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ON THE ZEROS OF EISENSTEIN SERIES FOR $\Gamma_0^*(p)$ AND $\Gamma_0(p)$ OF LOW LEVELS

JUNICHI SHIGEZUMI

Abstract. We decide the locating of all the zeros of Eisenstein series associated with the Fricke groups $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$ in their fundamental domains with applying the method of F. K. C. Rankin and H. P. F. Swinnerton-Dyer [RSD]. Also, we give some more consideration on $\Gamma_0^*(p)$ and $\Gamma_0(p)$ of low levels.

1. Introduction

The motive of this research is to decide the locating the zeros of modular forms from codes and lattices. Then Eisenstein series seems to be one of the most important modular forms. For example, $\mathrm{SL}_2(\mathbb{Z})$ is generated by E_4 and E_6, which are Eisenstein series associated with $\mathrm{SL}_2(\mathbb{Z})$.

In [RSD], F. K. C. Rankin and H. P. F. Swinnerton-Dyer considered the locating the zeros of Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$. They proved, in 1970, that for $k = 12n + s$ ($s = 4, 6, 8, 10, 0,$ *and* 14), then n zeros in their fundamental domain are on

$$A := \{z \in \mathbb{C}\,;\, |z| = 1,\ \pi/2 < Arg(z) < 2\pi/3\}.$$

We calculated the locating of the zeros of some modular forms from codes in computers. For some codes, all the zeros seems to be on A. However, for the other codes, it does not hold.

In last May, we (Tsuyoshi Miezaki, Hiroshi Nozaki, and I) were introduced Fricke group $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$ by Professor Eiichi Bannai. Then he adviced us to try to consider the locating the zeros of them. We applied the method of F. K. C. Rankin and H. P. F. Swinnerton-Dyer (RSD Method) to the Eisenstein series $E_{k,p}^*$ associated with $\Gamma_0^*(p)$ for $p = 2, 3$. Define

$$A_2^* := \{z \in \mathbb{C}\,;\, |z| = 1/\sqrt{2},\ \pi/2 < Arg(z) < 3\pi/4\},$$
$$A_3^* := \{z \in \mathbb{C}\,;\, |z| = 1/\sqrt{3},\ \pi/2 < Arg(z) < 5\pi/6\}.$$

Then we have $\overline{A_2^*} = A_2^* \cup \{i/\sqrt{2}, e^{3\pi/4}/\sqrt{2}\}$, and $\overline{A_3^*} = A_3^* \cup \{i/\sqrt{3}, e^{5\pi/6}/\sqrt{3}\}$.

We proved the next theorems.

Theorem 1. *Let $k \geqslant 4$ be an even integer. $E_{k,2}^*(z)$ has all zeros on $\overline{A_2^*}$.*

Theorem 2. *Let $k \geqslant 4$ be an even integer. $E_{k,3}^*(z)$ has all zeros on $\overline{A_3^*}$.*

After that we tried for $\Gamma_0^*(p)$ of upper levels and for $\Gamma_0(p)$ of low levels. I succeeded to decide almost all the zeros (exactly, all the zeros except for at most 2 zeros) of Eisenstein series for $\Gamma_0^*(5)$ and $\Gamma_0^*(7)$, and decided all zeros of Eisensitein series of low weights (such that $4 \leqslant k \leqslant 40$) for $\Gamma_0(2)$ and $\Gamma_0(3)$.

In section 2, we recall definitions concerning modular group and some groups, then in section 3, we recall classical methods for modular group. Section 4 gives definitions concerning $\Gamma_0(p)$ as a preliminaries for $\Gamma_0^*(p)$. In section 5, we give the proof of the above theorems, and consider about $\Gamma_0^*(p)$ of low levels. Finally, section 6 gives the results for $\Gamma_0(p)$ of low levels, where we decide the locating of many zeros. However, it is far from "complete", which means to decide locating of all zeros.

Date: January 16, 2005.

2. General theory

2.1. **The modular group and some groups.**

2.1.1. *The modular group.* (See §[VII.1] [SE], [I-I] [SI], and §[III.1] [KO])
We have a *special linear group* defined by following:

$$\mathrm{SL}_2(\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \forall a,b,c,d \in \mathbb{R}\ s.t.\ ad-bc=1 \right\}. \tag{1}$$

Write $\mathbb{H} := \{z \in \mathbb{C}\ ;\ Im(z) > 0\}$, which is complex upper half-plane. We consider a action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H} \cup \{\infty\}$ in the following way:

For every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and every $z \in \mathbb{H}$, we put

$$\gamma z := \frac{az+b}{cz+d}. \tag{2}$$

Now we have $-\gamma z = \gamma z$ for every $z \in \mathbb{H}$. So we may consider $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ instead of $\mathrm{SL}_2(\mathbb{R})$, where $I := -\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. But we discuss $\mathrm{SL}_2(\mathbb{R})$ in this note.
Futhermore, note that

$$Im(\gamma z) = Im(z)/|cz+d|^2. \tag{3}$$

In this note, we consider some discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$.
For example, we have

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})\ ; \forall a,b,c,d \in \mathbb{Z} \right\}, \tag{4}$$

which is called the *(full) modular group.* (Sometimes, $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ is called the modular group instead.) This group is a classical *Fuchsian group of the first kind.*

2.1.2. *Congruence subgroup.* (See §[III.1] [KO])
For another example of discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$, for a positive integer N, we have

$$\Gamma(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\ ;\ a \equiv d \equiv 1,\ b \equiv c \equiv 0 \pmod{N} \right\}. \tag{5}$$

This group is a subgroup of the modular group $\mathrm{SL}_2(\mathbb{Z})$, and it is called the *principal congruence subgroup of level N.*

Also, if Γ' is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma' \supset \Gamma(N)$, then Γ' is called a *congruence subgroup of level N.* Here are some examples:

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\ ;\ c \equiv 0 \pmod{N} \right\}, \tag{6}$$

$$\Gamma_1(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\ ;\ a \equiv d \equiv 1,\ c \equiv 0 \pmod{N} \right\}. \tag{7}$$

These groups are the most important examples of congruence subgroups.

2.1.3. *Fricke group.* (See [KR], [Q])
For a positive integer N, we consider the *Fricke group* $\Gamma_0^*(N)$. We define the following;

$$\Gamma_0^*(N) := \Gamma_0(N) \cup \Gamma_0(N)\, W_N, \quad W_N := \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}. \tag{8}$$

This group is not subgroup of the modular group $\mathrm{SL}_2(\mathbb{Z})$, but it is discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ and commensurable with $\mathrm{SL}_2(\mathbb{Z})$. This is as an important and interesting group as the modular group and congruence subgroups.

2.2. **Fundamental domain.** (See §[VII.1] [SE], §[I-I.4] [SI])

We refer to [SI] for definitions, and refer to [SE] for proofs of propositions.

Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$.

Definition 2.1. *$\mathbb{F}_\Gamma$ is a fundamental domain if and only if it satisfies following conditions:*

(FD1) *For every $z \in \mathbb{H}$, there exists $\gamma \in \Gamma$ such that $\gamma z \in \mathbb{F}_\Gamma$.*

(FD2) *For every two distinct points z_1, $z_2 \in \mathbb{F}_\Gamma$, there does not exist $\gamma \in \Gamma$ such that $\gamma z_1 = z_2$.*

Define $T(x) := \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ ($\in \mathrm{SL}_2(\mathbb{R})$) and $P := \{\pm T(x)\,;\, x \in \mathbb{R}\}$. In many cases, at least for the subgroups in this paper, we have

$$\Gamma \cap P \setminus \{\pm I\} \neq \phi. \tag{9}$$

We assume that the above inequality holds.

Put $u := min\{x > 0\,;\, T(x) \in \Gamma\}$, then we define

$$\mathbb{F}_{0,\Gamma} := \left\{z \in \mathbb{H}\,;\, -u/2 < Re(z) < u/2\,,\ |cz+d| > 1 \text{ for } \forall\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \setminus P\right\}. \tag{10}$$

Also, we have

$$\overline{\mathbb{F}_{0,\Gamma}} = \left\{z \in \mathbb{H}\,;\, -u/2 \leqslant Re(z) \leqslant u/2\,,\ |cz+d| \geqslant 1 \text{ for } \forall\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \setminus P\right\}.$$

(See Theorem 1.7 and 1.15, §[I-I.4 & 5] [SI])

Now, we have the following fact:

Proposition 2.1.

(i) $\overline{\mathbb{F}_{0,\Gamma}}$ *satisfies the condition* (FD1).

(ii) $\mathbb{F}_{0,\Gamma}$ *satisfies the condition* (FD2).

Proof. (*cf.* proof of Theorem 1, §[VII.1] [SE])

(i) Let $z \in \mathbb{H}$ be a fixed point. Recall that $Im(\gamma z) = Im(z)/|cz+d|^2$ for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ (See formula (3)).

Considering that Γ is a discrete group, numbers $|cz+d|$ have a lower bound greater than 0. Thus there exists $\gamma \in \Gamma$ which maximizes $Im(\gamma z)$. Furthermore, there exists $v \in u\mathbb{Z}$ such that $-u/2 \leqslant Re(T(v)\gamma z) \leqslant u/2$ (*i.e.* $v = -u\,\lfloor Re(\gamma z)/u + 1/2 \rfloor$).

Suppose $T(v)\gamma z \notin \overline{\mathbb{F}_{0,\Gamma}}$, then there exists $\gamma' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \Gamma$ such that $|c'T(v)\gamma z + d'| < 1$. Now, $Im(\gamma' T(v)\gamma z) = Im(T(v)\gamma z)/|c'T(v)\gamma z + d'|^2 > Im(T(v)\gamma z) = Im(\gamma z)$. This contradicts the way of choosing γ. Thus $T(v)\gamma z \in \overline{\mathbb{F}_{0,\Gamma}}$.

(ii) By the deffinition of $\mathbb{F}_{0,\Gamma}$, for every $z \in \mathbb{F}_{0,\Gamma}$ and every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, $Im(\gamma z) = Im(z)/|cz+d|^2 < Im(z)$. Let z_1, z_2 be distinct points of Γ.

Suppose that there exists $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ such that $\gamma z_1 = z_2$, then $Im(z_2) = Im(\gamma z_1) < Im(z_1)$. Note that $\gamma^{-1} z_2 = z_1$ and $\gamma^{-1} \in \Gamma$, then $Im(z_1) = Im(\gamma^{-1} z_2) < Im(z_2)$. These facts contradict each other. Thus there does not exist $\gamma \in \Gamma$ such that $\gamma z_1 = z_2$.

□

Here, $\overline{\mathbb{F}_{0,\Gamma}}$ does not satisfy the condition (FD2). We can, however, remove points from $\overline{\mathbb{F}_{0,\Gamma}}$ to satisfy (FD2). Define

$$\mathbb{F}_{1,\Gamma} := \left\{ z \in \mathbb{H}\,;\, -\frac{u}{2} \leqslant Re(z) < \frac{u}{2}\,,\ \begin{array}{c} |cz+d| \geqslant 1 \quad \text{if } Re(z) \leqslant -d/c, \\ |cz+d| > 1 \quad \text{if } Re(z) > -d/c \\ \text{for } \forall\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \setminus P \end{array} \right\}. \tag{11}$$

Then we have $\mathbb{F}_{0,\Gamma} \subset \mathbb{F}_{1,\Gamma} \subset \overline{\mathbb{F}_{0,\Gamma}}$. Moreover, we have following;

Lemma 2.1. *Let $z \in \partial\mathbb{F}_{0,\Gamma}$. (i.e. $|cz+d| \geqslant 1$ for $\forall\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \setminus P$)*

(i) *If $|cz+d| = 1$ for some $\gamma_0 = \left(\begin{smallmatrix} a_0 & b_0 \\ c & d \end{smallmatrix}\right) \in \Gamma \setminus P$, then there exist $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \setminus P$ such that $\gamma z \in \partial\mathbb{F}_{0,\Gamma}$ uniquely.*

(ii) *For every $\gamma' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \Gamma \setminus P$ such that $|c'z + d'| > 1$, $\gamma' z \notin \overline{\mathbb{F}_{0,\Gamma}}$.*

Proof.

(i) Write $z = (e^{i\theta} - d)/c$ for $\theta \in [0, \pi]$, then we have $\gamma z = (e^{i(\pi-\theta)} + a)/c$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \setminus P$. Let $n = -\lfloor Re(\gamma z)/u + 1/2 \rfloor$, then $-u/2 \leqslant Re(T(nu)\gamma z) < u/2$. Here, we have $T(nu)\gamma = \left(\begin{smallmatrix} a+cnu & b+dnu \\ c & d \end{smallmatrix}\right)$. Thus we can choose "$a$" such that $-u/2 \leqslant Re(\gamma z) < u/2$. We may assume $-u/2 \leqslant Re(\gamma z) < u/2$.

Also, write $z' = \gamma z (= (e^{i(\pi-\theta)} + a)/c)$. We have $|-cz' + a| = 1/|cz + d| = 1$, so $z' \notin \mathbb{F}_{0,\Gamma}$.

Suppose $z' \notin \partial\mathbb{F}_{0,\Gamma}$ (*i.e.* $z' \notin \overline{\mathbb{F}_{0,\Gamma}}$), then $|c'z' + d'| < 1$ for some $\gamma' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \Gamma \setminus P$. For $\gamma'\gamma = \left(\begin{smallmatrix} aa'+b'c & a'b+b'd \\ ac'+cd' & bc'+dd' \end{smallmatrix}\right)$,

$$\begin{aligned} |(ac' + cd')z + (bc' + dd')| &= \left| \frac{(ac' + cd')e^{i\theta} - c'}{c} \right| \\ &= \left| \frac{c'e^{i(\pi-\theta)} + (ac' + cd')}{c} \right| \left| \frac{1}{e^{-i\theta}} \right| \\ &= |c'z' + d'| \quad < 1. \end{aligned}$$

This inequality contradicts $z \in \overline{\mathbb{F}_{0,\Gamma}}$. Thus $z' \in \partial\mathbb{F}_{0,\Gamma}$.

Now, the rest of the question is uniqueness of γ. Suppose $\gamma''z \in \partial\mathbb{F}_{0,\Gamma}$ for some $\gamma'' = \left(\begin{smallmatrix} a'' & b'' \\ c & d \end{smallmatrix}\right) \in \Gamma \setminus P$, then $\gamma''\gamma^{-1} = \left(\begin{smallmatrix} 1 & ab''-a''b \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$. Furthermore, $ab'' - a''b = n'u$ for some $n' \in \mathbb{Z}$. So we have $\gamma''z = z' + n'u$. If $n' \neq 0$, then $Re(\gamma'z) \notin [-u/2, u/2]$, and then $\gamma'z \notin \overline{\mathbb{F}_{0,\Gamma}}$. Thus $n' = 0$. In conclusion, we have $\gamma' = \gamma$.

(ii) Let $\gamma' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \Gamma \setminus P$ satisfy $|c'z + d'| > 1$, and write $z'' = \gamma'z$. Then we have $Im(z'') = Im(\gamma'z) = Im(z)/|c'z + d'|^2 < Im(z)$.

Suppose $\gamma'z \in \overline{\mathbb{F}_{0,\Gamma}}$, then $|-c'z'' + a'| \geqslant 1$. Also, we have $Im(z) = Im(\gamma'^{-1}z'') = Im(z'')/|-c'z'' + a'|^2 \leqslant Im(z'')$. These facts contradict each other.

□

Let $z, z' \in \partial\mathbb{F}_{0,\Gamma}$. By Lemma2.1, if $\gamma z = z'$ for some $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \setminus P$, then $|cz + d| = 1$. Also, we have $z = (e^{i\theta} - d)/c$ and $z' = (e^{i(\pi-\theta)} + a)/c$. So we can remove the points with $\theta \in [0, \pi/2)$ from $\partial\mathbb{F}_{0,\Gamma}$. Thus we have following;

Proposition 2.2. $\mathbb{F}_{1,\Gamma}$ *satisfies the condition* (FD2).

However, if

$$\begin{aligned} &|c_1 z + d_1| = 1,\ Re(z) \leqslant -d_1/c_1 \quad \text{for some } \gamma_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right) \in \Gamma \setminus P, \quad \text{and} \\ &|c_2 z + d_2| = 1,\ Re(z) > -d_2/c_2 \quad \text{for some } \gamma_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right) \in \Gamma \setminus P, \end{aligned}$$

then we can write $z = (e^{i\theta_1} - d_1)/c_1 = (e^{i\theta_2} - d_2)/c_2$. Because $Re(z) \leqslant -d_1/c_1$, we have $\theta \in [\pi/2, \pi]$, so we should not remove z from $\partial\mathbb{F}_{0,\Gamma}$. On the other hand, because $Re(z) > -d_2/c_2$, we have $\theta \in [0, \pi/2)$, so we should remove z. These contradict each other. We need another consideration on these points.

We define

$$\begin{aligned} V_0 := &\left\{ z \in \partial\mathbb{F}_{0,\Gamma}\ ;\ \begin{matrix} |c_1 z + d_1| = 1,\ Re(z) \leqslant -d_1/c_1 & \text{for } \exists\gamma_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right) \in \Gamma \setminus P \\ |c_2 z + d_2| = 1,\ Re(z) > -d_2/c_2 & \text{for } \exists\gamma_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right) \in \Gamma \setminus P \end{matrix} \right\} \\ &\cup \{z \in \partial\mathbb{F}_{0,\Gamma}\ ;\ Re(z) = -1/2\}. \end{aligned} \tag{12}$$

Furthermore, we can remove points from V_0 to satisfy the condition (FD2). Let V_Γ be the subset of V_0 which satisfies (FD2). Then we have following theorem:

Theorem 2.1. $\mathbb{F}_{1,\Gamma} \cup V_\Gamma$ *is a fundamental domain of* Γ.

Let Γ^1 be a set of representatives of $\Gamma/(\Gamma \cap P)$, then $\Gamma = \Gamma^1(\Gamma \cap P)$, and define

$$\Gamma^0 := \{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma^1\ ;\ |cz + d| = 1 \text{ for } \exists z \in \mathbb{F}_{1,\Gamma}\}.$$

Now, let G be the subgroup of Γ generated by $\Gamma^0 \cup \{T(u)\}$, then $\mathbb{F}_{1,\Gamma} \cup V_\Gamma$ is a fundamental domain of G. For every $\gamma \in \Gamma$ and a fixed point $z \in \mathbb{F}_{0,\Gamma}$, there exist some $\gamma' \in G$ such that $\gamma'\gamma z \in \mathbb{F}_{1,\Gamma}$. Because $\gamma'\gamma \in \Gamma$, we have $\gamma'\gamma z = z$. Put $\gamma'\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $|cz + d| = 1$. Thus $\gamma = \gamma'^{-1}$ or $-\gamma'^{-1} \in G$. In conclusion, the next corollary follows:

Corollary 2.1.1. *If* $\Gamma \ni -I$, *then* $\Gamma^0 \cup \{T(u), -I\}$ *generates* Γ. *On the other hand, if* $\Gamma \not\ni -I$, $\Gamma^0 \cup \{T(u)\}$ *generates* Γ.

2.3. Modular forms.

2.3.1. *Preliminaries.* (See §[III.2, 3] [KO] and §[VII.1] [SE])

Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. For a cusp κ of Γ, we define

$$\Gamma_\kappa := \{\gamma \in \Gamma\,;\, \gamma\kappa = \kappa\}.$$

In the previous section, we assume that $\Gamma \cap P \setminus \{\pm I\} \neq \phi$. Then ∞ is one of the cusps of Γ, and we have $\Gamma_\infty = \Gamma \cap P$. Furthermore, there exist some $\gamma_\kappa \in \mathrm{SL}_2(\mathbb{R})$ such that $\gamma_\kappa\infty = \kappa$ and

$$\Gamma_\kappa = \gamma_\kappa \Gamma_\infty \gamma_\kappa^{-1}.$$

Let f be a function on $\mathbb{H}$. The relation

$$f(\gamma z) = (cz+d)^k f(z) \quad \text{for every } z \in \mathbb{H} \text{ and every } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \tag{13}$$

is called the *transformation rule* for Γ.

Incidentally, since $\mathrm{SL}_2(\mathbb{Z}) = \langle \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$ (See (37)), transformation rule for $\mathrm{SL}_2(\mathbb{Z})$ is equivalent to the following two equations:

$$f(z+1) = f(z), \tag{14}$$

$$f\left(-\frac{1}{z}\right) = z^k f(z). \tag{15}$$

We also have the following Fourier expansion for every cusp κ of Γ:

$$(cz+d)^{-k} f(\gamma_\kappa z) = \sum_{n \in \mathbb{Z}} a_{\kappa,n} q^n \quad \text{for } \gamma_\kappa = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \quad \text{where } q = e^{2\pi i z}. \tag{16}$$

We say f is *meromorphic at* κ if $a_{\kappa,n}$ is zero for n small enough. Also, we call f *holomorphic at at* κ if $a_{\kappa,n}$ is zero for every negative integer n.

Definition 2.2. *Let f be a meromorphic function on $\mathbb{H}$. f is called a modular function for Γ if f is meromorphic at every cusp and satisfies transformation rule for Γ.*

When f is holomorphic at a cusp κ, $f(\kappa) = 0$ if and only if $a_{\kappa,0} = 0$.

Definition 2.3. *Let f be a modular function for Γ which is holomorphic on $\mathbb{H}$. f is called modular form for Γ if f is holomorphic at its evry cusp. In addition, if f is equal to 0 at its every cusp, we call f cusp form for Γ.*

For a function f, let $v_p(f)$ be the order of f at $p \in \mathbb{H}$. In addition, we also define the order of f at a cusp κ:

$$v_\kappa(f) := \min\{n \in \mathbb{Z}\,;\, a_{\kappa,n} \neq 0\}.$$

Finally, we have following facts:

Proposition 2.3. *Let f be a modular form for Γ such that every coefficient of Fourier expansion is real. Then we have*

$$f(-\overline{z}) = \overline{f(z)}. \tag{17}$$

Proof. (See [G]) Let f be a modular form for Γ, and let

$$f(z) = \sum_{n \geqslant 0} a_n e^{2\pi i n z}$$

be a Fourier expansion of f with real coefficients a_n. Put $z = x + yi$, then we have

$$f(z) = \sum_{n \geqslant 0} a_n e^{2\pi n(-y+xi)},$$

$$f(-\overline{z}) = \sum_{n \geqslant 0} a_n e^{2\pi n(-y-xi)}.$$

□

Corollary 2.1.2. *Let f be a modular form for Γ such that every coefficient of Fourier expansion is real. Then we have*

$$v_\rho(f) = v_{\overline{\rho}}(f). \tag{18}$$

Proof. Let f be a modular form for Γ such that every coefficient of Fourier expansion is real, and let r be a positive number such that $\{z\,;\,0 < |z-\rho| \leqslant r\}$ has no zeros of f. Then

$$\begin{aligned} v_\rho(f) &= \frac{1}{2\pi i}\int_{\{|z-\rho|=r\}} \frac{df(z)}{f(z)} = -\frac{1}{2\pi i}\int_{\{|z-(-\overline{\rho})|=r\}} \frac{df(-\overline{z})}{f(-\overline{z})} \\ &= \overline{\frac{1}{2\pi i}\int_{\{|z-(-\overline{\rho})|=r\}} \frac{df(z)}{f(z)}} = \overline{v_{\overline{\rho}}(f)} = v_{\overline{\rho}}(f). \end{aligned}$$

□

2.3.2. *Eisenstein series.* (See [SU])

Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$.

Definition 2.4. *For $z \in \mathbb{H}$,*

$$E_{k,\kappa} := e \sum_{\gamma \in \Gamma_\kappa \setminus \Gamma} j\left(\gamma_\kappa^{-1}\gamma, z\right)^{-k} \quad (e\ :\ \text{fixed number}) \tag{19}$$

is the Eisenstein series associated with Γ for a cusp κ, where $j(\gamma, z) := cz + d$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. e is often selected so that the constant term of $E_{k,\kappa}$ is 1.

Note that Eisenstein series $E_{k,\kappa}$ is modular function for Γ of weight k.

For the cusp ∞, any elements of Γ_∞ stabilize (c, d) of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Thus we have only to consider about the pairs (c, d) as representatives of $\Gamma_\infty \setminus \Gamma$.

For example, let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, then we have only ∞ as the cusp of $\mathrm{SL}_2(\mathbb{Z})$. Now, for an even integer $k \geqslant 4$, we have

$$E_k(z) := \frac{1}{2}\sum_{(c,d)=1} (cz + d)^{-k} \tag{20}$$

as the Eisenstein series associated with $\mathrm{SL}_2(\mathbb{Z})$.

2.3.3. *Some notations.* For a prime p, we define

$$D_{k,p}(z) := \frac{1}{2}\sum_{\substack{(c,d)=1 \\ p\nmid cd}} (cz + d)^{-k}, \tag{21}$$

$$B_{k,p}(z) := \frac{1}{2}\sum_{\substack{(c,d)=1 \\ p\mid c}} (cz + d)^{-k}, \quad C_{k,p}(z) := \frac{1}{2}\sum_{\substack{(c,d)=1 \\ p\mid d}} (cz + d)^{-k}. \tag{22}$$

Then we have

$$B_{k,p}(z) = \frac{1}{2}\sum_{n\in\mathbb{N}}\sum_{\substack{(c,d)=1 \\ p\nmid cd}} (p^n cz + d)^{-k} = \sum_{n\in\mathbb{N}} D_{k,p}(p^n z),$$

$$C_{k,p}(z) = \frac{1}{2}\sum_{n\in\mathbb{N}}\sum_{\substack{(c,d)=1 \\ p\nmid cd}} (cz + p^n d)^{-k} = \sum_{n\in\mathbb{N}} p^{-kn} D_{k,p}\left(\frac{z}{p^n}\right).$$

Furthermore

$$\begin{aligned} E_k(z) &= D_{k,p}(z) + C_{k,p}(z) + B_{k,p}(z) \\ &= D_{k,p}(z) + \sum_{n\in\mathbb{N}} D_{k,p}(p^n z) + \sum_{n\in\mathbb{N}} p^{-kn} D_{k,p}\left(\frac{z}{p^n}\right). \end{aligned}$$

Conversely, let us consider to write $B_{k,p}$ and $C_{k,p}$ with E_k.

$$E_k(pz) = p^{-k} D_{k,p}(z) + \sum_{n\in\mathbb{N}} D_{k,p}(p^n z) + \sum_{n\in\mathbb{N}} p^{-k(n+1)} D_{k,p}\left(\frac{z}{p^n}\right).$$

Then $E_k(z) - p^k E_k(pz) = (1-p^k)\sum_{n\in\mathbb{N}} D_{k,p}(p^n z) = (1-p^k)B_{k,p}(z)$. In conclusion,

$$B_{k,p}(z) = \frac{1}{1-p^k}\left(E_k(z) - p^k E_k(pz)\right). \tag{23}$$

Similarly,

$$C_{k,p}(z) = \frac{1}{1-p^k}\left(E_k(z) - E_k\left(\frac{z}{p}\right)\right). \tag{24}$$

Furthermore, similar to equations(14) and (15),

$$\begin{aligned} D_{k,p}(z+p) &= D_{k,p}(z) \qquad D_{k,p}(-1/z) = z^k D_{k,p}(z),\\ B_{k,p}(z+1) &= B_{k,p}(z) \qquad B_{k,p}(-1/z) = z^k B_{k,p}(z),\\ C_{k,p}(z+p) &= C_{k,p}(z) \qquad C_{k,p}(-1/z) = z^k C_{k,p}(z). \end{aligned} \tag{25}$$

In the last part of this subsection, for $0 \leqslant n \leqslant p-1$, we define

$$D_{k,p}^n(z) := \frac{1}{2} \sum_{\substack{(c,d)=1\\ p\nmid cd\\ d\equiv nc \pmod p}} (cz+d)^{-k}. \tag{26}$$

Then we have $D_{k,p}^0(z) = C_{k,p}(z)$, $\sum_{1\leqslant n\leqslant p-1} D_{k,p}^n(z) = D_{k,p}(z)$, and

$$D_{k,p}^n(z+1) = D_{k,p}^{n+1}(z). \tag{27}$$

2.3.4. *Eta function.* (See §[III.2] [KO])
We put

$$\Delta(z) := \frac{1}{1728}\left((E_4(z))^3 - (E_6(z))^2\right). \tag{28}$$

Δ is a cusp form for $\mathrm{SL}_2(\mathbb{Z})$ of weight 12. Now, we have

Theorem 2.2 (Jacobi's product formula)**.**

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} \quad \textit{where } q = e^{2\pi i z}. \tag{29}$$

Also, we have

$$\eta(z) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n), \tag{30}$$

which is called the *Dedekind η-function.* Then we have

$$\eta(z+1) = e^{\frac{2\pi i}{24}}\eta(z) \quad \text{and} \quad \eta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{i}}\eta(z) \text{ (See [KO])}, \tag{31}$$

where $\sqrt{\cdot}$ denote a square root which has nonnegative real part. Furthermore, we have

$$\eta\left(\frac{az+b}{cz+d}\right) = \epsilon\sqrt{\frac{cz+d}{i}}\eta(z) \quad \text{for } \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}), \tag{32}$$

where ϵ is one of the 24th-roots of 1 which depends on a, b, c, and d.

3. Modular group $\mathrm{SL}_2(\mathbb{Z})$

3.1. **Eisenstein series.** Let k be an even integer greater than 4. We have Eisenstein series associated with $\mathrm{SL}_2(\mathbb{Z})$:

$$E_k(z) = \frac{1}{2}\sum_{(c,d)=1}(cz+d)^{-k}.$$

We also have its Fourier Expansion:

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n, \tag{33}$$

where $q := e^{2\pi iz}$, $\sigma_k(n) := \sum_{d|n} d^k$ which is called *divisor function*, and B_k are *Bernoulli number* which are defined by

$$\frac{x}{e^x-1} = \sum_{k=0}^{\infty}B_k\frac{x^k}{k!}.$$

For example:

$$E_4(z) = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \cdots$$
$$E_6(z) = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 - \cdots$$

It is easy to show that $E_k(z)$ is modular form (not cusp form) for $\mathrm{SL}_2(\mathbb{Z})$ for every $k \geqslant 4$.

For $k = 2$, we can define $E_2(z)$ as $E_k(z)$ for $k \geqslant 4$.

$$\begin{aligned} E_2(z) &:= \frac{1}{2}\sum_{(c,d)=1}(cz+d)^{-2} \quad = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n \\ &= 1 - 24q - 72q^2 - 96q^3 - 168q^4 - \cdots \end{aligned} \tag{34}$$

However, $E_2(z)$ is not modular form for $\mathrm{SL}_2(\mathbb{Z})$. $E_2(z)$ satisfies transformation rule (14), but it does not satisfy (15). We have

$$E_2\left(-\frac{1}{z}\right) = z^2E_2(z) + \frac{12}{2\pi i}z. \tag{35}$$

3.2. **Fundamental domain.** For deciding a fundamental domain, we consider the following condition:

$$|cz+d| > 1,\ -1/2 < Re(z) < 1/2 \quad \text{for } \forall\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \setminus P. \tag{C}$$

By $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have the condition $|z| > 1$. In addition, by $-1/2 < Re(z) < 1/2$, we have $Im(z) > \sqrt{3}/2$. If $|c| \geqslant 2$ or $d \neq 0$, then we have $Im(z) < \sqrt{3}/2$ for $z \in \mathbb{H}$ such that $|cz+d| = 1$ and $-1/2 < Re(z) < 1/2$. Thus the condition

$$|z| > 1,\ -1/2 < Re(z) < 1/2 \tag{C_0}$$

is a sufficient condition for (C). Now, we have a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ as follows:

$$\mathbb{F} := \left\{|z| \geqslant 1,\ -\frac{1}{2} \leqslant Re(z) \leqslant 0\right\} \bigcup \left\{|z| > 1,\ 0 \leqslant Re(z) < \frac{1}{2}\right\} \tag{36}$$

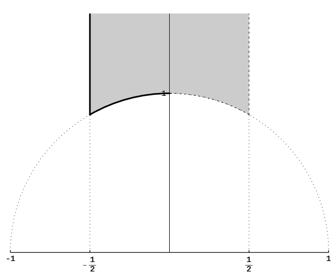


Figure 1. $\mathrm{SL}_2(\mathbb{Z})$

We have $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)^2 = -I$. Thus, by corollary2.1.1, we have

$$\mathrm{SL}_2(\mathbb{Z}) = \langle \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \rangle. \tag{37}$$

3.3. **RSD Method.** (See [RSD])

At the beginning of the proof of [RSD], they considered the following;

$$F_k(\theta) := e^{ik\theta/2} E_k\left(e^{i\theta}\right), \tag{38}$$

which is real for $\forall \theta \in \mathbb{R}$. Considering the four terms with $c^2 + d^2 = 1$, they proved

$$F_k(\theta) = 2\cos(k\theta/2) + R_1, \tag{39}$$

where R_1 is the rest of the series (*i.e.* $c^2 + d^2 > 1$). Moreover they showed

$$|R_1| \leqslant 1 + \left(\frac{1}{2}\right)^{k/2} + 4\left(\frac{2}{5}\right)^{k/2} + \frac{20\sqrt{2}}{k-3}\left(\frac{9}{2}\right)^{(3-k)/2}. \tag{40}$$

They computed the value of the right-hand side of (40) at $k = 12$ whose value is 1.03562. At this point, it is obviously monotonically decreasing in k. Thus they could show $|R_1| < 2$ for $\forall k \geqslant 12$.

For $\pi/2 \leqslant \theta \leqslant 2\pi/3$, we obtain $\frac{k}{4}\pi \leqslant k\theta/2 \leqslant \frac{k}{3}\pi$. So for any integer $m \in \left[\frac{k}{4}, \frac{k}{3}\right]$, if m is even or odd, then $\cos(k\theta/2)$ is $+1$ or -1, respectively; and $F_k(2m\pi/k)$ is positive or negative, respectively.

How many integers are there in $\left[\frac{k}{4}, \frac{k}{3}\right]$? If k is indivisible by 4, *i.e.* $k = 4l + 2$ ($\exists l \in \mathbb{N}$), then $\left[\frac{k}{4}, \frac{k}{3}\right] = \left[l + \frac{1}{2}, \frac{4l+2}{3}\right]$. Then it has $\left\lfloor \frac{k}{12} - \frac{1}{2} \right\rfloor + 1$ integers. On the other hand, if k is divisible by 4, then by the same consideration, it has $\left\lfloor \frac{k}{12} \right\rfloor + 1$ integer points. Write $m(k) := \left\lfloor \frac{k}{12} - \frac{t}{4} \right\rfloor$, where $t = 0$ or 2, such that $t \equiv k \pmod 4$. Then $k = 12m(k) + s$ ($s = 4, 6, 8, 10, 0,$ *and* 14).

In conclusion, they proved that $m(k)$ zeros were in A.

Remark 3.1. *(i) $F_k(\theta)$ is real. (ii) $\left[\frac{k}{4}, \frac{k}{3}\right]$ has $m(k)$ integers. (iii) For any integer $m \in \left[\frac{k}{4}, \frac{k}{3}\right]$, if m is even or odd, then $F_k(2m\pi/k)$ is positive or negative, respectively.*

They said in the last part of their paper [RSD], "This method can equally well be applied to Eisenstein series associated with subgroup of the modular group." However, it seems unclear how widely this claim holds.

3.4. **Valence formula.** In order to decide the locating of all zeros of $E_k(z)$, we need the *valence formula*:

Proposition 3.1 (valence formula)**.** *Let f be a modular function of weight k for $\mathrm{SL}_2(\mathbb{Z})$, which is not identically zero. We have*

$$v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\rho(f) + \sum_{\substack{p \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \\ p \neq i,\, \rho}} v_p(f) = \frac{k}{12}, \tag{41}$$

where $v_p(f)$ is the order of f at p, and $\rho := e^{2\pi/3}$. (See [SE]*)*

Proof. Let f be a nonzero modular function of weight k for $\mathrm{SL}_2(\mathbb{Z})$, and let $\mathscr{C}$ be a contour of $\mathbb{F}$ represented in Figure 2., whose interior contains every zero and pole of f except for i and ρ. By the *Residue theorem*, we have

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{df}{f} = \sum_{p \in \mathbb{F}\backslash\{i,\rho\}} v_p(f).$$

(i) For the arc EA, we have

$$\frac{1}{2\pi i}\int_E^A \frac{df}{f} = \frac{1}{2\pi i}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f'(u+Ki)}{f(u+Ki)}\,du = \frac{1}{2\pi i}\int_{w=\{|q|=e^{-2\pi K}\}} \frac{f'(q)}{f(q)}\,dq = -v_\infty(f).$$

(ii) For the arcs BB' which is a part of the circle around ρ, when the radius of the arc tends to 0, the angle of it tends to $\pi/3$. Then we have

$$\frac{1}{2\pi i}\int_B^{B'} \frac{df}{f} \to -\frac{1}{6}v_\rho(f).$$

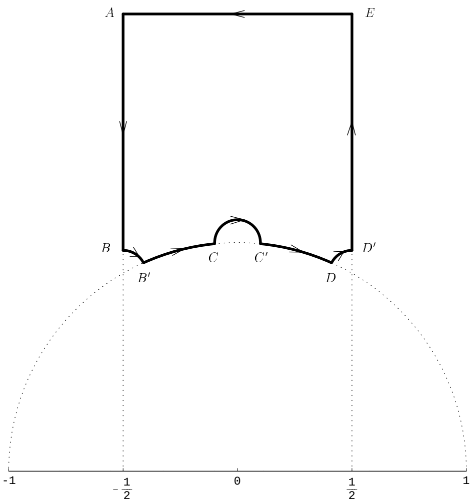


FIGURE 2

Similarly, when the radii of the arcs CC' and DD' tend to 0, the angles of them tend to π and $\pi/3$, respectively. We have

$$\frac{1}{2\pi i}\int_C^{C'} \frac{df}{f} \to -\frac{1}{2}v_i(f) \quad \text{and} \quad \frac{1}{2\pi i}\int_D^{D'} \frac{df}{f} \to -\frac{1}{6}v_\rho(f).$$

(iii) For the arcs AB and $D'E$, since $f(Tz) = f(z)$ for $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$,

$$\frac{1}{2\pi i}\int_A^B \frac{df}{f} + \frac{1}{2\pi i}\int_{D'}^E \frac{df}{f} = \frac{1}{2\pi i}\int_A^B \frac{df}{f} + \frac{1}{2\pi i}\int_B^A \frac{df}{f} = 0.$$

(iv) For the arcs $B'C$ and $C'D$, since $f(Sz) = z^k f(z)$ for $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we have

$$\frac{df(Sz)}{dz} = kz^{k-1}f(z) + z^k\frac{df(z)}{dz}, \quad \frac{df(Sz)}{f(Sz)} = k\frac{dz}{z} + \frac{df(z)}{f(z)}.$$

When the radii of the arcs BB', CC', DD' tend to 0, the angle of the arc $B'C$ tend to $\pi/6$. Thus we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{B'}^{C} \frac{df(z)}{f(z)} + \frac{1}{2\pi i}\int_{C'}^{D} \frac{df(z)}{f(z)} &= \frac{1}{2\pi i}\int_{B'}^{C} \frac{df(z)}{f(z)} + \frac{1}{2\pi i}\int_{C}^{B'} \frac{df(Sz)}{f(Sz)} \\ &= \frac{1}{2\pi i}\int_{B'}^{C} \left(\frac{df(z)}{f(z)} - \frac{df(Sz)}{f(Sz)}\right) \\ &= \frac{1}{2\pi i}\int_{B'}^{C} \left(-k\frac{dz}{z}\right) \quad \to \frac{k}{12}.\end{aligned}$$

(v) Finally, when f has a zero or pole on $\partial\mathbb{F} \setminus \{i, \rho\}$, we can transform $\mathscr{C}$ as Figure 3. Then we can show in a similar way to Figure 2.

□

By previous subsection (RSD Method), $E_k(z)$ has $m(k)$ zeros on A, so

$$\sum_{\substack{p \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \\ p \neq i,\, \rho}} v_p(E_k) \geqslant m(k).$$

If $k \equiv 4, 6, 8, 10,$ and $0 \pmod{12}$, then $k/12 - m(k) < 1$, so any other zero does not exist except for i and ρ. Thus all zeros of $E_k(z)$ is on $A \cup \{i, \rho\}$.

But if $k \equiv 2 \pmod{12}$, we need another consideration because we have $k/12 - m(k) > 1$.

Recall that $E_k(z)$ is a *modular form* of weight k for $\mathrm{SL}_2(\mathbb{Z})$. By the equation (15), substituting i for z, we have $E_k(i) = i^k E_k(i)$. Because $k \not\equiv 0 \pmod 4$, $E_k(i) = 0$. Thus i is a zero of $E_k(z)$, *i.e.* $v_i(E_k) \geqslant 1$, then we have $k/12 - m(k) - v_i(E_k)/2 < 1$.

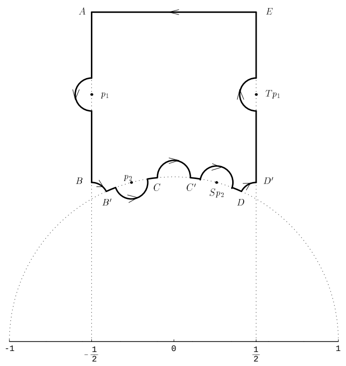


FIGURE 3

In conclusion, for every even integer $k \geqslant 4$, all zeros of $E_k(z)$ are on $A \cup \{i, \rho\}$.

3.5. **The space of modular forms.** Let M_k be the space of modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of weight k, and let M_k^0 be the space of cusp forms for $\mathrm{SL}_2(\mathbb{Z})$ of weight k. Because $\dim(M_k/M_k^0) \leqslant 1$, $M_k = \mathbb{C}E_k \oplus M_k^0$. As a classical result, we have the following theorem(See [SE]):

Theorem 3.1. *Let k be an even integer, and let $\Delta := \frac{1}{1728}(E_4^3 - E_6^2)$.*

(1) *For $k < 0$ and $k = 2$, $M_k = 0$.*
(2) *For $k = 0, 4, 6, 8, 10,$ and 14, we have $M_k^0 = 0$, and $\dim(M_k) = 1$ with a base E_k.*
(3) *$M_k^0 = \Delta M_{k-12}$.*

Furthermore, for a non-negative integer k, $\dim(M_k) = \lfloor k/12 \rfloor$ if $k \equiv 2 \pmod{12}$, and $\dim(M_k) = \lfloor k/12 \rfloor + 1$ if $k \not\equiv 2 \pmod{12}$.

Let k be an even integer $k \geqslant 4$. Write $n := \dim(M_k) - 1$, then $k - 12n = 0, 4, 6, 8, 10$, or 14. Because $E_k - E_{k-12n}E_{12n} \in M_k^0$, we have $M_k = \mathbb{C}E_{k-12n}E_{12n} \oplus M_k^0$. Then

$$M_k = E_{k-12n}(\mathbb{C}E_{12n} \oplus \mathbb{C}E_{12(n-1)}\Delta \oplus \cdots \oplus \mathbb{C}\Delta^n)$$

Thus, for every $p \in \mathbb{H}$ and for every $f \in M_k$, $v_p(f) \geqslant v_p(E_{k-12n})$.

In the previous subsection, we have $v_\infty(E_k) = \sum_{\substack{p \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \\ p \neq i,\, \rho}} v_p(E_k) = 0$ for $k = 0, 4, 6, 8, 10$, or 14. Thus, by the valence formula, $v_i(f)/2 + v_\rho(f)/3 = k/12$. In conclusion, the next proposision follows:

Proposition 3.2. *Let $k \geqslant 4$ be an even integer. For every $f \in M_k$, we have*

$$\begin{aligned} &v_i(f) \geqslant s_k \quad (s_k = 0, 1 \text{ such that } 2s_k \equiv k \pmod 4), \\ &v_\rho(f) \geqslant t_k \quad (t_k = 0, 1, 2 \text{ such that } -2t_k \equiv k \pmod 6). \end{aligned} \tag{42}$$

In paticular, if f is a constant multiple of E_k, then the equalities hold.

Remark 3.2. *Every modular form for $\mathrm{SL}_2(\mathbb{Z})$ is generated by*

$$E_4 \quad \textit{and} \quad E_6.$$

3.6. **On $E_k(i)$ and $E_k(\rho)$.** We consider the bound for $|R_1|$ again. Let $k \geqslant 4$ be an even integer.

Define $v_k(c, d, \theta) := |ce^{i\theta/2} + de^{-i\theta/2}|^{-k}$, then $v_k(c, d, \theta) = 1/(c^2 + d^2 + 2cd\cos\theta)^{k/2}$ and $v_k(c, d, \theta) = v_k(-c, -d, \theta)$.

Now we will consider the next three cases, namely $N = 2, 5$, and $N \geqslant 10$. Considering $\theta \in [\pi/2, 2\pi/3]$, we have the following:
When $N = 2$,

$$v_k(1, 1, \theta) = \left(\frac{1}{2 + 2\cos\theta}\right)^{k/2} \leqslant 1, \quad v_k(1, -1, \theta) = \left(\frac{1}{2 - 2\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{2}\right)^{k/2}.$$

When $N = 5$,

$$v_k(1,2,\theta) = \left(\frac{1}{5+4\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{3}\right)^{k/2}, \quad v_k(1,-2,\theta) = \left(\frac{1}{5-4\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{5}\right)^{k/2}.$$

When $N \geqslant 10$,

$$\begin{aligned}|ce^{i\theta/2} \pm de^{-i\theta/2}|^2 &\geqslant c^2 + d^2 - 2|cd||\cos\theta| \\ &= \frac{1}{2}\left(|c| - |d|\right)^2 + |cd|\left(1 - 2|\cos\theta|\right) + \frac{1}{2}(c^2+d^2) \\ &\geqslant \frac{1}{2}(c^2+d^2) \quad = \frac{1}{2}N,\end{aligned}$$

and the rest of the question is about the number of terms with $c^2+d^2 = N$. The number of $|c|$ is not more than $N^{1/2}$, and we consider four terms $(\pm(c,d), \pm(c,-d))$ and the number $1/2$ which is the coefficient of the summation. Thus the number of terms is not more than $2N^{1/2}$. Then

$$\begin{aligned}|R_1|_{N\geqslant 10} = \sum_{N=10}^{\infty} 2N^{1/2}\left(\frac{1}{2}N\right)^{-k/2} &\leqslant 2\sqrt{2}\int_9^{\infty}\left(\frac{1}{2}x\right)^{(1-k)/2} dx \\ = \frac{8\sqrt{2}}{k-3}\left(\frac{2}{9}\right)^{(k-3)/2} &= \frac{108}{k-3}\left(\frac{2}{9}\right)^{k/2}.\end{aligned}$$

Thus

$$\begin{aligned}|R_1| &\leqslant 1 + \left(\frac{1}{2}\right)^{k/2} + 2\left(\frac{1}{3}\right)^{k/2} + 2\left(\frac{1}{5}\right)^{k/2} + \frac{108}{k-3}\left(\frac{2}{9}\right)^{k/2} \\ &\leqslant 1.61013... \quad (k \geqslant 6)\end{aligned}$$

Similarly, for $\theta = \pi/2$, we have the following:
When $N = 2$,

$$v_k(1,1,\pi/2) = v_k(1,-1,\pi/2) = \left(\frac{1}{2}\right)^{k/2}.$$

When $N = 5$,

$$v_k(1,2,\pi/2) = v_k(1,-2,\pi/2) = \left(\frac{1}{5}\right)^{k/2}.$$

When $N \geqslant 10$,

$$|ce^{i(\pi/2)/2} \pm de^{-i(\pi/2)/2}|^2 = c^2 + d^2 = N,$$

and the number of terms is not more than $2N^{1/2}$. Then

$$|R_1|_{N\geqslant 10,\, \theta=\pi/2} = 2\sum_{N=10}^{\infty} N^{(1-k)/2} \leqslant \frac{108}{k-3}\left(\frac{2}{9}\right)^{k/2}.$$

Thus

$$|R_1|_{\theta=\pi/2} \leqslant 2\left(\frac{1}{2}\right)^{k/2} + 4\left(\frac{1}{5}\right)^{k/2} + \frac{108}{k-3}\left(\frac{1}{3}\right)^{k} \leqslant 1.99333... \quad (k \geqslant 4)$$

Now, we have

$$\begin{aligned}F_{4k}(\pi/2) &= e^{ik\pi}E_{4k}(i) = 2\cos(k\pi) + R_1, \\ F_{6k}(2\pi/3) &= e^{i2k\pi}E_{6k}(\rho) = 2\cos(2k\pi) + R_1.\end{aligned}$$

For both equations, we have the bound $|R_1| < 2$. Thus we have following proposition:

Proposition 3.3. *Let $k \geqslant 4$ be an even integer. We have*

$$E_k(i)\begin{cases} > 0 & k \equiv 0 \pmod 4 \\ = 0 & k \equiv 2 \pmod 4\end{cases},$$

$$E_k(\rho)\begin{cases} > 0 & k \equiv 0 \pmod 6 \\ = 0 & k \not\equiv 0 \pmod 6\end{cases}.$$

4. Preliminaries for Congruence subgroups $\Gamma_0(p)$

4.1. **Fandamental domain.** Let p be a prime. We consider the following condition for every $m \in \mathbb{N}$:

$$(C_m) \qquad |z - n/mp| > 1/mp,\ -1/2 < Re(z) < 1/2 \quad \text{for } \forall n \in \mathbb{N} \text{ such that } (mp, n) = 1.$$

It is easy to show that $|z - n/mp| > 1/mp$ is a necessary condition for $|z - \lfloor n/m \rfloor /p| > 1/p$. Thus the condition

$$(C_0) \qquad |z - n/p| > 1/p,\ -1/2 < Re(z) < 1/2 \quad \text{for } \forall n \in \mathbb{N} \text{ such that } 1 \leqslant |n| \leqslant p/2.$$

is sufficient condition for (C_m) for every $m \in \mathbb{N}$.

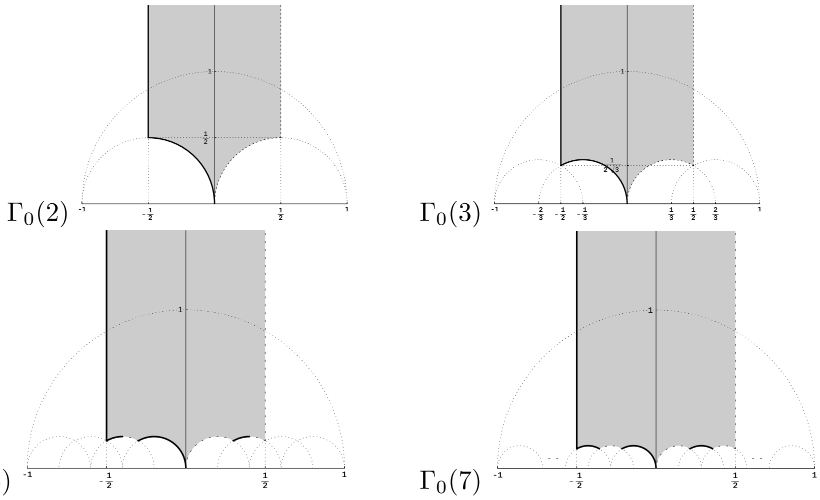


Figure 4. Congruence subgroups $\Gamma_0(p)$

By corollary2.1.1, we have

$$\Gamma_0(p) = \langle -I,\ \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} a & b \\ p & n \end{smallmatrix}\right) \in \Gamma_0(p)/(\Gamma_0(p) \cap P)\ ;\ 1 \leqslant |n| \leqslant p/2 \rangle. \tag{43}$$

For example: (these are for subsection4.3, *cf.* [RD])

$$\begin{aligned}
\Gamma_0(2) &= \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) \rangle \\
\Gamma_0(3) &= \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ -\left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right) \rangle \\
\Gamma_0(5) &= \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 1 & 0 \\ 5 & 1 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right) \rangle \\
\Gamma_0(7) &= \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ -\left(\begin{smallmatrix} 1 & 0 \\ 7 & 1 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 4 & 1 \\ 7 & 2 \end{smallmatrix}\right) \rangle \\
\Gamma_0(11) &= \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ -\left(\begin{smallmatrix} 1 & 0 \\ 11 & 1 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 4 & 1 \\ 11 & 3 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 3 & 1 \\ 11 & 4 \end{smallmatrix}\right) \rangle \\
\Gamma_0(13) &= \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 1 & 0 \\ 13 & 1 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 7 & 1 \\ 13 & 2 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} -5 & -2 \\ 13 & 5 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 5 & -2 \\ 13 & -5 \end{smallmatrix}\right) \rangle \\
\Gamma_0(17) &= \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 1 & 0 \\ 17 & 1 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 9 & 1 \\ 17 & 2 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 6 & 1 \\ 17 & 3 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 3 & 1 \\ 17 & 6 \end{smallmatrix}\right) \rangle \\
\Gamma_0(19) &= \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ -\left(\begin{smallmatrix} 1 & 0 \\ 19 & 1 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 10 & 1 \\ 19 & 2 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 5 & 1 \\ 19 & 4 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 4 & 1 \\ 19 & 5 \end{smallmatrix}\right) \rangle \\
\Gamma_0(23) &= \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ -\left(\begin{smallmatrix} 1 & 0 \\ 23 & 1 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 12 & 1 \\ 23 & 2 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 6 & 1 \\ 23 & 4 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 4 & 1 \\ 23 & 6 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 3 & 1 \\ 23 & 8 \end{smallmatrix}\right) \rangle
\end{aligned}$$

Calculation.

For $p = 2$, we have $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -1 & -1 \\ 2 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -1 & -1 \\ 2 & 1 \end{smallmatrix}\right)^2 = -I$, and $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right)$.

For $p = 3$, we have $(-\left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right))\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -1 & 1 \\ -3 & 2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -1 & 1 \\ -3 & 2 \end{smallmatrix}\right)^3 = -I$, and $(-\left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right))^{-1} = \left(\begin{smallmatrix} -1 & 0 \\ 3 & -1 \end{smallmatrix}\right)$.

For $p = 5$, we have $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -2 & -1 \\ 5 & 2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -2 & -1 \\ 5 & 2 \end{smallmatrix}\right)^2 = -I$, $\left(\begin{smallmatrix} 1 & 0 \\ 5 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -5 & 1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -5 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -2 & 1 \\ -5 & 2 \end{smallmatrix}\right)$.

For $p = 7$, we have $(-\left(\begin{smallmatrix} 1 & 0 \\ 7 & 1 \end{smallmatrix}\right))\left(\begin{smallmatrix} 4 & 1 \\ 7 & 2 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} -2 & 1 \\ -7 & 3 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -2 & 1 \\ -7 & 3 \end{smallmatrix}\right)^3 = -I$, $(-\left(\begin{smallmatrix} 1 & 0 \\ 7 & 1 \end{smallmatrix}\right))^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -7 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -2 & 1 \\ -7 & 3 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 3 & -1 \\ 7 & -2 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 4 & 1 \\ 7 & 2 \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 1 \\ -7 & -3 \end{smallmatrix}\right)$.

For $p = 11$, we have $\left(\begin{smallmatrix} 4 & 1 \\ 11 & 3 \end{smallmatrix}\right)\left(\begin{smallmatrix} 3 & 1 \\ 11 & 4 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 5 & -1 \\ 11 & -2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 4 & 1 \\ 11 & 3 \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} 3 & 1 \\ 11 & 4 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -2 & -1 \\ 11 & 5 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 5 & -1 \\ 11 & -2 \end{smallmatrix}\right)(-\left(\begin{smallmatrix} 1 & 0 \\ 11 & 1 \end{smallmatrix}\right))\left(\begin{smallmatrix} -2 & -1 \\ 11 & 5 \end{smallmatrix}\right) = -I$, $(-\left(\begin{smallmatrix} 1 & 0 \\ 11 & 1 \end{smallmatrix}\right))^{-1} = \left(\begin{smallmatrix} -1 & 0 \\ 11 & -1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -2 & -1 \\ 11 & 5 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 5 & 1 \\ -11 & -2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 5 & -1 \\ 11 & -2 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} -2 & 1 \\ -11 & 5 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 3 & 1 \\ 11 & 4 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 4 & -1 \\ -11 & 3 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 4 & 1 \\ 11 & 3 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 3 & -1 \\ -11 & 4 \end{smallmatrix}\right)$.

For $p = 13$, we have $\left(\begin{smallmatrix} -5 & -2 \\ 13 & 5 \end{smallmatrix}\right)^2 = -I$, $\left(\begin{smallmatrix} 1 & 0 \\ 13 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -13 & 1 \end{smallmatrix}\right)$,
$\left(\begin{smallmatrix} -5 & -2 \\ 13 & 5 \end{smallmatrix}\right)^2 \left(\begin{smallmatrix} 7 & 1 \\ 13 & 2 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ -13 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 6 & -1 \\ 13 & -2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 7 & 1 \\ 13 & 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -6 & -1 \\ 13 & 2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -\pm 6 & -1 \\ 13 & \mp 2 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} \mp 2 & 1 \\ -13 & \pm 6 \end{smallmatrix}\right)$,
$\left(\begin{smallmatrix} -5 & -2 \\ 13 & 5 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 7 & 1 \\ 13 & 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 4 & 1 \\ -13 & -3 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 5 & -2 \\ 13 & -5 \end{smallmatrix}\right) \left(\begin{smallmatrix} 7 & 1 \\ 13 & 2 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ -13 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -4 & 1 \\ -13 & 3 \end{smallmatrix}\right)$,
and $\left(\begin{smallmatrix} \pm 4 & 1 \\ -13 & \mp 3 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} \mp 3 & -1 \\ 13 & \pm 4 \end{smallmatrix}\right)$.

For $p = 17$, we have $\left(\begin{smallmatrix} 6 & 1 \\ 17 & 3 \end{smallmatrix}\right) \left(\begin{smallmatrix} 9 & 1 \\ 17 & 2 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 3 & 1 \\ 17 & 6 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 4 & -1 \\ 17 & -4 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 4 & -1 \\ 17 & -4 \end{smallmatrix}\right)^2 = -I$,
$\left(\begin{smallmatrix} 1 & 0 \\ 17 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -17 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 6 & 1 \\ 17 & 3 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 9 & 1 \\ 17 & 2 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ -17 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 3 & 1 \\ 17 & 6 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -4 & -1 \\ 17 & 4 \end{smallmatrix}\right)$,
$\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 9 & 1 \\ 17 & 2 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ -17 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 9 & -1 \\ -17 & 2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 9 & \pm 1 \\ \pm 17 & 2 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & \pm 1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & \pm 1 \\ \mp 17 & -8 \end{smallmatrix}\right)$,
$\left(\begin{smallmatrix} 3 & 1 \\ 17 & 6 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 6 & -1 \\ -17 & 3 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 6 & 1 \\ 17 & 3 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 3 & -1 \\ -17 & 6 \end{smallmatrix}\right)$,
$\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 9 & 1 \\ 17 & 2 \end{smallmatrix}\right) \left(\begin{smallmatrix} 6 & 1 \\ 17 & 3 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} -7 & 2 \\ 17 & -5 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 9 & 1 \\ 17 & 2 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ -17 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 3 & 1 \\ 17 & 6 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -7 & -2 \\ -17 & -5 \end{smallmatrix}\right)$,
and $\left(\begin{smallmatrix} -7 & \pm 2 \\ \pm 17 & -5 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} -5 & \mp 2 \\ \mp 17 & -7 \end{smallmatrix}\right)$.

For $p = 19$, we have $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 10 & 1 \\ 19 & 2 \end{smallmatrix}\right) \left(\begin{smallmatrix} 5 & 1 \\ 19 & 4 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 4 & 1 \\ 19 & 5 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 8 & 3 \\ -19 & -7 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 8 & 3 \\ -19 & -7 \end{smallmatrix}\right)^3 = -I$,
$\left(-\left(\begin{smallmatrix} 1 & 0 \\ 19 & 1 \end{smallmatrix}\right)\right)^{-1} = \left(\begin{smallmatrix} -1 & 0 \\ 19 & -1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 10 & 1 \\ 19 & 2 \end{smallmatrix}\right) \left(\begin{smallmatrix} -1 & 0 \\ 19 & -1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 4 & 1 \\ 19 & 5 \end{smallmatrix}\right) \left(\begin{smallmatrix} 5 & 1 \\ 19 & 4 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} -8 & 3 \\ -19 & 7 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \pm 8 & 3 \\ -19 & \mp 7 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} \mp 7 & -3 \\ 19 & \pm 8 \end{smallmatrix}\right)$,
$\left(\begin{smallmatrix} 10 & 1 \\ 19 & 2 \end{smallmatrix}\right) \left(\begin{smallmatrix} -1 & 0 \\ 19 & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 9 & -1 \\ 19 & -2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 10 & 1 \\ 19 & 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -9 & -1 \\ 19 & 2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \pm 9 & -1 \\ 19 & \mp 2 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} \mp 2 & 1 \\ -19 & \pm 9 \end{smallmatrix}\right)$,
$\left(\begin{smallmatrix} 5 & 1 \\ 19 & 4 \end{smallmatrix}\right) \left(\begin{smallmatrix} 4 & 1 \\ 19 & 5 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 6 & -1 \\ 19 & -3 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 4 & 1 \\ 19 & 5 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 5 & 1 \\ 19 & 4 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 6 & 1 \\ -19 & -3 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 6 & \mp 1 \\ \pm 19 & -3 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} -3 & \pm 1 \\ \mp 19 & 6 \end{smallmatrix}\right)$,
$\left(\begin{smallmatrix} 4 & 1 \\ 19 & 5 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 5 & -1 \\ -19 & 4 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 5 & 1 \\ 19 & 4 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 4 & -1 \\ -19 & 5 \end{smallmatrix}\right)$.

For $p = 23$, we have
$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 3 & 1 \\ 23 & 8 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 4 & 1 \\ 23 & 6 \end{smallmatrix}\right) \left(\begin{smallmatrix} 6 & 1 \\ 23 & 4 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 12 & 1 \\ 23 & 2 \end{smallmatrix}\right) \left(-\left(\begin{smallmatrix} 1 & 0 \\ 23 & 1 \end{smallmatrix}\right)\right)^{-1} \left(\begin{smallmatrix} 3 & 1 \\ 23 & 8 \end{smallmatrix}\right) \left(\begin{smallmatrix} 4 & 1 \\ 23 & 6 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 6 & 1 \\ 23 & 4 \end{smallmatrix}\right) \left(\begin{smallmatrix} 12 & 1 \\ 23 & 2 \end{smallmatrix}\right)^{-1} = -I$,
$\left(-\left(\begin{smallmatrix} 1 & 0 \\ 23 & 1 \end{smallmatrix}\right)\right)^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -23 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 12 & 1 \\ 23 & 2 \end{smallmatrix}\right) \left(-\left(\begin{smallmatrix} 1 & 0 \\ 23 & 1 \end{smallmatrix}\right)\right)^{-1} = \left(\begin{smallmatrix} -12 & 1 \\ 23 & -2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \pm 12 & 1 \\ 23 & \pm 2 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} \pm 2 & -1 \\ -23 & \pm 12 \end{smallmatrix}\right)$,
$\pm \left(\begin{smallmatrix} 2 & \mp 1 \\ \mp 23 & 12 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & \pm 1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \pm 2 & 1 \\ -23 & \mp 11 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 12 & 1 \\ 23 & 2 \end{smallmatrix}\right) \left(-\left(\begin{smallmatrix} 1 & 0 \\ 23 & 1 \end{smallmatrix}\right)\right)^{-1} \left(\begin{smallmatrix} 3 & 1 \\ 23 & 8 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 10 & 3 \\ 23 & 7 \end{smallmatrix}\right)$,
$\left(\begin{smallmatrix} 3 & 1 \\ 23 & 8 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 8 & -1 \\ -23 & 3 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 12 & 1 \\ 23 & 2 \end{smallmatrix}\right) \left(-\left(\begin{smallmatrix} 1 & 0 \\ 23 & 1 \end{smallmatrix}\right)\right)^{-1} \left(\begin{smallmatrix} 3 & 1 \\ 23 & 8 \end{smallmatrix}\right) = -\left(\begin{smallmatrix} 8 & 1 \\ 23 & 3 \end{smallmatrix}\right)$,
$\left(\begin{smallmatrix} 4 & 1 \\ 23 & 6 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 6 & -1 \\ -23 & 4 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 6 & 1 \\ 23 & 4 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 4 & -1 \\ -23 & 6 \end{smallmatrix}\right)$,
$\left(\begin{smallmatrix} 10 & 3 \\ 23 & 7 \end{smallmatrix}\right) \left(\begin{smallmatrix} 4 & 1 \\ 23 & 6 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} -9 & 2 \\ -23 & 5 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 3 & 1 \\ 23 & 8 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 4 & 1 \\ 23 & 6 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 9 & 2 \\ -23 & -5 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \pm 9 & 2 \\ -23 & \mp 5 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} \mp 5 & -2 \\ 23 & \pm 9 \end{smallmatrix}\right)$,
$\left(\begin{smallmatrix} 3 & 1 \\ 23 & 8 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 4 & 1 \\ 23 & 6 \end{smallmatrix}\right) \left(\begin{smallmatrix} 6 & 1 \\ 23 & 4 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} -10 & 3 \\ 23 & -7 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} \pm 10 & 3 \\ 23 & \pm 7 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} \pm 7 & -3 \\ -23 & \pm 10 \end{smallmatrix}\right)$.

□

4.2. **Eisenstein series.** Let p be a prime, and let $\Gamma = \Gamma_0(p)$. $\Gamma_0(p)$ has two cusps ∞ and 0.

For the cusp ∞, we have only to consider about the pairs (c, d) of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$ as representatives of $\Gamma_\infty \setminus \Gamma_0(p)$. Then we have

$$E^{\infty}_{k,p}(z) := \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p \mid c}} (cz + d)^{-k} \tag{44}$$

as the Eisenstein series associated with $\Gamma_0(p)$ for the cusp ∞. Incidentally, we have $E^{\infty}_{k,p}(z) = B_{k,p}(z)$. Then we can write

$$E^{\infty}_{k,p}(z) = \frac{1}{1 - p^k} \left(E_k(z) - p^k E_k(pz) \right). \tag{45}$$

For the cusp 0, let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$. If $\gamma \in \Gamma_0$, then we have $b = 0$, and then we have

$$\Gamma_0 = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ np & 1 \end{smallmatrix}\right) \ ; \ n \in \mathbb{Z} \right\} = W_p \Gamma_\infty W_p^{-1}.$$

Thus $\gamma_0 = W_p$, then $j(W_p^{-1}\gamma, z) = -\sqrt{p}(az + b)$. Because any elements of Γ_0 stabilize the pair (a, b), we have only to consider about the pairs (a, b) as representatives of $\Gamma_0 \setminus \Gamma_0(p)$. Then we have

$$E^{0}_{k,p}(z) := \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c}} (cz + d)^{-k} \tag{46}$$

as the Eisenstein series associated with $\Gamma_0(p)$ for the cusp 0. Incidentally, we have $E^{\infty}_{k,p}(z) + E^{0}_{k,p}(z) = E_k(z)$. Then we can write

$$E^{0}_{k,p}(z) = \frac{1}{1 - p^{-k}} \left(E_k(z) - E_k(pz) \right). \tag{47}$$

It is easy to show that $E_{k,p}^{\infty}(z)$ and $E_{k,p}^{0}(z)$ are holomorphic on $\mathbb{H}$ and at ∞ by equations (45), (47). For $\gamma_0 = W_p$, we have

$$(\sqrt{p}z)^{-k} E_{k,p}^{\infty}(W_p z) = (p^{-k/2}) E_{k,p}^{0}(z),$$
$$(\sqrt{p}z)^{-k} E_{k,p}^{0}(W_p z) = (p^{k/2}) E_{k,p}^{\infty}(z).$$

Thus both of them are holomorphic at cusp 0, *i.e.* $E_{k,p}^{\infty}(z)$ and $E_{k,p}^{0}(z)$ are modular form for $\Gamma_0(p)$.

Furthermore, we have $E_{k,p}^{\infty}(0) = E_{k,p}^{0}(\infty) = 0$, $E_{k,p}^{\infty}(\infty) \neq 0$ and $E_{k,p}^{0}(0) \neq 0$. Then, $E_{k,p}^{\infty}(z)$ and $E_{k,p}^{0}(z)$ are not cusp form for $\Gamma_0(p)$.

4.3. **Eta function.** Let p be a prime, and let k be a minimum positive even integer such that $24 \mid k(p+1)$ (*i.e.* $k = 24/(p+1, 12)$). We put

$$\Delta_p(z) := \eta^k(z)\eta^k(pz), \tag{48}$$

then for $T = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ by the equations(31), we have $T : z \mapsto z + 1$ and

$$\Delta_p(Tz) = \eta^k(z+1)\eta^k(pz+p) = e^{2\pi i \cdot k(p+1)/24}\eta^k(z)\eta^k(pz) = \Delta_p(z). \tag{49}$$

Also, for $W_p = \left(\begin{smallmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{smallmatrix}\right)$, we have $W_p, W_p^{-1} : z \mapsto -1/pz$ and

$$\Delta_p(W_p z) = \eta^k\left(-\frac{1}{pz}\right)\eta^k\left(-\frac{1}{z}\right) = \frac{(\sqrt{p}z)^k}{i^k}\Delta_p(z). \tag{50}$$

Then for $S_p := \left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right)$, because $S_p = W_p T^{-1} W_p^{-1}$ and k is even, we have

$$\Delta_p(S_p z) = (pz+1)^k \Delta_p(z). \tag{51}$$

Furthermore, we have next proposition:

Proposition 4.1. *Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a element of $\Gamma_0(p)$. If $c = bp$, then we have*

$$\Delta_p(\gamma z) = (cz+d)^k \Delta_p(z). \tag{52}$$

Proof. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$, if $c = bp$, then $\gamma W_p = \left(\begin{smallmatrix} b\sqrt{p} & -a/\sqrt{p} \\ d\sqrt{p} & -b\sqrt{p} \end{smallmatrix}\right)$, and put $\gamma' := \gamma W_p$. In addition, we put $\gamma_1 := \left(\begin{smallmatrix} b & -a \\ d & -bp \end{smallmatrix}\right)$ and $\gamma_2 := \left(\begin{smallmatrix} bp & -a \\ d & -b \end{smallmatrix}\right)$, then we have $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$, $\gamma' z = \gamma_1(pz)$, $p \cdot \gamma' z = \gamma_2 z$. Moreover, because $\gamma_1\gamma_2 = -I$, $\gamma_1 w = z$ for $w := \gamma_2 z$. By the equation(31), for some ϵ_1, ϵ_2 : 24th-roots of 1, we have

$$\eta(\gamma_1 z) = \epsilon_1 \sqrt{\frac{dz - bp}{i}}\eta(z), \quad \eta(\gamma_2 z) = \epsilon_2 \sqrt{\frac{dz-b}{i}}\eta(z)$$

Now,

$$\eta(z) = \eta(\gamma_1 w) = \epsilon_1 \sqrt{\frac{dw - bp}{i}}\eta(w) = \epsilon_1\epsilon_2 \sqrt{\frac{dw-bp}{i}}\sqrt{\frac{dz-b}{i}}\eta(z).$$

Furthermore,

$$dw - bp = -\frac{1}{dz-b} = \frac{-\overline{(dz-b)}}{|dz-b|^2},$$
$$\sqrt{\frac{dw-bp}{i}}\sqrt{\frac{dz-b}{i}} = \sqrt{\overline{\left(\frac{dz-b}{i|dz-b|}\right)}}\sqrt{\frac{dz-b}{i|dz-b|}} = 1.$$

Thus we have $\eta(z) = \epsilon_1\epsilon_2\eta(z)$, and $\epsilon_1\epsilon_2 = 1$.

