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On the Rigidity of Spherical t-Designs that are Orbits of Reflection Groups E_8 and H_4

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Abstract

The concept of rigid spherical t-designs was introduced by Eiichi Bannai. We want to find examples of rigid but not tight spherical designs. Sali investigated the case when X is an orbit of a finite reflection group and proved that X is rigid if and only if tight for the groups A_n, B_n, C_n, D_n, E_6, E_7, F_4, H_3. There are two cases left open, namely the group E_8 and the isometry group H_4 of the four dimensional regular polytope, the 600-cell. In this paper, we study the rigidity of spherical t-designs X that are orbits of a finite reflection groups E_8 and H_4, and prove that X is rigid if and only if tight or the 600-cell.

1 Introduction

Spherical t-designs were introduced by Delsarte, Goethals and Seidel [10]. A finite nonempty set X in the unit sphere

$$\mathbb{S}^d := \{x = (x_1, x_2, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} \mid x_1^2 + x_2^2 + \cdots + x_{d+1}^2 = 1\}$$

is called a *spherical t-design* in $\mathbb{S}^d$ if and only if the equality

$$\frac{1}{|\mathbb{S}^d|} \int_{\mathbb{S}^d} f(x) d\omega(x) = \frac{1}{|X|} \sum_{x \in X} f(x)$$

holds for all polynomials $f(x) = f(x_1, x_2, \ldots, x_{d+1})$ of degree at most t. Here, the left-hand side involves integration on the unit sphere, and $|\mathbb{S}^d|$ denotes the volume of the sphere $\mathbb{S}^d$.

It is known [10] that there is a lower bound (Fischer-type inequality) for the size of a spherical t-design in $\mathbb{S}^d$.

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Theorem 1.1 (Delsarte-Goethals-Seidel). *Let X be a spherical t-design in $\mathbb{S}^d$. Then*

$$|X| \geq \begin{cases} \binom{d+t/2}{d} + \binom{d+t/2-1}{d}, & \text{if } t \text{ is even} \\ 2\binom{d+(t-1)/2}{d}, & \text{if } t \text{ is odd} \end{cases}$$

If equality holds, then X is called *tight* spherical t-design.

The concept of the rigidity was introduced by Bannai [1]. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a spherical t-design in $\mathbb{S}^d$. X is said to be *non-rigid* or *deformable*, if for any given $\epsilon > 0$ there exist another spherical t-design $X' = \{x'_1, x'_2, \ldots, x'_n\}$ such that $|x_i - x'_i| < \epsilon$ for $1 \leq i \leq n$, and there exists no orthogonal transformation $g \in O(d+1)$ with $g(x_i) = x'_i$. X is said to be *rigid* if it is not non-rigid.

If X, X_1 and X_2 are spherical t-designs in $\mathbb{S}^d$, then the following hold.
(1) For any $\sigma \in O(d+1)$, $X^\sigma := \{x^\sigma \mid x \in X\}$ is spherical t-design in $\mathbb{S}^d$.
(2) If $X_1 \cap X_2 = \emptyset$, then $X_1 \cup X_2$ is spherical t-design in $\mathbb{S}^d$.

The property (2) means that we can make many spherical t-designs from given spherical t-designs. However spherical t-designs, that are disjoint union of spherical t-designs, are not "new" spherical t-designs. Such spherical t-designs is clearly non-rigid. Therefore rigid spherical t-designs are essential objects of study of spherical t-designs.

Bannai conjectured the following two propositions about rigid spherical t-design.

Conjecture 1.1 (Bannai, [1]). *There exist a function $f(d,t)$ such that if X is a spherical t-design in $\mathbb{S}^d$ such that $|X| > f(d,t)$, then X is non-rigid.*

Conjecture 1.2 (Bannai, [1]). *For each fixed pair d and t, there are only finitely many rigid spherical t-design in $\mathbb{S}^d$ up to orthogonal transformations.*

Lyubich and Vaserstein proved that Conjecture 1.1 and 1.2 are equivalent [12]. These conjecture are supported by the fact that the known rigid t-designs are very rare. Bannai proves this for dimension 1, by showing that any rigid spherical t-design X in $\mathbb{S}^1$ consists of the vertices of a regular $(k+1)$-gon with $t \leq k \leq 2t$.

