|\nabla \triangle_{-1} u\|_{L^2} \leq C\|\triangle_{-1}u\|_{L^2}$.

(*ii*) $C^{-1}2^j\|\dot{\triangle}_j u\|_{L^2} \leq \|\nabla\dot{\triangle}_j u\|_{L^2} \leq C2^j\|\dot{\triangle}_j u\|_{L^2} \quad (j \in \mathbb{Z})$.

(*iii*) $\|\nabla S_j u\|_{L^2} \leq C2^j\|S_j u\|_{L^2} \quad (j \geq 0)$.

(*iv*) $\|\tilde{S}_j u\|_{L^2} \leq C2^{-j}\|\nabla\tilde{S}_j u\|_{L^2} \quad (j \geq 0)$.

Lemma 2.7 easily follows from the Plancherel theorem.

Remark 2.8. *For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, we have*

(*i*) $C^{-1}\big(\sum_{k\leq j-1} 2^{srk}\|\dot{\triangle}u\|_{L^p}^r\big)^{\frac{1}{r}} \leq \|\dot{S}_j u\|_{\dot{B}^s_{p,r}} \leq C\big(\sum_{k\leq j-1} 2^{srk}\|\dot{\triangle}u\|_{L^p}^r\big)^{\frac{1}{r}}$

(*ii*) $C^{-1}\big(\sum_{k\geq j} 2^{srk}\|\dot{\triangle}u\|_{L^p}^r\big)^{\frac{1}{r}} \leq \|\tilde{S}_j u\|_{\dot{B}^s_{p,r}} \leq C\big(\sum_{k\geq j} 2^{srk}\|\dot{\triangle}u\|_{L^p}^r\big)^{\frac{1}{r}}$

One can easily prove Remark 2.8.

Lemma 2.9. *The following properties hold:*

(*i*) $C^{-1}\|u\|_{\dot{B}^s_{p,r}} \leq \|\nabla u\|_{\dot{B}^{s-1}_{p,r}} \leq C\|u\|_{\dot{B}^s_{p,r}}$.

(*ii*) $\|\nabla u\|_{B^{s-1}_{p,r}} \leq C\|u\|_{B^s_{p,r}}$.

(*iii*) *If $s' > s$ or if $s' = s$ and $r_1 \leq r$ then $B^{s'}_{p,r_1} \subset B^s_{p,r}$.*

(*iv*) *If $r_1 \leq r$ then $\dot{B}^s_{p,r_1} \subset \dot{B}^s_{p,r}$.*

(*v*) *Let $\Lambda := \sqrt{-\Delta}$ and $t \in \mathbb{R}$. Then the operator Λ^t is an isomorphism from $\dot{B}^s_{2,1}$ to $\dot{B}^{s-t}_{2,1}$.*

See, e.g., [2], [3] and [8] for a proof of Lemma 2.9.

Lemma 2.10. *The following properties hold:*

(i) $\|u\|_{L^\infty} \le C\|u\|_{\dot{B}^{\frac{n}{2}}_{2,1}} \qquad \big(\dot{B}^{\frac{n}{2}}_{2,1} \subset L^\infty\big)$.

(ii) $\dot{B}^0_{1,1} \subset L^1 \subset \dot{B}^0_{1,\infty}$.

(iii) $B^s_{2,2} = H^s$.

(iv) $B^s_{p,r} \subset \dot{B}^s_{p,r}$ $(s > 0)$.

See, e.g., [2], [3] and [8] for a proof of Lemma 2.10.

Lemma 2.11. *Let $1 \le p \le q \le \infty$. Assume that $f \in L^p(\mathbb{R}^n)$. Then for any $\alpha \in (\mathbb{N} \cup \{0\})^n$, there exist constants C_1, C_2 independent of f, j such that*

$$\operatorname{Supp} \hat{f} \subseteq \{|\xi| \le A_0 2^j\} \Longrightarrow \|\partial_x^\alpha f\|_{L^q} \le C_1 2^{j|\alpha| + jn(\frac{1}{p} - \frac{1}{q})}\|f\|_{L^p},$$

$$\operatorname{Supp} \hat{f} \subseteq \{A_1 2^j \le |\xi| \le A_2 2^j\} \Longrightarrow \|f\|_{L^p} \le C_2 2^{-j|\alpha|} \sup_{|\beta| = |\alpha|} \|\partial_x^\beta f\|_{L^p}.$$

See, e.g., [1] for a proof of Lemma 2.11.

By Lemma 2.11, we see that

$$\sum_{j\in\mathbb{Z}} \|\dot{\triangle} f\|_{L^n} \le C \sum_{j\in\mathbb{Z}} 2^{j(\frac{n}{2}-1)}\|\dot{\triangle} f\|_{L^2}, \tag{9}$$

hence, we obtain $\dot{B}^{\frac{n}{2}-1}_{2,1} \subset \dot{B}^0_{n,1}$.

Remark 2.12. *Let $s \ge 0$ and $1 \le p < 2$. Then*

$$\dot{B}^s_{2,1} \cap \dot{B}^0_{p,\infty} \subset B^s_{2,1} \subset L^2.$$

Proof. By using Lemma 2.11, we have

$$\begin{aligned}
\|u\|_{L^2} &= \big(\sum_{j\in\mathbb{Z}} \|\dot{\triangle}_j u\|_{L^2}^2\big)^{\frac{1}{2}} \le \big(\sum_{j<0} \|\dot{\triangle}_j u\|_{L^2}^2\big)^{\frac{1}{2}} + \big(\sum_{j\ge0} \|\dot{\triangle}_j u\|_{L^2}^2\big)^{\frac{1}{2}} \\
&\le C\big(\sum_{j<0} 2^{2jn(\frac{1}{p}-\frac{1}{2})}\|\dot{\triangle}_j u\|_{L^p}^2\big)^{\frac{1}{2}} + \sum_{j\ge0} 2^{js}\|\dot{\triangle}_j u\|_{L^2} \\
&\le C \sup_{j<0} \|\dot{\triangle}_j u\|_{L^p} \big(\sum_{j<0} 2^{2jn(\frac{1}{p}-\frac{1}{2})}\big)^{\frac{1}{2}} + \sum_{j\ge0} 2^{js}\|\dot{\triangle}_j u\|_{L^2}.
\end{aligned}$$

This completes the proof. □

3 Decay estimate for the compressible Navier-Stokes equation with potential force in Sobolev space

.

In sections 3-5 we prove Theorem 1.1. We consider the compressible Navier-Stokes equation with potential force.

In this section, we first state the existence of stationary solution $(\rho_*, 0)$ and some estimates on ρ_* which were obtained in Matsumura-Nishida [20]. We then rewrite system (1) into the one for the perturbation.

Proposition 3.1 (Matsumura-Nishida [20])**.** *There exist positive constants ϵ and C such that if*

$$\|\Phi\|_{H^{s_0+1}} + \|\big(1+|x|\big)\nabla\Phi\|_2 \le \epsilon,$$

the problem $(1)_1 - (1)_2$ has a stationary solution $\big(\rho_*, u\big) = \big(\rho_*(x), 0\big)$ in a small neighborhood of $(\bar\rho, 0)$; and it satisfies

$$\begin{aligned} &\|\rho_*(x) - \bar\rho\|_{H^{s_0+1}} + \|\big(1+|x|\big)\nabla\rho_*(x)\|_2 \\ \le\ & C\Big(\|\Phi\|_{H^{s_0+1}} + \|\big(1+|x|\big)\nabla\Phi\|_2\Big), \end{aligned}$$

$$|\rho_*(x) - \bar\rho| < \frac{1}{2}\bar\rho.$$

Let us rewrite the problem (1). By the change of variables,

$$\tilde\rho(t,x) = \rho(t,x) - \rho_*(x), \quad \tilde u(t,x) = u(t,x),$$

problem (1) is transformed into

$$\begin{cases} \partial_t\tilde\rho + \nabla\cdot(\rho_*\tilde u) = \tilde F_1, \\ \partial_t\tilde u - \frac{\mu}{\rho_*}\Delta\tilde u - \frac{\mu+\mu'}{\rho_*}\nabla\nabla\cdot\tilde u + \frac{P'(\rho_*)}{\rho_*}\nabla\tilde\rho + \big(\frac{P''(\rho_*)}{\rho_*} - \frac{P'(\rho_*)}{\rho_*^2}\big)\nabla\rho_*\tilde\rho = \tilde F_2, \\ (\tilde\rho, \tilde u)(0,x) = (\rho_0 - \rho_*, u_0)(x) \longrightarrow (0,0) \quad (|x|\to\infty), \end{cases}$$

where

$$\tilde F_1 = -\nabla\cdot(\tilde\rho\tilde u),$$

$$\begin{aligned} \tilde F_2 \ =\ & -(\tilde u\cdot\nabla)\tilde u - \mu\frac{\tilde\rho}{\rho_*(\tilde\rho+\rho_*)}\Delta\tilde u - (\mu+\mu')\frac{\tilde\rho}{\rho_*(\tilde\rho+\rho_*)}\nabla(\nabla\cdot\tilde u) \\ & +\Big(\frac{P'(\rho_*)}{\rho_*(\tilde\rho+\rho_*)} - \frac{1}{\tilde\rho+\rho_*}\int_0^1 P''(s\tilde\rho+\rho_*)ds\Big)\tilde\rho\nabla\tilde\rho \\ & +\Big(\frac{P''(\rho_*)}{\rho_*(\tilde\rho+\rho_*)}\nabla\rho_* - \frac{P'(\rho_*)}{\rho_*^2(\tilde\rho+\rho_*)}\nabla\rho_* - \frac{\nabla\rho_*}{\tilde\rho+\rho_*}\int_0^1(1-s)P'''(s\tilde\rho+\rho_*)ds\Big)\tilde\rho^2. \end{aligned}$$

Next, we define μ_1, μ_2 and γ by

$$\mu_1 = \frac{\mu}{\bar\rho},\ \mu_2 = \frac{\mu+\mu'}{\bar\rho},\ \gamma = \sqrt{P'(\bar\rho)}.$$

We also set

$$\bar\sigma = \rho_*(x) - \bar\rho.$$

By using the new unknown functions

$$\sigma(t,x) = \frac{1}{\bar\rho}\tilde\rho(t,x),\ w(t,x) = \frac{1}{\sqrt{P'(\bar\rho)}}\tilde u(t,x),$$

the initial value problem (1) is reformulated as

$$\left\{\begin{array}{l} \partial_t\sigma + \gamma\nabla\cdot w - B_1U = F_1(U), \\ \partial_t w - \mu_1\triangle w - \mu_2\nabla(\nabla\cdot w) + \gamma\nabla\sigma - B_2U = F_2(U), \\ (\sigma, w)(0,x) = (\sigma_0, w_0)(x), \end{array}\right. \tag{10}$$

where, $U = \begin{pmatrix} \sigma \\ w \end{pmatrix}$,

$$B_1U = -\frac{\gamma}{\bar\rho}(w\cdot\nabla\bar\sigma + \bar\sigma\nabla\cdot w),$$

$$\begin{aligned} B_2U &= -\mu_1\frac{\bar\sigma}{\rho_*}\Delta w - \mu_2\frac{\bar\sigma}{\rho_*}\nabla(\nabla\cdot w) + \gamma\frac{\bar\sigma}{\rho_*}\nabla\sigma \\ &\quad -\frac{\bar\sigma\bar\rho}{\gamma\rho_*}\nabla\sigma\int_0^1 P''(s\bar\sigma+\bar\rho)ds - \frac{\bar\rho\nabla\rho_*}{\gamma}\Big(\frac{P''(\rho_*)}{\rho_*} - \frac{P'(\rho_*)}{\rho_*^2}\Big)\sigma, \end{aligned}$$

$$F_1(U) = -\gamma(w\cdot\nabla\sigma + \sigma\nabla\cdot w),$$

$$\begin{aligned} F_2(U) &= -\gamma(w\cdot\nabla)w - \mu_1\frac{\bar\rho^2}{\rho_*(\bar\rho\sigma+\rho_*)}\sigma\Delta w - \mu_2\frac{\bar\rho^2}{\rho_*(\bar\rho\sigma+\rho_*)}\sigma\nabla(\nabla\cdot w) \\ &\quad +\frac{\bar\rho^2}{\gamma}\Big(\frac{P'(\rho_*)}{\rho_*(\bar\rho\sigma+\rho_*)} - \frac{1}{\bar\rho\sigma+\rho_*}\int_0^1 P''(s\bar\rho\sigma+\rho_*)ds\Big)\sigma\nabla\sigma \\ &\quad +\frac{\bar\rho^2\nabla\rho_*}{\gamma}\Big(\frac{P''(\rho_*)}{\rho_*(\bar\rho\sigma+\rho_*)} - \frac{P'(\rho_*)}{\rho_*^2(\bar\rho\sigma+\rho_*)} \\ &\qquad -\frac{1}{\bar\rho\sigma+\rho_*}\int_0^1(1-s)P'''(s\bar\rho\sigma+\rho_*)ds\Big)\sigma^2. \end{aligned}$$

For problem (50), Kawashita [14] proved the following global existence result.

Proposition 3.2 (Kawashita [14]). *Let $n \ge 2$ and let $U_0 = (\sigma_0, w_0) \in H^{s_0}$. There exist a positive constant ϵ_1 such that if*

$$\|U_0\|_{H^{s_0}} \le \epsilon_1,$$

$$\left\{\begin{array}{ll} \|\Phi\|_{H^{s_0+1}} + \|(1+|x|)\nabla\Phi\|_{L^2} \le \epsilon_1 & (n\ge 3), \\ \Phi = 0 & (n=2), \end{array}\right.$$

then problem (50) *has a unique global solution* U*:*

$$U = (\sigma, w) \in \bigcap_{j=0}^{1} C^j([0,\infty); H^{s_0-j}) \times C^j([0,\infty); H^{s_0-2j}),$$

$$w \in L^2(0,\infty; H^{s_0+1}) \cap H^1(0,\infty; H^{s_0-1}).$$

Proposition 3.2 were proved for the case $\Phi = 0$ in [14]. In a similar manner one can see that Proposition 3.2 holds for $\Phi \neq 0$ satisfying the smallness condition of Proposition 3.2 when $n \geq 3$. In terms of U, Theorem 1.1 is restated as follows:

Theorem 3.3. *Assume that* $n \geq 3$*. Let* (σ, w) *be a global solution in* H^{s_0} *with* $s_0 = [\frac{n}{2}] + 1$*, to the problem* (10)*. Then there exist* $\epsilon > 0$ *such that if* $(\sigma_0, w_0) \in H^{s_0} \cap L^1$ *and*

$$\|(\sigma_0, w_0)\|_{H^{s_0} \cap L^1} \leq \epsilon$$

$$\|\Phi\|_{H^{s_0+1}} + \|(1+|x|)\nabla\Phi\|_{L^2} \leq \epsilon$$

then, the estimates

$$\|\nabla^k(\sigma, w)(t)\|_{L^2} \leq C_0(1+t)^{-\frac{n}{4}-\frac{k}{2}}, \quad k = 0, 1, \tag{11}$$

hold for $t \geq 0$*.*

4 Decomposition of solution

In this section we introduce a decomposition of solutions to prove Theorem 3.3.

We set

$$U = \begin{pmatrix} \sigma \\ w \end{pmatrix}, \; U_0 = \begin{pmatrix} \sigma_0 \\ w_0 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & -\gamma\nabla\cdot \\ -\gamma\nabla & \mu_1\Delta + \mu_2\nabla\nabla\cdot \end{pmatrix}.$$

Then problem (50) is written as

$$\partial_t U - AU - BU = F(U), \quad U|_{t=0} = U_0, \tag{12}$$

where

$$BU = \begin{pmatrix} B_1 U \\ B_2 U \end{pmatrix}, \;\; F(U) = \begin{pmatrix} F_1(U) \\ F_2(U) \end{pmatrix}.$$

We next decompose a solution U of (12) into low and high frequency parts. Let $\hat{\chi_1}$ be a cutoff function defined by

$$\hat{\chi_1}(\xi) = \begin{cases} 1 & \big(|\xi| < r\big) \\ 0 & \big(|\xi| \geq r\big) \end{cases}, \quad \hat{\chi_\infty}(\xi) = 1 - \hat{\chi_1}(\xi).$$

Here $r = \frac{\gamma}{\sqrt{\mu_1+\mu_2}}$. (As for the number r, see Lemma 5.1 below.)

We define operator $Q_j (j = 1, \infty)$ on L^2 by

$$Q_j u := \mathfrak{F}^{-1}(\hat{\chi}_j \hat{u}) \quad (j = 1, \infty), \quad u \in L^2.$$

The operators $Q_j (j = 1, \infty)$ have the following properties.

Lemma 4.1. *$Q_j (j = 1, \infty)$ satisfy the following relations.*

(*i*) $Q_1 + Q_\infty = I$.

(*ii*) $Q_j^2 = Q_j$.

(*iii*) $Q_1 Q_\infty = 0$.

(*iv*) $(Q_j u, v) = (u, Q_j v)$ *for* $u, v \in L^2$.

Lemma 4.1 can be easily verified; and we omit the proof.

We next state boundedness properties of Q_j $(j = 1, \infty)$.

Lemma 4.2. (*i*) *For each nonnegative integer k, $Q_j (j = 1, \infty)$ are bounded linear operator on H^k.*

(*ii*) *For each nonnegative integer k, it holds that* $\|\nabla^k Q_1 u\|_2 \le \|u\|_2 \quad (u \in L^2)$.

(*iii*) *For each nonnegative integer k, it holds that* $\|\nabla^k Q_1 u\|_\infty \le C\|u\|_2 \quad (u \in L^2)$.

(*iv*) *Q_∞ satisfies* $\|\nabla Q_\infty u\|_2 \ge C\|Q_\infty u\|_2 \quad (u \in H^1)$.

The assertions $(i), (ii), (iv)$ easily follow from the Plancherel theorem. The inequality (iii) is obtained by (ii) and the Sobolev inequality.

In terms of Q_1 and Q_∞, we decompose a solution $U(t)$ of (12) as

$$U(t) = U_1(t) + U_\infty(t),\ U_j(t) = Q_j U(t) \quad (j = 1, \infty).$$

It then follows that $U_1(t)$ and $U_\infty(t)$ are governed by equations (18) and (19) given in Proposition 4.3 below.

To state Proposition 4.3 we introduce a semigroup associated with a low frequency part of A. We set

$$E_1(t)u := \mathfrak{F}^{-1}[\hat{\chi}_1 e^{\hat{A}(\xi)t}\hat{u}] \quad \text{for } u \in L^2,$$

where

$$\hat{A}(\xi) = \begin{pmatrix} 0 & -i\gamma\xi^{\mathrm{t}} \\ -i\gamma\xi & -\mu_1|\xi|^2 I_n - \mu_2 \xi\xi^{\mathrm{t}} \end{pmatrix}.$$

Here and in what follows the superscript $\cdot^{\mathrm{t}}$ means the transposition.