In conclusion,

$$\eta(\gamma' z)\eta(p \cdot \gamma' z) = \eta(\gamma_1(pz))\eta(\gamma_2 z) = \epsilon_1\epsilon_2 \sqrt{\frac{dpz - bp}{i}}\sqrt{\frac{dz-b}{i}}\eta(z)\eta(pz)$$
$$= \frac{d\sqrt{p}z - b\sqrt{p}}{i}\eta(z)\eta(pz),$$

and

$$\Delta_p(\gamma' z) = \frac{(d\sqrt{p}z - b\sqrt{p})^k}{i^k}\Delta_p(z).$$

Thus

$$\begin{aligned}\Delta_p(\gamma z) &= \Delta_p(\gamma' W_p^{-1} z) = \Delta_p\left(\gamma'\left(-\frac{1}{pz}\right)\right)\\ &= \frac{(-d/\sqrt{p}z - b\sqrt{p})^k}{i^k}\Delta_p\left(-\frac{1}{pz}\right) = \frac{(-d/\sqrt{p}z - b\sqrt{p})^k}{i^k}\cdot\frac{(\sqrt{p}z)^k}{i^k}\Delta_p(z)\\ &= (cz+d)^k\Delta_p(z).\end{aligned}$$

□

Proposition 4.2. *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *be a element of* $\Gamma_0(p)$*. If* $d = -a$*, then we have*

$$\Delta_p(\gamma z) = (cz+d)^k \Delta_p(z). \tag{53}$$

Proof. $k \equiv 0 \pmod 4$ (*i.e.* $\gamma^2 = -I$) This proof is similar to that of Proposition4.1.

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$, asume that $d = -a$, and write $c = c'p$ for some $c' \in \mathbb{Z}$. Then we have $\gamma := \begin{pmatrix} a & b \\ c'p & -a \end{pmatrix}$ and $|\gamma| = -a^2 - bc'p = 1$. Here, we have $a^2 \equiv -1 \pmod p$, *i.e.* -1 is a quadratic residue modulo p. Then, by the *Euler's criterion*, we have $p \equiv 1 \pmod 4$. Thus $k \equiv 0 \pmod 4$.

We put $\gamma' := \begin{pmatrix} a & bp \\ c' & -a \end{pmatrix}$, then we have $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$, $p \cdot \gamma z = \gamma'(pz)$. Moreover, because $\gamma^2 = -I$, $\gamma w = z$ for $w := \gamma z$. By the equation(31), for some ϵ_1, ϵ_2 : 24th-roots of 1, we have

$$\eta(\gamma z) = \epsilon_1\sqrt{\frac{c'pz - a}{i}}\eta(z), \quad \eta(\gamma' z) = \epsilon_2\sqrt{\frac{c'z - a}{i}}\eta(z)$$

Now, we have

$$\eta(z) = \epsilon_1^2\sqrt{\frac{c'pw - a}{i}}\sqrt{\frac{c'pz - a}{i}}\eta(z),$$

and $c'pw - a = -1/(c'pz - a)$, $\sqrt{(c'pw - a)/i}\sqrt{(c'pz - a)/i} = 1$. Thus we have $\epsilon_1^2 = 1$. Also, because ${\gamma'}^2 = -I$, we have $\epsilon_2^2 = 1$.

In conclusion, $\eta^4(\gamma z)\eta^4(p\cdot\gamma z) = \epsilon_1^4\epsilon_2^4\left((c'pz - a)/i\right)^4\eta^4(z)\eta^4(pz) = (cz+d)^4\eta^4(z)\eta^4(pz)$. Thus, because $k \equiv 0 \pmod 4$,

$$\Delta_p(\gamma z) = (cz+d)^k \Delta_p(z).$$

□

Remark 4.1. *For definition of the integer k for prime p, we need the condition* $24 \mid k(p+1)$ *for the transformation rule for* T*. The other condition* k *: even is for* $-I$*, which is a element of* $\Gamma_0(p)$*.*

In the subsection4.1, we have the basis of $\Gamma_0(p)$ for $2 \leqslant p \leqslant 23$. We have $c = bp$ or $d = -a$ for all the bases except for T and $\pm S_p$, which is the condition for the Proposition4.1 or 4.2, respectively. Thus Δ_p satisfies the transformation rule for $\Gamma_0(p)$. In addition, because $(\Delta_p(z))^{p+1} = \Delta(z)\Delta(pz)$ or $(\Delta(z)\Delta(pz))^2$, it is easy to show that Δ_p is a cusp form for $\Gamma_0(p)$ for $2 \leqslant p \leqslant 23$.

Finally, we have the following proposition:

Proposition 4.3. (*See* [KO])

For $p = 2, 3, 5,$ *and* 11*, every nonzero cusp form for* $\Gamma_0(p)$ *of weight* k *is a constant multiple of* Δ_p*.*

Proof. For $p = 2, 3, 5,$ and 11, let f be a nonzero cusp form for $\Gamma_0(p)$ of weight k. Because $(\Delta_p(z))^{p+1} = \Delta(z)\Delta(pz)$, and because $v_z(\Delta) = 0$ for every $z \in \mathbb{H}$, we have $v_z(\Delta_p) = 0$ for every $z \in \mathbb{H}$. In addition, by the definition of Δ_p, $v_\infty(\Delta_p) = v_0(\Delta_p) = 1$. Thus f/Δ_p is a modular form of weight 0, then it is clear that $f/\Delta_p \in \mathbb{C}$. □

Remark 4.2. *By the equation(50), if* $p = 2$ *and* $p \equiv 1 \pmod 4$*, then* $k \equiv 0 \pmod 4$ *and* Δ_p *is a cusp form of* $\Gamma_0^*(p)$*. On the other hand, if* $p \equiv 3 \pmod 4$*, then* $k \equiv 2 \pmod 4$ *and* $(\Delta_p)^2$ *is a cusp form of* $\Gamma_0^*(p)$*.*

Furthermore, similar to Proposition4.3, for $p = 2$ *and* 5*, every nonzero cusp form for* $\Gamma_0^*(p)$ *of weight* k *is a constant multiple of* Δ_p*.*

5. Fricke group $\Gamma_0^*(p)$

5.1. **Preliminaries.**

5.1.1. *Fandamental domain.* Let p be a prime. Similarly to $\Gamma_0(p)$, we consider the following condition:

(C_0) $\quad |z - n/p| > 1/p, \ -1/2 < Re(z) < 1/2 \quad$ for $\forall n \in \mathbb{N}$ such that $1 \leqslant |n| \leqslant p/2$.

In addition, we need to consider the following condition for every $m \in \mathbb{N}$ such that $(m,p) = 1$ for $\Gamma_0(p)W_p$:

$(C_{p,m})$ $\quad |z - n/m| > 1/m\sqrt{p}, \ -1/2 < Re(z) < 1/2 \quad$ for $\forall n \in \mathbb{N}$ such that $(m,n) = 1$.

By W_p, we have the condition $|z| > 1/\sqrt{p}$. In addition, by (C_0), we have $Im(z) > \sqrt{3}/2p$. Thus we need $(C_{p,m})$ only for $m < \sqrt{4p/3}$. In conclusion, we have following condition:

$$(C_p) \qquad \begin{aligned} &|z - n/m| > 1/m\sqrt{p}, \ -1/2 < Re(z) < 1/2 \\ &\quad \text{for } \forall m \in \mathbb{N} \text{ such that } m < \sqrt{4p/3} \\ &\quad\quad \forall n \in \mathbb{Z} \text{ such that } (m,n) = 1, \ |n| < m/2 + 1/\sqrt{p}. \end{aligned}$$

It seems that (C_p) is a sufficient condition for (C_0), but it is not clear.

5.1.2. *Eisenstein series.* Let p be a prime. $\Gamma_0^*(p)$ has only ∞ as a cusp. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$, $\gamma W_p = \left(\begin{smallmatrix} b\sqrt{p} & a/\sqrt{p} \\ d\sqrt{p} & c/\sqrt{p} \end{smallmatrix}\right)$. Thus we have only to consider about the pairs (c,d) and $(d\sqrt{p}, c/\sqrt{p})$. Then we have

$$E_{k,p}^*(z) := \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p|c}} (cz+d)^{-k} + \frac{p^{k/2}}{2} \sum_{\substack{(c,d)=1 \\ p|d}} (c(pz)+d)^{-k} \tag{54}$$

as the Eisenstein series associated with $\Gamma_0^*(p)$ for the cusp ∞. Furhtermore, we have $E_{k,p}^*(z) = B_{k,p}(z) + p^{k/2}C_{k,p}(pz)$. By the equations (23) and (24), we have

$$E_{k,p}^*(z) = \frac{1}{p^{k/2}+1} \left(p^{k/2} E_k(pz) + E_k(z) \right). \tag{55}$$

We can use each of the expression as a deffinition.

5.2. $\Gamma_0^*(2)$ (Proof of Theorem1).

5.2.1. *Preliminaries.* In the previous subsection, we have two conditions for a fundamental domain for $\Gamma_0^*(2)$:

$$(C_0) \qquad |z \pm 1/2| > 1/2,\ -1/2 < Re(z) < 1/2.$$

$$(C_p) \qquad |z - n| > 1/\sqrt{2},\ -1/2 < Re(z) < 1/2 \quad \text{for } n = 0,\ \pm 1.$$

By the condition $|z| > 1/\sqrt{2}$ and $-1/2 < Re(z) < 1/2$, we have $Im(z) > 1/2$. For $z \in \mathbb{H}$ such that $|z \pm 1/2| = 1/2$ or $|z \pm 1| = 1/\sqrt{2}$, and $-1/2 < Re(z) < 1/2$, we have $Im(z) < 1/2$. Thus

$$|z| > 1/\sqrt{2},\ -1/2 < Re(z) < 1/2$$

is a sufficient condition for (C_0) and (C_p). Furthermore, we have the following transformation:

$$W_2 : e^{i\theta}/\sqrt{2} \mapsto e^{i(\pi-\theta)}/\sqrt{2}$$

Then we have $V_{\Gamma_0^*(2)} = \{e^{3\pi/4}/\sqrt{2}\}$ (Theorem2.1).

Now, we have a fundamental domain for $\Gamma_0^*(2)$ as follows:

$$\mathbb{F}^*(2) := \left\{ |z| \geqslant 1/\sqrt{2},\ -\frac{1}{2} \leqslant Re(z) \leqslant 0 \right\} \bigcup \left\{ |z| > 1/\sqrt{2},\ 0 \leqslant Re(z) < \frac{1}{2} \right\}. \tag{56}$$

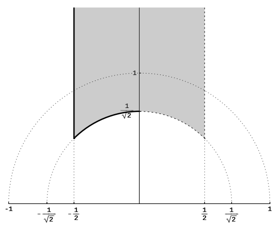


FIGURE 5. $\Gamma_0^*(2)$

5.2.2. *The function $F_{k,2}^*(\theta)$.* We give the next definition;

$$F_{k,2}^*(\theta) := e^{ik\theta/2} E_{k,2}^* \left(e^{i\theta}/\sqrt{2} \right). \tag{57}$$

Before proving Theorem1, we consider an expansion of $F_{k,2}^*(\theta)$.

In the definition of $E_{k,2}^*(z)$ (*cf.* (54)), when $2 \mid c$, then we can write $c = 2c'$ for $\exists c' \in \mathbb{Z}$, and have $2 \nmid d$. Also, when $2 \mid d$, then we have $2 \nmid c$ and $d = 2d'$ for $\exists d' \in \mathbb{Z}$. Then

$$\begin{aligned} F_{k,2}^*(z) &= \frac{e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1 \\ 2|c}} (ce^{i\theta}/\sqrt{2} + d)^{-k} + \frac{2^{k/2} e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1 \\ 2|d}} (c(2e^{i\theta}/\sqrt{2}) + d)^{-k} \\ &= \frac{e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1 \\ 2\nmid d}} (\sqrt{2}c' e^{i\theta} + d)^{-k} + \frac{2^{k/2} e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1 \\ 2\nmid c}} (\sqrt{2}ce^{i\theta} + 2d')^{-k} \\ &= \frac{1}{2} \sum_{\substack{(c,d)=1 \\ 2\nmid d}} (de^{-i\theta/2} + \sqrt{2}c' e^{i\theta/2})^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ 2\nmid c}} (ce^{i\theta/2} + \sqrt{2}d' e^{-i\theta/2})^{-k} \end{aligned}$$

Thus we can write as follows;

$$F_{k,2}^*(\theta) = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ c:odd}} (ce^{i\theta/2} + \sqrt{2}de^{-i\theta/2})^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ c:odd}} (ce^{-i\theta/2} + \sqrt{2}de^{i\theta/2})^{-k}. \tag{58}$$

Hence we use this expression as a definition.

In the last part of this section, we compare the two series in this expression. Note that for any pair (c, d), $(ce^{i\theta/2} + \sqrt{2}de^{-i\theta/2})^{-k}$ and $(ce^{-i\theta/2} + \sqrt{2}de^{i\theta/2})^{-k}$ are conjugates of each other. The next lemma follows.

Lemma 5.1. *$F^*_{k,2}(\theta)$ is real, for $\forall\theta \in \mathbb{R}$.*

Define $m_2(k) := \lfloor \frac{k}{8} - \frac{t}{4} \rfloor$, where $t = 0, 2$ is chosen so that $t \equiv k \pmod 4$, and $\lfloor n \rfloor$ is the largest integer not more than n.

Remark 5.1. *By Lemma5.1, $F^*_{k,2}(\theta)$ is real. Also, it can easily be shown that $\left[\frac{k}{4}, \frac{3k}{8}\right]$ has $m_2(k) + 1$ integers. Furthermore, for any integer $m \in \left[\frac{k}{4}, \frac{3k}{8}\right]$, if m is even or odd, then $F^*_{k,2}(2m\pi/k)$ is positive or negative, respectively. (cf. Remark3.1)*

5.2.3. *Application of the RSD Method.* We will apply the method of F. K. C. Rankin and H. P. F. Swinnerton-Dyer (RSD Method) to the Eisenstein series associated with $\Gamma_0^*(2)$. We note that $N := c^2+d^2$.

Firstly, we consider the case $N = 1$. Because c is odd, there are two cases, $(c, d) = (1, 0)$ and $(c, d) = (-1, 0)$. Then

$$\frac{1}{2}\left((e^{i\theta/2})^{-k} + (-e^{i\theta/2})^{-k} + (e^{-i\theta/2})^{-k} + (-e^{-i\theta/2})^{-k}\right) = 2\cos(k\theta/2).$$

So we can write;

$$F^*_{k,2}(\theta) = 2\cos(k\theta/2) + R^*_2, \tag{59}$$

where

$$R^*_2 = \frac{1}{2}\sum_{\substack{(c,d)=1\\ c:odd\\ N>1}} (ce^{i\theta/2} + \sqrt{2}de^{-i\theta/2})^{-k} + \frac{1}{2}\sum_{\substack{(c,d)=1\\ c:odd\\ N>1}} (ce^{-i\theta/2} + \sqrt{2}de^{i\theta/2})^{-k}.$$

Now,

$$\begin{aligned}|R^*_2| &\leqslant \frac{1}{2}\sum_{\substack{(c,d)=1\\ c:odd\\ N>1}} |ce^{i\theta/2} + \sqrt{2}de^{-i\theta/2}|^{-k} + \frac{1}{2}\sum_{\substack{(c,d)=1\\ c:odd\\ N>1}} |ce^{-i\theta/2} + \sqrt{2}de^{i\theta/2}|^{-k}\\ &= \sum_{\substack{(c,d)=1\\ c:odd\\ N>1}} |ce^{i\theta/2} + \sqrt{2}de^{-i\theta/2}|^{-k}.\end{aligned}$$

Let $v_k(c, d, \theta) := |ce^{i\theta/2} + \sqrt{2}de^{-i\theta/2}|^{-k}$, then

$$\begin{aligned}v_k(c, d, \theta) &= \left(|ce^{i\theta/2} + \sqrt{2}de^{-i\theta/2}|^{-2}\right)^{k/2}\\ &= \left(\frac{1}{c^2 + 2d^2 + 2\sqrt{2}cd\cos\theta}\right)^{k/2},\end{aligned}$$

and $v_k(c, d, \theta) = v_k(-c, -d, \theta)$.

Now we will consider the next three cases, namely $N = 2, 5$, and $N \geqslant 10$. Considering $\theta \in [\pi/2, 3\pi/4]$, we have the following:
When $N = 2$,

$$\begin{aligned}v_k(1, 1, \theta) &= \left(\frac{1}{3 + 2\sqrt{2}\cos\theta}\right)^{k/2} \leqslant 1,\\ v_k(1, -1, \theta) &= \left(\frac{1}{3 - 2\sqrt{2}\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{3}\right)^{k/2}.\end{aligned}$$

When $N = 5$,

$$\begin{aligned}v_k(1, 2, \theta) &= \left(\frac{1}{9 + 4\sqrt{2}\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{5}\right)^{k/2},\\ v_k(1, -2, \theta) &= \left(\frac{1}{9 - 4\sqrt{2}\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{3}\right)^{k}.\end{aligned}$$

When $N \geqslant 10$,

$$\begin{aligned}|ce^{i\theta/2} \pm \sqrt{2}de^{-i\theta/2}|^2 &\geqslant c^2 + 2d^2 - 2\sqrt{2}|cd||\cos\theta| \\ &= \frac{1}{3}\left(\sqrt{2}|c| - \sqrt{5}|d|\right)^2 + 2|cd|\left(\sqrt{10}/3 - \sqrt{2}|\cos\theta|\right) + \frac{1}{3}(c^2+d^2) \\ &\geqslant \frac{1}{3}(c^2+d^2) \quad = \frac{1}{3}N,\end{aligned}$$

and the rest of the question is about the number of terms with $c^2 + d^2 = N$. Because c is odd, $|c| = 1, 3, ..., 2N' - 1 \leqslant N^{1/2}$, so the number of $|c|$ is not more than $(N^{1/2}+1)/2$. Thus the number of terms with $c^2 + d^2 = N$ is not more than $2(N^{1/2}+1) \leqslant 3N^{1/2}$, for $N \geqslant 5$. Then

$$\begin{aligned}\sum_{\substack{(c,d)=1 \\ c:odd \\ N \geq 10}} |ce^{i\theta/2} + \sqrt{2}de^{-i\theta/2}|^{-k} &= \sum_{N=10}^{\infty} 3N^{1/2}\left(\frac{1}{3}N\right)^{-k/2} \leqslant 3\sqrt{3}\int_9^{\infty}\left(\frac{1}{3}x\right)^{(1-k)/2} dx \\ &= \frac{18\sqrt{3}}{k-3}\left(\frac{1}{3}\right)^{(k-3)/2} = \frac{162}{k-3}\left(\frac{1}{3}\right)^{k/2}.\end{aligned}$$

Thus

$$|R_2^*| \leqslant 2 + 2\left(\frac{1}{3}\right)^{k/2} + 2\left(\frac{1}{5}\right)^{k/2} + 2\left(\frac{1}{3}\right)^{k} + \frac{162}{k-3}\left(\frac{1}{3}\right)^{k/2}. \tag{60}$$

Recalling the previous section (RSD Method), we want to show that $|R_2^*| < 2$. But the right-hand side is greater than 2, so this bound is not good. The case when $(c,d) = \pm(1,1)$ gives us bound equal to 2. We will consider the expansion of the method in the following sections.

5.2.4. *Expansion of the RSD Method (1).* In the previous subsection, we could not get a good bound for $|R_2^*|$. The point was the case $(c,d) = \pm(1,1)$. Notice that "$v_k(1,1,\theta) = 1 \Leftrightarrow \theta = 3\pi/4$". Furthermore, "$v_k(1,1,\theta) < 1 \Leftrightarrow \theta < 3\pi/4$". So we can easily expect that we get a good bound for $\theta \in [\pi/2, 3\pi/4 - x]$ for small $x > 0$. But if $k = 8n$, we need $|R_2^*| < 2$ for $\theta = 3\pi/4$ in this method. We will consider the case when $k = 8n, \theta = 3\pi/4$ in the next section.

Let $k = 8n + s$ $(n = m_2(k),\ s = 4, 6, 0,\ \text{and } 10)$. We may assume that $k \geqslant 8$.

The first problem is how small x should be. We consider each of the cases $s = 4, 6, 0,$ *and* 10.

When $s = 4$, for $\pi/2 \leqslant \theta \leqslant 3\pi/4$, $(2n+1)\pi \leqslant k\theta/2$ $(= (4n+2)\theta) \leqslant (3n+1)\pi + \pi/2$. So the last integer point(*i.e.* ± 1) is $k\theta/2 = (3n+1)\pi$, then $\theta = \frac{3n+1}{4n+2}\pi = 3\pi/4 - \pi/k$. Similarly, when $s = 6$, *and* 10, the last integer points are $\theta = 3\pi/4 - \pi/2k$, $3\pi/4 - 3\pi/2k$, respectively. When $s = 0$, the second to the last integer point is $\theta = 3\pi/4 - \pi/k$.

Thus we need $x \leqslant \pi/2k$.

Lemma 5.2. *Let $k \geqslant 8$. For $\forall\theta \in [\pi/2, 3\pi/4 - x]$ $(x = \pi/2k)$, $|R_2^*| < 2$.*

Before proving the above lemma, we need the following preliminaries.

Proposition 5.1.

(1) *If $0 \leqslant x \leqslant \pi/2$, then $\sin x \geqslant 1 - \cos x$.*
(2) *If $0 \leqslant x \leqslant \pi/16$, then $1 - \cos x \geqslant \frac{31}{64}x^2$.*

This proposition is easily proved. The number $\frac{31}{64}$ in Proposition 5.1 (2) is near to and less than $\frac{1}{2}\cos(\pi/16)$. We use the previous proposition for the following proof:

Proof of Lemma 5.2. Let $k \geqslant 8$ and $x = \pi/2k$, then $0 \leqslant x \leqslant \pi/16$.

$$\begin{aligned}|e^{i\theta/2} + \sqrt{2}e^{-i\theta/2}|^2 &= 3 + 2\sqrt{2}\cos\theta \geqslant 3 + 2\sqrt{2}\cos(3\pi/4 - x) \\ &= 1 + 2(1 - \cos x) + 2\sin x \\ &\geqslant 1 + 4(1 - \cos x) \quad (Prop. 5.1(1)) \\ &\geqslant 1 + \frac{31}{16}x^2. \quad (Prop. 5.1(2))\end{aligned}$$

$$\begin{aligned}|e^{i\theta/2}+\sqrt{2}e^{-i\theta/2}|^k &\geqslant \left(1+\frac{31}{16}x^2\right)^{k/2}\\ &= 1+\frac{k}{2}\frac{31}{16}x^2+\binom{k/2}{2}\left(\frac{31}{16}\right)^2x^4+...\\ &\geqslant 1+\frac{k}{2}\frac{31}{16}x^2 \ \geqslant 1+\frac{31}{4}x^2 \ (k\geqslant 8).\end{aligned}$$

$$\begin{aligned}v_k(1,1,\theta) &\leqslant \frac{1}{1+(31/4)x^2} \ = 1-\frac{(31/4)x^2}{1+(31/4)x^2}\\ &\leqslant 1-\frac{(31/4)}{1+(31/4)(\pi/16)^2}x^2 \ = 1-\frac{31\times 256}{31\pi^2+1024}x^2.\end{aligned}$$

Thus

$$\begin{aligned}2v_k(1,1,\theta) &\leqslant 2-\frac{31\times 512}{31\pi^2+1024}x^2 \ = 2-\frac{31\times 512}{31\pi^2+1024}\left(\frac{\pi}{2k}\right)^2\\ &= 2-\frac{31\times 128\pi^2}{31\pi^2+1024}\frac{1}{k^2} \ \leqslant 2-\frac{265}{9}\frac{1}{k^2}.\end{aligned}$$

In inequality(60), replace 2 with the bound $2-\frac{265}{9}\frac{1}{k^2}$. Then

$$|R_2^*| \leqslant 2-\frac{265}{9}\frac{1}{k^2}+2\left(\frac{1}{3}\right)^{k/2}+2\left(\frac{1}{5}\right)^{k/2}+2\left(\frac{1}{3}\right)^{k}+\frac{162}{k-3}\left(\frac{1}{3}\right)^{k/2}.$$

Furthermore,

$$\begin{aligned}&2\left(\frac{1}{3}\right)^{k/2}+2\left(\frac{1}{5}\right)^{k/2}+2\left(\frac{1}{3}\right)^{k}+\frac{162}{k-3}\left(\frac{1}{3}\right)^{k/2}\\ &\qquad = \left(2+2\left(\frac{3}{5}\right)^{k/2}+2\left(\frac{1}{3}\right)^{k/2}+\frac{162}{k-3}\right)\left(\frac{1}{3}\right)^{k/2} \leqslant 35\left(\frac{1}{3}\right)^{k/2} \qquad (k\geqslant 8).\end{aligned}$$

Now if we can show that

$$35\left(\frac{1}{3}\right)^{k/2} < \frac{265}{9}\frac{1}{k^2} \quad \textit{or} \quad \frac{3^{k/2}}{35} > \frac{9}{265}k^2,$$

then the bound is less than 2. Then the proof will be complete.

Put $f(x) := (1/35)3^{x/2}-\frac{9}{265}x^2$, then $f'(x)=(\log 3/70)3^{x/2}-\frac{18}{265}x$, $f''(x)=((\log 3)^2/140)3^{x/2}-\frac{18}{265}$. Firstly, f'' is monotonically increasing for $x\geqslant 8$, and $f''(8)=0.63038...>0$, so $f''>0$ for $x\geqslant 8$. Secondly, f' is monotonically increasing for $x\geqslant 8$, and $f'(8)=0.72785...>0$, so $f'>0$ for $x\geqslant 8$. Finally, f is monotonically increasing for $x\geqslant 8$, and $f(8)=0.14070...>0$, so $f>0$ for $x\geqslant 8$. □

5.2.5. *Expansion of the RSD Method (2).* For the case "$k=8n, \theta=3\pi/4$", we need the next lemma.

Lemma 5.3. *Let k be an integer such that $k=8n$ for $\exists n\in\mathbb{N}$. If n is even, then $F_{k,2}^*(3\pi/4)>0$. On the other hand if n is odd, then $F_{k,2}^*(3\pi/4)<0$.*

If we can show this lemma, then we consequently show that for any integer $m\in\left[\frac{k}{4},\frac{3k}{8}\right]$, if m is even or odd, then $F_{k,2}^*(2m\pi/k)$ is positive or negative, respectively. (Remark 5.1)

Proof of Lemma 5.3. Let $k=8n$ $(n\geqslant 1)$. By the definition of $E_{k,2}^*(z), F_{k,2}^*(z)$ (*cf.* (54),(57)), we have

$$F_{k,2}^*(\theta)=\frac{e^{ik\theta/2}}{2^{k/2}+1}\left(2^{k/2}E_k\left(\sqrt{2}e^{i\theta}\right)+E_k\left(e^{i\theta}/\sqrt{2}\right)\right).$$

$$F_{k,2}^*(3\pi/4)=\frac{e^{i3(k/8)\pi}}{2^{k/2}+1}\left(2^{k/2}E_k(-1+i)+E_k\left(\frac{-1+i}{2}\right)\right).$$

By using the equations (14) and (15),

$$E_k(-1+i)=E_k(i),$$

$$E_k\left(\frac{-1+i}{2}\right) = E_k\left(-\frac{1}{1+i}\right) = (1+i)^k E_k(1+i) = 2^{k/2}E_k(i).$$

Then

$$\begin{aligned} F^*_{k,2}(3\pi/4) &= 2e^{i3(k/8)\pi}\frac{2^{k/2}}{2^{k/2}+1}E_k(i) \\ &= 2e^{i(k/8)\pi}\frac{2^{k/2}}{2^{k/2}+1}e^{ik(\pi/2)/2}E_k\left(e^{i(\pi/2)}\right) \\ &= 2e^{i(k/8)\pi}\frac{2^{k/2}}{2^{k/2}+1}F_k(\pi/2). \quad (cf.(38)) \end{aligned}$$

When $k = 8n$,

$$F^*_{8n,2}(3\pi/4) = 2e^{in\pi}\frac{2^{4n}}{2^{4n}+1}F_{8n}(\pi/2). \tag{61}$$

Here $\frac{2^{4n}}{2^{4n}+1} > 0$. By proposition 3.3, we have $F_{8n}(\pi/2) = e^{i2n\pi}E_{8n}(i) > 0$. So the sign($\pm$) of $F^*_{k,2}(3\pi/4)$ is that of $e^{in\pi}$. Thus the proof is complete. □

5.2.6. *Valence formula for* $\Gamma^*_0(2)$. By previous subsections, $E^*_{k,2}(z)$ has $m_2(k)$ zeros on A^*_2. In order to decide the locating of all zeros of $E^*_{k,2}(z)$, we need the valence formula for $\Gamma^*_0(2)$:

Proposition 5.2. *Let f be a modular function of weight k for $\Gamma^*_0(2)$, which is not identically zero. We have*

$$v_\infty(f) + \frac{1}{2}v_{i/\sqrt{2}}(f) + \frac{1}{4}v_{\rho_2}(f) + \sum_{\substack{p\in\Gamma^*_0(2)\backslash\mathbb{H} \\ p\neq i/\sqrt{2},\ \rho_2}} v_p(f) = \frac{k}{8}, \tag{62}$$

where $\rho_2 := e^{3\pi/4}/\sqrt{2}$.

The proof of this proposition is similar to Proposition 3.1 because the figure of fundamental domain of $\Gamma^*_0(2)$ is similar to that of $\mathrm{SL}_2(\mathbb{Z})$ (*cf.* Figure 1. and 5., and see [SE]). The angle of the arc around ρ_2 (BB' in Figure2.) tends to $\pi/4$ when radius of it tends to 0, thus the coefficient of $v_{\rho_2}(f)$ is 1/4. Furthermore, because the angle of the arc A^*_2 is $\pi/4$, the right-hand side is $k/8$.

If $k \equiv 4, 6,$ and $0 \pmod 8$, then $k/8 - m_2(k) < 1$. Thus all zeros of $E^*_{k,2}(z)$ are on $A^*_2 \cup \{i/\sqrt{2}, \rho_2\}$.

But if $k \equiv 2 \pmod 8$, we need another consideration because we have $k/8 - m_2(k) > 1$.

Recall that $E^*_{k,2}(z)$ is the *modular form* of weight k for $\Gamma^*_0(2)$. Any weight k modular form for $\Gamma^*_0(2)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \tag{63}$$

for every $z \in \mathbb{H}$, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma^*_0(2)$. When $z = i/\sqrt{2}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = W_2$, we have $E^*_{k,2}(i/\sqrt{2}) = i^k E^*_{k,2}(i/\sqrt{2})$. Because $k \not\equiv 0 \pmod 4$, $E^*_{k,2}(i/\sqrt{2}) = 0$. Thus we have $k/8 - m_2(k) - v_{i/\sqrt{2}}(E^*_{k,2})/2 < 1$.

In conclusion, for every even integer $k \geqslant 4$, all zeros of $E^*_{k,2}(z)$ is on $A^*_2 \cup \{i/\sqrt{2}, \rho_2\}$.

5.2.7. *The space of modular forms.* Let $M^*_{k,2}$ be the space of modular forms for $\Gamma^*_0(2)$ of weight k, and let $M^{*0}_{k,2}$ be the space of cusp forms for $\Gamma^*_0(2)$ of weight k. When we consider the map $M^*_{k,2} \ni f \mapsto f(\infty) \in \mathbb{C}$, the kernel of the map is $M^{*0}_{k,2}$. So $\dim(M^*_{k,2}/M^{*0}_{k,2}) \leqslant 1$, and $M^*_{k,2} = \mathbb{C}E^*_{k,2} \oplus M^{*0}_{k,2}$. Recall that $\Delta_2 = \eta^8(z)\eta^8(2z)$. We have following theorem:

Theorem 5.1. *Let k be an even integer.*

(1) *For $k < 0$ and $k = 2$, $M^*_{k,2} = 0$.*
(2) *For $k = 0, 4, 6,$ and 10, we have $M^{*0}_{k,2} = 0$, and* $\dim(M^*_{k,2}) = 1$ *with a base $E^*_{k,2}$.*
(3) $M^{*0}_{k,2} = \Delta_2 M^*_{k-8,2}$.

Proof. Let f be a nonzero function of $M_{k,2}^*$, then $v_p(f) \geqslant 0$ for every $p \in \mathbb{H}$. By the valence formula for $\Gamma_0^*(2)$(Proposition5.2), we have $k \geqslant 0$.

For every $f \in M_{k,2}^{*0}$, we have $v_p(f/\Delta_2) \geqslant 0$ for every $p \in \mathbb{H}$. Thus $f/\Delta_2 \in M_{k-8,2}^*$. This proves (3).

By (3) and $M_{k,2}^* = 0$ for $k < 0$, we have $M_{k,2}^{*0} = 0$ for $k = 0, 2, 4$, and 6. We also have $\dim(M_{k,2}^*) = 1$ for $k = 0, 4, 6$ with a base $E_{k,2}^*$.

Let f be a nonzero function of $M_{2,2}^*$. By the valence formula, we have $v_{\rho_2}(f) = 1$ and $v_p(f) = 0$ for every $p \neq \rho_2$. Because $f^3 \in M_{6,2}^*$, $f^3 = cE_{6,2}^*$ for some $c \in \mathbb{C}$. So $f^3(i/\sqrt{2}) = cE_{6,2}^*(i/\sqrt{2}) = 0$. This contradicts $v_{i/\sqrt{2}}(f) = 0$. Thus $M_{2,2}^* = 0$. This proves (1).

Moreover, by (3), $M_{10,2}^{*0} = 0$ and $\dim(M_{10,2}^*) = 1$ with a base $E_{10,2}^*$. This makes the proof of this theorem complete. □

Furthermore, for a non-negative integer k, $\dim(M_{k,2}^*) = \lfloor k/8 \rfloor$ if $k \equiv 2 \pmod 8$, and $\dim(M_{k,2}^*) = \lfloor k/8 \rfloor + 1$ if $k \not\equiv 2 \pmod 8$.

Let k be an even integer such that $k \geqslant 4$. Write $n := \dim(M_{k,2}^*) - 1$, then $k - 8n = 0, 4, 6$, or 10. Because $E_{k,2}^* - E_{k-8n,2}^*(E_{4,2}^*)^{2n} \in M_{k,2}^{*0}$, we have $M_{k,2}^* = \mathbb{C}E_{k-8n,2}^*(E_{4,2}^*)^{2n} \oplus M_{k,2}^{*0}$. Then

$$
\begin{aligned}
M_{k,2}^* &= \mathbb{C}E_{k-8n,2}^*(E_{4,2}^*)^{2n} \oplus \Delta_2 M_{k-8,2}^* \\
&= \mathbb{C}E_{k-8n,2}^*(E_{4,2}^*)^{2n} \oplus \mathbb{C}E_{k-8n,2}^*(E_{4,2}^*)^{2(n-1)}\Delta_2 \oplus \Delta_2^2 M_{k-16,2}^* \\
&\cdots \\
&= E_{k-8n,2}^*(\mathbb{C}(E_{4,2}^*)^{2n} \oplus \mathbb{C}(E_{4,2}^*)^{2(n-1)}\Delta_2 \oplus \cdots \oplus \mathbb{C}\Delta_2^n)
\end{aligned}
$$

Thus, for every $p \in \mathbb{H}$ and for every $f \in M_{k,2}^*$, $v_p(f) \geqslant v_p(E_{k-8n,2}^*)$.

By the valence formla and equation(63), we have $v_{i/\sqrt{2}}(E_{4,2}^*) = 0$, $v_{\rho_2}(E_{4,2}^*) = 2$, and $v_{i/\sqrt{2}}(E_{6,2}^*) = v_{\rho_2}(E_{6,2}^*) = 1$. For $k = 10$, we have $E_{4,2}^* E_{6,2}^* \in M_{10,2}^* = \mathbb{C}E_{10,2}^*$. Thus $E_{10,2}^* = E_{4,2}^* E_{6,2}^*$, and $v_{i/\sqrt{2}}(E_{10,2}^*) = 1$, $v_{\rho_2}(E_{10,2}^*) = 3$.

In conclusion, the next proposision follows:

Proposition 5.3. *Let $k \geqslant 4$ be an even integer. For every $f \in M_{k,2}^*$, we have*

$$
\begin{aligned}
& v_{i/\sqrt{2}}(f) \geqslant s_k \quad (s_k = 0, 1 \text{ such that } 2s_k \equiv k \pmod 4), \\
& v_{\rho_2}(f) \geqslant t_k \quad (t_k = 0, 1, 2, 3 \text{ such that } -2t_k \equiv k \pmod 8)).
\end{aligned} \tag{64}
$$

In paticular, if f is a constant multiple of $E_{k,2}^*$, then the equalities hold.

Remark 5.2. *Every modular form for $\Gamma_0^*(2)$ is generated by*

$$E_{4,2}^*, \quad E_{6,2}^*, \quad \text{and} \quad \Delta_2. \tag{65}$$

5.3. $\Gamma_0^*(3)$ **(Proof of Theorem2).**

5.3.1. *Preliminaries.* In the subsection5.1, we have two conditions for a fundamental domain for $\Gamma_0^*(3)$:

$$(C_0) \qquad |z \pm 1/3| > 1/3,\ -1/2 < Re(z) < 1/2.$$

$$(C_p) \qquad |z - n| > 1/\sqrt{3},\ -1/2 < Re(z) < 1/2 \quad \text{for } n = 0,\ \pm 1.$$

It is easy to show that

$$|z| > 1/\sqrt{3},\ -1/2 < Re(z) < 1/2$$

is a sufficient condition for (C_0) and (C_p). Furthermore, we have the following transformation:

$$W_3 : e^{i\theta}/\sqrt{3} \mapsto e^{i(\pi-\theta)}/\sqrt{3}$$

Then we have $V_{\Gamma_0^*(3)} = \{e^{5\pi/6}/\sqrt{3}\}$ (Theorem2.1).

Now, we have a fundamental domain for $\Gamma_0^*(3)$ as follows:

$$\mathbb{F}^*(3) := \left\{|z| \geqslant 1/\sqrt{3},\ -\frac{1}{2} \leqslant Re(z) \leqslant 0\right\} \bigcup \left\{|z| > 1/\sqrt{3},\ 0 \leqslant Re(z) < \frac{1}{2}\right\}. \tag{66}$$

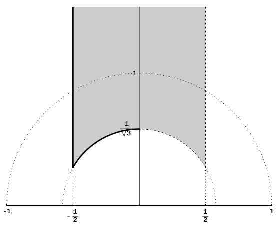


FIGURE 6. $\Gamma_0^*(3)$

5.3.2. *The function $F_{k,3}^*(\theta)$.* We give the next definition;

$$F_{k,3}^*(\theta) := e^{ik\theta/2} E_{k,3}^*\left(e^{i\theta}/\sqrt{3}\right). \tag{67}$$

Before proving Theorem1, we consider an expansion of $F_{k,3}^*(\theta)$.

In the definition of $E_{k,3}^*(z)$ (*cf.* (54)), when $3 \mid c$, then we can write $c = 3c'$ for $\exists c' \in \mathbb{Z}$, and have $3 \nmid d$. Also, when $3 \mid d$, then we have $3 \nmid c$ and $d = 3\exists d'$ for $\exists d' \in \mathbb{Z}$. Then

$$\begin{aligned} F_{k,2}^*(z) &= \frac{e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1 \\ 3|c}} (ce^{i\theta}/\sqrt{3} + d)^{-k} + \frac{3^{k/2}e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1 \\ 3|d}} (c(3e^{i\theta}/\sqrt{3}) + d)^{-k} \\ &= \frac{1}{2} \sum_{\substack{(c,d)=1 \\ 3\nmid d}} (de^{-i\theta/2} + \sqrt{3}c'e^{i\theta/2})^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ 3\nmid c}} (ce^{i\theta/2} + \sqrt{3}d'e^{-i\theta/2})^{-k} \end{aligned}$$

Thus we can write as follows;

$$F_{k,3}^*(\theta) = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ 3\nmid c}} (ce^{i\theta/2} + \sqrt{3}de^{-i\theta/2})^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ 3\nmid c}} (ce^{-i\theta/2} + \sqrt{3}de^{i\theta/2})^{-k}. \tag{68}$$

Hence we use this expression as a definition.

Note that for any pair (c, d), $(ce^{i\theta/2} + \sqrt{3}de^{-i\theta/2})^{-k}$ and $(ce^{-i\theta/2} + \sqrt{3}de^{i\theta/2})^{-k}$ are conjugates of each other. The next lemma follows.

Lemma 5.4. *$F_{k,3}^*(\theta)$ is real, for $\forall \theta \in \mathbb{R}$.*

Define $m_3(k) := \left\lfloor \frac{k}{6} - \frac{t}{4} \right\rfloor$, where $t = 0, 2$ is chosen so that $t \equiv k \pmod 4$.

Remark 5.3. *By Lemma5.4, $F_{k,3}^*(\theta)$ is real. Also, it can easily be shown that $\left[\frac{k}{4}, \frac{5k}{12}\right]$ has $m_3(k) + 1$ integers. Furthermore, for any integer $m \in \left[\frac{k}{4}, \frac{5k}{12}\right]$, if m is even or odd, then $F_{k,3}^*(2m\pi/k)$ is positive or negative, respectively.* (*cf.* *Remark3.1*)

5.3.3. *Application of the RSD Method.* We note that $N := c^2 + d^2$.

Firstly, we consider the case $N = 1$. Then we can write;

$$F_{k,3}^*(\theta) = 2\cos(k\theta/2) + R_3^*, \tag{69}$$

where

$$R_3^* = \frac{1}{2}\sum_{\substack{(c,d)=1\\ 3\nmid c\\ N>1}} (ce^{i\theta/2} + \sqrt{3}de^{-i\theta/2})^{-k} + \frac{1}{2}\sum_{\substack{(c,d)=1\\ 3\nmid c\\ N>1}} (ce^{-i\theta/2} + \sqrt{3}de^{i\theta/2})^{-k}.$$

Now,

$$|R_3^*| \leqslant \sum_{\substack{(c,d)=1\\ 3\nmid c\\ N>1}} |ce^{i\theta/2} + \sqrt{3}de^{-i\theta/2}|^{-k}.$$

Let $v_k(c,d,\theta) := |ce^{i\theta/2} + \sqrt{3}de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\theta) = \left(\frac{1}{c^2+3d^2+2\sqrt{3}cd\cos\theta}\right)^{k/2}$, and $v_k(c,d,\theta) = v_k(-c,-d,\theta)$.

Now we will consider the next cases, namely $N = 2, 5, 10, 13, 17$, and $N \geqslant 25$. Considering $\theta \in [\pi/2, 5\pi/6]$, we have the following:

When $N = 2$,

$$v_k(1,1,\theta) = \left(\frac{1}{4+2\sqrt{3}\cos\theta}\right)^{k/2} \leqslant 1,$$
$$v_k(1,-1,\theta) = \left(\frac{1}{4-2\sqrt{3}\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{2}\right)^{k}.$$

When $N = 5$,

$$v_k(1,2,\theta) = \left(\frac{1}{13+4\sqrt{3}\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{7}\right)^{k/2},$$
$$v_k(1,-2,\theta) = \left(\frac{1}{13-4\sqrt{3}\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{13}\right)^{k/2},$$
$$v_k(2,1,\theta) = \left(\frac{1}{7+4\sqrt{3}\cos\theta}\right)^{k/2} \leqslant 1,$$
$$v_k(2,-1,\theta) = \left(\frac{1}{7-4\sqrt{3}\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{7}\right)^{k/2}.$$

When $N = 10$,

$$v_k(1,3,\theta) = \left(\frac{1}{28+6\sqrt{3}\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{19}\right)^{k/2},$$
$$v_k(1,-3,\theta) = \left(\frac{1}{28-6\sqrt{3}\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{28}\right)^{k/2}.$$

When $N = 13$,

$$v_k(2,3,\theta) = \left(\frac{1}{31+12\sqrt{3}\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{13}\right)^{k/2},$$
$$v_k(2,-3,\theta) = \left(\frac{1}{31-12\sqrt{3}\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{31}\right)^{k/2}.$$

When $N=17$,

$$\begin{aligned} v_k(1,4,\theta) &= \left(\frac{1}{49+8\sqrt{3}\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{37}\right)^{k/2}, \\ v_k(1,-4,\theta) &= \left(\frac{1}{49-8\sqrt{3}\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{7}\right)^{k}, \\ v_k(4,1,\theta) &= \left(\frac{1}{19+8\sqrt{3}\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{7}\right)^{k/2}, \\ v_k(4,-1,\theta) &= \left(\frac{1}{19-8\sqrt{3}\cos\theta}\right)^{k/2} \leqslant \left(\frac{1}{19}\right)^{k/2}. \end{aligned}$$

When $N \geqslant 25$,

$$\begin{aligned} |ce^{i\theta/2} \pm \sqrt{3}de^{-i\theta/2}|^2 &\geqslant c^2+3d^2-2\sqrt{3}|cd||\cos\theta| \\ &= \frac{1}{6}\left(\sqrt{5}|c|-\sqrt{17}|d|\right)^2 + 2|cd|\left(\sqrt{85}/6-\sqrt{3}|\cos\theta|\right) + \frac{1}{6}(c^2+d^2) \\ &\geqslant \frac{1}{6}N, \end{aligned}$$

and the rest of the question is about the number of terms with $c^2+d^2=N$. Because $3 \nmid c$, $|c| = 1,2,4,5,7,... \leqslant N^{1/2}$, so the number of $|c|$ is not more than $(2/3)N^{1/2}+1$. Thus the number of terms with $c^2+d^2=N$ is not more than $4((2/3)N^{1/2}+1) \leqslant (11/3)N^{1/2}$, for $N \geqslant 16$. Then

$$\begin{aligned} \sum_{\substack{(c,d)=1 \\ 3\nmid c \\ N\geq 25}} |ce^{i\theta/2}+\sqrt{3}de^{-i\theta/2}|^{-k} &= \sum_{N=25}^{\infty} \frac{11}{3}N^{1/2}\left(\frac{1}{6}N\right)^{-k/2} = \frac{11\sqrt{6}}{3}\sum_{N=25}^{\infty}\left(\frac{1}{6}N\right)^{(1-k)/2} \\ &\leqslant \frac{11\sqrt{6}}{3}\int_{24}^{\infty}\left(\frac{1}{6}x\right)^{(1-k)/2} dx = \frac{352\sqrt{6}}{k-3}\left(\frac{1}{2}\right)^k. \end{aligned}$$

Thus

$$\begin{aligned} (70) \quad |R_3^*| \leqslant 4 &+ 2\left(\frac{1}{2}\right)^k + 6\left(\frac{1}{7}\right)^{k/2} + 4\left(\frac{1}{13}\right)^{k/2} + 4\left(\frac{1}{19}\right)^{k/2} \\ &+ 2\left(\frac{1}{28}\right)^{k/2} + 2\left(\frac{1}{31}\right)^{k/2} + 2\left(\frac{1}{37}\right)^{k/2} + 2\left(\frac{1}{7}\right)^k + \frac{352\sqrt{6}}{k-3}\left(\frac{1}{2}\right)^k. \end{aligned}$$

Recalling the "RSD Method" subsection, we want to show that $|R_3^*| < 2$. But the right-hand side is much greater than 2, so this bound is not good. The cases $(c,d) = \pm(1,1)$, $\pm(2,1)$ give us bound equal to 4. We will consider the expansion of the method in the following sections.

5.3.4. *Expansion of the RSD Method (1).* In the previous subsection, we could not get a good bound for $|R_3^*|$. The points were the cases $(c,d) = \pm(1,1)$, $\pm(2,1)$. Notice that "$v_k(1,1,\theta) < 1 \Leftrightarrow \theta < 5\pi/6$", and "$v_k(2,1,\theta) < 1 \Leftrightarrow \theta < 5\pi/6$". So we can easily expect that we get a good bound for $\theta \in [\pi/2, 5\pi/6 - x]$ for small $x > 0$. But if $k = 12n$, we need $|R_3^*| < 2$ for $\theta = 5\pi/6$. We will consider the case when $k = 12n$, $\theta = 5\pi/6$ in the next subsection.

How small should x be? Let $k = 12m_3(k) + s$ ($s = 4, 6, 8, 10, 0,$ and 14). We may assume that $k \geqslant 8$. When $s = 4, 6, 8, 10$ *and* 14, the last integer points are $\theta = 5\pi/6 - 4\pi/3k$, $5\pi/6 - \pi/k$, $5\pi/6 - 2\pi/3k$, $5\pi/6 - \pi/3k$, and $5\pi/6 - 5\pi/3k$, respectively. When $s = 0$, the second to the last integer point is $\theta = 5\pi/6 - 2\pi/k$. Thus we need $x \leqslant \pi/3k$.

Lemma 5.5. *Let $k \geqslant 8$. For $\forall\theta \in [\pi/2, 5\pi/6 - x]$ $(x = \pi/3k)$, $|R_3^*| < 2$.*

Proof. Let $k \geqslant 8$ and $x = \pi/3k$, then $0 \leqslant x \leqslant \pi/24$.

We try to prove Lemma 5.5 in a similar way to the proof of Lemma 5.2. Firstly, we have

$$\begin{aligned}2\left(\frac{1}{2}\right)^k+6\left(\frac{1}{7}\right)^{k/2}+4\left(\frac{1}{13}\right)^{k/2}+4\left(\frac{1}{19}\right)^{k/2}&\\ +2\left(\frac{1}{28}\right)^{k/2}+2\left(\frac{1}{31}\right)^{k/2}+2\left(\frac{1}{37}\right)^{k/2}+2\left(\frac{1}{7}\right)^{k}+\frac{352\sqrt{6}}{k-3}\left(\frac{1}{2}\right)^k&\\ \leqslant 176\left(\frac{1}{2}\right)^k. \quad (k\geqslant 8)&\end{aligned}$$

Here, we want to prove

$$|R_3^*|\leqslant 2-a_1\frac{\pi^2}{9k^2}+176\left(\frac{1}{2}\right)^k<2$$

for some positive integer a_1. Then we have to show

$$176\left(\frac{1}{2}\right)^k<\frac{a_1\pi^2}{9}\frac{1}{k^2}\quad \text{or}\quad \frac{2^k}{176}>\frac{9}{a_1\pi^2}k^2\quad \text{for } \forall k\geqslant 8.$$

Put $f(x):=(1/176)2^x-\frac{9}{a_1\pi^2}x^2$, then $f'(x)=(\log 2/176)2^x-\frac{18}{a_1\pi^2}x$, $f''(x)=((\log 2)^2/176)2^x-\frac{18}{a_1\pi^2}$. If we can show $f(x)>0$ for $x\geqslant 8$, then we can prove the above bound. f'' is monotonically increasing for $x\geqslant 8$, and

$$\begin{aligned}f(8)>0&\Leftrightarrow a_1>\frac{99}{\pi^2}\cdot 4,\\ f'(8)>0&\Leftrightarrow a_1>\frac{99}{\pi^2}\cdot\frac{1}{\log 2},\\ f''(8)>0&\Leftrightarrow a_1>\frac{99}{\pi^2}\cdot\frac{1}{8(\log 2)^2}.\end{aligned}$$

Thus we need $a_1>\frac{99}{\pi^2}\cdot 4$, then we can define $a_1:=\frac{321}{8}$. Now, we have only to prove

$$v_k(1,1,\theta)+v_k(2,1,\theta)\leqslant 1-\frac{321}{16}x^2\quad \text{for } \forall\theta\in[\pi/2,5\pi/6-x]\ (x=\pi/3k).$$

In addition, because we have $v_k(1,1,\theta)=(1+(3+2\sqrt{3}\cos\theta))^{-k/2}$ and $v_k(2,1,\theta)=(1+2(3+2\sqrt{3}\cos\theta))^{-k/2}$, we expect the following bounds:

$$\begin{aligned}v_k(1,1,\theta)&\leqslant\frac{2}{3}-\frac{107}{8}x^2,\\ v_k(2,1,\theta)&\leqslant\frac{1}{3}-\frac{107}{16}x^2.\end{aligned}$$

To prove the former bound, we consider the following sufficient conditions:

$$\begin{aligned}(4+2\sqrt{3}\cos\theta)^{k/2}&\geqslant\frac{3}{2}+a_2x^2\\ 4+2\sqrt{3}\cos\theta&\geqslant a_3+a_4x^2\end{aligned}$$

For the number a_2, we want to show

$$\frac{1}{3/2+a_2x^2}\leqslant\frac{2}{3}-\frac{107}{8}x^2.$$

Then

$$\begin{aligned}\frac{2}{3}-\frac{1}{3/2+a_2x^2}&\geqslant\frac{4a_2}{3(3+2a_2(\pi/3k)^2)}x^2\geqslant\frac{107}{8}x^2,\\ a_2&\geqslant\frac{9\cdot 107}{32-6\cdot 107\cdot(\pi/24)^2}\quad \text{for } k\geqslant 8.\end{aligned}$$

Thus we can define $a_2:=\frac{64\times 7\times 13}{127}$.

For the numbers a_3 and a_4, we want to show

$$(a_3+a_4x^2)^{\frac{k}{2}}\geqslant\frac{3}{2}+\frac{64\times 7\times 13}{127}x^2.$$

Then

$$(a_3+a_4x^2)^{\frac{k}{2}}\geqslant a_3^{\frac{k}{2}}+\frac{k}{2}\cdot a_3^{\frac{k}{2}-1}\cdot a_4x^2\geqslant\frac{3}{2}+\frac{64\times 7\times 13}{127}x^2.$$

Thus we can define $a_3 := \left(\frac{3}{2}\right)^{2/k}$ and $a_4 := \frac{256\times 7\times 13}{3\times 127k}\left(\frac{3}{2}\right)^{2/k}$.

In conclusion, we have

$$\begin{aligned}4+2\sqrt{3}\cos\theta &\geqslant \left(\frac{3}{2}\right)^{2/k}\left(1+\frac{256\times 7\times 13}{3\times 127k}x^2\right)\\ &\Rightarrow (4+2\sqrt{3}\cos\theta)^{k/2}\geqslant \frac{3}{2}+\frac{64\times 7\times 13}{127}x^2\\ &\Rightarrow v_k(1,1,\theta)\leqslant \frac{2}{3}-\frac{107}{8}x^2.\end{aligned}$$

Similarly, we have

$$\begin{aligned}7+4\sqrt{3}\cos\theta &\geqslant 3^{2/k}\left(1+\frac{256\times 7\times 13}{3\times 127k}x^2\right)\\ &\Rightarrow (7+4\sqrt{3}\cos\theta)^{k/2}\geqslant 3+\frac{128\times 7\times 13}{127}x^2\\ &\Rightarrow v_k(2,1,\theta)\leqslant \frac{1}{3}-\frac{107}{16}x^2.\end{aligned}$$

Finally, we need the following preliminaries.

Proposition 5.4.

(1) *For* $k\geqslant 8$, $\left(\frac{3}{2}\right)^{2/k}\leqslant 1+\left(2\log\frac{3}{2}\right)\frac{1}{k}+\frac{1}{2}\left(2\log\frac{3}{2}\right)^2\left(\frac{3}{2}\right)^{2/k}\frac{1}{k^2}$.

(2) *For* $k\geqslant 8$, $3+2\sqrt{3}\cos\left(\frac{5\pi}{6}-\frac{\pi}{3k}\right)\geqslant \frac{\pi}{\sqrt{3}}\frac{1}{k}$.

Proof.

(1) For $k\geqslant 8$,

$$\begin{aligned}\left(\frac{3}{2}\right)^{2/k} &= \sum_{n=0}^{\infty}\frac{(2\log 3/2)^n}{n!}\frac{1}{k^n}\\ &= 1+(2\log 3/2)\frac{1}{k}+\sum_{n=2}^{\infty}\frac{(2\log 3/2)^n}{n!}\frac{1}{k^n}\\ &= 1+(2\log 3/2)\frac{1}{k}+\sum_{n=0}^{\infty}\frac{(2\log 3/2)^{n+2}}{(n+2)!}\frac{1}{k^{n+2}}\\ &= 1+(2\log 3/2)\frac{1}{k}+\frac{1}{2}(2\log 3/2)^2\frac{1}{k^2}\sum_{n=2}^{\infty}\frac{2}{(n+1)(n+2)}\frac{(2\log 3/2)^n}{n!}\frac{1}{k^n}\\ &\leqslant 1+(2\log 3/2)\frac{1}{k}+\frac{1}{2}(2\log 3/2)^2\frac{1}{k^2}\sum_{n=2}^{\infty}\frac{(2\log 3/2)^n}{n!}\frac{1}{k^n}\\ &= 1+\left(2\log\frac{3}{2}\right)\frac{1}{k}+\frac{1}{2}\left(2\log\frac{3}{2}\right)^2\left(\frac{3}{2}\right)^{2/k}\frac{1}{k^2}.\end{aligned}$$

(2) Let $k\geqslant 8$, and put

$$g(k) := 3+2\sqrt{3}\cos\left(\frac{5\pi}{6}-\frac{\pi}{3k}\right)-\frac{\pi}{\sqrt{3}}\frac{1}{k}.$$

Then $g'(k)=\frac{\pi}{\sqrt{3}}\frac{1}{k^2}\left(1-2\sin\left(\frac{5\pi}{6}-\frac{\pi}{3k}\right)\right)\leqslant 0$ (by $k\geqslant 8$). Thus $g(k)$ is monotonically decreasing, and $\lim_{k\to\infty} g(k)=0$, so $g(k)\geqslant 0$.

□

Proposition 5.5.

(1) *For* $k\geqslant 8$, $3^{2/k}\leqslant 1+(2\log 3)\frac{1}{k}+\frac{1}{2}(2\log 3)^2 3^{2/k}\frac{1}{k^2}$.

(2) *For* $k\geqslant 8$, $6+4\sqrt{3}\cos\left(\frac{5\pi}{6}-\frac{\pi}{3k}\right)\geqslant \frac{2\pi}{\sqrt{3}}\frac{1}{k}$.

The proof of above proposition is similar to that of Proposition 5.4.

Then, write

$$f_1(k) := 4 + 2\sqrt{3}\cos\left(\frac{5\pi}{6} - \frac{\pi}{3k}\right) - \left(\frac{3}{2}\right)^{2/k}\left(1 + \frac{256\times 7\times 13\times\pi^2}{27\times 127}\frac{1}{k^3}\right).$$

If $k = 8$, then $f_1(8) = 0.00012876... > 0$. Next, if $k \geqslant 10$, then

$$\begin{aligned}
f_1(k) &= 1 + \left(3 + 2\sqrt{3}\cos\left(\frac{5\pi}{6} - \frac{\pi}{3k}\right)\right) - \left(\frac{3}{2}\right)^{2/k} \\
&\quad - \frac{256\times 7\times 13\times\pi^2}{27\times 127}\left(\frac{3}{2}\right)^{2/k}\frac{1}{k^3} \\
&\geqslant 1 + \frac{\pi}{\sqrt{3}}\frac{1}{k} - \left\{1 + \left(2\log\frac{3}{2}\right)\frac{1}{k} + \frac{1}{2}\left(2\log\frac{3}{2}\right)^2\left(\frac{3}{2}\right)^{2/k}\frac{1}{k^2}\right\} \\
&\quad - \frac{256\times 7\times 13\times\pi^2}{27\times 127}\left(\frac{3}{2}\right)^{2/k}\frac{1}{k^3} \quad \text{(by (1), (2))} \\
&= \frac{1}{k}\left\{\frac{\pi}{\sqrt{3}} - 2\log\frac{3}{2} - \frac{1}{2}\left(2\log\frac{3}{2}\right)^2\left(\frac{3}{2}\right)^{2/k}\frac{1}{k} - \frac{256\times 7\times 13\times\pi^2}{27\times 127}\left(\frac{3}{2}\right)^{2/k}\frac{1}{k^2}\right\} \\
&\geqslant \frac{1}{k}\left\{\frac{\pi}{\sqrt{3}} - 2\log\frac{3}{2} - \frac{1}{20}\left(2\log\frac{3}{2}\right)^2\left(\frac{3}{2}\right)^{1/5} - \frac{64\times 7\times 13\times\pi^2}{27\times 25\times 127}\left(\frac{3}{2}\right)^{1/5}\right\} \\
&\geqslant \frac{1}{k}\times 0.24004... \quad (k \geqslant 10) \quad \geqslant 0.
\end{aligned}$$

Similarly, we write

$$f_2(k) := 7 + 4\sqrt{3}\cos\left(\frac{5\pi}{6} - \frac{\pi}{3k}\right) - 3^{2/k}\left(1 + \frac{256\times 7\times 13\times\pi^2}{27\times 127}\frac{1}{k^3}\right).$$

If $k = 8$, then $f_2(8) = 0.015057... > 0$. Next, if $k \geqslant 10$, then

$$\begin{aligned}
f_2(k) &= 1 + \left(6 + 4\sqrt{3}\cos\left(\frac{5\pi}{6} - \frac{\pi}{3k}\right)\right) - 3^{2/k} \\
&\quad - \frac{256\times 7\times 13\times\pi^2}{27\times 127}3^{2/k}\frac{1}{k^3} \\
&\geqslant \frac{1}{k}\left\{\frac{\pi}{\sqrt{3}} - 2\log 3 - \frac{1}{2}(2\log 3)^2 3^{2/k}\frac{1}{k} - \frac{256\times 7\times 13\times\pi^2}{27\times 127}3^{2/k}\frac{1}{k^2}\right\} \\
&\geqslant \frac{1}{k}\times 0.29437... \quad \geqslant 0.
\end{aligned}$$

□

5.3.5. *Expansion of the RSD Method (2).* For the case "$k = 12n, \theta = 5\pi/6$", we need the next lemma.

Lemma 5.6. *Let k be the integer such that $k = 12n$ for $\exists n \in \mathbb{N}$. If n is even, then $F_{k,3}^*(5\pi/6) > 0$. On the other hand, if n is odd, then $F_{k,3}^*(5\pi/6) < 0$.*

If we can show this lemma, then we consequently show that for any integer $m \in \left[\frac{k}{4}, \frac{5k}{12}\right]$, if m is even or odd, then $F_{k,3}^*(2m\pi/k)$ is positive or negative, respectively. (Remark 5.3)

Proof of Lemma 5.6. Let $k = 12n$ $(n \geqslant 1)$. By the definition of $E_{k,3}^*(z), F_{k,3}^*(z)$ (*cf.* (54),(67)), we have

$$F_{k,3}^*(\theta) = \frac{e^{ik\theta/2}}{3^{k/2}+1}\left(3^{k/2}E_k\left(\sqrt{3}e^{i\theta}\right) + E_k\left(e^{i\theta}/\sqrt{3}\right)\right).$$

$$F_{k,3}^*(5\pi/6) = \frac{e^{i5(k/12)\pi}}{3^{k/2}+1}\left(3^{k/2}E_k\left(\frac{-3+\sqrt{3}i}{2}\right) + E_k\left(\frac{-\sqrt{3}+i}{2\sqrt{3}}\right)\right).$$

By using the equations (14) and (15),

$$E_k\left(\frac{-3+\sqrt{3}i}{2}\right)=E_k\left(\frac{-1+\sqrt{3}i}{2}\right),$$

$$E_k\left(\frac{-\sqrt{3}+i}{2\sqrt{3}}\right)=E_k\left(-\frac{1}{\frac{3+\sqrt{3}i}{2}}\right)=\left(\frac{3+\sqrt{3}i}{2}\right)^k E_k\left(\frac{3+\sqrt{3}i}{2}\right)=3^{-k/2}E_k\left(\frac{-1+\sqrt{3}i}{2}\right).$$

Then

$$\begin{aligned} F_k^*(5\pi/6) &= 2e^{i5(k/12)\pi}\frac{3^{k/2}}{3^{k/2}+1}E_k\left(\frac{-1+\sqrt{3}i}{2}\right) \\ &= 2e^{i(k/12)\pi}\frac{3^{k/2}}{3^{k/2}+1}F_k(2\pi/3) \quad (cf.(38)) \end{aligned}$$

When $k=12n$,

$$F_{12n,3}^*(5\pi/6)=2e^{in\pi}\frac{3^{6n}}{3^{6n}+1}F_{12n}(2\pi/3),$$

where $\frac{3^{6n}}{3^{6n}+1}>0$, $F_{12n}(2\pi/3)=e^{i4n\pi}E_{12n}(\rho)>0$ (Proposition 3.3). So the sign($\pm$) of $F_{k,3}^*(5\pi/6)$ is that of $e^{in\pi}$. Thus the proof is complete. □

5.3.6. *Valence formula for* $\Gamma_0^*(3)$. By previous subsections, $E_{k,3}^*(z)$ has $m_3(k)$ zeros on A_3^*. In order to decide the locating of all zeros of $E_{k,3}^*(z)$, we need the valence formula for $\Gamma_0^*(3)$:

Proposition 5.6. *Let f be a modular function of weight k for $\Gamma_0^*(3)$, which is not identically zero. We have*

$$v_\infty(f)+\frac{1}{2}v_{i/\sqrt{3}}(f)+\frac{1}{6}v_{\rho_3}(f)+\sum_{\substack{p\in\Gamma_0^*(3)\backslash\mathbb{H}\\ p\neq i/\sqrt{3},\ \rho_3}}v_p(f)=\frac{k}{6}, \tag{71}$$

where $\rho_3:=e^{5\pi/6}/\sqrt{3}$.

The proof of this proposition is similar to Proposition 3.1, 5.2. (See [SE])

If $k\equiv 4,8,10$ and $0 \pmod{12}$, then $k/6-m_3(k)<1$. Thus all zeros of $E_{k,3}^*(z)$ are on $A_3^*\cup\{i/\sqrt{3},\rho_3\}$.

But if $k\equiv 2,\ 6 \pmod{12}$, we need another consideration because we have $k/6-m_3(k)>1$. Because $E_{k,3}^*(z)$ is the *modular form* of weight k for $\Gamma_0^*(3)$, we have $E_{k,3}^*(i/\sqrt{3})=i^kE_{k,3}^*(i/\sqrt{3})$. Because $k\not\equiv 0 \pmod 4$, $E_{k,3}^*(i/\sqrt{3})=0$. Thus we have $k/6-m_3(k)-v_{i/\sqrt{3}}(E_{k,3}^*)/2<1$.

In conclusion, for every even integer $k\geqslant 4$, all zeros of $E_{k,3}^*(z)$ is on $A_3^*\cup\{i/\sqrt{3},\rho_3\}$.

5.3.7. *The space of modular forms.* Let $M_{k,3}^*$ be the space of modular forms for $\Gamma_0^*(3)$ of weight k, and let $M_{k,3}^{*0}$ be the space of cusp forms for $\Gamma_0^*(3)$ of weight k. Because $\dim(M_{k,3}^*/M_{k,3}^{*0})\leqslant 1$, $M_{k,3}^*=\mathbb{C}E_{k,3}^*\oplus M_{k,3}^{*0}$. Rcall that $\Delta_3=\eta^6(z)\eta^6(3z)$. We have following theorem:

Theorem 5.2. *Let k be an even integer.*

1. *For $k<0$ and $k=2$, $M_{k,3}^*=0$.*
2. *For $k=0,4,6$, we have $M_{k,3}^{*0}=0$, and $\dim(M_{k,3}^*)=1$ with a base $E_{k,3}^*$.*
3. *Let $\Delta_{3,8}:=\frac{41}{1728}((E_{4,3}^*)^2-E_{8,3}^*)$. We have $M_{8,3}^{*0}=\mathbb{C}\Delta_{3,8}$.*
4. *Let $\Delta_{3,10}:=\frac{61}{432}(E_{4,3}^*E_{6,3}^*-E_{10,3}^*)$. We have $M_{10,3}^{*0}=\mathbb{C}\Delta_{3,10}$.*
5. *Let $\Delta_{3,12}^0:=(\Delta_3)^2$, and $\Delta_{3,12}^1:=\Delta_{3,8}E_{4,3}^*$. We have $M_{12,3}^{*0}=\mathbb{C}\Delta_{3,12}^0\oplus\mathbb{C}\Delta_{3,12}^1$.*
6. *Let $\Delta_{3,14}:=\Delta_{3,10}E_{4,3}^*$. We have $M_{14,3}^{*0}=\mathbb{C}\Delta_{3,14}$.*
7. *$M_{k,3}^{*0}=M_{12,3}^{*0}M_{k-12,3}^*$.*

Before proving above theorem, we decide the order of $E_{k,3}^*$ at $i/\sqrt{3}$ and ρ_3 for $k = 0, 4, 6, 8$, and 10.