Because the distances between points of a tight spherical design are described by a theorem of Delsarte-Goethals-Seidel [10], we have the following proposition.

Proposition 1.1. *A tight spherical t-design is rigid.*

Unfortunately, tight spherical t-designs rarely exist [5], and it was proved that if a tight spherical t-design in $\mathbb{S}^d$ with $d \geq 2$ exists, then necessarily either $t \leq 5$, or $t = 7, 11$ [3, 4]. We want to find examples of rigid but not tight spherical t-designs.

The following theorem, which was proved by Delsarte-Goethals-Seidel, is very useful for getting examples of spherical t-designs.

Theorem 1.2 (Delsarte-Goethals-Seidel). *For a finite subgroup G of $O(d+1)$ the following conditions are equivalent:*

1. *every G-orbit is a spherical t-design in $\mathbb{S}^d$,*
2. *there are no G-invariant harmonic polynomials of degree $1, 2, \ldots, t$.*

Let q_i be the dimension of the space of G-invariant harmonic polynomials of degree i. If we know the eigenvalue of each $g \in G$, then we determine t by the harmonic Molien series

$$\sum_{i=0}^{\infty} q_i \lambda^i = \frac{1}{|G|} \sum_{g \in G} \frac{1-\lambda^2}{\det (I_{d+1} - \lambda g)}$$

where I_{d+1} is the $(d+1) \times (d+1)$ identity matrix [14, 11, Corollary 6.4].

Let W be a finite irreducible reflection group in $\mathbb{R}^{d+1}$. It is known that finite irreducible reflection groups are classified completely [6]. Let integers $1 = m_1 \leq m_2 \leq \cdots \leq m_{d+1}$ be the exponents of W (please see [6, Ch.V, §6]). The exponents of W is important for the following theorem [7, Ch.VIII, §8, Corollary 1].

Theorem 1.3. *Let W be a finite reflection group. Let q_i be the dimension of the space of W-invariant harmonic polynomials of degree i. Then we have*

$$\sum_{i=0}^{\infty} q_i \lambda^i = \prod_{i=2}^{d+1} \frac{1}{1-\lambda^{1+m_i}}.$$

Therefore every orbit $X = \{x^w \mid w \in W\}$ is a spherical m_2-design in $\mathbb{S}^d$.

If $\alpha_1, \alpha_2, \ldots, \alpha_{d+1}$ are the fundamental roots, then the **corner vectors** $v_1, v_2, \ldots, v_{d+1}$ are defined by $v_i \perp \alpha_j$ if and only if $i \neq j$. The following proposition is immediate.

Proposition 1.2 (Sali, [13, Proposition 1.13])**.** *If $X = \{x^w \mid w \in W\}$ is such that x is not a corner vector of W, then X is non-rigid spherical m_2-design.*

The following lemma is useful for proving the non-rigidity.

Lemma 1.1 (Sali, [13, Lemma 2.3])**.** *Suppose that $X \subset \mathbb{S}^d$ is a spherical t-design. Let $Y \subset X$ satisfy $Y \subset U^r \cup \mathbb{S}^d$ where U^r is an r-dimensional affine subspace of $\mathbb{R}^{d+1}(1 < r \leq d+1)$. That is, $U^r = \{z_0 + x \mid x \in T^r\}$ where T^r is a linear subspace of $\mathbb{R}^{d+1}$. Furthermore, let us assume that*

$$\tilde{Y} = \left\{ \frac{y - z_0}{|y - z_0|} \;\middle|\; y \in Y \right\}$$

forms a t-design in $\mathbb{S}^{r-1}$. If $X \setminus Y$ spans $\mathbb{R}^{d+1}$, then X is non-rigid.

Sali proved the following theorem by finding sub-t-designs in affine subspaces.

Theorem 1.4 (Sali, [13, Theorem 1.4])**.** *Let W be any of the following reflection groups.*

1. A_n *for* $n = 3, 4 \ldots$
2. B_n *for* $n = 3, 4, \ldots$
3. C_n *for* $n = 3, 4, \ldots$
4. D_n *for* $n = 4, 5, \ldots$
5. E_6, E_7, F_4, H_3

Then the orbit $X = \{x_0^w \mid w \in W\}$ for a corner vector x_0 is a rigid spherical m_2-design if and only if it is tight.