Proposition 4.3. *Let $T > 0$ and let $U = (\sigma, w)^{\mathrm{t}}$ be a solution of problem* (12) *on $[0, T]$ such that*

$$U = (\sigma, w)^{\mathrm{t}} \in \bigcap_{j=0}^{1} C^j([0,T]; H^{s_0 - j}) \times C^j([0,T]; H^{s_0 - 2j}), \tag{13}$$

$$w \in L^2(0,T;H^{s_0+1}) \cap H^1(0,T;H^{s_0-1}), \tag{14}$$

and let

$$U_j = Q_j U, \quad \sigma_j = Q_j \sigma,\ w_j = Q_j w\ (j = 1, \infty).$$

Then,

$$U_1 \in C^1([0,T];H^k), \quad \forall k = 0,1,2,\cdots, \tag{15}$$

$$U_\infty \in \bigcap_{j=0}^{1} C^j([0,T];H^{s_0-j}) \times C^j([0,T];H^{s_0-2j}), \tag{16}$$

$$w_\infty \in L^2(0,T;H^{s_0+1}) \cap H^1(0,T;H^{s_0-1}). \tag{17}$$

Furthermore $U_1(t)$ *and* $U_\infty(t)$ *satisfy*

$$U_1(t) = E_1(t)U_{01} + \int_0^t E_1(t-s)Q_1\big(B(U_1+U_\infty)(s) + F(U_1+U_\infty)(s)\big)ds \tag{18}$$

and

$$\partial_t U_\infty - AU_\infty - Q_\infty B(U_1+U_\infty) = Q_\infty F(U_1+U_\infty), \tag{19}$$

$$U_\infty|_{t=0} = U_{0\infty}, \tag{20}$$

where $U_{0j} = Q_j U_0$ $(j = 1, \infty)$.

Proof. Let $U(t) = (\sigma, w)^{\mathrm{t}}$ be a solution of (12) satisfying (13) and (14). It then follows from Lemma 4.2 that $U_1(t)$ and $U_\infty(t)$ satisfy (15), (16) and (17), respectively.

Since $Q_j AU = AQ_j U$ for $U \in H^{s_0}$ $(j = 1, \infty)$, applying Q_j to (12), we obtain

$$\begin{cases} \partial_t U_1 - AU_1 - Q_1 B(U_1+U_\infty) = Q_1 F(U_1+U_\infty),\ U_1|_{t=0} = U_{01}, \\ \partial_t U_\infty - AU_\infty - Q_\infty B(U_1+U_\infty) = Q_\infty F(U_1+U_\infty),\ U_\infty|_{t=0} = U_{0\infty}. \end{cases} \tag{21}$$

Taking the Fourier transform of $(21)_1$, we have

$$\hat{\chi}_1 \partial_t \hat{U} = \hat{\chi}_1 \hat{A}\hat{U} + \hat{\chi}_1 \widehat{BU} + \hat{\chi}_1 \widehat{F(U)}. \tag{22}$$

It follows from (22) that

$$\hat{\chi}_1 \hat{U}(t) = e^{\hat{A}t}\hat{\chi}_1 \hat{U}(0) + \int_0^t e^{\hat{A}(t-s)}\big(\hat{\chi}_1 \widehat{BU} + \hat{\chi}_1 \widehat{F(U)}\big)(s)ds.$$

We thus obtain

$$U_1(t) = E_1(t)U_{01} + \int_0^t E_1(t-s)Q_1\big(B(U_1+U_\infty) + F(U_1+U_\infty)\big)(s)ds.$$

This completes the proof.

$\square$

5 Apriori estimate with time weight

In this section we prove apriori estimate with time weight. In subsections 5.1 and 5.2 we establish the necessary estimates for $U_1(t)$ and $U_\infty(t)$, respectively. In subsection 5.3 we derive the a priori estimate to complete the proof of Theorem 3.3.

Set

$$M_1(t) := \sup_{0\le\tau\le t}\sum_{k=0}^{1}(1+\tau)^{\frac{n}{4}+\frac{k}{2}}\|\nabla^k U_1(\tau)\|_2,$$

$$M_\infty(t) := \sup_{0\le\tau\le t}(1+\tau)^{\frac{n}{4}+\frac{1}{2}}\|U_\infty(\tau)\|_{H^{s_0}},$$

$$M(t) := M_1(t) + M_\infty(t).$$

We also set $\delta = \frac{\bar\rho}{4C_s}$, where C_s is constant such that $\|f\|_\infty \le C_s\|f\|_{H^{s_0}}$ for all $f\in H^{s_0}$. Hereafter, we assume that

$$\sup_{0\le t\le T}\|\sigma(t)\|_{H^{s_0}} \le \delta.$$

Then we have

$$\|\sigma(t)\|_\infty \le C_s\|\sigma(t)\|_{H^{s_0}} \le \frac{\bar\rho}{4}.$$

5.1 Estimate of $U_1(t)$

In this subsection we derive the estimate of $U_1(t)$, in other words, we estimate $M_1(t)$.

Lemma 5.1 (Matsumura-Nishida [19]). *(i) The set of all eigenvalues of $\hat A(\xi)$ consists of $\lambda_i(\xi)$ $(i=1,2,3)$, where*

$$\begin{cases} \lambda_1(\xi) = \frac{-(\mu_1+\mu_2)|\xi|^2 + i|\xi|\sqrt{4\gamma^2-(\mu_1+\mu_2)|\xi|^2}}{2}, \\ \lambda_2(\xi) = \overline{\lambda_1(\xi)}, \\ \lambda_3(\xi) = -\mu_1|\xi|^2, \end{cases}$$

for $|\xi|\le r$, where $r = \frac{\gamma}{\sqrt{\mu_1+\mu_2}}$. Here $\overline{\lambda_1(\xi)}$ denotes the complex conjugate of $\lambda_1(\xi)$.

(ii) $e^{t\hat A(\xi)}$ has the spectral resolution

$$e^{t\hat A(\xi)} = \sum_{j=1}^{3} e^{t\lambda_j(\xi)}P_j(\xi),$$

for all $|\xi| \ne \frac{2\gamma}{\sqrt{\mu_1+\mu_2}}$, where $P_j(\xi)$ is the eigenprojection for $\lambda_j(\xi)$.

For $|\xi| = \frac{2\gamma}{\sqrt{\mu_1+\mu_2}}$, we have $\lambda_1(\xi)=\lambda_2(\xi) = -\frac{\mu_1+\mu_2}{2}|\xi|^2$ and

$$e^{t\hat A(\xi)} = e^{t\lambda_1(\xi)}\big(I + t(\hat A(\xi)-\lambda_1 I)\big)P_1 + e^{t\lambda_3(\xi)}P_3$$

where $P_1(\xi), P_3(\xi)$ is the eigenprojection for $\lambda_1(\xi), \lambda_3(\xi)$.

Remark 5.2. *For each $M > 0$ there exist $C_2 = C_2(M) > 0$ and $\beta_2 = \beta_2(M) > 0$ such that the estimate*

$$\|e^{t\hat{A}(\xi)}\| \leq C_2 e^{-\beta_2|\xi|^2 t}$$

holds for $|\xi| \leq M$ and $t > 0$.

$E_1(t)$ satisfies the following estimate:

Lemma 5.3. *Let k be a nonnegative integer. Then there holds*

$$\|\nabla^k E_1(t) Q_1 U_0\|_2 \leq C(1+t)^{-(\frac{n}{4}+\frac{k}{2})}\|U_0\|_1$$

for $t \geq 0$.

Proof. By Lemma 5.1 (i) we see that there exists a constant $\beta > 0$ such that

$$e^{2Re\lambda_j(\xi)t} \leq Ce^{-\beta|\xi|^2 t} \qquad (1 \leq j \leq 3).$$

Therefore, by Plancherel's theorem and Lemma 5.1 (ii), we have

$$\begin{aligned} \|\nabla^k E_1(t) Q_1 U_0(t)\|_2 &\leq C\Big(\int_{|\xi|\leq r} |\xi|^{2k} \big|e^{\hat{A}(\xi)t}\hat{U_0}\big|^2 d\xi\Big)^{\frac{1}{2}} \\ &\leq C\big(\int_{\mathbb{R}^n} |\xi|^{2k} e^{-\beta|\xi|^2 t}|\hat{U}_0(\xi)|^2 d\xi\big)^{\frac{1}{2}} \\ &\leq Ct^{-(\frac{n}{4}+\frac{k}{2})}\|U_0\|_1. \end{aligned} \tag{23}$$

We also find that

$$\begin{aligned} \|\nabla^k E_1(t) Q_1 U_0\|_2 &\leq C\|\hat{U_0}\|_\infty \big(\int_{|\xi|<r} e^{-\beta|\xi|^2 t} d\xi\big)^{\frac{1}{2}} \\ &\leq C\|U_0\|_1. \end{aligned} \tag{24}$$

The estimate of Lemma 5.3 follows from (23) and (24). □

As for $M_1(t)$, we show the following estimate.

Proposition 5.4. *Let $n \geq 3$. There exists a $\epsilon > 0$ such that if*

$$\|\Phi\|_{H^{s_0+1}} + \|(1+|x|)\nabla\Phi\|_{L^2} \leq \epsilon,$$

$$\sup_{0\leq t\leq T} \|\sigma(t)\|_{H^{s_0}} \leq \delta,$$

and

$$M(t) \leq 1$$

for $t \in [0, T]$, then there exists a constant $C > 0$ independent of T such that

$$M_1(t) \leq C\|U_0\|_1 + C\epsilon M(t) + CM^2(t)$$

for $t \in [0, T]$

To prove Proposition 5.4, we will use the following estimates on $B(U)$ and $F(U)$.

Lemma 5.5. *Let $n \geq 3$. There exists a $\epsilon > 0$ such that if*

$$\|\Phi\|_{H^{s_0+1}} + \|(1+|x|)\nabla\Phi\|_{L^2} \leq \epsilon,$$

and

$$M(t) \leq 1$$

for $t \in [0,T]$, then there exists a constant $C > 0$ independent of T such that

$$\|B(U_1(t)+U_\infty(t))\|_1 \leq C\epsilon(1+t)^{-\frac{n+2}{4}}M(t)$$

for $t \in [0,T]$.

Lemma 5.6. *Let $n \geq 3$. There exists a $\epsilon > 0$ such that if*

$$M(t) \leq 1$$

and

$$\|\Phi\|_{H^{s_0+1}} + \|(1+|x|)\nabla\Phi\|_{L^2} \leq \epsilon$$

for $t \in [0,T]$, then there exists a constant $C > 0$ independent of T such that

$$\|F(U_1(t)+U_\infty(t))\|_1 \leq C(1+t)^{-\frac{n+1}{2}}M^2(t)$$

for $t \in [0,T]$.

We will prove Lemma 5.5 and Lemma 5.6 later. Now we prove Proposition 5.4.

Proof of Proposition 5.4. By Lemma 5.3 and (18), we see that

$$\begin{aligned}
&\|\nabla^k U_1(\tau)\|_2 \leq \|\nabla^k E_1(\tau)U_{01}\|_2 \\
&+\int_0^\tau \|\nabla^k E_1(\tau-s)\big(Q_1B(U_1(s)+U_\infty(s))+Q_1F(U_1(s)+U_\infty(s))\big)\|_2 ds \\
\leq\ & C(1+\tau)^{-(\frac{n}{4}+\frac{k}{2})}\|U_0\|_1 \\
&+\int_0^t (1+\tau-s)^{-(\frac{n}{4}+\frac{k}{2})}\big(\|B(U_1(s)+U_\infty(s))\|_1 \\
&\qquad\qquad +\|F(U_1(s)+U_\infty(s))\|_1\big)ds. \qquad (25)
\end{aligned}$$

Using Lemma 5.5 and Lemma 5.6, we have

$$\begin{aligned}
&\int_0^\tau (1+\tau-s)^{-(\frac{n}{4}+\frac{k}{2})}\big(\|B(U_1(s)+U_\infty(s))\|_1+\|F(U_1(s)+U_\infty(s))\|_1\big)ds \\
&\leq C\int_0^\tau (1+\tau-s)^{-(\frac{n}{4}+\frac{k}{2})}\{\epsilon(1+s)^{-\frac{n+2}{4}}M(t)+(1+s)^{-\frac{n+1}{2}}M^2(t)\}ds \\
&\leq C\epsilon M(t)\int_0^\tau (1+\tau-s)^{-(\frac{n}{4}+\frac{k}{2})}(1+s)^{-\frac{n+2}{4}}ds \\
&+CM^2(t)\int_0^\tau (1+\tau-s)^{-(\frac{n}{4}+\frac{k}{2})}(1+s)^{-\frac{n+1}{2}}ds \\
&\leq C\epsilon(1+\tau)^{-(\frac{n}{4}+\frac{k}{2})}M(t)+C(1+\tau)^{-(\frac{n}{4}+\frac{k}{2})}M^2(t). \qquad (26)
\end{aligned}$$

Here we used $\frac{n+2}{4} > 1$ for $n \geq 3$ to handle the term $\epsilon(1+s)^{-\frac{n+2}{4}} M(t)$. By (25) and (26), we obtain

$$\|\nabla^k U_1(\tau)\|_2 \leq C(1+\tau)^{-(\frac{n}{4}+\frac{k}{2})}\|U_0\|_1 + C\epsilon(1+\tau)^{-(\frac{n}{4}+\frac{k}{2})}M(t) + C(1+\tau)^{-(\frac{n}{4}+\frac{k}{2})}M^2(t),$$

and hence,

$$(1+\tau)^{\frac{n}{4}+\frac{k}{2}}\|\nabla^k U_1(\tau)\|_2 \leq C\|U_0\|_1 + C\epsilon M(t) + CM^2(t).$$

Taking the supremum in $\tau \in [0,t]$, we obtain the desired estimate for $n \geq 3$.

□

It remains to prove Lemma 5.5 and Lemma 5.6.

Proof of Lemma 5.5. By Lemma 2.1 and Proposition 3.1, we have

$$\begin{aligned}\|w \cdot \nabla\bar{\sigma}\|_1 &\leq \|(1+|x|)\nabla\bar{\sigma}\|_2 \|\frac{1}{1+|x|}(w_1 + w_\infty)\|_2 \\ &\leq \epsilon(\|\nabla w_1\|_2 + \|\nabla w_\infty\|_2),\end{aligned}$$

$$\begin{aligned}&\|-\frac{\bar{\rho}\nabla\rho_*}{\gamma}(\frac{P''(\rho_*)}{\rho_*} + \frac{P'(\rho_*)}{\rho_*^2})\sigma\|_1 \\ \leq\ & C\|\frac{P''(\rho_*)}{\rho_*} + \frac{P'(\rho_*)}{\rho_*^2}\|_\infty \|(1+|x|)\nabla\rho_*\|_2 \|\frac{1}{1+|x|}\sigma\|_2 \\ \leq\ & C\epsilon\big(\|\nabla\sigma_1\|_2 + \|\nabla\sigma_\infty\|_2\big).\end{aligned}$$

By using the Hölder inequality and Lemma 4.2, one can see that the L^1 norms of the others terms are bounded by $C\epsilon\big(\|\nabla\sigma_1\|_2 + \|\nabla\sigma_\infty\|_2\big)$. We thus conclude that

$$\begin{aligned}\|B(U_1 + U_\infty)\|_1 &\leq C\epsilon\big(\|\nabla U_1\|_2 + \|\nabla U_\infty\|_{H^1}\big) \\ &\leq C\epsilon(1+s)^{-\frac{n+2}{4}}M(t).\end{aligned}$$

This completes the proof. □

Proof of Lemma 5.6.

When $n \geq 3$, we see from Lemma 2.2 that

$$\begin{aligned}&\|\nabla\rho_*\Big(\frac{P''(\rho_*)}{\rho_*(\bar{\rho}\sigma + \rho_*)} - \frac{P'(\rho_*)}{\rho_*^2(\bar{\rho}\sigma + \rho_*)} \\ &\qquad -\frac{1}{\bar{\rho}\sigma + \rho_*}\int_0^1 (1-s)P'''(s\bar{\rho}\sigma + \rho_*)ds\Big)\sigma^2\|_1 \\ \leq\ & C\|\nabla\rho_*\|_2\|\sigma\|_\infty\|\sigma\|_2 \\ \leq\ & C\|\nabla\sigma\|_{H^{s_0-1}}\|\sigma\|_2 \\ \leq\ & C(1+s)^{-\frac{n+1}{2}}M^2(t).\end{aligned}$$

The L^1 norm of the other terms are estimated by using the Hölder inequality, and bounded by $C(1+s)^{-\frac{n+1}{2}}M^2(t)$. Hence, we have

$$\|F(U_1 + U_\infty)\|_1 \leq C(1+s)^{-\frac{n+1}{2}}M^2(t).$$

This completes the proof. □

5.2 Estimate of $U_\infty(t)$

We next derive estimates for U_∞. The system (19) is written as

$$\begin{cases} \partial_t\sigma_\infty + \gamma\nabla\cdot w_\infty = Q_\infty(B_1U + F_1(U)), \\ \partial_t w_\infty - \mu_1\Delta w_\infty - \mu_2\nabla\cdot(\nabla w_\infty) + \gamma\nabla\sigma_\infty = Q_\infty(B_2U + F_2(U)). \end{cases} \tag{27}$$

Proposition 5.7. *There holds*

$$\sum_{0\le|\alpha|\le s_0} \frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha U_\infty(t)\|_2^2 + \mu_1\|\nabla\partial_x^\alpha w_\infty\|_2^2 + \mu_2\|\nabla\cdot\partial_x^\alpha w_\infty(t)\|_2^2 = \sum_{i=1}^4 I_i \tag{28}$$

for a.e. $t\in[0,T]$. *Here,*

$$I_1 := \sum_{0\le|\alpha|\le s_0}(\partial_x^\alpha B_1U, \partial_x^\alpha\sigma_\infty),$$

$$\begin{aligned} I_2 \ :=\ & \sum_{0\le|\alpha|\le s_0-1}(\partial_x^\alpha F_1(U), \partial_x^\alpha\sigma_\infty) \\ & -\sum_{|\alpha|=s_0}(\frac{\gamma}{\rho_\infty}[\partial_x^\alpha, w\cdot]\nabla\sigma, \partial_x^\alpha\sigma_\infty) - \sum_{|\alpha|=s_0}(\frac{\gamma}{\rho_\infty}\partial_x^\alpha(\sigma\nabla\cdot w), \partial_x^\alpha\sigma_\infty) \\ & -\frac{1}{2}\sum_{|\alpha|=s_0}\big(\nabla\cdot w, |\partial_x^\alpha\sigma_\infty|^2\big) + \sum_{|\alpha|=s_0}(w\cdot\nabla\partial_x^\alpha\sigma_1, \partial_x^\alpha\sigma_\infty), \end{aligned}$$

$$I_3 := -\sum_{|\alpha|=s_0}\sum_{|\gamma|=1}(\partial_x^{\alpha-\gamma}B_2U, \partial_x^{\alpha+\gamma}w_\infty) + \sum_{0\le|\alpha|\le s_0-1}(\partial_x^\alpha B_2U, \partial_x^\alpha w_\infty),$$

$$I_4 := -\sum_{|\alpha|=s_0}\sum_{|\gamma|=1}(\partial_x^{\alpha-\gamma}F_2(U), \partial_x^{\alpha+\gamma}w_\infty) + \sum_{0\le|\alpha|\le s_0-1}(\partial_x^\alpha F_2(U), \partial_x^\alpha w_\infty).$$