By section**??**, we have $v_p(E_{k,3}^*) = 0$ for every $p \neq i/\sqrt{3}$ and ρ_3 and for $k = 0, 4, 6, 8$, or 10. Thus, by the valence formula for $\Gamma_0^*(3)$ (Proposition5.6), $v_{i/\sqrt{3}}(E_{k,3}^*)/2 + v_{\rho_3}(E_{k,3}^*)/6 = k/6$.

Recall that any weight k modular form for $\Gamma_0^*(3)$ satisfies $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for every $z \in \mathbb{H}$ and every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^*(3)$. When $z = i/\sqrt{3}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = W_3$, we have $E_{k,3}^*(i/\sqrt{3}) = i^k E_{k,3}^*(i/\sqrt{3})$. Then $v_{i/\sqrt{3}}(E_{k,3}^*) \geqslant 1$ if $k \not\equiv 0 \pmod 4$. Also, when $z = \rho_3$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) W_3$, we have $E_{k,2}^*(\rho_3) = (e^{5\pi/6})^k E_{k,3}^*(\rho_3)$. Then $v_{\rho_3}(E_{k,3}^*) \geqslant 1$ if $k \not\equiv 0 \pmod{12}$.

Moreover, by the definition of $E_{k,3}^*(z)$ (*cf.* (54)), we have $E_{k,3}^*(i/\sqrt{3}) = \frac{1}{3^{k/2}+1}(3^{k/2}E_k(\sqrt{3}i) + E_k(i/\sqrt{3}))$. And by the equation(15), if $k \equiv 0 \pmod 4$, then $E_k(i/\sqrt{3}) = 3^{k/2}E_k(\sqrt{3}i)$. Thus, for $k \equiv 0 \pmod 4$, $E_{k,3}^*(i/\sqrt{3}) = \frac{2\cdot 3^{k/2}}{3^{k/2}+1}E_k(\sqrt{3}i) \neq 0$, *i.e.* $v_{i/\sqrt{3}}(E_{k,3}^*) = 0$.

In conclusion, we have the following table:

k	$v_{i/\sqrt{3}}(E_{k,3}^*)$	$v_{\rho_3}(E_{k,3}^*)$
0	0	0
14	1	5
4	0	4
6	1	3
8	0	2
10	1	1

We can not decide the orders of $E_{14,3}^*$ without above theorem. We decide it after the proof of the theorem.

proof of Theorem 5.2.

(1), (2) Let f be a nonzero function of $M_{k,3}^*$, then $v_p(f) \geqslant 0$ for every $p \in \mathbb{H}$. By the valence formula for $\Gamma_0^*(3)$(Proposition5.6), we have $k \geqslant 0$.

Let f be a nonzero function of $M_{k,3}^{*0}$, then $v_\infty(f) \geqslant 1$. By the valence formula, we have $M_{k,3}^{*0} = 0$ for $k = 0, 2, 4$. We also have $\dim(M_{k,3}^*) = 1$ for $k = 0, 4$ with a base $E_{k,3}^*$.

$\Delta_{3,10}$ is a cusp form of weight 10 with $v_{i/\sqrt{3}}(\Delta_{3,10}) \geqslant 1$ and $v_{\rho_3}(\Delta_{3,10}) = 1$ by the definition, and we have $v_\infty(\Delta_{3,10}) = v_{i/\sqrt{3}}(\Delta_{3,10}) = v_{\rho_3}(\Delta_{3,10}) = 1$ and $v_p(\Delta_{3,10}) = 0$ for every $p \neq i/\sqrt{3}, \rho_3, \infty$ by the valence formula.

Let Δ_3 be a nonzero cusp form of weight 6, then by $v_\infty(\Delta_3) \geqslant 1$ and the valence formula, $v_\infty(\Delta_3) = 1$ and $v_p(\Delta_3) = 0$ for every $p \neq \infty$. Then we have $v_p(\Delta_{3,10}/\Delta_3) \geqslant 0$ for every $p \in \mathbb{H}$, thus $\Delta_{3,10}/\Delta_3 \in M_{4,3}^* = \mathbb{C}E_{4,3}^*$, and $\Delta_{3,10} \in \mathbb{C}E_{4,3}^*\Delta_3$. However, we have $v_{i/\sqrt{3}}(\Delta_{3,10}) = 1$, $v_{i/\sqrt{3}}(E_{4,3}^*\Delta_3) = 0$. These are contradict each other. In conclusion, we have $M_{6,3}^{*0} = 0$, and $\dim(M_{6,3}^*) = 1$ with a base $E_{6,3}^*$.

Let f be a nonzero function of $M_{2,3}$. By the valence formula, we have $v_{\rho_3}(f) = 2$ and $v_p(f) = 0$ for every $p \neq \rho_3$. Because $f^3 \in M_{6,3}$, $f^3 = cE_{6,3}^*$ for some $c \in \mathbb{C}$. So $f^3(i/\sqrt{3}) = cE_{6,3}^*(i/\sqrt{3}) = 0$. It contradicts $f(i/\sqrt{3}) \neq 0$. Thus $M_{2,3} = 0$.

(3) $\Delta_{3,8}$ is a cusp form of weight 8 with $v_{\rho_3}(\Delta_{3,8}) = 2$ by the definition, and we have $v_\infty(\Delta_{3,8}) = 1$ and $v_p(\Delta_{3,8}) = 0$ for every $p \neq \rho_3, \infty$ by the valence formula. Let f be a nonzero function of $M_{8,3}^{*0}$, then we also have $v_{\rho_3}(f) = 2$, $v_\infty(f) = 1$, and $v_p(f) = 0$ for every $p \neq \rho_3, \infty$ by the valence formula. Thus $f/\Delta_{3,8} \in M_{0,3}^* = \mathbb{C}$, and $f \in \mathbb{C}\Delta_{3,8}$. Note that

$$v_{i/\sqrt{3}}(\Delta_{3,8}) = v_{i/\sqrt{3}}(E_{8,3}^*) = 0,$$
$$v_{\rho_3}(\Delta_{3,8}) = v_{\rho_3}(E_{8,3}^*) = 2.$$

(4) Let f be a nonzero function of $M_{10,3}^{*0}$, then we also have $v_\infty(f) = 1$ and $v_p(f) = 0$ for every $p \neq i/\sqrt{3}, \rho_3, \infty$ by the valence formula. However, for the other others, we can consider two cases, (*i*) $v_{i/\sqrt{3}}(f) = v_{\rho_3}(f) = 1$, and (*ii*) $v_{i/\sqrt{3}}(f) = 0$ and $v_{\rho_3}(f) = 4$.

For the first case(*i*), we have $f/\Delta_{3,10} \in M_{0,3}^* = \mathbb{C}$. For the second case(*ii*), define $g := f/E_{4,3}^*$. Then we have $v_\infty(g) = 1$ and $v_p(g) = 0$ for every $p \neq \infty$. Thus $g \in M_{6,3}^{*0}$. However, $M_{6,3}^{*0} = 0$. It contracicts that f is nonzero function. In conclusion, we have $v_{i/\sqrt{3}}(f) = v_{\rho_3}(f) = 1$, and $f \in \mathbb{C}\Delta_{3,10}$. Note that

$$v_{i/\sqrt{3}}(\Delta_{3,10}) = v_{i/\sqrt{3}}(E_{10,3}^*) = 1,$$
$$v_{\rho_3}(\Delta_{3,10}) = v_{\rho_3}(E_{10,3}^*) = 1.$$

(5) $\Delta^0_{3,12}$ is a cusp form of weight 12 with

$$v_\infty(\Delta^0_{3,12}) = 2, \quad v_p(\Delta^0_{3,12}) = 0 \quad \text{for every } p \neq \infty$$

by the definition, and $\Delta^1_{3,12}$ is a cusp form of weight 12 with

$$v_\infty(\Delta^1_{3,12}) = 1, \quad v_{\rho_3}(\Delta^1_{3,12}) = 6, \quad v_p(\Delta^1_{3,12}) = 0 \quad \text{for every } p \neq \rho_3, \infty$$

by the definition. Because $\dim(M^{*0}_{12,3}) \leqslant 2$, we have $M^{*0}_{12,3} = \mathbb{C}\Delta^0_{3,12} \oplus \mathbb{C}\Delta^1_{3,12}$.

(6) $\Delta_{3,14}$ is a cusp form of weight 14 with $v_\infty(\Delta_{3,14}) = v_{i/\sqrt{3}}(\Delta_{3,14}) = 1$, $v_{\rho_3}(\Delta_{3,14}) = 5$, and $v_p(\Delta_{3,14}) = 0$ for every $p \neq i/\sqrt{3}, \rho_3, \infty$ by the definition. Let f be a nonzero function of $M^{*0}_{14,3}$. If $v_\infty(f) \geqslant 2$, then $v_\infty(f) = 2$, $v_{\rho_3}(f) = 2$, and $v_p(f) = 0$ for every $p \neq \rho_3, \infty$ by the valence formula. Define $g := f/\Delta_{3,8}$, then $v_\infty(g) = 1$ and $v_p(f) = 0$ for every $p \neq \infty$. Thus $g \in M^{*0}_{6,3} = 0$. It contradicts that f is nonzero function. Now, we have $v_\infty(f) = 1$. Then we can write $f = a_1 q + a_2 q^2 + \cdots$. Also, we can write $\Delta_{3,14} = q + b_2 q^2 + \cdots$. Thus, $f - a_1\Delta_{3,14} = (a_2 - a_1 b_2)q^2 + \cdots \in M^{*0}_{14,3}$. Because $v_\infty(f - a_1\Delta_{3,14}) \neq 2$, we have $f - a_1\Delta_{3,14} = 0$. Furthermore, $f \in \mathbb{C}\Delta_{3,14}$. Thus $M^{*0}_{14,3} = \mathbb{C}\Delta_{3,14}$.

Furthermore, because $E^*_{14,3} - E^*_{10,3}E^*_{4,3} \in M^{*0}_{14,3}$, we have $E^*_{14,3} \in M^*_{14,3} = \mathbb{C}E^*_{10,3}E^*_{4,3} \oplus \mathbb{C}\Delta_{3,14}$. In conclusion, we have

$$v_{i/\sqrt{3}}(\Delta_{3,14}) = v_{i/\sqrt{3}}(E^*_{14,3}) = 1,$$
$$v_{\rho_3}(\Delta_{3,14}) = v_{\rho_3}(E^*_{14,3}) = 5.$$

(7) Let f be a nonzero function of $M^{*0}_{k,3}$. When $v_\infty(f) \geqslant 2$, $v_\infty(f/\Delta^0_{3,12}) = v_\infty(f) - 2 \geqslant 0$ and $v_p(f/\Delta^0_{3,12}) = v_p(f) \geqslant 0$ for every $p \neq \infty$. Thus $f/\Delta^0_{3,12} \in M^*_{k-12,3}$, and $f \in \Delta^0_{3,12}M^*_{k-12,3}$. On the other hand, when $v_\infty(f) = 1$, we can write $f = a_1 q + a_2 q^2 + \cdots$ for some $a_1 \neq 0$. Also, we can write $E^*_{k-12,3}\Delta^1_{3,12} = q + b_2 q^2 + \cdots$. Then $f - a_1 E^*_{k-12,3}\Delta^1_{3,12} = (a_2 - a_1 b_2)q^2 + \cdots$, and $v_\infty(f - a_1 E^*_{k-12,3}\Delta^1_{3,12}) \geqslant 2$. Thus $f - a_1 E^*_{k-12,3}\Delta^1_{3,12} \in \Delta^0_{3,12}M^*_{k-12,3}$, and $f \in \mathbb{C}E^*_{k-12,3}\Delta^1_{3,12} \oplus \Delta^0_{3,12}M_{k-12,3}$. In conclusion, $M^{*0}_{k,3} \subset \mathbb{C}E^*_{k-12,3}\Delta^1_{3,12} \oplus \Delta^0_{3,12}M^*_{k-12,3} \subset M^{*0}_{12,3}M^*_{k-12,3}$. This makes the proof of this theorem complete.

□

Furthermore, for a non-negative integer k, $\dim(M^*_{k,3}) = \lfloor k/6 \rfloor$ if $k \equiv 2, 6 \pmod{12}$, and $\dim(M^*_{k,3}) = \lfloor k/6 \rfloor + 1$ if $k \not\equiv 2, 6 \pmod{12}$.

Let k be an even integer $k \geqslant 16$. Write $n := \lfloor k/12 \rfloor$ - 1 if $k \equiv 2 \pmod{12}$, and $n := \lfloor k/12 \rfloor$ if $k \not\equiv 2 \pmod{12}$. Then $n \geqslant 1$ and $k - 12n = 0, 4, 6, 8, 10$, or 14. Because $E^*_{k,3} - E^*_{k-12n,3}(E^*_{4,3})^{3n} \in M^{*0}_{k,3}$, we have

$$\begin{aligned} M^*_{k,3} &= \mathbb{C}E^*_{k-12n,3}(E^*_{4,3})^{3n} \oplus M^{*0}_{12,3}M^*_{k-12,3} \\ &= E^*_{k-12n,3}\left\{\mathbb{C}(E^*_{4,3})^{3n} \oplus (E^*_{4,3})^{3(n-1)}M^{*0}_{12,3} \oplus (E^*_{4,3})^{3(n-2)}(M^{*0}_{12,3})^2 \oplus \cdots \oplus (M^{*0}_{12,3})^n\right\} \\ &\quad \oplus M^{*0}_{k-12n,3}(M^{*0}_{12,3})^n \end{aligned}$$

If $k - 12n = 4, 6, 0$, then $M^{*0}_{k-12n,3} = 0$. On the other hand, If $k - 12n = 8, 10, 14$, then $M^{*0}_{k-12n,3} = \mathbb{C}\Delta_{3,k-12n}$. Furthermore, we have $v_{i/\sqrt{3}}(\Delta_{3,k-12n}) = v_{i/\sqrt{3}}(E^*_{k-12n,3})$ and $v_{\rho_3}(\Delta_{3,k-12n}) = v_{\rho_3}(E^*_{k-12n,3})$. Thus, for $p = i/\sqrt{3}, \rho_3$ and for every $f \in M^*_{k,3}$, $v_p(f) \geqslant v_p(E^*_{k-12n,3})$.

In conclusion, we have next proposition:

Proposition 5.7. *Let $k \geqslant 4$ be an even integer. For every $f \in M^*_{k,3}$, we have*

$$\begin{aligned} v_{i/\sqrt{3}}(f) &\geqslant s_k \quad (s_k = 0, 1 \text{ such that } 2s_k \equiv k \pmod 4), \\ v_{\rho_3}(f) &\geqslant t_k \quad (t_k = 0, 1, 2, 3, 4, 5 \text{ such that } -2t_k \equiv k \pmod{12}). \end{aligned} \tag{72}$$

In paticular, if f is a constant multiple of $E^*_{k,3}$, then the equalities hold.

Remark 5.4. *Every modular form for $\Gamma^*_0(3)$ is generated by*

$$E^*_{4,3}, \quad E^*_{6,3}, \quad \Delta_{3,8}, \quad \Delta_{3,10}, \quad \text{and} \quad (\Delta_3)^2. \tag{73}$$

5.4. **Fundamental domains.** In the subsection5.1, we have two conditions for a fundamental domain for $\Gamma_0^*(p)$:

$$(C_0) \qquad |z-n/p| > 1/p,\ -1/2 < Re(z) < 1/2 \quad \text{for } \forall n \in \mathbb{N} \text{ such that } 1 \leqslant |n| \leqslant p/2.$$

$$(C_p) \qquad \begin{aligned} &|z-n/m| > 1/m\sqrt{p},\ -1/2 < Re(z) < 1/2 \\ &\quad \text{for } \forall m \in \mathbb{N} \text{ such that } m < \sqrt{4p/3} \\ &\quad\quad \forall n \in \mathbb{Z} \text{ such that } (m,n)=1,\ |n| < m/2 + 1/\sqrt{p}. \end{aligned}$$

5.4.1. $\Gamma_0^*(5)$. We have the folloing conditions for a fundamental domain for $\Gamma_0^*(5)$:

$$(C_0) \qquad |z \pm 1/5| > 1/5,\ |z \pm 2/5| > 1/5, \quad -1/2 < Re(z) < 1/2.$$

$$(C_5) \qquad |z| > 1/\sqrt{5},\ |z \pm 1/2| > 1/2\sqrt{5}, \quad -1/2 < Re(z) < 1/2.$$

Now,

$$\begin{aligned} &|z| > 1/\sqrt{5} && \Rightarrow \quad |z \pm 1/5| > 1/5, \\ &|z| > 1/\sqrt{5} \quad \text{and} \quad |z \pm 1/2| > 1/2\sqrt{5} && \Rightarrow \quad |z \pm 2/5| > 1/5. \end{aligned}$$

Thus (C_5) is a sufficient condition for (C_0). Furthermore, we have the following transformation:

$$\begin{aligned} W_5: &\qquad \frac{e^{i\theta}}{\sqrt{5}} \mapsto \frac{e^{i(\pi-\theta)}}{\sqrt{5}}, \\ \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix} W_5: &\quad \frac{e^{i\theta}}{2\sqrt{5}} + \frac{1}{2} \mapsto \frac{e^{i(\pi-\theta)}}{2\sqrt{5}} - \frac{1}{2}. \end{aligned}$$

Then we have $V_{\Gamma_0^*(5)} = \{i/2\sqrt{5},\ -2/5 + i/5\}$ (*cf.* Theorem2.1).

A fundamental domain for $\Gamma_0^*(5)$ is represented as Figure 7.

5.4.2. $\Gamma_0^*(7)$. We have the folloing conditions for a fundamental domain for $\Gamma_0^*(7)$:

$$(C_0) \qquad |z \pm 1/7| > 1/7,\ |z \pm 2/7| > 1/7,\ |z \pm 3/7| > 1/7, \quad -1/2 < Re(z) < 1/2.$$

$$(C_7) \qquad |z| > 1/\sqrt{7},\ |z \pm 1/2| > 1/2\sqrt{7},\ |z \pm 1/3| > 1/3\sqrt{7}, \quad -1/2 < Re(z) < 1/2.$$

Now,

$$\begin{aligned} &|z| > 1/\sqrt{7} \quad \text{and} \quad |z \pm 1/2| > 1/2\sqrt{7} && \Rightarrow \quad |z \pm 1/3| > 1/3\sqrt{7}, \\ &|z| > 1/\sqrt{7} && \Rightarrow \quad |z \pm 1/7| > 1/7, \\ &|z| > 1/\sqrt{7} \quad \text{and} \quad |z \pm 1/2| > 1/2\sqrt{7} && \Rightarrow \quad |z \pm n/7| > 1/7 \quad (n = 2,3). \end{aligned}$$

Thus

$$(C_{7,0}) \qquad |z| > 1/\sqrt{7},\ |z \pm 1/2| > 1/2\sqrt{7}, \quad -1/2 < Re(z) < 1/2$$

is a sufficient condition for (C_0) and (C_7). Furthermore, we have the following transformation:

$$\begin{aligned} W_7: &\qquad \frac{e^{i\theta}}{\sqrt{7}} \mapsto \frac{e^{i(\pi-\theta)}}{\sqrt{7}}, \\ \begin{pmatrix} -3 & -1 \\ 7 & 2 \end{pmatrix} W_7: &\quad \frac{e^{i\theta}}{2\sqrt{7}} + \frac{1}{2} \mapsto \frac{e^{i(\pi-\theta)}}{2\sqrt{7}} - \frac{1}{2}. \end{aligned}$$

Then we have $V_{\Gamma_0^*(7)} = \{i/2\sqrt{7},\ -5/14 + \sqrt{3}i/14\}$. (Figure 7)

5.4.3. $\Gamma_0^*(11)$. We have the folloing conditions for a fundamental domain for $\Gamma_0^*(11)$:

$$(C_0) \qquad |z \pm n/11| > 1/11 \quad (1 \leqslant n \leqslant 5), \quad -1/2 < Re(z) < 1/2.$$

$$(C_{11}) \qquad |z| > 1/\sqrt{11},\ |z \pm 1/2| > 1/2\sqrt{11},\ |z \pm 1/3| > 1/3\sqrt{11}, \quad -1/2 < Re(z) < 1/2.$$

Now,

$$\begin{aligned}
&|z| > 1/\sqrt{11} && \Rightarrow \quad |z \pm n/11| > 1/11 \quad (n = 1, 2),\\
&|z \pm 1/2| > 1/2\sqrt{11} && \Rightarrow \quad |z \pm 5/11| > 1/11,\\
&|z| > 1/\sqrt{11} \quad \text{and} \quad |z \pm 1/2| > 1/2\sqrt{11} && \Rightarrow \quad |z \pm 3/11| > 1/11,\\
&|z \pm 1/2| > 1/2\sqrt{11} \quad \text{and} \quad |z \pm 1/3| > 1/3\sqrt{11} && \Rightarrow \quad |z \pm 4/11| > 1/11.
\end{aligned}$$

Thus (C_{11}) is a sufficient condition for (C_0). Furthermore, we have the following transformation:

$$\begin{aligned}
W_{11}: &\quad \frac{e^{i\theta}}{\sqrt{11}} \mapsto \frac{e^{i(\pi-\theta)}}{\sqrt{11}},\\
\begin{pmatrix} -5 & -1 \\ 11 & 2 \end{pmatrix} W_{11}: &\quad \frac{e^{i\theta}}{2\sqrt{11}} + \frac{1}{2} \mapsto \frac{e^{i(\pi-\theta)}}{2\sqrt{11}} - \frac{1}{2},\\
\begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix} W_{11}: &\quad \frac{e^{i\theta}}{3\sqrt{11}} + \frac{1}{3} \mapsto \frac{e^{i(\pi-\theta)}}{3\sqrt{11}} + \frac{1}{3},\\
\begin{pmatrix} -4 & 1 \\ 11 & -3 \end{pmatrix} W_{11}: &\quad \frac{e^{i\theta}}{3\sqrt{11}} - \frac{1}{3} \mapsto \frac{e^{i(\pi-\theta)}}{3\sqrt{11}} - \frac{1}{3}.
\end{aligned}$$

Then we have $V_{\Gamma_0^*(11)} = \{i/2\sqrt{11},\ -25/66 + \sqrt{35}i/66\}$. (Figure 7)

5.4.4. $\Gamma_0^*(13)$. We have the folloing conditions for a fundamental domain for $\Gamma_0^*(13)$:

$$(C_0) \qquad |z \pm n/13| > 1/13 \quad (1 \leqslant n \leqslant 6), \quad -1/2 < Re(z) < 1/2.$$

$$(C_{13}) \qquad |z| > 1/\sqrt{13}, \quad |z \pm 1/m| > 1/m\sqrt{13} \quad (2 \leqslant m \leqslant 4), \quad -1/2 < Re(z) < 1/2.$$

Now,

$$\begin{aligned}
&|z| > 1/\sqrt{13} \quad \text{and} \quad |z \pm 1/3| > 1/3\sqrt{13} && \Rightarrow \quad |z \pm 1/4| > 1/4\sqrt{13},\\
&|z| > 1/\sqrt{13} && \Rightarrow \quad |z \pm n/13| > 1/13 \quad (n = 1, 2),\\
&|z \pm 1/2| > 1/2\sqrt{13} && \Rightarrow \quad |z \pm 6/13| > 1/13,\\
&|z| > 1/\sqrt{13} \quad \text{and} \quad |z \pm 1/3| > 1/3\sqrt{13} && \Rightarrow \quad |z \pm n/13| > 1/13 \quad (n = 3, 4),\\
&|z \pm 1/2| > 1/2\sqrt{13} \quad \text{and} \quad |z \pm 1/3| > 1/3\sqrt{13} && \Rightarrow \quad |z \pm 5/13| > 1/13.
\end{aligned}$$

Thus

$$(C_{13,0}) \qquad |z| > 1/\sqrt{13},\ |z \pm 1/2| > 1/2\sqrt{13},\ |z \pm 1/3| > 1/3\sqrt{13}, \quad -1/2 < Re(z) < 1/2$$

is a sufficient condition for (C_0) and (C_{13}). Furthermore, we have the following transformation:

$$\begin{aligned}
W_{13}: &\quad \frac{e^{i\theta}}{\sqrt{13}} \mapsto \frac{e^{i(\pi-\theta)}}{\sqrt{13}},\\
\begin{pmatrix} -6 & -1 \\ 13 & 2 \end{pmatrix} W_{13}: &\quad \frac{e^{i\theta}}{2\sqrt{13}} + \frac{1}{2} \mapsto \frac{e^{i(\pi-\theta)}}{2\sqrt{13}} - \frac{1}{2},\\
\begin{pmatrix} -4 & -1 \\ 13 & 3 \end{pmatrix} W_{13}: &\quad \frac{e^{i\theta}}{3\sqrt{13}} + \frac{1}{3} \mapsto \frac{e^{i(\pi-\theta)}}{3\sqrt{13}} - \frac{1}{3}.
\end{aligned}$$

Then we have $V_{\Gamma_0^*(13)} = \{i/2\sqrt{13},\ -7/26 + \sqrt{3}i/26,\ -5/13 + i/13\}$. (Figure 7)

5.4.5. $\Gamma_0^*(17)$. We have the folloing conditions for a fundamental domain for $\Gamma_0^*(17)$:

$$(C_0) \qquad |z \pm n/17| > 1/17 \quad (1 \leqslant n \leqslant 8), \quad -1/2 < Re(z) < 1/2.$$

$$(C_{17}) \qquad |z| > 1/\sqrt{17}, \quad |z \pm 1/m| > 1/m\sqrt{17} \quad (2 \leqslant m \leqslant 4), \quad -1/2 < Re(z) < 1/2.$$

Now,

$$\begin{aligned}
&|z| > 1/\sqrt{17} && \Rightarrow \ |z \pm n/17| > 1/17 \quad (n = 1,2,3),\\
&|z \pm 1/2| > 1/2\sqrt{17} && \Rightarrow \ |z \pm 8/17| > 1/17,\\
&|z \pm 1/3| > 1/3\sqrt{17} && \Rightarrow \ |z \pm 6/17| > 1/17,\\
&|z| > 1/\sqrt{17} \quad \text{and} \quad |z \pm 1/4| > 1/4\sqrt{17} && \Rightarrow \ |z \pm 4/17| > 1/17,\\
&|z \pm 1/4| > 1/4\sqrt{17} \quad \text{and} \quad |z \pm 1/3| > 1/3\sqrt{17} && \Rightarrow \ |z \pm 5/17| > 1/17.\\
&|z \pm 1/3| > 1/3\sqrt{17} \quad \text{and} \quad |z \pm 1/2| > 1/2\sqrt{17} && \Rightarrow \ |z \pm 7/17| > 1/17.
\end{aligned}$$

Thus (C_{17}) is a sufficient condition for (C_0). Furthermore, we have the following transformation:

$$\begin{aligned}
W_{17}: &\quad \frac{e^{i\theta}}{\sqrt{17}} \mapsto \frac{e^{i(\pi-\theta)}}{\sqrt{17}},\\
\begin{pmatrix} -8 & -1 \\ 17 & 2 \end{pmatrix} W_{17}: &\quad \frac{e^{i\theta}}{2\sqrt{17}} + \frac{1}{2} \mapsto \frac{e^{i(\pi-\theta)}}{2\sqrt{17}} - \frac{1}{2},\\
\begin{pmatrix} 6 & 1 \\ 17 & 3 \end{pmatrix} W_{17}: &\quad \frac{e^{i\theta}}{3\sqrt{17}} + \frac{1}{3} \mapsto \frac{e^{i(\pi-\theta)}}{3\sqrt{17}} + \frac{1}{3},\\
\begin{pmatrix} -6 & 1 \\ 17 & -3 \end{pmatrix} W_{17}: &\quad \frac{e^{i\theta}}{3\sqrt{17}} - \frac{1}{3} \mapsto \frac{e^{i(\pi-\theta)}}{3\sqrt{17}} - \frac{1}{3},\\
\begin{pmatrix} -4 & -1 \\ 17 & 4 \end{pmatrix} W_{17}: &\quad \frac{e^{i\theta}}{4\sqrt{17}} + \frac{1}{4} \mapsto \frac{e^{i(\pi-\theta)}}{4\sqrt{17}} - \frac{1}{4}.
\end{aligned}$$

Then we have $V_{\Gamma_0^*(17)} = \{i/2\sqrt{17},\ -20/51 + 2\sqrt{2}i/51,\ -4/17 + i/17\}$. (Figure 7)

5.4.6. $\Gamma_0^*(19)$. We have the folloing conditions for a fundamental domain for $\Gamma_0^*(19)$:

$$(C_0) \qquad |z \pm n/19| > 1/19 \quad (1 \leqslant n \leqslant 8), \quad -1/2 < Re(z) < 1/2.$$

$$(C_{19}) \qquad \begin{aligned} &|z| > 1/\sqrt{19}, \quad |z \pm 1/m| > 1/m\sqrt{19}\ (2 \leqslant m \leqslant 5), \quad |z \pm 2/5| > 1/5\sqrt{19}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -1/2 < Re(z) < 1/2.\end{aligned}$$

Now,

$$\begin{aligned}
&|z| > 1/\sqrt{19} \quad \text{and} \quad |z \pm 1/4| > 1/4\sqrt{19} && \Rightarrow \ |z \pm 1/5| > 1/5\sqrt{19},\\
&|z \pm 1/3| > 1/3\sqrt{19} \quad \text{and} \quad |z \pm 1/2| > 1/2\sqrt{19} && \Rightarrow \ |z \pm 2/5| > 1/5\sqrt{19}.\\
&|z| > 1/\sqrt{19} && \Rightarrow \ |z \pm n/19| > 1/19 \quad (n = 1,2,3),\\
&|z \pm 1/2| > 1/2\sqrt{19} && \Rightarrow \ |z \pm 9/19| > 1/19,\\
&|z \pm 1/3| > 1/3\sqrt{19} && \Rightarrow \ |z \pm 6/19| > 1/19,\\
&|z| > 1/\sqrt{19} \quad \text{and} \quad |z \pm 1/4| > 1/4\sqrt{19} && \Rightarrow \ |z \pm 4/19| > 1/19,\\
&|z \pm 1/4| > 1/4\sqrt{19} \quad \text{and} \quad |z \pm 1/3| > 1/3\sqrt{19} && \Rightarrow \ |z \pm 5/19| > 1/19.\\
&|z \pm 1/3| > 1/3\sqrt{19} \quad \text{and} \quad |z \pm 1/2| > 1/2\sqrt{19} && \Rightarrow \ |z \pm n/19| > 1/19 \quad (n = 7,8).
\end{aligned}$$

Thus

$$(C_{19,0}) \qquad |z| > 1/\sqrt{19}, \quad |z \pm 1/m| > 1/m\sqrt{19} \quad (2 \leqslant m \leqslant 4), \quad -1/2 < Re(z) < 1/2.$$

is a sufficient condition for (C_0) and (C_{19}). Furthermore, we have the following transformation:

$$
\begin{aligned}
W_{19}: &\quad \frac{e^{i\theta}}{\sqrt{19}} \mapsto \frac{e^{i(\pi-\theta)}}{\sqrt{19}},\\
\begin{pmatrix} -9 & -1 \\ 19 & 2 \end{pmatrix} W_{19}: &\quad \frac{e^{i\theta}}{2\sqrt{19}} + \frac{1}{2} \mapsto \frac{e^{i(\pi-\theta)}}{2\sqrt{19}} - \frac{1}{2},\\
\begin{pmatrix} -6 & -1 \\ 19 & 3 \end{pmatrix} W_{19}: &\quad \frac{e^{i\theta}}{3\sqrt{19}} + \frac{1}{3} \mapsto \frac{e^{i(\pi-\theta)}}{3\sqrt{19}} - \frac{1}{3},\\
\begin{pmatrix} 5 & 1 \\ 19 & 4 \end{pmatrix} W_{19}: &\quad \frac{e^{i\theta}}{4\sqrt{19}} + \frac{1}{4} \mapsto \frac{e^{i(\pi-\theta)}}{4\sqrt{19}} + \frac{1}{4},\\
\begin{pmatrix} -5 & 1 \\ 19 & -4 \end{pmatrix} W_{19}: &\quad \frac{e^{i\theta}}{4\sqrt{19}} - \frac{1}{4} \mapsto \frac{e^{i(\pi-\theta)}}{4\sqrt{19}} - \frac{1}{4}.
\end{aligned}
$$

Then we have $V_{\Gamma_0^*(19)} = \{i/2\sqrt{19},\ -15/38 + \sqrt{3}i/38,\ -21/76 + \sqrt{15}i/76\}$.

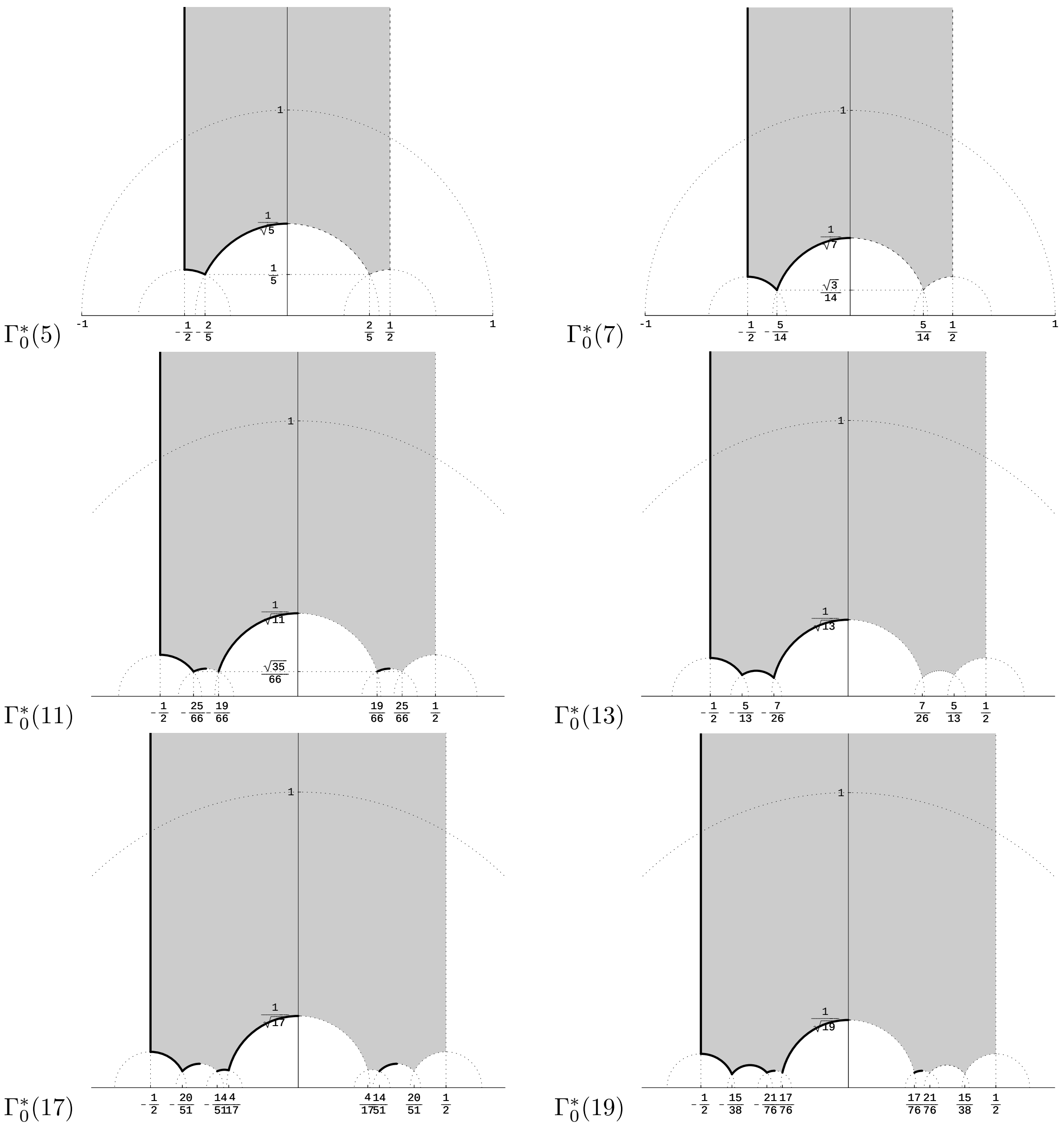


FIGURE 7. $\Gamma_0^*(p)$

5.5. **The function $F_{k,p,m}^*(\theta)$.** Let p be a prime. If $p \geqslant 5$, then the figure of a fundamental domain of $\Gamma_0^*(p)$ is more complex than $\Gamma_0^*(2)$, $\Gamma_0^*(3)$, and $\mathrm{SL}_2(\mathbb{Z})$. We expect all the zeros are on the arcs $e^{i\theta}/\sqrt{p}$, $e^{i\theta}/2\sqrt{p} \pm 1/2$, $e^{i\theta}/3\sqrt{p} \pm 1/3$, $\cdots$, which form the boundary of the fundamental domain defined in the sense of Theorem 2.1 (Figure 7). We will begin with consider the function $F_{k,p,m}^*(\theta)$.

Again, Eisenstein series associated with $\Gamma_0^*(p)$ is denoted by

$$E_{k,p}^*(z) = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p|c}} (cz+d)^{-k} + \frac{p^{k/2}}{2} \sum_{\substack{(c,d)=1 \\ p|d}} (c(pz)+d)^{-k}.$$

5.5.1. *For the arc $e^{i\theta}/\sqrt{p}$.* We give the next definition;

$$F_{k,p,1}^*(\theta) := e^{ik\theta/2} E_{k,p}^* \left(e^{i\theta}/\sqrt{p}\right). \tag{74}$$

We consider an expansion of $F_{k,p,1}^*(\theta)$. Similarly to $F_{k,2}^*(\theta)$, we have

$$\begin{aligned} F_{k,p,1}^*(z) &= \frac{e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1 \\ p|c}} (ce^{i\theta}/\sqrt{p}+d)^{-k} + \frac{p^{k/2} e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1 \\ p|d}} (c(pe^{i\theta}/\sqrt{p})+d)^{-k} \\ &= \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p\nmid d}} (de^{-i\theta/2} + \sqrt{p}c' e^{i\theta/2})^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c}} (ce^{i\theta/2} + \sqrt{p}d' e^{-i\theta/2})^{-k}. \end{aligned}$$

Thus we can write as follows;

$$F_{k,p,1}^*(\theta) = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c}} (ce^{i\theta/2} + \sqrt{p}de^{-i\theta/2})^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c}} (ce^{-i\theta/2} + \sqrt{p}de^{i\theta/2})^{-k}. \tag{75}$$

Hence we use this expression as a definition. Note that for any pair (c,d), $(ce^{i\theta/2} + \sqrt{p}de^{-i\theta/2})^{-k}$ and $(ce^{-i\theta/2} + \sqrt{p}de^{i\theta/2})^{-k}$ are conjugates of each other. The next proposition follows.

Proposition 5.8. *$F_{k,p,1}^*(\theta)$ is real, for $\forall\theta \in \mathbb{R}$.*

5.5.2. *For the arcs $e^{i\theta}/2\sqrt{p} \pm 1/2$.* Let p be a prime such that $p \geqslant 5$. Then we can write $p = 2n+1$ for $\exists n \in \mathbb{Z}$, and we have the following transformation:

$$\begin{pmatrix} -n & -1 \\ p & 2 \end{pmatrix} W_p : \quad \frac{e^{i\theta}}{2\sqrt{p}} + \frac{1}{2} \mapsto \frac{e^{i(\pi-\theta)}}{2\sqrt{p}} - \frac{1}{2}.$$

Because we have the condition $-1/2 \leqslant Re(z) < 1/2$, we have only $e^{i\theta}/2\sqrt{p} - 1/2$ in our fundamental domain. Then we give the next definition:

$$F_{k,p,2}^*(\theta) := e^{ik\theta/2} E_{k,p}^* \left(e^{i\theta}/2\sqrt{p} - 1/2\right). \tag{76}$$

We consider an expansion of $F^*_{k,p,2}(\theta)$. When $p \mid c$, then we can write $c = c'p$ for $\exists c' \in \mathbb{Z}$, and have $p \nmid d$. Also, when $p \mid d$, then we have $p \nmid c$ and $d = d'p$ for $\exists d' \in \mathbb{Z}$. Thus

$$\begin{aligned}
F^*_{k,p,2}(z) &= \frac{e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1\\ p|c}} \left(c\left(\frac{e^{i\theta}}{2\sqrt{p}} - \frac{1}{2}\right) + d\right)^{-k} + \frac{p^{k/2}e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1\\ p|d}} \left(cp\left(\frac{e^{i\theta}}{2\sqrt{p}} - \frac{1}{2}\right) + d\right)^{-k} \\
&= \frac{e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1\\ p\nmid d}} \left(c'\left(\frac{\sqrt{p}e^{i\theta}}{2} - \frac{p}{2}\right) + d\right)^{-k} + \frac{p^{k/2}e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1\\ p\nmid c}} \left(c\sqrt{p}\left(\frac{e^{i\theta}}{2} - \frac{\sqrt{p}}{2}\right) + d'p\right)^{-k} \\
&= \frac{e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1\\ p\nmid d}} \left(\frac{c'}{2}\sqrt{p}e^{i\theta} + \frac{2d-c'p}{2}\right)^{-k} + \frac{e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1\\ p\nmid c}} \left(\frac{c}{2}e^{i\theta} + \frac{2d'-c}{2}\sqrt{p}\right)^{-k} \\
&= \frac{1}{2} \sum_{\substack{(c,d)=1\\ p\nmid d}} \left(\frac{2d-c'p}{2}e^{-i\theta/2} + \frac{c'}{2}\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1\\ p\nmid c}} \left(\frac{c}{2}e^{i\theta/2} + \frac{2d'-c}{2}\sqrt{p}e^{-i\theta/2}\right)^{-k}.
\end{aligned}$$

Now, we split terms in two cases, namely $2 \mid c$ or $2 \nmid c$. Note that the parities of c and c' are same. For the case $2 \mid c$, we can write $c' = 2c''$ and $c = 2c'''$ for $\exists c'', c''' \in \mathbb{Z}$. Then

$$\begin{aligned}
&\frac{1}{2} \sum_{\substack{(c,d)=1\\ p\nmid d\\ 2|c'}} \left(\frac{2d-c'p}{2}e^{-i\theta/2} + \frac{c'}{2}\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1\\ p\nmid c\\ 2|c}} \left(\frac{c}{2}e^{i\theta/2} + \frac{2d'-c}{2}\sqrt{p}e^{-i\theta/2}\right)^{-k} \\
=&\frac{1}{2} \sum_{\substack{(c,d)=1\\ p\nmid d}} \left((d-c''p)e^{-i\theta/2} + c''\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1\\ p\nmid c}} \left(c'''e^{i\theta/2} + (d'-c''')\sqrt{p}e^{-i\theta/2}\right)^{-k}.
\end{aligned}$$

Then we have $(d - c''p, c'') = 1$, $p \nmid d - c''p$, $2 \mid (d - c'')c''$, and $(c''', d' - c''') = 1$, $p \nmid c'''$, $2 \mid c'''(d' - c''')$. Thus we can write above terms as follows:

$$\frac{1}{2} \sum_{\substack{(c,d)=1\\ p\nmid c\\ 2|cd}} \left(ce^{-i\theta/2} + d\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1\\ p\nmid c\\ 2|cd}} \left(ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k}.$$

For the other case $2 \nmid c$,

$$\begin{aligned}
&\frac{1}{2} \sum_{\substack{(c,d)=1\\ p\nmid d\\ 2\nmid c'}} \left(\frac{2d-c'p}{2}e^{-i\theta/2} + \frac{c'}{2}\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1\\ p\nmid c\\ 2\nmid c}} \left(\frac{c}{2}e^{i\theta/2} + \frac{2d'-c}{2}\sqrt{p}e^{-i\theta/2}\right)^{-k} \\
=&\frac{2^k}{2} \sum_{\substack{(c,d)=1\\ p\nmid d}} \left((2d-c'p)e^{-i\theta/2} + c'\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{2^k}{2} \sum_{\substack{(c,d)=1\\ p\nmid c}} \left(ce^{i\theta/2} + (2d'-c)\sqrt{p}e^{-i\theta/2}\right)^{-k}.
\end{aligned}$$

Then we have $(2d - c'p, c') = 1$, $p \nmid 2d - c'p$, $2 \nmid (2d - c')c'$, and $(c, 2d' - c) = 1$, $p \nmid c$, $2 \nmid c(d' - c)$. Thus we can write above terms as follows:

$$\frac{2^k}{2} \sum_{\substack{(c,d)=1\\ p\nmid c\\ 2\nmid cd}} \left(ce^{-i\theta/2} + d\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{2^k}{2} \sum_{\substack{(c,d)=1\\ p\nmid c\\ 2\nmid cd}} \left(ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k}.$$

In conclusion, we can write as follows;

$$
\begin{aligned}
F_{k,p,2}^*(\theta) = &\frac{1}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c \\ 2|cd}} \left(ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c \\ 2|cd}} \left(ce^{-i\theta/2} + d\sqrt{p}e^{i\theta/2}\right)^{-k} \\
&+ \frac{2^k}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c \\ 2\nmid cd}} \left(ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k} + \frac{2^k}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c \\ 2\nmid cd}} \left(ce^{-i\theta/2} + d\sqrt{p}e^{i\theta/2}\right)^{-k}.
\end{aligned} \tag{77}
$$

Hence we use this expression as a definition. Note that for any pair (c,d), $(ce^{i\theta/2} + \sqrt{p}de^{-i\theta/2})^{-k}$ and $(ce^{-i\theta/2} + \sqrt{p}de^{i\theta/2})^{-k}$ are conjugates of each other. The next proposition follows.

Proposition 5.9. *$F_{k,p,2}^*(\theta)$ is real, for $\forall\theta \in \mathbb{R}$.*

5.5.3. *For the arcs $e^{i\theta}/3\sqrt{p} \pm 1/3$.* Let p be a prime such that $p \geqslant 11$.

If $p \equiv 1 \pmod 3$, then we can write $p = 3n+1$ for $\exists n \in \mathbb{Z}$, and we have the following transformation:

$$
\begin{pmatrix} -n & -1 \\ p & 3 \end{pmatrix} W_p : \quad \frac{e^{i\theta}}{3\sqrt{p}} + \frac{1}{3} \mapsto \frac{e^{i(\pi-\theta)}}{3\sqrt{p}} - \frac{1}{3}.
$$

Thus we have only $e^{i\theta}/3\sqrt{p} - 1/3$ in our fundamental domain.

On the other hand, if $p \equiv -1 \pmod 3$, then we can write $p = 3n-1$ for $\exists n \in \mathbb{Z}$, and we have the following transformation:

$$
\begin{aligned}
\begin{pmatrix} n & 1 \\ p & 3 \end{pmatrix} W_p : &\quad \frac{e^{i\theta}}{3\sqrt{p}} + \frac{1}{3} \mapsto \frac{e^{i(\pi-\theta)}}{3\sqrt{p}} + \frac{1}{3}, \\
\begin{pmatrix} -n & 1 \\ p & -3 \end{pmatrix} W_p : &\quad \frac{e^{i\theta}}{3\sqrt{p}} - \frac{1}{3} \mapsto \frac{e^{i(\pi-\theta)}}{3\sqrt{p}} - \frac{1}{3}.
\end{aligned}
$$

Thus we have both $e^{i\theta}/3\sqrt{p} \pm 1/3$ in our fundamental domain.

Now, we give the next definition:

$$
F_{k,p,3}^*(\theta) := e^{ik\theta/2} E_{k,p}^* \left(e^{i\theta}/3\sqrt{p} - 1/3\right), \tag{78}
$$

$$
F_{k,p,3}^{*,+}(\theta) := e^{ik\theta/2} E_{k,p}^* \left(e^{i\theta}/3\sqrt{p} + 1/3\right). \tag{79}
$$

We consider an expansion of $F_{k,p,3}^*(\theta)$. When $p \mid c$, then we can write $c = c'p$ for $\exists c' \in \mathbb{Z}$, and have $p \nmid d$. Also, when $p \mid d$, then we have $p \nmid c$ and $d = d'p$ for $\exists d' \in \mathbb{Z}$. Similar to $F_{k,p,2}^*(\theta)$,

$$
\begin{aligned}
F_{k,p,3}^*(z) &= \frac{e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1 \\ p|c}} \left(c\left(\frac{e^{i\theta}}{3\sqrt{p}} - \frac{1}{3}\right) + d\right)^{-k} + \frac{p^{k/2}e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1 \\ p|d}} \left(cp\left(\frac{e^{i\theta}}{3\sqrt{p}} - \frac{1}{3}\right) + d\right)^{-k} \\
&= \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p\nmid d}} \left(\frac{3d - c'p}{3}e^{-i\theta/2} + \frac{c'}{3}\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c}} \left(\frac{c}{3}e^{i\theta/2} + \frac{3d' - c}{3}\sqrt{p}e^{-i\theta/2}\right)^{-k}.
\end{aligned}
$$

Now, we split terms in two cases, namely $3 \mid c$ or $3 \nmid c$. Note that the parities of c and c' are same.

For the case $3 \mid c$, we can write $c' = 3c''$ and $c = 3c'''$ for $\exists c'', c''' \in \mathbb{Z}$. Then

$$
\begin{aligned}
&\frac{1}{2} \sum_{\substack{(c,d)=1 \\ p\nmid d \\ 3|c'}} \left(\frac{3d - c'p}{3}e^{-i\theta/2} + \frac{c'}{3}\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c \\ 3|c}} \left(\frac{c}{3}e^{i\theta/2} + \frac{3d' - c}{3}\sqrt{p}e^{-i\theta/2}\right)^{-k} \\
=&\frac{1}{2} \sum_{\substack{(c,d)=1 \\ p\nmid d}} \left((d - c''p)e^{-i\theta/2} + c''\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c}} \left(c'''e^{i\theta/2} + (d' - c''')\sqrt{p}e^{-i\theta/2}\right)^{-k}.
\end{aligned}
$$

Then we have $(d - c''p, c'') = 1$, $p \nmid d - c''p$, $d - c''p \not\equiv -p \cdot c'' \pmod 3$, and $(c''', d' - c''') = 1$, $p \nmid c'''$, $c''' \not\equiv -(d' - c''') \pmod 3$. Thus we can write above terms as follows:

$$\frac{1}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \\ c \not\equiv -pd\ (3)}} \left(ce^{-i\theta/2} + d\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \\ c \not\equiv -d\ (3)}} \left(ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k}.$$

For the other case $3 \nmid c$,

$$\begin{aligned} &\frac{1}{2} \sum_{\substack{(c,d)=1 \\ p \nmid d \\ 3 \nmid c'}} \left(\frac{3d - c'p}{3}e^{-i\theta/2} + \frac{c'}{3}\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \\ 3 \nmid c}} \left(\frac{c}{3}e^{i\theta/2} + \frac{3d' - c}{3}\sqrt{p}e^{-i\theta/2}\right)^{-k} \\ =&\frac{3^k}{2} \sum_{\substack{(c,d)=1 \\ p \nmid d}} \left((3d - c'p)e^{-i\theta/2} + c'\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{3^k}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c}} \left(ce^{i\theta/2} + (3d' - c)\sqrt{p}e^{-i\theta/2}\right)^{-k}. \end{aligned}$$

Then we have $(3d - c'p, c') = 1$, $p \nmid 3d - c'p$, $3d - c'p \equiv -p \cdot c' \not\equiv 0 \pmod 3$, and $(c, 3d' - c) = 1$, $p \nmid c$, $c \equiv -(3d' - c) \not\equiv 0 \pmod 3$. Thus we can write above terms as follows:

$$\frac{3^k}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \\ c \equiv -pd \not\equiv 0\ (3)}} \left(ce^{-i\theta/2} + d\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{3^k}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \\ c \equiv -d \not\equiv 0\ (3)}} \left(ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k}.$$

In conclusion, we can write as follows;

$$\begin{aligned} F^*_{k,p,3}(\theta) = &\frac{1}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \\ c \not\equiv -d\ (3)}} \left(ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \\ c \not\equiv -pd\ (3)}} \left(ce^{-i\theta/2} + d\sqrt{p}e^{i\theta/2}\right)^{-k} \\ &+ \frac{3^k}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \\ c \equiv -d \not\equiv 0\ (3)}} \left(ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k} + \frac{3^k}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \\ c \equiv -pd \not\equiv 0\ (3)}} \left(ce^{-i\theta/2} + d\sqrt{p}e^{i\theta/2}\right)^{-k}. \end{aligned} \tag{80}$$

Again, if $p \equiv 1 \pmod 3$, then the condition "$c \not\equiv -pd \pmod 3$" is equivalent to "$c \not\equiv -d \pmod 3$", and "$c \equiv -pd \not\equiv 0 \pmod 3$" is equivalant to "$c \equiv -d \not\equiv 0 \pmod 3$". In addition, for any pair (c, d), $(ce^{i\theta/2} + \sqrt{p}de^{-i\theta/2})^{-k}$ and $(ce^{-i\theta/2} + \sqrt{p}de^{i\theta/2})^{-k}$ are conjugates of each other. Thus $F^*_{k,p,3}(\theta)$ is real.

On the other hand, if $p \equiv -1 \pmod 3$, then the condition "$c \not\equiv -pd \pmod 3$" is equivalent to "$c \not\equiv d \pmod 3$", and "$c \equiv -pd \not\equiv 0 \pmod 3$" is equivalant to "$c \equiv d \not\equiv 0 \pmod 3$". Thus we need more consideration.

Similarly to $F^*_{k,p,3}(\theta)$, we have another expansion, which is for $F^{*,+}_{k,p,3}(\theta)$:

$$\begin{aligned} F^{*,+}_{k,p,3}(\theta) = &\frac{1}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \\ c \not\equiv d\ (3)}} \left(ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \\ c \not\equiv pd\ (3)}} \left(ce^{-i\theta/2} + d\sqrt{p}e^{i\theta/2}\right)^{-k} \\ &+ \frac{3^k}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \\ c \equiv d \not\equiv 0\ (3)}} \left(ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k} + \frac{3^k}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c \\ c \equiv pd \not\equiv 0\ (3)}} \left(ce^{-i\theta/2} + d\sqrt{p}e^{i\theta/2}\right)^{-k}. \end{aligned} \tag{81}$$

Because $p \equiv -1 \pmod 3$, we have $\overline{F^{*,+}_{k,p,3}(\theta)} = F^*_{k,p,3}(\theta)$.

Thus next proposition follows:

Proposition 5.10. *If $p \equiv 1 \pmod 3$, then $F^*_{k,p,3}(\theta)$ is real. On the other hand, if $p \equiv -1 \pmod 3$, then we have $\overline{F^{*,+}_{k,p,3}(\theta)} = F^*_{k,p,3}(\theta)$.*

5.5.4. *For the arcs* $e^{i\theta}/4\sqrt{p}\pm 1/4$. Let p be a prime such that $p\geqslant 17$, then p is odd.

If $p\equiv 1 \pmod 4$, then we can write $p=4n+1$ for $\exists n\in\mathbb{Z}$, and we have the following transformation:

$$\begin{pmatrix}-n & -1\\ p & 4\end{pmatrix} W_p: \quad \frac{e^{i\theta}}{4\sqrt{p}}+\frac{1}{4}\mapsto \frac{e^{i(\pi-\theta)}}{4\sqrt{p}}-\frac{1}{4}.$$

Thus we have only $e^{i\theta}/4\sqrt{p}-1/4$ in our fundamental domain.

On the other hand, if $p\equiv -1 \pmod 4$, then we can write $p=4n-1$ for $\exists n\in\mathbb{Z}$, and we have the following transformation:

$$\begin{pmatrix}n & 1\\ p & 4\end{pmatrix} W_p: \quad \frac{e^{i\theta}}{4\sqrt{p}}+\frac{1}{4}\mapsto \frac{e^{i(\pi-\theta)}}{4\sqrt{p}}+\frac{1}{4},$$

$$\begin{pmatrix}-n & 1\\ p & -4\end{pmatrix} W_p: \quad \frac{e^{i\theta}}{4\sqrt{p}}-\frac{1}{4}\mapsto \frac{e^{i(\pi-\theta)}}{4\sqrt{p}}-\frac{1}{4}.$$

Thus we have both $e^{i\theta}/4\sqrt{p}\pm 1/4$ in our fundamental domain.

Now, we give the next definition:

$$F_{k,p,4}^*(\theta) := e^{ik\theta/2}E_{k,p}^*\left(e^{i\theta}/4\sqrt{p}-1/4\right), \tag{82}$$

$$F_{k,p,4}^{*,+}(\theta) := e^{ik\theta/2}E_{k,p}^*\left(e^{i\theta}/4\sqrt{p}+1/4\right). \tag{83}$$

We consider an expansion of $F_{k,p,4}^*(\theta)$. When $p\mid c$, then we can write $c=c'p$ for $\exists c'\in\mathbb{Z}$, and have $p\nmid d$. Also, when $p\mid d$, then we have $p\nmid c$ and $d=d'p$ for $\exists d'\in\mathbb{Z}$. Similar to $F_{k,p,2}^*(\theta)$,

$$\begin{aligned} F_{k,p,4}^*(z) &= \frac{e^{ik\theta/2}}{2}\sum_{\substack{(c,d)=1\\ p\mid c}}\left(c\left(\frac{e^{i\theta}}{4\sqrt{p}}-\frac{1}{4}\right)+d\right)^{-k} + \frac{p^{k/2}e^{ik\theta/2}}{2}\sum_{\substack{(c,d)=1\\ p\mid d}}\left(cp\left(\frac{e^{i\theta}}{4\sqrt{p}}-\frac{1}{4}\right)+d\right)^{-k}\\ &= \frac{1}{2}\sum_{\substack{(c,d)=1\\ p\nmid d}}\left(\frac{4d-c'p}{4}e^{-i\theta/2}+\frac{c'}{4}\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{1}{2}\sum_{\substack{(c,d)=1\\ p\nmid c}}\left(\frac{c}{4}e^{i\theta/2}+\frac{4d'-c}{4}\sqrt{p}e^{-i\theta/2}\right)^{-k}. \end{aligned}$$

Now, we split terms in three cases, namely "$4\mid c$" or "$4\nmid c$ and $2\mid c$" or "$2\nmid c$". Note that the parities of c and c' are same.

For the case "$4\mid c$", we can write $c'=4c_1$ and $c=4c_1'$ for $\exists c_1, c_1'\in\mathbb{Z}$. Then

$$\begin{aligned} &\frac{1}{2}\sum_{\substack{(c,d)=1\\ p\nmid d\\ 4\mid c'}}\left(\frac{4d-c'p}{4}e^{-i\theta/2}+\frac{c'}{4}\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{1}{2}\sum_{\substack{(c,d)=1\\ p\nmid c\\ 4\mid c}}\left(\frac{c}{4}e^{i\theta/2}+\frac{4d'-c}{4}\sqrt{p}e^{-i\theta/2}\right)^{-k}\\ =&\frac{1}{2}\sum_{\substack{(c,d)=1\\ p\nmid d}}\left((d-c_1p)e^{-i\theta/2}+c_1\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{1}{2}\sum_{\substack{(c,d)=1\\ p\nmid c}}\left(c_1'e^{i\theta/2}+(d'-c_1')\sqrt{p}e^{-i\theta/2}\right)^{-k}. \end{aligned}$$

Then we have $(d-c_1p, c_1)=1$, $p\nmid d-c_1p$, $2\mid (d-c_1p)\cdot c_1$, and $(c_1', d'-c_1')=1$, $p\nmid c_1'$, $2\mid c_1'\cdot(d'-c_1')$. Thus we can write above terms as follows:

$$\frac{1}{2}\sum_{\substack{(c,d)=1\\ p\nmid c\\ 2\mid cd}}\left(ce^{-i\theta/2}+d\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{1}{2}\sum_{\substack{(c,d)=1\\ p\nmid c\\ 2\mid cd}}\left(ce^{i\theta/2}+d\sqrt{p}e^{-i\theta/2}\right)^{-k}.$$

For another case "$4\nmid c$ and $2\mid c$", we can write $c'=2c_2$ and $c=2c_2'$ for $\exists c_2, c_2'\in\mathbb{Z}$, which are odd integers. Then

$$\begin{aligned} &\frac{1}{2}\sum_{\substack{(c,d)=1\\ p\nmid d\\ 4\nmid c',\, 2\mid c'}}\left(\frac{4d-c'p}{4}e^{-i\theta/2}+\frac{c'}{4}\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{1}{2}\sum_{\substack{(c,d)=1\\ p\nmid c\\ 4\nmid c,\, 2\mid c}}\left(\frac{c}{4}e^{i\theta/2}+\frac{4d'-c}{4}\sqrt{p}e^{-i\theta/2}\right)^{-k}\\ =&\frac{2^k}{2}\sum_{\substack{(c,d)=1\\ p\nmid d}}\left((2d-c_2p)e^{-i\theta/2}+c_2\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{2^k}{2}\sum_{\substack{(c,d)=1\\ p\nmid c}}\left(c_2'e^{i\theta/2}+(2d'-c_2')\sqrt{p}e^{-i\theta/2}\right)^{-k}. \end{aligned}$$

Then we have $(2d-c_2p, c_2)=1$, $p\nmid 2d-c_2p$, $2d-c_2p\equiv p\cdot c_2\equiv\pm1 \pmod 4$, and $(c_2', 2d'-c_2')=1$, $p\nmid c_2'$, $c_2'\equiv 2d'-c_2'\equiv\pm1 \pmod 4$ because $2\nmid d$, $2d\equiv 2 \pmod 4$. Thus we can write above terms as follows:

$$\frac{2^k}{2}\sum_{\substack{(c,d)=1\\ p\nmid c\\ 2\nmid cd\\ c\equiv pd\,(4)}}\left(ce^{-i\theta/2}+d\sqrt{p}e^{i\theta/2}\right)^{-k}+\frac{2^k}{2}\sum_{\substack{(c,d)=1\\ p\nmid c\\ 2\nmid cd\\ c\equiv d\,(4)}}\left(ce^{i\theta/2}+d\sqrt{p}e^{-i\theta/2}\right)^{-k}.$$

For the other case "$2\nmid c$",

$$\begin{aligned}&\frac{1}{2}\sum_{\substack{(c,d)=1\\ p\nmid d\\ 4\nmid c'}}\left(\frac{4d-c'p}{4}e^{-i\theta/2}+\frac{c'}{4}\sqrt{p}e^{i\theta/2}\right)^{-k}+\frac{1}{2}\sum_{\substack{(c,d)=1\\ p\nmid c\\ 4\nmid c}}\left(\frac{c}{4}e^{i\theta/2}+\frac{4d'-c}{4}\sqrt{p}e^{-i\theta/2}\right)^{-k}\\ =&\frac{4^k}{2}\sum_{\substack{(c,d)=1\\ p\nmid d}}\left((4d-c'p)e^{-i\theta/2}+c'\sqrt{p}e^{i\theta/2}\right)^{-k}+\frac{4^k}{2}\sum_{\substack{(c,d)=1\\ p\nmid c}}\left(ce^{i\theta/2}+(4d'-c)\sqrt{p}e^{-i\theta/2}\right)^{-k}.\end{aligned}$$

Then we have $(4d-c'p, c')=1$, $p\nmid 4d-c'p$, $2\nmid(4d-c'p)c'$, $4d-c'p\equiv -p\cdot c'\not\equiv 0 \pmod 4$, and $(c, 4d'-c)=1$, $p\nmid c$, $2\nmid c(4d'-c)$, $c\equiv-(4d'-c)\not\equiv 0 \pmod 4$. Thus we can write above terms as follows:

$$\frac{4^k}{2}\sum_{\substack{(c,d)=1\\ p\nmid c\\ 2\nmid cd\\ c\equiv -pd\,(4)}}\left(ce^{-i\theta/2}+d\sqrt{p}e^{i\theta/2}\right)^{-k}+\frac{4^k}{2}\sum_{\substack{(c,d)=1\\ p\nmid c\\ 2\nmid cd\\ c\equiv -d\,(4)}}\left(ce^{i\theta/2}+d\sqrt{p}e^{-i\theta/2}\right)^{-k}.$$

In conclusion, we can write as follows;

$$\begin{aligned}F^*_{k,p,4}(\theta)=&\frac{1}{2}\sum_{\substack{(c,d)=1\\ p\nmid c\\ 2\mid cd}}\left(ce^{i\theta/2}+d\sqrt{p}e^{i\theta/2}\right)^{-k}+\frac{1}{2}\sum_{\substack{(c,d)=1\\ p\nmid c\\ 2\mid cd}}\left(ce^{-i\theta/2}+d\sqrt{p}e^{-i\theta/2}\right)^{-k}\\ &+\frac{2^k}{2}\sum_{\substack{(c,d)=1\\ p\nmid c\\ 2\nmid cd\\ c\equiv d\,(4)}}\left(ce^{i\theta/2}+d\sqrt{p}e^{i\theta/2}\right)^{-k}+\frac{2^k}{2}\sum_{\substack{(c,d)=1\\ p\nmid c\\ 2\nmid cd\\ c\equiv pd\,(4)}}\left(ce^{-i\theta/2}+d\sqrt{p}e^{-i\theta/2}\right)^{-k}\\ &+\frac{4^k}{2}\sum_{\substack{(c,d)=1\\ p\nmid c\\ 2\nmid cd\\ c\equiv -d\,(4)}}\left(ce^{i\theta/2}+d\sqrt{p}e^{i\theta/2}\right)^{-k}+\frac{4^k}{2}\sum_{\substack{(c,d)=1\\ p\nmid c\\ 2\nmid cd\\ c\equiv -pd\,(4)}}\left(ce^{-i\theta/2}+d\sqrt{p}e^{-i\theta/2}\right)^{-k}.\end{aligned} \tag{84}$$

Again, if $p\equiv 1 \pmod 4$, then the condition "$c\equiv pd \pmod 4$" is equivalent to "$c\equiv d \pmod 4$", and "$c\equiv -pd \pmod 4$" is equivalant to "$c\equiv -d \pmod 4$". In addition, for any pair (c,d), $(ce^{i\theta/2}+\sqrt{p}de^{-i\theta/2})^{-k}$ and $(ce^{-i\theta/2}+\sqrt{p}de^{i\theta/2})^{-k}$ are conjugates of each other. Thus $F^*_{k,p,4}(\theta)$ is real.

On the other hand, if $p\equiv -1 \pmod 4$, then the condition "$c\equiv pd \pmod 4$" is equivalent to "$c\equiv -d \pmod 4$", and "$c\equiv -pd \pmod 4$" is equivalant to "$c\equiv d \pmod 4$". Thus we need more consideration.

Similaly to $F^*_{k,p,4}(\theta)$, we have another expansion, which is for $F^{*,+}_{k,p,4}(\theta)$:

$$\begin{aligned} F^{*,+}_{k,p,4}(\theta) = {} & \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c \\ 2|cd}} \left(ce^{i\theta/2} + d\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c \\ 2|cd}} \left(ce^{-i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k} \\ & + \frac{2^k}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c \\ 2\nmid cd \\ c\equiv -d\ (4)}} \left(ce^{i\theta/2} + d\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{2^k}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c \\ 2\nmid cd \\ c\equiv -pd\ (4)}} \left(ce^{-i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k} \\ & + \frac{4^k}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c \\ 2\nmid cd \\ c\equiv d\ (4)}} \left(ce^{i\theta/2} + d\sqrt{p}e^{i\theta/2}\right)^{-k} + \frac{4^k}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c \\ 2\nmid cd \\ c\equiv pd\ (4)}} \left(ce^{-i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k}. \end{aligned} \tag{85}$$

Because $p \equiv -1 \pmod 4$, we have $\overline{F^{*,+}_{k,p,4}(\theta)} = F^*_{k,p,4}(\theta)$.

Thus next proposition follows:

Proposition 5.11. *If $p \equiv -1 \pmod 4$, then $F^*_{k,p,4}(\theta)$ is real. On the other hand, if $p \equiv -1 \pmod 4$, then we have $\overline{F^{*,+}_{k,p,4}(\theta)} = F^*_{k,p,4}(\theta)$.*

5.6. **Application of the RSD Method.**

5.6.1. $\Gamma_0^*(5)$. We note that $N := c^2 + d^2$.

Firstly, we consider the case $N = 1$. Then we can write;

$$F^*_{k,5,1}(\theta) = 2\cos(k\theta/2) + R^*_{5,1}, \tag{86}$$

$$F^*_{k,5,2}(\theta) = 2\cos(k\theta/2) + R^*_{5,2} \tag{87}$$

where $R^*_{5,1}$ and $R^*_{5,2}$ are the terms such that $N > 1$ of $F^*_{k,5,1}$ and $F^*_{k,5,2}$, respectively.