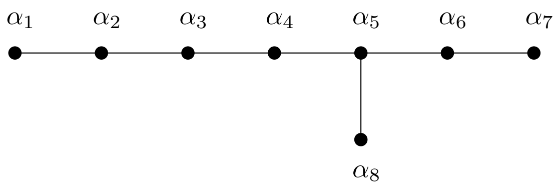
There were two cases left open, namely the group E_8 and the isometry group H_4 of the four dimensional regular polytope, the 600-cell. In this paper, we investigate the case of the group E_8 and H_4, and prove the following theorems.

Theorem 1.5. *Let $W(E_8)$ be the reflection groups of E_8. Then the orbit $X = \{x_0^w \mid w \in W(E_8)\}$ for a corner vector x_0 is a rigid spherical 7-design if and only if it is tight (i.e. $x_0 = v_1$).*

Theorem 1.6. *Let $W(H_4)$ be the reflection group H_4. Then the orbit $X = \{x_0^w \mid w \in W(H_4)\}$ for a corner vector x_0 is a rigid spherical 11-design if and only if it is the 600-cell (i.e. $x_0 = v_1$).*

2 Group E_8

Space $\mathbb{R}^8$
Dynkin diagram



Exponents $1, 7, 11, 13, 17, 19, 23, 29$.
Reflection Group The order is $2^{14}\, 3^5\, 5^2\, 7$.

Fundamental Roots	**Corner vector**
$\alpha_1 = [-2, 2, 0, 0, 0, 0, 0, 0]$	$v_1 = [-1, 1, 1, 1, 1, 1, 1, 1]$
$\alpha_2 = [0, -2, 2, 0, 0, 0, 0, 0]$	$v_2 = [0, 0, 1, 1, 1, 1, 1, 1]$
$\alpha_3 = [0, 0, -2, 2, 0, 0, 0, 0]$	$v_3 = [1, 1, 1, 3, 3, 3, 3, 3]$
$\alpha_4 = [0, 0, 0, -2, 2, 0, 0, 0]$	$v_4 = [1, 1, 1, 1, 2, 2, 2, 2]$
$\alpha_5 = [0, 0, 0, 0, -2, 2, 0, 0]$	$v_5 = [3, 3, 3, 3, 3, 5, 5, 5]$
$\alpha_6 = [0, 0, 0, 0, 0, -2, 2, 0]$	$v_6 = [1, 1, 1, 1, 1, 1, 2, 2]$
$\alpha_7 = [0, 0, 0, 0, 0, 0, -2, 2]$	$v_7 = [1, 1, 1, 1, 1, 1, 1, 3]$
$\alpha_8 = [1, 1, 1, 1, 1, -1, -1, -1]$	$v_8 = [1, 1, 1, 1, 1, 1, 1, 1]$

By computer search, using GAP, we get the orbits of v_i for $i = 1, 2, \ldots, 8$ as following.

	Cardinality	Vectors
v_1	240	$2^2\,0^6$, D 1^8
v_2	6720	$2\,1^2 0^5$, $1^6\,0^2$, D $(3/2)^2\,(1/2)^6$
v_3	60480	$6\,2^3\,0^4$, $4^3\,0^5$, $4^2\,2^4\,0^2$, E $3^5\,1^3$, D $5\,3^2\,1^5$
v_4	241920	$4\,1^4\,0^3$, $2^5\,0^3$, $1^3\,2^2\,3\,0^2$,
		E $2^4\,1^4$, E $(5/2)^3\,(1/2)^5$, E $(5/2)^2\,(3/2)^3\,(1/2)^3$, D $(7/2)\,(3/2)^3\,(1/2)^4$
v_5	483840	$(10)\,2^5\,0^2$, $6^2\,4^3\,0^3$, $6^3\,2^3\,0^2$, $8\,4^3\,2^2\,0^2$,
		E $5^3\,3^5$, E $6^2\,4^2\,2^4$, E $7\,5^2\,3^2\,1^3$, D $9\,3^4\,1^3$
v_6	69120	$3\,1^5\,0^2$, $2^3\,1^2\,0^3$, E $2^2\,1^6$, E $(7/2)\,(1/2)^7$, E $(5/2)\,(3/2)^3\,(1/2)^4$
v_7	2160	$4\,0^7$, $2^4\,0^4$, E $3\,1^7$
v_8	17280	$2\,1^4\,0^3$, E 1^8, E $(3/2)^3\,(1/2)^5$, D $(5/2)\,(1/2)^7$

The full list of vectors is obtained by applying arbitrary permutations and signs to the vectors in the table, except that if the vector is prefixed by an E (resp. D) then an even (resp. odd) number of minus signs are required.