Proof. Let $\eta\in C_0^\infty(\mathbb{R}^n)$ satisfying $\eta\ge 0$, $\operatorname{supp}\eta\subset\{x;|x|\le 1\}$, $\eta(-x)=\eta(x)$ and $\int\eta(x)dx = 1$. Set $\eta_\epsilon(x) = \epsilon^{-n}\eta(\frac{x}{\epsilon})$. Note that due to $\eta(-x)=\eta(x)$ we have

$$(\eta_\epsilon * f, g) = (f, \eta_\epsilon * g).$$

Let $\varphi\in C_0^\infty$ and $|\alpha| = s_0$. We take the inner product of $(27)_1$ with $\partial_x^\alpha(\eta_\epsilon*\varphi)$ to obtain

$$\begin{aligned} & (\partial_t\sigma_\infty, \partial_x^\alpha\eta_\epsilon*\varphi) + (\gamma\nabla\cdot w_\infty, \partial_x^\alpha\eta_\epsilon*\varphi) \\ =\ & \big(Q_\infty(B_1U + F_1(U)), \partial_x^\alpha(\eta_\epsilon*\varphi)\big) \\ =\ & (Q_\infty B_1U, \partial_x^\alpha\eta_\epsilon*\varphi) - \gamma\{\big(Q_\infty(\sigma\nabla\cdot w), \partial_x^\alpha\eta_\epsilon*\varphi\big) \\ & +\big(w\cdot\nabla\sigma, \partial_x^\alpha\eta_\epsilon*\varphi\big) - \big(Q_1(w\cdot\nabla\sigma), \partial_x^\alpha\eta_\epsilon*\varphi\big)\}. \end{aligned} \tag{29}$$

By integration by parts, we have

$$\begin{aligned}
&(\partial_t(\eta_\epsilon * \partial_x^\alpha \sigma_\infty), \varphi) + \gamma(\eta_\epsilon * \nabla \cdot \partial_x^\alpha w_\infty, \varphi) \\
= \; &(\eta_\epsilon * \partial_x^\alpha Q_\infty B_1 U, \varphi) - \gamma\{\big(\eta_\epsilon * \partial_x^\alpha Q_\infty(\sigma \nabla \cdot w), \varphi\big) \\
&+\big(\eta_\epsilon * ([\partial_x^\alpha, w]\nabla)\sigma, \varphi\big) - \big([\eta_\epsilon *, w \cdot \nabla]\partial_x^\alpha \sigma, \varphi)\big) \\
&-\big(w \cdot \nabla \eta_\epsilon * \partial_x^\alpha \sigma, \varphi\big) - \big(\eta_\epsilon * \partial_x^\alpha Q_1(w \cdot \nabla \sigma), \varphi\big)\}. \qquad (30)
\end{aligned}$$

Next, we multiply (30) by $h \in C_0^\infty(0,T)$ and take $\varphi = \eta_\epsilon * \partial_x^\alpha \sigma_\infty \in C^\infty \cap L^2$. Integrating the resulting equation over $[0,T]$, we obtain

$$\begin{aligned}
-\frac{1}{2}\int_0^T \|\eta_\epsilon * \partial_x^\alpha \sigma_\infty\|_2^2 \frac{d}{dt} h dt + \int_0^T \big(\eta_\epsilon * \partial_x^\alpha(\gamma \nabla \cdot w_\infty), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
= \int_0^T -\big(\eta_\epsilon * (\partial_x^\alpha Q_\infty B_1 U), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
+\gamma \int_0^T \big(w \cdot \nabla(\eta_\epsilon * \partial_x^\alpha \sigma), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
+\gamma \int_0^T \big([\eta_\epsilon *, w \cdot \nabla]\partial_x^\alpha \varphi, \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
+\gamma \int_0^T \big(\eta_\epsilon * [\partial_x^\alpha, w], \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
-\gamma \int_0^T \big(\eta_\epsilon * \partial_x^\alpha Q_\infty(\sigma \nabla \cdot w), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
+\gamma \int_0^T \big(\eta_\epsilon * \partial_x^\alpha Q_1(w \cdot \nabla \sigma), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt.
\end{aligned}$$

We rewrite this equality to let $\epsilon \to 0$. The second term on the right hand-side is written as

$$\begin{aligned}
&\big(w \cdot \nabla(\eta_\epsilon * \partial_x^\alpha \sigma), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) \\
= \; &\big(w \cdot \nabla(\eta_\epsilon * \partial_x^\alpha \sigma_\infty), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) + \big(w \cdot \nabla(\eta_\epsilon * \partial_x^\alpha \sigma_1), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) \\
= \; &\frac{1}{2}\big(w, \nabla|\eta_\epsilon * \partial_x^\alpha \sigma_\infty|^2\big) + \big(w \cdot \nabla(\eta_\epsilon * \partial_x^\alpha \sigma_1), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) \\
= \; &-\frac{1}{2}\big(\nabla \cdot w, |\eta_\epsilon * \partial_x^\alpha \sigma_\infty|^2\big) + \big(w \cdot \nabla(\eta_\epsilon * \partial_x^\alpha \sigma_1), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big).
\end{aligned}$$

Hence, we obtain

$$
\begin{aligned}
-\frac{1}{2}\int_0^T & \|\eta_\epsilon * \partial_x^\alpha \sigma_\infty\|_2^2 \frac{d}{dt} h dt + \gamma \int_0^T (\eta_\epsilon * \nabla \cdot \partial_x^\alpha w_\infty, \eta_\epsilon * \partial_x^\alpha \sigma_\infty) h dt \\
&= -\int_0^T (\eta_\epsilon * \partial_x^\alpha Q_\infty B_1 U, \eta_\epsilon * \partial_x^\alpha \sigma_\infty) h dt \\
&\quad -\gamma \int_0^T \frac{1}{2}\big(\nabla \cdot w, |\eta_\epsilon * \partial_x^\alpha \sigma_\infty|^2\big) h dt \\
&\quad +\gamma \int_0^T \big(w \cdot \nabla(\eta_\epsilon * \partial_x^\alpha \sigma_1), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
&\quad +\gamma \int_0^T \big([\eta_\epsilon *, w \cdot \nabla]\partial_x^\alpha \varphi, \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
&\quad +\gamma \int_0^T \big(\eta_\epsilon * [\partial_x^\alpha, w]\nabla \sigma, \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
&\quad -\gamma \int_0^T \big(\eta_\epsilon * \partial_x^\alpha Q_\infty(\sigma \nabla \cdot w), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt \\
&\quad +\int_0^T \big(\eta_\epsilon * \partial_x^\alpha Q_1(w \cdot \nabla \sigma), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) h dt
\end{aligned} \tag{31}
$$

Letting $\epsilon \to 0$ in (31), we can obtain

$$
\begin{aligned}
& \frac{1}{2}\int_0^T \frac{d}{dt}\|\partial_x^\alpha \sigma_\infty\|_2^2 h dt + \gamma \int_0^T (\nabla \cdot \partial_x^\alpha w_\infty, \partial_x^\alpha \sigma_\infty) h dt \\
= \; & -\int_0^T (\partial_x^\alpha B_1 U, \partial_x^\alpha \sigma_\infty) h dt \\
& -\gamma \int_0^T \frac{1}{2}\big(\nabla \cdot w, |\partial_x^\alpha \sigma_\infty|^2\big) h dt \\
& +\gamma \int_0^T \big(w \cdot \nabla \partial_x^\alpha \sigma_1, \partial_x^\alpha \sigma_\infty\big) h dt \\
& +\gamma \int_0^T \big([\partial_x^\alpha, w]\nabla \sigma, \partial_x^\alpha \sigma_\infty\big) h dt \\
& -\gamma \int_0^T \big(\partial_x^\alpha(\sigma \nabla \cdot w), \partial_x^\alpha \sigma_\infty\big) h dt.
\end{aligned} \tag{32}
$$

In fact, as for the third term on the right hand-side of (31), by Lemma 4.2, we see that

$$
\begin{aligned}
& \big(w \cdot \nabla(\eta_\epsilon * (\partial_x^\alpha \sigma_1)), \eta_\epsilon * \partial_x^\alpha \sigma_\infty\big) \\
\leq \; & \|w\|_\infty \|\nabla \partial_x^\alpha \sigma_1\|_2 \|\partial_x^\alpha \sigma_\infty\|_2 \\
\leq \; & \|w\|_{H^{s_0}} \|\nabla \sigma_1\|_2 \|\partial_x^\alpha \sigma_\infty\|_2 \; \in L^1(0,T).
\end{aligned}
$$

Hence, we have

$$\begin{aligned}
&\int_0^T \big(w\cdot\nabla(\eta_\epsilon*(\partial_x^\alpha\sigma_1)),\eta_\epsilon*\partial_x^\alpha\sigma_\infty\big)dt\\
\longrightarrow\ &\int_0^T \big(w\cdot\nabla(\partial_x^\alpha\sigma_1),\partial_x^\alpha\sigma_\infty\big)dt.
\end{aligned}$$

The fourth term on the right hand-side of (31) can be shown to go to zero by using Lemma 2.3. In fact, since $\partial_x^\alpha\sigma\in C([0,T];L^2), w\in L^2(0,T;H^{s_0+1})\subset L^2(0,T;BC^1)$, applying Lemma 2.3, we have

$$\|[\eta_\epsilon*,w\cdot\nabla]\partial_x^\alpha\sigma\|_2\longrightarrow 0\quad(\epsilon\to 0),$$

for a.e. $t\in(0,T)$. We thus obtain

$$\begin{aligned}
&\big|\big([\eta_\epsilon*,w\cdot\nabla]\partial_x^\alpha\sigma,\eta_\epsilon*\partial_x^\alpha\sigma_\infty\big)h\big|\\
\le\ & C\left\{\begin{array}{l}\|\nabla w(t)\|_\infty\|\partial_x^\alpha\sigma(t)\|_2^2|h(t)|\\ \|[\eta_\epsilon*,w\cdot\nabla]\partial_x^\alpha\sigma\|_2\|\eta_\epsilon*\partial_x^\alpha\sigma_\infty\|_2|h(t)|\longrightarrow 0\quad(\epsilon\to 0).\end{array}\right.
\end{aligned}$$

for a.e. $t\in(0,T)$. Since, $\|\nabla w(t)\|_\infty\le C\|w(t)\|_{H^{s_0+1}}\in L^2(0,T)$, we see that

$$\int_0^T\big([\eta_\epsilon*,w\cdot\nabla]\partial_x^\alpha\sigma,\eta_\epsilon*\partial_x^\alpha\sigma_\infty\big)hdt\longrightarrow 0.$$

As for the seventh term on the right hand-side of (31), by Lemma 4.2 and the dominated convergence theorem, we have

$$\begin{aligned}
&\int_0^T\big(\eta_\epsilon*\partial_x^\alpha Q_1(w\cdot\nabla\sigma),\eta_\epsilon*\partial_x^\alpha\sigma_\infty\big)hdt\\
\to\ &\int_0^T\big(\partial_x^\alpha Q_1(w\cdot\nabla\sigma),\partial_x^\alpha\sigma_\infty\big)hdt=0.
\end{aligned}$$

Here we have used $\big(\partial_x^\alpha Q_1(w\cdot\nabla\sigma),\partial_x^\alpha\sigma_\infty\big)=\big(Q_1\partial_x^\alpha Q_1(w\cdot\nabla\sigma),\partial_x^\alpha\sigma_\infty\big)=\big(\partial_x^\alpha Q_1(w\cdot\nabla\sigma),\partial_x^\alpha Q_1\sigma_\infty\big)=0$.

For the other terms of (31), one can apply the dominated convergence theorem to pass the limit and we obtain (32). It then follows from (32) that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha\sigma_\infty(t)\|_2^2+\gamma(\nabla\cdot\partial_x^\alpha w_\infty,\partial_x^\alpha\sigma_\infty)\ =\ &(\partial_x^\alpha B_1U,\partial_x^\alpha\sigma_\infty)-\gamma\frac{1}{2}(\nabla\cdot w,|\partial_x^\alpha\sigma|^2)\\
&+\gamma\big(w\cdot\nabla\partial_x^\alpha\sigma_1,\partial_x^\alpha\sigma_\infty\big)\\
&+\gamma\big([\partial_x^\alpha,w]\nabla\sigma,\partial_x^\alpha\sigma_\infty\big)\\
&-\gamma\big(\partial_x^\alpha(\sigma\nabla\cdot w),\partial_x^\alpha\sigma_\infty\big)
\end{aligned}\tag{33}$$

for a.e. $t\in(0,T)$ and $|\alpha|=s_0$.

When $|\alpha|\le s_0-1$, by simply taking the inner product of $\partial_x^\alpha(27)_1$ with $\partial_x^\alpha\sigma_\infty$, we have

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha\sigma_\infty(t)\|_2^2+\gamma(\nabla\cdot\partial_x^\alpha w_\infty,\partial_x^\alpha\sigma_\infty)=(\partial_x^\alpha B_1U,\partial_x^\alpha\sigma_\infty)+(\partial_x^\alpha F_1U,\partial_x^\alpha\sigma_\infty).\tag{34}$$

We see from (33) and (34) that

$$
\begin{aligned}
&\sum_{0\le|\alpha|\le s_0} \frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha \sigma_\infty(t)\|_2^2 + \gamma(\nabla\cdot\partial_x^\alpha w_\infty, \partial_x^\alpha \sigma_\infty)\\
=\;& \sum_{0\le|\alpha|\le s_0} (\partial_x^\alpha B_1 U, \partial_x^\alpha \sigma_\infty) + \sum_{0\le|\alpha|\le s_0-1} (\partial_x^\alpha F_1(U), \partial_x^\alpha \sigma_\infty)\\
&+\sum_{|\alpha|=s_0} (-\frac{\gamma}{\bar\rho}[\partial_x^\alpha, w\cdot]\nabla\sigma, \partial_x^\alpha\sigma_\infty) + \sum_{|\alpha|=s_0}(-\frac{\gamma}{\bar\rho}\partial_x^\alpha(\sigma\nabla\cdot w)\partial_x^\alpha\sigma_\infty)\\
&-\frac{1}{2}\sum_{|\alpha|=s_0}\big(\nabla\cdot w, |\partial_x^\alpha\sigma_\infty|^2\big) + \sum_{|\alpha|=s_0}(w\cdot\nabla\partial_x^\alpha\sigma_1, \partial_x^\alpha\sigma_\infty) \qquad (35)
\end{aligned}
$$

for a.e. $t\in(0,T)$.

We next consider $(27)_2$. Let $\varphi\in C_0^\infty$ and let $|\alpha| = s_0$. We take the inner-product of $(27)_2$ with $\partial_x^\alpha\varphi$ to obtain

$$
\begin{aligned}
&(\partial_t w_\infty, \partial_x^\alpha\varphi) - \mu_1(\Delta w_\infty, \partial_x^\alpha\varphi) - \mu_2\big(\nabla(\nabla\cdot w_\infty), \partial_x^\alpha\varphi\big)\\
&+\gamma(\nabla\sigma_\infty, \partial_x^\alpha\varphi) = (Q_\infty B_2 U, \partial_x^\alpha\varphi) + (Q_\infty F_2(U), \partial_x^\alpha\varphi).
\end{aligned}
$$

Integrating by parts, we obtain

$$
\begin{aligned}
&\big\langle \partial_x^\alpha\partial_t w_\infty, \varphi\big\rangle + \mu_1(\nabla\partial_x^\alpha w_\infty, \nabla\varphi)\\
&+\mu_2(\nabla\cdot\partial_x^\alpha w_\infty, \nabla\cdot\varphi) - \gamma(\partial_x^\alpha\sigma_\infty, \nabla\cdot\varphi)\\
=\;& -\sum_{|\gamma|=1}(\partial_x^{\alpha-\gamma}Q_\infty B_2 U, \partial_x^\gamma\varphi) - \sum_{|\gamma|=1}(\partial_x^{\alpha-\gamma}Q_\infty F_2(U), \partial_x^\gamma\varphi).
\end{aligned}
$$

Here we have used the fact that $\sum_{|\gamma|=1}\partial_x^{\alpha-\gamma}Q_\infty F_2 \in L^2$, which can be seen from the proof of Proposition 5.8 below. By density, we can set $\varphi = \partial_x^\alpha w_\infty$. So we obtain by Lemma 2.4,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha w_\infty\|_2^2 + \mu_1\|\nabla\partial_x^\alpha w_\infty\|_2^2 + \mu_2\|\nabla\cdot\partial_x^\alpha w_\infty\|_2^2 - \gamma(\partial_x^\alpha\sigma_\infty, \nabla\cdot\partial_x^\alpha w_\infty)\\
=\;& -\sum_{|\gamma|=1}(\partial_x^{\alpha-\gamma}B_2 U, \partial_x^\gamma\partial_x^\alpha w_\infty) - \sum_{|\gamma|=1}(\partial_x^{\alpha-\gamma}F_2(U), \partial_x^\gamma\partial_x^\alpha w_\infty). \qquad (36)
\end{aligned}
$$

for a.e. $t\in(0,T)$.

When $|\alpha|\le s_0-1$, in a similar way as above we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha w_\infty\|_2^2 + \mu_1\|\nabla\partial_x^\alpha w_\infty\|_2^2 + \mu_2\|\nabla\cdot\partial_x^\alpha w_\infty\|_2^2 - \gamma(\partial_x^\alpha\sigma_\infty, \nabla\cdot\partial_x^\alpha w_\infty)\\
=\;& \sum_{0\le|\alpha|\le s_0-1}\big\{(\partial_x^\alpha B_2 U, \partial_x^\alpha w_\infty) + (\partial_x^\alpha F_2(U), \partial_x^\alpha w_\infty)\big\}. \qquad (37)
\end{aligned}
$$

We see from (36) and (37) that

$$\begin{aligned}
&\sum_{0\le|\alpha|\le s_0} \frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha w_\infty\|_2^2+\mu_1\|\partial_x^\alpha\nabla w_\infty\|_2^2+\mu_2\|\nabla\cdot\partial_x^\alpha w_\infty\|_2^2 \\
&-\gamma(\partial_x^\alpha\sigma_\infty,\nabla\cdot\partial_x^\alpha w_\infty) \\
= \; & -\sum_{|\alpha|=s_0}\sum_{|\gamma|=1}(\partial_x^{\alpha-\gamma}B_2U,\partial_x^\gamma\partial_x^\alpha w_\infty)-\sum_{|\alpha|=s_0}\sum_{|\gamma|=1}(\partial_x^{\alpha-\gamma}F_2(U),\partial_x^\gamma\partial_x^\alpha w_\infty). \\
&+\sum_{0\le|\alpha|\le s_0-1}\big\{(\partial_x^\alpha B_2U,\partial_x^\alpha w_\infty)+(\partial_x^\alpha F_2(U),\partial_x^\alpha w_\infty)\big\}. \qquad (38)
\end{aligned}$$

A linear combination of (35) and (38) yields the desired result. □

We next estimate I_1 and I_3.