For $F^*_{k,5,1}(\theta)$,

$$|R^*_{5,1}| \leqslant \sum_{\substack{(c,d)=1 \\ 5\nmid c \\ N>1}} |ce^{i\theta/2} + \sqrt{5}de^{-i\theta/2}|^{-k}.$$

Let $v_k(c,d,\theta) := |ce^{i\theta/2} + \sqrt{5}de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\theta) = 1/\left(c^2 + 5d^2 + 2\sqrt{5}cd\cos\theta\right)^{k/2}$, and $v_k(c,d,\theta) = v_k(-c,-d,\theta)$. Now we will consider the next cases, namely $N = 2, 5, 10$, and $N \geqslant 13$. Considering $-2/\sqrt{5} \leqslant \cos\theta \leqslant 0$, we have the following:

When $N = 2$,	$v_k(1,1,\theta) \leqslant (1/2)^{k/2}$,	$v_k(1,-1,\theta) \leqslant (1/6)^{k/2}$.
When $N = 5$,	$v_k(1,2,\theta) \leqslant (1/13)^{k/2}$,	$v_k(1,-2,\theta) \leqslant (1/21)^{k/2}$,
	$v_k(2,1,\theta) \leqslant 1$,	$v_k(2,-1,\theta) \leqslant (1/3)^k$.
When $N = 10$,	$v_k(1,3,\theta) \leqslant (1/34)^{k/2}$,	$v_k(1,-3,\theta) \leqslant (1/46)^{k/2}$,
	$v_k(3,1,\theta) \leqslant (1/2)^{k/2}$,	$v_k(3,-1,\theta) \leqslant (1/14)^{k/2}$.
When $N \geqslant 13$,	$\lvert ce^{i\theta/2} \pm \sqrt{5}de^{-i\theta/2}\rvert^2 \geqslant N/6$,	

and the rest of the question is about the number of terms with $c^2 + d^2 = N$. Because $5 \nmid c$, the number of $|c|$ is not more than $(4/5)N^{1/2} + 1$. Thus the number of terms with $c^2 + d^2 = N$ is not more than $4((4/5)N^{1/2} + 1) \leqslant (21/5)N^{1/2}$ for $N \geqslant 13$. Then

$$|R^*_{5,1}|_{N\geqslant 13} = \frac{21\sqrt{6}}{5} \sum_{N=13}^{\infty} \left(\frac{1}{6}N\right)^{(1-k)/2} \leqslant \frac{1008\sqrt{6}}{5(k-3)} \left(\frac{1}{2}\right)^{k/2}.$$

On the other hand, for $F^*_{k,5,2}(\theta)$,

$$|R^*_{5,2}| \leqslant \sum_{\substack{(c,d)=1\\ 5\nmid c\\ 2|cd\\ N>1}} |ce^{i\theta/2}+d\sqrt{5}e^{-i\theta/2}|^{-k} + 2^k \sum_{\substack{(c,d)=1\\ 5\nmid c\\ 2\nmid cd}} |ce^{i\theta/2}+d\sqrt{5}e^{-i\theta/2}|^{-k}.$$

Now we will consider the next cases, namely $N = 2, 5, 10, 13, 17$, and $N \geqslant 25$. Considering $0 \leqslant \cos\theta \leqslant 1/\sqrt{5}$, we have the following:

When $N = 2$,	$2^k \cdot v_k(1,1,\theta) \leqslant (2/3)^{k/2}$,	$2^k \cdot v_k(1,-1,\theta) \leqslant 1$.
When $N = 5$,	$v_k(1,2,\theta) \leqslant (1/21)^{k/2}$,	$v_k(1,-2,\theta) \leqslant (1/17)^{k/2}$,
	$v_k(2,1,\theta) \leqslant (1/3)^{k}$,	$v_k(2,-1,\theta) \leqslant (1/5)^{k/2}$.
When $N = 10$,	$2^k \cdot v_k(1,3,\theta) \leqslant (2/23)^{k/2}$,	$2^k \cdot v_k(1,-3,\theta) \leqslant (1/10)^{k/2}$,
	$2^k \cdot v_k(3,1,\theta) \leqslant (2/7)^{k/2}$,	$2^k \cdot v_k(3,-1,\theta) \leqslant (1/2)^{k/2}$.
When $N = 13$,	$v_k(2,3,\theta) \leqslant (1/7)^{k}$,	$v_k(2,-3,\theta) \leqslant (1/37)^{k/2}$,
	$v_k(3,2,\theta) \leqslant (1/29)^{k/2}$,	$v_k(3,-2,\theta) \leqslant (1/17)^{k/2}$.
When $N = 17$,	$v_k(1,4,\theta) \leqslant (1/9)^{k}$,	$v_k(1,-4,\theta) \leqslant (1/73)^{k/2}$,
	$v_k(4,1,\theta) \leqslant (1/21)^{k/2}$,	$v_k(4,-1,\theta) \leqslant (1/13)^{k/2}$.
When $N \geqslant 25$,	$|ce^{i\theta/2} \pm \sqrt{5}de^{-i\theta/2}|^2 \geqslant N/2$,	

and the number of terms with $c^2+d^2 = N$ is not more than $4N^{1/2}$ for $N \geqslant 25$. Then

$$|R^*_{5,2}|_{N\geqslant 25} = 8\sqrt{2}\sum_{N=25}^{\infty}\left(\frac{1}{8}N\right)^{(1-k)/2} \leqslant \frac{378\sqrt{6}}{k-3}\left(\frac{1}{3}\right)^{k/2}.$$

Thus

$$|R^*_{5,1}| \leqslant 2 + 4\left(\frac{1}{2}\right)^{k/2} + 2\left(\frac{1}{3}\right)^{k/2} + \cdots + 2\left(\frac{1}{46}\right)^{k/2} + \frac{1008\sqrt{6}}{5(k-3)}\left(\frac{1}{2}\right)^{k/2}, \tag{88}$$

$$|R^*_{5,2}| \leqslant 2 + 2\left(\frac{2}{3}\right)^{k/2} + 2\left(\frac{1}{2}\right)^{k/2} + \cdots + 2\left(\frac{1}{9}\right)^{k} + \frac{378\sqrt{6}}{k-3}\left(\frac{1}{3}\right)^{k/2}. \tag{89}$$

Recalling the "RSD Method" subsection, we want to show that $|R^*_{5,1}| < 2$ and $|R^*_{5,2}| < 2$. But the right-hand sides of both bounds are greater than 2, so these bounds are not good. The cases $(c,d) = \pm(2,1)$ give us bound equal to 2 for $|R^*_{5,1}|$, and the cases $(c,d) = \pm(1,-1)$ give us bound equal to 2 for $|R^*_{5,2}|$.

5.6.2. $\Gamma^*_0(7)$. We note that $N := c^2 + d^2$.

Firstly, we consider the case $N = 1$. Then we can write;

$$F^*_{k,7,1}(\theta) = 2\cos(k\theta/2) + R^*_{7,1}, \tag{90}$$

$$F^*_{k,7,2}(\theta) = 2\cos(k\theta/2) + R^*_{7,2} \tag{91}$$

where $R^*_{7,1}$ and $R^*_{7,2}$ are the terms such that $N > 1$ of $F^*_{k,7,1}$ and $F^*_{k,7,2}$, respectively.

For $F^*_{k,7,1}(\theta)$,

$$|R^*_{7,1}| \leqslant \sum_{\substack{(c,d)=1\\ 7\nmid c\\ N>1}} |ce^{i\theta/2}+\sqrt{7}de^{-i\theta/2}|^{-k}.$$

Let $v_k(c,d,\theta) := |ce^{i\theta/2}+\sqrt{7}de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\theta) = 1/\left(c^2+7d^2+2\sqrt{7}cd\cos\theta\right)^{k/2}$, and $v_k(c,d,\theta) = v_k(-c,-d,\theta)$. Now we will consider the next cases, namely $N = 2, 5, \cdots, 29$, and $N \geqslant 34$. Considering $-5/(2\sqrt{7}) \leqslant \cos\theta \leqslant 0$, we have the following:

When $N = 2$,	$v_k(1,1,\theta) \leqslant (1/3)^{k/2}$,	$v_k(1,-1,\theta) \leqslant (1/8)^{k/2}$.
When $N = 5$,	$v_k(1,2,\theta) \leqslant (1/19)^{k/2}$,	$v_k(1,-2,\theta) \leqslant (1/29)^{k/2}$,

	$v_k(2,1,\theta) \leqslant 1,$	$v_k(2,-1,\theta) \leqslant (1/11)^{k/2}.$
When $N = 10$,	$v_k(1,3,\theta) \leqslant (1/7)^k,$	$v_k(1,-3,\theta) \leqslant (1/8)^k,$
	$v_k(3,1,\theta) \leqslant 1,$	$v_k(3,-1,\theta) \leqslant (1/4)^k.$
When $N = 13$,	$v_k(2,3,\theta) \leqslant (1/39)^{k/2},$	$v_k(2,-3,\theta) \leqslant (1/69)^{k/2},$
	$v_k(3,2,\theta) \leqslant (1/7)^{k/2},$	$v_k(3,-2,\theta) \leqslant (1/37)^{k/2}.$
When $N = 17$,	$v_k(1,4,\theta) \leqslant (1/93)^{k/2},$	$v_k(1,-4,\theta) \leqslant (1/113)^{k/2},$
	$v_k(4,1,\theta) \leqslant (1/3)^{k/2},$	$v_k(4,-1,\theta) \leqslant (1/23)^{k/2}.$
When $N = 25$,	$v_k(3,4,\theta) \leqslant (1/61)^{k/2},$	$v_k(3,-4,\theta) \leqslant (1/11)^k,$
	$v_k(4,3,\theta) \leqslant (1/19)^{k/2},$	$v_k(4,-3,\theta) \leqslant (1/79)^{k/2}.$
When $N = 26$,	$v_k(1,5,\theta) \leqslant (1/151)^{k/2},$	$v_k(1,-5,\theta) \leqslant (1/176)^{k/2},$
	$v_k(5,1,\theta) \leqslant (1/7)^{k/2},$	$v_k(5,-1,\theta) \leqslant (1/32)^{k/2}.$
When $N = 29$,	$v_k(2,5,\theta) \leqslant (1/129)^{k/2},$	$v_k(2,-5,\theta) \leqslant (1/179)^{k/2},$
	$v_k(5,2,\theta) \leqslant (1/3)^{k/2},$	$v_k(5,-2,\theta) \leqslant (1/53)^{k/2}.$
When $N \geqslant 34$,	$\lvert ce^{i\theta/2} \pm \sqrt{7}de^{-i\theta/2}\rvert^2 \geqslant N/11,$	

and the rest of the question is about the number of terms with $c^2+d^2=N$. Because $7 \nmid c$, the number of $|c|$ is not more than $(6/7)N^{1/2}+1$. Thus the number of terms with $c^2+d^2=N$ is not more than $4((6/7)N^{1/2}+1) \leqslant (29/7)N^{1/2}$ for $N \geqslant 34$. Then

$$|R^*_{7,1}|_{N\geqslant 34} = \frac{29\sqrt{11}}{7}\sum_{N=34}^{\infty}\left(\frac{1}{11}N\right)^{(1-k)/2} \leqslant \frac{1914\sqrt{33}}{7(k-3)}\left(\frac{1}{3}\right)^{k/2}.$$

On the other hand, for $F^*_{k,7,2}(\theta)$,

$$|R^*_{7,2}| \leqslant \sum_{\substack{(c,d)=1\\ 7\nmid c\\ 2|cd\\ N>1}} |ce^{i\theta/2}+d\sqrt{7}e^{-i\theta/2}|^{-k} + 2^k \sum_{\substack{(c,d)=1\\ 7\nmid c\\ 2\nmid cd}} |ce^{i\theta/2}+d\sqrt{7}e^{-i\theta/2}|^{-k}.$$

Now we will consider the next cases, namely $N = 2, 5, 10, 13, 17$, and $N \geqslant 25$. Considering $0 \leqslant \cos\theta \leqslant 2/\sqrt{7}$, we have the following:

When $N = 2$,	$2^k \cdot v_k(1,1,\theta) \leqslant (1/2)^{k/2},$	$2^k \cdot v_k(1,-1,\theta) \leqslant 1.$
When $N = 5$,	$v_k(1,2,\theta) \leqslant (1/29)^{k/2},$	$v_k(1,-2,\theta) \leqslant (1/21)^{k/2},$
	$v_k(2,1,\theta) \leqslant (1/11)^{k/2},$	$v_k(2,-1,\theta) \leqslant (1/3)^{k/2}.$
When $N = 10$,	$2^k \cdot v_k(1,3,\theta) \leqslant (1/4)^k,$	$2^k \cdot v_k(1,-3,\theta) \leqslant (1/13)^{k/2},$
	$2^k \cdot v_k(3,1,\theta) \leqslant (1/2)^k,$	$2^k \cdot v_k(3,-1,\theta) \leqslant 1.$
When $N = 13$,	$v_k(2,3,\theta) \leqslant (1/69)^{k/2},$	$v_k(2,-3,\theta) \leqslant (1/45)^{k/2},$
	$v_k(3,2,\theta) \leqslant (1/37)^{k/2},$	$v_k(3,-2,\theta) \leqslant (1/14)^{k/2}.$
When $N = 17$,	$v_k(1,4,\theta) \leqslant (1/113)^{k/2},$	$v_k(1,-4,\theta) \leqslant (1/97)^{k/2},$
	$v_k(4,1,\theta) \leqslant (1/23)^{k/2},$	$v_k(4,-1,\theta) \leqslant (1/7)^{k/2}.$
When $N \geqslant 25$,	$\lvert ce^{i\theta/2} \pm \sqrt{7}de^{-i\theta/2}\rvert^2 \geqslant N/3,$	

and the number of terms with $c^2+d^2=N$ is not more than $(29/7)N^{1/2}$ for $N \geqslant 25$. Then

$$|R^*_{7,2}|_{N\geqslant 25} = \frac{58\sqrt{3}}{7}\sum_{N=25}^{\infty}\left(\frac{1}{12}N\right)^{(1-k)/2} \leqslant \frac{2784\sqrt{6}}{7(k-3)}\left(\frac{1}{2}\right)^{k/2}.$$

Thus

$$|R^*_{7,1}| \leqslant 4 + 6\left(\frac{1}{3}\right)^{k/2} + 4\left(\frac{1}{7}\right)^{k/2} + \cdots + 2\left(\frac{1}{179}\right)^{k/2} + \frac{1914\sqrt{33}}{7(k-3)}\left(\frac{1}{3}\right)^{k/2}, \tag{92}$$

$$|R^*_{7,2}| \leqslant 4 + 2\left(\frac{1}{2}\right)^{k/2} + 2\left(\frac{1}{3}\right)^{k/2} + \cdots + 2\left(\frac{1}{113}\right)^{k/2} + \frac{2784\sqrt{6}}{7(k-3)}\left(\frac{1}{2}\right)^{k/2}. \tag{93}$$

We want to show that $|R^*_{7,1}| < 2$ and $|R^*_{7,2}| < 2$. But the right-hand sides of both bounds are much greater than 2. The cases $(c,d) = \pm(2,1)$ and $\pm(3,1)$ give us bound equal to 4 for $|R^*_{7,1}|$, and the cases $(c,d) = \pm(1,-1)$ and $\pm(3,-1)$ give us bound equal to 4 for $|R^*_{7,2}|$.

5.6.3. $\Gamma^*_0(11)$. We note that $N := c^2 + d^2$.

Firstly, we consider the case $N = 1$. Then we can write;

$$F^*_{k,11,1}(\theta) = 2\cos(k\theta/2) + R^*_{11,1}, \tag{94}$$

$$F^*_{k,11,2}(\theta) = 2\cos(k\theta/2) + R^*_{11,2} \tag{95}$$

where $R^*_{11,1}$ and $R^*_{11,2}$ are the terms such that $N > 1$ of $F^*_{k,11,1}$ and $F^*_{k,11,2}$, respectively.

For $F^*_{k,11,1}(\theta)$,

$$|R^*_{11,1}| \leqslant \sum_{\substack{(c,d)=1\\ 11\nmid c\\ N>1}} |ce^{i\theta/2} + \sqrt{11}de^{-i\theta/2}|^{-k}.$$

Let $v_k(c,d,\theta) := |ce^{i\theta/2} + \sqrt{11}de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\theta) = 1/\left(c^2 + 11d^2 + 2\sqrt{11}cd\cos\theta\right)^{k/2}$, and $v_k(c,d,\theta) = v_k(-c,-d,\theta)$.

Now we will consider the next cases, namely $N = 2, 5, 10, 13, 17, 25$, and $N \geqslant 26$. Considering $-19/(6\sqrt{11}) \leqslant \cos\theta \leqslant 0$, we have the following:

When $N = 2$,	$v_k(1,1,\theta) \leqslant (3/17)^{k/2}$,	$v_k(1,-1,\theta) \leqslant (1/12)^{k/2}$.
When $N = 5$,	$v_k(1,2,\theta) \leqslant (3/97)^{k/2}$,	$v_k(1,-2,\theta) \leqslant (1/45)^{k/2}$,
	$v_k(2,1,\theta) \leqslant (3/7)^{k/2}$,	$v_k(2,-1,\theta) \leqslant (1/15)^{k/2}$.
When $N = 10$,	$v_k(1,3,\theta) \leqslant (1/9)^{k}$,	$v_k(1,-3,\theta) \leqslant (1/10)^{k}$,
	$v_k(3,1,\theta) \leqslant 1$,	$v_k(3,-1,\theta) \leqslant (1/20)^{k/2}$.
When $N = 13$,	$v_k(2,3,\theta) \leqslant (1/65)^{k/2}$,	$v_k(2,-3,\theta) \leqslant (1/103)^{k/2}$,
	$v_k(3,2,\theta) \leqslant (1/15)^{k/2}$,	$v_k(3,-2,\theta) \leqslant (1/53)^{k/2}$.
When $N = 17$,	$v_k(1,4,\theta) \leqslant (3/455)^{k/2}$,	$v_k(1,-4,\theta) \leqslant (1/177)^{k/2}$,
	$v_k(4,1,\theta) \leqslant (3/5)^{k/2}$,	$v_k(4,-1,\theta) \leqslant (1/27)^{k/2}$.
When $N = 25$,	$v_k(3,4,\theta) \leqslant (1/109)^{k/2}$,	$v_k(3,-4,\theta) \leqslant (1/185)^{k/2}$,
	$v_k(4,3,\theta) \leqslant (1/39)^{k/2}$,	$v_k(4,-3,\theta) \leqslant (1/115)^{k/2}$.
When $N \geqslant 26$,	$\lvert ce^{i\theta/2} \pm \sqrt{11}de^{-i\theta/2}\rvert^2 \geqslant 2N/25$,	

and the rest of the question is about the number of terms with $c^2 + d^2 = N$. Because $11 \nmid c$, the number of $|c|$ is not more than $(10/11)N^{1/2} + 1$. Thus the number of terms with $c^2 + d^2 = N$ is not more than $4((10/11)N^{1/2} + 1) \leqslant (48/11)N^{1/2}$ for $N \geqslant 26$. Then

$$|R^*_{11,1}|_{N\geqslant 26} = \frac{120\sqrt{2}}{11}\sum_{N=26}^{\infty}\left(\frac{2}{25}N\right)^{(1-k)/2} \leqslant \frac{12000}{11(k-3)}\left(\frac{1}{2}\right)^{k/2}.$$

On the other hand, for $F^*_{k,11,2}(\theta)$,

$$|R^*_{11,2}| \leqslant \sum_{\substack{(c,d)=1\\ 11\nmid c\\ 2\mid cd\\ N>1}} |ce^{i\theta/2} + d\sqrt{11}e^{-i\theta/2}|^{-k} + 2^k \sum_{\substack{(c,d)=1\\ 11\nmid c\\ 2\nmid cd}} |ce^{i\theta/2} + d\sqrt{11}e^{-i\theta/2}|^{-k}.$$

Now we will consider the next cases, namely $N = 2, 5, 10, 13, 17$, and $N \geqslant 25$. Considering $0 \leqslant \cos\theta \leqslant 8/(3\sqrt{11})$, we have the following:

When $N = 2$,	$2^k \cdot v_k(1,1,\theta) \leqslant (1/3)^{k/2}$,	$2^k \cdot v_k(1,-1,\theta) \leqslant (3/5)^{k/2}$.
When $N = 5$,	$v_k(1,2,\theta) \leqslant (1/45)^{k/2}$,	$v_k(1,-2,\theta) \leqslant (3/103)^{k/2}$,
	$v_k(2,1,\theta) \leqslant (1/15)^{k/2}$,	$v_k(2,-1,\theta) \leqslant (3/13)^{k/2}$.
When $N = 10$,	$2^k \cdot v_k(1,3,\theta) \leqslant (1/5)^{k}$,	$2^k \cdot v_k(1,-3,\theta) \leqslant (1/21)^{k/2}$,
	$2^k \cdot v_k(3,1,\theta) \leqslant (1/5)^{k/2}$,	$2^k \cdot v_k(3,-1,\theta) \leqslant 1$.
When $N = 13$,	$v_k(2,3,\theta) \leqslant (1/103)^{k/2}$,	$v_k(2,-3,\theta) \leqslant (1/71)^{k/2}$,
	$v_k(3,2,\theta) \leqslant (1/53)^{k/2}$,	$v_k(3,-2,\theta) \leqslant (1/21)^{k/2}$.
When $N = 17$,	$v_k(1,4,\theta) \leqslant (1/177)^{k/2}$,	$v_k(1,-4,\theta) \leqslant (3/467)^{k/2}$,
	$v_k(4,1,\theta) \leqslant (1/27)^{k/2}$,	$v_k(4,-1,\theta) \leqslant (3/17)^{k/2}$.
When $N \geqslant 25$,	$\lvert ce^{i\theta/2} \pm \sqrt{11}de^{-i\theta/2}\rvert^2 \geqslant \frac{1}{3}N$,	

and the number of terms with $c^2 + d^2 = N$ is not more than $(48/11)N^{1/2}$ for $N \geqslant 25$. Then

$$|R^*_{11,2}|_{N\geqslant 25} = \frac{96\sqrt{3}}{11}\sum_{N=25}^{\infty}\left(\frac{1}{12}N\right)^{(1-k)/2} \leqslant \frac{4608\sqrt{6}}{11(k-3)}\left(\frac{1}{2}\right)^{k/2}.$$

Thus

$$|R^*_{11,1}| \leqslant 2 + 2\left(\frac{3}{5}\right)^{k/2} + 2\left(\frac{3}{7}\right)^{k/2} + \cdots + 2\left(\frac{1}{185}\right)^{k/2} + \frac{12000}{11(k-3)}\left(\frac{1}{2}\right)^{k/2}, \tag{96}$$

$$|R^*_{11,2}| \leqslant 2 + 2\left(\frac{3}{5}\right)^{k/2} + 2\left(\frac{1}{3}\right)^{k/2} + \cdots + 2\left(\frac{1}{177}\right)^{k/2} + \frac{4608\sqrt{6}}{11(k-3)}\left(\frac{1}{2}\right)^{k/2}. \tag{97}$$

We want to show that $|R^*_{11,1}| < 2$ and $|R^*_{11,2}| < 2$. But the right-hand sides of both bounds are greater than 2. The cases $(c,d) = \pm(3,1)$ give us bound equal to 2 for $|R^*_{11,1}|$, and the cases $(c,d) = \pm(3,-1)$ give us bound equal to 2 for $|R^*_{11,2}|$.

5.6.4. $\Gamma_0^*(13)$. We note that $N := c^2 + d^2$.

Firstly, we consider the case $N = 1$. Then we can write;

$$F^*_{k,13,1}(\theta) = 2\cos(k\theta/2) + R^*_{13,1}, \tag{98}$$

$$F^*_{k,13,2}(\theta) = 2\cos(k\theta/2) + R^*_{13,2}, \tag{99}$$

$$F^*_{k,13,3}(\theta) = 2\cos(k\theta/2) + R^*_{13,3}, \tag{100}$$

where $R^*_{13,1}$, $R^*_{13,2}$, and $R^*_{13,3}$ are the terms such that $N > 1$ of $F^*_{k,13,1}$, $F^*_{k,13,2}$, and $F^*_{k,13,3}$, respectively.

Firstly, for $F^*_{k,13,1}(\theta)$,

$$|R^*_{13,1}| \leqslant \sum_{\substack{(c,d)=1\\ 13\nmid c\\ N>1}} |ce^{i\theta/2} + \sqrt{13}de^{-i\theta/2}|^{-k}.$$

Let $v_k(c,d,\theta) := |ce^{i\theta/2} + \sqrt{13}de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\theta) = 1/\left(c^2 + 13d^2 + 2\sqrt{13}cd\cos\theta\right)^{k/2}$, and $v_k(c,d,\theta) = v_k(-c,-d,\theta)$. Now we will consider the next cases, namely $N = 2, 5, \cdots, 37$, and $N \geqslant 41$. Considering $-7/(2\sqrt{13}) \leqslant \cos\theta \leqslant 0$, we have the following:

When $N = 2$,	$v_k(1,1,\theta) \leqslant (1/7)^{k/2}$,	$v_k(1,-1,\theta) \leqslant (1/14)^{k/2}$.
When $N = 5$,	$v_k(1,2,\theta) \leqslant (1/39)^{k/2}$,	$v_k(1,-2,\theta) \leqslant (1/53)^{k/2}$,
	$v_k(2,1,\theta) \leqslant (1/3)^{k/2}$,	$v_k(2,-1,\theta) \leqslant (1/17)^{k/2}$.
When $N = 10$,	$v_k(1,3,\theta) \leqslant (1/97)^{k/2}$,	$v_k(1,-3,\theta) \leqslant (1/118)^{k/2}$,

	$v_k(3,1,\theta) \leqslant 1,$	$v_k(3,-1,\theta) \leqslant (1/22)^{k/2}.$
When $N=13$,	$v_k(2,3,\theta) \leqslant (1/79)^{k/2},$	$v_k(2,-3,\theta) \leqslant (1/11)^{k},$
	$v_k(3,2,\theta) \leqslant (1/19)^{k/2},$	$v_k(3,-2,\theta) \leqslant (1/61)^{k/2}.$
When $N=17$,	$v_k(1,4,\theta) \leqslant (1/181)^{k/2},$	$v_k(1,-4,\theta) \leqslant (1/209)^{k/2},$
	$v_k(4,1,\theta) \leqslant 1,$	$v_k(4,-1,\theta) \leqslant (1/29)^{k/2}.$
When $N=25$,	$v_k(3,4,\theta) \leqslant (1/133)^{k/2},$	$v_k(3,-4,\theta) \leqslant (1/217)^{k/2},$
	$v_k(4,3,\theta) \leqslant (1/39)^{k/2},$	$v_k(4,-3,\theta) \leqslant (1/123)^{k/2}.$
When $N=26$,	$v_k(1,5,\theta) \leqslant (1/291)^{k/2},$	$v_k(1,-5,\theta) \leqslant (1/326)^{k/2},$
	$v_k(5,1,\theta) \leqslant (1/3)^{k/2},$	$v_k(5,-1,\theta) \leqslant (1/38)^{k/2}.$
When $N=29$,	$v_k(2,5,\theta) \leqslant (1/255)^{k/2},$	$v_k(2,-5,\theta) \leqslant (1/329)^{k/2},$
	$v_k(5,2,\theta) \leqslant (1/7)^{k/2},$	$v_k(5,-2,\theta) \leqslant (1/77)^{k/2}.$
When $N=34$,	$v_k(3,5,\theta) \leqslant (1/229)^{k/2},$	$v_k(3,-5,\theta) \leqslant (1/334)^{k/2},$
	$v_k(5,3,\theta) \leqslant (1/37)^{k/2},$	$v_k(5,-3,\theta) \leqslant (1/142)^{k/2}.$
When $N=37$,	$v_k(1,6,\theta) \leqslant (1/426)^{k/2},$	$v_k(1,-6,\theta) \leqslant (1/468)^{k/2},$
	$v_k(6,1,\theta) \leqslant (1/7)^{k/2},$	$v_k(6,-1,\theta) \leqslant (1/7)^{k}.$
When $N \geqslant 41$,	$\lvert ce^{i\theta/2} \pm \sqrt{13}de^{-i\theta/2}\rvert^2 \geqslant N/20,$	

and the rest of the question is about the number of terms with $c^2+d^2=N$. Because $13 \nmid c$, the number of $|c|$ is not more than $(12/13)N^{1/2}+1$. Thus the number of terms with $c^2+d^2=N$ is not more than $4((12/13)N^{1/2}+1) \leqslant (30/7)N^{1/2}$ for $N \geqslant 41$. Then

$$|R^*_{13,1}|_{N\geqslant 41} = \frac{60\sqrt{5}}{7}\sum_{N=41}^{\infty}\left(\frac{1}{20}N\right)^{(1-k)/2} \leqslant \frac{4800\sqrt{10}}{13(k-3)}\left(\frac{1}{2}\right)^{k/2}.$$

Secondly, for $F^*_{k,13,2}(\theta)$,

$$|R^*_{13,2}| \leqslant \sum_{\substack{(c,d)=1\\ 13\nmid c\\ 2|cd\\ N>1}} v_k(c,d,\theta) + 2^k \sum_{\substack{(c,d)=1\\ 13\nmid c\\ 2\nmid cd}} v_k(c,d,\theta).$$

Now we will consider the next cases, namely $N=2,5,\cdots,26$, and $N \geqslant 29$. Considering $0 \leqslant \cos\theta \leqslant 3/\sqrt{13}$, we have the following:

When $N=2$,	$2^k \cdot v_k(1,1,\theta) \leqslant (2/7)^{k/2},$	$2^k \cdot v_k(1,-1,\theta) \leqslant (1/2)^{k/2}.$
When $N=5$,	$v_k(1,2,\theta) \leqslant (1/53)^{k/2},$	$v_k(1,-2,\theta) \leqslant (1/41)^{k/2},$
	$v_k(2,1,\theta) \leqslant (1/17)^{k/2},$	$v_k(2,-1,\theta) \leqslant (1/5)^{k/2}.$
When $N=10$,	$2^k \cdot v_k(1,3,\theta) \leqslant (2/59)^{k/2},$	$2^k \cdot v_k(1,-3,\theta) \leqslant (1/5)^{k},$
	$2^k \cdot v_k(3,1,\theta) \leqslant (2/11)^{k/2},$	$2^k \cdot v_k(3,-1,\theta) \leqslant 1.$
When $N=13$,	$v_k(2,3,\theta) \leqslant (1/11)^{k},$	$v_k(2,-3,\theta) \leqslant (1/85)^{k/2},$
	$v_k(3,2,\theta) \leqslant (1/61)^{k/2},$	$v_k(3,-2,\theta) \leqslant (1/5)^{k}.$
When $N=17$,	$v_k(1,4,\theta) \leqslant (1/209)^{k/2},$	$v_k(1,-4,\theta) \leqslant (1/285)^{k/2},$
	$v_k(4,1,\theta) \leqslant (1/29)^{k/2},$	$v_k(4,-1,\theta) \leqslant (1/5)^{k/2}.$
When $N=25$,	$v_k(3,4,\theta) \leqslant (1/217)^{k/2},$	$v_k(3,-4,\theta) \leqslant (1/145)^{k/2},$
	$v_k(4,3,\theta) \leqslant (1/123)^{k/2},$	$v_k(4,-3,\theta) \leqslant (1/51)^{k/2}.$
When $N=26$,	$2^k \cdot v_k(1,5,\theta) \leqslant (2/163)^{k/2},$	$2^k \cdot v_k(1,-5,\theta) \leqslant (1/74)^{k/2},$

$$2^k \cdot v_k(5,1,\theta) \leqslant (2/19)^{k/2}, \qquad 2^k \cdot v_k(5,-1,\theta) \leqslant (1/2)^{k/2}.$$

When $N \geqslant 29$, $\quad |ce^{i\theta/2} \pm \sqrt{13}de^{-i\theta/2}|^2 \geqslant 2N/7$,

and the number of terms with $c^2+d^2=N$ is not more than $(22/5)N^{1/2}$ for $N \geqslant 29$. Then

$$|R^*_{13,2}|_{N\geqslant 29} = \frac{22\sqrt{14}}{5}\sum_{N=29}^{\infty}\left(\frac{1}{14}N\right)^{(1-k)/2} \leqslant \frac{2464\sqrt{7}}{5(k-3)}\left(\frac{1}{2}\right)^{k/2}.$$

Finally, for $F^*_{k,13,3}(\theta)$,

$$|R^*_{13,3}| \leqslant \sum_{\substack{(c,d)=1\\ 13\nmid c\\ 3|cd\\ N>1}} v_k(c,d,\theta) + \sum_{\substack{(c,d)=1\\ 13\nmid c\\ 3\nmid cd\\ c\equiv d\ (3)}} v_k(c,d,\theta) + 3^k \sum_{\substack{(c,d)=1\\ 13\nmid c\\ 3\nmid cd\\ c\equiv -d\ (3)}} v_k(c,d,\theta).$$

Now we will consider the next cases, namely $N = 2, 5, \cdots, 37$, and $N \geqslant 41$. Considering $-2/\sqrt{13} \leqslant \cos\theta \leqslant 5/(2\sqrt{13})$, we have the following:

When $N=2$, $\quad v_k(1,1,\theta) \leqslant (1/10)^{k/2}, \quad 3^k \cdot v_k(1,-1,\theta) \leqslant 1.$

When $N=5$, $\quad 3^k \cdot v_k(1,2,\theta) \leqslant (1/5)^{k/2}, \quad v_k(1,-2,\theta) \leqslant (1/43)^{k/2},$
$\quad 3^k \cdot v_k(2,1,\theta) \leqslant 1, \quad v_k(2,-1,\theta) \leqslant (1/7)^{k/2}.$

When $N=10$, $\quad v_k(1,3,\theta) \leqslant (1/106)^{k/2}, \quad v_k(1,-3,\theta) \leqslant (1/103)^{k/2},$
$\quad v_k(3,1,\theta) \leqslant (1/10)^{k/2}, \quad v_k(3,-1,\theta) \leqslant (1/7)^{k/2}.$

When $N=13$, $\quad v_k(2,3,\theta) \leqslant (1/97)^{k/2}, \quad v_k(2,-3,\theta) \leqslant (1/91)^{k/2},$
$\quad v_k(3,2,\theta) \leqslant (1/37)^{k/2}, \quad v_k(3,-2,\theta) \leqslant (1/31)^{k/2}.$

When $N=17$, $\quad v_k(1,4,\theta) \leqslant (1/193)^{k/2}, \quad 3^k \cdot v_k(1,-4,\theta) \leqslant (1/21)^{k/2},$
$\quad v_k(4,1,\theta) \leqslant (1/13)^{k/2}, \quad 3^k \cdot v_k(4,-1,\theta) \leqslant 1.$

When $N=25$, $\quad v_k(3,4,\theta) \leqslant (1/13)^{k}, \quad v_k(3,-4,\theta) \leqslant (1/157)^{k/2},$
$\quad v_k(4,3,\theta) \leqslant (1/75)^{k/2}, \quad v_k(4,-3,\theta) \leqslant (1/63)^{k/2}.$

When $N=26$, $\quad 3^k \cdot v_k(1,5,\theta) \leqslant (1/34)^{k/2}, \quad v_k(1,-5,\theta) \leqslant (1/301)^{k/2},$
$\quad 3^k \cdot v_k(5,1,\theta) \leqslant (1/2)^{k/2}, \quad v_k(5,-1,\theta) \leqslant (1/13)^{k/2}.$

When $N=29$, $\quad v_k(2,5,\theta) \leqslant (1/13)^{k}, \quad 3^k \cdot v_k(2,-5,\theta) \leqslant (1/31)^{k/2},$
$\quad v_k(5,2,\theta) \leqslant (1/37)^{k/2}, \quad 3^k \cdot v_k(5,-2,\theta) \leqslant (1/3)^{k/2}.$

When $N=34$, $\quad v_k(3,5,\theta) \leqslant (1/274)^{k/2}, \quad v_k(3,-5,\theta) \leqslant (1/259)^{k/2},$
$\quad v_k(5,3,\theta) \leqslant (1/82)^{k/2}, \quad v_k(5,-3,\theta) \leqslant (1/67)^{k/2}.$

When $N=37$, $\quad v_k(1,6,\theta) \leqslant (1/444)^{k/2}, \quad v_k(1,-6,\theta) \leqslant (1/438)^{k/2},$
$\quad v_k(6,1,\theta) \leqslant (1/5)^{k}, \quad v_k(6,-1,\theta) \leqslant (1/19)^{k/2}.$

When $N \geqslant 41$, $\quad |ce^{i\theta/2} \pm \sqrt{13}de^{-i\theta/2}|^2 \geqslant 9N/20$,

and the number of terms with $c^2+d^2=N$ is not more than $(30/7)N^{1/2}$ for $N \geqslant 41$. Then

$$|R^*_{13,3}|_{N\geqslant 41} = \frac{60\sqrt{5}}{7}\sum_{N=41}^{\infty}\left(\frac{1}{20}N\right)^{(1-k)/2} \leqslant \frac{4800\sqrt{10}}{7(k-3)}\left(\frac{1}{2}\right)^{k/2}.$$

Thus

$$|R^*_{13,1}| \leqslant 4 + 4\left(\frac{1}{3}\right)^{k/2} + 6\left(\frac{1}{7}\right)^{k/2} + \cdots + 2\left(\frac{1}{468}\right)^{k/2} + \frac{4800\sqrt{10}}{7(k-3)}\left(\frac{1}{2}\right)^{k/2}, \tag{101}$$

$$|R^*_{13,2}| \leqslant 2 + 4\left(\frac{1}{2}\right)^{k/2} + 2\left(\frac{2}{7}\right)^{k/2} + \cdots + 2\left(\frac{1}{285}\right)^{k/2} + \frac{2464\sqrt{7}}{5(k-3)}\left(\frac{1}{2}\right)^{k/2}, \tag{102}$$

$$|R^*_{13,3}| \leqslant 6 + 2\left(\frac{1}{2}\right)^{k/2} + 2\left(\frac{1}{3}\right)^{k/2} + \cdots + 2\left(\frac{1}{444}\right)^{k/2} + \frac{4800\sqrt{10}}{7(k-3)}\left(\frac{1}{2}\right)^{k/2}. \tag{103}$$

We want to show that $|R^*_{13,1}| < 2$, $|R^*_{13,2}| < 2$, and $|R^*_{13,3}| < 2$. But the right-hand side of every bound is greater than 2. The cases $(c,d) = \pm(3,1)$ and $\pm(4,1)$ give us bound equal to 4 for $|R^*_{13,1}|$, and the cases $(c,d) = \pm(3,-1)$ give us bound equal to 2 for $|R^*_{13,2}|$, and the cases $(c,d) = \pm(1,-1)$, $\pm(2,1)$, and $\pm(4,-1)$ give us bound equal to 6 for $|R^*_{13,3}|$.

5.6.5. $\Gamma^*_0(17)$. We note that $N := c^2 + d^2$.

Firstly, we consider the case $N = 1$. Then we can write;

$$F^*_{k,17,1}(\theta) = 2\cos(k\theta/2) + R^*_{17,1}, \tag{104}$$

$$F^*_{k,17,2}(\theta) = 2\cos(k\theta/2) + R^*_{17,2}, \tag{105}$$

$$F^*_{k,17,4}(\theta) = 2\cos(k\theta/2) + R^*_{17,4}, \tag{106}$$

where $R^*_{17,1}$, $R^*_{17,2}$, and $R^*_{17,2}$, and $R^*_{17,4}$ are the terms such that $N > 1$ of $F^*_{k,17,1}$, $F^*_{k,17,2}$, and $F^*_{k,17,4}$, respectively.

Firstly, for $F^*_{k,17,1}(\theta)$,

$$|R^*_{17,1}| \leqslant \sum_{\substack{(c,d)=1 \\ 17\nmid c \\ N>1}} |ce^{i\theta/2} + \sqrt{17}de^{-i\theta/2}|^{-k}.$$

Let $v_k(c,d,\theta) := |ce^{i\theta/2} + \sqrt{17}de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\theta) = 1/\left(c^2 + 17d^2 + 2\sqrt{17}cd\cos\theta\right)^{k/2}$, and $v_k(c,d,\theta) = v_k(-c,-d,\theta)$. Now we will consider the next cases, namely $N = 2, 5, \cdots, 34$, and $N \geqslant 37$. Considering $-4/\sqrt{17} \leqslant \cos\theta \leqslant 0$, we have the following:

When $N = 2$,	$v_k(1,1,\theta) \leqslant (1/10)^{k/2}$,	$v_k(1,-1,\theta) \leqslant (1/18)^{k/2}$.
When $N = 5$,	$v_k(1,2,\theta) \leqslant (1/53)^{k/2}$,	$v_k(1,-2,\theta) \leqslant (1/69)^{k/2}$,
	$v_k(2,1,\theta) \leqslant (1/5)^{k/2}$,	$v_k(2,-1,\theta) \leqslant (1/21)^{k/2}$.
When $N = 10$,	$v_k(1,3,\theta) \leqslant (1/130)^{k/2}$,	$v_k(1,-3,\theta) \leqslant (1/154)^{k/2}$,
	$v_k(3,1,\theta) \leqslant (1/2)^{k/2}$,	$v_k(3,-1,\theta) \leqslant (1/26)^{k/2}$.
When $N = 13$,	$v_k(2,3,\theta) \leqslant (1/109)^{k/2}$,	$v_k(2,-3,\theta) \leqslant (1/157)^{k/2}$,
	$v_k(3,2,\theta) \leqslant (1/29)^{k/2}$,	$v_k(3,-2,\theta) \leqslant (1/77)^{k/2}$.
When $N = 17$,	$v_k(1,4,\theta) \leqslant (1/241)^{k/2}$,	$v_k(1,-4,\theta) \leqslant (1/273)^{k/2}$,
	$v_k(4,1,\theta) \leqslant 1$,	$v_k(4,-1,\theta) \leqslant (1/33)^{k/2}$.
When $N = 25$,	$v_k(3,4,\theta) \leqslant (1/185)^{k/2}$,	$v_k(3,-4,\theta) \leqslant (1/281)^{k/2}$,
	$v_k(4,3,\theta) \leqslant (1/73)^{k/2}$,	$v_k(4,-3,\theta) \leqslant (1/13)^{k}$.
When $N = 26$,	$v_k(1,5,\theta) \leqslant (1/386)^{k/2}$,	$v_k(1,-5,\theta) \leqslant (1/426)^{k/2}$,
	$v_k(5,1,\theta) \leqslant (1/2)^{k/2}$,	$v_k(5,-1,\theta) \leqslant (1/42)^{k/2}$.
When $N = 29$,	$v_k(2,5,\theta) \leqslant (1/349)^{k/2}$,	$v_k(2,-5,\theta) \leqslant (1/429)^{k/2}$,
	$v_k(5,2,\theta) \leqslant (1/23)^{k/2}$,	$v_k(5,-2,\theta) \leqslant (1/103)^{k/2}$.
When $N = 34$,	$v_k(3,5,\theta) \leqslant (1/314)^{k/2}$,	$v_k(3,-5,\theta) \leqslant (1/434)^{k/2}$,
	$v_k(5,3,\theta) \leqslant (1/42)^{k/2}$,	$v_k(5,-3,\theta) \leqslant (1/162)^{k/2}$.

When $N \geqslant 37$, $\quad |ce^{i\theta/2} \pm \sqrt{17}de^{-i\theta/2}|^2 \geqslant N/18,$

and the rest of the question is about the number of terms with $c^2+d^2=N$. Because $17 \nmid c$, the number of $|c|$ is not more than $(16/17)N^{1/2}+1$. Thus the number of terms with $c^2+d^2=N$ is not more than $4((16/17)N^{1/2}+1) \leqslant (22/5)N^{1/2}$ for $N \geqslant 37$. Then

$$|R^*_{17,1}|_{N\geqslant 37} = \frac{66\sqrt{2}}{5}\sum_{N=37}^{\infty}\left(\frac{1}{18}N\right)^{(1-k)/2} \leqslant \frac{9504}{5(k-3)}\left(\frac{1}{2}\right)^{k/2}.$$

Secondly, for $F^*_{k,17,2}(\theta)$,

$$|R^*_{17,2}| \leqslant \sum_{\substack{(c,d)=1\\ 17\nmid c\\ 2|cd\\ N>1}} v_k(c,d,\theta) + 2^k \sum_{\substack{(c,d)=1\\ 17\nmid c\\ 2\nmid cd}} v_k(c,d,\theta).$$

Now we will consider the next cases, namely $N = 2, 5, \cdots, 34$, and $N \geqslant 37$. Considering $0 \leqslant \cos\theta \leqslant 11/(3\sqrt{17})$, we have the following:

When $N=2$,	$2^k \cdot v_k(1,1,\theta) \leqslant (2/9)^{k/2}$,	$2^k \cdot v_k(1,-1,\theta) \leqslant (3/8)^{k/2}$.
When $N=5$,	$v_k(1,2,\theta) \leqslant (1/69)^{k/2}$,	$v_k(1,-2,\theta) \leqslant (3/163)^{k/2}$,
	$v_k(2,1,\theta) \leqslant (1/21)^{k/2}$,	$v_k(2,-1,\theta) \leqslant (1/19)^{k/2}$.
When $N=10$,	$2^k \cdot v_k(1,3,\theta) \leqslant (2/77)^{k/2}$,	$2^k \cdot v_k(1,-3,\theta) \leqslant (1/33)^{k/2}$,
	$2^k \cdot v_k(3,1,\theta) \leqslant (2/13)^{k/2}$,	$2^k \cdot v_k(3,-1,\theta) \leqslant 1$.
When $N=13$,	$v_k(2,3,\theta) \leqslant (1/157)^{k/2}$,	$v_k(2,-3,\theta) \leqslant (1/113)^{k/2}$,
	$v_k(3,2,\theta) \leqslant (1/77)^{k/2}$,	$v_k(3,-2,\theta) \leqslant (1/33)^{k/2}$.
When $N=17$,	$v_k(1,4,\theta) \leqslant (1/273)^{k/2}$,	$v_k(1,-4,\theta) \leqslant (3/731)^{k/2}$,
	$v_k(4,1,\theta) \leqslant (1/33)^{k/2}$,	$v_k(4,-1,\theta) \leqslant (3/11)^{k/2}$.
When $N=25$,	$v_k(3,4,\theta) \leqslant (1/281)^{k/2}$,	$v_k(3,-4,\theta) \leqslant (1/193)^{k/2}$,
	$v_k(4,3,\theta) \leqslant (1/13)^{k}$,	$v_k(4,-3,\theta) \leqslant (1/9)^{k}$.
When $N=26$,	$2^k \cdot v_k(1,5,\theta) \leqslant (2/213)^{k/2}$,	$2^k \cdot v_k(1,-5,\theta) \leqslant (3/292)^{k/2}$,
	$2^k \cdot v_k(5,1,\theta) \leqslant (2/21)^{k/2}$,	$2^k \cdot v_k(5,-1,\theta) \leqslant (3/4)^{k/2}$.
When $N=29$,	$v_k(2,5,\theta) \leqslant (1/429)^{k/2}$,	$v_k(2,-5,\theta) \leqslant (3/1067)^{k/2}$,
	$v_k(5,2,\theta) \leqslant (1/103)^{k/2}$,	$v_k(5,-2,\theta) \leqslant (3/89)^{k/2}$.
When $N=34$,	$2^k \cdot v_k(3,5,\theta) \leqslant (2/217)^{k/2}$,	$2^k \cdot v_k(3,-5,\theta) \leqslant (1/9)^{k}$,
	$2^k \cdot v_k(5,3,\theta) \leqslant (2/81)^{k/2}$,	$2^k \cdot v_k(5,-3,\theta) \leqslant (1/13)^{k/2}$.
When $N \geqslant 37$,	$|ce^{i\theta/2} \pm \sqrt{17}de^{-i\theta/2}|^2 \geqslant 16N/81$,	

and the number of terms with $c^2+d^2=N$ is not more than $(22/5)N^{1/2}$ for $N \geqslant 37$. Then

$$|R^*_{17,2}|_{N\geqslant 37} = \frac{99}{5}\sum_{N=37}^{\infty}\left(\frac{4}{81}N\right)^{(1-k)/2} \leqslant \frac{9504}{5(k-3)}\left(\frac{3}{4}\right)^{k}.$$

Finally, for $F^*_{k,17,4}(\theta)$,

$$|R^*_{17,4}| \leqslant \sum_{\substack{(c,d)=1\\ 17\nmid c\\ 2|cd\\ N>1}} v_k(c,d,\theta) + 2^k \sum_{\substack{(c,d)=1\\ 17\nmid c\\ 2\nmid cd\\ c\equiv d\,(4)}} v_k(c,d,\theta) + 4^k \sum_{\substack{(c,d)=1\\ 17\nmid c\\ 2\nmid cd\\ c\equiv -d\,(4)}} v_k(c,d,\theta).$$

Now we will consider the next cases, namely $N = 2, 5, \cdots, 34$, and $N \geqslant 37$. Considering $-5/(3\sqrt{17}) \leqslant \cos\theta \leqslant 1/\sqrt{17}$, we have the following:

When $N=2$, $2^k\cdot v_k(1,1,\theta)\leqslant(3/11)^{k/2}$, $4^k\cdot v_k(1,-1,\theta)\leqslant 1$.

When $N=5$, $v_k(1,2,\theta)\leqslant(3/187)^{k/2}$, $v_k(1,-2,\theta)\leqslant(1/65)^{k/2}$,

$v_k(2,1,\theta)\leqslant(3/43)^{k/2}$, $v_k(2,-1,\theta)\leqslant(1/17)^{k/2}$.

When $N=10$, $4^k\cdot v_k(1,3,\theta)\leqslant(1/3)^{k}$, $2^k\cdot v_k(1,-3,\theta)\leqslant(1/37)^{k/2}$,

$4^k\cdot v_k(3,1,\theta)\leqslant 1$, $2^k\cdot v_k(3,-1,\theta)\leqslant(1/5)^{k/2}$.

When $N=13$, $v_k(2,3,\theta)\leqslant(1/137)^{k/2}$, $v_k(2,-3,\theta)\leqslant(1/145)^{k/2}$,

$v_k(3,2,\theta)\leqslant(1/57)^{k/2}$, $v_k(3,-2,\theta)\leqslant(1/65)^{k/2}$.

When $N=17$, $v_k(1,4,\theta)\leqslant(3/779)^{k/2}$, $v_k(1,-4,\theta)\leqslant(1/265)^{k/2}$,

$v_k(4,1,\theta)\leqslant(3/59)^{k/2}$, $v_k(4,-1,\theta)\leqslant(1/5)^{k}$.

When $N=25$, $v_k(3,4,\theta)\leqslant(1/241)^{k/2}$, $v_k(3,-4,\theta)\leqslant(1/257)^{k/2}$,

$v_k(4,3,\theta)\leqslant(1/129)^{k/2}$, $v_k(4,-3,\theta)\leqslant(1/145)^{k/2}$.

When $N=26$, $2^k\cdot v_k(1,5,\theta)\leqslant(3/307)^{k/2}$, $4^k\cdot v_k(1,-5,\theta)\leqslant(1/26)^{k/2}$,

$2^k\cdot v_k(5,1,\theta)\leqslant(3/19)^{k/2}$, $4^k\cdot v_k(5,-1,\theta)\leqslant(1/2)^{k/2}$.

When $N=29$, $v_k(2,5,\theta)\leqslant(3/1187)^{k/2}$, $v_k(2,-5,\theta)\leqslant(1/409)^{k/2}$,

$v_k(5,2,\theta)\leqslant(3/209)^{k/2}$, $v_k(5,-2,\theta)\leqslant(1/83)^{k/2}$.

When $N=34$, $4^k\cdot v_k(3,5,\theta)\leqslant(1/24)^{k/2}$, $2^k\cdot v_k(3,-5,\theta)\leqslant(1/101)^{k/2}$,

$4^k\cdot v_k(5,3,\theta)\leqslant(1/7)^{k/2}$, $2^k\cdot v_k(5,-3,\theta)\leqslant(1/33)^{k/2}$.

When $N\geqslant 37$, $|ce^{i\theta/2}\pm\sqrt{17}de^{-i\theta/2}|^2\geqslant 4N/5$,

and the number of terms with $c^2+d^2=N$ is not more than $(22/5)N^{1/2}$ for $N\geqslant 41$. Then

$$|R^*_{17,4}|_{N\geqslant 37}=\frac{44\sqrt{5}}{5}\sum_{N=37}^{\infty}\left(\frac{1}{20}N\right)^{(1-k)/2}\leqslant\frac{9504}{5(k-3)}\left(\frac{5}{9}\right)^{k/2}.$$

Thus

$$|R^*_{17,1}|\leqslant 2+4\left(\frac{1}{2}\right)^{k/2}+2\left(\frac{1}{5}\right)^{k/2}+\cdots+2\left(\frac{1}{434}\right)^{k/2}+\frac{9504}{5(k-3)}\left(\frac{1}{2}\right)^{k/2},\tag{107}$$

$$|R^*_{17,2}|\leqslant 2+2\left(\frac{3}{4}\right)^{k/2}+2\left(\frac{3}{8}\right)^{k/2}+\cdots+2\left(\frac{1}{429}\right)^{k/2}+\frac{9504}{5(k-3)}\left(\frac{3}{4}\right)^{k},\tag{108}$$

$$|R^*_{17,4}|\leqslant 4+2\left(\frac{1}{2}\right)^{k/2}+2\left(\frac{1}{3}\right)^{k/2}+\cdots+2\left(\frac{1}{409}\right)^{k/2}+\frac{9504}{5(k-3)}\left(\frac{5}{9}\right)^{k/2}.\tag{109}$$

We want to show that $|R^*_{17,1}|<2$, $|R^*_{17,2}|<2$, and $|R^*_{17,4}|<2$. But the right-hand side of every bound is greater than 2. The cases $(c,d)=\pm(4,1)$ give us bound equal to 2 for $|R^*_{17,1}|$, and the cases $(c,d)=\pm(3,-1)$ give us bound equal to 2 for $|R^*_{17,2}|$, and the cases $(c,d)=\pm(1,-1)$ and $\pm(3,1)$ give us bound equal to 4 for $|R^*_{17,4}|$.

5.6.6. $\Gamma_0^*(19)$. We note that $N:=c^2+d^2$.

Firstly, we consider the case $N=1$. Then we can write;

$$F^*_{k,19,1}(\theta)=2\cos(k\theta/2)+R^*_{19,1},\tag{110}$$

$$F^*_{k,19,2}(\theta)=2\cos(k\theta/2)+R^*_{19,2},\tag{111}$$

$$F^*_{k,19,3}(\theta)=2\cos(k\theta/2)+R^*_{19,3},\tag{112}$$

where $R^*_{19,1}$, $R^*_{19,2}$, and $R^*_{19,3}$ are the terms such that $N>1$ of $F^*_{k,19,1}$, $F^*_{k,19,2}$, and $F^*_{k,19,3}$, respectively.

Firstly, for $F^*_{k,19,1}(\theta)$,

$$|R^*_{19,1}| \leqslant \sum_{\substack{(c,d)=1\\ 19\nmid c\\ N>1}} |ce^{i\theta/2} + \sqrt{19}de^{-i\theta/2}|^{-k}.$$

Let $v_k(c,d,\theta) := |ce^{i\theta/2} + \sqrt{19}de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\theta) = 1/\left(c^2+19d^2+2\sqrt{19}cd\cos\theta\right)^{k/2}$, and $v_k(c,d,\theta) = v_k(-c,-d,\theta)$. Now we will consider the next cases, namely $N = 2, 5, \cdots, 41$, and $N \geqslant 50$. Considering $-17/(4\sqrt{19}) \leqslant \cos\theta \leqslant 0$, we have the following:

When $N=2$,	$v_k(1,1,\theta) \leqslant (2/23)^{k/2}$,	$v_k(1,-1,\theta) \leqslant (1/20)^{k/2}$.
When $N=5$,	$v_k(1,2,\theta) \leqslant (1/60)^{k/2}$,	$v_k(1,-2,\theta) \leqslant (1/77)^{k/2}$,
	$v_k(2,1,\theta) \leqslant (1/6)^{k/2}$,	$v_k(2,-1,\theta) \leqslant (1/23)^{k/2}$.
When $N=10$,	$v_k(1,3,\theta) \leqslant (2/293)^{k/2}$,	$v_k(1,-3,\theta) \leqslant (1/172)^{k/2}$,
	$v_k(3,1,\theta) \leqslant (2/5)^{k/2}$,	$v_k(3,-1,\theta) \leqslant (1/28)^{k/2}$.
When $N=13$,	$v_k(2,3,\theta) \leqslant (1/124))^{k/2}$,	$v_k(2,-3,\theta) \leqslant (1/175)^{k/2}$,
	$v_k(3,2,\theta) \leqslant (1/34)^{k/2}$,	$v_k(3,-2,\theta) \leqslant (1/85)^{k/2}$.
When $N=17$,	$v_k(1,4,\theta) \leqslant (1/271)^{k/2}$,	$v_k(1,-4,\theta) \leqslant (1/305)^{k/2}$,
	$v_k(4,1,\theta) \leqslant 1$,	$v_k(4,-1,\theta) \leqslant (1/35)^{k/2}$.
When $N=25$,	$v_k(3,4,\theta) \leqslant (1/211)^{k/2}$,	$v_k(3,-4,\theta) \leqslant (1/313)^{k/2}$,
	$v_k(4,3,\theta) \leqslant (1/85)^{k/2}$,	$v_k(4,-3,\theta) \leqslant (1/187)^{k/2}$.
When $N=26$,	$v_k(1,5,\theta) \leqslant (2/867)^{k/2}$,	$v_k(1,-5,\theta) \leqslant (1/476)^{k/2}$,
	$v_k(5,1,\theta) \leqslant (2/3)^{k/2}$,	$v_k(5,-1,\theta) \leqslant (1/44)^{k/2}$.
When $N=29$,	$v_k(2,5,\theta) \leqslant (1/394)^{k/2}$,	$v_k(2,-5,\theta) \leqslant (1/479)^{k/2}$,
	$v_k(5,2,\theta) \leqslant (1/4)^{k}$,	$v_k(5,-2,\theta) \leqslant (1/101)^{k/2}$.
When $N=34$,	$v_k(3,5,\theta) \leqslant (2/715)^{k/2}$,	$v_k(3,-5,\theta) \leqslant (1/485)^{k/2}$,
	$v_k(5,3,\theta) \leqslant (2/137)^{k/2}$,	$v_k(5,-3,\theta) \leqslant (1/13)^{k}$.
When $N=37$,	$v_k(1,6,\theta) \leqslant (1/634)^{k/2}$,	$v_k(1,-6,\theta) \leqslant (1/685)^{k/2}$,
	$v_k(6,1,\theta) \leqslant (1/4)^{k/2}$,	$v_k(6,-1,\theta) \leqslant (1/55)^{k/2}$.
When $N=41$,	$v_k(4,5,\theta) \leqslant (1/321)^{k/2}$,	$v_k(4,-5,\theta) \leqslant (1/491)^{k/2}$,
	$v_k(5,4,\theta) \leqslant (1/159)^{k/2}$,	$v_k(5,-4,\theta) \leqslant (1/329)^{k/2}$.
When $N \geqslant 50$,	$\lvert ce^{i\theta/2} \pm \sqrt{19}de^{-i\theta/2}\rvert^2 \geqslant \frac{1}{22}N$,	

and the rest of the question is about the number of terms with $c^2+d^2=N$. Because $19 \nmid c$, the number of $|c|$ is not more than $(18/19)N^{1/2}+1$. Thus the number of terms with $c^2+d^2=N$ is not more than $4((18/19)N^{1/2}+1) \leqslant (13/3)N^{1/2}$ for $N \geqslant 50$. Then

$$|R^*_{19,1}|_{N\geqslant 50} = \frac{13\sqrt{22}}{3}\sum_{N=50}^{\infty}\left(\frac{1}{22}N\right)^{(1-k)/2} \leqslant \frac{4459\sqrt{462}}{33(k-3)}\left(\frac{22}{49}\right)^{k/2}.$$

Secondly, for $F^*_{k,19,2}(\theta)$,

$$|R^*_{19,2}| \leqslant \sum_{\substack{(c,d)=1\\ 19\nmid c\\ 2|cd\\ N>1}} v_k(c,d,\theta) + 2^k \sum_{\substack{(c,d)=1\\ 19\nmid c\\ 2\nmid cd}} v_k(c,d,\theta).$$

Now we will consider the next cases, namely $N = 2, 5, \cdots, 41$, and $N \geqslant 50$. Considering $0 \leqslant \cos\theta \leqslant 4/\sqrt{19}$, we have the following:

When $N=2$,	$2^k\cdot v_k(1,1,\theta)\leqslant(1/5)^{k/2}$,	$2^k\cdot v_k(1,-1,\theta)\leqslant(1/3)^{k/2}$.
When $N=5$,	$v_k(1,2,\theta)\leqslant(1/77)^{k/2}$,	$v_k(1,-2,\theta)\leqslant(1/61)^{k/2}$,
	$v_k(2,1,\theta)\leqslant(1/23)^{k/2}$,	$v_k(2,-1,\theta)\leqslant(1/7)^{k/2}$.
When $N=10$,	$2^k\cdot v_k(1,3,\theta)\leqslant(1/43)^{k/2}$,	$2^k\cdot v_k(1,-3,\theta)\leqslant(1/37)^{k/2}$,
	$2^k\cdot v_k(3,1,\theta)\leqslant(1/7)^{k/2}$,	$2^k\cdot v_k(3,-1,\theta)\leqslant 1$.
When $N=13$,	$v_k(2,3,\theta)\leqslant(1/175)^{k/2}$,	$v_k(2,-3,\theta)\leqslant(1/127)^{k/2}$,
	$v_k(3,2,\theta)\leqslant(1/85)^{k/2}$,	$v_k(3,-2,\theta)\leqslant(1/37)^{k/2}$.
When $N=17$,	$v_k(1,4,\theta)\leqslant(1/305)^{k/2}$,	$v_k(1,-4,\theta)\leqslant(1/273)^{k/2}$,
	$v_k(4,1,\theta)\leqslant(1/35)^{k/2}$,	$v_k(4,-1,\theta)\leqslant(1/3)^{k/2}$.
When $N=25$,	$v_k(3,4,\theta)\leqslant(1/313)^{k/2}$,	$v_k(3,-4,\theta)\leqslant(1/217)^{k/2}$,
	$v_k(4,3,\theta)\leqslant(1/187)^{k/2}$,	$v_k(4,-3,\theta)\leqslant(1/91)^{k/2}$.
When $N=26$,	$2^k\cdot v_k(1,5,\theta)\leqslant(1/119)^{k/2}$,	$2^k\cdot v_k(1,-5,\theta)\leqslant(1/109)^{k/2}$,
	$2^k\cdot v_k(5,1,\theta)\leqslant(1/11)^{k/2}$,	$2^k\cdot v_k(5,-1,\theta)\leqslant 1$.
When $N=29$,	$v_k(2,5,\theta)\leqslant(1/479)^{k/2}$,	$v_k(2,-5,\theta)\leqslant(1/399)^{k/2}$,
	$v_k(5,2,\theta)\leqslant(1/101)^{k/2}$,	$v_k(5,-2,\theta)\leqslant(1/21)^{k/2}$.
When $N=34$,	$2^k\cdot v_k(3,5,\theta)\leqslant(4/485)^{k/2}$,	$2^k\cdot v_k(3,-5,\theta)\leqslant(4/405)^{k/2}$,
	$2^k\cdot v_k(5,3,\theta)\leqslant(2/13)^{k}$,	$2^k\cdot v_k(5,-3,\theta)\leqslant(1/29)^{k/2}$.
When $N=37$,	$v_k(1,6,\theta)\leqslant(1/685)^{k/2}$,	$v_k(1,-6,\theta)\leqslant(1/637)^{k/2}$,
	$v_k(6,1,\theta)\leqslant(1/55)^{k/2}$,	$v_k(6,-1,\theta)\leqslant(1/7)^{k/2}$.
When $N=41$,	$v_k(4,5,\theta)\leqslant(1/491)^{k/2}$,	$v_k(4,-5,\theta)\leqslant(1/331)^{k/2}$,
	$v_k(5,4,\theta)\leqslant(1/329)^{k/2}$,	$v_k(5,-4,\theta)\leqslant(1/13)^{k}$.
When $N\geqslant 50$,	$\lvert ce^{i\theta/2}\pm\sqrt{19}de^{-i\theta/2}\rvert^2\geqslant 3N/20$,	

and the number of terms with $c^2+d^2=N$ is not more than $(13/3)N^{1/2}$ for $N\geqslant 50$. Then

$$|R^*_{19,2}|_{N\geqslant 50}=\frac{52\sqrt{15}}{9}\sum_{N=50}^{\infty}\left(\frac{3}{80}N\right)^{(1-k)/2}\leqslant\frac{4459}{3(k-3)}\left(\frac{80}{147}\right)^{k/2}.$$

Finally, for $F^*_{k,19,3}(\theta)$,

$$|R^*_{19,3}|\leqslant\sum_{\substack{(c,d)=1\\ 19\nmid c\\ 3|cd\\ N>1}}v_k(c,d,\theta)+\sum_{\substack{(c,d)=1\\ 19\nmid c\\ 3\nmid cd\\ c\equiv d\ (3)}}v_k(c,d,\theta)+3^k\sum_{\substack{(c,d)=1\\ 19\nmid c\\ 3\nmid cd\\ c\equiv -d\ (3)}}v_k(c,d,\theta).$$

Now we will consider the next cases, namely $N=2,5,\cdots,41$, and $N\geqslant 50$. Considering $-7/(2\sqrt{19})\leqslant\cos\theta\leqslant 13/(4\sqrt{19})$, we have the following:

When $N=2$,	$v_k(1,1,\theta)\leqslant(1/13)^{k/2}$,	$3^k\cdot v_k(1,-1,\theta)\leqslant(2/3)^{k/2}$.
When $N=5$,	$3^k\cdot v_k(1,2,\theta)\leqslant(1/7)^{k/2}$,	$v_k(1,-2,\theta)\leqslant(1/8)^{k}$,
	$3^k\cdot v_k(2,1,\theta)\leqslant 1$,	$v_k(2,-1,\theta)\leqslant(1/10)^{k/2}$.
When $N=10$,	$v_k(1,3,\theta)\leqslant(1/151)^{k/2}$,	$v_k(1,-3,\theta)\leqslant(2/305)^{k/2}$,
	$v_k(3,1,\theta)\leqslant(1/7)^{k/2}$,	$v_k(3,-1,\theta)\leqslant(2/17)^{k/2}$.
When $N=13$,	$v_k(2,3,\theta)\leqslant(1/133)^{k/2}$,	$v_k(2,-3,\theta)\leqslant(1/136)^{k/2}$,

$$v_k(3,2,\theta) \leqslant (1/43)^{k/2}, \qquad v_k(3,-2,\theta) \leqslant (1/46)^{k/2}.$$

When $N = 17$,
$$v_k(1,4,\theta) \leqslant (1/277)^{k/2}, \qquad 3^k \cdot v_k(1,-4,\theta) \leqslant (1/31)^{k/2},$$
$$v_k(4,1,\theta) \leqslant (1/7)^{k/2}, \qquad 3^k \cdot v_k(4,-1,\theta) \leqslant 1.$$

When $N = 25$,
$$v_k(3,4,\theta) \leqslant (1/229)^{k/2}, \qquad v_k(3,-4,\theta) \leqslant (1/235)^{k/2},$$
$$v_k(4,3,\theta) \leqslant (1/103)^{k/2}, \qquad v_k(4,-3,\theta) \leqslant (1/109)^{k/2}.$$

When $N = 26$,
$$3^k \cdot v_k(1,5,\theta) \leqslant (1/7)^{k}, \qquad v_k(1,-5,\theta) \leqslant (2/887)^{k/2},$$
$$3^k \cdot v_k(5,1,\theta) \leqslant 1, \qquad v_k(5,-1,\theta) \leqslant (2/23)^{k/2}.$$

When $N = 29$,
$$v_k(2,5,\theta) \leqslant (1/409)^{k/2}, \qquad 3^k \cdot v_k(2,-5,\theta) \leqslant (1/46)^{k/2},$$
$$v_k(5,2,\theta) \leqslant (1/31)^{k/2}, \qquad 3^k \cdot v_k(5,-2,\theta) \leqslant (1/2)^{k}.$$

When $N = 34$,
$$v_k(3,5,\theta) \leqslant (1/380)^{k/2}, \qquad v_k(3,-5,\theta) \leqslant (2/775)^{k/2},$$
$$v_k(5,3,\theta) \leqslant (1/91)^{k/2}, \qquad v_k(5,-3,\theta) \leqslant (2/197)^{k/2}.$$

When $N = 37$,
$$v_k(1,6,\theta) \leqslant (1/643)^{k/2}, \qquad v_k(1,-6,\theta) \leqslant (1/646)^{k/2},$$
$$v_k(6,1,\theta) \leqslant (1/13)^{k/2}, \qquad v_k(6,-1,\theta) \leqslant (1/4)^{k}.$$

When $N = 41$,
$$3^k \cdot v_k(4,5,\theta) \leqslant (1/39)^{k/2}, \qquad v_k(4,-5,\theta) \leqslant (1/19)^{k},$$
$$3^k \cdot v_k(5,4,\theta) \leqslant (1/21)^{k/2}, \qquad v_k(5,-4,\theta) \leqslant (1/199)^{k/2}.$$

When $N \geqslant 50$,
$$|ce^{i\theta/2} \pm \sqrt{19}de^{-i\theta/2}|^2 \geqslant 27N/80,$$

and the number of terms with $c^2 + d^2 = N$ is not more than $(13/3)N^{1/2}$ for $N \geqslant 50$. Then

$$|R^*_{19,3}|_{N\geqslant 50} = \frac{52\sqrt{15}}{9} \sum_{N=50}^{\infty} \left(\frac{3}{80}N\right)^{(1-k)/2} \leqslant \frac{4459}{3(k-3)} \left(\frac{80}{147}\right)^{k/2}.$$

Thus

$$|R^*_{19,1}| \leqslant 2 + 2\left(\frac{2}{3}\right)^{k/2} + 2\left(\frac{2}{5}\right)^{k/2} + \cdots + 2\left(\frac{1}{685}\right)^{k/2} + \frac{4459\sqrt{462}}{33(k-3)}\left(\frac{22}{49}\right)^{k/2}, \tag{113}$$

$$|R^*_{19,2}| \leqslant 4 + 4\left(\frac{1}{3}\right)^{k/2} + 2\left(\frac{1}{5}\right)^{k/2} + \cdots + 2\left(\frac{1}{685}\right)^{k/2} + \frac{4459}{3(k-3)}\left(\frac{80}{147}\right)^{k/2}, \tag{114}$$

$$|R^*_{19,3}| \leqslant 6 + 2\left(\frac{2}{3}\right)^{k/2} + 2\left(\frac{1}{2}\right)^{k} + \cdots + 2\left(\frac{1}{646}\right)^{k/2} + \frac{4459}{3(k-3)}\left(\frac{80}{147}\right)^{k/2}. \tag{115}$$

We want to show that $|R^*_{19,1}| < 2$, $|R^*_{19,2}| < 2$, and $|R^*_{19,3}| < 2$. But the right-hand side of every bound is greater than 2. The cases $(c,d) = \pm(4,1)$ give us bound equal to 2 for $|R^*_{19,1}|$, and the cases $(c,d) = \pm(3,-1)$ and $\pm(5,-1)$ give us bound equal to 4 for $|R^*_{19,2}|$, and the cases $(c,d) = \pm(2,1)$, $\pm(4,-1)$, and $\pm(5,1)$ give us bound equal to 6 for $|R^*_{19,3}|$.

5.7. **Valence formula.** In order to decide the locating of zeros of $E^*_{k,p}(z)$, we need the valence formula for $\Gamma^*_0(p)$:

5.7.1. *Valence formula for* $\Gamma^*_0(5)$.