These orbits are spherical 7-designs in $\mathbb{S}^7$ because the exponent $m_2 = 7$. By Fischer-type inequality, a spherical 7-designs in $\mathbb{S}^7$ has at least 240 points. Therefore the orbit of v_1, which is the E_8 root system, is tight 7-design in $\mathbb{S}^7$. We shall find the subset Y in Lemma 1.1 to prove that other orbits are non-rigid. Indeed, the orbit of v_i for $i = 2, 3, \dots, 8$ contains the E_8 root system which is tight 7-design in $\mathbb{S}^7$. The E_8 root system, which contained in the orbit, has the following fundamental roots.

The orbit of v_2

$\alpha_1 = [-2, -1, -1, 0, 0, 0, 0, 0]$
$\alpha_2 = [0, 1, 2, -1, 0, 0, 0, 0]$
$\alpha_3 = [0, 1, -1, 2, 0, 0, 0, 0]$
$\alpha_4 = [1/2, -3/2, 1/2, -1/2, -3/2, -1/2, -1/2, 1/2]$
$\alpha_5 = [0, 0, 0, 0, 1, 0, 1, -2]$
$\alpha_6 = [0, 0, 0, 0, 0, 1, 1, 2]$
$\alpha_7 = [0, 0, 0, 0, 0, 1, -2, -1]$
$\alpha_8 = [0, 1, -1, -1, 0, -1, -1, 1]$

The orbit of v_3

$\alpha_1 = [-6, -2, -2, -2, 0, 0, 0, 0]$
$\alpha_2 = [0, 4, 4, 4, 0, 0, 0, 0]$
$\alpha_3 = [2, -6, -2, 2, 0, 0, 0, 0]$
$\alpha_4 = [0, 2, 2, -4, -4, -2, -2, 0]$
$\alpha_5 = [0, 0, 0, 0, 2, 2, 6, -2]$
$\alpha_6 = [0, 0, 0, 0, 0, 4, -4, 4]$
$\alpha_7 = [0, 0, 0, 0, 2, -6, 2, 2]$
$\alpha_8 = [0, 2, -4, 2, 0, -2, -4, -2]$

The orbit of v_4

$\alpha_1 = [-4, -1, -1, -1, -1, 0, 0, 0]$
$\alpha_2 = [1/2, 3/2, 5/2, 5/2, 3/2, -3/2, -1/2, 1/2]$
$\alpha_3 = [1/2, -3/2, -3/2, -1/2, 3/2, 5/2, 1/2, -5/2]$
$\alpha_4 = [0, 1, 0, 0, -1, 1, 1, 4]$
$\alpha_5 = [0, 1, 1, -3, 1, -2, 0, -2]$
$\alpha_6 = [0, 0, 1, 2, -3, 2, -1, -1]$
$\alpha_7 = [0, 1, -3, 0, 2, -1, -2, 1]$
$\alpha_8 = [1/2, -5/2, -3/2, 5/2, -1/2, -3/2, 3/2, 1/2]$

The orbit of v_5

$\alpha_1 = [-10, -2, -2, -2, -2, -2, 0, 0]$
$\alpha_2 = [2, 2, 2, 6, 6, 4, -4, -2]$
$\alpha_3 = [0, 0, 4, -4, -2, 2, 8, 4]$
$\alpha_4 = [1, 1, -1, 5, -3, -7, -5, 3]$
$\alpha_5 = [0, 0, 2, -2, -2, 2, 2, -10]$
$\alpha_6 = [0, 2, -8, 2, 4, 0, 4, 4]$
$\alpha_7 = [0, 0, 2, -2, -6, 6, -6, 2]$
$\alpha_8 = [1, -7, 1, -3, 5, -1, -3, 5]$