Proposition 5.8. *Let $n\ge 3$. There exists a constant $\epsilon>0$ such that if*

$$\|\Phi\|_{H^{s_0+1}}+\|(1+|x|)\nabla\Phi\|_{L^2}\le\epsilon,$$

$$\sup_{0\le t\le T}\|\sigma(t)\|_{H^{s_0}}\le\delta,$$

and

$$M(t)\le 1$$

for $t\in[0,T]$, then

$$|I_1|+|I_3|\le C\epsilon\{(1+t)^{-(\frac{n+1}{2})}M^2(t)+\|\nabla^{s_0+1}w_\infty(t)\|_2^2\}$$

for $t\in[0,T]$. Here $C>0$ is a constant independent of T.

Proof. First we show the estimate of I_1. We have

$$\begin{aligned}
|I_1| &= \Big|\sum_{0\le|\alpha|\le s_0}\big\{(\partial_x^\alpha(w\cdot\nabla\bar\sigma),\partial_x^\alpha\sigma_\infty)+(\partial_x^\alpha(\bar\sigma\cdot\nabla w),\partial_x^\alpha\sigma_\infty)\big\}\Big| \\
&\le \sum_{0\le|\alpha|\le s_0}(\|\partial_x^\alpha(w\cdot\nabla\bar\sigma)\|_2+\|\partial_x^\alpha(\bar\sigma\nabla\cdot w)\|_2)\|\partial_x^\alpha\sigma_\infty\|_2. \qquad (39)
\end{aligned}$$

By Lemma 2.2 and Lemma 2.6, the terms on the right-hand side of (39) is estimated as

$$\begin{aligned}
\|\partial_x^\alpha(w\cdot\nabla\bar\sigma)\|_2 &\le C\{\|w\|_\infty\|\nabla\bar\sigma\|_{H^{s_0}}+\|\nabla w\|_{s_0-1}\|\nabla\bar\sigma\|_{s_0}\} \\
&\le C\epsilon(1+t)^{-(\frac{n}{4}+\frac{1}{2})}M(t),
\end{aligned}$$

$$\begin{aligned}
\|\partial_x^\alpha(\bar\sigma\nabla\cdot w)\|_2 &\le C\|\bar\rho\|_{H^{s_0}}\|\nabla w\|_{H^{s_0}} \\
&\le C\epsilon(1+t)^{-(\frac{n}{4}+\frac{1}{2})}M(t)+C\epsilon\|\nabla^{s_0+1}w_\infty\|_2.
\end{aligned}$$

Hence, we obtain the estimate of I_1.

Let us next consider I_3:

$$\begin{aligned}|I_3| &= |-\sum_{|\alpha|=s_0}\sum_{|\gamma|=1}(\partial_x^{\alpha-\gamma}B_2U,\partial_x^{\alpha+\gamma}w_\infty)+\sum_{0\le|\alpha|\le s_0-1}(\partial_x^\alpha B_2U,\partial_x^\alpha w_\infty)| \\ &\le C\big(\sum_{|\alpha|\le s_0-1}\|\partial_x^\alpha B_2U\|_2\big)\|\nabla w_\infty\|_{H^{s_0}}\end{aligned}$$

We estimate $\|\partial_x^\alpha B_2U\|_2$ ($|\alpha|\le s_0-1$). We write B_2U as

$$B_2U = G_1(\bar\sigma,x)\triangle w+G_2(\bar\sigma,x)\nabla(\nabla\cdot w)+G_3(\bar\sigma,x)\nabla\sigma+G_4(x)\sigma,$$

where

$$G_1(\bar\sigma,x)=-\mu_1\frac{\bar\sigma}{\rho_*}$$

$$G_2(\bar\sigma,x)=-\mu_2\frac{\bar\sigma}{\rho_*}$$

$$G_3(\bar\sigma,x)=-\gamma\frac{\bar\sigma}{\rho_*}+\frac{\bar\sigma\bar\rho}{\gamma\rho_*}\int_0^1P''(s\bar\sigma+\bar\rho)ds,$$

$$G_4(x)=-\frac{\bar\rho\nabla\rho_*}{\gamma}\big(\frac{P''(\rho_*)}{\rho_*}-\frac{P'(\rho_*)}{\rho_*^2}\big).$$

We thus obtain by Lemma 2.2 and Lemma 2.6

$$\begin{aligned}\|\partial_x^\alpha B_2U\|_2 &\le C\{\|\bar\sigma\|_{H^{s_0}}\|\nabla^2w\|_{H^{s_0-1}}+\|\bar\sigma\|_{H^{s_0+1}}\|\sigma\|_{H^{s_0}}+\|\partial_x^\alpha G_4(x)\|_2\|\sigma\|_\infty\} \\ &\le C\epsilon\{(1+t)^{-\frac{n}{4}-\frac12}M(t)+\|\nabla^{s_0+1}w_\infty\|_2\}.\end{aligned}$$

Hence, we have

$$|I_3| \le C\epsilon(1+t)^{-(\frac{n}{2}+1)}M^2(t)+C\epsilon\|\nabla^{s_0+1}w_\infty\|_2^2.$$

This completes the proof. □

Proposition 5.9. *Let $n\ge3$. There exists a constant $\epsilon>0$ such that if*

$$\|\Phi\|_{H^{s_0+1}}+\|(1+|x|)\nabla\Phi\|_{L^2}\le\epsilon$$

$$\sup_{0\le t\le T}\|\sigma(t)\|_{H^{s_0}}\le\delta,$$

and

$$M(t)\le1$$

for $t\in[0,T]$, then

$$|I_2|+|I_4|\le C(1+t)^{-\frac{n}{4}}M(t)\{(1+t)^{-(\frac{n}{2}+1)}M^2(t)+\|\nabla^{s_0+1}w_\infty(t)\|_2^2\}$$

for $t\in[0,T]$. Here $C>0$ is a constant independent of T.

Proof. Let us estimate I_2. For the first term of I_2, by Lemma 2.6 and Lemma 4.2 we have

$$
\begin{aligned}
& \Big|\sum_{0\le|\alpha|\le s_0-1}(\partial_x^\alpha F_1(U),\partial_x^\alpha\sigma_\infty)\Big| \\
\le\ & C(\|\nabla\sigma\|_{H^{s_0-1}}\|w\|_{H^{s_0}}+\|\sigma\|_{H^{s_0}}\|\nabla w\|_{H^{s_0-1}})\|\sigma_\infty\|_{H^{s_0}} \\
\le\ & C\{(1+t)^{-(\frac{3n}{4}+1)}M^3(t)+(1+t)^{-\frac{n}{4}}M(t)\|\nabla^{s_0+1}w_\infty\|_2\}.
\end{aligned}
$$

By Lemma 2.6, the second term of I_2 is estimated as

$$
\Big|\sum_{|\alpha|=s_0}(-\frac{\gamma}{\rho_\infty}[\partial_x^\alpha, w\cdot]\nabla\sigma,\partial_x^\alpha\sigma_\infty)\Big| \le C\|\nabla w\|_{H^{s_0}}\|\nabla\sigma\|_{H^{s_0-1}}\|\sigma_\infty\|_{H^{s_0}}
$$

We finally, we consider I_4 :

$$
\begin{aligned}
|I_4| &= \Big|\sum_{|\alpha|=s_0}\sum_{|\gamma|=1}(\partial_x^{\alpha-\gamma}F_2(U),\partial_x^{\alpha+\gamma}w_\infty)+\sum_{0\le|\alpha|\le s_0-1}(\partial_x^\alpha F_2(U),\partial_x^\alpha w_\infty)\Big| \\
&\le \sum_{|\alpha|=s_0}\sum_{|\gamma|=1}\|\partial_x^{\alpha-\gamma}F_2(U)\|_2\|\partial_x^{\alpha+\gamma}w_\infty\|_2+\sum_{0\le|\alpha|\le s_0-1}\|\partial_x^\alpha F_2(U)\|_2\|\partial_x^\alpha w_\infty\|_2 \\
&\le \Big(\sum_{0\le|\alpha|\le s_0-1}\|\partial_x^\alpha F_2(U)\|_2\Big)\|\nabla w_\infty\|_{H^{s_0}}
\end{aligned}
$$

Let us estimate $\|\partial_x^\alpha F_2(U)\|_2$ for $|\alpha|\le s_0-1$. $F_2(U)$ is written as

$$
\begin{aligned}
F_2(U) &= R_0(w)\cdot\nabla w+R_1(\sigma,x)\Delta w+R_2(\sigma,x)\nabla(\nabla\cdot w) \\
&\quad +R_3(\sigma,x)\sigma+R_4(\sigma,x)\nabla\sigma,
\end{aligned}
$$

where

$$
R_0(w)=-\gamma w
$$

$$
R_1(\sigma,x)=-\mu_1\frac{\bar\rho^2}{\rho_*(\bar\rho\sigma+\rho_*)}\sigma,\quad R_2(\sigma,x)=-\mu_2\frac{\bar\rho^2}{\rho_*(\bar\rho\sigma+\rho_*)}\sigma,
$$

$$
\begin{aligned}
R_3(\sigma,x) &= \frac{\bar\rho^2\nabla\rho_*}{\gamma}\Big(\frac{P''(\rho_*)}{\rho_*(\bar\rho\sigma+\rho_*)}-\frac{P'(\rho_*)}{\rho_*^2(\bar\rho\sigma+\rho_*)} \\
&\quad -\frac{1}{\bar\rho\sigma+\rho_*}\int_0^1(1-s)P'''(s\bar\rho\sigma+\rho_*)ds\Big)\sigma,
\end{aligned}
$$

$$
R_4(\sigma,x) = \frac{\bar\rho^2}{\gamma}\Big(\frac{P'(\rho_*)}{\rho_*(\bar\rho\sigma+\rho_*)}-\frac{1}{\bar\rho\sigma+\rho_*}\int_0^1 P''(s\bar\rho\sigma+\rho_*)ds\Big)\sigma.
$$

From Lemma 2.2 and Lemma 2.6, we have

$$
\|\partial_x^\alpha\big(R_0(w)\cdot\nabla w\big)\|_2\le C\|\nabla w\|_{H^{s_0-1}}^2,
$$

$$
\|\partial_x^\alpha\big(R_1(\sigma,x)\Delta w\big)\|_2\le C\|\nabla\sigma\|_{H^{s_0-1}}\|\Delta w\|_{H^{s_0-1}},
$$

$$\|\partial_x^\alpha\big(R_2(\sigma,x)\nabla(\nabla\cdot w)\big)\|_2 \le C\|\nabla\sigma\|_{H^{s_0-1}}\|\nabla(\nabla\cdot w)\|_{H^{s_0-1}},$$
$$\|\partial_x^\alpha\big(R_3(\sigma,x)\sigma\big)\|_2 \le C\|\nabla\sigma\|_{H^{s_0-1}}\|\sigma\|_{H^{s_0-1}},$$
$$\|\partial_x^\alpha\big(R_4(\sigma,x)\nabla\sigma\big)\|_2 \le C\|\nabla\sigma\|_{H^{s_0-1}}\|\nabla\sigma\|_{H^{s_0-1}}.$$

We thus obtain

$$|I_4| \le C(1+t)^{-\frac{n}{4}}M(t)\{(1+t)^{-(\frac{n}{2}+1)}M^2(t)+\|\nabla^{s_0+1}w_\infty(t)\|_2^2\}.$$

This completes the proof. □

Proposition 5.10. *There holds the inequality*

$$\sum_{0\le|\alpha|\le s_0-1}\frac{d}{dt}\big(\partial_x^\alpha w_\infty(t),\partial_x^\alpha\nabla\sigma_\infty(t)\big)+\frac{\gamma}{2}\|\partial_x^\alpha\nabla\sigma_\infty(t)\|_2^2 \le C\|\nabla w_\infty\|_{H^{s_0}}^2+\sum_{i=1}^4 J_i \quad (40)$$

for a.e. $t\in(0,T)$, *where,*

$$J_1=\sum_{0\le|\alpha|\le s_0-1}|(\partial_x^\alpha Q_\infty B_1U,\partial_x^\alpha\nabla\cdot w_\infty)|,\quad J_2=\sum_{0\le|\alpha|\le s_0-1}|(\partial_x^\alpha Q_\infty F_1(U),\partial_x^\alpha\nabla\cdot w_\infty)|,$$

$$J_3=C\sum_{0\le|\alpha|\le s_0-1}|(\partial_x^\alpha B_2U,\partial_x^\alpha\nabla\sigma_\infty)|,\quad J_4=\sum_{0\le|\alpha|\le s_0-1}|(\partial_x^\alpha Q_\infty F_2(U),\partial_x^\alpha\nabla\sigma_\infty)|.$$

Proof. Let $|\alpha|\le s_0-1$. We take the inner-product of $\partial_x^\alpha(27)_2$ with $\partial_x^\alpha\nabla\sigma_\infty$ to obtain

$$\begin{aligned}
&(\partial_x^\alpha\partial_t w_\infty,\partial_x^\alpha\nabla\sigma_\infty)+\gamma\|\nabla\partial_x^\alpha\sigma_\infty\|_2^2\\
=\;&\mu_1(\partial_x^\alpha\Delta w_\infty,\partial_x^\alpha\nabla\sigma_\infty)+\mu_2\big(\partial_x^\alpha\nabla\cdot(\nabla w_\infty),\partial_x^\alpha\nabla\sigma_\infty\big)\\
&+(\partial_x^\alpha Q_\infty F_2(U),\partial_x^\alpha\nabla\sigma_\infty)-(\partial_x^\alpha Q_\infty B_2U,\partial_x^\alpha\nabla\sigma_\infty).
\end{aligned}\quad (41)$$

We next take the inner-product of ∂_x^α $(27)_1$ with $-\partial_x^\alpha\nabla\cdot w_\infty$ to obtain

$$\begin{aligned}
-(\partial_x^\alpha\partial_t\sigma_\infty,\partial_x^\alpha\nabla\cdot w_\infty)=&+\gamma(\partial_x^\alpha(\nabla\cdot w_\infty),\partial_x^\alpha\nabla\cdot w_\infty)\\
&-(\partial_x^\alpha Q_\infty B_1U,\partial_x^\alpha\nabla\cdot w_\infty)-(\partial_x^\alpha Q_\infty F_1(U),\partial_x^\alpha\nabla\cdot w_\infty),
\end{aligned}\quad (42)$$

Since

$$\mu_1(\partial_x^\alpha\Delta w_\infty,\partial_x^\alpha\nabla\sigma_\infty)\le C\|\partial_x^\alpha\Delta w_\infty\|_2^2+\frac{\gamma}{4}\|\partial_x^\alpha\nabla\sigma_\infty\|_2^2,$$

and

$$\mu_2\big(\partial_x^\alpha\nabla\cdot(\nabla w_\infty),\partial_x^\alpha\nabla\sigma_\infty\big)\le C\|\partial_x^\alpha\triangle w_\infty\|_2^2+\frac{\gamma}{4}\|\partial_x^\alpha\nabla\sigma_\infty\|_2^2,$$

by adding (41) and (42), we obtain the desired inequality

$$\begin{aligned}
&\sum_{0\le|\alpha|\le s_0-1}\frac{d}{dt}\big(\partial_x^\alpha w_\infty(t),\partial_x^\alpha\nabla\sigma_\infty(t)\big)+\frac{\gamma}{2}\|\nabla\partial_x^\alpha\sigma_\infty(t)\|_2^2\\
\le\;&C\Big(\|\nabla w\|_{H^{s_0}}^2+\sum_{0\le|\alpha|\le s_0-1}|(\partial_x^\alpha Q_\infty B_1U,\partial_x^\alpha\nabla\cdot w_\infty)|+|(\partial_x^\alpha Q_\infty F_1(U),\partial_x^\alpha\nabla\cdot w_\infty)|\\
&+|(\partial_x^\alpha B_2U,\partial_x^\alpha\nabla\sigma_\infty)|+|(\partial_x^\alpha Q_\infty F_2(U),\partial_x^\alpha\nabla\sigma_\infty)|\Big)
\end{aligned}\quad (43)$$

for a.e. $t \in [0,T]$. In fact, let $h \in C_0^\infty(0,T)$ and let η_ϵ is standard Friedrichs mollifier, as for the first term on the left hand side of (43)

$$\begin{aligned}
&\int_0^T (\partial_x^\alpha \partial_t w_\infty, \partial_x^\alpha \nabla \eta_\epsilon * \sigma_\infty) h dt \\
= &\int_0^T \frac{d}{dt}(\partial_x^\alpha w_\infty, \partial_x^\alpha \nabla \eta_\epsilon * \sigma_\infty) h dt - \int_0^T (\partial_x^\alpha w_\infty, \partial_x^\alpha \partial_t (\nabla \eta_\epsilon * \sigma_\infty)) h dt \\
= &-\int_0^T (\partial_x^\alpha w_\infty, \partial_x^\alpha \nabla(\eta_\epsilon * \sigma_\infty)) \frac{d}{dt} h dt + \int_0^T (\partial_x^\alpha \nabla \cdot w_\infty, \eta_\epsilon * \partial_t \partial_x^\alpha \sigma_\infty) h dt. \quad (44)
\end{aligned}$$

Since $\partial_x^\alpha w_\infty, \partial_x^\alpha \nabla \cdot w_\infty, \partial_x^\alpha \partial_t \sigma$ and $\partial_x^\alpha \partial_t \sigma \in C([0,T];L^2)$ for $|\alpha| \le s_0 - 1$, letting $\epsilon \to 0$ in (44) we can obtain by similar to proof of Lemma 5.7

$$(\partial_x^\alpha \partial_t w_\infty, \partial_x^\alpha \nabla \sigma_\infty) = \frac{d}{dt}(\partial_x^\alpha w_\infty, \partial_x^\alpha \nabla \sigma_\infty) + (\partial_x^\alpha \nabla \cdot w_\infty, \partial_x^\alpha \partial_t \sigma_\infty)$$

for a.e. $t \in [0,T]$.