Proposition 5.12. *Let f be a modular function of weight k for $\Gamma^*_0(5)$, which is not identically zero. We have*

$$v_\infty(f) + \frac{1}{2}v_{i/\sqrt{5}}(f) + \frac{1}{2}v_{\rho_{5,1}}(f) + \frac{1}{2}v_{\rho_{5,2}}(f) + \sum_{\substack{p\in\Gamma^*_0(5)\backslash\mathbb{H} \\ p\neq i/\sqrt{5},\rho_{5,1},\rho_{5,2}}} v_p(f) = \frac{k}{4}, \tag{116}$$

where $\rho_{5,1} := -1/2 + i/\left(2\sqrt{5}\right)$, $\rho_{5,2} := -2/5 + i/5$.

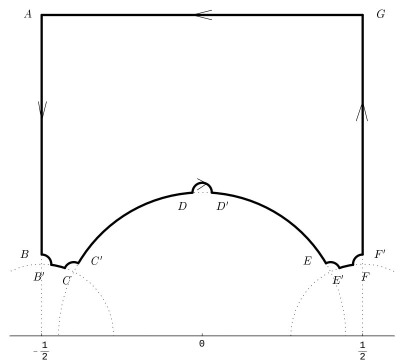


FIGURE 8

Proof. Let f be a nonzero modular function of weight k for $\Gamma^*_0(5)$, and let $\mathscr{C}$ be a contour of its fundamental domain $\mathbb{F}^*(5)$ represented in Figure 8, whose interior contains every zero and pole of f except for $i/\sqrt{5}$, $\rho_{5,1}$, and $\rho_{5,2}$. By the *Residue theorem*, we have

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{df}{f} = \sum_{\substack{p\in\Gamma^*_0(5)\backslash\mathbb{H} \\ p\neq i/\sqrt{5},\rho_{5,1},\rho_{5,2}}} v_p(f).$$

Similar to Proposition 3.1, (See [SE])

(i) For the arc GA, we have

$$\frac{1}{2\pi i}\int_G^A \frac{df}{f} = -v_\infty(f).$$

(ii) For the arcs BB', CC', DD', EE', and FF', when the radii of each arc tends to 0, then we have

$$\frac{1}{2\pi i}\int_B^{B'} \frac{df}{f} = \frac{1}{2\pi i}\int_F^{F'} \frac{df}{f} \to -\frac{1}{4}v_{\rho_{5,1}}(f),$$
$$\frac{1}{2\pi i}\int_C^{C'} \frac{df}{f} = \frac{1}{2\pi i}\int_E^{E'} \frac{df}{f} \to -\frac{1}{4}v_{\rho_{5,2}}(f),$$
$$\frac{1}{2\pi i}\int_D^{D'} \frac{df}{f} \to -\frac{1}{2}v_{i/\sqrt{5}}(f).$$

(iii) For the arcs AB and $F'G$, since $f(Tz) = f(z)$ for $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$,

$$\frac{1}{2\pi i}\int_A^B \frac{df}{f} + \frac{1}{2\pi i}\int_{F'}^G \frac{df}{f} = 0.$$

(iv) For the arcs $C'D$ and $D'E$, since $f(W_5z) = (\sqrt{5}z)^k f(z)$, we have

$$\frac{df(W_5z)}{f(W_5z)} = k\frac{dz}{z} + \frac{df(z)}{f(z)}.$$

When the radii of the arcs CC', DD', EE' tend to 0, then

$$\frac{1}{2\pi i}\int_{C'}^{D}\frac{df(z)}{f(z)} + \frac{1}{2\pi i}\int_{D'}^{E}\frac{df(z)}{f(z)} = \frac{1}{2\pi i}\int_{C'}^{D}\left(-k\frac{dz}{z}\right) \to k\frac{\theta_1}{2\pi},$$

where $\tan\theta_1 = 2$.

Similarly, for the arcs $B'C$ and $E'F$, since $f\left(\left(\begin{smallmatrix}-2&1\\-5&2\end{smallmatrix}\right)W_5z\right) = (2\sqrt{5}z+\sqrt{5})^k f(z)$, we have

$$\frac{df\left(\left(\begin{smallmatrix}-2&1\\-5&2\end{smallmatrix}\right)W_5z\right)}{f\left(\left(\begin{smallmatrix}-2&1\\-5&2\end{smallmatrix}\right)W_5z\right)} = k\frac{dz}{z+1/2} + \frac{df(z)}{f(z)}.$$

When the radii of the arcs CC', DD', EE' tend to 0, then

$$\frac{1}{2\pi i}\int_{B'}^{C}\frac{df(z)}{f(z)} + \frac{1}{2\pi i}\int_{E'}^{F}\frac{df(z)}{f(z)} = \frac{1}{2\pi i}\int_{B'}^{C}\left(-k\frac{dz}{z+1/2}\right) \to k\frac{\theta_2}{2\pi},$$

where $\tan\theta_1 = 1/2$.

Thus, since $\theta_1 + \theta_2 = \pi/2$,

$$k\frac{\theta_1}{2\pi} + k\frac{\theta_2}{2\pi} = \frac{k}{4}$$

□

5.7.2. *Valence formula for $\Gamma_0^*(7)$.*

Proposition 5.13. *Let f be a modular function of weight k for $\Gamma_0^*(7)$, which is not identically zero. We have*

$$v_\infty(f) + \frac{1}{2}v_{i/\sqrt{7}}(f) + \frac{1}{2}v_{\rho_{7,1}}(f) + \frac{1}{3}v_{\rho_{7,2}}(f) + \sum_{\substack{p\in\Gamma_0^*(7)\backslash\mathbb{H}\\ p\neq i/\sqrt{7},\rho_{7,1},\rho_{7,2}}} v_p(f) = \frac{k}{3}, \tag{117}$$

where $\rho_{7,1} := -1/2 + i/\left(2\sqrt{7}\right)$, $\rho_{7,2} := -5/14 + \sqrt{3}i/14$.

The proof of this proposition is similar to Proposition 3.1, 5.12.

5.7.3. *Valence formula for $\Gamma_0^*(11)$.*

Proposition 5.14. *Let f be a modular function of weight k for $\Gamma_0^*(11)$, which is not identically zero. We have*

$$\begin{aligned} v_\infty(f) &+ \frac{1}{2}v_{i/\sqrt{11}}(f) + \frac{1}{2}v_{\rho_{11,1}}(f) + \frac{1}{2}v_{\rho_{11,2}}(f) + \frac{1}{2}v_{\rho_{11,3}}(f) \\ &+ \sum_{\substack{p\in\Gamma_0^*(11)\backslash\mathbb{H}\\ p\neq i/\sqrt{11},\rho_{11,1},\rho_{11,2},\rho_{11,3}}} v_p(f) = \frac{k}{2}, \end{aligned} \tag{118}$$

where $\rho_{11,1} := -1/2 + i/\left(2\sqrt{11}\right)$, $\rho_{11,2} := -1/3 + i/\left(3\sqrt{11}\right)$, and $\rho_{11,3} := 1/3 + i/\left(3\sqrt{11}\right)$.

Proof. Let f be a nonzero modular function of weight k for $\Gamma_0^*(11)$, and let $\mathscr{C}$ be a contour of its fundamental domain $\mathbb{F}^*(11)$ (Figure 7), whose interior contains every zero and pole of f except for $i/\sqrt{11}$, $\rho_{11,1}$, $\rho_{11,2}$, $\rho_{11,3}$, and $\rho_{11,4} := -25/66 + \sqrt{35}i/66$ (*cf.* Figure 8). By the *Residue theorem*, we have

$$\frac{1}{2\pi i}\int_{\mathscr{C}}\frac{df}{f} = \sum_{\substack{p\in\Gamma_0^*(11)\backslash\mathbb{H}\\ p\neq i/\sqrt{11},\rho_{11,1},\cdots,\rho_{11,4}}} v_p(f).$$

(i) For the arc around ∞, we have $-v_\infty(f)$.

(ii) For the arcs around $i/\sqrt{11}$, $\rho_{11,1}$, $\cdots$, $\rho_{11,4}$, when the radii of each arc tends to 0, then we have

$$-\frac{1}{2}v_{i/\sqrt{11}}(f),\quad -\frac{1}{2}v_{\rho_{11,1}}(f),\quad -\frac{1}{2}v_{\rho_{11,2}}(f),\quad -\frac{1}{2}v_{\rho_{11,3}}(f),$$
$$-v_{\rho_{11,4}}(f).$$

(iii) For the arcs on $\{z\,;\, Re(z)=-1/2\}$ and $\{z\,;\, Re(z)=1/2\}$, since $f(Tz)=f(z)$ for $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, we have 0.

(iv) For the arcs on $\{z\,;\,|z|=1/\sqrt{11}\}$, since $f(W_{11}z)=(\sqrt{11}z)^k f(z)$, we have $k\theta_1/(2\pi)$, where $\tan\theta_1 = 19/\sqrt{35}$. For the arcs on $\{z\,;\,|z\pm 1/2|=1/(2\sqrt{11})\}$, since $f\left(\left(\begin{smallmatrix}-5&1\\-11&2\end{smallmatrix}\right)W_{11}z\right)=(2\sqrt{11}z+\sqrt{11})^k f(z)$, we have $k\theta_2/(2\pi)$, where $\tan\theta_2 = 8/\sqrt{35}$.

Furthermore, for the arcs on $\{z; |z+1/3|=1/(3\sqrt{11})\}$, since $f\left(\left(\begin{smallmatrix}-4&1\\11&-3\end{smallmatrix}\right)W_{11}z\right)=(3\sqrt{11}z+\sqrt{11})^k f(z)$, we have

$$\frac{df\left(\left(\begin{smallmatrix}-4&1\\11&-3\end{smallmatrix}\right)W_{11}z\right)}{f\left(\left(\begin{smallmatrix}-4&1\\11&-3\end{smallmatrix}\right)W_{11}z\right)}=k\frac{dz}{z+1/3}+\frac{df(z)}{f(z)},\quad\text{and}\quad k\frac{\theta_3}{2\pi},$$

where $\tan\theta_3 = 3/\sqrt{35}$.

Similarly, for the arcs on $\{z\,;\,|z-1/3|=1/(3\sqrt{11})\}$, since $f\left(\left(\begin{smallmatrix}4&1\\11&3\end{smallmatrix}\right)W_{11}z\right)=(3\sqrt{11}z-\sqrt{11})^k f(z)$, we have $k\theta_3/(2\pi)$.

Thus, since $\theta_1+\theta_2+2\theta_3=\pi$,

$$k\frac{\theta_1}{2\pi}+k\frac{\theta_2}{2\pi}+2\cdot k\frac{\theta_3}{2\pi}=\frac{k}{2}.$$

□

5.7.4. *Valence formula for* $\Gamma_0^*(13)$.

Proposition 5.15. *Let f be a modular function of weight k for $\Gamma_0^*(13)$, which is not identically zero. We have*

$$v_\infty(f)+\frac{1}{2}v_{i/\sqrt{13}}(f)+\frac{1}{2}v_{\rho_{13,1}}(f)+\frac{1}{2}v_{\rho_{13,2}}(f)+\frac{1}{3}v_{\rho_{13,3}}(f) + \sum_{\substack{p\in\Gamma_0^*(13)\backslash\mathbb{H}\\ p\neq i/\sqrt{13},\rho_{13,1},\rho_{13,2},\rho_{13,3}}} v_p(f)=\frac{7}{12}k, \tag{119}$$

where $\rho_{13,1}:=-1/2+i/\left(2\sqrt{13}\right)$, $\rho_{13,2}:=-5/13+i/13$, and $\rho_{13,3}:=-7/26+\sqrt{3}i/26$.

Proof. Let f be a nonzero modular function of weight k for $\Gamma_0^*(13)$, and let $\mathscr{C}$ be a contour of its fundamental domain $\mathbb{F}^*(13)$ (Figure 7), whose interior contains every zero and pole of f except for $i/\sqrt{13}$, $\rho_{13,1}$, $\rho_{13,2}$, and $\rho_{13,3}$ (*cf.* Figure 8). By the *Residue theorem*, we have

$$\frac{1}{2\pi i}\int_{\mathscr{C}}\frac{df}{f}=\sum_{\substack{p\in\Gamma_0^*(13)\backslash\mathbb{H}\\ p\neq i/\sqrt{13},\rho_{13,1},\rho_{13,2},\rho_{13,3}}} v_p(f).$$

(i) For the arc around ∞, we have $-v_\infty(f)$.

(ii) For the arcs around $i/\sqrt{13}$, $\rho_{13,1}$, $\rho_{13,2}$, and $\rho_{13,3}$, when the radii of each arc tends to 0, then we have

$$-\frac{1}{2}v_{i/\sqrt{13}}(f),\quad -\frac{1}{2}v_{\rho_{13,1}}(f),\quad -\frac{1}{2}v_{\rho_{13,2}}(f),\quad -\frac{1}{3}v_{\rho_{13,3}}(f).$$

(iii) For the arcs on $\{z\,;\, Re(z)=-1/2\}$ and $\{z\,;\, Re(z)=1/2\}$, since $f(Tz)=f(z)$ for $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, we have 0.

(iv) For the arcs on $\{z\,;\,|z|=1/\sqrt{13}\}$, since $f(W_{13}z)=(\sqrt{13}z)^k f(z)$, we have $k\theta_1/(2\pi)$, where $\tan\theta_1 = 7/\sqrt{3}$. For the arcs on $\{z\,;\,|z\pm 1/2|=1/(2\sqrt{13})\}$, since $f\left(\left(\begin{smallmatrix}-6&1\\-13&2\end{smallmatrix}\right)W_{13}z\right)=(2\sqrt{13}z+\sqrt{13})^k f(z)$, we have $k\theta_2/(2\pi)$, where $\tan\theta_2 = 3/2$.

Furthermore, for the arcs on $\{z; |z\pm 1/3|=1/(3\sqrt{13})\}$, since $f\left(\left(\begin{smallmatrix}-4&1\\-13&3\end{smallmatrix}\right)W_{13}z\right)=(3\sqrt{13}z+\sqrt{13})^k f(z)$, we have $k(\theta_3+{\theta_3}')/(2\pi)$, where $\tan\theta_3=5/(3\sqrt{3})$ and $\tan{\theta_3}'=2/3$.

Thus, since $\theta_1+\theta_2+\theta_3+{\theta_3}'=7\pi/6$,

$$k\frac{\theta_1}{2\pi}+k\frac{\theta_2}{2\pi}+k\frac{\theta_3+{\theta_3}'}{2\pi}=\frac{7}{12}k.$$

□

5.7.5. *Valence formula for $\Gamma_0^*(17)$.*

Proposition 5.16. *Let f be a modular function of weight k for $\Gamma_0^*(17)$, which is not identically zero. We have*

$$(120)\quad v_\infty(f) + \frac{1}{2}v_{i/\sqrt{17}}(f) + \frac{1}{2}v_{\rho_{17,1}}(f) + \frac{1}{2}v_{\rho_{17,2}}(f) + \frac{1}{2}v_{\rho_{17,3}}(f) + \frac{1}{2}v_{\rho_{17,4}}(f) + \sum_{\substack{p\in\Gamma_0^*(17)\backslash\mathbb{H} \\ p\neq i/\sqrt{17},\rho_{17,1},\rho_{17,2},\rho_{17,3},\rho_{17,4}}} v_p(f) = \frac{3}{4}k,$$

where $\rho_{17,1} := -1/2 + i/\left(2\sqrt{17}\right)$, $\rho_{17,2} := -1/3 + i/\left(3\sqrt{17}\right)$, $\rho_{17,3} := -4/17 + i/17$, and $\rho_{17,4} := 1/3 + i/\left(3\sqrt{17}\right)$.

Proof. Let f be a nonzero modular function of weight k for $\Gamma_0^*(17)$, and let $\mathscr{C}$ be a contour of its fundamental domain $\mathbb{F}^*(17)$ (Figure 7), whose interior contains every zero and pole of f except for $i/\sqrt{17}$, $\rho_{17,1}, \cdots, \rho_{17,4}$, and $\rho_{17,5} := -20/51 + 2\sqrt{2}i/51$ (*cf.* Figure 8). By the *Residue theorem*, we have

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{df}{f} = \sum_{\substack{p\in\Gamma_0^*(17)\backslash\mathbb{H} \\ p\neq i/\sqrt{17},\rho_{17,1},\cdots,\rho_{17,5}}} v_p(f).$$

(i) For the arc around ∞, we have $-v_\infty(f)$.
(ii) For the arcs around $i/\sqrt{17}$, $\rho_{17,1}, \cdots, \rho_{17,5}$, when the radii of each arc tends to 0, then we have

$$-\frac{1}{2}v_{i/\sqrt{17}}(f),\quad -\frac{1}{2}v_{\rho_{17,1}}(f),\quad -\frac{1}{2}v_{\rho_{17,2}}(f),\quad -\frac{1}{2}v_{\rho_{17,3}}(f),\quad -\frac{1}{2}v_{\rho_{17,4}}(f),\quad -v_{\rho_{17,5}}(f).$$

(iii) For the arcs on $\{z\,;\, Re(z) = -1/2\}$ and $\{z\,;\, Re(z) = 1/2\}$, since $f(Tz) = f(z)$ for $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, we have 0.
(iv) For the arcs on $\{z\,;\, |z| = 1/\sqrt{17}\}$, since $f(W_{17}z) = (\sqrt{17}z)^k f(z)$, we have $k\theta_1/(2\pi)$, where $\tan\theta_1 = 4$. For the arcs on $\{z\,;\, |z \pm 1/2| = 1/(2\sqrt{17})\}$, since $f\left(\left(\begin{smallmatrix}-8&1\\-17&2\end{smallmatrix}\right)W_{17}z\right) = (2\sqrt{17}z + \sqrt{17})^k f(z)$, we have $k\theta_2/(2\pi)$, where $\tan\theta_2 = 11/(4\sqrt{2})$.

For the arcs on $\{z\,;\, |z + 1/3| = 1/(3\sqrt{17})\}$, since $f\left(\left(\begin{smallmatrix}-6&1\\17&-3\end{smallmatrix}\right)W_{17}z\right) = (3\sqrt{17}z + \sqrt{17})^k f(z)$, we have $k\theta_3/(2\pi)$, where $\tan\theta_3 = 3/(2\sqrt{2})$. Similarly, for the arcs on $\{z\,;\, |z - 1/3| = 1/(3\sqrt{17})\}$, since $f\left(\left(\begin{smallmatrix}6&1\\17&3\end{smallmatrix}\right)W_{17}z\right) = (3\sqrt{17}z - \sqrt{17})^k f(z)$, we have $k\theta_3/(2\pi)$.

Furthermore, for the arcs on $\{z; |z\pm1/4| = 1/(4\sqrt{17})\}$, since $f\left(\left(\begin{smallmatrix}-4&1\\-17&4\end{smallmatrix}\right)W_{17}z\right) = (4\sqrt{17}z+\sqrt{17})^k f(z)$, we have

$$\frac{df\left(\left(\begin{smallmatrix}-4&1\\-17&4\end{smallmatrix}\right)W_{17}z\right)}{f\left(\left(\begin{smallmatrix}-4&1\\-17&4\end{smallmatrix}\right)W_{17}z\right)} = k\frac{dz}{z+1/4} + \frac{df(z)}{f(z)},\quad \text{and}\quad k\frac{\theta_4+\theta_4{}'}{2\pi},$$

where $\tan\theta_4 = 1/4$ and $\tan\theta_4{}' = 5/(8\sqrt{2})$..

Thus, since $\theta_1 + \theta_2 + 2\theta_3 + \theta_4 + \theta_4{}' = 3\pi/2$,

$$k\frac{\theta_1}{2\pi} + k\frac{\theta_2}{2\pi} + 2\cdot k\frac{\theta_3}{2\pi} + k\frac{\theta_4+\theta_4{}'}{2\pi} = \frac{3}{4}k.$$

□

5.7.6. *Valence formula for $\Gamma_0^*(19)$.*

Proposition 5.17. *Let f be a modular function of weight k for $\Gamma_0^*(19)$, which is not identically zero. We have*

$$(121)\quad v_\infty(f) + \frac{1}{2}v_{i/\sqrt{19}}(f) + \frac{1}{2}v_{\rho_{19,1}}(f) + \frac{1}{3}v_{\rho_{19,2}}(f) + \frac{1}{2}v_{\rho_{19,3}}(f) + \frac{1}{2}v_{\rho_{19,4}}(f) + \sum_{\substack{p\in\Gamma_0^*(19)\backslash\mathbb{H} \\ p\neq i/\sqrt{19},\rho_{19,1},\rho_{19,2},\rho_{19,3},\rho_{19,4}}} v_p(f) = \frac{5}{6}k,$$

where $\rho_{19,1} := -1/2 + i/\left(2\sqrt{19}\right)$, $\rho_{19,2} := -15/38 + \sqrt{3}i/38$, $\rho_{19,3} := -1/4 + i/\left(4\sqrt{19}\right)$, *and* $\rho_{19,4} := 1/4 + i/\left(4\sqrt{19}\right)$.

Proof. Let f be a nonzero modular function of weight k for $\Gamma_0^*(19)$, and let $\mathscr{C}$ be a contour of its fundamental domain $\mathbb{F}^*(19)$ (Figure 7), whose interior contains every zero and pole of f except for $i/\sqrt{19}$, $\rho_{19,1}, \cdots, \rho_{19,4}$, and $\rho_{19,5} := -21/76 + \sqrt{15}i/76$ (*cf.* Figure 8). By the *Residue theorem*, we have

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{df}{f} = \sum_{\substack{p \in \Gamma_0^*(19)\backslash\mathbb{H} \\ p \neq i/\sqrt{19}, \rho_{19,1}, \cdots, \rho_{19,5}}} v_p(f).$$

(i) For the arc around ∞, we have $-v_\infty(f)$.
(ii) For the arcs around $i/\sqrt{19}$, $\rho_{19,1}, \cdots, \rho_{19,5}$, when the radii of each arc tends to 0, then we have

$$-\frac{1}{2}v_{i/\sqrt{19}}(f), \quad -\frac{1}{2}v_{\rho_{19,1}}(f), \quad -\frac{1}{3}v_{\rho_{19,2}}(f), \quad -\frac{1}{2}v_{\rho_{19,3}}(f), \quad -\frac{1}{2}v_{\rho_{19,4}}(f),$$
$$-v_{\rho_{19,5}}(f).$$

(iii) For the arcs on $\{z\,;\, Re(z) = -1/2\}$ and $\{z\,;\, Re(z) = 1/2\}$, since $f(Tz) = f(z)$ for $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, we have 0.
(iv) For the arcs on $\{z\,;\, |z| = 1/\sqrt{19}\}$, since $f(W_{19}z) = (\sqrt{19}z)^k f(z)$, we have $k\theta_1/(2\pi)$, where $\tan\theta_1 = 17/\sqrt{15}$. For the arcs on $\{z\,;\, |z \pm 1/2| = 1/(2\sqrt{19})\}$, since $f\left(\left(\begin{smallmatrix} -9 & 1 \\ -19 & 2 \end{smallmatrix}\right) W_{19}z\right) = (2\sqrt{19}z + \sqrt{19})^k f(z)$, we have $k\theta_2/(2\pi)$, where $\tan\theta_2 = 4/\sqrt{3}$. For the arcs on $\{z;\, |z \pm 1/3| = 1/(3\sqrt{19})\}$, since $f\left(\left(\begin{smallmatrix} -6 & 1 \\ -19 & 3 \end{smallmatrix}\right) W_{19}z\right) = (3\sqrt{19}z + \sqrt{19})^k f(z)$, we have $k(\theta_3 + {\theta_3}')/(2\pi)$, where $\tan\theta_3 = 13/(3\sqrt{15})$ and $\tan{\theta_3}' = 7/(3\sqrt{3})$.

Furthermore, for the arcs on $\{z;\, |z+1/4| = 1/(4\sqrt{19})\}$, since $f\left(\left(\begin{smallmatrix} -5 & 1 \\ 19 & -4 \end{smallmatrix}\right) W_{19}z\right) = (4\sqrt{19}z + \sqrt{19})^k f(z)$, we have $k\theta_4/(2\pi)$, where $\tan\theta_4 = 2/\sqrt{15}$. Similarly, for the arcs on $\{z\,;\, |z - 1/4| = 1/(4\sqrt{19})\}$, since $f\left(\left(\begin{smallmatrix} 5 & 1 \\ 19 & 4 \end{smallmatrix}\right) W_{19}z\right) = (4\sqrt{19}z - \sqrt{19})^k f(z)$, we have $k\theta_4/(2\pi)$.

Thus, since $\theta_1 + \theta_2 + \theta_3 + {\theta_3}' + 2\theta_4 = 5\pi/3$,

$$k\frac{\theta_1}{2\pi} + k\frac{\theta_2}{2\pi} + k\frac{\theta_3 + {\theta_3}'}{2\pi} + 2 \cdot k\frac{\theta_4}{2\pi} = \frac{5}{6}k.$$

□

5.8. $\Gamma_0^*(5)$.

5.8.1. *Preliminaries.* Let $\alpha_5 \in [0, \pi/2]$ such that $\tan\alpha_5 = 2$, then we denote

$$A_{5,1}^* := \{z\,;\,|z| = 1/\sqrt{5},\ \pi/2 < Arg(z) < \pi/2 + \alpha_5\},$$
$$A_{5,2}^* := \{z\,;\,|z+1/2| = 1/(2\sqrt{5}),\ \alpha_5 < Arg(z) < \pi/2\}.$$

Now we have

$$\rho_{5,2} = -\frac{2}{5} + \frac{i}{5} = \frac{e^{i(\pi/2+\alpha_5)}}{\sqrt{5}} = \frac{e^{i\alpha_5}}{2\sqrt{5}} - \frac{1}{2},$$

and we have

$$F_{k,5,1}^*(\pi/2+\alpha_5) = e^{ik(\pi/2+\alpha_5)/2} E_{k,5}^*(\rho_{5,2}),$$
$$F_{k,5,2}^*(\alpha_5) = e^{ik\alpha_5/2} E_{k,5}^*(\rho_{5,2}).$$

Now, we define

$$F_{k,5}^*(\theta) = \begin{cases} F_{k,5,1}^*(\theta) & \pi/2 \leqslant \theta \leqslant \pi/2+\alpha_5 \\ F_{k,5,2}^*(\theta-\pi/2) & \pi/2+\alpha_5 \leqslant \theta \leqslant \pi \end{cases}.$$

Then $F_{k,5}^*$ is continuous in the interval $[\pi/2, \pi]$. Note that $F_{k,5,1}^*(\pi/2+\alpha_5) = e^{i(\pi/2)k/2} F_{k,5,2}^*(\alpha_5)$.

Let f be a modular form for $\Gamma_0^*(5)$ of weight k, and let $k \equiv 2 \pmod 4$. Then we have

$$f(i/\sqrt{5}) = f(W_5\, i/\sqrt{5}) = i^k f(i/\sqrt{5}) = -f(i/\sqrt{5}),$$
$$f(\rho_{5,1}) = f(\left(\begin{smallmatrix}1 & -1\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}-2 & 1\\ -5 & 2\end{smallmatrix}\right) W_5\, \rho_{5,1}) = i^k f(\rho_{5,1}) = -f(\rho_{5,1}),$$
$$f(\rho_{5,2}) = f(\left(\begin{smallmatrix}-2 & 1\\ -5 & 2\end{smallmatrix}\right)\ \rho_{5,2}) = i^k f(\rho_{5,2}) = -f(\rho_{5,2}).$$

Thus $f(i/\sqrt{5}) = f(\rho_{5,1}) = f(\rho_{5,2}) = 0$, then we have $v_{i/\sqrt{5}}(f) \geqslant 1$, $v_{\rho_{5,1}}(f) \geqslant 1$, and $v_{\rho_{5,2}}(f) \geqslant 1$.

Let k be an even integer such that $k \equiv 0 \pmod 4$. Then we have

$$E_{k,5}^*\left(\frac{i}{\sqrt{5}}\right) = \frac{2\cdot 5^{k/2}}{5^{k/2}+1} E_k(\sqrt{5}i) \neq 0$$
$$E_{k,5}^*(\rho_{5,1}) = \frac{2\cdot 5^{k/2}}{5^{k/2}+1} E_k\left(-\frac{1}{2} + \frac{\sqrt{5}}{2}i\right) \neq 0$$
$$E_{k,5}^*(\rho_{5,2}) = \frac{1}{5^{k/2}+1}(5^{k/2} + (2+i)^k) E_k(i) \neq 0.$$

Thus $v_{i/\sqrt{5}}(E_{k,5}^*) = v_{\rho_{5,1}}(E_{k,5}^*) = v_{\rho_{5,2}}(E_{k,5}^*) = 0$.

5.8.2. $E_{k,5}^*$ *of low weights.*
$F_{k,5}^*(\pi/2)$. Let $k \geqslant 4$ be an even integer divisible by 4. We note that $N := c^2 + d^2$.

Firstly, we consider the case $N = 1$. Then we can write;

$$F_{k,5}^*(\pi/2) = F_{k,5,1}^*(\pi/2) = 2\cos(k\pi/4) + R_{5,\pi/2}^*$$

where

$$|R_{5,\pi/2}^*| \leqslant \sum_{\substack{(c,d)=1\\ 5\nmid d\\ N>1}} |ce^{i\pi/4} + \sqrt{5}de^{-i\pi/4}|^{-k}.$$

Let $v_k(c,d,\theta) := |ce^{i\theta/2} + \sqrt{5}de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\pi/2) = 1/(c^2+5d^2)^{k/2}$. Now we will consider the next cases, namely $N = 2, 5, 10, 13, 17$, and $N \geqslant 25$. We have the following:

When $N = 2$,	$v_k(1,1,\pi/2) \leqslant (1/6)^{k/2}$.	
When $N = 5$,	$v_k(1,2,\pi/2) \leqslant (1/21)^{k/2}$,	$v_k(2,1,\pi/2) \leqslant (1/3)^k$.
When $N = 10$,	$v_k(1,3,\pi/2) \leqslant (1/46)^{k/2}$,	$v_k(3,1,\pi/2) \leqslant (1/14)^{k/2}$.
When $N = 13$,	$v_k(2,3,\pi/2) \leqslant (1/7)^k$,	$v_k(3,2,\pi/2) \leqslant (1/29)^{k/2}$.
When $N = 17$,	$v_k(1,4,\pi/2) \leqslant (1/21)^{k/2}$,	$v_k(4,1,\pi/2) \leqslant (1/3)^{2k}$.

When $N \geqslant 25$, $c^2 + 5d^2 \geqslant N$,

and the number of terms with $c^2 + d^2 = N$ is not more than $(96/25)N^{1/2}$ for $N \geqslant 25$. Then

$$|R^*_{5,\pi/2}|_{N \geqslant 25} \leqslant \frac{192}{25(k-3)} \left(\frac{1}{24}\right)^{(k-3)/2}.$$

Furthermore,

$$\begin{aligned} |R^*_{5,\pi/2}| &\leqslant 4\left(\frac{1}{6}\right)^{k/2} + 4\left(\frac{1}{3}\right)^{k} + \cdots + 4\left(\frac{1}{3}\right)^{2k} + \frac{192}{25(k-3)}\left(\frac{1}{24}\right)^{(k-3)/2}, \\ &\leqslant 1.77563... \quad (k \geqslant 4) \end{aligned}$$

In conclusion, we have following:

Lemma 5.7. *For an even integer $k \geqslant 4$,*

$$F^*_{k,5}(\pi/2) \begin{cases} > 0 & k \equiv 0 \pmod 8 \\ < 0 & k \equiv 4 \pmod 8 \\ = 0 & k \equiv 2 \pmod 4 \end{cases}.$$

$F^*_{k,5}(\pi/2 + \alpha_5)$. Let $k \geqslant 4$ be an even integer divisible by 4.

$$\begin{aligned} F^*_{k,5}(\pi/2 + \alpha_5) &= e^{ik(\pi/2+\alpha_5)/2} E^*_{k,5}(\rho_{5,2}) \\ &= \frac{e^{ik(\pi/2+\alpha_5)/2}}{5^{k/2}+1}(5^{k/2} + (2+i)^k) E_k(i). \end{aligned}$$

Here

$$\begin{aligned} e^{ik(\pi/2+\alpha_5)/2}(5^{k/2} + (2+i)^k) &= 5^{k/2} e^{ik(\pi/2+\alpha_5)/2}(1 + e^{-ik(\pi/2+\alpha_5)}) \\ &= 2 \cdot 5^{k/2} \cos(k(\pi/2+\alpha_5)/2). \end{aligned}$$

In conclusion, we have following:

Lemma 5.8. *For an even integer $k \geqslant 4$,*

$$F^*_{k,5}(\pi/2 + \alpha_5) = \begin{cases} \frac{2 \cdot 5^{k/2}}{5^{k/2}+1} \cos(k(\pi/2+\alpha_5)/2) E_k(i) & k \equiv 0 \pmod 4 \\ 0 & k \equiv 2 \pmod 4 \end{cases}.$$

Furthermore, by Proposition 3.3, we have $E_k(i) > 0$ for every $k \geqslant 4$ such that $k \equiv 0 \pmod 4$.

$F^*_{k,5}(\pi)$. Let $k \geqslant 8$ be an even integer divisible by 4. We note that $N := c^2 + d^2$.

Firstly, we consider the case $N = 1$. Then we can write;

$$F^*_{k,5}(\pi) = F^*_{k,5,2}(\pi/2) = 2\cos(k\pi/4) + R^*_{5,\pi}.$$

where

$$|R^*_{5,\pi}| \leqslant \sum_{\substack{(c,d)=1 \\ 5 \nmid c \\ 2 | cd \\ N>1}} |ce^{i\pi/4} + d\sqrt{5}e^{-i\pi/4}|^{-k} + 2^k \sum_{\substack{(c,d)=1 \\ 5 \nmid c \\ 2 \nmid cd}} |ce^{i\pi/4} + d\sqrt{5}e^{-i\pi/4}|^{-k}.$$

Let $v_k(c,d,\theta) := |ce^{i\theta/2} + \sqrt{5}de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\pi/2) = 1/(c^2+5d^2)^{k/2}$. Now we will consider the next cases, namely $N = 2, 5, 10, 13, 17$, and $N \geqslant 25$. We have the following:

When $N = 2$,	$v_k(1,1,\pi/2) \leqslant (2/3)^{k/2}$.	
When $N = 5$,	$v_k(1,2,\pi/2) \leqslant (1/21)^{k/2}$,	$v_k(2,1,\pi/2) \leqslant (1/3)^k$.
When $N = 10$,	$v_k(1,3,\pi/2) \leqslant (2/23)^{k/2}$,	$v_k(3,1,\pi/2) \leqslant (2/7)^{k/2}$.
When $N = 13$,	$v_k(2,3,\pi/2) \leqslant (1/7)^k$,	$v_k(3,2,\pi/2) \leqslant (1/29)^{k/2}$.
When $N = 17$,	$v_k(1,4,\pi/2) \leqslant (1/21)^{k/2}$,	$v_k(4,1,\pi/2) \leqslant (1/3)^{2k}$.

When $N \geqslant 25$, $c^2 + 5d^2 \geqslant N$,

and the number of terms with $c^2 + d^2 = N$ is not more than $(96/25)N^{1/2}$ for $N \geqslant 25$. Then

$$|R^*_{5,\pi}|_{N\geqslant 25} \leqslant \frac{1536}{25(k-3)}\left(\frac{1}{6}\right)^{(k-3)/2}.$$

Furthermore,

$$\begin{aligned}|R^*_{5,\pi}| &\leqslant 4\left(\frac{2}{3}\right)^{k/2} + \cdots + 4\left(\frac{1}{3}\right)^{2k} + \frac{1536}{25(k-3)}\left(\frac{1}{6}\right)^{(k-3)/2}, \\ &\leqslant 0.95701... \quad (k \geqslant 8)\end{aligned}$$

In conclusion, we have following:

Lemma 5.9. *For an even integer $k \geqslant 8$,*

$$F^*_{k,5}(\pi)\begin{cases} > 0 & k \equiv 0 \pmod 8 \\ < 0 & k \equiv 4 \pmod 8 \\ = 0 & k \equiv 2 \pmod 4 \end{cases}.$$

$F^*_{k,5}(\theta)$ for $\pi/2 \leqslant \theta \leqslant 5\pi/6$. Let $k \geqslant 10$ be an even integer. We note that $N := c^2 + d^2$.

Firstly, we consider the case $N = 1$. Since $5\pi/6 < \pi/2 + \alpha_5$, we can write;

$$F^*_{k,5}(\theta) = F^*_{k,5,1}(\theta) = 2\cos(k\theta/2) + R^*_{5,5\pi/6} \quad \text{for } \theta \in [\pi/2, 5\pi/6],$$

where

$$|R^*_{5,5\pi/6}| \leqslant \sum_{\substack{(c,d)=1 \\ 5\nmid d \\ N>1}} |ce^{i\theta/2} + \sqrt{5}de^{-i\theta/2}|^{-k}.$$

Let $v_k(c,d,\theta) := |ce^{i\theta/2} + \sqrt{5}de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\theta) = 1/\left(c^2 + 5d^2 + 2\sqrt{5}cd\cos\theta\right)^{k/2}$. Now we will consider the next cases, namely $N = 2, 5, 10$, and $N \geqslant 13$. Considering $-\sqrt{3}/2 \leqslant \cos\theta \leqslant 0$, we have the following:

When $N = 2$, $v_k(1,1,\theta) \leqslant 1/\left(6 - \sqrt{15}\right)^{k/2}$, $v_k(1,-1,\theta) \leqslant (1/6)^{k/2}$.

When $N = 5$, $v_k(1,2,\theta) \leqslant 1/\left(21 - 2\sqrt{15}\right)^{k/2}$, $v_k(1,-2,\theta) \leqslant (1/21)^{k/2}$,

$v_k(2,1,\theta) \leqslant 1/\left(9 - 2\sqrt{15}\right)^{k/2}$, $v_k(2,-1,\theta) \leqslant (1/3)^{k}$.

When $N = 10$, $v_k(1,3,\theta) \leqslant 1/\left(46 - 3\sqrt{15}\right)^{k/2}$, $v_k(1,-3,\theta) \leqslant (1/46)^{k/2}$,

$v_k(3,1,\theta) \leqslant 1/\left(14 - 3\sqrt{15}\right)^{k/2}$, $v_k(3,-1,\theta) \leqslant (1/14)^{k/2}$.

When $N \geqslant 13$, $|ce^{i\theta/2} \pm \sqrt{5}de^{-i\theta/2}|^2 \geqslant N/5$,

and the number of terms with $c^2 + d^2 = N$ is not more than $(21/5)N^{1/2}$ for $N \geqslant 13$. Then

$$|R^*_{5,5\pi/6}|_{N\geqslant 13} \leqslant \frac{1008\sqrt{3}}{5(k-3)}\left(\frac{5}{12}\right)^{(k-3)/2}.$$

Furthermore,

$$\begin{aligned}|R^*_{5,\pi/2}| &\leqslant 2\left(\frac{1}{9 - 2\sqrt{15}}\right)^{k/2} + \cdots + 2\left(\frac{1}{46}\right)^{k/2} + \frac{1008\sqrt{3}}{5(k-3)}\left(\frac{5}{12}\right)^{(k-3)/2}, \\ &\leqslant 1.34372... \quad (k \geqslant 10)\end{aligned}$$

In conclusion, we have following:

Lemma 5.10. *For an even integer $k \geqslant 10$,*

$$F^*_{k,5}(\theta) = 2\cos(k\theta/2) + R^*_{5,5\pi/6} \quad \text{for}\, \theta \in [\pi/2, 5\pi/6],$$

where $|R^*_{5,5\pi/6}| < 2$.

The locating zeros of $E^*_{4,5}$. We have $F^*_{4,5}(\pi/2) < 0$ by Lemma 5.7, and we have $F^*_{4,5}(\pi/2 + \alpha_5) > 0$ by Lemma 5.8 because $\cos(2(\pi/2 + \alpha_5)) = 3/5 > 0$. Thus $E^*_{4,5}$ has at least one zero in $A^*_{5,1}$. Furthermore, by the valence formula for $\Gamma^*_0(5)$ (Proposition 5.12), $E^*_{4,5}$ has no other zero. Thus we have following:

Lemma 5.11. *$E^*_{4,5}$ has only one zero in $A^*_{5,1}$, and we have $v_{i/\sqrt{5}}(E^*_{4,5}) = v_{\rho_{5,1}}(E^*_{4,5}) = v_{\rho_{5,2}}(E^*_{4,5}) = 0$.*

The locating zeros of $E^*_{6,5}$. By previous subsubsection, we have $v_{i/\sqrt{5}}(E^*_{6,5}) \geqslant 1$, $v_{\rho_{5,1}}(E^*_{6,5}) \geqslant 1$, and $v_{\rho_{5,2}}(E^*_{6,5}) \geqslant 1$. Furthermore, by the valence formula for $\Gamma^*_0(5)$, $E^*_{6,5}$ has no other zero. Thus we have following:

Lemma 5.12. *We have $v_{i/\sqrt{5}}(E^*_{6,5}) = v_{\rho_{5,1}}(E^*_{6,5}) = v_{\rho_{5,2}}(E^*_{6,5}) = 1$, and $E^*_{6,5}$ has no other zero.*

The locating zeros of $E^*_{8,5}$. We have $F^*_{8,5}(\pi/2) > 0$ by Lemma 5.7, and we have $F^*_{8,5}(\pi/2 + \alpha_5) > 0$ by Lemma 5.8 because $\cos(4(\pi/2 + \alpha_5)) = -7/25 < 0$, and we have $F^*_{8,5}(\pi) > 0$ by Lemma 5.9. Thus $E^*_{8,5}$ has at least two zeros in each arc $A^*_{5,1}$ and $A^*_{5,2}$. Furthermore, by the valence formula for $\Gamma^*_0(5)$, $E^*_{8,5}$ has no other zero. Thus we have following:

Lemma 5.13. *$E^*_{8,5}$ has only two zeros in each arc $A^*_{5,1}$ and $A^*_{5,2}$, and we have $v_{i/\sqrt{5}}(E^*_{8,5}) = v_{\rho_{5,1}}(E^*_{8,5}) = v_{\rho_{5,2}}(E^*_{8,5}) = 0$.*

The locating zeros of $E^*_{10,5}$.
We have $F^*_{10,5}(3\pi/5) < 0$ and $F^*_{10,5}(4\pi/5) > 0$ by Lemma 5.10. Thus $E^*_{10,5}$ has at least one zero in $A^*_{5,1}$. In addition, by previous subsubsection, we have $v_{i/\sqrt{5}}(E^*_{10,5}) \geqslant 1$, $v_{\rho_{5,1}}(E^*_{10,5}) \geqslant 1$, and $v_{\rho_{5,2}}(E^*_{10,5}) \geqslant 1$. Furthermore, by the valence formula for $\Gamma^*_0(5)$, $E^*_{10,5}$ has no other zero. Thus we have following:

Lemma 5.14. *$E^*_{10,5}$ has only one zero in $A^*_{5,1}$, and we have $v_{i/\sqrt{5}}(E^*_{10,5}) = v_{\rho_{5,1}}(E^*_{10,5}) = v_{\rho_{5,2}}(E^*_{10,5}) = 1$, and $E^*_{10,5}$ has no other zero.*

5.8.3. *The space of modular forms.* Let $M^*_{k,5}$ be the space of modular forms for $\Gamma^*_0(5)$ of weight k, and let $M^{*0}_{k,5}$ be the space of cusp forms for $\Gamma^*_0(5)$ of weight k. When we consider the map $M^*_{k,5} \ni f \mapsto f(\infty) \in \mathbb{C}$, the kernel of the map is $M^{*0}_{k,5}$. So $\dim(M^*_{k,5}/M^{*0}_{k,5}) \leqslant 1$, and $M^*_{k,5} = \mathbb{C}E^*_{k,5} \oplus M^{*0}_{k,5}$.

Recall that

$$\Delta_5 = \eta^4(z)\eta^4(5z)$$

is a cusp form for $\Gamma^*_0(5)$ of weight 4 (Remark 4.2). We have following theorem:

Theorem 5.3. *Let k be an even integer.*

1. *For $k < 0$ and $k = 2$, $M^*_{k,5} = 0$.*
2. *For $k = 0$ and 6, we have $M^{*0}_{k,5} = 0$, and $\dim(M^*_{k,5}) = 1$ with a base $E^*_{k,5}$.*
3. *$M^{*0}_{k,5} = \Delta_5 M^*_{k-4,5}$.*

The proof of this theorem is very similar to Theorem 3.1 and 5.1. Furthermore, for an even integer $k \geqslant 4$, $\dim(M^*_{k,5}) = (k-2)/4$ if $k \equiv 2 \pmod 4$, and $\dim(M^*_{k,5}) = k/4 + 1$ if $k \equiv 0 \pmod 4$.

Let k be an even integer $k \geqslant 4$. Write $n := \dim(M^*_{k,5}) - 1$, then $k - 4n = 0$ or 6. Because $E^*_{k,5} - E^*_{k-4n,5}(E^*_{4,5})^n \in M^{*0}_{k,5}$, we have $M^*_{k,5} = \mathbb{C}E^*_{k-4n,5}(E^*_{4,5})^n \oplus M^{*0}_{k,5}$. Then

$$M^*_{4n,5} = \mathbb{C}(E^*_{4,5})^n \oplus \mathbb{C}(E^*_{4,5})^{n-1}\Delta_5 \oplus \cdots \oplus \mathbb{C}\Delta_5^n$$
$$M^*_{4n+6,5} = E^*_{6,5}((E^*_{4,5})^n \oplus \mathbb{C}(E^*_{4,5})^{n-1}\Delta_5 \oplus \cdots \oplus \mathbb{C}\Delta_5^n)$$

Thus, for every $p \in \mathbb{H}$ and for every $f \in M^*_{k,5}$, $v_p(f) \geqslant v_p(E^*_{k-4n,5})$.

In conclusion, the next proposision follows:

Proposition 5.18. *Let $k \geqslant 4$ be an even integer. For every $f \in M_{k,5}^*$, we have*

$$v_{i/\sqrt{5}}(f) \geqslant s_k, \quad v_{\rho_{5,1}}(f) \geqslant s_k, \quad v_{\rho_{5,2}}(f) \geqslant s_k \tag{122}$$
$$(s_k = 0, 1 \text{ such that } 2s_k \equiv k \pmod 4).$$

Remark 5.5. *Every modular form for $\Gamma_0^*(5)$ is generated by*

$$E_{4,5}^*, \quad E_{6,5}^*, \quad \textit{and} \quad \Delta_5.$$

Now, we have following conjecture:

Conjecture 5.1. *Let $k \geqslant 4$ be an even integer. $E_{k,5}^*$ has $k/4$ zeros in $A_{5,1}^*$ and $A_{5,2}^*$ if $k \equiv 0 \pmod 4$, and $E_{k,5}^*$ has $(k-6)/4$ zeros in $A_{5,1}^*$ and $A_{5,2}^*$ if $k \equiv 2 \pmod 4$. Furthermore, in Proposition 5.18, the equality hold if f is equal to $E_{k,5}^*$ or its constant multiple.*

5.8.4. *Observation on Conjecture 5.1.* To prove Conjecture 5.1 is much more difficult than the proof of Theorem 1 and 2. The most difficult point is the argument $Arg(\rho_{5,2})$, which is not a product of rational number and π. When $p = 2$ and 3, for $\Gamma_0^*(p)$, the arguments of ρ_p are $3\pi/4$ and $5\pi/6$, respectively. Then, in Lemma 5.2 and 5.5, we removed the angle $\pi/2k$ and $\pi/3k$ from the angles of A_2^* and A_3^*, respectively. However, for $\rho_{5,2}$, we can not decide the angle corresponding to $\pi/2k$ and $\pi/3k$. We need more radical expansion to prove this conjecture.

As a prelude to prove Conjecture 5.1, we omit a few zeros. We consider the interval $[\pi/2, \pi/2+\alpha_5-\pi/k]$ and $[\pi/2+\alpha_5+\pi/k, \pi]$ for $F_{k,5}^*$. Now, we will prove next lemmas in the next subsubsection:

Lemma 5.15. *Let $k \geqslant 12$. For $\forall\theta \in [\pi/2, \pi/2+\alpha_5-x]$ $(x = \pi/k)$, $|R_{5,1}^*| < 2$.*

Lemma 5.16. *Let $k \geqslant 12$. For $\forall\theta \in [\alpha_5+x, \pi/2]$ $(x = \pi/k)$, $|R_{5,2}^*| < 2$.*

By above lemmas, we can easily show that $E_{k,5}^*$ has at least $k/4-2$ zeros in $A_{5,1}^*$ and $A_{5,2}^*$ if $k \equiv 0 \pmod 4$, and $E_{k,5}^*$ has at least $(k-6)/4-2$ zeros in $A_{5,1}^*$ and $A_{5,2}^*$ if $k \equiv 2 \pmod 4$. Thus, we can prove Conjecture 5.1 except for at most 2 zeros.

5.8.5. *Expansion of the RSD Method.* Before proving the above lemmas, we need the following preliminaries.

Proposition 5.19.

(1) *If $0 \leqslant x \leqslant \pi/2$, then $\sin x \geqslant 1-\cos x$.*
(2) *If $0 \leqslant x \leqslant \pi/12$, then $1-\cos x \geqslant \frac{23}{48}x^2$.*

The proofs of Lemma 5.9.4 and 5.9.4 are similar to that of Lemma 5.2. We use the previous proposition for the following proofs:

Proof of Lemma 5.9.4. Let $k \geqslant 12$ and $x = \pi/k$, then $0 \leqslant x \leqslant \pi/12$.

$$|e^{i\theta/2}+\sqrt{5}e^{-i\theta/2}|^2 \geqslant 9+4\sqrt{5}\cos(\pi/2+\alpha_5-x) \geqslant 1+\frac{23}{4}x^2. \quad (Prop.5.19)$$
$$|e^{i\theta/2}+\sqrt{5}e^{-i\theta/2}|^k \geqslant 1+\frac{69}{2}x^2 \ (k \geqslant 12).$$
$$2v_k(1,1,\theta) \leqslant 2-\frac{6624\pi^2}{23\pi^2+96}\frac{1}{k^2}.$$

In inequality(88), replace 2 with the bound $2-\frac{6624\pi^2}{23\pi^2+96}\frac{1}{k^2}$. Then

$$|R_{5,1}^*| \leqslant 2-\frac{6624\pi^2}{23\pi^2+96}\frac{1}{k^2}+4\left(\frac{1}{2}\right)^{k/2}+\cdots+2\left(\frac{1}{46}\right)^{k/2}+\frac{1008\sqrt{6}}{5(k-3)}\left(\frac{1}{2}\right)^{k/2}.$$

Furthermore, $(1/2)^{k/2}$ is more rapidly decreasing in k than $1/k^2$, and for $k = 12$, we have

$$|R_{5,1}^*| \leqslant 1.26593...$$

□

Proof of Lemma 5.9.4. Let $k \geqslant 12$ and $x = \pi/k$, then $0 \leqslant x \leqslant \pi/12$.

$$\frac{1}{4}|e^{i\theta/2} + \sqrt{5}e^{-i\theta/2}|^2 \geqslant \frac{1}{4}(6 + 2\sqrt{5}\cos(\alpha_5 + x)) \geqslant 1 + \frac{69}{96}x^2. \quad (Prop.5.19)$$

$$\frac{1}{2^k}|e^{i\theta/2} + \sqrt{5}e^{-i\theta/2}|^k \geqslant 1 + \frac{69}{16}x^2 \ (k \geqslant 12).$$

$$2^k \cdot 2v_k(1,1,\theta) \leqslant 2 - \frac{6624\pi^2}{23\pi^2 + 768}\frac{1}{k^2}.$$

In inequality(89), replace 2 with the bound $2 - \frac{6624\pi^2}{23\pi^2+768}\frac{1}{k^2}$. Then

$$|R^*_{5,2}| \leqslant 2 - \frac{6624\pi^2}{23\pi^2 + 768}\frac{1}{k^2} + 2\left(\frac{2}{3}\right)^{k/2} + \cdots + 2\left(\frac{1}{9}\right)^k + \frac{378\sqrt{6}}{k-3}\left(\frac{1}{3}\right)^{k/2}.$$

Furthermore, $(2/3)^{k/2}$ is more rapidly decreasing in k than $1/k^2$, and for $k = 12$, we have

$$|R^*_{5,2}| \leqslant 1.89789...$$

□

5.9. $\Gamma_0^*(7)$.

5.9.1. *Preliminaries.* Let $\alpha_7 \in [0, \pi/2]$ such that $\tan\alpha_7 = 2$, then we denote
$$A_{7,1}^* := \{z\,;\, |z| = 1/\sqrt{7},\ \pi/2 < Arg(z) < \pi/2 + \alpha_7\},$$
$$A_{7,2}^* := \{z\,;\, |z+1/2| = 1/(2\sqrt{7}),\ \alpha_7 - \pi/6 < Arg(z) < \pi/2\}.$$

Now we have
$$\rho_{7,2} = -\frac{5}{14} + \frac{\sqrt{3}i}{14} = \frac{e^{i(\pi/2+\alpha_7)}}{\sqrt{7}} = \frac{e^{i(\alpha_7-\pi/6)}}{2\sqrt{7}} - \frac{1}{2},$$
and we have
$$F_{k,7,1}^*(\pi/2+\alpha_7) = e^{ik(\pi/2+\alpha_7)/2} E_{k,7}^*(\rho_{7,2}),$$
$$F_{k,7,2}^*(\alpha_7-\pi/6) = e^{ik(\alpha_7-\pi/6)/2} E_{k,7}^*(\rho_{7,2}).$$
Now, we define
$$F_{k,7}^*(\theta) = \begin{cases} F_{k,7,1}^*(\theta) & \pi/2 \leqslant \theta \leqslant \pi/2+\alpha_7 \\ F_{k,7,2}^*(\theta - 2\pi/3) & \pi/2+\alpha_7 \leqslant \theta \leqslant 7\pi/6 \end{cases}.$$
Then $F_{k,7}^*$ is continuous in the interval $[\pi/2, 7\pi/6]$. Note that $F_{k,7,1}^*(\pi/2+\alpha_7) = e^{i(2\pi/3)k/2} F_{k,7,2}^*(\alpha_7-\pi/6)$.

Let f be a modular form for $\Gamma_0^*(7)$ of weight k, and let $k \equiv 2 \pmod 4$. Then we have
$$f(i/\sqrt{7}) = f(W_7\, i/\sqrt{7}) = i^k f(i/\sqrt{7}) = -f(i/\sqrt{7}),$$
$$f(\rho_{7,1}) = f(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}-3&-1\\7&2\end{smallmatrix}\right) W_7\, \rho_{7,1}) = i^k f(\rho_{7,1}) = -f(\rho_{7,1}).$$
Thus $f(i/\sqrt{7}) = f(\rho_{7,1}) = 0$, then we have $v_{i/\sqrt{7}}(f) \geqslant 1$ and $v_{\rho_{7,1}}(f) \geqslant 1$. On the other hand, let $k \not\equiv 0 \pmod 6$. Then we have
$$f(\rho_{7,2}) = f(\left(\begin{smallmatrix}-3&-1\\7&2\end{smallmatrix}\right)\ \rho_{7,2}) = (e^{i2\pi/3})^k f(\rho_{7,2}).$$
Thus $f(\rho_{7,2}) = 0$, then we have $v_{\rho_{7,2}}(f) \geqslant 1$.

Let k be an even integer such that $k \equiv 0 \pmod 4$. Then we have
$$E_{k,7}^*\left(\frac{i}{\sqrt{7}}\right) = \frac{2\cdot 7^{k/2}}{7^{k/2}+1} E_k(\sqrt{7}i) \neq 0,$$
$$E_{k,7}^*(\rho_{7,1}) = \frac{2\cdot 7^{k/2}}{7^{k/2}+1} E_k\left(-\frac{1}{2} + \frac{\sqrt{7}}{2}i\right) \neq 0.$$
Thus $v_{i/\sqrt{7}}(E_{k,7}^*) = v_{\rho_{7,1}}(E_{k,7}^*) = 0$. On the other hand, let k be an even integer such that $k \equiv 0 \pmod 6$. Then we have
$$E_{k,7}^*(\rho_{7,2}) = \frac{1}{7^{k/2}+1}\left(7^{k/2} + \left(\frac{5+\sqrt{3}i}{2}\right)^k\right) E_k(\rho) \neq 0.$$
Thus $v_{\rho_{7,2}}(E_{k,7}^*) = 0$.

5.9.2. $E_{k,7}^*$ *of low weights.*
$F_{k,7}^*(\pi/2)$. Let $k \geqslant 4$ be an even integer divisible by 4. We note that $N := c^2 + d^2$.

Firstly, we consider the case $N = 1$. Then we can write;
$$F_{k,7}^*(\pi/2) = F_{k,7,1}^*(\pi/2) = 2\cos(k\pi/4) + R_{7,\pi/2}^*$$
where
$$|R_{7,\pi/2}^*| \leqslant \sum_{\substack{(c,d)=1\\ 7\nmid d\\ N>1}} |ce^{i\pi/4} + \sqrt{7}de^{-i\pi/4}|^{-k}.$$

Let $v_k(c,d,\theta) := |ce^{i\theta/2} + \sqrt{7}de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\pi/2) = 1/(c^2+7d^2)^{k/2}$. Now we will consider the next cases, namely $N = 2, 5, 10, 13, 17$, and $N \geqslant 25$. We have the following:

When $N = 2$, $\quad v_k(1,1,\pi/2) \leqslant (1/8)^{k/2}.$

When $N=5$, $v_k(1,2,\pi/2)\leqslant(1/29)^{k/2}$, $v_k(2,1,\pi/2)\leqslant(1/11)^{k/2}$.
When $N=10$, $v_k(1,3,\pi/2)\leqslant(1/8)^{k}$, $v_k(3,1,\pi/2)\leqslant(1/4)^{k}$.
When $N=13$, $v_k(2,3,\pi/2)\leqslant(1/69)^{k/2}$, $v_k(3,2,\pi/2)\leqslant(1/37)^{k/2}$.
When $N=17$, $v_k(1,4,\pi/2)\leqslant(1/113)^{k/2}$, $v_k(4,1,\pi/2)\leqslant(1/23)^{k/2}$.
When $N\geqslant 25$, $c^2+7d^2\geqslant N$,

and the number of terms with $c^2+d^2=N$ is not more than $(144/35)N^{1/2}$ for $N\geqslant 25$. Then

$$|R^*_{7,\pi/2}|_{N\geqslant 25}\leqslant\frac{288}{35(k-3)}\left(\frac{1}{24}\right)^{(k-3)/2}.$$

Furthermore,

$$|R^*_{7,\pi/2}|\leqslant 4\left(\frac{1}{8}\right)^{k/2}+4\left(\frac{1}{11}\right)^{k/2}+\cdots+4\left(\frac{1}{113}\right)^{k/2}+\frac{288}{35(k-3)}\left(\frac{1}{24}\right)^{(k-3)/2},$$
$$\leqslant 1.80820\ldots\quad(k\geqslant 4)$$

In conclusion, we have following:

Lemma 5.17. *For an even integer $k\geqslant 4$,*

$$F^*_{k,7}(\pi/2)\begin{cases}>0 & k\equiv 0\pmod 8\\ <0 & k\equiv 4\pmod 8\\ =0 & k\equiv 2\pmod 4\end{cases}.$$

$F^*_{4,7}(5\pi/6)$. To decide the zeros of $E^*_{4,7}$, we need the value of $F^*_{4,7}(5\pi/6)$. We note that $N:=c^2+d^2$.

Firstly, we consider the case $N=1$. Then we can write;

$$F^*_{4,7}(5\pi/6)=2\cos(10\pi/3)+R^*_{7,4}=1+R^*_{7,4}$$

where

$$R^*_{7,4}\leqslant\frac{1}{2}\sum_{\substack{(c,d)=1\\ 7\nmid d\\ N>1}}(ce^{i5\pi/12}+\sqrt{7}de^{-i5\pi/12})^{-4}+\frac{1}{2}\sum_{\substack{(c,d)=1\\ 7\nmid d\\ N>1}}(ce^{-i5\pi/12}+\sqrt{7}de^{i5\pi/12})^{-4}.$$

We want to prove $F^*_{4,7}(5\pi/6)>0$, but it is too difficult to prove that $|R^*_{7,4}|<1$. However, we have only to prove $R^*_{7,4}>-1$.

Let $u_0(c,d):=(ce^{i5\pi/12}+\sqrt{7}de^{-i5\pi/12})^{-4}+(ce^{-i5\pi/12}+\sqrt{7}de^{i5\pi/12})^{-4}$, and let $u(c,d):=u_0(c,d)+u_0(c,-d)+u_0(d,c)+u_0(d,-c)$. Now we will consider the next cases, namely $N=2,5,\cdots,197$, and $N\geqslant 202$. We have the following:

When $N=2$, $u_0(1,1)+u_0(1,-1)\geqslant-0.08151$.
When $N=5$, $u(1,2)\geqslant-0.19373$. When $N=10$, $u(1,3)\geqslant 0.24147$.
When $N=13$, $u(2,3)\geqslant-0.02162$. When $N=17$, $u(1,4)\geqslant-0.07736$.
When $N=25$, $u(3,4)\geqslant-0.00313$. When $N=26$, $u(1,5)\geqslant-0.02262$.
When $N=29$, $u(2,5)\geqslant 0.03569$. When $N=34$, $u(3,5)\geqslant-0.00503$.
When $N=37$, $u(1,6)\geqslant-0.00586$. When $N=41$, $u(4,5)\geqslant-0.00083$.
When $N=50$, $u(1,7)\geqslant-0.00168$. When $N=53$, $u(2,7)\geqslant-0.00400$.
When $N=58$, $u(3,7)\geqslant 0.00491$. When $N=61$, $u(5,6)\geqslant-0.00033$.
When $N=65$, $u(1,8)+u(4,7)\geqslant-0.00211$.
When $N=73$, $u(3,8)\geqslant 0.00692$. When $N=74$, $u(5,7)\geqslant-0.00048$.
When $N=82$, $u(1,9)\geqslant-0.00014$.
When $N=85$, $u(2,9)+u(6,7)\geqslant-0.00295$.
When $N=89$, $u(5,8)\geqslant-0.00064$. When $N=97$, $u(4,9)\geqslant 0.00099$.

When $N = 101$, $u(1,10) \geqslant -0.00003$. When $N = 106$, $u(5,9) \geqslant -0.00064$.
When $N = 109$, $u(3,10) \geqslant -0.00014$. When $N = 113$, $u(7,8) \geqslant -0.00009$.
When $N = 122$, $u(1,11) \geqslant 0.00002$. When $N = 125$, $u(2,11) \geqslant -0.00073$.
When $N = 130$, $u(3,11) + u(7,9) \geqslant -0.00116$.
When $N = 137$, $u(4,11) \geqslant 0.00195$.
When $N = 145$, $u(1,12) + u(8,9) \geqslant -0.00003$.
When $N = 146$, $u(5,11) \geqslant 0.00025$. When $N = 149$, $u(7,10) \geqslant -0.00015$.
When $N = 157$, $u(6,11) \geqslant -0.00030$. When $N = 169$, $u(1,2) \geqslant 0.00077$.
When $N = 170$, $u(1,13) + u(7,11) \geqslant -0.00016$.
When $N = 173$, $u(2,13) \geqslant -0.00021$. When $N = 178$, $u(3,13) \geqslant -0.00068$.
When $N = 181$, $u(9,10) \geqslant -0.00004$.
When $N = 185$, $u(4,13) + u(8,11) \geqslant 0.00003$.
When $N = 193$, $u(7,12) \geqslant -0.00019$. When $N = 194$, $u(5,13) \geqslant 0.00093$.
When $N = 197$, $u(1,14) \geqslant 0.00002$.

When $N \geqslant 202$, for the case $cd < 0$, we put $X + Yi = (ce^{i5\pi/12} + \sqrt{7}de^{-i5\pi/12})^2 = -(\sqrt{3}/2)(c^2 + 7d^2) + 2\sqrt{7}cd + (1/2)(c^2 - 7d^2)i$. Then $|Y| - |X| < -\frac{\sqrt{3}-1}{2}c^2 - \frac{\sqrt{3}-1}{2}d^2 + 2\sqrt{7}cd < 0$. Thus $(ce^{i5\pi/12} + \sqrt{7}de^{-i5\pi/12})^{-4} + (ce^{-i5\pi/12} + \sqrt{7}de^{i5\pi/12})^{-4} > 0$.

For the case $cd > 0$, we have $c^2 + 7d^2 - \sqrt{21}|cd| > (2/9)N$, and the number of terms with $c^2 + d^2 = N$ is not more than $(13/7)N^{1/2}$ for $N \geqslant 144$. However, this bound is too large. We must consider some cases.

For the case $|c| < |d|$, we have $c^2 + 7d^2 - \sqrt{21}|cd| > (3/2)N$ and $|c| < (1/\sqrt{2})N^{1/2}$, $1/\sqrt{2} > 7/10$. For the case $|d| \leqslant |c| < (6/\sqrt{21})|d|$, we have $c^2 + 7d^2 - \sqrt{21}|cd| > N$ and $|c| < (6/\sqrt{57})N^{1/2}$, $6/\sqrt{57} > 7/9$. For the case and $(6/\sqrt{21})|d| \leqslant |c| < \sqrt{7/3}|d|$, we have $c^2+7d^2-\sqrt{21}|cd| > (1/2)N$ and $|c| < \sqrt{7/10}N^{1/2}$, $\sqrt{7/10} > 5/6$. For the case $cd > 0$ and $\sqrt{7/3}|d| \leqslant |c| < (22/3\sqrt{21})|d|$, we have $c^2+7d^2-\sqrt{21}|cd| > (1/3)N$ and $|c| < (22/\sqrt{673})N^{1/2}$, $22/\sqrt{673} > 22/25$.

In conclusion, we have

$$\begin{aligned}
R^*_{7,4}\Big|_{N\geqslant 202, cd>0}
&\geqslant -\frac{13}{7}\Big(\frac{7}{10}N^{1/2}\sum_{N\geqslant 202}\left(\frac{3}{2}N\right)^{-k/2} + \frac{7}{90}N^{1/2}\sum_{N\geqslant 202}N^{-k/2} + \frac{1}{18}N^{1/2}\sum_{N\geqslant 202}\left(\frac{1}{2}N\right)^{-k/2}\\
&\quad + \frac{7}{150}N^{1/2}\sum_{N\geqslant 202}\left(\frac{1}{3}N\right)^{-k/2} + \frac{3}{25}N^{1/2}\sum_{N\geqslant 202}\left(\frac{2}{9}N\right)^{-k/2}\Big)\\
&= -\frac{13}{7}\left(\frac{7}{10}\cdot\frac{4}{9} + \frac{7}{90}\cdot 1 + \frac{1}{18}\cdot 4 + \frac{7}{150}\cdot 9 + \frac{3}{25}\cdot\frac{81}{4}\right)\sum_{N\geqslant 202}N^{(1-k)/2}\\
&= -\frac{7579}{630\sqrt{201}}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
R^*_{7,4} &\geqslant -0.08152 - 0.19373 + \cdots + 0.00002 - \frac{7579}{630\sqrt{201}}\\
&= -0.98316... \quad (k \geqslant 4)
\end{aligned}$$

In conclusion, we have following:

Lemma 5.18.

$$F^*_{4,7}(5\pi/6) > 0.$$

$F^*_{k,7}(\theta)$ for $\pi/2 \leqslant \theta \leqslant 2\pi/3$. Let $k \geqslant 6$ be an even integer. We note that $N := c^2 + d^2$.

Firstly, we consider the case $N = 1$. Since $2\pi/3 < \pi/2 + \alpha_7$, we can write;

$$F^*_{k,7}(\theta) = F^*_{k,7,1}(\theta) = 2\cos(k\theta/2) + R^*_{7,2\pi/3} \quad \text{for } \theta \in [\pi/2, 2\pi/3],$$

where

$$|R^*_{7,2\pi/3}| \leqslant \sum_{\substack{(c,d)=1 \\ 7\nmid d \\ N>1}} |ce^{i\theta/2} + \sqrt{7}de^{-i\theta/2}|^{-k}.$$

Let $v_k(c,d,\theta) := |ce^{i\theta/2} + \sqrt{7}de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\theta) = 1/\left(c^2 + 7d^2 + 2\sqrt{7}cd\cos\theta\right)^{k/2}$. Now we will consider the next cases, namely $N = 2$ and $N \geqslant 5$. Considering $-1/2 \leqslant \cos\theta \leqslant 0$, we have the following:

When $N = 2$, $\quad v_k(1,1,\pi/2) \leqslant (1/(8-\sqrt{7}))^{k/2}, \quad v_k(1,-1,\pi/2) \leqslant (1/8)^{k/2}$.

When $N \geqslant 5$, $\quad |ce^{i\theta/2} \pm \sqrt{7}de^{-i\theta/2}|^2 \geqslant 5N/7$,

and the number of terms with $c^2 + d^2 = N$ is not more than $(36/7)N^{1/2}$ for $N \geqslant 5$. Then

$$|R^*_{7,2\pi/3}|_{N\geqslant 5} \leqslant \frac{576}{7(k-3)} \left(\frac{7}{20}\right)^{(k-3)/2}.$$

Furthermore,

$$\begin{aligned} |R^*_{7,2\pi/3}| &\leqslant 2\left(\frac{1}{8-\sqrt{7}}\right)^{k/2} + 2\left(\frac{1}{8}\right)^{k/2} + \frac{576}{7(k-3)}\left(\frac{7}{20}\right)^{(k-3)/2}, \\ &\leqslant 1.19293\ldots \quad (k \geqslant 6) \end{aligned}$$

In conclusion, we have following:

Lemma 5.19. *For an even integer $k \geqslant 6$,*

$$F^*_{k,7}(\theta) = 2\cos(k\theta/2) + R^*_{7,2\pi/3} \quad \textit{for } \theta \in [\pi/2, 2\pi/3],$$

where $|R^*_{7,2\pi/3}| < 2$.

$F^*_{k,7}(\theta)$ for $\pi \leqslant \theta \leqslant 7\pi/6$. Let $k \geqslant 6$ be an even integer. We note that $N := c^2 + d^2$.