The orbit of v_6
$\alpha_1 = [-7/2, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2]$
$\alpha_2 = [1, 0, 0, 0, 1, 2, 2, 2]$
$\alpha_3 = [0, 0, 1, 2, 1, -2, -2, 0]$
$\alpha_4 = [0, 0, 1, 0, -2, 1, 2, -2]$
$\alpha_5 = [0, 1, 0, -2, 2, 1, -2, 0]$
$\alpha_6 = [0, 1, -2, 1, 0, -2, 2, 0]$
$\alpha_7 = [0, 0, 0, 1, -2, 2, -2, 1]$
$\alpha_8 = [1/2, -5/2, 1/2, -1/2, -3/2, -3/2, 1/2, 3/2]$

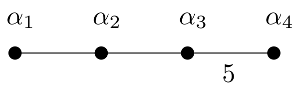
The orbit of v_7
$\alpha_1 = [-4, 0, 0, 0, 0, 0, 0, 0]$
$\alpha_2 = [2, -2, -2, -2, 0, 0, 0, 0]$
$\alpha_3 = [0, 0, 0, 4, 0, 0, 0, 0]$
$\alpha_4 = [0, 0, 2, -2, -2, -2, 0, 0]$
$\alpha_5 = [0, 0, 0, 0, 0, 4, 0, 0]$
$\alpha_6 = [0, 0, 0, 0, 2, -2, -2, -2]$
$\alpha_7 = [0, 0, 0, 0, 0, 0, 0, 4]$
$\alpha_8 = [0, 2, -2, 0, 0, -2, 2, 0]$

The orbit of v_8
$\alpha_1 = [-5/2, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2, 1/2]$
$\alpha_2 = [1/2, -1/2, 3/2, 1/2, 1/2, 1/2, 3/2, -3/2]$
$\alpha_3 = [0, 1, -2, 0, 1, 1, -1, 0]$
$\alpha_4 = [0, 0, 1, 1, -1, 0, 1, 2]$
$\alpha_5 = [0, 0, 1, 0, 1, -1, -2, -1]$
$\alpha_6 = [0, 1, -1, -1, 0, -1, 2, 0]$
$\alpha_7 = [0, 0, 0, 1, -2, 1, -1, -1]$
$\alpha_8 = [1/2, -3/2, -1/2, -3/2, -1/2, 3/2, 1/2, 1/2]$

It is well known that the E_8 root system is linear combination of the fundamental roots with integer coefficients all of the same sign (all non-negative or all non-positive). Moreover, by seeing [6, PLATE VII], we easily get the E_8 root system which is contained in the orbit. Therefore the orbits of the group E_8 are non-rigid spherical 7-designs except the E_8 root system.

3 Group H_4

This group is the isometry group of the 600-cell acting on $\mathbb{R}^4$.
Space $\mathbb{R}^4$
Dynkin diagram



Exponents $1, 11, 19, 29$
Reflection Group The order is 14400.

Fundamental roots	**Corner vectors**
$\alpha_1 = [-2, 2, 0, 0]$	$v_1 = [3 - \sqrt{5}, 1 + \sqrt{5}, 1 + \sqrt{5}, -1 - \sqrt{5}]$
$\alpha_2 = [0, -2, 2, 0]$	$v_2 = [2 - 2\sqrt{5}, 2 - 2\sqrt{5}, -4, 4]$
$\alpha_3 = [0, 0, -2, -2]$	$v_3 = [10, 10, 10, -6\sqrt{5}]$
$\alpha_4 = [1, 1, 1, \sqrt{5}]$	$v_4 = [4, 4, 4, -4]$

We get the orbits of v_i for $i = 1, 2, 3, 4$ as following.