This completes the proof. □

Proposition 5.11. *Let $n \ge 3$. There exists a $\epsilon > 0$ such that if*

$$\|\Phi\|_{H^{s_0+1}} + \|(1+|x|)\nabla\Phi\|_{L^2} \le \epsilon,$$

$$\sup_{0 \le t \le T} \|\sigma(t)\|_{H^{s_0}} \le \delta,$$

and

$$M(t) \le 1$$

for $t \in [0,T]$, then there holds

$$|J_1| + |J_3| \le C\epsilon\{(1+t)^{-\frac{n+2}{2}} M^2(t) + \|\nabla^{s_0+1} w_\infty(t)\|_2^2\}.$$

for $t \in [0,T]$. Here $C > 0$ is a constant independent of T.

The proof is similar to that of Proposition 5.8 . We omit it.

Proposition 5.12. *Let $n \ge 3$. There exists a $\epsilon > 0$ such that if*

$$\|\Phi\|_{H^{s_0+1}} + \|(1+|x|)\nabla\Phi\|_{L^2} \le \epsilon$$

$$\sup_{0 \le t \le T} \|\sigma(t)\|_{H^{s_0}} \le \delta,$$

and

$$M(t) \le 1$$

for $t \in [0,T]$, then there hold

$$|J_2| + |J_4| \le C(1+t)^{-\frac{n}{4}} M(t)\{(1+t)^{-\frac{n+2}{2}} M^2(t) + \|\nabla^{s_0+1} w_\infty(t)\|_2^2\}$$

fir $t \in [0,T]$. Here $C > 0$ is a constant independent of T.

The proof is similar to that of Proposition 5.9. We omit it.

Proposition 5.13. *Let $n \geq 3$. There exists a $\epsilon > 0$ such that if*

$$\|\Phi\|_{H^{s_0}} + \|(1+|x|)\nabla\Phi\|_{L^2} \leq \epsilon$$

$$\sup_{0\leq t\leq T} \|\sigma(t)\|_{H^{s_0}} \leq \delta,$$

and

$$M(t) \leq 1$$

for $t \in [0,T]$, then there holds

$$\begin{aligned} &\frac{d}{dt}E_\infty(t) + C_1 E_\infty(t) + C_2 D_\infty(t) \leq C\epsilon(1+t)^{-\frac{n}{2}-\frac{1}{2}}M^2(t) \\ &\quad +C(1+t)^{-\frac{3n+4}{4}}M^3(t) + C(1+t)^{-\frac{n}{4}}M(t)D_\infty(t) \end{aligned} \tag{45}$$

for $t \in [0,T]$. Here, $E_\infty(t)$ and $D_\infty(t)$ are equivalent to $\|U_\infty(t)\|^2_{H^{s_0}}$ and $\|\nabla w_\infty(t)\|^2_{H^{s_0}} + \|\nabla\sigma_\infty(t)\|^2_{H^{s_0-1}}$ respectively. That is, there exist $d_1, d_2 > 0$ such that

$$\frac{1}{d_1}E_\infty(t) \leq \|U_\infty(t)\|^2_{H^{s_0}} \leq d_1 E_\infty(t),$$

$$\frac{1}{d_2}D_\infty(t) \leq \|\nabla w_\infty(t)\|^2_{H^{s_0}} + \|\nabla\sigma_\infty(t)\|^2_{H^{s_0-1}} \leq d_2 D_\infty(t).$$

Proof. We add $\kappa \times$ (28) to (40) with a constant $\kappa > 0$ to be determined later. Then, by Proposition 5.8 and Proposition 5.12, we obtain

$$\begin{aligned} &\frac{d}{dt}\Big(\frac{\kappa}{2}\|U_\infty\|^2_{H^{s_0}} + \sum_{0\leq|\alpha|\leq s_0-1}(\partial_x^\alpha w_\infty, \partial_x^\alpha \nabla\sigma_\infty)\Big) \\ &+\kappa\Big(\sum_{0\leq|\alpha|\leq s_0} \mu_1\|\nabla\partial_x^\alpha w_\infty\|_2^2 + \mu_2\|\nabla\cdot\partial_x^\alpha w_\infty(t)\|_2^2\Big) + \frac{\gamma}{2}\|\nabla\sigma_\infty\|^2_{H^{s_0-1}} \\ \leq\ & C\sum_{0\leq|\alpha|\leq s_0-1}\|\partial_x^\alpha\nabla w_\infty\|^2_{H^{s_0-1}} + C\epsilon\{(1+t)^{-\frac{n+1}{2}}M(t)^2 + \|\nabla^{s_0+1}w_\infty\|_2^2\} \\ &+C(1+t)^{-\frac{n}{4}}M(t)\big((1+t)^{-\frac{n+2}{2}}M(t)^2 + \|\nabla^{s_0+1}w_\infty\|_2^2\big). \end{aligned} \tag{46}$$

We set

$$E_\infty(t) = \frac{\kappa}{2}\|U_\infty(t)\|^2_{H^{s_0}} + \sum_{0\leq|\alpha|\leq s_0-1}(\partial_x^\alpha w_\infty(t), \partial_x^\alpha\nabla\sigma_\infty(t)),$$

$$D_\infty(t) = \frac{\kappa}{2}\sum_{0\leq|\alpha|\leq s_0}(\mu_1\|\nabla\partial_x^\alpha w_\infty(t)\|_2^2 + \mu_2\|\nabla\cdot\partial_x^\alpha w_\infty(t)\|_2^2) + \frac{\gamma}{2}\|\nabla\sigma_\infty(t)\|^2_{H^{s_0-1}}.$$

For each $\kappa > 0$, $D_\infty(t)$ and $\|\nabla w_\infty(t)\|^2_{H^{s_0}} + \|\nabla\sigma_\infty(t)\|^2_{H^{s_0-1}}$ are equivalent. Since

$$\Big|\sum_{0\leq|\alpha|\leq s_0}(\partial_x^\alpha w_\infty(t), \partial_x^\alpha\nabla\sigma_\infty(t))\Big| \leq C'\|U_\infty(t)\|^2_{H^{s_0}},$$

if κ is fixed in such a way that $\kappa > 2C'$, then one can see that $E_\infty(t)$ and $\|U_\infty(t)\|^2_{H^{s_0}}$ are equivalent. With this $\kappa > 0$, we see from (46) that

$$\begin{aligned}&\frac{d}{dt}E_\infty(t) + 2C_2 D_\infty \le C\epsilon(1+t)^{-\frac{n}{2}-\frac{1}{2}}M^2(t)\\&\quad +C(1+t)^{-\frac{3n+4}{4}}M^3(t) + C(1+t)^{-\frac{n}{4}}M(t)D_\infty(t).\end{aligned}\tag{47}$$

By Lemma 4.2, we have

$$E_\infty(t) \le CD_\infty(t).$$

This, together with (47), gives the desired inequality (45). □

5.3 Proof of Theorem 3.3.

Proposition 5.14. *There exists a constant $\epsilon_2 > 0$ such that if*

$$\|U_0\|^2_{H^{s_0}\cap L^1} \le \epsilon_2,$$

then there holds

$$M(t) \le C\|U_0\|_{H^{s_0}\cap L^1}$$

for $0 \le t \le T$, where the constant C does not depend on T.

Proof. By (45) we have

$$\begin{aligned}&E_\infty(t) + C_2\int_0^t e^{-C_1(t-\tau)}D_\infty(\tau)d\tau\\ \le\ & e^{-C_1 t}E_\infty(0) + C\epsilon M^2(t)\int_0^t e^{-C_1(t-\tau)}(1+\tau)^{-\frac{n+2}{2}}d\tau\\ &+C\big\{M^3(t)\int_0^t (1+\tau)^{-\frac{3n+4}{4}}e^{-C_1(t-\tau)}d\tau\\ &+M(t)\int_0^t (1+\tau)^{-\frac{n}{4}}e^{-C_1(t-\tau)}D_\infty(\tau)d\tau\big\}\\ \le\ & e^{-C_1 t}E_\infty(0) + C\epsilon(1+t)^{-\frac{n+2}{2}}M^2(t) + C(1+t)^{-\frac{3n+4}{4}}M^3(t)\\ &+CM(t)\int_0^t e^{-C_1(t-\tau)}D_\infty(\tau)d\tau.\end{aligned}\tag{48}$$

We set $\mathcal{D}_\infty(t) := (1+t)^{\frac{n+2}{2}}\int_0^t e^{-C_1(t-\tau)}D_\infty(\tau)d\tau$. Since $\frac{3n+4}{4} > \frac{n+2}{2}$, we see from (48) that

$$M^2_\infty(t) + C_2\mathcal{D}_\infty(t) \le C\big(E_\infty(0) + \epsilon M^2(t) + M^3(t) + CM(t)\mathcal{D}_\infty(t)\big).$$

This, together with Proposition 5.4, gives

$$M^2(t) + C_2\mathcal{D}_\infty(t) \le C\big(E_\infty(0) + \|U_0\|^2_1 + M^4(t) + M^3(t) + M(t)\mathcal{D}_\infty(t) + \epsilon M^2(t)\big).$$

By taking $\epsilon > 0$ suitable small, we obtain

$$M^2(t) + C_2'\mathcal{D}_\infty(t) \le C_3\big(\|U_0\|^2_{H^{s_0}\cap L^1} + M^3(t) + M(t)\mathcal{D}_\infty(t)\big).\tag{49}$$

We observe that there exists a constant $C_4 > 0$ such that

$$M(0) \le C_4 \|U_0\|_{H^{s_0}\cap L^1}.$$

Since $M(t)$ is continuous in t, there exists $t_0 > 0$ such that

$$M(t) < 2C_4 \|U_0\|_{H^{s_0}\cap L^1}$$

for all $t \in [0, t_0]$. Moreover there exists constants $C_6 > 0$ and C_7 such that

$$\|\sigma_0\|_{H^{s_0}} + \|w_0\|_{H^{s_0}} \le C_6 M(0).$$

We set $C_5 := \max\{\sqrt{\frac{C_3}{2}}, C_4\}$, and take ϵ_2 in such a way that $0 < \epsilon_2 < \min\{\frac{1}{4C_5^2}, \frac{\delta}{4C_5^2}, \frac{\epsilon_1^2}{C_4^2 C_6^2}, \frac{1}{16C_3^2C_5^2}, \frac{C_2'}{16C_3^2C_5^2}\}$. We will show $M(t) < 2C_5\|U_0\|_{H^{s_0}\cap L^1},\ 0 \le t \le T$.

Assume that there exists $t_1 \in (t_0, T]$ such that

$$M(t) < 2C_5 \|U_0\|_{H^{s_0}\cap L^1}$$

for $0 \le t < t_1$ and

$$M(t_1) = 2C_5 \|U_0\|_{H^{s_0}\cap L^1}.$$

It then follows from (49) that

$$\begin{aligned} M^2(t) + C_2'\mathcal{D}_\infty(t) &\le C_3\|U_0\|^2_{H^{s_0}\cap L^1} + C_3 M(t)\big(M^2(t) + \mathcal{D}_\infty(t)\big) \\ &< C_3\|U_0\|^2_{H^{s_0}\cap L^1} + \frac{1}{2}\big(M^2(t) + C_2'\mathcal{D}_\infty(t)\big) \end{aligned}$$

for $t \in [0, t_1]$, and hence,

$$\begin{aligned} M^2(t) + C_2'\mathcal{D}_\infty(t) &< 2C_3\big(E_\infty(0) + \|U_0\|^2_{L^1\cap L^2}\big) \\ &\le 4C_5^2\|U_0\|^2_{H^{s_0}\cap L^1} \end{aligned}$$

for $t \in [0, t_1]$. But this contradicts to $M(t_1) = 2C_5\|U_0\|_{H^{s_0}\cap L^1}$. We thus conclude that

$$M(t) < 2C_5 \|U_0\|_{H^{s_0}\cap L^1}$$

for all $0 \le t \le T$. □

It follows from Theorem 3.2 and Proposition 5.14 that

$$M(t) \le C_0 \qquad \text{for all } t.$$

Hence we obtain the desired decay estimate in Theorem 3.3.

6 Decay estimate of strong solutions in critical spaces

.

In sections 6-7 we prove Theorem 1.3. We consider the compressible Navier-Stokes equation in critical space.

We first rewrite system (1) into the one for the perturbation. We then introduce some auxiliary lemmas which will be useful in the proof of the main result.

Let us rewrite the problem (1). We define μ_1, μ_2 and γ by

$$\mu_1 = \frac{\mu}{\bar{\rho}},\ \mu_2 = \frac{\mu+\mu'}{\bar{\rho}},\ \gamma = \sqrt{P'(\bar{\rho})}.$$

By using the new unknown function

$$\sigma(t,x) = \frac{\rho(t,x)-\bar{\rho}}{\bar{\rho}}, \quad w(t,x) = \frac{1}{\gamma}u(t,x),$$

the initial value problem (1) is reformulated as

$$\begin{cases} \partial_t\sigma + \gamma\nabla\cdot w = F_1(U), \\ \partial_t w - \mu_1\Delta w - \mu_2\nabla(\nabla\cdot w) + \gamma\nabla\sigma = F_2(U), \\ (\sigma, w)(0,x) = (\sigma_0, w_0)(x), \end{cases} \tag{50}$$

where, $U = \begin{pmatrix} \sigma \\ w \end{pmatrix}$,

$$F_1(U) = -\gamma(w\cdot\nabla\sigma + \sigma\nabla\cdot w),$$

$$\begin{aligned} F_2(U) &= -\gamma(w\cdot\nabla)w - \mu_1\frac{\sigma}{\sigma+1}\Delta w - \mu_2\frac{\sigma}{\sigma+1}\nabla(\nabla\cdot w) \\ &\quad + \Big(\frac{\bar{\rho}\gamma}{\sigma+1} - \frac{\bar{\rho}}{\gamma}\frac{\int_0^1 P''(s\bar{\rho}\sigma+\bar{\rho})ds}{\sigma+1}\Big)\sigma\nabla\sigma. \end{aligned}$$

We set

$$A = \begin{pmatrix} 0 & -\gamma\nabla\cdot \\ -\gamma\nabla & \mu_1\Delta + \mu_2\nabla\nabla\cdot \end{pmatrix}.$$

By using operator A, problem (50) is written as

$$\partial_t U - AU = F(U), \quad U|_{t=0} = U_0, \tag{51}$$

where

$$F(U) = \begin{pmatrix} F_1(U) \\ F_2(U) \end{pmatrix}, \quad U_0 = \begin{pmatrix} \sigma_0 \\ w_0 \end{pmatrix}.$$

In terms of U, Theorem 1.3 is restated as follows:

Theorem 6.1. *Assume that $n \geq 2$ and $\Phi = 0$ and $1 \leq p < \frac{2n}{n+1}$. Then there exists $\epsilon > 0$ such that if*

$$\sigma_0 \in \dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^0_{p,\infty},\ w_0 \in \dot{B}^{\frac{n}{2}-1}_{2,1} \cap \dot{B}^0_{p,\infty}$$

and

$$\|\sigma_0\|_{\dot{B}^{\frac{n}{2}}_{2,1}\cap\dot{B}^0_{p,\infty}} + \|w_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}\cap\dot{B}^0_{p,\infty}} \leq \epsilon,$$

then problem (1) *has a unique global solution* (ρ, u) *satisfying*

$$(\sigma, w) \in C([0,\infty); B_{2,1}^{\frac{n}{2}}) \times \big(C([0,\infty); B_{2,1}^{\frac{n}{2}-1}) \cap L^1(0,\infty; \dot{B}_{2,1}^{\frac{n}{2}+1})\big).$$

Furthermore, there exists a constant $C_0 > 0$ *such that the estimates*

$$\|(\sigma, w)(t)\|_{L^2} \le C_0(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}, \tag{52}$$

$$\|w(t)\|_{\dot{B}_{2,1}^{s_1}} \le C_0(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{s_1}{2}}, \qquad \textit{for} \quad 0 \le s_1 \le \tfrac{n}{2}-1 \tag{53}$$

$$\|\sigma(t)\|_{\dot{B}_{2,1}^{s_2}} \le C_0(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{s_2}{2}}, \qquad \textit{for} \quad 0 \le s_2 \le \tfrac{n}{2} \tag{54}$$

hold for $t \ge 0$.

We introduce a semigroup generated by A. We set

$$E(t)u := \mathfrak{F}^{-1}[e^{\hat{A}(\xi)t}\hat{u}] \quad \text{for } u \in L^2,$$

where

$$\hat{A}(\xi) = \begin{pmatrix} 0 & -i\gamma\xi^{\mathrm{t}} \\ -i\gamma\xi & -\mu_1|\xi|^2 I_n - \mu_2\xi\xi^{\mathrm{t}} \end{pmatrix}.$$

Here and in what follows the superscript $\cdot^{\mathrm{t}}$ means the transposition.

We next state some basic lemmas.

Lemma 6.2. *Let* $s_1, s_2 \le \frac{n}{2}$ *such that* $s_1 + s_2 > 0$*; and let* $u \in \dot{B}_{2,1}^{s_1}$ *and* $v \in \dot{B}_{2,1}^{s_2}$*. Then* $uv \in \dot{B}_{2,1}^{s_1+s_2-\frac{n}{2}}$ *and*

$$\|uv\|_{\dot{B}_{2,1}^{s_1+s_2-\frac{n}{2}}} \le C\|u\|_{\dot{B}_{2,1}^{s_1}}\|v\|_{\dot{B}_{2,1}^{s_2}}.$$

See, e.g., [1], for a proof of Lemma 6.2.

Lemma 6.3. *Let* $s > 0$ *and let* $u \in \dot{B}_{2,1}^{s} \cap L^\infty$*. Let* $F \in W_{loc}^{[s]+2,\infty}(\mathbb{R}^n)$ *such that* $F(0) = 0$*. Then* $F(u) \in \dot{B}_{2,1}^{s}$*. Moreover, there exists a function* C_1 *of one variable depending only on* s, n *and* F *such that*

$$\|F(u)\|_{\dot{B}_{2,1}^{s}} \le C_1(\|u\|_{L^\infty})\|u\|_{\dot{B}_{2,1}^{s}}.$$

See, e.g., [2], for a proof of Lemma 6.3.

Lemma 6.4. (*i*) *Let* $a, b > 0$ *satisfying* $\max\{a, b\} > 1$*. Then*

$$\int_0^t (1+s)^{-a}(1+t-s)^{-b}ds \le C(1+t)^{-\min\{a,b\}}, \quad t \ge 0.$$

(*ii*) *Let* $f \in L^p(0,\infty)$ *and* $a, b > 0$ *satisfying* $\max\{a,b\} > \frac{1}{p'}$ *for* $1 \le p \le \infty$ *and* p' *is the conjugate exponent to* p. *Then*

$$\int_0^t (1+s)^{-a}(1+t-s)^{-b} f ds \le C(1+t)^{-\min\{a,b\}}\Big(\int_0^t |f|^p ds\Big)^{\frac{1}{p}}, \quad t \ge 0.$$

For a proof of (i), see [19]. Proof of (ii) is given by using Hölder inequality; we omit it.