Firstly, we consider the case $N = 1$. Then we can write;

$$F^*_{k,7}(\theta) = F^*_{k,7,2}(\theta - 2\pi/3) = 2\cos(k(\theta - 2\pi/3)/2) + R^*_{7,\pi} \quad \text{for } \theta \in [\pi, 7\pi/6],$$

where

$$|R^*_{7,\pi}| \leqslant \sum_{\substack{(c,d)=1 \\ 7\nmid c \\ 2\mid cd \\ N>1}} |ce^{i\theta/2} + d\sqrt{7}e^{-i\theta/2}|^{-k} + 2^k \sum_{\substack{(c,d)=1 \\ 7\nmid c \\ 2\nmid cd}} |ce^{i\theta/2} + d\sqrt{7}e^{-i\theta/2}|^{-k}.$$

Let $v_k(c,d,\theta) := |ce^{i\theta/2} + \sqrt{7}de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\theta) = 1/\left(c^2 + 7d^2 + 2\sqrt{7}cd\cos\theta\right)^{k/2}$. Now we will consider the next cases, namely $N = 2, 5, \cdots, 82$, and $N \geqslant 85$. Considering $0 \leqslant \cos\theta \leqslant 1/2$, we have the following:

When $N = 2$, $\quad 2^k \cdot v_k(1,1,\theta) \leqslant (1/2)^{k/2}, \quad 2^k \cdot v_k(1,-1,\theta) \leqslant \left(4/\left(8-\sqrt{7}\right)\right)^{k/2}$.

When $N = 5$, $\quad v_k(1,2,\theta) \leqslant (1/29)^{k/2}, \quad v_k(1,-2,\theta) \leqslant 1/\left(29 - 2\sqrt{7}\right)^{k/2}$,

$\quad v_k(2,1,\theta) \leqslant (1/11)^{k/2}, \quad v_k(2,-1,\theta) \leqslant 1/\left(11 - 2\sqrt{7}\right)^{k/2}$.

When $N = 10$, $\quad 2^k \cdot v_k(1,3,\theta) \leqslant (1/4)^{k}, \quad 2^k \cdot v_k(1,-3,\theta) \leqslant \left(4/\left(64 - 3\sqrt{7}\right)\right)^{k/2}$,

$\quad 2^k \cdot v_k(3,1,\theta) \leqslant (1/2)^{k}, \quad 2^k \cdot v_k(3,-1,\theta) \leqslant \left(4/\left(16 - 3\sqrt{7}\right)\right)^{k/2}$.

When $N = 13$, $\quad v_k(2,3,\theta) \leqslant (1/69)^{k/2}, \quad v_k(2,-3,\theta) \leqslant 1/\left(69 - 6\sqrt{7}\right)^{k/2}$,

$v_k(3,2,\theta) \leqslant (1/37)^{k/2}$, $\quad v_k(3,-2,\theta) \leqslant 1/\left(37-6\sqrt{7}\right)^{k/2}$.

When $N=17$, $\quad v_k(1,4,\theta) \leqslant (1/113)^{k/2}$, $\quad v_k(1,-4,\theta) \leqslant 1/\left(113-4\sqrt{7}\right)^{k/2}$,

$v_k(4,1,\theta) \leqslant (1/23)^{k/2}$, $\quad v_k(4,-1,\theta) \leqslant 1/\left(23-4\sqrt{7}\right)^{k/2}$.

When $N=25$, $\quad v_k(3,4,\theta) \leqslant (1/11)^{k}$, $\quad v_k(3,-4,\theta) \leqslant 1/\left(121-12\sqrt{7}\right)^{k/2}$,

$v_k(4,3,\theta) \leqslant (1/79)^{k/2}$, $\quad v_k(4,-3,\theta) \leqslant 1/\left(79-12\sqrt{7}\right)^{k/2}$.

When $N=26$, $\quad 2^k\cdot v_k(1,5,\theta) \leqslant (1/44)^{k/2}$, $\quad 2^k\cdot v_k(1,-5,\theta) \leqslant \left(4/\left(176-5\sqrt{7}\right)\right)^{k/2}$,

$2^k\cdot v_k(5,1,\theta) \leqslant (1/8)^{k/2}$, $\quad 2^k\cdot v_k(5,-1,\theta) \leqslant \left(4/\left(32-5\sqrt{7}\right)\right)^{k/2}$.

When $N=29$, $\quad v_k(2,5,\theta) \leqslant (1/179)^{k/2}$, $\quad v_k(2,-5,\theta) \leqslant 1/\left(179-10\sqrt{7}\right)^{k/2}$,

$v_k(5,2,\theta) \leqslant (1/53)^{k/2}$, $\quad v_k(5,-2,\theta) \leqslant 1/\left(53-10\sqrt{7}\right)^{k/2}$.

When $N=34$, $\quad 2^k\cdot v_k(3,5,\theta) \leqslant (1/46)^{k/2}$, $\quad 2^k\cdot v_k(3,-5,\theta) \leqslant \left(4/\left(184-15\sqrt{7}\right)\right)^{k/2}$,

$2^k\cdot v_k(5,3,\theta) \leqslant (1/22)^{k}$, $\quad 2^k\cdot v_k(5,-3,\theta) \leqslant \left(4/\left(88-15\sqrt{7}\right)\right)^{k/2}$.

When $N=37$, $\quad v_k(1,6,\theta) \leqslant (1/253)^{k/2}$, $\quad v_k(1,-6,\theta) \leqslant 1/\left(253-6\sqrt{7}\right)^{k/2}$,

$v_k(6,1,\theta) \leqslant (1/43)^{k/2}$, $\quad v_k(6,-1,\theta) \leqslant 1/\left(43-6\sqrt{7}\right)^{k/2}$.

When $N=41$, $\quad v_k(4,5,\theta) \leqslant (1/191)^{k/2}$, $\quad v_k(4,-5,\theta) \leqslant 1/\left(191-20\sqrt{7}\right)^{k/2}$,

$v_k(5,4,\theta) \leqslant (1/137)^{k/2}$, $\quad v_k(5,-4,\theta) \leqslant 1/\left(137-20\sqrt{7}\right)^{k/2}$.

When $N=50$, $\quad 2^k\cdot v_k(1,7,\theta) \leqslant (1/86)^{k/2}$, $\quad 2^k\cdot v_k(1,-7,\theta) \leqslant \left(4/\left(344-7\sqrt{7}\right)\right)^{k/2}$,

$2^k\cdot v_k(7,1,\theta) \leqslant (1/14)^{k/2}$, $\quad 2^k\cdot v_k(7,-1,\theta) \leqslant \left(4/\left(56-7\sqrt{7}\right)\right)^{k/2}$.

When $N=53$, $\quad v_k(2,7,\theta) \leqslant (1/347)^{k/2}$, $\quad v_k(2,-7,\theta) \leqslant 1/\left(347-14\sqrt{7}\right)^{k/2}$,

$v_k(7,2,\theta) \leqslant (1/77)^{k/2}$, $\quad v_k(7,-2,\theta) \leqslant 1/\left(77-14\sqrt{7}\right)^{k/2}$.

When $N=58$, $\quad 2^k\cdot v_k(3,7,\theta) \leqslant (1/88)^{k/2}$, $\quad 2^k\cdot v_k(3,-7,\theta) \leqslant \left(4/\left(352-21\sqrt{7}\right)\right)^{k/2}$,

$2^k\cdot v_k(7,3,\theta) \leqslant (1/28)^{k}$, $\quad 2^k\cdot v_k(7,-3,\theta) \leqslant \left(4/\left(112-21\sqrt{7}\right)\right)^{k/2}$.

When $N=61$, $\quad v_k(5,6,\theta) \leqslant (1/277)^{k/2}$, $\quad v_k(5,-6,\theta) \leqslant 1/\left(277-30\sqrt{7}\right)^{k/2}$,

$v_k(6,5,\theta) \leqslant (1/211)^{k/2}$, $\quad v_k(6,-5,\theta) \leqslant 1/\left(211-30\sqrt{7}\right)^{k/2}$.

When $N=65$, $\quad v_k(1,8,\theta) \leqslant (1/449)^{k/2}$, $\quad v_k(1,-8,\theta) \leqslant 1/\left(449-8\sqrt{7}\right)^{k/2}$,

$v_k(8,1,\theta) \leqslant (1/71)^{k/2}$, $\quad v_k(8,-1,\theta) \leqslant 1/\left(71-8\sqrt{7}\right)^{k/2}$,

$v_k(4,7,\theta) \leqslant (1/359)^{k/2}$, $\quad v_k(4,-7,\theta) \leqslant 1/\left(359-28\sqrt{7}\right)^{k/2}$,

$v_k(7,4,\theta) \leqslant (1/245)^{k/2}$, $\quad v_k(7,-4,\theta) \leqslant 1/\left(245-28\sqrt{7}\right)^{k/2}$.

When $N=73$, $v_k(3,8,\theta) \leqslant (1/457)^{k/2}$, $v_k(3,-8,\theta) \leqslant 1/\left(457-24\sqrt{7}\right)^{k/2}$,

$v_k(8,3,\theta) \leqslant (1/127)^{k/2}$, $v_k(8,-3,\theta) \leqslant 1/\left(127-24\sqrt{7}\right)^{k/2}$.

When $N=74$, $2^k \cdot v_k(5,7,\theta) \leqslant (1/92)^{k/2}$, $2^k \cdot v_k(5,-7,\theta) \leqslant \left(4/\left(368-35\sqrt{7}\right)\right)^{k/2}$,

$2^k \cdot v_k(7,5,\theta) \leqslant (1/56)^{k/2}$, $2^k \cdot v_k(7,-5,\theta) \leqslant \left(4/\left(224-35\sqrt{7}\right)\right)^{k/2}$.

When $N=82$, $2^k \cdot v_k(1,9,\theta) \leqslant (1/142)^{k/2}$, $2^k \cdot v_k(1,-9,\theta) \leqslant \left(4/\left(568-9\sqrt{7}\right)\right)^{k/2}$,

$2^k \cdot v_k(9,1,\theta) \leqslant (1/22)^{k/2}$, $2^k \cdot v_k(9,-1,\theta) \leqslant \left(4/\left(88-9\sqrt{7}\right)\right)^{k/2}$.

When $N \geqslant 85$, $|ce^{i\theta/2} \pm \sqrt{7}de^{-i\theta/2}|^2 \geqslant 5N/7$,

and the number of terms with $c^2+d^2=N$ is not more than $(27/7)N^{1/2}$ for $N \geqslant 64$. Then

$$|R^*_{7,\pi}|_{N\geqslant 85} \leqslant \frac{1296\sqrt{21}}{k-3}\left(\frac{1}{15}\right)^{(k-3)/2}.$$

Furthermore,

$$\begin{aligned}|R^*_{7,\pi}| &\leqslant 2\left(\frac{4}{8-\sqrt{7}}\right)^{k/2}+\cdots+2\left(\frac{1}{457}\right)^{k/2}+\frac{1296\sqrt{21}}{k-3}\left(\frac{1}{15}\right)^{(k-3)/2},\\ &\leqslant 1.98849\ldots \quad (k\geqslant 6)\end{aligned}$$

In conclusion, we have following:

Lemma 5.20. *For an even integer $k \geqslant 6$,*

$$F^*_{k,7}(\theta) = 2\cos(k\theta/2) + R^*_{7,\pi} \quad \textit{for}\, \theta \in [\pi, 7\pi/6],$$

where $|R^*_{7,\pi}| < 2$.

$F^*_{k,7}(\theta)$ for $\pi/2 \leqslant \theta \leqslant 5\pi/6$. Let $k \geqslant 8$ be an even integer. We note that $N := c^2+d^2$.

Firstly, we consider the case $N=1$. Since $5\pi/6 < \pi/2+\alpha_5$, we can write;

$$F^*_{k,7}(\theta) = F^*_{k,7,1}(\theta) = 2\cos(k\theta/2) + R^*_{7,5\pi/6} \quad \text{for } \theta \in [\pi/2, 5\pi/6],$$

where

$$|R^*_{7,5\pi/6}| \leqslant \sum_{\substack{(c,d)=1\\ 7\nmid d\\ N>1}} |ce^{i\theta/2}+\sqrt{7}de^{-i\theta/2}|^{-k}.$$

Let $v_k(c,d,\theta) := |ce^{i\theta/2}+\sqrt{7}de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\theta) = 1/\left(c^2+7d^2+2\sqrt{7}cd\cos\theta\right)^{k/2}$. Now we will consider the next cases, namely $N=2,5,10$, and $N \geqslant 13$. Considering $-\sqrt{3}/2 \leqslant \cos\theta \leqslant 0$, we have the following:

When $N=2$, $v_k(1,1,\theta) \leqslant 1/\left(8-\sqrt{21}\right)^{k/2}$, $v_k(1,-1,\theta) \leqslant (1/8)^{k/2}$.

When $N=5$, $v_k(1,2,\theta) \leqslant 1/\left(29-2\sqrt{21}\right)^{k/2}$, $v_k(1,-2,\theta) \leqslant (1/29)^{k/2}$,

$v_k(2,1,\theta) \leqslant 1/\left(11-2\sqrt{21}\right)^{k/2}$, $v_k(2,-1,\theta) \leqslant (1/11)^{k/2}$.

When $N=10$, $v_k(1,3,\theta) \leqslant 1/\left(64-3\sqrt{21}\right)^{k/2}$, $v_k(1,-3,\theta) \leqslant (1/8)^{k}$,

$v_k(3,1,\theta) \leqslant 1/\left(16-3\sqrt{21}\right)^{k/2}$, $v_k(3,-1,\theta) \leqslant (1/4)^{k}$.

When $N \geqslant 13$, $|ce^{i\theta/2} \pm \sqrt{7}de^{-i\theta/2}|^2 \geqslant 2N/9$, and the number of terms with $c^2+d^2=N$ is not more than $(36/7)N^{1/2}$ for $N \geqslant 13$. Then

$$|R^*_{7,5\pi/6}|_{N\geqslant 13} \leqslant \frac{1728\sqrt{3}}{7(k-3)}\left(\frac{3}{8}\right)^{(k-3)/2}.$$

Furthermore,

$$\begin{aligned}|R^*_{7,\pi/2}| &\leqslant 2\left(\frac{1}{11-2\sqrt{21}}\right)^{k/2}+\cdots+2\left(\frac{1}{8}\right)^{k}+\frac{1728\sqrt{3}}{7(k-3)}\left(\frac{3}{8}\right)^{(k-3)/2},\\ &\leqslant 1.96057... \quad (k\geqslant 8)\end{aligned}$$

In conclusion, we have following:

Lemma 5.21. *For an even integer $k \geqslant 8$,*

$$F^*_{k,7}(\theta) = 2\cos(k\theta/2) + R^*_{7,5\pi/6} \quad \textit{for}\, \theta \in [\pi/2, 5\pi/6],$$

where $|R^*_{7,5\pi/6}| < 2$.

The locating zeros of $E^*_{4,7}$. We have $F^*_{4,7}(\pi/2) < 0$ by Lemma 5.17, and we have $F^*_{4,7}(5\pi/6) > 0$ by Lemma 5.18. Thus $E^*_{4,7}$ has at least one zero in $A^*_{7,1}$. In addition, by the previous subsubsections, we have $v_{\rho_{7,2}}(E^*_{4,7}) \geqslant 1$. Furthermore, by the valence formula for $\Gamma_0^*(7)$ (Proposition 5.13), $E^*_{4,7}$ has no other zero. Thus we have following:

Lemma 5.22. *$E^*_{4,7}$ has only one zero in $A^*_{7,1}$, and we have $v_{i/\sqrt{7}}(E^*_{4,7}) = v_{\rho_{7,1}}(E^*_{4,7}) = 0$ and $v_{\rho_{7,2}}(E^*_{4,7}) = 1$.*

The locating zeros of $E^*_{6,7}$. We have $F^*_{6,7}(2\pi/3) > 0$ by Lemma 5.19, and we have $F^*_{6,7}(\pi) < 0$ by Lemma 5.20, thus $E^*_{6,7}$ has at least one zero in $A^*_{7,1} \cup A^*_{7,2}$. By previous subsubsection, we have $v_{i/\sqrt{7}}(E^*_{6,7}) \geqslant 1$ and $v_{\rho_{7,1}}(E^*_{6,7}) \geqslant 1$. Furthermore, by the valence formula for $\Gamma_0^*(7)$, $E^*_{6,7}$ has no other zero. Thus we have following:

Lemma 5.23. *$E^*_{6,7}$ has only one zero in $A^*_{7,1} \cup A^*_{7,2}$, and we have $v_{i/\sqrt{7}}(E^*_{6,7}) = v_{\rho_{7,1}}(E^*_{6,7}) = 1$ and $v_{\rho_{7,2}}(E^*_{4,7}) = 0$.*

The locating zeros of $E^*_{12,7}$. We have $F^*_{12,7}(\pi/2) < 0$, $F^*_{12,7}(2\pi/3) > 0$, $F^*_{12,7}(5\pi/6) < 0$ by Lemma 5.21, and we have $F^*_{12,7}(\pi) > 0$, $F^*_{12,7}(7\pi/6) < 0$ by Lemma 5.20. Thus $E^*_{12,7}$ has at least four zeros in $A^*_{7,1} \cup A^*_{7,2}$. Furthermore, by the valence formula for $\Gamma_0^*(7)$, $E^*_{12,7}$ has no other zero. Thus we have following:

Lemma 5.24. *$E^*_{12,7}$ has just four zeros in $A^*_{7,1} \cup A^*_{7,2}$, and we have $v_{i/\sqrt{7}}(E^*_{12,7}) = v_{\rho_{7,1}}(E^*_{12,7}) = v_{\rho_{7,2}}(E^*_{12,7}) = 0$.*

5.9.3. *The space of modular forms.* Let $M^*_{k,7}$ be the space of modular forms for $\Gamma_0^*(7)$ of weight k, and let $M^{*0}_{k,7}$ be the space of cusp forms for $\Gamma_0^*(7)$ of weight k. When we consider the map $M^*_{k,7} \ni f \mapsto f(\infty) \in \mathbb{C}$, the kernel of the map is $M^{*0}_{k,7}$. So $\dim(M^*_{k,7}/M^{*0}_{k,7}) \leqslant 1$, and $M^*_{k,7} = \mathbb{C}E^*_{k,7} \oplus M^{*0}_{k,7}$.

Recall that

$$\Delta_7 = \eta^6(z)\eta^6(7z)$$

is a cusp form for $\Gamma_0(7)$ of weight 6, and $(\Delta_7)^2$ is a cusp form for $\Gamma_0^*(7)$ of weight 12 (Remark 4.2).

We also have

$$E_{2,7}{}'(z) = \frac{1}{6}(7E_2(7z) - E_2(z)),$$

which is a modular form for $\Gamma_0(7)$ with $v_{\rho_{7,2}}(E_{2,7}{}') = 2$ and $v_p(E_{2,7}{}') = 0$ for every $p \neq \rho_{7,2}$. Furthermore, because we have $E_{2,7}{}'(W_7 z) = -(\sqrt{7}z)^2 E_{2,7}{}'(z)$, $(E_{2,7}{}')^2$ is a modular form for $\Gamma_0^*(7)$ of weight 4. (See Section 6)

We have following theorem:

Theorem 5.4. *Let k be an even integer, and let $\Delta_{7,4} := (5/16)((E_{2,7}{}')^2 - E^*_{4,7})$, $\Delta^0_{7,10} := (559/690)((41065/137592)(E^*_{4,7}E^*_{6,7} - E^*_{10,7}) - E^*_{6,7}\Delta_{7,4})$.*

(1) *For* $k<0$ *and* $k=2$, $M_{k,7}^*=0$. *We have* $M_{0,7}^*=\mathbb{C}$.
(2) *We have* $M_{4,7}^{*0}=\mathbb{C}\Delta_{7,4}$.
(3) *Let* $\Delta_{7,6}:=\Delta_{7,10}^0/\Delta_{7,4}$. *We have* $M_{6,7}^{*0}=\mathbb{C}\Delta_{7,6}$.
(4) *Let* $\Delta_{7,8}^0:=(\Delta_{7,4})^2$ *and* $\Delta_{7,8}^1:=E_{4,7}^*\Delta_{7,4}$. *We have* $M_{8,7}^{*0}=\mathbb{C}\Delta_{7,8}^0\oplus\mathbb{C}\Delta_{7,8}^1$.
(5) *Let* $\Delta_{7,10}^1:=E_{6,7}^*\Delta_{7,4}$. *We have* $M_{10,7}^{*0}=\mathbb{C}\Delta_{7,10}^0\oplus\mathbb{C}\Delta_{7,10}^1$.
(6) *Let* $\Delta_{7,12}^0:=(\Delta_7)^2$, $\Delta_{7,12}^1:=(\Delta_{7,4})^3$, $\Delta_{7,12}^2:=E_{4,7}^*(\Delta_{7,4})^2$, *and* $\Delta_{7,12}^3:=(E_{4,7}^*)^2\Delta_{7,4}$. *We have* $M_{12,7}^{*0}=\mathbb{C}\Delta_{7,12}^0\oplus\mathbb{C}\Delta_{7,12}^1\oplus\mathbb{C}\Delta_{7,12}^2\oplus\mathbb{C}\Delta_{7,12}^3$.
(7) *Let* $\Delta_{7,14}^0:=\Delta_{7,4}\Delta_{7,10}^0$, $\Delta_{7,14}^1:=E_{6,7}^*(\Delta_{7,4})^2$, *and* $\Delta_{7,14}^2:=E_{4,7}^*E_{6,7}^*\Delta_{7,4}$. *We have* $M_{14,7}^{*0}=\mathbb{C}\Delta_{7,14}^0\oplus\mathbb{C}\Delta_{7,14}^1\oplus\mathbb{C}\Delta_{7,14}^2$.
(8) $M_{k,7}^{*0}=M_{12,7}^{*0}M_{k-12,7}^*$.

Let k be an even integer $k\geqslant 4$. Define $m_7(k):=\left\lfloor\frac{k}{3}-\frac{t}{2}\right\rfloor$, where $t=0,2$ is chosen so that $t\equiv k \pmod 4$, and $\lfloor n\rfloor$ is the largest integer not more than n. Define $f_0(k):=(E_{4,7}^*)^{k/4}$ if $k\equiv 0\pmod 4$, and $f_0(k):=E_{6,7}^*(E_{4,7}^*)^{(k-6)/4}$ if $k\equiv 2\pmod 4$.

The proof of this theorem is similar to Theorem 5.2. For every $f\in M_{k,7}^{*0}$, it is easy to show that there exist some a_1, a_2, a_3 such that $f+a_1\Delta_{7,12}^1+a_2\Delta_{7,12}^2+a_3\Delta_{7,12}^3=b_4q^4+\cdots$. Then $(f+a_1\Delta_{7,12}^1+a_2\Delta_{7,12}^2+a_3\Delta_{7,12}^3)/\Delta_{7,12}^0\in M_{k-12,7}^*$. This proves (8).

The table of orders of zeros of basis for $M_{k,7}^*$ is following:

k	f	v_∞	$v_{i/\sqrt{7}}$	$v_{\rho_{7,1}}$	$v_{\rho_{7,2}}$	$\lvert$zeros on $A_7^*\rvert$
4	$E_{4,7}^*$	0	0	0	1	1
	$({E_{2,7}}')^2$	0	0	0	4	0
	$\Delta_{7,4}$	1	0	0	1	0
6	$E_{6,7}^*$	0	1	1	0	1
	$\Delta_{7,6}$	1	1	1	0	0
8	$(E_{4,7}^*)^2$	0	0	0	2	2
	$\Delta_{7,8}^0$	2	0	0	2	0
	$\Delta_{7,8}^1$	1	0	0	2	1
10	$E_{4,7}^*E_{6,7}^*$	0	1	1	1	2
	$\Delta_{7,10}^0$	2	1	1	1	0
	$\Delta_{7,10}^1$	1	1	1	1	1
12	$E_{12,7}^*$	0	0	0	0	4
	$\Delta_{7,12}^0$	4	0	0	0	0
	$\Delta_{7,12}^1$	3	0	0	3	0
	$\Delta_{7,12}^2$	2	0	0	3	1
	$\Delta_{7,12}^3$	1	0	0	3	2
14	$(E_{4,7}^*)^2E_{6,7}^*$	0	1	1	2	3
	$\Delta_{7,14}^0$	3	1	1	2	0
	$\Delta_{7,14}^1$	2	1	1	2	1
	$\Delta_{7,14}^2$	1	1	1	2	2
k	f	v_∞	$v_{i/\sqrt{7}}$	$v_{\rho_{7,1}}$	$v_{\rho_{7,2}}$	$\lvert$zeros on $A_7^*\rvert$

Then we have $\dim(M_{k,7}^{*0})=m_7(k)$ and $\dim(M_{k,7}^*)=m_7(k)+1$.

Write $n=m_7(k)$, then $k-12n=0$, 4, 6, 8, 10 or 14. Because $E_{k,7}-E_{k-12n,7}^*(E_{4,7}^*)^n\in M_{k,7}^{*0}$, we have $M_{k,7}^*=\mathbb{C}E_{k-4n,7}^*(E_{4,7}^*)^{3n}\oplus M_{k,7}^{*0}$. Then

$$M_{k,7}^*=E_{k-12n,7}^*\left\{\mathbb{C}(E_{4,7}^*)^{3n}\oplus(E_{4,7}^*)^{3(n-1)}M_{12,7}^{*0}\oplus(E_{4,7}^*)^{3(n-2)}(M_{12,7}^{*0})^2\oplus\cdots\oplus(M_{12,7}^{*0})^n\right\}$$
$$\oplus M_{k-12n,7}^{*0}(M_{12,7}^{*0})^n$$

Thus, for every $p\in\mathbb{H}$ and for every $f\in M_{k,7}^*$, $v_p(f)\geqslant v_p(E_{k-12n,7}^*)$.

In conclusion, the next proposision follows:

Proposition 5.20. *Let* $k\geqslant 4$ *be an even integer. For every* $f\in M_{k,7}^*$, *we have*

$$v_{i/\sqrt{7}}(f)\geqslant s_k,\quad v_{\rho_{7,1}}(f)\geqslant s_k\quad (s_k=0,1\ \textit{such that}\ 2s_k\equiv k\pmod 4),$$
$$v_{\rho_{7,2}}(f)\geqslant t_k\quad (s_k=0,1,2\ \textit{such that}\ -2t_k\equiv k\pmod 6).\qquad(123)$$

In addition, we have $(\Delta_7)^2 = (38/45)\left((5/114)\left(E_{4,7}^*(\Delta_{7,4})^2 - (\Delta_{7,6})^2\right) - (\Delta_{7,4})^3\right)$. Then

Remark 5.6. *Every modular form for $\Gamma_0^*(7)$ is generated by*

$$E_{4,7}^*, \quad E_{6,7}^*, \quad \Delta_{7,4}, \quad \textit{and} \quad \Delta_{7,6}.$$

Now, we have following conjecture:

Conjecture 5.2. *Let $k \geqslant 4$ be an even integer. $E_{k,7}^*$ has $m_7(k)$ zeros in $A_{7,1}^*$ and $A_{7,2}^*$. Furthermore, in Proposition 5.20, the equality hold if f is equal to $E_{k,7}^*$ or its constant multiple.*

5.9.4. *Observation on Conjecture 5.2.* Similar to $\Gamma_0^*(5)$, to prove Conjecture 5.2 is very difficult. The most difficult point is also the argument $Arg(\rho_{7,2})$. We need more radical expansion to prove this conjecture.

By the Lemma 5.20 and 5.21, $E_{k,7}^*$ has at least $m_7(k) - 2$ zeros in $A_{7,1}^*$ and $A_{7,2}^*$ for $k \leqslant 24$. Then, imilarly to Lemma and , we will prove next lemmas in the next subsubsection:

Lemma 5.25. *Let $k \geqslant 26$. For $\forall\theta \in [\pi/2, \pi/2 + \alpha_7 - x]$ $(x = \pi/k)$, $|R_{7,1}^*| < 2$.*

Lemma 5.26. *Let $k \geqslant 26$. For $\forall\theta \in [\alpha_7 - \pi/6 + x, \pi/2]$ $(x = \pi/k)$, $|R_{7,2}^*| < 2$.*

By above lemmas, we can easily show that $E_{k,7}^*$ has at least $m_7(k) - 2$ zeros in $A_{7,1}^*$ and $A_{7,2}^*$. Thus, we can prove Conjecture 5.2 except for at most 2 zeros.

5.9.5. *Expansion of the RSD Method.* The proofs of Lemma 5.25 and 5.26 are similar to that of Lemma 5.5. We need following preliminaries.

Proposition 5.21 (for Lemma 5.25)**.**

(1) *For $k \geqslant 26$, $\left(\frac{5}{3}\right)^{2/k} \leqslant 1 + \left(2\log\frac{5}{3}\right)\frac{1}{k} + \frac{1}{2}\left(2\log\frac{5}{3}\right)^2\left(\frac{5}{3}\right)^{2/k}\frac{1}{k^2}$,*
$10 + 4\sqrt{7}\cos\left(\frac{\pi}{2} + \alpha_7 - \frac{\pi}{k}\right) \geqslant 2\sqrt{3}\pi\frac{1}{k}$.

(2) *For $k \geqslant 26$, $\left(\frac{5}{2}\right)^{2/k} \leqslant 1 + \left(2\log\frac{5}{2}\right)\frac{1}{k} + \frac{1}{2}\left(2\log\frac{5}{2}\right)^2\left(\frac{5}{2}\right)^{2/k}\frac{1}{k^2}$,*
$15 + 3\sqrt{7}\cos\left(\frac{\pi}{2} + \alpha_7 - \frac{\pi}{k}\right) \geqslant 3\sqrt{3}\pi\frac{1}{k}$.

Proposition 5.22 (for Lemma 5.26)**.**

(1) *For $k \geqslant 26$, $\left(\frac{4}{3}\right)^{2/k} \leqslant 1 + \left(2\log\frac{4}{3}\right)\frac{1}{k} + \frac{1}{2}\left(2\log\frac{4}{3}\right)^2\left(\frac{4}{3}\right)^{2/k}\frac{1}{k^2}$,*
$1 + \frac{\sqrt{7}}{2}\cos\left(\alpha_7 - \frac{\pi}{6} - \frac{\pi}{k}\right) \geqslant \pi\frac{1}{k}$.

(2) *For $k \geqslant 26$, $4^{2/k} \leqslant 1 + (4\log 2)\frac{1}{k} + \frac{1}{2}(4\log 2)^2 4^{2/k}\frac{1}{k^2}$,*
$3 + \frac{3\sqrt{7}}{2}\cos\left(\alpha_7 - \frac{\pi}{6} - \frac{\pi}{k}\right) \geqslant 3\pi\frac{1}{k}$.

Proof of Lemma 5.25. Let $k \geqslant$ and $x = \pi/k$, then $0 \leqslant x \leqslant \pi/26$.

$$\begin{aligned}
11{+}4\sqrt{7}\cos&\left(\frac{\pi}{2} + \alpha_7 - \frac{\pi}{k}\right) - \left(\frac{5}{3}\right)^{2/k}\left(1 + \frac{12}{5k}x^2\right) \\
&\geqslant \frac{1}{k}\left\{2\sqrt{3}\pi - 2\log\frac{5}{3} - \frac{1}{2}\left(2\log\frac{5}{3}\right)^2\left(\frac{5}{3}\right)^{2/k}\frac{1}{k} - \frac{12\pi^2}{5}\left(\frac{5}{3}\right)^{2/k}\frac{1}{k^2}\right\} \\
&\geqslant \frac{1}{k} \times 9.81047... \quad (k \geqslant 26)
\end{aligned}$$

By Proposition 5.21,

$$\begin{aligned}
&|2e^{i\theta/2} + \sqrt{7}e^{-i\theta/2}|^2 \geqslant \left(\frac{5}{3}\right)^{2/k}\left(1 + \frac{12}{5k}x^2\right) \\
&\Rightarrow |2e^{i\theta/2} + \sqrt{7}e^{-i\theta/2}|^k \geqslant \frac{5}{3} + 2x^2 \\
&\Rightarrow v_k(2,1,\theta) \leqslant \frac{3}{5} - \frac{3}{5}x^2.
\end{aligned}$$

Similarly,

$$16 + 6\sqrt{7}\cos\left(\frac{\pi}{2} + \alpha_7 - \frac{\pi}{k}\right) - \left(\frac{5}{2}\right)^{2/k}\left(1 + \frac{12}{5k}x^2\right) \geqslant \frac{1}{k} \times 14.39532... \quad (k \geqslant 26)$$

$$\begin{aligned}
&|3e^{i\theta/2}+\sqrt{7}e^{-i\theta/2}|^2 \geqslant \left(\frac{5}{2}\right)^{2/k}\left(1+\frac{12}{5k}x^2\right)\\
&\Rightarrow |2e^{i\theta/2}+\sqrt{7}e^{-i\theta/2}|^k \geqslant \frac{5}{2}+3x^2\\
&\Rightarrow v_k(3,1,\theta) \leqslant \frac{2}{5}-\frac{2}{5}x^2.
\end{aligned}$$

Furthermore,

$$6\left(\frac{1}{3}\right)^{k/2}+4\left(\frac{1}{7}\right)^{k/2}+\cdots+2\left(\frac{1}{179}\right)^{k/2}+\frac{1914\sqrt{33}}{7(k-3)}\left(\frac{1}{3}\right)^{k/2} \leqslant 25\left(\frac{1}{3}\right)^{k/2}$$

Thus

$$|R^*_{7,1}| \leqslant 2-\frac{2\pi^2}{k^2}+25\left(\frac{1}{3}\right)^{k/2}$$

Now, $(1/3)^{k/2}$ is more rapidly decreasing in k than $1/k^2$, and for $k=26$, we have

$$|R^*_{7,1}| \leqslant 1.97081...$$

□

Proof of Lemma 5.26. Let $k \geqslant$ and $x=\pi/k$, then $0 \leqslant x \leqslant \pi/26$.

$$2-\frac{\sqrt{7}}{2}\cos\left(\alpha_7-\frac{\pi}{6}-\frac{\pi}{k}\right)-\left(\frac{4}{3}\right)^{2/k}\left(1+\frac{9}{5k}x^2\right) \geqslant \frac{1}{k}\times 2.47345... \quad (k \geqslant 26)$$

By Proposition 5.22,

$$\begin{aligned}
&2^{-2}\cdot|e^{i\theta/2}-\sqrt{7}e^{-i\theta/2}|^2 \geqslant \left(\frac{4}{3}\right)^{2/k}\left(1+\frac{9}{4k}x^2\right)\\
&\Rightarrow 2^{-k}\cdot|e^{i\theta/2}-\sqrt{7}e^{-i\theta/2}|^k \geqslant \frac{4}{3}+\frac{3}{2}x^2\\
&\Rightarrow 2^k\cdot v_k(1,-1,\theta) \leqslant \frac{3}{4}-\frac{3}{4}x^2.
\end{aligned}$$

Similarly,

$$4-\frac{3\sqrt{7}}{2}\cos\left(\alpha_7-\frac{\pi}{6}-\frac{\pi}{k}\right)-4^{2/k}\left(1+\frac{12}{5k}x^2\right) \geqslant \frac{1}{k}\times 14.39532... \quad (k \geqslant 26)$$

$$\begin{aligned}
&2^{-2}\cdot|3e^{i\theta/2}-\sqrt{7}e^{-i\theta/2}|^2 \geqslant 4^{2/k}\left(1+\frac{9}{4k}x^2\right)\\
&\Rightarrow 2^{-k}\cdot|3e^{i\theta/2}-\sqrt{7}e^{-i\theta/2}|^k \geqslant 4+\frac{9}{2}x^2\\
&\Rightarrow 2^k\cdot v_k(3,-1,\theta) \leqslant \frac{1}{4}-\frac{1}{4}x^2.
\end{aligned}$$

Furthermore,

$$2\left(\frac{1}{2}\right)^{k/2}+2\left(\frac{1}{3}\right)^{k/2}+\cdots+2\left(\frac{1}{113}\right)^{k/2}+\frac{2784\sqrt{6}}{7(k-3)}\left(\frac{1}{2}\right)^{k/2} \leqslant 41\left(\frac{1}{2}\right)^{k/2}$$

Thus

$$|R^*_{7,2}| \leqslant 2-\frac{2\pi^2}{k^2}+41\left(\frac{1}{2}\right)^{k/2}$$

Now, $(1/2)^{k/2}$ is more rapidly decreasing in k than $1/k^2$, and for $k=26$, we have

$$|R^*_{7,2}| \leqslant 1.97580...$$

□

5.10. **Conclusion of $\Gamma_0^*(p)$.** Let $k \geqslant 4$ be an even integer.

In section 3, we have following result:

$k \pmod{12}$	$v_i(E_k)$	$v_\rho(E_k)$	$\lvert$zeros on $A\rvert$
0	0	0	$k/12$
2	1	2	$(k-14)/12$
4	0	1	$(k-4)/12$
6	1	0	$(k-6)/12$
8	0	2	$(k-8)/12$
10	1	1	$(k-10)/12$

Table for E_k : $\mathrm{SL}_2(\mathbb{Z})$

Also, by the theorem 1, 2, we have following:

$k \pmod 8$	$v_{i/\sqrt{2}}(E_{k,2}^*)$	$v_{\rho_2}(E_{k,2}^*)$	$\lvert$zeros on $A_2^*\rvert$
0	0	0	$k/8$
2	1	3	$(k-10)/8$
4	0	2	$(k-4)/8$
6	1	1	$(k-6)/8$

Table for $E_{k,2}^*$: $\Gamma_0^*(2)$

$k \pmod{12}$	$v_{i/\sqrt{3}}(E_{k,3}^*)$	$v_{\rho_3}(E_{k,3}^*)$	$\lvert$zeros on $A_3^*\rvert$
0	0	0	$k/6$
2	1	5	$(k-8)/6$
4	0	4	$(k-4)/6$
6	1	3	$(k-6)/6$
8	0	2	$(k-2)/6$
10	1	1	$(k-4)/6$

Table for $E_{k,3}^*$: $\Gamma_0^*(3)$

Above results are proved in this section.

For a prime $p \geqslant 5$, let $A_p^* := \partial\mathbb{F}^*(p) \setminus \{z\,;\, Re(z) = -1/2\}$, where $\mathbb{F}^*(p)$ is the fundamental domain of $\Gamma_0^*(p)$ represented in Figure 7.

In addition, in Conjecture 5.1, we expect following:

$k \pmod 4$	$v_{i/\sqrt{5}}$	$v_{\rho_{5,1}}$	$v_{\rho_{5,2}}$	$\lvert$zeros on $A_5^*\rvert$
0	0	0	0	$k/4$
2	1	1	1	$(k-6)/4$

Table for $E_{k,5}^*$: $\Gamma_0^*(5)$

We proved except for at most 2 zeros.

Also, in Conjecture 5.2, we expect following:

$k \pmod{12}$	$v_{i/\sqrt{7}}$	$v_{\rho_{7,1}}$	$v_{\rho_{7,2}}$	$\lvert$zeros on $A_7^*\rvert$
0	0	0	0	$k/3$
2	1	1	2	$(k-5)/3$
4	0	0	1	$(k-1)/3$
6	1	1	0	$(k-3)/3$
8	0	0	2	$(k-2)/3$
10	1	1	1	$(k-4)/3$

Table for $E_{k,7}^*$

We proved except for at most 2 zeros.

Finally, we expect followings:

Conjecture 5.3 ($\Gamma_0^*(11)$)**.** *Let $k \geqslant 4$ be an even integer.*

$k \pmod 4$	$v_{i/\sqrt{11}}$	$v_{\rho_{11,1}}$	$v_{\rho_{11,2}}$	$v_{\rho_{11,3}}$	$\lvert$*zeros on* $A_{11}^*\rvert$
0	0	0	0	0	$k/2$
2	1	1	1	1	$(k-4)/2$

Table for $E_{k,11}^*$

Conjecture 5.4 ($\Gamma_0^*(13)$)**.** *Let $k \geqslant 4$ be an even integer.*

$k \pmod{12}$	$v_{i/\sqrt{13}}$	$v_{\rho_{13,1}}$	$v_{\rho_{13,2}}$	$v_{\rho_{13,3}}$	$\lvert$*zeros on* $A_{13}^*\rvert$
0	0	0	0	0	$7k/12$
2	1	1	1	2	$(7k-26)/12$
4	0	0	0	1	$(7k-4)/12$
6	1	1	1	0	$(7k-18)/12$
8	0	0	0	2	$(7k-8)/12$
10	1	1	1	1	$(7k-22)/12$

Table for $E_{k,13}^*$

Conjecture 5.5 ($\Gamma_0^*(17)$)**.** *Let $k \geqslant 4$ be an even integer.*

$k \pmod 4$	$v_{i/\sqrt{17}}$	$v_{\rho_{17,1}}$	$v_{\rho_{17,2}}$	$v_{\rho_{17,3}}$	$v_{\rho_{17,4}}$	$\lvert$*zeros on* $A_{17}^*\rvert$
0	0	0	0	0	0	$3k/4$
2	1	1	1	1	1	$(3k-10)/4$

Table for $E_{k,17}^*$

Conjecture 5.6 ($\Gamma_0^*(19)$)**.** *Let $k \geqslant 4$ be an even integer.*

$k \pmod{12}$	$v_{i/\sqrt{19}}$	$v_{\rho_{19,1}}$	$v_{\rho_{19,2}}$	$v_{\rho_{19,3}}$	$v_{\rho_{19,4}}$	$\lvert$*zeros on* $A_{19}^*\rvert$
0	0	0	0	0	0	$5k/6$
2	1	1	2	1	1	$(5k-16)/6$
4	0	0	1	0	0	$(5k-2)/6$
6	1	1	0	1	1	$(5k-12)/6$
8	0	0	2	0	0	$(5k-4)/6$
10	1	1	1	1	1	$(5k-14)/6$

Table for $E_{k,19}^*$

By the application of the RSD Method, we have many zeros on some arcs in A_p^*. But we have some arcs in A_p^*, in which we do not know whether $E_{k,p}^*$ has zeros or not.

6. Congruence subgroup $\Gamma_0(p)$

6.1. $\Gamma_0(2)$.

We have the following transformation:

$$\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}: \quad \frac{e^{i\theta}+1}{2} \mapsto \frac{e^{i(\pi-\theta)}-1}{2}.$$

6.1.1. *Valence formula.* In order to decide the locating of all zeros of $E_{k,2}^{\infty}(z)$ and $E_{k,2}^{0}(z)$, we need the *valence formula for* $\Gamma_0(2)$:

Proposition 6.1. *Let f be a modular function of weight k for $\Gamma_0(2)$, which is not identically zero. We have*

$$v_\infty(f) + v_0(f) + \frac{1}{2}v_{\rho_2}(f) + \sum_{\substack{p\in\Gamma_0(2)\backslash\mathbb{H} \\ p\neq\rho_2}} v_p(f) = \frac{k}{4}, \tag{124}$$

where $\rho_2 := -1/2 + i/2$. (See [KO] *and* [SE]*)*

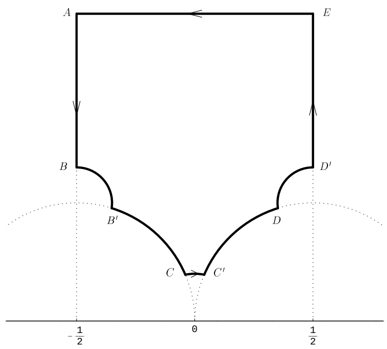


Figure 9

Proof. Let f be a nonzero modular function of weight k for $\Gamma_0(2)$, and let $\mathscr{C}$ be a contour of $\mathbb{F}(2)$ which is a fundamental domain of $\Gamma_0(2)$ represented in Figure 9, whose interior contains every zero and pole of f except for ρ_2. By the *Residue theorem*, we have

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{df}{f} = \sum_{p\in\mathbb{F}(2)\backslash\{\rho_2\}} v_p(f).$$

Similar to Proposition 3.1,

(i) For the arc EA, we have

$$\frac{1}{2\pi i}\int_E^A \frac{df}{f} = -v_\infty(f).$$

(ii) For the arc CC', without loss of generallity, we can define arc CC' so that it equals the image of EA by the transformation W_2. Define $f_0(z) := (\sqrt{2}z)^{-k}f(W_2 z)$, then we can write

$$f_0(z) = \sum_{n\in\mathbb{N}} a_{0,n}q^n, \quad \text{where } q = e^{2\pi i z}. \quad \text{(See equation (16))}$$

Furthermore, we have $f_0(W_2^{-1}z) = (\sqrt{2}z)^k f(z)$ and

$$\frac{df_0(W_2^{-1}z)}{f_0(W_2^{-1}z)} = \frac{df(z)}{f(z)} + k\frac{dz}{z}.$$

Thus

$$\frac{1}{2\pi i}\int_C^{C'} \frac{df(z)}{f(z)} = \frac{1}{2\pi i}\int_E^A \frac{df_0(z)}{f_0(z)} - \frac{1}{2\pi i}\int_C^{C'} k\frac{dz}{z}.$$

Now, when the arcs CC' tend to 0, we have

$$\frac{1}{2\pi i}\int_{C}^{C'} k\frac{dz}{z} \to 0.$$

In addition,

$$\frac{1}{2\pi i}\int_{E}^{A} \frac{df_0}{f_0} = -v_\infty(f_0) = -v_0(f).$$

In conclusion, we have

$$\frac{1}{2\pi i}\int_{C}^{C'} \frac{df}{f} \to -v_0(f).$$

(iii) For the arcs BB' and DD', when the radii of the each arc tends to 0, then we have

$$\frac{1}{2\pi i}\int_{B}^{B'} \frac{df}{f} = \frac{1}{2\pi i}\int_{D}^{D'} \frac{df}{f} \to -\frac{1}{4}v_{\rho_2}(f).$$

(iv) For the arcs AB and $D'E$, since $f(Tz) = f(z)$ for $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$,

$$\frac{1}{2\pi i}\int_{A}^{B} \frac{df}{f} + \frac{1}{2\pi i}\int_{D'}^{E} \frac{df}{f} = 0.$$

(v) For the arcs $B'C$ and $C'D$, since $f(S_2 z) = (2z+1)^k f(z)$ for $S_2 := \left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$, we have

$$\frac{df(S_2z)}{f(S_2z)} = k\frac{dz}{z+1/2} + \frac{df(z)}{f(z)}.$$

When the radii of the arcs BB', CC', DD' tend to 0, the angle of the arc $B'C$ tend to $\pi/2$. Thus we have

$$\frac{1}{2\pi i}\int_{B'}^{C} \frac{df}{f} + \frac{1}{2\pi i}\int_{C'}^{D} \frac{df}{f} = \frac{1}{2\pi i}\int_{B'}^{C} \left(-k\frac{dz}{z+1/2}\right) \quad \to \frac{k}{4}.$$

□

6.1.2. *Modular forms of weight* 2. We define

$$E_{2,2}'(z) := 2E_2(2z) - E_2(z). \tag{125}$$

Note that $E_{2,2}'$ is generated by Eisensitein series for $\mathrm{SL}_2(\mathbb{Z})$, but it is not Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ nor $\Gamma_0(2)$.

It is easy to show that $E_{2,2}'$ satisfies transformation rule (14):

$$E_{2,2}'\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)z\right) = E_{2,2}'(z+1) = E_{2,2}'(z).$$

Since $\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right) = W_2\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)^{-1}W_2^{-1}$, and

$$E_2\left(2\left(-\frac{1}{2z}\right)\right) = z^2E_2(z) + \frac{12}{2\pi i}\cdot z,$$

$$E_2\left(-\frac{1}{2z}\right) = 4z^2E_2(2z) + \frac{12}{2\pi i}\cdot 2z,$$

we have

$$E_{2,2}'(W_2z) = E_{2,2}'\left(-\frac{1}{2z}\right) = 2z^2E_2(z) - 4z^2E_2(2z) = -(\sqrt{2}z)^2E_{2,2}'(z), \tag{126}$$

then

$$E_{2,2}'\left(\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)z\right) = (2z+1)^2E_{2,2}'(z).$$

Recall that $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$ generate $\Gamma_0(2)$ (in Section 4), then we can show that $E_{2,2}'$ satisfies transformation rule for $\Gamma_0(2)$.

Furthermore, by the definition, it is easy to show that $E_{2,2}'$ is holomorphic on $\mathbb{H}$ and at ∞. In addition, by the equation (126), we have

$$(\sqrt{2}z)^{-2}E_{2,2}'(W_2z) = -E_{2,2}'(z).$$

Thus $E_{2,2}'$ is holomorphic at cusp 0. (See equation (16)) Now, we prove $E_{2,2}'$ is a modular form for $\Gamma_0(2)$ of weight 2.

How about the locating zeros of $E_{2,2}'$? By the valence formula for $\Gamma_0(2)$ (Proposition 6.1), we have

$$v_{\rho_2}(E_{2,2}') = 1, \qquad v_p(E_{2,2}') = 0 \quad \text{for every } p \neq \rho_2.$$

Incidentally, let f be a modular form for $\Gamma_0(2)$ of weight 2. Then, By the valence formula for $\Gamma_0(2)$, we also have

$$v_{\rho_2}(f) = 1, \qquad v_p(f) = 0 \quad \text{for every } p \neq \rho_2.$$

Thus $f/{E_{2,2}}'$ is a modular form of weight 0, then $f/{E_{2,2}}' \in \mathbb{C}$. In conclusion, f is a constant multiple of ${E_{2,2}}'$.

6.1.3. *Preliminaries.* Let f be a modular form for $\Gamma_0(2)$ of weight k, and let $k \equiv 2 \pmod 4$. Then we have

$$f(\rho_2) = f(\left(\begin{smallmatrix} -1 & -1 \\ 2 & 1 \end{smallmatrix}\right) \rho_2) = i^k f(\rho_2) = -f(\rho_2).$$

Thus $f(\rho_2) = 0$ and $v_{\rho_2}(f) \geqslant 1$.

Let k be an even integer such that $k \equiv 0 \pmod 4$. Then we have

$$E_{k,2}^{\infty}(\rho_2) = \frac{1}{1-2^k}((1+i)^k - 2^k)E_k(i) \neq 0,$$
$$E_{k,2}^{0}(\rho_2) = \frac{1}{1-2^{-k}}((1+i)^k - 1)E_k(i) \neq 0.$$

Thus $v_{\rho_2}(E_{k,2}^{\infty}) = v_{\rho_2}(E_{k,2}^{0}) = 0$.

Recall that $v_0(E_{k,2}^{\infty}) = v_\infty(E_{k,2}^{0}) = 1$ and $v_\infty(E_{k,2}^{\infty}) = v_0(E_{k,2}^{0}) = 0$. (Section 4)

Finally, we study the locating zeros of $E_{k,2}^{\infty}$ and $E_{k,2}^{0}$ of weight 4 and 6.

For $k = 4$, we have $v_0(E_{4,2}^{\infty}) = v_\infty(E_{4,2}^{0}) = 1$. By the valence formula, $E_{4,2}^{\infty}$ and $E_{4,2}^{0}$ do not have any other zeros.

For $k = 6$, we have $v_0(E_{6,2}^{\infty}) = v_\infty(E_{6,2}^{0}) = 1$, $v_{\rho_2}(E_{6,2}^{\infty}) \geqslant 1$, and $v_{\rho_2}(E_{6,2}^{0}) \geqslant 1$. By the valence formula, we have $v_{\rho_2}(E_{6,2}^{\infty}) = v_{\rho_2}(E_{6,2}^{0}) = 1$, and they do not have any other zeros.

6.1.4. *The space of modular forms.* Let $M_{k,2}$ be the space of modular forms for $\Gamma_0(2)$ of weight k, and let $M_{k,2}^0$ be the space of cusp forms for $\Gamma_0(2)$ of weight k. When we consider the map $M_{k,2} \ni f \mapsto (f(\infty), f(0)) \in \mathbb{C} \times \mathbb{C}$, the kernel of the map is $M_{k,2}^0$. So $\dim(M_{k,2}/M_{k,2}^0) \leqslant 2$, and $M_{k,2} = \mathbb{C}E_{k,2}^{\infty} \oplus \mathbb{C}E_{k,2}^{0} \oplus M_{k,2}^0$. Recall that $\Delta_2 = \eta^8(z)\eta^8(2z)$. We have following theorem:

Theorem 6.1. *Let k be an even integer.*

(1) *For $k < 0$, $M_{k,2} = 0$.*
(2) *For $k = 0, 2, 4,$ and 6, we have $M_{k,2}^0 = 0$. Furthemore, we have $M_{0,2} = \mathbb{C}$, $M_{2,2} = \mathbb{C}{E_{2,2}}'$, and $M_{k,2} = \mathbb{C}E_{k,2}^{\infty} \oplus \mathbb{C}E_{k,2}^{0}$ for $k = 4$ and 6.*
(3) *$M_{k,2}^0 = \Delta_2 M_{k-8,2}$.*

Proof. Let f be a nonzero function of $M_{k,2}$, then $v_p(f) \geqslant 0$ for every $p \in \mathbb{H}$. By the valence formula for $\Gamma_0(2)$(Proposition6.1), we have $k \geqslant 0$. This proves (1).

In Section 4, we have $v_\infty(\Delta_2) = v_0(\Delta_2) = 1$ and $v_p(\Delta_2) = 0$ for every $p \in \mathbb{H}$. Then, for every $f \in M_{k,2}^0$, we have $v_p(f/\Delta_2) \geqslant 0$ for every $p \in \mathbb{H} \cup \{\infty, 0\}$. Thus $f/\Delta_2 \in M_{k-8,2}$. This proves (3).

By (3) and $M_{k,2} = 0$ for $k < 0$, we have $M_{k,2}^0 = 0$ for $k = 0, 2, 4,$ and 6. By previous subsubsections, we can prove (2). □

Furthermore, we have $\dim(M_{k,2}) = \lfloor k/4 \rfloor + 1$ for $k \geqslant 0$, and $\dim(M_{k,2}^0) = \lfloor k/4 \rfloor - 1$ for $k \geqslant 8$.

Let k be an even integer such that $k \geqslant 4$ and $k \equiv 2 \pmod 4$. For $f \in M_{k,2}$, by previous subsubsections, we have $v_p(f/{E_{2,2}}') \geqslant 0$ for every $p \in \mathbb{H} \cup \{\infty, 0\}$. Then $f/{E_{2,2}}' \in M_{k-2,2}$. Thus $M_{k,2} = {E_{2,2}}' M_{k-2,2}$, and $k - 2 \equiv 0 \pmod 4$.

On the other hand, let k be an even integer such that $k \geqslant 4$ and $k \equiv 0 \pmod 4$. Write $n := \lfloor k/8 \rfloor$, then $k - 8n = 0$ or 4. Now, we have $v_0(E_{k,2}^{\infty}) = 1$ and $v_\infty(E_{k,2}^{\infty} - E_{k-8n,2}^{\infty}(E_{4,2}^{\infty})^{2n}) \geqslant 1$. Thus $E_{k,2}^{\infty} - E_{k-8n,2}^{\infty}(E_{4,2}^{\infty})^{2n} \in M_{k,2}^0$. Similarly, we have $E_{k,2}^{0} - E_{k-8n,2}^{0}(E_{4,2}^{0})^{2n} \in M_{k,2}^0$. In conclusion, we have

$M_{k,2} = \mathbb{C}E^{\infty}_{k-8n,2}(E^{\infty}_{4,2})^{2n} \oplus \mathbb{C}E^{0}_{k-8n,2}(E^{0}_{4,2})^{2n} \oplus M^{0}_{k,2}$. Then

$$\begin{aligned} M_{k,2} &= \mathbb{C}E^{\infty}_{k-8n,2}(E^{\infty}_{4,2})^{2n} \oplus \mathbb{C}E^{0}_{k-8n,2}(E^{0}_{4,2})^{2n} \oplus \Delta_2 M_{k-8,2} \\ &\cdots \\ &= E^{\infty}_{k-8n,2}(\mathbb{C}(E^{\infty}_{4,2})^{2n} \oplus \mathbb{C}(E^{\infty}_{4,2})^{2(n-1)}\Delta_2 \oplus \cdots \oplus \mathbb{C}\Delta_2^n) \\ &\quad \oplus E^{0}_{k-8n,2}(\mathbb{C}(E^{0}_{4,2})^{2n} \oplus \mathbb{C}(E^{0}_{4,2})^{2(n-1)}\Delta_2 \oplus \cdots \oplus \mathbb{C}\Delta_2^n) \end{aligned}$$

Thus, the next proposision follows:

Proposition 6.2. *Let $k \geqslant 4$ be an even integer. For every $f \in M_{k,2}$, we have*

$$v_{\rho_2}(f) \geqslant t_k \quad (t_k = 0, 1 \text{ such that } 2t_k \equiv k \pmod 4). \tag{127}$$

In addition, we have $E^0_{4,2} = 4(({E_{2,2}}')^2 - E^{\infty}_{4,2})$ and $\Delta_2 = E^{\infty}_{4,2} \cdot E^0_{4,2}/256$. Then

Remark 6.1. *Every modular form for $\Gamma_0(2)$ is generated by*

$$ {E_{2,2}}' \quad \text{and} \quad E^{\infty}_{4,2}.$$

Finally, define

$$A_2 := \{z\,;\, |z + 1/2| = 1/2,\ 0 < Arg(z) < \pi/2\}, \tag{128}$$

$$A^0_2 := \{z\,;\, Re(z) = -1/2,\ Im(z) > 1/2\}. \tag{129}$$

Then we have following:

Conjecture 6.1. *Let $k \geqslant 4$ be an even integer. $E^{\infty}_{k,2}$ has $\lfloor k/4 \rfloor - 1$ zeros in A_2, and $E^0_{k,2}$ has $\lfloor k/4 \rfloor - 1$ zeros in A^0_2. Furthermore, in Proposition 6.2, the equality hold if f is equal to $E^{\infty}_{k,2}$ or $E^0_{k,2}$.*

Now, we have the following transformation:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} W_2 : \frac{e^{i\theta} - 1}{2} \mapsto -\frac{1}{2} + \frac{i}{2}\frac{1}{\tan\theta/2}. \tag{130}$$

This transform A_2 to A^0_2. Moreover,

$$E^0_{k,2}\left(\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) W_2\, z\right) = (2z)^k E^{\infty}_{k,2}(z) \quad \text{for every } z \in A_2.$$

Then

Remark 6.2. *If $E^{\infty}_{k,2}$ has $\lfloor k/4 \rfloor - 1$ zeros in A_2, then $E^0_{k,2}$ has $\lfloor k/4 \rfloor - 1$ zeros in A^0_2.*

6.1.5. *The function $F_{k,2}$.* We give the next definition;

$$F_{k,2}(\theta) := e^{ik\theta/2} E^{\infty}_{k,2}(e^{i\theta}/2 - 1/2). \tag{131}$$

Again, $E^{\infty}_{k,2}$ is denoted by

$$E^{\infty}_{k,2}(z) = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ 2|c}} (cz + d)^{-k}.$$

Since $2 \mid c$, we can write $c = 2c'$ for $\exists c' \in \mathbb{Z}$, and have $2 \nmid d$.

$$\begin{aligned} F_{k,2}(\theta) &= \frac{e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1 \\ 2|c}} \left(c\frac{e^{i\theta} - 1}{2} + d \right)^{-k} = \frac{e^{ik\theta/2}}{2} \sum_{\substack{(c,d)=1 \\ 2\nmid d}} (c'e^{i\theta} + (-c' + d))^{-k} \\ &= \frac{1}{2} \sum_{\substack{(c,d)=1 \\ 2\nmid d}} (c'e^{i\theta/2} + (-c' + d)e^{-i\theta/2})^{-k}. \end{aligned}$$

Then we have $(c', -c' + d) = 1$, $2 \mid c'(-c' + d)$. Thus we can write as following:

$$F_{k,2}(\theta) = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ 2|cd}} (ce^{i\theta/2} + de^{-i\theta/2})^{-k}.$$

Note that for any pair (c, d), $(ce^{i\theta/2} + de^{i\theta/2})^{-k}$ and $(de^{i\theta/2} + ce^{-i\theta/2})^{-k}$ are conjugates of each other. The next proposition follows.

Proposition 6.3. *$F_{k,2}(\theta)$ is real for every $\theta \in \mathbb{R}$.*

6.1.6. *Application of the RSD Method* (0). We note that $N := c^2 + d^2$.

Firstly, we consider the case $N = 1$. Then we can write:

$$F_{k,2}(\theta) = 2\cos(k\theta/2) + R_2, \tag{132}$$

where

$$R_2 = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ 2|cd \\ N>1}} (ce^{i\theta/2} + de^{-i\theta/2})^{-k}.$$

Now, we have

$$|R_2| \leqslant \frac{1}{2} \sum_{\substack{(c,d)=1 \\ 2|cd \\ N>1}} |ce^{i\theta/2} + de^{-i\theta/2}|^{-k}.$$

Let $v_k(c, d, \theta) := |ce^{i\theta/2} + de^{-i\theta/2}|^{-k}$, then $v_k(c, d, \theta) = 1/(c^2 + d^2 + 2cd\cos\theta)^{k/2}$ and $v_k(c, d, \theta) = v_k(-c, -d, \theta) = v_k(\pm d, \pm c, \theta)$.

However, for every $n \in \mathbb{N}$, we have $(2n, -2n - 1) = 1$, $2 \mid 2n(-2n - 1)$, and

$$v_k(2n, -2n - 1, \theta) \leqslant 1 \quad \text{for } 0 \leqslant \theta \leqslant \pi/2.$$

Here, the number of the pairs $(2n, -2n - 1)$ is infinite. Thus we have the bound

$$|R_2| \leqslant \infty,$$

which does not make sense.

For $0 \leqslant \theta \leqslant \pi/2$, we have $0 \leqslant k\theta/2 \leqslant k\pi/4$. Here the interval $[0, k/4]$ has $\lfloor k/4 \rfloor + 1$ integers, but we need at most $\lfloor k/4 \rfloor$ integers for $\lfloor k/4 \rfloor - 1$ zeros. Now, $|R_2|$ tends to ∞ when θ tend to 0. Thus we expect to remove the integer 0 from the interval $[0, k/4]$. Then we need

$$|R_2| < 2 \qquad \text{for every } \theta \in [2\pi/k, \pi/2]. \tag{133}$$

However, it is still difficult to prove above bound.

For the first step, we will prove following in the next subsubsections:

$$|R_2| < 2 \qquad \text{for every } \theta \in [\pi/6, \pi/2]. \tag{134}$$

$$|R_2| < 2 \qquad \text{for every } \theta \in [\pi/12, \pi/2]. \tag{135}$$

$$|R_2| < 2 \qquad \text{for every } \theta \in [\pi/20, \pi/2]. \tag{136}$$

6.1.7. *Application of the RSD Method* (1) : $[\pi/6, \pi/2]$. In this subsubsection, we prove the bound (134). In privious subsubsections, $E_{k,2}^{\infty}$ and $E_{k,2}^{0}$ of weight $k < 8$ has no zeros other than ∞, 0, and ρ_2. Thus we may assume $k \geqslant 8$.

Now we will consider the next cases, namely $N = 5, 13, 17$, and $N \geqslant 25$. Considering $0 \leqslant \cos\theta \leqslant 7/8$ for the interval $[\pi/6, \pi/2]$, we have the following:

When $N = 5$,	$v_k(1, 2, \theta) \leqslant (1/5)^{k/2}$,	$v_k(1, -2, \theta) \leqslant (2/3)^{k/2}$.
When $N = 13$,	$v_k(2, 3, \theta) \leqslant (1/13)^{k/2}$,	$v_k(2, -3, \theta) \leqslant (2/5)^{k/2}$.
When $N = 17$,	$v_k(1, 4, \theta) \leqslant (1/17)^{k/2}$,	$v_k(1, -4, \theta) \leqslant (1/10)^{k/2}$.
When $N \geqslant 25$,	$\lvert ce^{i\theta/2} \pm de^{-i\theta/2} \rvert^2 \geqslant N/8$,	

and the rest of the question is about the number of terms with $c^2+d^2=N$. The number of $|c|$ is not more than $N^{1/2}$, and we consider four terms $(\pm(c,d), \pm(c,-d))$ and the number $1/2$ which is the coefficient of the summation. Thus the number of terms is not more than $2N^{1/2}$. Then

$$|R_2|_{N\geqslant 25} = 4\sqrt{2}\sum_{N=25}^{\infty}\left(\frac{1}{8}N\right)^{(1-k)/2} \leqslant \frac{252\sqrt{6}}{k-3}\left(\frac{1}{3}\right)^{k/2}.$$

Thus

$$|R_2| \leqslant 2\left(\frac{2}{3}\right)^{k/2} + 2\left(\frac{2}{5}\right)^{k/2} + \cdots + 2\left(\frac{1}{17}\right)^{k/2} + \frac{252\sqrt{6}}{k-3}\left(\frac{1}{3}\right)^{k/2}, \quad \leqslant 1.61099\ldots \quad (k\geqslant 8) \tag{137}$$

In conclusion,

Remark 6.3. *We proved Conjecture 6.1 for* $4\leqslant k\leqslant 12$.

6.1.8. *Application of the RSD Method* (2) : $[\pi/12, \pi/2]$. In this subsubsection, we prove the bound (135). In privious subsubsections, we proved for $k\leqslant 12$. Thus we may assume $k\geqslant 14$.

Now we will consider the next cases, namely $N=5,13,\cdots,61$, and $N\geqslant 65$. Considering $0\leqslant\cos\theta\leqslant 29/30$ for the interval $[\pi/12,\pi/2]$, we have the following:

When $N=5$, $v_k(1,2,\theta)\leqslant(1/5)^{k/2}$, $v_k(1,-2,\theta)\leqslant(15/17)^{k/2}$.
When $N=13$, $v_k(2,3,\theta)\leqslant(1/13)^{k/2}$, $v_k(2,-3,\theta)\leqslant(5/7)^{k/2}$.
When $N=17$, $v_k(1,4,\theta)\leqslant(1/17)^{k/2}$, $v_k(1,-4,\theta)\leqslant(15/139)^{k/2}$.
When $N=25$, $v_k(3,4,\theta)\leqslant(1/5)^{k}$, $v_k(3,-4,\theta)\leqslant(5/9)^{k/2}$.
When $N=29$, $v_k(2,5,\theta)\leqslant(1/29)^{k/2}$, $v_k(2,-5,\theta)\leqslant(3/29)^{k/2}$.
When $N=37$, $v_k(1,6,\theta)\leqslant(1/37)^{k/2}$, $v_k(1,-6,\theta)\leqslant(5/127)^{k/2}$.
When $N=41$, $v_k(4,5,\theta)\leqslant(1/41)^{k/2}$, $v_k(4,-5,\theta)\leqslant(3/7)^{k/2}$.
When $N=53$, $v_k(2,7,\theta)\leqslant(1/53)^{k/2}$, $v_k(2,-7,\theta)\leqslant(15/389)^{k/2}$.
When $N=61$, $v_k(5,6,\theta)\leqslant(1/61)^{k/2}$, $v_k(5,-6,\theta)\leqslant(1/3)^{k/2}$.
When $N\geqslant 65$, $|ce^{i\theta/2}\pm de^{-i\theta/2}|^2\geqslant N/30$,

and the number of terms with $c^2+d^2=N$ is not more than $2N^{1/2}$. Then

$$|R_2|_{N\geqslant 65} = 2\sqrt{30}\sum_{N=65}^{\infty}\left(\frac{1}{30}N\right)^{(1-k)/2} \leqslant \frac{2048}{k-3}\left(\frac{15}{32}\right)^{k/2}.$$

Thus

$$|R_2| \leqslant 2\left(\frac{5}{7}\right)^{k/2} + 2\left(\frac{5}{9}\right)^{k/2} + \cdots + 2\left(\frac{1}{61}\right)^{k/2} + \frac{2048}{k-3}\left(\frac{15}{32}\right)^{k/2}, \quad \leqslant 1.98724\ldots \quad (k\geqslant 14) \tag{138}$$

In conclusion,

Remark 6.4. *We proved Conjecture 6.1 for* $14\leqslant k\leqslant 24$.

6.1.9. *Application of the RSD Method* (3) : $[\pi/20, \pi/2]$. In this subsubsection, we prove the bound (136). In privious subsubsections, we proved for $k\leqslant 24$. Thus we may assume $k\geqslant 26$.

Now we will consider the next cases, namely $N=5,13,\cdots,125$, and $N\geqslant 137$. Considering $0\leqslant \cos\theta\leqslant 81/82$ for the interval $[\pi/20,\pi/2]$, we have the following:

When $N=5$, $v_k(1,2,\theta)\leqslant(1/5)^{k/2}$, $v_k(1,-2,\theta)\leqslant(41/43)^{k/2}$.
When $N=13$, $v_k(2,3,\theta)\leqslant(1/13)^{k/2}$, $v_k(2,-3,\theta)\leqslant(41/47)^{k/2}$.

When $N = 17$, $v_k(1,4,\theta) \leqslant (1/17)^{k/2}$, $v_k(1,-4,\theta) \leqslant (41/373)^{k/2}$.

When $N = 25$, $v_k(3,4,\theta) \leqslant (1/5)^{k}$, $v_k(3,-4,\theta) \leqslant (41/53)^{k/2}$.

When $N = 29$, $v_k(2,5,\theta) \leqslant (1/29)^{k/2}$, $v_k(2,-5,\theta) \leqslant (41/379)^{k/2}$.

When $N = 37$, $v_k(1,6,\theta) \leqslant (1/37)^{k/2}$, $v_k(1,-6,\theta) \leqslant (41/1031)^{k/2}$.

When $N = 41$, $v_k(4,5,\theta) \leqslant (1/41)^{k/2}$, $v_k(4,-5,\theta) \leqslant (41/61)^{k/2}$.

When $N = 53$, $v_k(2,7,\theta) \leqslant (1/53)^{k/2}$, $v_k(2,-7,\theta) \leqslant (41/1039)^{k/2}$.

When $N = 61$, $v_k(5,6,\theta) \leqslant (1/61)^{k/2}$, $v_k(5,-6,\theta) \leqslant (41/71)^{k/2}$.