Let us now introduce a few bilinear estimates in Besov spaces. We will use the Bony decomposition

$$uv = T_u v + T_v u + R(u,v), \tag{55}$$

with

$$T_u v = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} u \dot{\triangle}_j v, \quad R(u,v) = \sum_{j\in\mathbb{Z}} \dot{\triangle}_j u \tilde{\triangle}_j v, \quad \tilde{\triangle}_j v = \dot{\triangle}_{j-1} v + \dot{\triangle}_j v + \dot{\triangle}_{j+1} v.$$

Lemma 6.5. *It holds that*

(*i*)
$$\sup_{j<0} \|\dot{\triangle}_j T_g f\|_{L^1} \le C\|\dot{S}_4 f\|_{L^2}\|\dot{S}_4 g\|_{L^2},$$
$$\sup_{j<0} \|\dot{\triangle}_j R(f,g)\|_{L^1} \le C\big(\|\dot{S}_3 f\|_{L^2}\|\dot{S}_3 g\|_{L^2} + \|\tilde{S}_0 f\|_{L^2}\|\tilde{S}_0 g\|_{L^2}\big).$$

(*ii*) *If* $0 \le s_1, s_2, s_3, s_4 \le \frac{n}{2}$, *then*

$$\sum_{j\ge0} 2^{s_1 j}\|\triangle_j T_g f\|_{L^2} \le C\big(\|\dot{S}_{-5} g\|_{\dot{B}^{\frac{n}{2}-s_2}_{2,1}}\|\tilde{S}_{-5} f\|_{\dot{B}^{s_1+s_2}_{2,1}} + \|\tilde{S}_{-5} g\|_{\dot{B}^{\frac{n}{2}-s_3}_{2,1}}\|\tilde{S}_{-5} f\|_{\dot{B}^{s_1+s_3}_{2,1}}\big),$$

$$\sum_{j\ge0} 2^{s_1 j}\|\triangle_j R(f,g)\|_{L^2} \le C\|\tilde{S}_{-4} f\|_{\dot{B}^{\frac{n}{2}-s_4}_{2,1}}\|\tilde{S}_{-4} g\|_{\dot{B}^{s_1+s_4}_{2,1}}.$$

Remark 6.6. *By Lemma* 6.5, *we have*

(*i*)
$$\sup_{j<0} \|\dot{\triangle} uv\|_{L^1} \le C(\|\dot{S}_4 u\|_{L^2}\|\dot{S}_4 v\|_{L^2} + \|\tilde{S}_0 u\|_{L^2}\|\tilde{S}_0 v\|_{L^2}).$$

(*ii*) *If* $0 \le s_1, s_2, s_3, s_4 \le \frac{n}{2}$, *then*

$$\begin{aligned}\sum_{j\ge0} 2^{s_1 j}\|\triangle_j uv\|_{L^2} \quad \le \quad & C\big(\|\dot{S}_{-5} u\|_{\dot{B}^{\frac{n}{2}-s_2}_{2,1}}\|\tilde{S}_{-5} v\|_{\dot{B}^{s_1+s_2}_{2,1}} + \|\dot{S}_{-5} v\|_{\dot{B}^{\frac{n}{2}-s_3}_{2,1}}\|\tilde{S}_{-5} v\|_{\dot{B}^{s_1+s_3}_{2,1}} \\ & + \|\tilde{S}_{-5} u\|_{\dot{B}^{\frac{n}{2}-s_4}_{2,1}}\|\tilde{S}_{-5} v\|_{\dot{B}^{s_1+s_4}_{2,1}}\big).\end{aligned}$$

Proof of Lemma 6.5**.** We have

$$\dot{\triangle}_j T_g f = \sum_{|j'-j|\le4} \dot{\triangle}_j(\dot{S}_{j'-1} g \dot{\triangle}_{j'} f), \quad \dot{\triangle}_j R(f,g) = \sum_{j'\ge j-3} \dot{\triangle}_j(\dot{\triangle}_{j'} f \tilde{\triangle}_{j'} g).$$

For any $j<0$, by the Hölder inequality, we have

$$\begin{aligned}\|\dot{\triangle}_j T_g f\|_{L^1} &\le C\sum_{|j'-j|\le 4}\|\dot{S}_{j'-1}g\dot{\triangle}_{j'}f\|_{L^1}\\ &\le C\|\dot{S}_4 g\|_{L^2}\|\dot{S}_4 f\|_{L^2},\end{aligned}$$

and

$$\begin{aligned}\|\dot{\triangle}_j R(f,g)\|_{L^1} &\le C\|\sum_{j'\ge j-3}\dot{\triangle}_j(\dot{\triangle}_{j'}f\tilde{\triangle}_{j'}g)\|_{L^1}\\ &\le C\sum_{j'\le 0}\|\dot{\triangle}_{j'}f\tilde{\triangle}_{j'}g\|_{L^1}+\sum_{j'\ge 1}\|\dot{\triangle}_{j'}f\tilde{\triangle}_{j'}g\|_{L^1}\\ &\le C(\|\dot{S}_3 f\|_{L^2}\|\dot{S}_3 g\|_{L^2}+\|\tilde{S}_0 f\|_{L^2}\|\tilde{S}_0 g\|_{L^2}).\end{aligned}$$

Taking the supremum in $j<0$, we obtain the desired estimates of (i).

We next prove (ii). Choose $s_1\in[0,\frac{n}{2}]$. We then obtain by Hölder inequality and Lemma 2.11 that

$$\begin{aligned}\sum_{j\ge 0}2^{s_1 j}\|\dot{\triangle}_j T_g f\|_{L^2} &\le C\sum_{j\ge 0}\sum_{|j'-j|\le 4}2^{s_1 j}\|\dot{\triangle}_j(\dot{S}_{j'-1}g\dot{\triangle}_{j'}f)\|_{L^2}\\ &\le C\sum_{j'\ge -4}2^{s_1 j'}\|\dot{S}_{j'-1}g\dot{\triangle}_{j'}f\|_{L^2}\\ &\le C\sum_{j'\ge -4}2^{s_1 j'}\|\{\dot{S}_{-5}g+(\dot{S}_{j'-1}-\dot{S}_{-5})g\}\dot{\triangle}_{j'}f\|_{L^2}\\ &\le C\sum_{j'\ge -4}2^{s_1 j'}\{\|\dot{S}_{-5}g\|_{L^{\frac{n}{s_2}}}\|\dot{\triangle}_{j'}f\|_{L^{\frac{2n}{n-2s_2}}}\\ &\quad+\|(\dot{S}_{j'-1}-\dot{S}_{-5})g\|_{L^{\frac{n}{s_3}}}\|\dot{\triangle}_{j'}f\|_{L^{\frac{2n}{n-2s_3}}}\}\\ &\le C(\|\dot{S}_{-5}g\|_{\dot{B}^{\frac{n}{2}-s_2}_{2,1}}\|\tilde{S}_{-5}g\|_{\dot{B}^{s_1+s_2}_{2,1}}+\|\tilde{S}_{-5}g\|_{\dot{B}^{\frac{n}{2}-s_3}_{2,1}}\|\tilde{S}_{-5}g\|_{\dot{B}^{s_1+s_3}_{2,1}}),\end{aligned}$$

$$\begin{aligned}\sum_{j\ge 0}2^{s_1 j}\|\triangle_j R(f,g)\|_{L^2} &\le C\sum_{j\ge 0}\sum_{j'\ge j-3}2^{s_1 j}\|\dot{\triangle}_j(\dot{\triangle}_{j'}f\tilde{\triangle}_{j'}g)\|_{L^2}\\ &\le C\sum_{j\ge 0}\sum_{j'\ge j-3}2^{(s_1+\frac{n}{2})j}\|\dot{\triangle}_j(\dot{\triangle}_{j'}f\tilde{\triangle}_{j'}g)\|_{L^1}\\ &\le C\sum_{j\ge 0}\sum_{j'\ge j-3}2^{(s_1+\frac{n}{2})(j-j')}2^{(\frac{n}{2}-s_4)j'}\|\dot{\triangle}_{j'}f\|_{L^2}2^{(s_1+s_4)j'}\|\tilde{\triangle}_{j'}g\|_{L^2}\\ &\le C\|\tilde{S}_{-4}f\|_{\dot{B}^{\frac{n}{2}-s_4}_{2,1}}\|\tilde{S}_{-4}g\|_{\dot{B}^{s_1+s_4}_{2,1}}.\end{aligned}$$

This completes the proof. $\square$

We now introduce commutator estimates.

Lemma 6.7. *Let $s\in\big(-\frac{n}{2},\frac{n}{2}+1\big]$. There exists a sequence $c_j\in l^1(\mathbb{Z})$ such that $\|c_j\|_{l^1}=1$ and a constant C depending only on n and s such that*

$$\forall j\in\mathbb{Z},\quad \|[f\cdot\nabla,\dot{\triangle}_j]g\|_{L^2}\le Cc_j2^{-sj}\|\nabla f\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|g\|_{\dot{B}^{s}_{2,1}}.$$

See, e.g., [1] for a proof of Lemma 6.7.

Lemma 6.8. *Let* $0 < p < q < r \le \infty$ *and set* $\theta = \frac{q^{-1}-r^{-1}}{p^{-1}-r^{-1}} \in (0,1)$. *Then it holds that*

(*i*) $L^p \cap L^r \subset L^q$ *and* $\|f\|_{L^q} \le \|f\|_{L^p}^{\theta}\|f\|_{L^r}^{1-\theta}$,

(*ii*) $\dot{B}^0_{p,l} \cap \dot{B}^0_{r,l} \subset \dot{B}^0_{q,l}$ *and* $\|f\|_{\dot{B}^0_{q,l}} \le \|f\|^{\theta}_{\dot{B}^0_{p,l}}\|f\|^{1-\theta}_{\dot{B}^0_{r,l}}$, *for* $1 \le l \le \infty$.

Proof. (*i*) is a well-known inequality. Let us prove (*ii*). Let $1 \le l < \infty$. Then by using (*i*) and Hölder inequality of sequence space, we have

$$\begin{aligned}
\|f\|_{\dot{B}^0_{q,l}} &= \big(\sum_{j\in\mathbb{Z}} \|\dot{\triangle}_j f\|^l_{L^q}\big)^{\frac{1}{l}} \\
&\le \big(\sum_{j\in\mathbb{Z}} \|\dot{\triangle}_j f\|^{l\theta}_{L^p}\|\dot{\triangle}_j f\|^{l(1-\theta)}_{L^r}\big)^{\frac{1}{l}} \\
&\le \big\{\big(\sum_{j\in\mathbb{Z}} \|\dot{\triangle}_j f\|^l_{L^p}\big)^{\theta}\big(\sum_{j\in\mathbb{Z}} \|\dot{\triangle}_j f\|^l_{L^r}\big)^{(1-\theta)}\big\}^{\frac{1}{l}} \\
&\le \|f\|^{\theta}_{\dot{B}^0_{p,l}}\|f\|^{1-\theta}_{\dot{B}^0_{r,l}}.
\end{aligned}$$

When $l = \infty$, we have

$$\|f\|_{\dot{B}^0_{q,\infty}} \le \sup_{j\in\mathbb{Z}} \|\dot{\triangle}_j f\|^{l\theta}_{L^p}\|\dot{\triangle}_j f\|^{l(1-\theta)}_{L^r} \le \|f\|^{\theta}_{\dot{B}^0_{p,\infty}}\|f\|^{1-\theta}_{\dot{B}^0_{r,\infty}}.$$

This completes the proof. □

7 Apriori estimate in critical space with time weight

In this section we prove Theorem 6.1. In subsections 7.1 and 7.2 we establish the necessary estimates for $\triangle_{-1}U(t)$ and $\triangle_j U(t)$ for $j \ge 0$, respectively. In subsection 7.3 we derive the a priori estimate to complete the proof of Theorem 1.3.

We first explain known results which are used to prove Theorem 1.3.

Danchin [2] proved the following global existence result in nonhomogeneous Besov space.

Proposition 7.1 (Danchin [2]). *Let* $n \ge 2$. *There are two positive constants* ϵ_1 *and* M *such that for all* (ρ_0, u_0) *with* $(\rho_0 - \bar{\rho}) \in \dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^{\frac{n}{2}-1}_{2,1}$, $u_0 \in \dot{B}^{\frac{n}{2}-1}_{2,1}$ *and*

$$\|\rho_0 - \bar{\rho}\|_{\dot{B}^{\frac{n}{2}}_{2,1}\cap\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|u_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \le \epsilon_1, \tag{56}$$

problem (1) *has a unique global solution* $(\rho, u) \in C(\mathbb{R}^+; \dot{B}^{\frac{n}{2}}_{2,1}\cap\dot{B}^{\frac{n}{2}-1}_{2,1}) \times \big(L^1(\mathbb{R}^+; \dot{B}^{\frac{n}{2}+1}_{2,1}) \cap C(\mathbb{R}^+; \dot{B}^{\frac{n}{2}-1}_{2,1})\big)$ *that satisfies the estimate*

$$\sup_{t\ge0}\{\|\rho(t)-\bar{\rho}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|u(t)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\}+\int_0^\infty \|u\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}dt \le M\big(\|\rho_0-\bar{\rho}\|_{\dot{B}^{\frac{n}{2}}_{2,1}\cap\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|u_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\big).$$

Haspot [8] proved the following local existence result in nonhomogeneous Besov space.

Proposition 7.2 (Haspot [8])**.** *Let $n \geq 2$ and $1 \leq p < 2n$. Let $u_0 \in B^{\frac{n}{p}-1}_{p,1}$ and $(\rho_0 - \bar{\rho}) \in B^{\frac{n}{p}}_{p,1}$ with $\frac{1}{\rho_0}$ bounded away from zero. Then there exist a constant $T > 0$ such that the problem (1) has a local solution (ρ, u) on $[0, T]$ with $\frac{1}{\rho} > 0$ bounded away from zero and:*

$$\rho - \bar{\rho} \in C([0,T]; B^{\frac{n}{p}}_{p,1}), \quad u \in \big(C([0,T]; B^{\frac{n}{p}-1}_{p,1}) \cap L^1(0,T; B^{\frac{n}{p}+1}_{p,1})\big).$$

Moreover, this solution is unique if

$$p \leq n.$$

Proposition 7.3. *Let $T > 0$ and let (σ, w) be a solution of problem (51) on $[0, T]$ such that*

$$\sigma \in C([0,T]; B^{\frac{n}{2}}_{2,1}), w \in C([0,T]; B^{\frac{n}{2}}_{2,1}) \cap L^1(0,T; B^{\frac{n}{2}+1}_{2,1}), \tag{57}$$

Then, $\triangle_j U(t) = (\triangle_j \sigma, \triangle_j w)^{\mathrm{t}}$ for $j \geq -1$ satisfy

$$\partial_t \triangle_j U - A\triangle_j U = \triangle_j F(U), \tag{58}$$

$$\triangle_j U|_{t=0} = \triangle_j U_0. \tag{59}$$

Moreover, $\triangle_{-1} U(t)$ satisfy

$$\triangle_{-1} U(t) \in C\big([0,T]; \dot{B}^k_{2,1}\big), \quad \forall k \in [0, \infty) \tag{60}$$

and

$$\triangle_{-1} U(t) = E(t)\triangle_{-1} U_0 + \int_0^t E(t-s)\triangle_{-1} F(U)(s) ds. \tag{61}$$

Proof. Let $U(t) = (\sigma, w)^{\mathrm{t}}$ be a solution of (51) satisfying (57). Since $\triangle_j AU = A\triangle_j U$, applying $\triangle_j$ to (51), we obtain (58) and (59). It then follows that

$$\triangle_j U(t) = E(t)\triangle_j U_0 + \int_0^t E(t-s)\triangle_j F(U)(s) ds.$$

We also have (60) from Lemma 2.7. This completes the proof. □

Set

$$\begin{aligned} M_1(t) \;:=\; & \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{2})} \|\triangle_{-1} U(\tau)\|_{L^2} \\ & + \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{2})+\frac{1}{2}} \sum_{j<0} 2^j \|\dot{\triangle}_j U(\tau)\|_{L^2} \\ & + \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{n}{2p}-\frac{1}{2}} \sum_{j<0} 2^{(\frac{n}{2}-1)j} \|\dot{\triangle}_j U(\tau)\|_{L^2} \\ & + \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{n}{2p}} \sum_{j<0} 2^{\frac{n}{2}j} \|\dot{\triangle}_j U(\tau)\|_{L^2}, \end{aligned}$$

$$M_\infty(t) := \sup_{0\le\tau\le t}(1+\tau)^{\frac{n}{2p}}\sum_{j=0}^{\infty}2^{(\frac{n}{2}-1)j}\{\|\triangle_j U(\tau)\|_{L^2}+2^j\|\triangle_j\sigma\|_{L^2}\},$$

$$M(t) := M_1(t)+M_\infty(t).$$

If we could obtain uniform estimates of $M_1(t)$ and $M_\infty(t)$, then Theorem 1.3 would be proved.

Remark 7.4. *$M_1(t)$ includes the $B^{\frac{n}{2}}_{2,1}$-norm of the low frequency part of perturbation with time weigh. Since any order of differentiation acts as a bounded operator on the low frequency part, we can treat $\dot{B}^{\frac{n}{2}}_{2,1}$-norm of the low frequency part of velocity, although the velocity itself belongs to $C([0,\infty);\dot{B}^{n/2-1}_{2,1})$. $M_\infty(t)$ is $\dot{B}^{\frac{n}{2}}_{2,1}\times\dot{B}^{\frac{n}{2}-1}_{2,1}$ -norm of the high frequency part of perturbation with time weigh. We note that the decay order of high frequency part is faster than the low frequency part. These facts are used to obtain decay estimates of nonlinear term.*

7.1 Estimate of low frequency parts

In this subsection we derive the estimate of $\triangle_{-1}U(t)$, in other words, we estimate $M_1(t)$.

Lemma 7.5. *Let $s\ge 0$ and let $1\le p\le 2$. Then $E(t)$ satisfies the estimates*

$$\|E(t)\triangle_{-1}U_0\|_{L^2}\le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}\sup_{j<0}\|\dot{\triangle}_j U_0\|_{L^p},$$

$$\sum_{j<0}2^{sj}\|E(t)\dot{\triangle}_j U_0\|_{L^2}\le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{s}{2}}\sup_{j<0}\|\dot{\triangle}_j U_0\|_{L^p}$$

for $t\ge 0$.

To prove Lemma 7.5, we will use the following inequalities.

Lemma 7.6. *Let $\alpha>0$, $p_0>0$ and $s>-\frac{n}{p_0}$. Then there holds the estimate*

$$\sum_{j<0}\Big(\int_{2^{j-1}<|\xi|<2^{j+2}}|\xi|^{p_0 s}e^{-p_0\alpha|\xi|^2 t}d\xi\Big)^{\frac{1}{p_0}}\le C(1+t)^{\frac{n}{2p_0}-\frac{s}{2}}$$

for all $t>0$.

We will prove Lemma 7.6 later. Now we prove Lemma 7.5.