When $N = 65$, $v_k(1,8,\theta) \leqslant (1/65)^{k/2}$, $v_k(1,-8,\theta) \leqslant (41/2017)^{k/2}$,
$v_k(4,7,\theta) \leqslant (1/65)^{k/2}$, $v_k(4,-7,\theta) \leqslant (41/397)^{k/2}$.

When $N = 73$, $v_k(3,8,\theta) \leqslant (1/73)^{k/2}$, $v_k(3,-8,\theta) \leqslant (41/1049)^{k/2}$.

When $N = 85$, $v_k(2,9,\theta) \leqslant (1/85)^{k/2}$, $v_k(2,-9,\theta) \leqslant (41/2027)^{k/2}$,
$v_k(6,7,\theta) \leqslant (1/85)^{k/2}$, $v_k(6,-7,\theta) \leqslant (41/83)^{k/2}$.

When $N = 89$, $v_k(5,8,\theta) \leqslant (1/89)^{k/2}$, $v_k(5,-8,\theta) \leqslant (41/409)^{k/2}$.

When $N = 97$, $v_k(4,9,\theta) \leqslant (1/97)^{k/2}$, $v_k(4,-9,\theta) \leqslant (41/1061)^{k/2}$.

When $N = 101$, $v_k(1,10,\theta) \leqslant (1/101)^{k/2}$, $v_k(1,-10,\theta) \leqslant (41/3331)^{k/2}$.

When $N = 109$, $v_k(3,10,\theta) \leqslant (1/109)^{k/2}$, $v_k(3,-10,\theta) \leqslant (41/2039)^{k/2}$.

When $N = 113$, $v_k(7,8,\theta) \leqslant (1/113)^{k/2}$, $v_k(7,-8,\theta) \leqslant (41/97)^{k/2}$.

When $N = 125$, $v_k(2,11,\theta) \leqslant (1/125)^{k/2}$, $v_k(2,-11,\theta) \leqslant (41/3343)^{k/2}$.

When $N \geqslant 137$, $|ce^{i\theta/2} \pm de^{-i\theta/2}|^2 \geqslant N/82$,

and the number of terms with $c^2 + d^2 = N$ is not more than $2N^{1/2}$. Then

$$|R_2|_{N\geqslant 137} = 2\sqrt{82} \sum_{N=137}^{\infty} \left(\frac{1}{82}N\right)^{(1-k)/2} \leqslant \frac{1088\sqrt{34}}{k-3} \left(\frac{41}{68}\right)^{k/2}.$$

Thus

$$|R_2| \leqslant 2\left(\frac{41}{43}\right)^{k/2} + 2\left(\frac{41}{47}\right)^{k/2} + \cdots + 2\left(\frac{1}{125}\right)^{k/2} + \frac{1088\sqrt{34}}{k-3}\left(\frac{41}{68}\right)^{k/2}, \quad (139)$$
$$\leqslant 1.88380... \quad (k \geqslant 26)$$

In conclusion,

Remark 6.5. *We proved Conjecture 6.1 for* $26 \leqslant k \leqslant 40$.

Now, by Remark 6.3, 6.4, and 6.5, we prove Conjecture 6.1 for $4 \leqslant k \leqslant 40$. However, for greater k, we prove only about 90% of Conjecture 6.1 by the sense of the interval $[\pi/20, \pi/2]$.

However, for greater k, it seems that we can not prove for all zeros with the same method. For example, for $k = 100$, we consider $0 \leqslant \cos\theta \leqslant 506/507$ for the interval $[\pi/50, \pi/2]$, we have

$$\begin{aligned} |R_2| &\leqslant 2v_k(1,-2,\theta) + 2v_k(2,-3,\theta) + 2v_k(3,-4,\theta) \\ &\leqslant 2\left(\frac{507}{511}\right)^{k/2} + 2\left(\frac{169}{173}\right)^{k/2} + 2\left(\frac{169}{177}\right)^{k/2} \\ &= 2.16912... \quad (k = 100) \end{aligned}$$

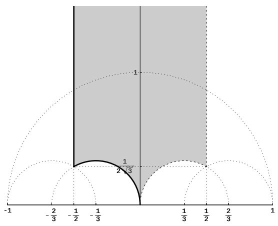
Thus we can not prove for $k \geqslant 100$ with this method. We will need some expansion.

6.2. $\Gamma_0(3)$.

We have the following transformation:

$$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix}: \quad \frac{e^{i\theta}+1}{3} \mapsto \frac{e^{i(\pi-\theta)}-1}{3}.$$

Then we have $V_{\Gamma_0(3)} = \{-1/2 + \sqrt{3}i/6\}$ (*cf.* Theorem2.1).



6.2.1. *Valence formula.* In order to decide the locating of all zeros of $E_{k,3}^{\infty}(z)$ and $E_{k,3}^{0}(z)$, we need the *valence formula for* $\Gamma_0(3)$:

Proposition 6.4. *Let f be a modular function of weight k for $\Gamma_0(3)$, which is not identically zero. We have*

$$v_\infty(f) + v_0(f) + \frac{1}{3}v_{\rho_3}(f) + \sum_{\substack{p \in \Gamma_0(3)\backslash\mathbb{H} \\ p \neq \rho_3}} v_p(f) = \frac{k}{3}, \tag{140}$$

where $\rho_3 := -1/2 + \sqrt{3}i/6$.

The proof of this proposition is similar to Proposition 6.1 because the figure of fundamental domain of $\Gamma_0(3)$ is similar to that of $\Gamma_0(2)$ (*cf.* Figure 4). The angle of the arc around ρ_3 (BB' in Figure9.) tends to $\pi/6$ when radius of it tends to 0, thus the coefficient of $v_{\rho_3}(f)$ is 1/3. Furthermore, since $f(S_3 z) = (-3z-1)^k f(z)$ for $S_3 := -\left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right)$, the right-hand side is $k/3$.

6.2.2. *Modular forms of weight* 2. We define

$$E_{2,3}{}'(z) := \frac{1}{2}(3E_2(3z) - E_2(z)). \tag{141}$$

Note that $E_{2,3}{}'$ is generated by Eisensitein series for $\mathrm{SL}_2(\mathbb{Z})$, but it is not Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ nor $\Gamma_0(3)$.

Similarly to $E_{2,2}{}'$, we have

$$E_{2,3}{}'\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) z\right) = E_{2,3}{}'(z),$$
$$E_{2,3}{}'\left(-\left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right) z\right) = (-3z-1)^2 E_{2,3}{}'(z).$$

Furthermore, because

$$E_{2,3}{}'(W_3 z) = -(\sqrt{3}z)^2 E_{2,3}{}'(z), \tag{142}$$

$E_{2,3}{}'$ is holomorphic at cusp 0. Now, we prove $E_{2,3}{}'$ is a modular form for $\Gamma_0(3)$ of weight 2.

By the valence formula for $\Gamma_0(3)$ (Proposition 6.4), we have

$$v_{\rho_3}(E_{2,3}{}') = 2, \qquad v_p(E_{2,3}{}') = 0 \quad \text{for every } p \neq \rho_3.$$

Incidentally, let f be a modular form for $\Gamma_0(3)$ of weight 2. Then, By the valence formula for $\Gamma_0(3)$, we also have $v_{\rho_3}(f) = 2$ and $v_p(f) = 0$ for every $p \neq \rho_3$. Thus $f/E_{2,3}{}'$ is a modular form of weight 0, then $f/E_{2,3}{}' \in \mathbb{C}$. In conclusion, f is a constant multiple of $E_{2,3}{}'$.

6.2.3. *Preliminaries.* Let f be a modular form for $\Gamma_0(3)$ of weight k, and let $k \not\equiv 0 \pmod 3$. Then we have

$$f(\rho_3) = f(\left(\begin{smallmatrix} -2 & -1 \\ 3 & 1 \end{smallmatrix}\right) \rho_3) = (e^{i2\pi/3})^k f(\rho_3).$$

Thus $f(\rho_3) = 0$ and $v_{\rho_3}(f) \geqslant 1$.

Let k be an even integer such that $k \equiv 0 \pmod 3$. Then we have

$$E_{k,3}^{\infty}(\rho_3) = \frac{1}{1-3^k}(3^{k/2}(e^{i\pi/6})^k - 3^k)E_k(\rho) \neq 0,$$
$$E_{k,3}^{0}(\rho_3) = \frac{1}{1-3^{-k}}(3^{k/2}(e^{i\pi/6})^k - 1)E_k(\rho) \neq 0.$$

(*cf.* Proposition 3.3) Thus $v_{\rho_3}(E_{k,3}^{\infty}) = v_{\rho_3}(E_{k,3}^{0}) = 0$.

Recall that $v_0(E_{k,3}^{\infty}) = v_\infty(E_{k,3}^{0}) = 1$ and $v_\infty(E_{k,3}^{\infty}) = v_0(E_{k,3}^{0}) = 0$. (Section 4)

In paticular, we have $v_0(E_{4,3}^{\infty}) = v_\infty(E_{4,3}^{0}) = 1$, $v_{\rho_3}(E_{4,3}^{\infty}) \geqslant 1$, and $v_{\rho_3}(E_{4,3}^{0}) \geqslant 1$. By the valence formula, we have $v_{\rho_3}(E_{4,3}^{\infty}) = v_{\rho_3}(E_{4,3}^{0}) = 1$, and they do not have any other zeros.

Finally, define

$$A_3 := \{z \,;\, |z+1/3| = 1/3,\ 0 < Arg(z) < 2\pi/3\}, \tag{143}$$
$$A_3^0 := \{z \,;\, Re(z) = -1/2,\ Im(z) > \sqrt{3}/6\}. \tag{144}$$

Now, we have the following transformation:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} W_3 : \frac{e^{i\theta}-1}{3} \mapsto -\frac{1}{2} + \frac{i}{2}\frac{1}{\tan\theta/2}. \tag{145}$$

This transform A_3 to A_3^0. Moreover,

$$E_{k,3}^{0}\left(\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) W_3\, z\right) = (3z)^k E_{k,3}^{\infty}(z) \quad \text{for every } z \in A_3.$$

Then

Remark 6.6. *The number of zeros of $E_{k,3}^{\infty}$ in A_3 is equal to that of zeros of $E_{k,3}^{0}$ in A_3^0.*

6.2.4. *The function $F_{k,3}$.* We give the next definition;

$$F_{k,3}(\theta) := e^{ik\theta/2} E_{k,3}^{\infty}(e^{i\theta}/3 - 1/3). \tag{146}$$

Again, $E_{k,3}^{\infty}$ is denoted by

$$E_{k,3}^{\infty}(z) = \frac{1}{2}\sum_{\substack{(c,d)=1 \\ 3|c}} (cz+d)^{-k}.$$

Since $3 \mid c$, we can write $c = 3c'$ for $\exists c' \in \mathbb{Z}$, and have $3 \nmid d$.

$$F_{k,3}(\theta) = \frac{e^{ik\theta/2}}{2}\sum_{\substack{(c,d)=1 \\ 3|c}} \left(c\frac{e^{i\theta}-1}{3} + d\right)^{-k}$$
$$= \frac{1}{2}\sum_{\substack{(c,d)=1 \\ 3\nmid d}} (c'e^{i\theta/2} + (-c'+d)e^{-i\theta/2})^{-k}.$$

Then we have $(c', -c'+d) = 1$. If $3 \nmid c'(-c'+d)$, then $c' \not\equiv 0, c' \not\equiv d \pmod 3$, and we have $c' \equiv -c'+d \pmod 3$. Thus we can write as following:

$$F_{k,3}(\theta) = \frac{1}{2}\sum_{\substack{(c,d)=1 \\ 3|cd}} (ce^{i\theta/2} + de^{-i\theta/2})^{-k} + \frac{1}{2}\sum_{\substack{(c,d)=1 \\ 3\nmid cd \\ c\equiv d(3)}} (ce^{i\theta/2} + de^{-i\theta/2})^{-k}.$$

Note that for any pair (c,d), $(ce^{i\theta/2} + de^{i\theta/2})^{-k}$ and $(de^{i\theta/2} + ce^{-i\theta/2})^{-k}$ are conjugates of each other. The next proposition follows.

Proposition 6.5. *$F_{k,3}(\theta)$ is real for every $\theta \in \mathbb{R}$.*

6.2.5. *Application of the RSD Method* (0). We note that $N := c^2 + d^2$.

Firstly, we consider the case $N = 1$. Then we can write:

$$F_{k,3}(\theta) = 2\cos(k\theta/2) + R_3, \tag{147}$$

where

$$R_3 = \frac{1}{2}\sum_{\substack{(c,d)=1\\ 3|cd\\ N>1}} (ce^{i\theta/2} + de^{-i\theta/2})^{-k} + \frac{1}{2}\sum_{\substack{(c,d)=1\\ 3\nmid cd\\ c\equiv d(3)}} (ce^{i\theta/2} + de^{-i\theta/2})^{-k}.$$

Now, we have

$$|R_3| \leqslant \frac{1}{2}\sum_{\substack{(c,d)=1\\ 3|cd\\ N>1}} |ce^{i\theta/2} + de^{-i\theta/2}|^{-k} + \frac{1}{2}\sum_{\substack{(c,d)=1\\ 3\nmid cd\\ c\equiv d(3)\\ N>1}} |ce^{i\theta/2} + de^{-i\theta/2}|^{-k}.$$

Let $v_k(c,d,\theta) := |ce^{i\theta/2} + de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\theta) = 1/(c^2 + d^2 + 2cd\cos\theta)^{k/2}$ and $v_k(c,d,\theta) = v_k(-c,-d,\theta) = v_k(\pm d, \pm c, \theta)$.

However, for every $n \in \mathbb{N}$, we have $(3n, -3n-1) = 1$, $3 \mid 3n(-3n-1)$, and

$$v_k(3n, -3n-1, \theta) \leqslant 1 \quad \text{for } 0 \leqslant \theta \leqslant 2\pi/3.$$

Here, the number of the pairs $(3n, -3n-1)$ is infinite. Thus we have the bound

$$|R_3| \leqslant \infty,$$

which does not make sense.

6.2.6. *Application of the RSD Method* (1) : $[\pi/4, 2\pi/3]$. In privious subsubsections, $E_{k,3}^{\infty}$ and $E_{k,3}^{0}$ of weight $k < 6$ has no zeros other than ∞, 0, and ρ_3. Thus we may assume $k \geqslant 6$. We consider two cases, namely $[\pi/2, 2\pi/3]$ and $[\pi/4, \pi/2]$.

For the interval $[\pi/2, 2\pi/3]$, we will consider the next cases, namely $N = 2, 5$, and $N \geqslant 10$. Considering $-1/2 \leqslant \cos\theta \leqslant 0$ for the interval $[\pi/2, 2\pi/3]$, we have the following:

When $N = 2$,	$v_k(1,1,\theta) \leqslant 1.$
When $N = 5$,	$v_k(1,-2,\theta) \leqslant (1/5)^{k/2}.$
When $N \geqslant 10$,	$\lvert ce^{i\theta/2} \pm de^{-i\theta/2}\rvert^2 \geqslant N/2,$

and the number of terms is not more than $2N^{1/2}$. Then

$$|R_3|_{N\geqslant 10} = 2\sqrt{2}\sum_{N=10}^{\infty}\left(\frac{1}{2}N\right)^{(1-k)/2} \leqslant \frac{108}{k-3}\left(\frac{2}{9}\right)^{k/2}.$$

Thus

$$|R_3| \leqslant 1 + 2\left(\frac{1}{5}\right)^{k/2} + \frac{108}{k-3}\left(\frac{2}{9}\right)^{k/2} \leqslant 1.41106... \quad (k \geqslant 6) \tag{148}$$

On the other hand, for the interval $[\pi/4, \pi/2]$, we will consider the next cases, namely $N = 2, 5, 10, 13$, and $N \geqslant 17$. Considering $0 \leqslant \cos\theta \leqslant 3/4$ for the interval $[\pi/4, \pi/2]$, we have the following:

When $N = 2$,	$v_k(1,1,\theta) \leqslant (1/2)^{k/2}.$	
When $N = 5$,	$v_k(1,2,\theta) \leqslant (1/2)^{k/2}.$	
When $N = 10$,	$v_k(1,3,\theta) \leqslant (1/10)^{k/2},$	$v_k(1,-3,\theta) \leqslant (2/11)^{k/2}.$
When $N = 13$,	$v_k(2,3,\theta) \leqslant (1/13)^{k/2},$	$v_k(2,-3,\theta) \leqslant (1/2)^{k}.$
When $N \geqslant 17$,	$\lvert ce^{i\theta/2} \pm de^{-i\theta/2}\rvert^2 \geqslant N/4,$	

and the number of terms is not more than $2N^{1/2}$. Then

$$|R_3|_{N\geqslant 17} = 4\sum_{N=17}^{\infty}\left(\frac{1}{4}N\right)^{(1-k)/2} \leqslant \frac{256}{k-3}\left(\frac{1}{2}\right)^k.$$

Thus

$$|R_3| \leqslant 3\left(\frac{1}{2}\right)^{k/2} + 2\left(\frac{1}{2}\right)^k + \cdots + 2\left(\frac{1}{13}\right)^{k/2} + \frac{256}{k-3}\left(\frac{1}{2}\right)^k, \quad (149)$$
$$\leqslant 1.75451... \quad (k \geqslant 6)$$

By above bounds, $E_{6,3}^{\infty}$ and $E_{8,3}^{\infty}$ have at least 1 zeros in A_3. Then by Remark 6.6, $E_{6,3}^0$ and $E_{8,3}^0$ have at least 1 zeros in A_3^0.

Recall that $v_0(E_{6,3}^{\infty}) = v_{\infty}(E_{6,3}^0) = 1$. By the valence formula, $E_{6,3}^{\infty}$ and $E_{6,3}^0$ have just 1 zero in A_3 and A_3^0, respectively. Furthermore, they have no other zeros.

Recall that $v_0(E_{8,3}^{\infty}) = v_{\infty}(E_{8,3}^0) = 1$, $v_{\rho_3}(E_{8,3}^{\infty}) \geqslant 1$, and $v_{\rho_3}(E_{8,3}^0) \geqslant 1$. By the valence formula, $E_{8,3}^{\infty}$ and $E_{8,3}^0$ have just 1 zero in A_3 and A_3^0, respectively. Furthermore, we have $v_{\rho_3}(E_{8,3}^{\infty}) = v_{\rho_3}(E_{8,3}^0) = 2$, and they have no other zeros.

6.2.7. *The space of modular forms.* Let $M_{k,3}$ be the space of modular forms for $\Gamma_0(3)$ of weight k, and let $M_{k,3}^0$ be the space of cusp forms for $\Gamma_0(3)$ of weight k. Because $\dim(M_{k,3}/M_{k,3}^0) \leqslant 2$, we have $M_{k,3} = \mathbb{C}E_{k,3}^{\infty} \oplus \mathbb{C}E_{k,3}^0 \oplus M_{k,3}^0$. Recall that $\Delta_3 = \eta^6(z)\eta^6(3z)$. We have following theorem:

Theorem 6.2. *Let k be an even integer.*

(1) *For $k < 0$, $M_{k,3} = 0$.*
(2) *For $k = 0, 2,$ and 4, we have $M_{k,3}^0 = 0$. Furthemore, we have $M_{0,2} = \mathbb{C}$, $M_{2,3} = \mathbb{C}{E_{2,3}}'$, and $M_{4,3} = \mathbb{C}E_{4,3}^{\infty} \oplus \mathbb{C}E_{4,3}^0$.*
(3) *$M_{k,3}^0 = \Delta_3 M_{k-6,3}$.*

The proof of this theorem is similar to that of Theorem 6.1. Furthermore, we have $\dim(M_{k,3}) = \lfloor k/3 \rfloor + 1$ for $k \geqslant 0$, and $\dim(M_{k,3}^0) = \lfloor k/3 \rfloor - 1$ for $k \geqslant 6$.

Let k be an even integer such that $k \geqslant 4$ and $k \equiv 2 \pmod 6$, then we have $M_{k,3} = {E_{2,3}}' M_{k-2,3}$, and $k - 2 \equiv 0 \pmod 6$.

On the other hand, let k be an even integer such that $k \geqslant 4$ and $k \equiv 0, 4 \pmod 6$. Write $n := \lfloor k/6 \rfloor$, then $k - 6n = 0$ or 4. Now, we have $E_{k,3}^{\infty} - E_{k-6n,3}^{\infty}(E_{6,3}^{\infty})^n \in M_{k,3}^0$ and $E_{k,3}^0 - E_{k-6n,3}^0(E_{6,3}^0)^n \in M_{k,3}^0$. Thus we have $M_{k,3} = \mathbb{C}E_{k-6n,3}^{\infty}(E_{6,3}^{\infty})^n \oplus \mathbb{C}E_{k-6n,3}^0(E_{6,3}^0)^n \oplus M_{k,3}^0$. Then

$$\begin{aligned} M_{k,3} = {} & E_{k-6n,3}^{\infty}(\mathbb{C}(E_{6,3}^{\infty})^n \oplus \mathbb{C}(E_{6,3}^{\infty})^{n-1}\Delta_3 \oplus \cdots \oplus \mathbb{C}\Delta_3^n) \\ & \oplus E_{k-6n,3}^0(\mathbb{C}(E_{6,3}^0)^n \oplus \mathbb{C}(E_{6,3}^0)^{n-1}\Delta_3 \oplus \cdots \oplus \mathbb{C}\Delta_3^n) \end{aligned}$$

Thus, the next proposision follows:

Proposition 6.6. *Let $k \geqslant 4$ be an even integer. For every $f \in M_{k,3}$, we have*

$$v_{\rho_3}(f) \geqslant t_k \quad (t_k = 0, 1, 2 \text{ such that } -2t_k \equiv k \pmod 6). \quad (150)$$

In addition, we have $E_{4,3}^0 = 9(({E_{2,3}}')^2 - E_{4,3}^{\infty})$, $E_{6,3}^{\infty} = {E_{2,3}}' E_{4,3}^{\infty} - (108/13)\Delta_3$, and $E_{6,3}^0 = (49/729){E_{2,3}}' E_{4,3}^0 - (50/3)\Delta_3$. Then

Remark 6.7. *Every modular form for $\Gamma_0(3)$ is generated by*

$${E_{2,3}}', \quad E_{4,3}^{\infty}, \quad \text{and} \quad \Delta_3.$$

Then we have following:

Conjecture 6.2. *Let $k \geqslant 4$ be an even integer. $E_{k,3}^{\infty}$ has $\lfloor k/3 \rfloor - 1$ zeros in A_3, and $E_{k,3}^0$ has $\lfloor k/3 \rfloor - 1$ zeros in A_3^0. Furthermore, in Proposition 6.6, the equality hold if f is equal to $E_{k,3}^{\infty}$ or $E_{k,3}^0$.*

Then we improve Remark 6.6 as following:

Remark 6.8. *If $E_{k,3}^{\infty}$ has $\lfloor k/3 \rfloor - 1$ zeros in A_3, then $E_{k,3}^0$ has $\lfloor k/3 \rfloor - 1$ zeros in A_3^0.*

For $0 \leqslant \theta \leqslant 2\pi/3$, we have $0 \leqslant k\theta/2 \leqslant k\pi/3$. Here we need at most $\lfloor k/3 \rfloor$ integers for $\lfloor k/3 \rfloor - 1$ zeros. We may remove the integer 0 from the interval $[0, k/3]$. We have already proved the bound $|R_3| < 2$ for the interval $[\pi/2, 2\pi/3]$ in the previous section. Thus we need

$$|R_3| < 2 \qquad \text{for every } \theta \in [2\pi/k, \pi/2]. \tag{151}$$

For the first step, we will prove following in the next subsubsection:

$$|R_3| < 2 \qquad \text{for every } \theta \in [\pi/8, \pi/2]. \tag{152}$$

$$|R_3| < 2 \qquad \text{for every } \theta \in [\pi/15, \pi/2]. \tag{153}$$

$$|R_3| < 2 \qquad \text{for every } \theta \in [\pi/20, \pi/2]. \tag{154}$$

6.2.8. *Application of the RSD Method* (2) : $[\pi/8, \pi/2]$. In this subsubsection, we prove the bound (152). In privious subsubsections, we proved for $k \leqslant 8$. Thus we may assume $k \geqslant 10$.

Now we will consider the next cases, namely $N = 2, 5, \cdots, 34$, and $N \geqslant 37$. Considering $0 \leqslant \cos\theta \leqslant 13/14$ for the interval $[\pi/8, \pi/2]$, we have the following:

When $N = 2$, $v_k(1, 1, \theta) \leqslant (1/2)^{k/2}$.

When $N = 5$, $v_k(1, -2, \theta) \leqslant (7/9)^{k/2}$.

When $N = 10$, $v_k(1, 3, \theta) \leqslant (1/10)^{k/2}$, $v_k(1, -3, \theta) \leqslant (7/31)^{k/2}$.

When $N = 13$, $v_k(2, 3, \theta) \leqslant (1/13)^{k/2}$, $v_k(2, -3, \theta) \leqslant (7/13)^{k/2}$.

When $N = 17$, $v_k(1, 4, \theta) \leqslant (1/17)^{k/2}$.

When $N = 25$, $v_k(3, 4, \theta) \leqslant (1/5)^{k}$, $v_k(3, -4, \theta) \leqslant (7/19)^{k/2}$.

When $N = 29$, $v_k(2, 5, \theta) \leqslant (1/29)^{k/2}$.

When $N = 34$, $v_k(3, 5, \theta) \leqslant (1/34)^{k/2}$, $v_k(3, -5, \theta) \leqslant (7/43)^{k/2}$.

When $N \geqslant 37$, $|ce^{i\theta/2} \pm de^{-i\theta/2}|^2 \geqslant N/30$,

and the number of terms with $c^2 + d^2 = N$ is not more than $2N^{1/2}$. Then

$$|R_3|_{N \geqslant 37} = 2\sqrt{14} \sum_{N=37}^{\infty} \left(\frac{1}{14}N\right)^{(1-k)/2} \leqslant \frac{864}{k-3}\left(\frac{7}{18}\right)^{k/2}.$$

Thus

$$\begin{aligned} |R_3| &\leqslant 2\left(\frac{7}{13}\right)^{k/2} + \left(\frac{1}{2}\right)^{k/2} + \cdots + 2\left(\frac{1}{34}\right)^{k/2} + \frac{864}{k-3}\left(\frac{7}{18}\right)^{k/2}, \\ &\leqslant 1.80389\ldots \quad (k \geqslant 10) \end{aligned} \tag{155}$$

In conclusion,

Remark 6.9. *We proved Conjecture 6.2 for* $4 \leqslant k \leqslant 16$.

6.2.9. *Application of the RSD Method* (3) : $[\pi/15, \pi/2]$. In this subsubsection, we prove the bound (153). In privious subsubsections, we proved for $k \leqslant 16$. Thus we may assume $k \geqslant 18$.

Now we will consider the next cases, namely $N = 2, 5, \cdots, 85$, and $N \geqslant 89$. Considering $0 \leqslant \cos\theta \leqslant 45/46$ for the interval $[\pi/15, \pi/2]$, we have the following:

When $N = 2$, $v_k(1, 1, \theta) \leqslant (1/2)^{k/2}$.

When $N = 5$, $v_k(1, -2, \theta) \leqslant (23/25)^{k/2}$.

When $N = 10$, $v_k(1, 3, \theta) \leqslant (1/10)^{k/2}$, $v_k(1, -3, \theta) \leqslant (23/95)^{k/2}$.

When $N = 13$, $v_k(2, 3, \theta) \leqslant (1/13)^{k/2}$, $v_k(2, -3, \theta) \leqslant (23/29)^{k/2}$.

When $N = 17$, $v_k(1, 4, \theta) \leqslant (1/17)^{k/2}$.

When $N=25$, $v_k(3,4,\theta) \leqslant (1/5)^k$, $v_k(3,-4,\theta) \leqslant (23/35)^{k/2}$.

When $N=26$, $v_k(1,-5,\theta) \leqslant (23/373)^{k/2}$.

When $N=29$, $v_k(2,5,\theta) \leqslant (1/29)^{k/2}$.

When $N=34$, $v_k(3,5,\theta) \leqslant (1/34)^{k/2}$, $v_k(3,-5,\theta) \leqslant (23/107)^{k/2}$.

When $N=37$, $v_k(1,6,\theta) \leqslant (1/37)^{k/2}$, $v_k(1,-6,\theta) \leqslant (23/581)^{k/2}$.

When $N=41$, $v_k(4,-5,\theta) \leqslant (23/43)^{k/2}$.

When $N=37$, $v_k(1,7,\theta) \leqslant (1/50)^{k/2}$.

When $N=53$, $v_k(2,-7,\theta) \leqslant (23/589)^{k/2}$.

When $N=58$, $v_k(3,7,\theta) \leqslant (1/58)^{k/2}$, $v_k(3,-7,\theta) \leqslant (23/389)^{k/2}$.

When $N=61$, $v_k(5,6,\theta) \leqslant (1/61)^{k/2}$, $v_k(5,-6,\theta) \leqslant (23/53)^{k/2}$.

When $N=65$, $v_k(1,-8,\theta) \leqslant (23/1135)^{k/2}$, $v_k(4,7,\theta) \leqslant (1/65)^{k/2}$.

When $N=73$, $v_k(3,8,\theta) \leqslant (1/73)^{k/2}$, $v_k(3,-8,\theta) \leqslant (23/599)^{k/2}$.

When $N=74$, $v_k(5,-7,\theta) \leqslant (23/126)^{k/2}$.

When $N=82$, $v_k(1,9,\theta) \leqslant (1/82)^{k/2}$, $v_k(1,-9,\theta) \leqslant (23/1481)^{k/2}$.

When $N=85$, $v_k(2,9,\theta) \leqslant (1/85)^{k/2}$, $v_k(2,-9,\theta) \leqslant (23/1145)^{k/2}$,

$v_k(6,7,\theta) \leqslant (1/85)^{k/2}$, $v_k(6,-7,\theta) \leqslant (23/65)^{k/2}$.

When $N \geqslant 89$, $|ce^{i\theta/2} \pm de^{-i\theta/2}|^2 \geqslant N/46$,

and the number of terms with $c^2+d^2=N$ is not more than $2N^{1/2}$. Then

$$|R_3|_{N\geqslant 89} = 2\sqrt{46}\sum_{N=89}^{\infty}\left(\frac{1}{46}N\right)^{(1-k)/2} \leqslant \frac{704\sqrt{22}}{k-3}\left(\frac{23}{44}\right)^{k/2}.$$

Thus

$$|R_3| \leqslant 2\left(\frac{23}{25}\right)^{k/2} + 2\left(\frac{23}{29}\right)^{k/2} + \cdots + 4\left(\frac{1}{85}\right)^{k/2} + \frac{704\sqrt{22}}{k-3}\left(\frac{23}{44}\right)^{k/2}, \tag{156}$$
$$\leqslant 1.89019\ldots \quad (k \geqslant 18)$$

In conclusion,

Remark 6.10. *We proved Conjecture 6.2 for* $18 \leqslant k \leqslant 30$.

6.2.10. *Application of the RSD Method* (4) : $[\pi/20, \pi/2]$. In this subsubsection, we prove the bound (154). In privious subsubsections, we proved for $k \leqslant 30$. Thus we may assume $k \geqslant 32$.

Now we will consider the next cases, namely $N = 2, 5, \cdots, 113$, and $N \geqslant 125$. Considering $0 \leqslant \cos\theta \leqslant 81/82$ for the interval $[\pi/20, \pi/2]$, we have the following:

When $N=2$, $v_k(1,1,\theta) \leqslant (1/2)^{k/2}$.

When $N=5$, $v_k(1,-2,\theta) \leqslant (41/43)^{k/2}$.

When $N=10$, $v_k(1,3,\theta) \leqslant (1/10)^{k/2}$, $v_k(1,-3,\theta) \leqslant (41/167)^{k/2}$.

When $N=13$, $v_k(2,3,\theta) \leqslant (1/13)^{k/2}$, $v_k(2,-3,\theta) \leqslant (41/47)^{k/2}$.

When $N=17$, $v_k(1,4,\theta) \leqslant (1/17)^{k/2}$.

When $N=25$, $v_k(3,4,\theta) \leqslant (1/5)^k$, $v_k(3,-4,\theta) \leqslant (41/53)^{k/2}$.

When $N=26$, $v_k(1,-5,\theta) \leqslant (41/661)^{k/2}$.

When $N=29$, $v_k(2,5,\theta) \leqslant (1/29)^{k/2}$.

When $N=34$, $v_k(3,5,\theta) \leqslant (1/34)^{k/2}$, $v_k(3,-5,\theta) \leqslant (41/179)^{k/2}$.

When $N = 37$, $v_k(1,6,\theta) \leqslant (1/37)^{k/2}$, $v_k(1,-6,\theta) \leqslant (41/1031)^{k/2}$.
When $N = 41$, $v_k(4,-5,\theta) \leqslant (41/61)^{k/2}$.
When $N = 50$, $v_k(1,7,\theta) \leqslant (1/50)^{k/2}$.
When $N = 53$, $v_k(2,-7,\theta) \leqslant (41/1039)^{k/2}$.
When $N = 58$, $v_k(3,7,\theta) \leqslant (1/58)^{k/2}$, $v_k(3,-7,\theta) \leqslant (41/677)^{k/2}$.
When $N = 61$, $v_k(5,6,\theta) \leqslant (1/61)^{k/2}$, $v_k(5,-6,\theta) \leqslant (41/71)^{k/2}$.
When $N = 65$, $v_k(1,-8,\theta) \leqslant (41/2017)^{k/2}$, $v_k(4,7,\theta) \leqslant (1/65)^{k/2}$.
When $N = 73$, $v_k(3,8,\theta) \leqslant (1/73)^{k/2}$, $v_k(3,-8,\theta) \leqslant (41/1049)^{k/2}$.
When $N = 74$, $v_k(5,-7,\theta) \leqslant (41/199)^{k/2}$.
When $N = 82$, $v_k(3,8,\theta) \leqslant (1/82)^{k/2}$, $v_k(3,-8,\theta) \leqslant (41/2633)^{k/2}$.
When $N = 85$, $v_k(2,9,\theta) \leqslant (1/85)^{k/2}$, $v_k(2,-9,\theta) \leqslant (41/2027)^{k/2}$,
$v_k(6,7,\theta) \leqslant (1/85)^{k/2}$, $v_k(6,-7,\theta) \leqslant (41/83)^{k/2}$.
When $N = 89$, $v_k(5,8,\theta) \leqslant (1/89)^{k/2}$.
When $N = 97$, $v_k(4,9,\theta) \leqslant (1/97)^{k/2}$, $v_k(4,-9,\theta) \leqslant (41/1061)^{k/2}$.
When $N = 101$, $v_k(1,10,\theta) \leqslant (1/101)^{k/2}$.
When $N = 106$, $v_k(5,9,\theta) \leqslant (1/106)^{k/2}$, $v_k(5,-9,\theta) \leqslant (41/701)^{k/2}$.
When $N = 109$, $v_k(3,10,\theta) \leqslant (1/109)^{k/2}$, $v_k(3,-10,\theta) \leqslant (41/2039)^{k/2}$.
When $N = 113$, $v_k(7,-8,\theta) \leqslant (41/97)^{k/2}$.
When $N \geqslant 125$, $|ce^{i\theta/2} \pm de^{-i\theta/2}|^2 \geqslant N/82$,

and the number of terms with $c^2 + d^2 = N$ is not more than $2N^{1/2}$. Then

$$|R_3|_{N \geqslant 125} = 2\sqrt{82} \sum_{N=125}^{\infty} \left(\frac{1}{82}N\right)^{(1-k)/2} \leqslant \frac{992\sqrt{31}}{k-3} \left(\frac{41}{62}\right)^{k/2}.$$

Thus

$$|R_3| \leqslant 2\left(\frac{41}{43}\right)^{k/2} + 2\left(\frac{41}{47}\right)^{k/2} + \cdots + 2\left(\frac{1}{109}\right)^{k/2} + \frac{992\sqrt{31}}{k-3}\left(\frac{41}{62}\right)^{k/2}, \tag{157}$$
$$\leqslant 1.69883... \quad (k \geqslant 30)$$

In conclusion,

Remark 6.11. *We proved Conjecture 6.2 for* $32 \leqslant k \leqslant 40$.

Now, by Remark 6.9, 6.10, and 6.11, we prove Conjecture 6.2 for $4 \leqslant k \leqslant 40$. However, for greater k, we prove only about 92.5% of Conjecture 6.2 by the sense of the interval $[\pi/20, \pi/2]$.

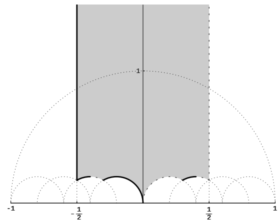
However, for greater k, it seems that we can not prove for all zeros with the same method.

6.3. $\Gamma_0(5)$.

We have the following transformation:

$$\begin{pmatrix}-1 & 0\\ 5 & -1\end{pmatrix}: \quad \frac{e^{i\theta}+1}{5} \mapsto \frac{e^{i(\pi-\theta)}-1}{5},$$
$$\begin{pmatrix}2 & -1\\ 5 & -2\end{pmatrix}: \quad \frac{e^{i\theta}-2}{5} \mapsto \frac{e^{i(\pi-\theta)}-2}{5},$$
$$\begin{pmatrix}-2 & -1\\ 5 & 2\end{pmatrix}: \quad \frac{e^{i\theta}+2}{5} \mapsto \frac{e^{i(\pi-\theta)}+2}{5}.$$

Then we have $V_{\Gamma_0(5)} = \{-3/10 + \sqrt{3}i/10\}$ (*cf.* Theorem2.1).



6.3.1. *Valence formula.*

Proposition 6.7. *Let f be a modular function of weight k for $\Gamma_0(5)$, which is not identically zero. We have*

$$v_\infty(f) + v_0(f) + \frac{1}{2}v_{\rho_{5,2}}(f) + \frac{1}{2}v_{\rho_{5,3}}(f) + \sum_{\substack{p\in\Gamma_0(5)\backslash\mathbb{H}\\ p\neq\rho_{5,2},\rho_{5,3}}} v_p(f) = \frac{k}{2}, \tag{158}$$

where $\rho_{5,2} = -2/5 + i/5$ *and* $\rho_{5,3} := 2/5 + i/5$.

Proof. Let f be a nonzero modular function of weight k for $\Gamma_0(5)$, and let $\mathscr{C}$ be a contour of its fundamental domain $\mathbb{F}(5)$ (Figure 4), whose interior contains every zero and pole of f except for $\rho_{5,2}$, $\rho_{5,3}$, and $\rho_{5,4} := -3/10 + \sqrt{3}i/10$ (*cf.* Figure 9). By the *Residue theorem*, we have

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{df}{f} = \sum_{\substack{p\in\Gamma_0(5)\backslash\mathbb{H}\\ p\neq\rho_{5,2},\rho_{5,3},\rho_{5,4}}} v_p(f).$$

(i) For the arc around ∞, we have $-v_\infty(f)$.
(ii) For the arc around 0, we have $-v_0(f)$.
(iii) For the arcs around $\rho_{5,2}$, $\rho_{5,3}$, $\rho_{5,4}$, when the radii of each arc tends to 0, then we have

$$-\frac{1}{2}v_{\rho_{5,2}}(f), \quad -\frac{1}{2}v_{\rho_{5,3}}(f), \quad \text{and} \quad -v_{\rho_{5,4}}(f).$$

(iv) For the arcs on $\{z\,;\, Re(z) = -1/2\}$ and $\{z\,;\, Re(z) = 1/2\}$, since $f(Tz) = f(z)$ for $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, we have 0.
(v) For the arcs on $\{z\,;\, |z \pm 1/5| = 1/5\}$, since $f(S_5 z) = (5z+5)^k f(z)$ for $S_5 := \left(\begin{smallmatrix}1&0\\5&1\end{smallmatrix}\right)$, we have $k/3$. Furthermore, for the arcs on $\{z\,;\, |z+2/5| = 1/5\}$, since $f\left(\left(\begin{smallmatrix}-2&-1\\5&2\end{smallmatrix}\right)z\right) = (5z+2)^k f(z)$, we have

$$\frac{df\left(\left(\begin{smallmatrix}-2&-1\\5&2\end{smallmatrix}\right)z\right)}{f\left(\left(\begin{smallmatrix}-2&-1\\5&2\end{smallmatrix}\right)z\right)} = k\frac{dz}{z+2/5} + \frac{df(z)}{f(z)}, \quad \text{and} \quad \frac{k}{12}.$$

Similarly, for the arcs on $\{z\,;\, |z-2/5| = 1/5\}$, since $f\left(\left(\begin{smallmatrix}2&-1\\5&-2\end{smallmatrix}\right)z\right) = (5z-2)^k f(z)$, we have $k/12$. Thus $k/3 + 2\cdot k/12 = k/2$.

□

6.3.2. *Modular forms of weight* 2. We define

$$E_{2,5}{}'(z) := \frac{1}{4}(5E_2(5z) - E_2(z)). \tag{159}$$

Note that $E_{2,5}{}'$ is generated by Eisensitein series for $\mathrm{SL}_2(\mathbb{Z})$, but it is not Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ nor $\Gamma_0(5)$.

Similarly to $E_{2,2}{}'$ and $E_{2,3}{}'$, we have

$$\begin{aligned} E_{2,5}{}'\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) z\right) &= E_{2,5}{}'(z), \\ E_{2,5}{}'\left(\left(\begin{smallmatrix} 1 & 0 \\ 5 & 1 \end{smallmatrix}\right) z\right) &= (5z+1)^2 E_{2,5}{}'(z). \end{aligned}$$

In addition, we have $\left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right) = TST^2ST^{-2}S^{-1}$ for $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. (See [RD]) Similarly, we have $5 \cdot \left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right) z = \left(\begin{smallmatrix} 3 & 5 \\ 1 & 2 \end{smallmatrix}\right) 5z$ and $\left(\begin{smallmatrix} 3 & 5 \\ 1 & 2 \end{smallmatrix}\right) = T^2S^{-1}T^{-1}ST$. Recall that we have following: (Section 3)

$$E_2(Tz) = E_2(z), \qquad E_2(Sz) = z^2E_2(z) + \frac{12}{2\pi i}z. \tag{160}$$

Then we have

$$\begin{aligned} E_2(TST^2ST^{-2}S^{-1}z) &= E_2(ST^2ST^{-2}S^{-1}z) \\ &= \left(\frac{5z+2}{2z+1}\right)^2 E_2(T^2ST^{-2}S^{-1}z) + \frac{12}{2\pi i}\frac{5z+2}{2z+1} \\ &= \left(\frac{5z+2}{2z+1}\right)^2 E_2(ST^{-2}S^{-1}z) + \frac{12}{2\pi i}\frac{5z+2}{2z+1} \\ &= \left(\frac{5z+2}{2z+1}\right)^2 \left(\left(\frac{2z+1}{z}\right)^2 E_2(T^{-2}S^{-1}z) - \frac{12}{2\pi i}\frac{2z+1}{z}\right) + \frac{12}{2\pi i}\frac{5z+2}{2z+1} \\ &= \frac{(5z+2)^2}{z^2}E_2(S^{-1}z) - \frac{12}{2\pi i}\frac{2(5z+2)}{z} \\ &= \frac{(5z+2)^2}{z^2}\left(z^2E_2(z) + \frac{12}{2\pi i}z\right) - \frac{12}{2\pi i}\frac{2(5z+2)}{z} \\ &= (5z+2)^2E_2(z) + \frac{12}{2\pi i}5(5z+2). \end{aligned}$$

Similarly,

$$\begin{aligned} E_2(T^2S^{-1}T^{-1}ST\,5z) &= \left(\frac{5z+2}{2z+1}\right)^2 E_2(ST\,5z) - \frac{12}{2\pi i}\frac{5z+2}{5z+1} \\ &= (5z+2)^2E_2(5z) + \frac{12}{2\pi i}(5z+2). \end{aligned}$$

Thus

$$E_{2,5}{}'\left(\left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right) z\right) = (5z+2)^2 E_{2,5}{}'(z).$$

Recall that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 5 & 1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right)$ generate $\Gamma_0(5)$ (in Section 4), then we can show that $E_{2,5}{}'$ satisfies transformation rule for $\Gamma_0(5)$.

Furthermore, because

$$E_{2,5}{}'(W_5z) = -(\sqrt{5}z)^2E_{2,5}{}'(z), \tag{161}$$

$E_{2,5}{}'$ is holomorphic at cusp 0. Now, we prove $E_{2,5}{}'$ is a modular form for $\Gamma_0(5)$ of weight 2.

6.3.3. *Preliminaries.* Let f be a modular form for $\Gamma_0(5)$ of weight k, and let $k \equiv 2 \pmod 4$. Then we have

$$\begin{aligned} f(\rho_{5,2}) &= f\left(\left(\begin{smallmatrix} -2 & -1 \\ 5 & 2 \end{smallmatrix}\right) \rho_{5,2}\right) = i^k f(\rho_{5,2}) = -f(\rho_{5,2}). \\ f(\rho_{5,3}) &= f\left(\left(\begin{smallmatrix} 2 & -1 \\ 5 & -2 \end{smallmatrix}\right) \rho_{5,3}\right) = i^k f(\rho_{5,3}) = -f(\rho_{5,3}). \end{aligned}$$

Thus $f(\rho_{5,2}) = f(\rho_{5,3}) = 0$, $v_{\rho_{5,2}}(f) \geqslant 1$, and $v_{\rho_{5,3}}(f) \geqslant 1$.

Let k be an even integer such that $k \equiv 0 \pmod 4$. Then we have

$$E_{k,5}^{\infty}(\rho_{5,2}) = \frac{1}{1-5^k}((2+i)^k - 5^k)E_k(i) \neq 0,$$
$$E_{k,5}^{0}(\rho_{5,2}) = \frac{1}{1-5^{-k}}((2+i)^k - 1)E_k(i) \neq 0.$$
$$E_{k,5}^{\infty}(\rho_{5,3}) = \frac{1}{1-5^k}((-2+i)^k - 5^k)E_k(i) \neq 0,$$
$$E_{k,5}^{0}(\rho_{5,3}) = \frac{1}{1-5^{-k}}((-2+i)^k - 1)E_k(i) \neq 0.$$

Thus $v_{\rho_{5,2}}(E_{k,5}^{\infty}) = v_{\rho_{5,2}}(E_{k,5}^{0}) = 0$, $v_{\rho_{5,3}}(E_{k,5}^{\infty}) = v_{\rho_{5,3}}(E_{k,5}^{0}) = 0$.

Recall that $v_0(E_{k,5}^{\infty}) = v_{\infty}(E_{k,5}^{0}) = 1$ and $v_{\infty}(E_{k,5}^{\infty}) = v_0(E_{k,5}^{0}) = 0$. (Section 4)

Finally, for $E_{2,5}{}'$, we have $v_{\rho_{5,2}}(E_{2,5}{}') \geqslant 1$ and $v_{\rho_{5,3}}(E_{2,5}{}') \geqslant 1$. By the valence formula for $\Gamma_0(5)$ (Proposition 6.7), we have

$$v_{\rho_{5,2}}(E_{2,5}{}') = v_{\rho_{5,3}}(E_{2,5}{}') = 1, \qquad v_p(E_{2,5}{}') = 0 \quad \text{for every } p \neq \rho_{5,2}, \rho_{5,3}.$$

Incidentally, let f be a modular form for $\Gamma_0(5)$ of weight 2. Then, By the valence formula for $\Gamma_0(5)$, we also have

$$v_{\rho_{5,2}}(f) = v_{\rho_{5,3}}(f) = 1, \qquad v_p(f) = 0 \quad \text{for every } p \neq \rho_{5,2}, \rho_{5,3}.$$

Thus $f/E_{2,5}{}'$ is a modular form of weight 0, then $f/E_{2,5}{}' \in \mathbb{C}$. In conclusion, f is a constant multiple of $E_{2,5}{}'$.

6.3.4. *The space of modular forms.* Let $M_{k,5}$ be the space of modular forms for $\Gamma_0(5)$ of weight k, and let $M_{k,5}^0$ be the space of cusp forms for $\Gamma_0(5)$ of weight k. Since $\dim(M_{k,5}/M_{k,5}^0) \leqslant 2$, we have $M_{k,5} = \mathbb{C}E_{k,5}^{\infty} \oplus \mathbb{C}E_{k,5}^{0} \oplus M_{k,5}^0$. Recall that $\Delta_5 = \eta^4(z)\eta^4(5z)$. We have following theorem:

Theorem 6.3. *Let k be an even integer.*

(1) *For $k < 0$, $M_{k,5} = 0$.*
(2) *For $k = 0$ and 2, we have $M_{k,5}^0 = 0$. Furthemore, we have $M_{0,5} = \mathbb{C}$, $M_{2,5} = \mathbb{C}E_{2,5}{}'$.*
(3) *$M_{k,5}^0 = \Delta_5 M_{k-4,5}$.*

The proof of this theorem is similar to that of Theorem 6.3. Furthermore, we have $\dim(M_{k,5}) = 2\lfloor k/4\rfloor + 1$ for $k \geqslant 0$, and $\dim(M_{k,5}^0) = 2\lfloor k/4\rfloor - 1$ for $k \geqslant 4$.

Let k be an even integer such that $k \geqslant 4$ and $k \equiv 2 \pmod 4$. For $f \in M_{k,5}$, by previous subsubsections, we have $v_p(f/E_{2,5}{}') \geqslant 0$ for every $p \in \mathbb{H} \cup \{\infty, 0\}$. Then $f/E_{2,5}{}' \in M_{k-2,5}$. Thus $M_{k,5} = E_{2,5}{}' M_{k-2,5}$, and $k - 2 \equiv 0 \pmod 4$.

On the other hand, let k be an even integer such that $k \geqslant 4$ and $k \equiv 0 \pmod 4$. Write $n := k/4$. Now, we have $E_{k,5}^{\infty} - (E_{4,5}^{\infty})^n \in M_{k,5}^0$ and $E_{k,5}^{0} - (E_{4,5}^{0})^n \in M_{k,5}^0$. In conclusion, we have $M_{k,5} = \mathbb{C}(E_{4,5}^{\infty})^n \oplus \mathbb{C}(E_{4,5}^{0})^n \oplus M_{k,5}^0$. Then

$$\begin{aligned} M_{k,5} = {} & (\mathbb{C}(E_{4,5}^{\infty})^n \oplus \mathbb{C}(E_{4,5}^{\infty})^{n-1}\Delta_5 \oplus \cdots \oplus \mathbb{C}\Delta_5^n) \\ & \oplus (\mathbb{C}(E_{4,5}^{0})^n \oplus \mathbb{C}(E_{4,5}^{0})^{n-1}\Delta_5 \oplus \cdots \oplus \mathbb{C}\Delta_5^n) \end{aligned}$$

Thus, the next proposision follows:

Proposition 6.8. *Let $k \geqslant 4$ be an even integer. For every $f \in M_{k,5}$, we have*

$$v_{\rho_{5,2}}(f) \geqslant t_k, \quad v_{\rho_{5,3}}(f) \geqslant t_k \quad (t_k = 0, 1 \text{ such that } 2t_k \equiv k \pmod 4)). \tag{162}$$

In addition, we have $E_{4,5}^{0} = 25((E_{2,5}{}')^2 - E_{4,5}^{\infty}) - (900/13)\Delta_5$. Then

Remark 6.12. *Every modular form for $\Gamma_0(5)$ is generated by*

$$E_{2,5}{}', \quad E_{4,5}^{\infty}, \quad \text{and} \quad \Delta_5.$$

Finally, define

$$A_{5,1} := \{z\,;\, |z+1/5| = 1/5,\ 0 < Arg(z) \leqslant 2\pi/3\}, \tag{163}$$

$$A_{5,1}^0 := \{z\,;\, Re(z) = -1/2,\ Im(z) \leqslant \sqrt{3}/6\}, \tag{164}$$

$$A_{5,2} := \{z\,;\, |z+2/5| = 1/5,\ \pi/3 \leqslant Arg(z) < \pi/2\}, \tag{165}$$

$$A_{5,2}^0 := \{z\,;\, |z+2/3| = 1/3,\ \alpha < Arg(z) \leqslant \pi/3\}, \tag{166}$$

$$A_{5,2}^+ := \{z\,;\, |z-2/5| = 1/5,\ \pi/2 < Arg(z) \leqslant 2\pi/3\}, \tag{167}$$

$$A_{5,2}^{+\,0} := \{z\,;\, |z-2/3| = 1/3,\ 2\pi/3 \leqslant Arg(z) < \pi-\alpha\}, \tag{168}$$

where $\alpha \in [0, \pi/2]$ such that $\tan\alpha = 3/4$. Furthermore, we define $A_5 := A_{5,1} \cup A_{5,2} \cup A_{5,2}^+$ and $A_5^0 := A_{5,1}^0 \cup A_{5,2}^0 \cup A_{5,2}^{+\,0}$. Then we have following:

Conjecture 6.3. *Let $k \geqslant 4$ be an even integer. $E_{k,5}^\infty$ has $2\lfloor k/4\rfloor - 1$ zeros in A_5, and $E_{k,5}^0$ has $2\lfloor k/4\rfloor - 1$ zeros in A_5^0. Furthermore, in Proposition 6.8, the equality hold if f is equal to $E_{k,5}^\infty$ or $E_{k,5}^0$.*

Now, we have the following transformations:

$$\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix} W_5 : A_{5,1} \ni \frac{e^{i\theta}-1}{5} \mapsto -\frac{1}{2} + \frac{i}{2}\frac{1}{\tan\theta/2} \in A_{5,1}^0, \tag{169}$$

$$W_5 : A_{5,2} \ni \frac{e^{i\theta}-2}{5} \mapsto \frac{2-\cos\theta + i\sin\theta}{5-4\cos\theta} \in A_{5,2}^0, \tag{170}$$

$$W_5 : A_{5,2}^+ \ni \frac{e^{i\theta}+2}{5} \mapsto \frac{-2-\cos\theta + i\sin\theta}{5+4\cos\theta} \in A_{5,2}^{+\,0}. \tag{171}$$

Furthermore,

$$E_{k,5}^0\left(\left(\begin{smallmatrix}1 & -1\\ 0 & 1\end{smallmatrix}\right) W_5\, z\right) = (5z)^k E_{k,5}^\infty(z) \quad \text{for every } z \in A_{5,1},$$
$$E_{k,5}^0(W_5\, z) = (5z)^k E_{k,5}^\infty(z) \quad \text{for every } z \in A_{5,2},\ A_{5,2}^+.$$

Then

Remark 6.13. *If $E_{k,5}^\infty$ has $2\lfloor k/4\rfloor - 1$ zeros in A_5, then $E_{k,5}^0$ has $2\lfloor k/4\rfloor - 1$ zeros in A_5^0.*

6.3.5. *The function $F_{k,5,1}(\theta)$, $F_{k,5,2}(\theta)$, and $F_{k,5,2}^+(\theta)$.* We give the next definition;

$$F_{k,5,1}(\theta) := e^{ik\theta/2} E_{k,5}^\infty(e^{i\theta}/5 - 1/5), \tag{172}$$

$$F_{k,5,2}(\theta) := e^{ik\theta/2} E_{k,5}^\infty(e^{i\theta}/5 - 2/5), \tag{173}$$

$$F_{k,5,2}^+(\theta) := e^{ik\theta/2} E_{k,5}^\infty(e^{i\theta}/5 + 2/5). \tag{174}$$

Again, $E_{k,5}^\infty$ is denoted by

$$\begin{aligned} E_{k,5}^\infty(z) &= \frac{1}{2}\sum_{\substack{(c,d)=1\\ 5|c}} (cz+d)^{-k} \\ &= B_{k,5}(z) = \sum_{n\in\mathbb{N}} D_{k,5}(5^n z). \quad \text{(See Section 2)} \end{aligned}$$

Then

$$\begin{aligned} E_{k,3}^\infty\left(\frac{e^{i\theta}-1}{5}\right) &= \sum_{n\in\mathbb{N}} D_{k,5}(5^{n-1}(e^{i\theta}-1)) \\ &= D_{k,5}(e^{i\theta}-1) + \sum_{n\in\mathbb{N}} D_{k,5}(5^n e^{i\theta} - 5^n) \\ &= \sum_{n=1}^{4} D_{k,5}^n(e^{i\theta}-1) + \sum_{n\in\mathbb{N}} D_{k,5}(5^n e^{i\theta}) \\ &= B_{k,5}(e^{i\theta}) + C_{k,5}(e^{i\theta}) + \sum_{n=1}^{3} D_{k,5}^n(e^{i\theta}). \end{aligned}$$

Thus

$$\begin{aligned}F_{k,5,1}(\theta) =&\frac{1}{2}\sum_{\substack{(c,d)=1\\5|c}}(ce^{i\theta/2}+de^{-i\theta/2})^{-k}+\frac{1}{2}\sum_{\substack{(c,d)=1\\5|c}}(ce^{-i\theta/2}+de^{i\theta/2})^{-k}\\&+\frac{1}{2}\sum_{\substack{(c,d)=1\\5\nmid cd\\c\equiv d\ (5)}}(ce^{i\theta/2}+de^{-i\theta/2})^{-k}\\&+\frac{1}{2}\sum_{\substack{(c,d)=1\\5\nmid cd\\d\equiv 2c\ (5)}}(ce^{i\theta/2}+de^{-i\theta/2})^{-k}+\frac{1}{2}\sum_{\substack{(c,d)=1\\5\nmid cd\\d\equiv 2c\ (5)}}(ce^{-i\theta/2}+de^{i\theta/2})^{-k}.\end{aligned}$$

Similarly,

$$\begin{aligned}E_{k,3}^{\infty}\left(\frac{e^{i\theta}-2}{5}\right)&=B_{k,5}(e^{i\theta})+C_{k,5}(e^{i\theta})+\sum_{n=1,2,4}D_{k,5}^{n}(e^{i\theta}),\\E_{k,3}^{\infty}\left(\frac{e^{i\theta}+2}{5}\right)&=B_{k,5}(e^{i\theta})+C_{k,5}(e^{i\theta})+\sum_{n=1,3,4}D_{k,5}^{n}(e^{i\theta}).\end{aligned}$$

Thus

$$\begin{aligned}F_{k,5,2}(\theta) =&\frac{1}{2}\sum_{\substack{(c,d)=1\\5|c}}(ce^{i\theta/2}+de^{-i\theta/2})^{-k}+\frac{1}{2}\sum_{\substack{(c,d)=1\\5|c}}(ce^{-i\theta/2}+de^{i\theta/2})^{-k}\\&+\frac{1}{2}\sum_{\substack{(c,d)=1\\5\nmid cd\\c\equiv d\ (5)}}(ce^{i\theta/2}+de^{-i\theta/2})^{-k}+\frac{1}{2}\sum_{\substack{(c,d)=1\\5\nmid cd\\c\equiv -d\ (5)}}(ce^{i\theta/2}+de^{-i\theta/2})^{-k}\\&+\frac{1}{2}\sum_{\substack{(c,d)=1\\5\nmid cd\\d\equiv 2c\ (5)}}(ce^{i\theta/2}+de^{-i\theta/2})^{-k}.\end{aligned}$$

$$\begin{aligned}F_{k,5,2}^{+}(\theta) =&\frac{1}{2}\sum_{\substack{(c,d)=1\\5|c}}(ce^{i\theta/2}+de^{-i\theta/2})^{-k}+\frac{1}{2}\sum_{\substack{(c,d)=1\\5|c}}(ce^{-i\theta/2}+de^{i\theta/2})^{-k}\\&+\frac{1}{2}\sum_{\substack{(c,d)=1\\5\nmid cd\\c\equiv d\ (5)}}(ce^{i\theta/2}+de^{-i\theta/2})^{-k}+\frac{1}{2}\sum_{\substack{(c,d)=1\\5\nmid cd\\c\equiv -d\ (5)}}(ce^{i\theta/2}+de^{-i\theta/2})^{-k}\\&+\frac{1}{2}\sum_{\substack{(c,d)=1\\5\nmid cd\\d\equiv 2c\ (5)}}(ce^{-i\theta/2}+de^{i\theta/2})^{-k}.\end{aligned}$$

Note that $(ce^{i\theta/2}+de^{i\theta/2})^{-k}$ and $(de^{i\theta/2}+ce^{-i\theta/2})^{-k}$ are conjugates of each other for any pair (c,d) such that $c\equiv\pm d \pmod 5$, and $(ce^{i\theta/2}+de^{i\theta/2})^{-k}$ and $(ce^{-i\theta/2}+de^{i\theta/2})^{-k}$ are conjugates of each other for any pair (c,d) such that $c\not\equiv d \pmod 5$.

The next proposition follows.

Proposition 6.9. *$F_{k,5,1}(\theta)$ is real for every $\theta\in\mathbb{R}$. On the other hand, $F_{k,5,2}(\theta)$ and $F_{k,5,2}(\theta)$ are conjugates of each other for every $\theta\in\mathbb{R}$.*

6.3.6. *Application of the RSD Method.* We note that $N:=c^2+d^2$. Let $v_k(c,d,\theta):=|ce^{i\theta/2}+de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\theta)=1/(c^2+d^2+2cd\cos\theta)^{k/2}$ and $v_k(c,d,\theta)=v_k(-c,-d,\theta)=v_k(\pm d,\pm c,\theta)$.

In this subsubsection, we consider an application of the RSD Method to $F_{k,5,1}(\theta)$. Firstly, we consider the case $N=1$. Then we can write:

$$F_{k,5,1}(\theta) = 2\cos(k\theta/2) + R_{5,1}, \tag{175}$$

where

$$|R_{5,1}| = \sum_{\substack{(c,d)=1 \\ 5|c \\ N>1}} v_k(c,d,\theta) + \frac{1}{2}\sum_{\substack{(c,d)=1 \\ 5\nmid cd \\ c\equiv d\,(5)}} v_k(c,d,\theta) + \sum_{\substack{(c,d)=1 \\ 5\nmid cd \\ d\equiv 2c\,(5)}} v_k(c,d,\theta).$$

We consider two cases, namely $[\pi/2, 2\pi/3]$ and $[\pi/4, \pi/2]$.

For the interval $[\pi/2, 2\pi/3]$, we will consider the next cases, namely $N=2,5$, and $N\geqslant 10$. Considering $-1/2 \leqslant \cos\theta \leqslant 0$ for the interval $[\pi/2, 2\pi/3]$, we have the following:

When $N=2$, $\quad v_k(1,1,\theta) \leqslant 1.$

When $N=5$, $\quad v_k(1,2,\theta) \leqslant (1/3)^{k/2}, \quad v_k(2,-1,\theta) \leqslant (1/5)^{k/2}.$

When $N\geqslant 10$, $\quad |ce^{i\theta/2} \pm de^{-i\theta/2}|^2 \geqslant N/2,$

and the number of terms is not more than $2N^{1/2}$. Then

$$|R_{5,1}|_{N\geqslant 10} = 2\sqrt{2}\sum_{N=10}^{\infty}\left(\frac{1}{2}N\right)^{(1-k)/2} \leqslant \frac{108}{k-3}\left(\frac{2}{9}\right)^{k/2}.$$

Thus

$$\begin{aligned} |R_{5,1}| &\leqslant 1 + 2\left(\frac{1}{3}\right)^{k/2} + 2\left(\frac{1}{5}\right)^{k/2} + \frac{108}{k-3}\left(\frac{2}{9}\right)^{k/2} \\ &\leqslant 1.41106... \quad (k\geqslant 6) \end{aligned} \tag{176}$$

On the other hand, for the interval $[\pi/6, \pi/2]$, we will consider the next cases, namely $N=2,5,10,13,17$, and $N\geqslant 25$. Considering $0\leqslant \cos\theta \leqslant 7/8$ for the interval $[\pi/6,\pi/2]$, we have the following:

When $N=2$, $\quad v_k(1,1,\theta) \leqslant (1/2)^{k/2}.$

When $N=5$, $\quad v_k(1,2,\theta) \leqslant (1/5)^{k/2}, \quad v_k(2,-1,\theta) \leqslant (2/3)^{k/2}.$

When $N=10$, $\quad v_k(3,1,\theta) \leqslant (1/10)^{k/2}, \quad v_k(1,-3,\theta) \leqslant (4/19)^{k/2}.$

When $N=13$, $\quad v_k(2,-3,\theta) \leqslant (2/5)^{k}.$

When $N=17$, $\quad v_k(1,-4,\theta) \leqslant (1/10)^{k}.$

When $N\geqslant 25$, $\quad |ce^{i\theta/2} \pm de^{-i\theta/2}|^2 \geqslant N/8,$

and the number of terms is not more than $2N^{1/2}$. Then

$$|R_{5,1}|_{N\geqslant 25} = 4\sqrt{2}\sum_{N=25}^{\infty}\left(\frac{1}{8}N\right)^{(1-k)/2} \leqslant \frac{192\sqrt{6}}{k-3}\left(\frac{1}{3}\right)^{k/2}.$$

Thus

$$\begin{aligned} |R_{5,1}| &\leqslant 2\left(\frac{2}{3}\right)^{k/2} + \left(\frac{1}{2}\right)^{k/2} + \cdots + 2\left(\frac{1}{10}\right)^{k/2} + \frac{192\sqrt{6}}{k-3}\left(\frac{1}{3}\right)^{k/2} \\ &\leqslant 1.67753... \quad (k\geqslant 8) \end{aligned} \tag{177}$$

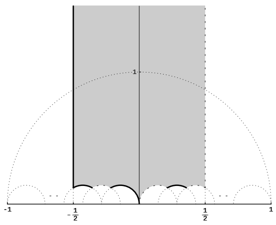
By the above bounds, we know that $E_{k,5}^{\infty}$ has many zeros in $A_{5,1}$. However, for the arcs $A_{5,2}$ and $A_{5,2}^{+}$, it is not clear.

6.4. $\Gamma_0(7)$.

We have the following transformation:

$$\begin{pmatrix} -1 & 0 \\ 7 & -1 \end{pmatrix} : \quad \frac{e^{i\theta}+1}{7} \mapsto \frac{e^{i(\pi-\theta)}-1}{7},$$
$$\begin{pmatrix} -3 & -1 \\ 7 & 2 \end{pmatrix} : \quad \frac{e^{i\theta}-2}{7} \mapsto \frac{e^{i(\pi-\theta)}-3}{7},$$
$$\begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix} : \quad \frac{e^{i\theta}+3}{7} \mapsto \frac{e^{i(\pi-\theta)}+2}{7}.$$

Then we have $V_{\Gamma_0(7)} = \{-3/14 + \sqrt{3}i/10,\ -5/14 + \sqrt{3}i/10\}$.



6.4.1. *Valence formula.*

Proposition 6.10. *Let f be a modular function of weight k for $\Gamma_0(7)$, which is not identically zero. We have*

$$v_\infty(f) + v_0(f) + \frac{1}{3}v_{\rho_{7,2}}(f) + \frac{1}{3}v_{\rho_{7,3}}(f) + \sum_{\substack{p\in\Gamma_0(7)\backslash\mathbb{H} \\ p\neq\rho_{7,2},\rho_{7,3}}} v_p(f) = \frac{k}{2}, \tag{178}$$

where $\rho_{7,2} = -5/14 + \sqrt{3}i/14$ and $\rho_{7,3} := 5/14 + \sqrt{3}i/14$.

Proof. Let f be a nonzero modular function of weight k for $\Gamma_0(7)$, and let $\mathscr{C}$ be a contour of its fundamental domain $\mathbb{F}(7)$ (Figure 4), whose interior contains every zero and pole of f except for $\rho_{7,2}$, $\rho_{7,3}$, and $\rho_{7,4} := -3/14 + \sqrt{3}i/14$ (*cf.* Figure 9). By the *Residue theorem*, we have

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{df}{f} = \sum_{\substack{p\in\Gamma_0(7)\backslash\mathbb{H} \\ p\neq\rho_{7,2},\rho_{7,3},\rho_{7,4}}} v_p(f).$$

(i) For the arc around ∞, we have $-v_\infty(f)$.
(ii) For the arc around 0, we have $-v_0(f)$.
(iii) For the arcs around $\rho_{7,2}$, $\rho_{7,3}$, $\rho_{7,4}$, when the radii of each arc tends to 0, then we have

$$-\frac{1}{3}v_{\rho_{7,2}}(f), \quad -\frac{1}{3}v_{\rho_{7,3}}(f), \quad \text{and} \quad -v_{\rho_{7,4}}(f).$$

(iv) For the arcs on $\{z\,;\,Re(z) = -1/2\}$ and $\{z\,;\,Re(z) = 1/2\}$, since $f(Tz) = f(z)$ for $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, we have 0.
(v) For the arcs on $\{z\,;\,|z \pm 1/7| = 1/7\}$, since $f(S_7 z) = (7z+1)^k f(z)$ for $S_7 := \left(\begin{smallmatrix} 1 & 0 \\ 7 & 1 \end{smallmatrix}\right)$, we have $k/3$.
Furthermore, for the arcs on $\{z\,;\,|z+2/7| = 1/7\}$ and $\{z\,;\,|z+3/7| = 1/7\}$, since $f\left(\left(\begin{smallmatrix} -2 & -1 \\ 7 & 3 \end{smallmatrix}\right)z\right) = (7z+3)^k f(z)$, we have $k/6$. Similarly, for the arcs on $\{z\,;\,|z-2/7| = 1/7\}$ and $\{z\,;\,|z-3/7| = 1/7\}$, since $f\left(\left(\begin{smallmatrix} 3 & -1 \\ 7 & -2 \end{smallmatrix}\right)z\right) = (7z-2)^k f(z)$, we have $k/6$.
Thus $k/3 + 2\cdot k/6 = 2k/3$.

□

6.4.2. *Modular forms of weight* 2. We define

$$E_{2,7}{}'(z) := \frac{1}{6}(7E_2(7z) - E_2(z)). \tag{179}$$

Note that $E_{2,7}{}'$ is generated by Eisensitein series for $\mathrm{SL}_2(\mathbb{Z})$, but it is not Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ nor $\Gamma_0(7)$.