Proof of Lemma 7.5. Let $1\le p<2$ and p' be the Hölder conjugate exponent to p. By Plancherel's theorem and Lemma 5.1 (ii), we have that there exists a constant $\beta' > 0$ such that

$$\begin{aligned}
\|E(t)\triangle_{-1}U_0(t)\|_{L^2} &\le C\Big(\int_{|\xi|\le 2}\big|e^{\hat{A}(\xi)t}\chi(\xi)\hat{U}_0(\xi)\big|^2 d\xi\Big)^{\frac{1}{2}} \\
&\le C\sup_{j<0}\|\phi_j(\xi)\hat{U}_0\|_{L^{p'}}\big(\sum_{j<0}\int_{2^{j-1}|\xi|<2^{j+2}} e^{-\frac{2p}{2-p}\beta'|\xi|^2t}d\xi\big)^{\frac{1}{p}-\frac{1}{2}} \\
&\le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}\sup_{j<0}\|\dot{\triangle}_jU_0\|_{L^p}, \qquad (62)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j<0}2^{sj}\|E(t)\dot{\triangle}_jU_0(t)\|_{L^2} &\le C\sum_{j<0}2^{sj}\Big(\int_{2^{j-1}<|\xi|<2^{j+2}}\big|e^{\hat{A}(\xi)t}\phi_j(\xi)\hat{U}_0(\xi)\big|^2 d\xi\Big)^{\frac{1}{2}} \\
&\le C\sum_{j<0}\big(\int_{2^{j-1}<|\xi|\le 2^{j+2}}|\xi|^{2s}e^{-2\beta'|\xi|^2t}|\phi_j(\xi)\hat{U}_0(\xi)|^2d\xi\big)^{\frac{1}{2}} \\
&\le C\sum_{j<0}\|\dot{\triangle}_jU_0\|_{L^p}\big(\int_{2^{j-1}<|\xi|\le 2^{j+2}}|\xi|^{\frac{2p}{2-p}s}e^{-\frac{2p}{2-p}\beta'|\xi|^2t}d\xi\big)^{\frac{1}{p}-\frac{1}{2}} \\
&\le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{s}{2}}\sup_{j<0}\|\dot{\triangle}_jU_0\|_{L^p}. \qquad (63)
\end{aligned}$$

Here we used Lemma 7.6.

The desired estimates of Lemma 7.5 for $1 \le p < 2$ follow from (62) and (63). We can easily prove for $p = 2$. □

It remains to prove Lemma 7.6.

Proof of Lemma 7.6. Let $\alpha > 0$, $p_0 > 0$ and $s > -\frac{n}{p_0}$. We have

$$\begin{aligned}
&\sum_{j<0}\big(\int_{2^{j-1}<|\xi|<2^{j+2}}|\xi|^{p_0s}e^{-p_0\alpha|\xi|^2t}d\xi\big)^{\frac{1}{p_0}} \\
\le\ & C\sum_{j<0}2^{js}\big(\int_{|\xi|<2^{j+2}}d\xi\big)^{\frac{1}{p_0}} \\
\le\ & C\sum_{j<0}2^{j(s+\frac{n}{p_0})} \le C. \qquad (64)
\end{aligned}$$

We will next show the the inequality

$$\sum_{j<0}\big(\int_{2^{j-1}<|\xi|<2^{j+2}}|\xi|^{p_0s}e^{-p_0\alpha|\xi|^2t}d\xi\big)^{\frac{1}{p_0}} \le Ct^{-\frac{n}{2p_0}-\frac{s}{2}}. \qquad (65)$$

By the substitution $\eta = t^{\frac{1}{2}}\xi$, we obtain

$$\begin{aligned}
&\sum_{j<0}\big(\int_{2^{j-1}<|\xi|<2^{j+2}}|\xi|^{p_0s}e^{-p_0\alpha|\xi|^2t}d\xi\big)^{\frac{1}{p_0}} \\
=\ & t^{-\frac{n}{2p_0}-\frac{s}{2}}\sum_{j<0}\big(\int_{2^{j-1}\sqrt{t}<|\xi|<2^{j+2}\sqrt{t}}|\eta|^{p_0s}e^{-p_0\alpha|\eta|^2}d\xi\big)^{\frac{1}{p_0}}.
\end{aligned}$$

If $t \leq 1$, we can easily prove (65).

We suppose $t > 1$. There exist an integer $J < 0$ such that $2^{-2J} < t < 2^{-2(J-1)}$. We have

$$\begin{aligned}
& \sum_{j<0} \Big(\int_{2^{j-1}\sqrt{t}<|\xi|<2^{j+2}\sqrt{t}} |\eta|^{p_0 s} e^{-p_0\alpha|\eta|^2} d\xi \Big)^{\frac{1}{p_0}} \\
\leq & \sum_{j\leq J} \Big(\int_{2^{j-J-1}<|\xi|<2^{j-J+3}} |\eta|^{p_0 s} e^{-p_0\alpha|\eta|^2} d\xi \Big)^{\frac{1}{p_0}} \\
& + \sum_{J<j<0} \Big(\int_{2^{j-J-1}<|\xi|<2^{j-J+3}} |\eta|^{p_0 s} e^{-p_0\alpha|\eta|^2} d\xi \Big)^{\frac{1}{p_0}} \\
=: & \; I_1 + I_2.
\end{aligned}$$

By the substitution $k = j - J$, we have

$$I_1 = \sum_{k\leq 0} \Big(\int_{2^{k-1}<|\xi|<2^{k+3}} |\eta|^{p_0 s} e^{-p_0\alpha|\eta|^2} d\xi \Big)^{\frac{1}{p_0}} < C,$$

and

$$\begin{aligned}
I_2 \;\leq\; & \sum_{k>0} \Big(\int_{2^{k-1}<|\xi|<2^{k+3}} |\eta|^{p_0 s} e^{-p_0\alpha|\eta|^2} d\xi \Big)^{\frac{1}{p_0}} \\
\leq\; & C \sum_{k>0} e^{-\frac{1}{2}2^k} \Big(\int_{2^{k-1}<|\xi|<2^{k+3}} |\eta|^{p_0 s} e^{-\frac{1}{2}p_0\alpha|\eta|^2} d\xi \Big)^{\frac{1}{p_0}} \\
\leq\; & C \sum_{k>0} e^{-\frac{1}{2}2^k} \leq C.
\end{aligned}$$

Hence we obtain (65). By (64) and (65) we have the desired inequality . □

As for $M_1(t)$, we show the following estimate.

Proposition 7.7. *Let $1 \leq p < \frac{2n}{n+1}$. Then there exists a constant $C > 0$ independent of T such that*

$$M_1(t) \leq C\|U_0\|_{\dot{B}^0_{p,\infty}} + CM(t) \int_0^t \|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} d\tau + CM^2(t)$$

for $t \in [0, T]$.

To prove Proposition 7.7, we will use the following estimate on $F(U)$.

Lemma 7.8. *Suppose that $1 \leq p < \frac{2n}{n+1}$. Then there exists a constant $C > 0$ independent of T such that*

$$\sum_{j<0} \|\dot{\triangle}_j F(U)\|_{L^1} \;\leq\; C(1+t)^{-\frac{n}{2p}} M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} + C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} M^2(t)$$

for $t \in [0, T]$.

We will prove Lemma 7.8 later. Now we prove Proposition 7.7.

Proof of Proposition 7.7. By Lemma 7.5 and (61), we see that

$$\begin{aligned}\|\triangle_{-1}U(\tau)\|_{L^2} &\leq \|E(\tau)\triangle_{-1}U_0\|_{L^2} + \int_0^\tau \|E(\tau-\tau')\triangle_{-1}F(U(\tau'))\|_{L^2}d\tau' \\ &\leq C(1+\tau)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}\sup_{j<0}\|\dot{\triangle}_jU_0\|_{L^p} \\ &\quad +\int_0^t (1+\tau-\tau')^{-\frac{n}{4}}\sup_{j<0}\|\dot{\triangle}_jF(U(\tau'))\|_{L^1}ds, \qquad (66)\end{aligned}$$

and

$$\begin{aligned}\sum_{j<0}2^{sj}\|\dot{\triangle}_jU(\tau)\|_{L^2} &\leq \sum_{j<0}\|E(\tau)\dot{\triangle}_jU_0\|_{L^2} + \int_0^\tau \sum_{j<0}\|E(\tau-\tau')\dot{\triangle}_jF(U(\tau'))\|_{L^2}d\tau' \\ &\leq C(1+\tau)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{s}{2}}\sup_{j<0}\|\dot{\triangle}_jU_0\|_{L^p} \\ &\quad +\int_0^\tau (1+\tau-\tau')^{-\frac{n}{4}-\frac{s}{2}}\sup_{j<0}\|\dot{\triangle}_jF(U(\tau'))\|_{L^1}d\tau' \qquad (67)\end{aligned}$$

for $s>0$.

Using Lemma 7.8, for $0\leq s\leq \frac{n}{2}$, we have

$$\begin{aligned}&\int_0^\tau (1+\tau-\tau')^{-\frac{n}{4}-\frac{s}{2}}\sup_{j<0}\|\dot{\triangle}_jF(U(\tau'))\|_{L^1}d\tau' \\ \leq\;& C\int_0^t (1+\tau-\tau')^{-\frac{n}{4}-\frac{s}{2}}\{(1+\tau')^{-\frac{n}{2p}}M(\tau')\|w(\tau')\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} + (1+\tau')^{-n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}M^2(\tau')\}d\tau' \\ \leq\;& CM(t)\int_0^\tau (1+\tau-\tau')^{-\frac{n}{4}-\frac{s}{2}}(1+\tau')^{-n(\frac{1}{p}-\frac{1}{2})}\|w(\tau')\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau' \\ &+CM^2(t)\int_0^\tau (1+\tau-\tau')^{-\frac{n}{4}-\frac{s}{2}}(1+\tau')^{-n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}d\tau' \\ \leq\;& C(1+\tau)^{-\frac{n}{4}-\frac{s}{2}}M(t)\int_0^\tau \|w(\tau')\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau' + C(1+\tau)^{-\frac{n}{4}-\frac{s}{2}}M^2(t). \qquad (68)\end{aligned}$$

Here we used Lemma 6.4 and the facts that $n(\frac{1}{p}-\frac{1}{2})+\frac{1}{2}>1$ for $n\geq 2$ and $1\leq p<\frac{2n}{n+1}$. By (66) and (68), we obtain

$$\begin{aligned}\|\triangle_{-1}U(\tau)\|_{L^2} &\leq C(1+\tau)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}\|U_0\|_{\dot{B}^0_{p,\infty}} \\ &\quad +C(1+\tau)^{-\frac{n}{4}}M(t)\int_0^t \|w(\tau')\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau' + C(1+\tau)^{-\frac{n}{4}}M^2(t),\end{aligned}$$

and hence,

$$(1+\tau)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}\|\triangle_{-1}U(\tau)\|_2 \leq C\|U_0\|_{\dot{B}^0_{p,\infty}} + CM(t)\int_0^t \|w(\tau')\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau' + CM^2(t).$$

Similarly, we get estimates

$$(1+\tau)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{2})+\frac{1}{2}}\sum_{j<0}2^j\|\dot{\triangle}_jU(\tau)\|_2 \le C\|U_0\|_{\dot{B}^0_{p,\infty}}+CM(t)\int_0^t\|w(\tau')\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau'+CM^2(t),$$

$$(1+\tau)^{\frac{n}{2p}-\frac{1}{2}}\sum_{j<0}2^{(\frac{n}{2}-1)j}\|\dot{\triangle}_jU(\tau)\|_2 \le C\|U_0\|_{\dot{B}^0_{p,\infty}}+CM(t)\int_0^t\|w(\tau')\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau'+CM^2(t),$$

$$(1+\tau)^{\frac{n}{2p}}\sum_{j<0}2^{\frac{n}{2}j}\|\dot{\triangle}_jU(\tau)\|_2 \le C\|U_0\|_{\dot{B}^0_{p,\infty}}+CM(t)\int_0^t\|w(\tau')\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau'+CM^2(t).$$

Taking the supremum in $\tau \in [0,t]$, we obtain the desired estimate. $\square$

It remains to prove Lemma 7.8.

Proof of Lemma 7.8. We consider each term of $F(U)$. By Lemma 6.5, we have

$$\begin{aligned}\sup_{j<0}\|\dot{\triangle}_j(w\cdot\nabla\sigma)\|_{L^1} &\le C\big\{\|\dot{S}_4w\|_{L^2}\|\dot{S}_4\nabla\sigma\|_{L^2}+\|\tilde{S}_0w\|_{L^2}\|\tilde{S}_0\nabla\sigma\|_{L^2}\big\}\\ &\le C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}M^2(t),\end{aligned}$$

$$\begin{aligned}\sup_{j<0}\|\dot{\triangle}_j(\sigma\nabla\cdot w)\|_{L^1} &\le C\{\|\dot{S}_4\sigma\|_{L^2}\|\dot{S}_4\nabla w\|_{L^2}+\|\tilde{S}_0\sigma\|_{L^2}\|\tilde{S}_0\nabla w\|_{L^2}\}\\ &\le C\{\|\dot{S}_4\sigma\|_{L^2}(\|\dot{S}_0\nabla w\|_{L^2}+\|\dot{\triangle}_0w\|_{L^2}+\|\dot{\triangle}_1w\|_{L^2}\\ &\quad+\|\dot{\triangle}_2w\|_{L^2}+\|\dot{\triangle}_3w\|_{L^2})+\|\tilde{S}_0\sigma\|_{L^2}\|\tilde{S}_0\nabla w\|_{L^2}\}\\ &\le C\{(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}M^2(t)+(1+t)^{-\frac{n}{2p}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\}.\end{aligned}$$

Similarly, we have

$$\sup_{j<0}\|\dot{\triangle}_j(w\cdot\nabla w)\|_{L^1} \le C\{(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}M^2(t)+(1+t)^{-\frac{n}{2p}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\}.$$

We obtain by Lemma 2.7, 6.3 and 6.5

$$\begin{aligned}\sup_{j<0}\|\dot{\triangle}_j(\frac{\sigma}{\sigma+1}\Delta w)\|_{L^1} &\le C\{\|\dot{S}_4(\frac{\sigma}{\sigma+1})\|_{L^2}\|\dot{S}_4\Delta w\|_{L^2}+\|\tilde{S}_0(\frac{\sigma}{\sigma+1})\|_{L^2}\|\tilde{S}_0\Delta w\|_{L^2}\}\\ &\le C\{\|\sigma\|_{L^2}\|\dot{S}_4w\|_{\dot{B}^1_{2,1}}++\|\tilde{S}_0(\frac{\sigma}{\sigma+1})\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\tilde{S}_0\Delta w\|_{L^2}\}\\ &\le C\{(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}M^2(t)+(1+t)^{-\frac{n}{2p}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\}.\end{aligned}$$

The other terms are estimated similarly, and we arrive at

$$\sup_{j<0}\|\dot{\triangle}F(U)\|_{L^1}\le C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}M^2(t)+C(1+t)^{-\frac{n}{2p}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}.$$

This completes the proof. $\square$

7.2 Estimate of high frequency parts

We next derive estimates for $M_\infty(t)$. The system (58) is written as

$$\begin{cases} \partial_t \triangle_j \sigma + \gamma \nabla \cdot \triangle_j w = \triangle_j F_1(U), \\ \partial_t \triangle_j w - \mu_1 \Delta \triangle_j w - \mu_2 \nabla(\nabla \cdot \triangle_j w) + \gamma \nabla \triangle_j \sigma = \triangle_j F_2(U). \end{cases} \tag{69}$$

Proposition 7.9. *Let $j \geq 0$. There holds*

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\triangle_j U(t)\|_{L^2}^2 + \mu_1 \|\nabla \triangle_j w(t)\|_{L^2}^2 + \mu_2 \|\nabla \cdot \triangle_j w(t)\|_{L^2}^2 \\ = \ &(\triangle_j F_1(U), \triangle_j \sigma) + (\triangle_j F_2(U), \triangle_j w) \end{aligned} \tag{70}$$

for a.e. $t \in [0, T]$.

Proof. We take the inner product of $(69)_1$ and $(69)_2$ with $\triangle_j \sigma$ and $\triangle_j w$ respectively, integrating by parts and then adding them together, we obtain our proposition. □

We recall that for $s \in \mathbb{R}$, Λ^s is defined by $\Lambda^s z := \mathfrak{F}^{-1}[|\xi|^s \hat{z}]$. Let $d = \Lambda^{-1}\nabla \cdot w$ be the "compressible part" of the velocity. Applying $\Lambda^{-1}\nabla\cdot$ to $(69)_2$, system (69) writes

$$\begin{cases} \partial_t \triangle_j \sigma + \gamma \Lambda \triangle_j d = \triangle_j F_1(U), \\ \partial_t \triangle_j d - \nu \Delta \triangle_j d - \gamma \Lambda \triangle_j \sigma = \Lambda^{-1} \nabla \cdot \triangle_j F_2(U), \end{cases} \tag{71}$$

where we denote $\nu = \mu_1 + \mu_2$.

Proposition 7.10. *Let $j \geq 0$. There holds*

$$\begin{aligned} &\frac{1}{2}\frac{\nu}{\gamma}\frac{d}{dt}\|\Lambda \triangle_j \sigma\|_{L^2}^2 - \frac{d}{dt}(\Lambda \triangle_j \sigma, \triangle_j d) + \|\Lambda \triangle_j \sigma\|_{L^2}^2 = \gamma \|\Lambda \triangle_j d\|_{L^2}^2 \\ &-(\Lambda \triangle_j F_1(U), \triangle_j d) - (\Lambda^{-1} \nabla \cdot \triangle_j F_2(U), \Lambda \triangle_j \sigma) + \frac{\nu}{\gamma}(\Lambda \triangle_j F_1(U), \Lambda \triangle_j \sigma) \end{aligned} \tag{72}$$

for a.e. $t \in [0, T]$.

Proof. We apply Λ to the equation $(71)_1$ and then take L^2 inner product with $\triangle_j d$. We take L^2 inner product of $(71)_2$ with $\Lambda \triangle_j \sigma$. We also apply Λ to the equation $(71)_1$ and take L^2 inner product with $\frac{\nu}{\gamma}\triangle_j \sigma$. By a suitable linear combination of them, we obtain the desired identity of the proposition. □

We introduce a lemma for estimates of the right-hand side of (72).

Lemma 7.11. *The following inequalities hold*

(i)

$$\left|\left(\Lambda \triangle_j (w \cdot \nabla \sigma), \Lambda \triangle_j \sigma\right)\right| \leq C \alpha_j 2^{-(\frac{n}{2}-1)j} \|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} \|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}} \|\Lambda \triangle_j \sigma\|_{L^2},$$

(ii)

$$
\begin{aligned}
&\big|\big(\Lambda\triangle_j(w\cdot\nabla\sigma),\triangle_j d\big)\big|\\
\leq\ & C\big\{\alpha_j 2^{-(\frac{n}{2}-1)j}\|w\|_{\dot B^{\frac{n}{2}+1}_{2,1}}\|\sigma\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2}\\
&+\|\nabla\triangle_j\sigma\|_{L^2}\big(2^j\|\dot S_0 w\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2}+2^{2j}\|\tilde S_0 w\|_{\dot B^{\frac{n}{2}-1}_{2,1}}\|\triangle_j d\|_{L^2}\big)\big\},
\end{aligned}
$$

where C is independ of $j\in\mathbb{Z}$ and $\{\alpha_j\}$ with $\|\{\alpha_j\}\|_{l^1}\leq 1$.

Proof. As for (i), see, e.g., [2].
Let us prove (ii). By using Lemma 6.7, we obtain

$$
\begin{aligned}
&\big|\big(\Lambda\triangle_j(w\cdot\nabla\sigma),\triangle_j d\big)\big|\\
\leq\ & \big|\big([w\cdot\nabla,\triangle_j]\sigma,\Lambda\triangle_j d\big)\big|+\big|\big(w\cdot\nabla\triangle_j\sigma,\Lambda\triangle_j d\big)\big|\\
\leq\ & C\{\alpha_j 2^{-(\frac{n}{2}-1)j}\|\nabla w\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\sigma\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2}\\
&+\|\nabla\triangle_j\sigma\|_{L^2}\big(\|\dot S_0 w\|_{L^\infty}\|\Lambda\triangle_j d\|_{L^2}+\|\tilde S_0 w\|_{L^n}\|\Lambda\triangle_j d\|_{L^{\frac{2n}{n-2}}}\big)\}\\
\leq\ & C\big\{\alpha_j 2^{-(\frac{n}{2}-1)j}\|\nabla w\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\sigma\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2}\\
&+\|\nabla\triangle_j\sigma\|_{L^2}\big(2^j\|\dot S_0 w\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2}+2^{2j}\|\tilde S_0 w\|_{\dot B^{\frac{n}{2}-1}_{2,1}}\|\triangle_j d\|_{L^2}\big)\big\}.
\end{aligned}
$$

This completes the proof. □

Proposition 7.12. *There holds*

$$
\begin{aligned}
&\frac{d}{dt}E_j(t)+c_0E_j(t)\\
\leq\ & C\{\alpha_j(1+t)^{-\frac{n}{2p}}M(t)\|w\|_{\dot B^{\frac{n}{2}+1}_{2,1}}+(1+t)^{-\frac{n}{2p}}2^{(\frac{n}{2}+1)j}\|\triangle_j d\|_{L^2}M(t)\\
&+2^{(\frac{n}{2}-1)j}\|\Lambda\triangle_j(\sigma\nabla\cdot w)\|_{L^2}+2^{(\frac{n}{2}-1)j}\|\triangle_j F_1(U)\|_{L^2}\\
&+2^{(\frac{n}{2}-1)j}\|\triangle_j F_2(U)\|_{L^2}\},
\end{aligned}
\tag{73}
$$

for $t\in[0,T]$ and $j\geq 1$, where $\sum_{j\in\mathbb{Z}}\alpha_j\leq 1$, and c_0 is a positive constant independent of j. Here, $E_j(t)$ is equivalent to $2^{(\frac{n}{2}-1)j}\|\triangle_j U(t)\|_{L^2}+2^{\frac{n}{2}j}\|\triangle_j\sigma(t)\|_{L^2}$. That is, there exists a positive constant D_1 such that

$$
\begin{aligned}
&\frac{1}{D_1}\big(2^{(\frac{n}{2}-1)j}\|\triangle_j U(t)\|_{L^2}+2^{\frac{n}{2}j}\|\triangle_j\sigma(t)\|_{L^2}\big)\\
\leq\ & E_j(t)\\
\leq\ & D_1\big(2^{(\frac{n}{2}-1)j}\|\triangle_j U(t)\|_{L^2}+2^{\frac{n}{2}j}\|\triangle_j\sigma(t)\|_{L^2}\big).
\end{aligned}
$$

Proof. We add (70) to $\kappa\times$ (72) with a constant $\kappa>0$ to be determined later. Then, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big\{\frac{1}{2}\|\triangle_j U\|^2_{L^2}+\frac{\kappa}{2}\frac{\nu}{\gamma}\|\Lambda\triangle_j\sigma\|^2_{L^2}-\kappa(\Lambda\triangle_j\sigma,\triangle_j d)\Big\}\\
&+\mu_1\|\nabla\triangle_j w\|^2_{L^2}+\mu_2\|\nabla\cdot\triangle_j w\|^2_{L^2}+\kappa\|\Lambda\triangle_j\sigma\|^2_{L^2}\\
=\ & \gamma\kappa\|\Lambda\triangle_j w\|^2_{L^2}+\big(\triangle_j F_1(U),\triangle_j\sigma\big)+\big(\triangle_j F_2(U),\triangle_j w\big)+\kappa\frac{\nu}{\gamma}\big(\Lambda\triangle_j F_1(U),\Lambda\triangle_j\sigma\big)\\
&-\kappa\big(\Lambda\triangle_j F_1(U),\triangle_j d\big)-\kappa\big(\Lambda^{-1}\nabla\cdot\triangle_j F_2(U),\Lambda\triangle_j\sigma\big).
\end{aligned}
\tag{74}
$$

We set

$$E_j^2(t) = 2^{2(\frac{n}{2}-1)j}\big\{\frac{1}{2}\|\triangle_j U\|_{L^2}^2 + \frac{\kappa}{2}\frac{\nu}{\gamma}\|\Lambda\triangle_j\sigma\|_{L^2}^2 - \kappa(\Lambda\triangle_j\sigma, \triangle_j d)\big\}.$$

It is not difficult to see that there exists $D_1 > 0$ such that, if $\kappa = \min\{D_1\frac{\mu}{\gamma}, 1\}$, then $E_j(t)$ is equivalent to $2^{(\frac{n}{2}-1)j}\|\triangle_j U(t)\|_{L^2} + 2^{\frac{n}{2}j}\|\triangle_j\sigma(t)\|_{L^2}$ and that there exists a $c_0 > 0$ such that

$$2c_0 E_j^2 \le 2^{2(\frac{n}{2}-1)j}\big\{\mu_1\|\nabla\triangle_j w\|_{L^2}^2 + \mu_1\|\nabla\cdot\triangle_j w\|_{L^2}^2 + \kappa\|\Lambda\triangle_j\sigma\|_{L^2}^2 - \gamma\kappa\|\Lambda\triangle_j w\|_{L^2}^2\big\}.$$

Let us next estimate the right-hand side of $2^{2(\frac{n}{2}-1)j}\times(74)$. By Hölder's inequality, we obtain

$$2^{2(\frac{n}{2}-1)j}\big(\triangle_j F_1(U), \triangle_j\sigma\big) \le 2^{(\frac{n}{2}-1)j}\|\triangle_j F_1(U)\|_{L^2}2^{(\frac{n}{2}-1)j}\|\triangle_j\sigma\|_{L^2},$$

$$2^{2(\frac{n}{2}-1)j}\big(\triangle_j F_2(U), \triangle_j w\big) \le 2^{(\frac{n}{2}-1)j}\|\triangle_j F_2(U)\|_{L^2}2^{(\frac{n}{2}-1)j}\|\triangle_j w\|_{L^2},$$

$$2^{2(\frac{n}{2}-1)j}\big(\Lambda^{-1}\nabla\cdot\triangle_j F_2(U), \triangle_j\sigma\big) \le 2^{(\frac{n}{2}-1)j}\|\triangle_j F_2(U)\|_{L^2}2^{(\frac{n}{2}-1)j}\|\triangle_j\sigma\|_{L^2}.$$

By Lemma 7.11 we have

$$\begin{aligned}
& 2^{2(\frac{n}{2}-1)j}\big(\Lambda\triangle_j F_1(U), \Lambda\triangle_j\sigma\big) \\
= {} & 2^{2(\frac{n}{2}-1)j}\big(\Lambda\triangle_j(w\cdot\nabla\sigma), \Lambda\triangle_j\sigma\big) + 2^{2(\frac{n}{2}-1)j}\big(\Lambda\triangle_j(\sigma\nabla\cdot w), \Lambda\triangle_j\sigma\big) \\
\le {} & C\alpha_j\|w\|_{\dot B^{\frac{n}{2}+1}_{2,1}}\|\sigma\|_{\dot B^{\frac{n}{2}}_{2,1}}2^{(\frac{n}{2}-1)j}\|\Lambda\triangle_j\sigma\|_{L^2} + 2^{2(\frac{n}{2}-1)j}\|\Lambda\triangle_j(\sigma\nabla\cdot w)\|_{L^2}\|\Lambda\triangle_j\sigma\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
& 2^{2(\frac{n}{2}-1)j}\big(\Lambda\triangle_j F_1(U), \triangle_j d\big) \\
= {} & 2^{2(\frac{n}{2}-1)j}\big(\Lambda\triangle_j(w\cdot\nabla\sigma), \triangle_j d\big) + 2^{2(\frac{n}{2}-1)j}\big(\Lambda\triangle_j(\sigma\nabla\cdot w), \triangle_j d\big) \\
\le {} & C\big\{\alpha_j 2^{-(\frac{n}{2}-1)j}\|w\|_{\dot B^{\frac{n}{2}+1}_{2,1}}\|\sigma\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2} \\
& +\|\nabla\triangle_j\sigma\|_{L^2}\big(2^j\|\dot S_0 w\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2} + 2^{2j}\|\tilde S_0 w\|_{\dot B^{\frac{n}{2}-1}_{2,1}}\|\triangle_j d\|_{L^2}\big)\big\} \\
& +2^{2(\frac{n}{2}-1)j}\|\Lambda\triangle_j(\sigma\nabla\cdot w)\|_{L^2}\|\triangle_j d\|_{L^2},
\end{aligned}$$

where $\sum_{j\in\mathbb{Z}}\alpha_j \le 1$. Hence we obtain

$$\begin{aligned}
& \frac{d}{dt}E_j^2 + 2c_0E_j^2 \le CE_j\big\{\alpha_j(1+t)^{-\frac{n}{2p}}M(t)\|w\|_{\dot B^{\frac{n}{2}+1}_{2,1}} \\
& \quad +(1+t)^{-\frac{n}{2p}}2^{(\frac{n}{2}+1)j}\|\triangle_j d\|_{L^2}M(t) + 2^{(\frac{n}{2}-1)j}\|\Lambda\triangle_j(\sigma\nabla\cdot w)\|_{L^2} \\
& \quad +2^{(\frac{n}{2}-1)j}\|\triangle_j F_1(U)\|_{L^2} + 2^{(\frac{n}{2}-1)j}\|\triangle_j F_2(U)\|_{L^2}\big\}. \qquad (75)
\end{aligned}$$

From (75) and dividing by E_j, we get the desired result. □

7.3 Proof of Theorem 6.1.

Proposition 7.13. *Let $1 \le p < \frac{2n}{n+1}$. There exists a constant $\epsilon_2 > 0$ such that if*

$$\|U_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}\cap \dot{B}^0_{p,\infty}} + \|\sigma_0\|_{\dot{B}^{\frac{n}{2}}_{2,1}} \le \epsilon_2,$$

then there holds

$$M(t) \le C\big\{\|U_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}\cap \dot{B}^0_{p,\infty}} + \|\sigma_0\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\big\}$$

for $0 \le t \le T$, where the constant C does not depend on T.

Proof. By (73) we have

$$\begin{aligned}
E_j(t) \le\ & e^{-c_0 t}E_j(0) \\
&+C\int_0^t e^{-c_0(t-\tau)}\{\alpha_j(1+\tau)^{-\frac{n}{2p}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} \\
&+(1+\tau)^{-\frac{n}{2p}}2^{(\frac{n}{2}+1)j}\|\triangle_j d\|_{L^2}M(t) \\
&+2^{(\frac{n}{2}-1)j}\|\Lambda\triangle_j(\sigma\nabla\cdot w)\|_{L^2} \\
&+2^{(\frac{n}{2}-1)j}\|\triangle_j F_1(U)\|_{L^2} + 2^{(\frac{n}{2}-1)j}\|\triangle_j F_2(U)\|_{L^2}\}d\tau, \qquad (76)
\end{aligned}$$

where $\sum_{j=0}^{\infty}\alpha_j \le 1$. Hence summing up on $j \ge 0$, by the monotone convergence theorem, we obtain

$$\begin{aligned}
&\sum_{j=0}^{\infty}E_j(t) \le e^{-c_0 t}\sum_{j=0}^{\infty}E_j(0) \\
&+C\int_0^t e^{-c_0(t-\tau)}\{(1+\tau)^{-\frac{n}{2p}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} + \sum_{j=0}^{\infty}2^{j\frac{n}{2}}\|\dot{\triangle}_j(\sigma\nabla\cdot w)\|_{L^2} \\
&+\sum_{j=0}^{\infty}2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j F_1(U)\|_{L^2} + \sum_{j=0}^{\infty}2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j F_2(U)\|_{L^2}\}d\tau. \qquad (77)
\end{aligned}$$

We next estimate the right-hand side of (77). From Lemma 6.2, we have

$$\sum_{j=0}^{\infty}2^{j\frac{n}{2}}\|\dot{\triangle}_j\sigma\nabla\cdot w\|_{L^2} \le \|\sigma\nabla\cdot w\|_{\dot{B}^{\frac{n}{2}}_{2,1}} \le C\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\nabla\cdot w\|_{\dot{B}^{\frac{n}{2}}_{2,1}} \le C(1+\tau)^{-\frac{n}{2p}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}.$$

Let us next consider the quantities $\sum_{j=0}^{\infty}2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j F_1(U)\|_{L^2}$:

$$\begin{aligned}
\sum_{j=0}^{\infty}2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j(w\cdot\nabla\sigma)\|_{L^2} &\le \|w\cdot\nabla\sigma\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\le C\|w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\nabla\sigma\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\le C\big(\|\dot{S}_0 w\|_{\dot{B}^{\frac{n}{2}}_{2,1}} + \|\tilde{S}_0 w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\big)\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}} \\
&\le C(1+\tau)^{-\frac{n}{p}}M^2(\tau) + C(1+\tau)^{-\frac{n}{2p}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}},
\end{aligned}$$

$$
\begin{aligned}
\sum_{j=0}^{\infty} 2^{j(\frac{n}{2}-1)} \|\dot{\triangle}_j(\sigma\nabla\cdot w)\|_{L^2} &\leq \|\sigma\nabla\cdot w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\leq C\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\nabla w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\leq C(1+\tau)^{-\frac{n}{p}}M^2(\tau)+C(1+\tau)^{-\frac{n}{2p}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}.
\end{aligned}
$$

Hence, we obtain the estimate of $\sum_{j=0}^{\infty} 2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j F_1(U)\|_{L^2}$. By using Lemma 6.2, Lemma 6.3 and Lemma 6.5, $\sum_{j=0}^{\infty} 2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j F_2(U)\|_{L^2}$ is estimated as

$$
\begin{aligned}
\sum_{j=0}^{\infty} 2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j(w\cdot\nabla)w\| &\leq C\{\|\dot{S}_{-5}w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\tilde{S}_{-5}\nabla w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\quad +\|\dot{S}_{-5}\nabla w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\|\tilde{S}_{-5}w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}+\|\tilde{S}_{-5}w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\|\tilde{S}_{-5}\nabla w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\} \\
&\leq C(1+\tau)^{-\frac{n}{2p}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}.
\end{aligned}
$$

Here we used

$$
\|\tilde{S}_{-5}w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \leq C\{\big(\sum_{j=-5}^{-1} 2^{j\frac{n}{2}}\|\dot{\triangle}_j w\|_{L^2}\big)+\|\tilde{S}_0 w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\} \leq C(1+\tau)^{-\frac{n}{2p}}M(\tau),
$$

$$
\|\tilde{S}_{-4}w\|_{\dot{B}^{\frac{n}{2}}_{2,1}} \leq C\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}.
$$

$$
\begin{aligned}
\sum_{j=0}^{\infty} 2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j(\frac{\sigma}{\sigma+1}\Delta w)\|_{L^2} &\leq \|\frac{\sigma}{\sigma+1}\Delta w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\leq C\|\frac{\sigma}{\sigma+1}\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\Delta w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\leq C\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} \\
&\leq C(1+\tau)^{-\frac{n}{2p}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j=0}^{\infty} 2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j(\frac{\sigma}{\sigma+1}\nabla\sigma)\|_{L^2} &\leq \|\frac{\sigma}{\sigma+1}\nabla\sigma\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\leq C\|\frac{\sigma}{\sigma+1}\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\nabla\sigma\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\leq C(1+\tau)^{-\frac{n}{p}}M^2(\tau).
\end{aligned}
$$

In the same way as above, we can obtain estimates of other terms on $\|F_2(U)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}$. Hence, by using Lemma 6.4, the integral of the right-hand side of (77) is estimated as

$$
\begin{aligned}
&\int_0^t e^{-c_0(t-\tau)}\big\{(1+\tau)^{-\frac{n}{p}}M^2(\tau)+(1+\tau)^{-\frac{n}{2p}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\big\}d\tau \\
\leq\ & M(t)\int_0^t e^{-c_0(t-\tau)}(1+\tau)^{-\frac{n}{2p}}\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau+M^2(t)\int_0^t e^{-c_0(t-\tau)}(1+\tau)^{-\frac{n}{p}}d\tau \\
\leq\ & C(1+t)^{-\frac{n}{2p}}\epsilon_2 M(t)+C(1+t)^{-\frac{n}{p}}M^2(t).
\end{aligned}
$$

Hence, we obtain

$$M_\infty(t) \le C\big(\|U_0\|_{\dot B^{\frac{n}{2}-1}_{2,1}} + \|\sigma_0\|_{\dot B^{\frac{n}{2}}_{2,1}}\big) + C\epsilon_2 M(t) + CM^2(t). \tag{78}$$

By Proposition 7.7 and (78), we have

$$M(t) \le C\big(\|U_0\|_{\dot B^{\frac{n}{2}-1}_{2,1}\cap \dot B^0_{p,\infty}} + \|\sigma_0\|_{\dot B^{\frac{n}{2}}_{2,1}}\big) + C\epsilon_2 M(t) + CM^2(t).$$

By taking $\epsilon_2 > 0$ suitably small, we obtain

$$M(t) \le C\big(\|U_0\|_{\dot B^{\frac{n}{2}-1}_{2,1}\cap \dot B^0_{p,\infty}} + \|\sigma_0\|_{\dot B^{\frac{n}{2}}_{2,1}}\big)$$

for all $0 \le t \le T$ with C independent of T. This completes the proof. □

It follows from Proposition 7.2 and Proposition 7.13 that

$$M(t) \le C_3 \qquad \text{for all } t,$$

if the initial perturbation is sufficiently small. Hence we obtain the desired decay estimate (52), (53) and (54) of Theorem 6.1.

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