Similarly to ${E_{2,2}}'$ and ${E_{2,3}}'$, we have

$$ {E_{2,7}}'\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)z\right)={E_{2,7}}'(z),\quad {E_{2,7}}'\left(-\left(\begin{smallmatrix}1&0\\7&1\end{smallmatrix}\right)z\right)=(-7z-1)^2{E_{2,7}}'(z). $$

In addition, we have $\left(\begin{smallmatrix}4&1\\7&2\end{smallmatrix}\right)=TST^2ST^{-3}S^{-1}$ for $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $S=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. (See [RD]) Similarly, we have $7\cdot\left(\begin{smallmatrix}4&1\\7&2\end{smallmatrix}\right)z=\left(\begin{smallmatrix}4&7\\1&2\end{smallmatrix}\right)7z$ and $\left(\begin{smallmatrix}4&7\\1&2\end{smallmatrix}\right)=T^3S^{-1}T^{-1}ST$. Recall that $E_2(Tz)=E_2(z)$, $E_2(Sz)=z^2E_2(z)+(12/2\pi i)z$. Similarly to ${E_{2,5}}'$

$$ \begin{aligned} E_2(TST^2ST^{-3}S^{-1}z)&=\left(\frac{7z+2}{3z+1}\right)^2E_2(ST^{-3}S^{-1}z)+\frac{12}{2\pi i}\frac{7z+2}{3z+1}\\ &=\frac{(7z+2)^2}{z^2}E_2(S^{-1}z)-\frac{12}{2\pi i}\frac{2(7z+2)}{z}\\ &=(7z+2)^2E_2(z)+\frac{12}{2\pi i}5(7z+2),\\ E_2(T^3S^{-1}T^{-1}ST\,5z)&=\left(\frac{7z+2}{7z+1}\right)^2E_2(ST\,7z)-\frac{12}{2\pi i}\frac{7z+2}{7z+1}\\ &=(7z+2)^2E_2(7z)+\frac{12}{2\pi i}(7z+2). \end{aligned} $$

Thus

$$ {E_{2,7}}'\left(\left(\begin{smallmatrix}4&1\\7&2\end{smallmatrix}\right)z\right)=(7z+2)^2{E_{2,7}}'(z). $$

Recall that $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $-\left(\begin{smallmatrix}1&0\\7&1\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}4&1\\7&2\end{smallmatrix}\right)$ generate $\Gamma_0(7)$ (in Section 4), then we can show that ${E_{2,7}}'$ satisfies transformation rule for $\Gamma_0(7)$.

Furthermore, because

$$ {E_{2,7}}'(W_7z)=-(\sqrt{7}z)^2{E_{2,7}}'(z), \tag{180} $$

${E_{2,7}}'$ is holomorphic at cusp 0. Now, we prove ${E_{2,7}}'$ is a modular form for $\Gamma_0(7)$ of weight 2.

6.4.3. *Preliminaries.* Let f be a modular form for $\Gamma_0(7)$ of weight k, and let $k\not\equiv 0 \pmod 6$. Then we have

$$ \begin{aligned} f(\rho_{7,2})&=f\left(\left(\begin{smallmatrix}-3&-1\\7&2\end{smallmatrix}\right)\rho_{7,2}\right)=(e^{i2\pi/3})^kf(\rho_{7,2}).\\ f(\rho_{7,3})&=f\left(\left(\begin{smallmatrix}2&-1\\7&-3\end{smallmatrix}\right)\rho_{7,3}\right)=(e^{i2\pi/3})^kf(\rho_{7,3}). \end{aligned} $$

Thus $f(\rho_{7,2})=f(\rho_{7,3})=0$, $v_{\rho_{7,2}}(f)\geqslant 1$, and $v_{\rho_{7,3}}(f)\geqslant 1$.

Let k be an even integer such that $k\equiv 0 \pmod 6$. Then we have

$$ \begin{aligned} E_{k,7}^{\infty}(\rho_{7,2})&=\frac{1}{1-7^k}\left(\left(\left(5+\sqrt{3}i\right)/2\right)^k-7^k\right)E_k(\rho)\neq 0,\\ E_{k,7}^{0}(\rho_{7,2})&=\frac{1}{1-7^{-k}}\left(\left(\left(5+\sqrt{3}i\right)/2\right)^k-1\right)E_k(\rho)\neq 0.\\ E_{k,7}^{\infty}(\rho_{7,3})&=\frac{1}{1-7^k}\left(\left(\left(-5+\sqrt{3}i\right)/2\right)^k-7^k\right)E_k(\rho)\neq 0,\\ E_{k,7}^{0}(\rho_{7,3})&=\frac{1}{1-7^{-k}}\left(\left(\left(-5+\sqrt{3}i\right)/2\right)^k-1\right)E_k(\rho)\neq 0. \end{aligned} $$

Thus $v_{\rho_{7,2}}(E_{k,7}^{\infty})=v_{\rho_{7,2}}(E_{k,7}^{0})=0$, $v_{\rho_{7,3}}(E_{k,7}^{\infty})=v_{\rho_{7,3}}(E_{k,7}^{0})=0$.

Recall that $v_0(E_{k,7}^{\infty})=v_\infty(E_{k,7}^{0})=1$ and $v_\infty(E_{k,7}^{\infty})=v_0(E_{k,7}^{0})=0$. (Section 4)

Finally, for ${E_{2,7}}'$, we have $v_{\rho_{7,2}}({E_{2,7}}')\geqslant 1$ and $v_{\rho_{7,3}}({E_{2,7}}')\geqslant 1$. By the valence formula for $\Gamma_0(7)$ (Proposition 6.10) and Corollary 2.1.2, we have

$$ v_{\rho_{7,2}}({E_{2,7}}')=v_{\rho_{7,3}}({E_{2,7}}')=2,\qquad v_p({E_{2,7}}')=0\quad\text{for every } p\neq\rho_{7,2},\rho_{7,3}. $$

Incidentally, let f be a modular form for $\Gamma_0(7)$ of weight 2. Then, By the valence formula for $\Gamma_0(7)$, we also have $v_{\rho_{7,2}}(f)+v_{\rho_{7,3}}(f)=4$. Then, if $v_{\rho_{7,2}}(f)\neq 2$, we have $f=a_0+a_1q+\cdots$ for some $a_0\neq 0$, $f-a_0{E_{2,7}}'=(a_1-4a_0)q+\cdots$, and $v_\infty(f-a_0{E_{2,7}}')\geqslant 1$. This contradicts $v_p(f)=0$ for every $p\neq\rho_{7,2},\rho_{7,3}$. Now

$$ v_{\rho_{7,2}}(f)=v_{\rho_{7,3}}(f)=2,\qquad v_p(f)=0\quad\text{for every } p\neq\rho_{7,2},\rho_{7,3}. $$

Thus $f/{E_{2,7}}'$ is a modular form of weight 0, then $f/{E_{2,7}}' \in \mathbb{C}$. In conclusion, f is a constant multiple of ${E_{2,7}}'$.

6.4.4. *The space of modular forms.* Let $M_{k,7}$ be the space of modular forms for $\Gamma_0(7)$ of weight k, and let $M_{k,7}^0$ be the space of cusp forms for $\Gamma_0(7)$ of weight k. Since $\dim(M_{k,7}/M_{k,7}^0) \leqslant 2$, we have $M_{k,7} = \mathbb{C}E_{k,7}^\infty \oplus \mathbb{C}E_{k,7}^0 \oplus M_{k,7}^0$. Recall that $\Delta_7 = \eta^6(z)\eta^6(7z)$, and $\Delta_{7,4}$, $\Delta_{7,6}$ are the cusp forms for $\Gamma_0^*(7)$ of weight 4, 6, respectively. We have following theorem:

Theorem 6.4. *Let k be an even integer.*

(1) *For $k < 0$, $M_{k,7} = 0$.*
(2) *For $k = 0$ and 2, we have $M_{k,7}^0 = 0$. Furthemore, we have $M_{0,7} = \mathbb{C}$, $M_{2,7} = \mathbb{C}{E_{2,7}}'$.*
(3) *$M_{4,7}^0 = \mathbb{C}\Delta_{7,4}$.*
(4) *Let $\Delta_{7,6}^- := (1/13)({E_{2,7}}'\Delta_{7,4} - \Delta_{7,6})$ and $\Delta_{7,6}^+ := -(1/13)({E_{2,7}}'\Delta_{7,4} + \Delta_{7,6})$. $M_{6,7}^0 = \mathbb{C}\Delta_7 \oplus \mathbb{C}\Delta_{7,6}^- \oplus \mathbb{C}\Delta_{7,6}$.*
(5) *$M_{k,7}^0 = M_{6,7}^0 M_{k-6,7}$.*

Let k be an even integer $k \geqslant 4$. Define $f_0(k) := (E_{4,7}^*)^{k/4}$ if $k \equiv 0 \pmod 4$, and $f_0(k) := E_{6,7}^*(E_{4,7}^*)^{(k-6)/4}$ if $k \equiv 2 \pmod 4$. Then we have $(\sqrt{7}z)^k f_0(k)\big|_{z=W_7 z} = f_0(k)$ because $f_0(k)$ is a modular form for $\Gamma_0^*(7)$.

The proof of this theorem is similar to that of Theorem 5.2. We have $\Delta_{7,6}^- = (\sqrt{7}z)^k \Delta_{7,6}^+$ and $\Delta_{7,6}^+ = (\sqrt{7}z)^k \Delta_{7,6}^-$. Because $\Delta_{7,6}^- = q^2 + \cdots$ and $\Delta_{7,6}^+ = -(2/13)q + \cdots$, we have $v_\infty(\Delta_{7,6}^-) = v_0(\Delta_{7,6}^+) = 2$ and $v_\infty(\Delta_{7,6}^+) = v_0(\Delta_{7,6}^-) = 1$.

For every $f \in M_{k,7}^0$, we can write $f(z) = a_1 q + \cdots$ and $(\sqrt{7}z)^{-k} f(W_7 z) = b_1 q + \cdots$. Now, put $g := f - (a_1\Delta_{7,6} - (13/2)(b_1 - a_1)\Delta_{7,6}^-)f_0(k-6)$, then it is easy to show that $v_\infty(g) \geqslant 2$ and $v_0(g) \geqslant 2$. Thus $g/\Delta_7 \in M_{k-6,7}$. This proves (5).

The table of orders of zeros of basis for $M_{k,7}^*$ is following:

k	f	v_∞	v_0	$v_{\rho_{7,2}}$	$v_{\rho_{7,3}}$	|other zeros|
2	${E_{2,7}}'$	0	0	2	2	0
4	$E_{4,7}^\infty$	0	1	1	1	1
	$E_{4,7}^0$	1	0	1	1	1
	$\Delta_{7,4}$	1	1	1	1	0
6	$E_{6,7}^\infty$	0	1	0	0	3
	$E_{6,7}^0$	1	0	0	0	3
	Δ_7	2	2	0	0	0
	$\Delta_{7,6}^-$	2	1	0	0	1
	$\Delta_{7,6}^+$	1	2	0	0	1
	$\Delta_{7,6}$	1	1	0	0	2

Furthermore, we have $\dim(M_{k,7}) = \lfloor 2k/3 \rfloor + 1$ for $k \geqslant 0$, and $\dim(M_{k,7}^0) = \lfloor 2k/3 \rfloor - 1$ for $k \geqslant 4$.

Let $k \geqslant 8$. Write $n := \lfloor k/6 \rfloor$. If $k \equiv 0, 4 \pmod 6$, then we have $E_{k,7}^\infty - E_{k-6n,7}^\infty (E_{6,7}^\infty)^n \in M_{k,7}^0$ and $E_{k,7}^0 - E_{k-6n,7}^0 (E_{6,7}^0)^n \in M_{k,7}^0$. In conclusion, we have $M_{k,7} = \mathbb{C}E_{k-6n,7}^\infty (E_{6,7}^\infty)^n \oplus \mathbb{C}E_{k-6n,7}^0 (E_{6,7}^0)^n \oplus M_{k,7}^0$. Similarly, if $k \equiv 2 \pmod 6$, then we have $M_{k,7} = \mathbb{C}{E_{2,7}}'(E_{6,7}^\infty)^n \oplus \mathbb{C}{E_{2,7}}'(E_{6,7}^0)^n \oplus M_{k,7}^0$. Thus we have following:

If $k \equiv 0, 4 \pmod 6$,

$$\begin{aligned} M_{k,7} = {} & E_{k-6n,7}^\infty (\mathbb{C}(E_{6,7}^\infty)^n \oplus \mathbb{C}(E_{6,7}^\infty)^{n-1} M_{6,7}^0 \oplus \cdots \oplus \mathbb{C}(M_{6,7}^0)^n) \\ & \oplus E_{k-6n,7}^0 (\mathbb{C}(E_{6,7}^0)^n \oplus \mathbb{C}(E_{6,7}^0)^{n-1} M_{6,7}^0 \oplus \cdots \oplus \mathbb{C}(M_{6,7}^0)^n) \\ & \oplus M_{k-6n,7}^0. \end{aligned}$$

If $k \equiv 2 \pmod 6$,

$$\begin{aligned} M_{k,7} = {E_{2,7}}' \Big(& (\mathbb{C}(E_{6,7}^\infty)^n \oplus \mathbb{C}(E_{6,7}^\infty)^{n-1} M_{6,7}^0 \oplus \cdots \oplus \mathbb{C}(M_{6,7}^0)^n) \\ & \oplus (\mathbb{C}(E_{6,7}^0)^n \oplus \mathbb{C}(E_{6,7}^0)^{n-1} M_{6,7}^0 \oplus \cdots \oplus \mathbb{C}(M_{6,7}^0)^n) \Big). \end{aligned}$$

Now, the next proposision follows:

Proposition 6.11. *Let $k \geqslant 4$ be an even integer. For every $f \in M_{k,7}$, we have*

$$v_{\rho_{7,2}}(f) \geqslant t_k, \quad v_{\rho_{7,3}}(f) \geqslant t_k \quad (t_k = 0, 1, 2 \text{ such that } -2t_k \equiv k \pmod 6). \tag{181}$$

In addition, we have $E_{4,7}^0 = 49((E_{2,7}{}')^2 - E_{4,7}^\infty) - (784/5)\Delta_{7,4}$, $E_{6,7}^\infty = E_{2,7}{}'(E_{4,7}^\infty - (348/95)\Delta_{7,4}) - (36/19)\Delta_7 - (10/43)\Delta_{7,6}$, and $E_{6,7}^0 = E_{2,7}{}'(343(E_{4,7}^\infty - (E_{2,7}{}')^2) + (223636/95)\Delta_{7,4}) + (12348/19)\Delta_7 - (3430/43)\Delta_{7,6}$. Then

Remark 6.14. *Every modular form for $\Gamma_0(7)$ is generated by*

$$E_{2,7}{}', \quad E_{4,7}^\infty, \quad \Delta_{7,4}, \quad \Delta_{7,6}, \quad \text{and} \quad \Delta_7.$$

APPENDIX.
On the zeros of Eisenstein Series for $\Gamma_0(4)$ and $\Gamma_0^*(4)$

In this paper, we consider Eisenstein Series for $\Gamma_0^*(p)$ and $\Gamma_0(p)$ for primes p. In general, $\Gamma_0^*(N)$ and $\Gamma_0(N)$ for a nonprime integer N have more cusps than $\Gamma_0^*(p)$ and $\Gamma_0(p)$ for a prime p. Then, it gets much more difficult to decide the locating of zeros.

In this appendix, we consider about $\Gamma_0^*(4)$ and $\Gamma_0(4)$. Note that $\Gamma_0(4)$ is a subgroup of $\Gamma_0(2)$. Interestingly, if we can decide all the zeros of Eisenstein Series for $\Gamma_0(2)$, then we can decide all the zeros of Eisenstein Series for $\Gamma_0^*(4)$ and $\Gamma_0(4)$. Thus, by the results of this paper, we decide all the zeros of Eisenstein Series of low weights $0 \leqslant k \leqslant 40$ for $\Gamma_0^*(4)$ and $\Gamma_0(4)$. We may say it is natural, though it is not clear for me.

Again,

$$\Gamma_0(4) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z})\, ;\, c \equiv 0 \pmod 4 \right\},$$
$$\Gamma_0^*(4) = \Gamma_0(4) \cup \Gamma_0(4)\, W_4, \quad W_4 = \left(\begin{smallmatrix} 0 & -1/2 \\ 2 & 0 \end{smallmatrix} \right).$$

Appendix A. Preliminaries

A.1. **Fundamental Domains.** For $\Gamma_0(4)$, similarly to $\Gamma_0(p)$ for prime p, we consider the following condition:

$$(C_0) \qquad |z \pm 1/4| > 1/4,\ -1/2 < Re(z) < 1/2.$$

Then, by corollary2.1.1, we have

$$(182) \qquad \Gamma_0(4) = \langle -I,\ \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right),\ \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix} \right) \rangle.$$

On the other hand, for $\Gamma_0(4)W_4$, we need to consider the following condition for every positive number m:

$$(C_{4,m}) \qquad |z - n/m| > 1/2m\sqrt{p},\ -1/2 < Re(z) < 1/2 \quad \text{for } \forall n \in \mathbb{N} \text{ such that } (m,n) = 1.$$

For every $z \in \mathbb{H}$ such that $|z - n/m| = 1/2m$, we have $(2n-1)/2m \leqslant Re(z) \leqslant (2n+1)/2m$. If $(2n-1)/2m < 1/2$, then we have $(2n-1) < m$, $(2n+1)/2m \leqslant 1/2$. Also, if $(2n+1)/2m > -1/2$, then we have $(2n-1)/2m \geqslant -1/2$. In addition, By W_4, we have the condition $|z| > 1/2$. Note that we have the condition $-1/2 < Re(z) < 1/2$, then we have following sufficient condition:

$$(C_4) \qquad |z| > 1/2,\ -1/2 < Re(z) < 1/2.$$

Moreover, (C_4) is a sufficient condition for (C_0).

Furthermore, we have the following transformations:

For $\Gamma_0(4)$,

$$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} : \frac{e^{i\theta} - 1}{4} \mapsto \frac{e^{i(\pi-\theta)} + 1}{4}.$$

For $\Gamma_0^*(4)$,

$$W_4 : \frac{e^{i\theta}}{2} \mapsto \frac{e^{i(\pi-\theta)}}{2}.$$

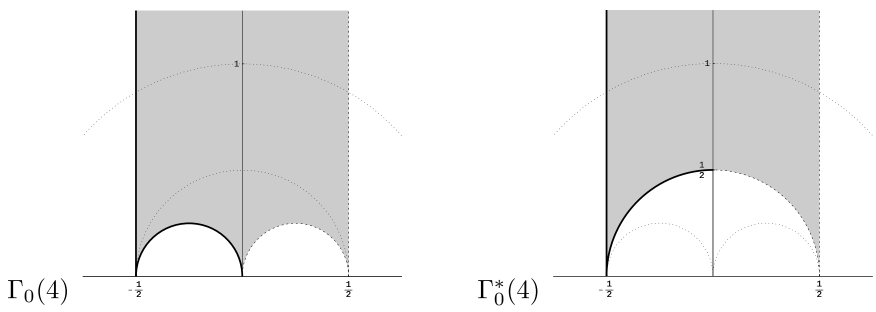


FIGURE 10. $\Gamma_0(4)$ and $\Gamma_0^*(4)$

A.2. $E_k(z+1/2)$**.** Let $k \geqslant 2$ be a even integer. In this subsection, we consider about $E_k(z+1/2)$. We will denote $E_k(z+1/2)$ by $E_k(z)$, $E_k(2z)$, and $E_k(4z)$.

Firstly, we have

$$E_k(z+1/2) = \frac{1}{2}\sum_{(c,d)=1}(c(z+1/2)+d)^{-k}.$$

If c is odd, then $(c(z+1/2)+d)^{-k} = 2^k(2cz+(c+2d))^{-k}$, and $(2c, c+2d) = 1$. If c is divisible by 4, then we can write $c = 4c'$ for some integer c', and we have $(c(z+1/2)+d)^{-k} = (4c'z+(2c'+d))^{-k}$, $(4c', 2c'+d) = 1$. And if c is even and not divisible by 4, then we can write $c = 2c''$ for some integer c'', and we have $(c(z+1/2)+d)^{-k} = (1/2^k)(c''z+(c''+d)/2)^{-k}$, $(c'', (c''+d)/2) = 1$. Here, we have

$$E_k(z+1/2) = \frac{1}{2}\sum_{\substack{(c,d)=1\\ 4|c}}(cz+d)^{-k} + 2^k\cdot\frac{1}{2}\sum_{\substack{(c,d)=1\\ c:even,4\nmid c}}(cz+d)^{-k} + \frac{1}{2^k}\cdot\frac{1}{2}\sum_{\substack{(c,d)=1\\ c:odd}}(cz+d)^{-k}.$$

Now, let us denote A, B, C as follows:

$$A := \frac{1}{2}\sum_{\substack{(c,d)=1\\ 4|c}}(cz+d)^{-k}, \quad B := \frac{1}{2}\sum_{\substack{(c,d)=1\\ c:even,4\nmid c}}(cz+d)^{-k}, \quad C := \frac{1}{2}\sum_{\substack{(c,d)=1\\ c:odd}}(cz+d)^{-k}.$$

Then

$$E_k(z+1/2) = A + 2^k\cdot B + \frac{1}{2^k}\cdot C.$$

Similarly,

$$E_k(z) = A + B + C.$$
$$E_k(2z) = A + B + \frac{1}{2^k}\cdot C.$$
$$E_k(4z) = A + \frac{1}{2^k}\cdot B + \frac{1}{4^k}\cdot C.$$

In conclusion,

(183) $$E_k(z+1/2) = -E_k(z) + (2^k+2)E_k(2z) - 2^kE_k(4z).$$

A.3. **Eisenstein Series.** Let $k \geqslant 4$ be an even integer. Recall Section 2, and we have

$$\Gamma_\kappa := \{\gamma \in \Gamma_0(4)\ ;\ \gamma\kappa = \kappa\}, \quad \Gamma_\kappa^* := \{\gamma \in \Gamma_0^*(4)\ ;\ \gamma\kappa = \kappa\}$$

for a cusp κ. Furthermore, $\gamma_\kappa, \gamma_\kappa^* \in \mathrm{SL}_2(\mathbb{R})$ satisfy $\gamma_\kappa\infty = \gamma_\kappa^*\infty = \kappa$ and

$$\Gamma_\kappa = \gamma_\kappa\Gamma_\infty\gamma_\kappa^{-1}, \quad \Gamma_\kappa^* = \gamma_\kappa^*\Gamma_\infty^*(\gamma_\kappa^*)^{-1}.$$

In addition, we denote the Eisenstein series associated with $\Gamma_0(4)$ and $\Gamma_0^*(4)$ for a cusp κ by $E_{k,4}^\kappa$ and $E_{k,4}^{*\,\kappa}$, respectively.

Now, $\Gamma_0(4)$ has three cusps ∞, 0, and $-1/2$. On the other hand, $\Gamma_0^*(4)$ has two cusps ∞ and $-1/2$.

A.3.1. *For the cusp* ∞. We have $\Gamma_\infty = \Gamma_\infty^* = \{\pm\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right)\ ;\ n \in \mathbb{Z}\}$. Thus, similarly to E_k, $E_{k,p}^*$, and $E_{k,p}^\infty$,

(184) $$E_{k,4}^\infty(z) = e\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} j(\gamma,z)^{-k} = \frac{1}{2}\sum_{\substack{(c,d)=1\\ 4|c}}(cz+d)^{-k},$$

$$E_{k,4}^{*\,\infty}(z) = e\sum_{\gamma\in\Gamma_\infty^*\backslash\Gamma_0(4)} j(\gamma,z)^{-k} + e\sum_{\gamma\in\Gamma_\infty^*\backslash\Gamma_0(4)W_4} j(\gamma,z)^{-k}$$

(185) $$= \frac{1}{2}\sum_{\substack{(c,d)=1\\ 4|c}}(cz+d)^{-k} + \frac{2^{-k}}{2}\sum_{\substack{(c,d)=1\\ c:odd}}(cz+d)^{-k}.$$

Also, we have

$$E_{k,4}^{\infty}(z) = B_{k,2}(2z) = \frac{1}{1-2^k}\left(E_k(2z) - 2^k E_k(4z)\right), \tag{186}$$

$$\begin{aligned} E_{k,4}^{*\,\infty}(z) &= B_{k,2}(2z) + 2^{-k}(E_k(z) - B_{k,2}(z)) \\ &= \frac{1}{1-2^k}\left(-E_k(z) + 2E_k(2z) - 2^k E_k(4z)\right). \end{aligned} \tag{187}$$

A.3.2. *For the cusp* 0 (*only for* $\Gamma_0(4)$). Similarly to $E_{k,p}^0$, we have $\Gamma_0 = \{\pm\left(\begin{smallmatrix}1 & 0\\ 4n & 1\end{smallmatrix}\right)\ ;\ n \in \mathbb{Z}\}$ and $\gamma_0 = W_4$. Thus

$$E_{k,4}^0(z) = e \sum_{\gamma\in\Gamma_0\backslash\Gamma_0(4)} j(\gamma_0^{-1}\gamma, z)^{-k} = \frac{1}{2}\sum_{\substack{(c,d)=1\\ c:odd}} (cz+d)^{-k}. \tag{188}$$

Also,

$$E_{k,4}^0(z) = E_k(z) - B_{k,2}(z) = \frac{1}{1-2^{-k}}\left(E_k(z) - E_k(2z)\right). \tag{189}$$

In addition, we have following:

Remark A.1. *Let $k \geqslant 4$ be an even integer. We have*

$$E_{k,4}^0(z) = E_{k,2}^0(z) \quad \textit{for every}\, z \in \mathbb{H}. \tag{190}$$

A.3.3. *For the cusp* $-1/2$. Firstly, we decide $\Gamma_{-1/2}$ and $\Gamma_{-1/2}^*$.

For a $\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(4)$, if $\gamma(-1/2) = -1/2$, then we have $c = 2a+4b+2d$. By $ad-bc=1$, we have $a = 2b\pm 1$, $d = 2b \pm 1$, and $c = -4b$. Thus we have

$$\Gamma_{-1/2} = \left\{\pm\left(\begin{smallmatrix}2n+1 & n\\ -4n & -2n+1\end{smallmatrix}\right)\ ;\ n\in\mathbb{Z}\right\}.$$

Furthermore, if $\gamma W_4(-1/2) = -1/2$, then we have $c = -2a-4b-2d$. Similarly, we have $\Gamma_{-1/2}^* \supset \left\{\pm\left(\begin{smallmatrix}2n & n-1/2\\ -4n+2 & -2n+2\end{smallmatrix}\right)\ ;\ n\in\mathbb{Z}\right\}$. Thus we have

$$\Gamma_{-1/2}^* = \left\{\pm\left(\begin{smallmatrix}n+1 & n/2\\ -2n & -n+1\end{smallmatrix}\right)\ ;\ n\in\mathbb{Z}\right\}.$$

Secondly, we decide $\gamma_{-1/2}$ and $\gamma_{-1/2}^*$.

For a $\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(4)$, if $\gamma\infty = -1/2$, then we have $c = -2a$ and $d = 1/a - 2b$. Then we have

$$\gamma\Gamma_\infty\gamma^{-1} = \left\{\pm\left(\begin{smallmatrix}2na^2+1 & na^2\\ -4na^2 & -2na^2+1\end{smallmatrix}\right)\ ;\ n\in\mathbb{Z}\right\}.$$

For $\Gamma_0(4)$, we need $a^2 = 1$, thus we can define as following:

$$\gamma_{-1/2} := \begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}.$$

On the other hand, for $\Gamma_0^*(4)$, we need $a^2 = 1/2$, thus we can define

$$\gamma_{-1/2}^* := \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}}\\ -\sqrt{2} & \frac{1}{\sqrt{2}}\end{pmatrix}.$$

Finally, we decide $E_{k,4}^{-1/2}$ and $E_{k,4}^{*\,-1/2}$.

For a $\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(4)$, we have $j((\gamma_{-1/2})^{-1}\gamma, z) = ((2a+c)z + (2b+d))$, and $\Gamma_{-1/2}$ stabilize the pair $(2a+c, 2b+d)$. Thus we have

$$E_{k,4}^{-1/2}(z) = \frac{1}{2}\sum_{\substack{(c,d)=1\\ c:even, 4\nmid c}} (cz+d)^{-k}. \tag{191}$$

Similarly, we have $j((\gamma_{-1/2})^{-1}\gamma W_4, z) = (1/\sqrt{2})((4b+2d)z + (a+c/2))$, and $\Gamma^*_{-1/2}$ stabilize the pair $(2a+c, 2b+d)$ and $(4b+2d, a+c/2)$. Thus we have

$$\begin{aligned} E^{*\,-1/2}_{k,4}(z) &= e \sum_{\substack{(c,d)=1 \\ c:even, 4\nmid c}} (cz+d)^{-k} + 2^{k/2} e \sum_{\substack{(c,d)=1 \\ c:even, 4\nmid c}} (cz+d)^{-k} \\ &= \frac{1}{2} \sum_{\substack{(c,d)=1 \\ c:even, 4\nmid c}} (cz+d)^{-k}. \end{aligned} \tag{192}$$

Then

Remark A.2. *Let $k \geqslant 4$ be an even integer. We have*

$$E^{*\,-1/2}_{k,4}(z) = E^{-1/2}_{k,4}(z) \quad \textit{for every } z \in \mathbb{H}. \tag{193}$$

Also,

$$\begin{aligned} E^{-1/2}_{k,4}(z) = E^{*\,-1/2}_{k,4}(z) &= B_{k,2}(z) - B_{k,2}(2z) \\ &= \frac{1}{1-2^k}\left(E_k(z) - (2^k+1)E_k(2z) + 2^k E_k(4z)\right). \end{aligned} \tag{194}$$

A.3.4. *The orders at cusps.* For $\Gamma_0(4)$, we have followings:

$$\begin{aligned} E^\infty_{k,4}(\gamma_0 z) &= \frac{1}{1-2^k}\left(E_k\left(-\frac{1}{2z}\right) - 2^k E_k\left(-\frac{1}{z}\right)\right) \\ &= z^k \frac{1}{1-2^{-k}}\left(E_k(z) - E_k(2z)\right) = z^k E^0_{k,4}(z), \end{aligned}$$

then

$$E^0_{k,4}(\gamma_0 z) = 4^k z^k E^\infty_{k,4}(z).$$

$$\begin{aligned} E^\infty_{k,4}(\gamma_{-1/2} z) &= \frac{1}{1-2^k}\left(E_k\left(\frac{2z}{-2z+1}\right) - 2^k E_k\left(\frac{4z}{-2z+1}\right)\right) \\ &= \frac{1}{1-2^k}\left(E_k\left(\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right) 2z\right) - 2^k E_k\left(-\frac{1}{1/2 - 1/4z}\right)\right) \\ &= (-2z+1)^k \frac{1}{1-2^k}\left(E_k(z) - (2^k+1)E_k(2z) + 2^k E_k(4z)\right) \\ &= (-2z+1)^k E^{-1/2}_{k,4}(z). \end{aligned}$$

Similarly,

$$E^{-1/2}_{k,4}(\gamma_{-1/2} z) = (-2z+1)^k E^\infty_{k,4}(z).$$

By $E^0_{k,4} = E^0_{k,2}$ and $\gamma_{-1/2} \in \Gamma_0(2)$,

$$E^0_{k,4}(\gamma_0 z) = (-2z+1)^k E^0_{k,4}(z).$$

$$\begin{aligned} E^{-1/2}_{k,4}(\gamma_0 z) &= \frac{1}{1-2^k}\left(E_k\left(-\frac{1}{4z}\right) - (2^k+1)E_k\left(-\frac{1}{2z}\right) + 2^k E_k\left(-\frac{1}{z}\right)\right) \\ &= (2z)^k \frac{1}{1-2^k}\left(E_k(z) - (2^k+1)E_k(2z) + 2^k E_k(4z)\right) \\ &= (2z)^k E^{-1/2}_{k,4}(z). \end{aligned}$$

Thus the orders at cusps are folloing:

	v_∞	v_0	$v_{-1/2}$
$E^\infty_{k,4}$	0	1	1
$E^0_{k,4}$	1	0	1
$E^{-1/2}_{k,4}$	1	1	0

Table for the orders at cusps

For $\Gamma_0^*(4)$, since

$$E_{k,4}^{*\,\infty}(z) = \frac{1}{1-2^k}\left(E_k(z \pm 1/2) - 2^k E_k(2z)\right),$$

we have

$$\begin{aligned} E_{k,4}^{*\,\infty}(\gamma_{-1/2}^* z) &= E_{k,4}^{*\,\infty}\left(\frac{2z+1}{-4z+2}\right) \\ &= \frac{1}{1-2^k}\left(E_k\left(-\frac{1}{2z-1}\right) - 2^k E_k\left(-1-\frac{1}{z-1/2}\right)\right) \\ &= (-2z+1)^k \frac{1}{1-2^k}\left(E_k(z) - (2^k+1)E_k(2z) + 2^k E_k(4z)\right) \\ &= (-2z+1)^k E_{k,4}^{*\,-1/2}(z). \end{aligned}$$

Similarly,

$$E_{k,4}^{*\,-1/2}(\gamma_{-1/2}^* z) = \frac{(-2z+1)^k}{2^k} E_{k,4}^{*\,\infty}(z).$$

Thus we have

$$\begin{aligned} v_\infty(E_{k,4}^{*\,\infty}) &= v_{-1/2}(E_{k,4}^{*\,-1/2}) = 0, \\ v_{-1/2}(E_{k,4}^{*\,\infty}) &= v_\infty(E_{k,4}^{*\,-1/2}) = 1. \end{aligned}$$

Appendix B. Modular forms for $\Gamma_0(4)$ and $\Gamma_0^*(4)$

B.1. **Valence formula for** $\Gamma_0(4)$**.** In order to decide the locating of zeros of Eisenstein series, we need the *valence formula for* $\Gamma_0(4)$:

Proposition B.1. *Let f be a modular function of weight k for $\Gamma_0(4)$, which is not identically zero. We have*

$$v_\infty(f) + v_0(f) + v_{-1/2}(f) + \sum_{p\in\Gamma_0(4)\backslash\mathbb{H}} v_p(f) = \frac{k}{2}, \tag{195}$$

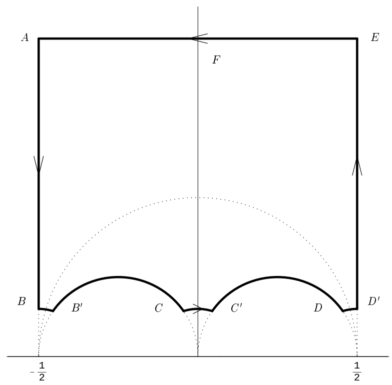


Figure 11

Proof. Let f be a nonzero modular function of weight k for $\Gamma_0(4)$, and let $\mathscr{C}$ be a contour of $\mathbb{F}(4)$ which is a fundamental domain of $\Gamma_0(4)$ represented in Figure 11, whose interior contains every zero and pole of f. By the *Residue theorem*, we have

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{df}{f} = \sum_{p\in\mathbb{F}(4)} v_p(f).$$

Similar to Proposition 6.1,

(i) For the arc EA, we have

$$\frac{1}{2\pi i}\int_E^A \frac{df}{f} = -v_\infty(f).$$

(ii) For the arc CC', define $f_0(z) := (\sqrt{4}z)^{-k} f(\gamma_0 z)$, then we have $f_0(\gamma_0^{-1}z) = (\sqrt{4}z)^k f(z)$ and

$$\frac{df_0(\gamma_0^{-1}z)}{f_0(\gamma_0^{-1}z)} = \frac{df(z)}{f(z)} + k\frac{dz}{z}.$$

Thus

$$\frac{1}{2\pi i}\int_C^{C'} \frac{df(z)}{f(z)} = \frac{1}{2\pi i}\int_E^A \frac{df_0(z)}{f_0(z)} - \frac{1}{2\pi i}\int_C^{C'} k\frac{dz}{z}.$$

Now, when the arc CC' tend to 0, we have

$$\frac{1}{2\pi i}\int_C^{C'} \frac{df}{f} \to -v_0(f).$$

(iii) For the arcs BB' and DD', without loss of generallity, we can define arcs BB' and DD' so that it equals the image of EF and FA by the transformation $\gamma_{-1/2}$ and $T\gamma_{-1/2}T$, respectively. Define $f_{-1/2}(z) := (-2z+1)^{-k} f(\gamma_{-1/2}z)$, then we have $f_{-1/2}(\gamma_0^{-1}z) = (2z+1)^k f(z)$ and

$$\frac{df_{-1/2}(\gamma_{-1/2}^{-1}z)}{f_{-1/2}(\gamma_{-1/2}^{-1}z)} = \frac{df(z)}{f(z)} + k\frac{dz}{z+1/2}.$$

Here, $f_{-1/2}$ is holomorphic at ∞. On the Fourier expansion, we have $q|_{z=z+1} = e^{2\pi i(z+1)} = e^{2\pi i z} = q$. Thus we have $f_{-1/2}(Tz) = f_{-1/2}(z)$ for $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$.

Now, when the arcs BB' and DD' tend to 0, we have

$$\frac{1}{2\pi i}\int_B^{B'} \frac{df(z)}{f(z)} = \frac{1}{2\pi i}\int_F^E \frac{df_{-1/2}(z)}{f_{-1/2}(z)} - \frac{1}{2\pi i}\int_B^{B'} k\frac{dz}{z+1/2} \to -\frac{1}{2}v_{-1/2}(f).$$

$$\frac{1}{2\pi i}\int_D^{D'} \frac{df(z)}{f(z)} = \frac{1}{2\pi i}\int_{T^{-1}D}^{T^{-1}D'} \frac{df(z)}{f(z)} = \frac{1}{2\pi i}\int_{TF}^{TA} \frac{df_{-1/2}(z)}{f_{-1/2}(z)} - \frac{1}{2\pi i}\int_{T^{-1}D}^{T^{-1}D'} k\frac{dz}{z+1/2}$$
$$\to \frac{1}{2\pi i}\int_F^A \frac{df_{-1/2}(z)}{f_{-1/2}(z)} = -\frac{1}{2}v_{-1/2}(f).$$

Thus

$$-\frac{1}{2}v_{-1/2}(f) - \frac{1}{2}v_{-1/2}(f) = -v_{-1/2}(f).$$

(iv) For the arcs AB and $D'E$, since $f(Tz) = f(z)$,

$$\frac{1}{2\pi i}\int_A^B \frac{df}{f} + \frac{1}{2\pi i}\int_{D'}^E \frac{df}{f} = 0.$$

(v) For the arcs $B'C$ and $C'D$, since $f(S_4 z) = (4z+1)^k f(z)$ for $S_4 := \left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)$, we have

$$\frac{df(S_4z)}{f(S_4z)} = k\frac{dz}{z+1/4} + \frac{df(z)}{f(z)} \quad \text{and} \quad \frac{k}{2}.$$

□

B.2. **Valence formula for $\Gamma_0^*(4)$.** In order to decide the locating of zeros of Eisenstein series, we need the *valence formula for* $\Gamma_0^*(4)$:

Proposition B.2. *Let f be a modular function of weight k for $\Gamma_0^*(4)$, which is not identically zero. We have*

$$v_\infty(f) + v_{-1/2}(f) + \frac{1}{2}v_{i/2}(f) + \sum_{\substack{p\in\Gamma_0^*(4)\backslash\mathbb{H}\\ p\neq i/2}} v_p(f) = \frac{k}{4}, \tag{196}$$

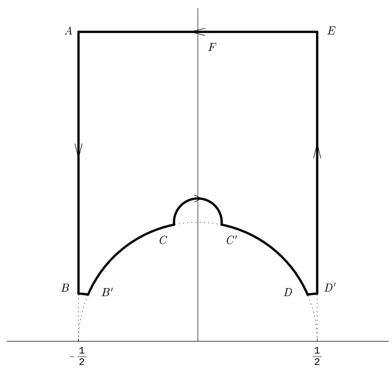


FIGURE 12

Proof. Let f be a nonzero modular function of weight k for $\Gamma_0^*(4)$, and let $\mathscr{C}$ be a contour of $\mathbb{F}^*(4)$ which is a fundamental domain of $\Gamma_0^*(4)$ represented in Figure 12, whose interior contains every zero and pole of f except for $i/2$. By the *Residue theorem*, we have

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{df}{f} = \sum_{\substack{p\in\mathbb{F}^*(4)\\ p\neq i/2}} v_p(f).$$

Similar to Proposition B.1,

(i) For the arc EA, we have

$$\frac{1}{2\pi i}\int_E^A \frac{df}{f} = -v_\infty(f).$$

(ii) For the arcs BB' and DD', define $f^*_{-1/2}(z) := (-\sqrt{2}z+\sqrt{2})^{-k} f(\gamma_{-1/2}z)$, then we have $f^*_{-1/2}(\gamma_0^{-1}z) = (\sqrt{2}z + 1/\sqrt{2})^k f(z)$ and

$$\frac{df^*_{-1/2}((\gamma^*_{-1/2})^{-1}z)}{f^*_{-1/2}((\gamma^*_{-1/2})^{-1}z)} = \frac{df(z)}{f(z)} + k\frac{dz}{z+1/2}.$$

Now, when the arcs BB' and DD' tend to 0, we have

$$\frac{1}{2\pi i}\int_B^{B'} \frac{df(z)}{f(z)} = \frac{1}{2\pi i}\int_F^E \frac{df^*_{-1/2}(z)}{f^*_{-1/2}(z)} - \frac{1}{2\pi i}\int_B^{B'} k\frac{dz}{z+1/2} \to -\frac{1}{2}v_{-1/2}(f).$$

$$\frac{1}{2\pi i}\int_D^{D'} \frac{df(z)}{f(z)} = \frac{1}{2\pi i}\int_{T^{-1}D}^{T^{-1}D'} \frac{df(z)}{f(z)} = \frac{1}{2\pi i}\int_{TF}^{TA} \frac{df^*_{-1/2}(z)}{f^*_{-1/2}(z)} - \frac{1}{2\pi i}\int_{T^{-1}D}^{T^{-1}D'} k\frac{dz}{z+1/2}$$
$$\to \frac{1}{2\pi i}\int_F^A \frac{df^*_{-1/2}(z)}{f^*_{-1/2}(z)} = -\frac{1}{2}v_{-1/2}(f).$$

Thus

$$-\frac{1}{2}v_{-1/2}(f) - \frac{1}{2}v_{-1/2}(f) = -v_{-1/2}(f).$$

(iii) For the arc CC', when the radius of the arc tends to 0, then we have

$$\frac{1}{2\pi i}\int_C^{C'} \frac{df}{f} \to -\frac{1}{2}v_{i/2}(f).$$

(iv) For the arcs AB and $D'E$, since $f(Tz) = f(z)$ for $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$,

$$\frac{1}{2\pi i}\int_A^B \frac{df}{f} + \frac{1}{2\pi i}\int_{D'}^E \frac{df}{f} = 0.$$

(v) For the arcs $B'C$ and $C'D$, since $f(W_4z) = (\sqrt{2}z)^k f(z)$, we have

$$\frac{df(W_4z)}{f(W_4z)} = k\frac{dz}{z} + \frac{df(z)}{f(z)} \quad \text{and} \quad \frac{k}{4}.$$

□

Let f be a modular form for $\Gamma_0(4)$ of weight k, and let $k \equiv 2 \pmod 4$. Then we have

$$f(i/2) = f(W_4\, i/2) = i^k f(i/2) = -f(i/2).$$

Thus $f(i/2) = 0$ and $v_{i/2}(f) \geqslant 1$.

Let k be an even integer such that $k \equiv 0 \pmod 4$. Then we have

$$E^{*\,\infty}_{k,4}(i/2) = \frac{1}{1-2^k}\left(E_k(i/2-1/2) - 2^k E_k(2(i/2-1/2))\right) = E^{\infty}_{k,2}(-1/2+i/2) \neq 0,$$
$$E^{*\,-1/2}_{k,4}(i/2) = \frac{1}{1-2^k}\left(E_k(2(i/2-1/2)) - E_k(i/2-1/2)\right) = 2^{-k} E^{0}_{k,2}(-1/2+i/2) \neq 0.$$

Thus $v_{i/2}(E^{*\,\infty}_{k,4}) = v_{i/2}(E^{*\,-1/2}_{k,4}) = 0$.

B.3. **Modlular forms of weight** 2. As a preliminaries, we have following:

$$\begin{aligned}
E_2(\gamma_0 z) &= 16z^2 E_2(4z) + (12/(2\pi i)) \cdot 4z,\\
E_2(2\cdot\gamma_0 z) &= 4z^2 E_2(2z) + (12/(2\pi i)) \cdot 2z,\\
E_2(4\cdot\gamma_0 z) &= z^2 E_2(z) + (12/(2\pi i)) \cdot z,\\
E_2(\gamma_{-1/2} z) &= E_2\left(-\frac{1}{2-1/z}\right) = (-2z+1)^2 E_2(z) + \frac{12}{2\pi i}\cdot 2(-2z+1),\\
E_2(2\cdot\gamma_{-1/2} z) &= E_2\left(-\frac{1}{1-1/2z}\right) = (-2z+1)^2 E_2(2z) + \frac{12}{2\pi i}\cdot (-2z+1),\\
E_2(4\cdot\gamma_{-1/2} z) &= E_2\left(-\frac{1}{1/2-1/4z}\right)\\
&= \frac{(-2z+1)^2}{4}(-E_2(z) + 6E_2(2z) - 4E_2(4z)) + \frac{12}{2\pi i}\cdot\frac{-2z+1}{2}.
\end{aligned}$$

Recall that

$$E_{2,2}{}'(z) = 2E_2(2z) - E_2(z) \tag{197}$$

be a modular form for $\Gamma_0(2)$. Since $\Gamma_0(4) \subset \Gamma_0(2)$, $E_{2,2}{}'$ satisfies transformation rule for $\Gamma_0(4)$ and is holomorphic in $\mathbb{H}$ and at ∞. In addition, we have

$$\begin{aligned}
E_{2,2}{}'(\gamma_0 z) &= -2(2z)^2 E_{2,2}{}'(2z),\\
E_{2,2}{}'(\gamma_{-1/2} z) &= (-2z+1)^2 E_{2,2}{}'(z).
\end{aligned}$$

Thus $E_{2,2}{}'$ is a modular form for $\Gamma_0(4)$, and we have

$$v_{-1/2+i/2}(E_{2,2}{}') = 1, \qquad v_p(E_{2,2}{}') = 0 \quad \text{for every } p \neq -1/2+i/2.$$

Similarly, we define

$$E_{2,4}{}'(z) := (4E_2(4z) - E_2(z))/3, \tag{198}$$

then we have

$$\begin{aligned}
E_{2,4}{}'(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) z) &= E_{2,4}{}'(z),\\
E_{2,4}{}'(\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right) z) &= (4z+1)^2 E_{2,4}{}'(z),\\
E_{2,4}{}'(\gamma_0 z) &= -(2z)^2 E_{2,4}{}'(z),\\
E_{2,4}{}'(\gamma_{-1/2} z) &= (-2z+1)^2 (E_{2,2}{}'(z) - E_{2,4}{}'(z)).
\end{aligned}$$

Thus $E_{2,4}{}'$ is a modular form for $\Gamma_0(4)$, and we have

$$v_{-1/2}(E_{2,4}{}') = 1, \qquad v_p(E_{2,4}{}') = 0 \quad \text{for every } p \neq -1/2.$$

For every $\alpha, \beta \in \mathbb{C}$, $\alpha E_{2,4}{}' + \beta E_{2,2}{}'$ is a modular form for $\Gamma_0(4)$. For example:

$$\begin{aligned}
v_\infty(-(E_{2,4}{}' - E_{2,2}{}')/16) &= 1, &\quad v_p(-(E_{2,4}{}' - E_{2,2}{}')/16) &= 0 \quad \text{for every } p \neq \infty,\\
v_0(2E_{2,4}{}' - E_{2,2}{}') &= 1, &\quad v_p(2E_{2,4}{}' - E_{2,2}{}') &= 0 \quad \text{for every } p \neq 0,\\
v_{-1/4+i/4}((3E_{2,4}{}' - E_{2,2}{}')/2) &= 1, &\quad v_p((3E_{2,4}{}' - E_{2,2}{}')/2) &= 0 \quad \text{for every } p \neq -1/4+i/4.
\end{aligned}$$

Furthermore, let f be a modular form for $\Gamma_0(4)$ of weight 2. If $v_\infty(f) = 1$, then f is a constant multiple of $-({E_{2,4}}' - {E_{2,2}}')/16$. Also, if $v_{-1/2}(f) = 1$, then f is a constant multiple of ${E_{2,4}}'$. Otherwise, $v_\infty(f - f(\infty){E_{2,4}}') = 1$, and $f - f(\infty){E_{2,4}}'$ is a constant multiple of $-({E_{2,4}}' - {E_{2,2}}')/16$. Thus every modular form for $\Gamma_0(4)$ of weight 2 is written as a linear combination of ${E_{2,2}}'$ and ${E_{2,4}}'$.

For $\Gamma_0^*(4)$, since

$$\alpha {E_{2,4}}'(W_2 z) + \beta {E_{2,2}}'(W_2 z) = (2z)^2\{(-\alpha - 3\beta){E_{2,4}}'(z) + \beta {E_{2,2}}'(z)\},$$

we need $2\alpha = -3\beta$. Now, we define

$$\text{(199)} \qquad {E_{2,4}}'^*(z) := 3{E_{2,4}}'(z) - 2{E_{2,2}}'(z).$$

We also have

$$\text{(200)} \qquad {E_{2,4}}'^*(z) = E_2(z) - 4E_2(2z) + 4E_2(4z) = 2E_2(2z) - E_2(z + 1/2).$$

Then

$$\begin{aligned} {E_{2,4}}'^*(\gamma_{-1/2}^* z) &= 2E_2\left(\frac{2z+1}{-2z+1}\right) - E_2\left(\frac{1}{-2z+1}\right) \\ &= 2E_2\left(-1 - \frac{1}{z - 1/2}\right) - E_2\left(-\frac{1}{2z-1}\right) \\ &= -(-\sqrt{2}z + 1/\sqrt{2})^2 {E_{2,4}}'^*(z). \end{aligned}$$

Thus ${E_{2,4}}'^*$ is a modular form for $\Gamma_0^*(4)$, and we have

$$v_{i/2}({E_{2,4}}'^*) = 1, \qquad v_p({E_{2,4}}'^*) = 0 \quad \text{for every } p \neq i/2.$$

Furthermore, let f be a modular form for $\Gamma_0^*(4)$ of weight 2. By the valence formula for $\Gamma_0^*(4)$, we have

$$v_{i/2}(f) = 1, \qquad v_p(f) = 0 \quad \text{for every } p \neq i/2.$$

Thus f is a constant multiple of ${E_{2,4}}'^*$.

B.4. **The space of modular forms for** $\Gamma_0(4)$**.** Let $M_{k,4}$ be the space of modular forms for $\Gamma_0(4)$ of weight k, and let $M_{k,4}^0$ be the space of cusp forms for $\Gamma_0(4)$ of weight k. When we consider the map $M_{k,4} \ni f \mapsto (f(\infty), f(0), f(-1/2)) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}$, the kernel of the map is $M_{k,4}^0$. So $\dim(M_{k,4}/M_{k,4}^0) \leqslant 3$, and $M_{k,4} = \mathbb{C}E_{k,4}^\infty \oplus \mathbb{C}E_{k,4}^0 \oplus \mathbb{C}E_{k,4}^{-1/2} \oplus M_{k,4}^0$. Define

$$\text{(201)} \qquad \Delta_4 := -\frac{1}{16}{E_{2,4}}'({E_{2,4}}' - {E_{2,2}}')(2{E_{2,4}}' - {E_{2,2}}').$$

We have following theorem:

Theorem B.1. *Let k be an even integer.*

1. *For $k < 0$, $M_{k,4} = 0$.*
2. *For $k = 0, 2,$ and 4, we have $M_{k,4}^0 = 0$. Furthemore, we have $M_{0,2} = \mathbb{C}$, $M_{2,2} = \mathbb{C}{E_{2,2}}' \oplus \mathbb{C}{E_{2,4}}'$, and $M_{4,4} = \mathbb{C}E_{4,4}^\infty \oplus \mathbb{C}E_{4,4}^0 \oplus \mathbb{C}E_{4,4}^{-1/2}$.*
3. *$M_{k,4}^0 = \Delta_4 M_{k-6,2}$.*

The proof of this theorem is similar to that of Theorem 6.1. The table of orders of zeros of basis for $M_{k,4}$ is following:

k	f	v_∞	v_0	$v_{-1/2}$	|other zeros|
2	${E_{2,2}}'$	0	0	0	1
	${E_{2,4}}'$	0	0	1	0
	${E_{2,4}}' - {E_{2,2}}'$	1	0	0	0
	$2{E_{2,4}}' - {E_{2,2}}'$	0	1	0	0
4	$E_{4,4}^{\infty}$	0	1	1	0
	$E_{4,4}^{0}$	1	0	1	0
	$E_{4,4}^{-1/2}$	1	1	0	0
6	$E_{6,4}^{\infty}$	0	1	1	0
	$E_{6,4}^{0}$	1	0	1	0
	$E_{6,4}^{-1/2}$	1	1	0	0
	Δ_4	1	1	1	0
8	Δ_2	1	2	1	0

Furthermore, we have $\dim(M_{k,3}) = k/2+1$ for $k \geqslant 0$, and $\dim(M_{k,3}^0) = k/2-2$ for $k \geqslant 6$. Similarly to $M_{k,2}$ and $M_{k,3}$, if $k \equiv 2 \pmod 6$, we have $M_{k,4} = M_{2,4}M_{k-2,4}$. On the other hand, if $k \equiv 0, 4 \pmod 6$, then let $n := \lfloor k/6 \rfloor$, and we have

$$\begin{aligned} M_{k,4} = {} & E_{k-6n,4}^{\infty}(\mathbb{C}(E_{6,4}^{\infty})^n \oplus \mathbb{C}(E_{6,4}^{\infty})^{n-1}\Delta_4 \oplus \cdots \oplus \mathbb{C}\Delta_4^n) \\ & \oplus E_{k-6n,4}^{0}(\mathbb{C}(E_{6,4}^{0})^n \oplus \mathbb{C}(E_{6,4}^{0})^{n-1}\Delta_4 \oplus \cdots \oplus \mathbb{C}\Delta_4^n) \\ & \oplus E_{k-6n,4}^{-1/2}(\mathbb{C}(E_{6,4}^{-1/2})^n \oplus \mathbb{C}(E_{6,4}^{-1/2})^{n-1}\Delta_4 \oplus \cdots \oplus \mathbb{C}\Delta_4^n). \end{aligned}$$

In addition we have followings:

$$\begin{aligned} E_{4,4}^{\infty} &= {E_{2,4}}'(2{E_{2,4}}' - {E_{2,2}}'), \\ E_{4,4}^{0} &= (-1/16){E_{2,4}}'({E_{2,4}}' - {E_{2,2}}'), \\ E_{4,4}^{-1/2} &= ({E_{2,4}}' - {E_{2,2}}')(2{E_{2,4}}' - {E_{2,2}}'). \\ E_{4,6}^{\infty} &= (1/2){E_{2,4}}'(2{E_{2,4}}' - {E_{2,2}}')(3{E_{2,4}}' - 2{E_{2,2}}'), \\ E_{4,6}^{0} &= 32{E_{2,4}}'{E_{2,2}}'(2{E_{2,4}}' - {E_{2,2}}'), \\ E_{4,6}^{-1/2} &= (-1/2)({E_{2,4}}' - {E_{2,2}}')(2{E_{2,4}}' - {E_{2,2}}')(3{E_{2,4}}' - 2{E_{2,2}}'). \end{aligned}$$

Then

Remark B.1. *Every modular form for $\Gamma_0(4)$ is generated by*

$$ {E_{2,2}}' \quad \text{and} \quad {E_{2,4}}'. $$

B.5. **The space of modular forms for $\Gamma_0^*(4)$.** Let $M_{k,4}^*$ be the space of modular forms for $\Gamma_0^*(4)$ of weight k, and let $M_{k,4}^{*0}$ be the space of cusp forms for $\Gamma_0^*(4)$ of weight k. When we consider the map $M_{k,4}^* \ni f \mapsto (f(\infty), f(-1/2)) \in \mathbb{C} \times \mathbb{C}$, the kernel of the map is $M_{k,4}^{*0}$. So $\dim(M_{k,4}^*/M_{k,4}^{*0}) \leqslant 2$, and $M_{k,4}^* = \mathbb{C}E_{k,4}^{*\,\infty} \oplus \mathbb{C}E_{k,4}^{*\,-1/2} \oplus M_{k,4}^{*0}$. Define

$$\Delta_4^* := -\frac{1}{16} E_{4,4}^{*\,\infty} E_{4,4}^{*\,-1/2}. \tag{202}$$

We have following theorem:

Theorem B.2. *Let k be an even integer.*

1. *For $k < 0$, $M_{k,4}^* = 0$.*
2. *For $k = 0, 2, 4$, and 6, we have $M_{k,4}^{*0} = 0$. Furthemore, we have $M_{0,2} = \mathbb{C}$, $M_{2,2} = \mathbb{C}{E_{2,4}}'^*$, and $M_{k,4}^* = \mathbb{C}E_{k,4}^{*\,\infty} \oplus \mathbb{C}E_{4,4}^{*\,-1/2}$ for $k = 4$ and 6.*
3. *$M_{k,4}^{*0} = \Delta_4^* M_{k-8,2}$.*

The proof of this theorem is similar to that of Theorem B.1. The table of orders of zeros of basis for $M_{k,4}^*$ is following:

k	f	v_∞	$v_{-1/2}$	$v_{i/2}$	$\lvert$other zeros$\rvert$
2	${E_{2,4}}'^*$	0	0	1	0
4	$E_{4,4}^{*\,\infty}$	0	1	0	0
	$E_{4,4}^{*\,-1/2}$	1	0	0	0
6	$E_{6,4}^{*\,\infty}$	0	1	1	0
	$E_{6,4}^{*\,-1/2}$	1	0	1	0
8	$E_{8,4}^{*\,\infty}$	0	1	0	1
	$E_{8,4}^{*\,-1/2}$	1	0	0	1
	Δ_4^*	1	1	0	0

Furthermore, we have $\dim(M_{k,3}) = \lfloor k/4 \rfloor + 1$ for $k \geqslant 0$, and $\dim(M_{k,3}^{*0}) = \lfloor k/4 \rfloor - 1$ for $k \geqslant 8$. Similarly to $M_{k,4}$, if $k \equiv 2 \pmod 8$, we have $M_{k,4}^* = {E_{2,4}}'^* M_{k-2,4}^*$. On the other hand, if $k \not\equiv 2 \pmod 8$, then let $n := \lfloor k/8 \rfloor$, and we have

$$\begin{aligned} M_{k,4}^* = & E_{k-8n,4}^{*\,\infty}(\mathbb{C}(E_{8,4}^{*\,\infty})^n \oplus \mathbb{C}(E_{8,4}^{*\,\infty})^{n-1}\Delta_4^* \oplus \cdots \oplus \mathbb{C}(\Delta_4^*)^n) \\ & \oplus E_{k-8n,4}^{*\,-1/2}(\mathbb{C}(E_{8,4}^{*\,-1/2})^n \oplus \mathbb{C}(E_{8,4}^{*\,-1/2})^{n-1}\Delta_4^* \oplus \cdots \oplus \mathbb{C}(\Delta_4^*)^n). \end{aligned}$$

Thus, the next proposision follows:

Proposition B.3. *Let $k \geqslant 4$ be an even integer. For every $f \in M_{k,4}^*$, we have*

$$v_{i/2}(f) \geqslant t_k \quad (t_k = 0, 1 \text{ such that } 2t_k \equiv k \pmod 4). \tag{203}$$

In addition we have $E_{4,4}^{*\,-1/2} = (1/4)(({E_{2,4}}'^*)^2 - E_{4,4}^{*\,\infty})$, $E_{4,6}^{*\,\infty} = {E_{2,4}}'^* E_{4,4}^{*\,\infty}$, $E_{4,6}^{*\,-1/2} = (-1/2){E_{2,4}}'^* E_{4,4}^{*\,-1/2}$. Then

Remark B.2. *Every modular form for $\Gamma_0(4)$ is generated by*

$$ {E_{2,4}}'^* \quad \text{and} \quad E_{4,4}^{*\,\infty}.$$

Appendix C. Locating the zeros of Eisenstein Series

C.1. **On $\Gamma_0(4)$.** Define

$$A_4 := \{z \,;\, |z + 1/4| = 1/4,\ 0 < Arg(z) < \pi\}, \tag{204}$$

$$A_4^0 := \{z \,;\, Re(z) = -1/2,\ Im(z) > 0\}, \tag{205}$$

$$A_4^{-1/2} := \{z \,;\, Re(z) = 0,\ Im(z) > 0\}. \tag{206}$$

Then we have following:

Conjecture C.1. *Let $k \geqslant 4$ be an even integer. $E_{k,4}^\infty$ has $k/2 - 1$ zeros in A_4, $E_{k,4}^0$ has $k/2 - 1$ zeros in A_4^0, and $E_{k,4}^{-1/2}$ has $k/2 - 1$ zeros in $A_4^{-1/2}$.*

Now, we have the following transformations:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} (\gamma_0)^{-1} : \frac{e^{i\theta} - 1}{4} \mapsto -\frac{1}{2} + \frac{\tan(\theta/2)}{2} i, \tag{207}$$

$$(\gamma_{-1/2})^{-1} : \frac{e^{i\theta} - 1}{4} \mapsto \frac{\tan(\theta/2)}{2} i. \tag{208}$$

This transform A_4 to A_4^0 and $A_4^{-1/2}$, respectively. Moreover, for every $z \in A_4$,

$$\begin{aligned} E_{k,4}^\infty\left(\gamma_0 \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) z\right) &= (z + 1)^k E_{k,4}^0(z), \\ E_{k,4}^\infty\left(\gamma_{-1/2}\, z\right) &= (-2z + 1)^k E_{k,4}^{-1/2}(z). \end{aligned}$$

Then

Remark C.1. *The number of the zeros of $E_{k,4}^\infty$ in A_4 equals to that of $E_{k,4}^0$ in A_4^0 and that of $E_{k,4}^{-1/2}$ in $A_4^{-1/2}$.*

Now, recall that Remark A.1, where we have $E_{k,4}^0 = E_{k,2}^0$. Moreover, we have $A_2^0 = \{z\,;\,Re(z) = -1/2, Im(z) > 1/2\}$ and write $A_2^{0\prime} := A_4^0 \setminus A_2^0 \cup \{-1/2+i/2\}$. Then we have the following transformation:

$$\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} : -\frac{1}{2} + \frac{y}{2}i \mapsto -\frac{1}{2} + \frac{1}{2y}i. \tag{209}$$

This transform A_2^0 to $A_2^{0\prime}$. Moreover,

$$E_{k,4}^0\left(\left(\begin{smallmatrix} -1 & -1 \\ 2 & 1 \end{smallmatrix}\right) W_2\, z\right) = (2z+1)^k E_{k,4}^0(z) \quad \text{for every } z \in A_2^0.$$

Then

Remark C.2. *If $E_{k,2}^0 = E_{k,4}^0$ has $\lfloor k/4 \rfloor - 1$ zeros in A_2^0, then $E_{k,4}^0$ has also $\lfloor k/4 \rfloor - 1$ zeros in $A_2^{0\prime}$.*

In addition, recall that $v_{-1/2+i/2}(E_{k,2}^0) \geqslant 1$ for $k \equiv 2 \pmod 4$.
In conclusion, we have following:

Remark C.3. *If Conjecture 6.1 is proved, then we can prove Conjecture C.1.*

C.2. **On $\Gamma_0^*(4)$.** Define

$$A_4^* := \{z\,;\,|z| = 1/2,\ \pi/2 < Arg(z) < \pi\}, \tag{210}$$

$$A_4^{*\,-1/2} := \{z\,;\,Re(z) = 0,\ Im(z) > 1/2\}. \tag{211}$$

Then we have following:

Conjecture C.2. *Let $k \geqslant 4$ be an even integer. $E_{k,4}^{*\,\infty}$ has $\lfloor k/4 \rfloor - 1$ zeros in A_4^*, and $E_{k,4}^{*\,-1/2}$ has $\lfloor k/4 \rfloor - 1$ zeros in $A_4^{*\,-1/2}$. Furthermore, in Proposition B.3, the equality hold if f is equal to $E_{k,4}^{*\,\infty}$ or $E_{k,4}^{*\,-1/2}$.*

Now, we have the following transformation:

$$(\gamma_{-1/2}^*)^{-1} : \frac{e^{i\theta}}{2} \mapsto \frac{\tan(\theta/2)}{2} i. \tag{212}$$

This transform A_4^* to $A_4^{*\,-1/2}$. Moreover,

$$E_{k,4}^{*\,\infty}(\gamma_{-1/2}^*\, z) = (-2z+1)^k E_{k,4}^{*\,-1/2}(z) \quad \text{for every } z \in A_4^{*\,-1/2}.$$

Then

Remark C.4. *$E_{k,4}^{*\,\infty}$ has $\lfloor k/4 \rfloor - 1$ zeros in A_4^*, if and only if $E_{k,4}^{*\,-1/2}$ has $\lfloor k/4 \rfloor - 1$ zeros in $A_4^{*\,-1/2}$.*

Now, recall that Remark A.2, where we have $E_{k,4}^{*\,-1/2} = E_{k,4}^{-1/2}$. Moreover, we write $A_4^{*\,-1/2\prime} := A_4^{-1/2} \setminus A_4^{*\,-1/2} \cup \{i/2\}$. Then we have the following transformation:

$$W_4 : \frac{y}{2} i \mapsto \frac{1}{2y} i. \tag{213}$$

This transform $A_4^{*\,-1/2}$ to $A_4^{*\,-1/2\prime}$. Moreover,

$$E_{k,4}^{-1/2}(W_4\, z) = (2z)^k E_{k,4}^{-1/2}(z) \quad \text{for every } z \in A_4^{*\,-1/2}.$$

Then

Remark C.5. *If $E_{k,4}^{*\,-1/2} = E_{k,4}^{-1/2}$ has $\lfloor k/4 \rfloor - 1$ zeros in $A_4^{*\,-1/2}$, then $E_{k,4}^{-1/2}$ has also $\lfloor k/4 \rfloor - 1$ zeros in $A_4^{*\,-1/2\prime}$.*

In addition, note that $v_{i/2}(E_{k,4}^{-1/2}) \geqslant 1$ for $k \equiv 2 \pmod 4$. Furthermore, recall Remark C.3.
In conclusion, we have following:

Remark C.6. *If Conjecture 6.1 is proved, then we can prove Conjecture C.2.*

Remark C.7. *Note that $E_{k,4}^{*\,\infty}(z \pm 1/2) = E_{k,2}^{\infty}(z)$ and $E_{k,4}^{*\,-1/2}(z \pm 1/2) = E_{k,2}^0(z)$. The Remark C.6 is natural result.*

Appendix D. Another consideration : Isomorphism

Let M_2^{pr} be the space of modular forms for $\Gamma(2)$, and M_4 be that for $\Gamma_0(4)$. By collorary 2.1.1, we have

$$\Gamma(2) = \langle -I,\ \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) \rangle, \quad \Gamma_0(4) := \langle -I,\ \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right) \rangle.$$

Define $V_2 := \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$. Then

$$V_2\, \Gamma_0(4)\, V_2^{-1} = \Gamma(2).$$

In addition, by the map $\varphi : M_2^{\mathrm{pr}} \ni f(z) \mapsto f(2z) \in M_4$, we have following:

Theorem D.1. *M_2^{pr} is isomorphic to M_4.*

Proof. For every $f \in M_4$ and every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2)$, we put $\gamma' := V_2 \gamma V_2^{-1} = \begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix}$, then

$$f(2 \cdot \gamma z) = f(2(V_2^{-1} \gamma' V_2 z)) = f(\gamma'(2z)) = (cz + d)^k f(2z).$$

□

Similarly, let M_2 be the space of modular forms for $\Gamma_0(2)$, and M_4^* be that for $\Gamma_0^*(4)$. By collorary 2.1.1, we have

$$\Gamma_0(2) = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) \rangle, \quad \Gamma_0^*(4) = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 0 & -1/2 \\ 2 & 0 \end{smallmatrix}\right) \rangle.$$

Thus

$$\left(\begin{smallmatrix} 1 & -1/2 \\ 0 & 1 \end{smallmatrix}\right)\ \Gamma_0^*(4)\ \left(\begin{smallmatrix} 1 & 1/2 \\ 0 & 1 \end{smallmatrix}\right) = \Gamma_0(2).$$

In addition, by the map $\varphi : M_2 \ni f(z) \mapsto f(z + 1/2) \in M_4^*$, we have following:

Theorem D.2. *M_2 is isomorphic to M_4^*.*

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References

[G] J. Getz, *A generalization of a theorem of Rankin and Swinnerton-Dyer on zeros of modular forms.*, Proc. Amer. Math. Soc., 132(2004), No. 8, 2221-2231.

[KO] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Graduate Texts in Mathematics, No. 97, Springer-Verlag, New York, 1984.

[KR] A. Krieg, *Modular Forms on the Fricke Group.*, Abh. Math. Sem. Univ. Hamburg, 65(1995), 293-299.

[Q] H. -G. Quebbemann, *Atkin-Lehner eigenforms and strongly modular lattices*, Enseign. Math. (2), 43(1997), No. 1-2, 55-65.

[RD] H. Rademacher, *Über die Erzeugenden von Kongruenzuntergruppen der Modulgruppe*, Abh. Math. Sem. Univ. Hamburg, 7(1929), 134-148.

[RSD] F. K. C. Rankin, H. P. F. Swinnerton-Dyer, *On the zeros of Eisenstein Series*, Bull. London Math. Soc., 2(1970), 169-170.

[SE] J. -P. Serre, *A Corse in Arithmetic*, Graduate Texts in Mathematics, No. 7, Springer-Verlag, New York-Heidelberg, 1973. (Translation of *Cours d'arithmétique* (*French*), Presses Univ. France, Paris, 1970.)

[SI] H. Shimizu, *Hokei kansu. I-III.* (*Japanese*) [*Automorphic functions. I-III*], Iwanami Shoten Kiso Sugaku [Iwanami Lectures on Fundamental Mathematics] 8, Iwanami Shoten Publishers, Tokyo, 1977-1978.

[SU] G. Shimura, *On Eisenstein Series*, Duke Math. J., 50(1983), No. 2, 417-476.