	Cardinality	Vectors
v_1	120	$4^2\,0^2$, $\mathrm{E}\,2^3\,(2\sqrt{5})$, $\mathrm{D}\,(1+\sqrt{5})^3\,(3-\sqrt{5})$
		$\mathrm{D}\,(-1+\sqrt{5})^3\,(3+\sqrt{5})$
v_2	720	$4^2\,(-2+2\sqrt{5})^2$, $\mathrm{D}\,2\,(2\sqrt{5})^2\,(-4+2\sqrt{5})$, $\mathrm{E}\,2^2\,6\,(-4+2\sqrt{5})$
		$0^2\,(6-2\sqrt{5})\,(2+2\sqrt{5})$, $\mathrm{E}\,(7-\sqrt{5})\,(3-\sqrt{5})\,(1+\sqrt{5})^2$
		$\mathrm{D}\,(5-\sqrt{5})^2\,(3+\sqrt{5})\,(-1+\sqrt{5})$, $\mathrm{E}\,(-1+\sqrt{5})^2\,(3+\sqrt{5})\,(-3+3\sqrt{5})$
		$\mathrm{D}\,(3-\sqrt{5})^2\,(-1+3\sqrt{5})\,(1+\sqrt{5})$
v_3	1200	$0^2\,(20)\,(4\sqrt{5})$, $\mathrm{D}\,(10)^3\,(6\sqrt{5})$, $\mathrm{E}\,(10)\,(2\sqrt{5})\,(10-4\sqrt{5})\,(10+4\sqrt{5})$
		$0\,(4\sqrt{5})^2\,(8\sqrt{5})$, $(10-2\sqrt{5})^2\,(10+2\sqrt{5})^2$, $\mathrm{E}\,(10)\,(2\sqrt{5})\,(6\sqrt{5})^2$,
		$\mathrm{E}\,(5+3\sqrt{5})^3\,(15-3\sqrt{5})$, $\mathrm{E}\,(15-\sqrt{5})\,(-5+3\sqrt{5})\,(5+\sqrt{5})\,(5+5\sqrt{5})$
		$\mathrm{D}\,(15+\sqrt{5})\,(-5+5\sqrt{5})\,(5-\sqrt{5})\,(5+3\sqrt{5})$, $\mathrm{E}\,(15+3\sqrt{5})\,(-5+3\sqrt{5})^3$
		$\mathrm{D}\,(5+\sqrt{5})^2\,(-5+7\sqrt{5})\,(5+5\sqrt{5})$, $\mathrm{E}\,(5-\sqrt{5})^2\,(5+7\sqrt{5})\,(-5+5\sqrt{5})$
v_4	600	$8\,0^3$, 4^4, $0\,4\,(-2+2\sqrt{5})\,(2+2\sqrt{5})$, $\mathrm{E}\,2\,(2\sqrt{5})^3$, $\mathrm{D}\,2^2\,6\,(2\sqrt{5})$
		$\mathrm{E}\,(3+\sqrt{5})^2\,(5-\sqrt{5})\,(-1+\sqrt{5})$, $\mathrm{E}\,(3-\sqrt{5})^2\,(5+\sqrt{5})\,(1+\sqrt{5})$
		$\mathrm{D}\,(1+\sqrt{5})^3\,(-1+3\sqrt{5})$, $\mathrm{E}\,(-1+\sqrt{5})^3\,(1+3\sqrt{5})$

These orbits are spherical 11-designs in $\mathbb{S}^3$ because the exponent $m_2 = 11$. The orbit of v_1 is the 600-cell which has 120 points. Boyvalenkov and Danev [8] proved that uniqueness of the 120 points spherical 11-design in $\mathbb{S}^3$. Of course, the uniqueness is stronger than the rigidity. The 600-cell is the first reported rigid non-tight t-design for $t \geq 3$ and $d \geq 2$.

Each orbit of v_i for $i = 2, 3, 4$ contains the 600-cell. Moreover the following proposition holds in the case of the group H_4.

Proposition 3.1. *Let $W(H_4)$ denote the reflection group H_4. Every $W(H_4)$-orbit is disjoint union of orthogonal transformations of the* 600*-cell.*

Proof. There exists the normal chain, such that

$$W(H_4) \rhd D(W(H_4)) \rhd N \rhd \{\pm I_4\}.$$

Here, $D(W(H_4)) := \langle x^{-1}y^{-1}xy \mid \forall x, y \in W(H_4) \rangle$ is the derived subgroup of $W(H_4)$ and N is isomorphic to $\mathbb{Z}_2 \cdot A_5$ (non-splitting semi-direct product) where A_5 is alternating group on five symbols. The cardinality of $D(W(H_4))$ is 7200 and that of N is 120.

Let q_i be the dimension of the space of N-invariant harmonic polynomials of degree i. The harmonic Molien series of N is

$$\begin{aligned}
\sum_{i=0}^{\infty} q_i\lambda^i &= \frac{1}{|N|}\sum_{g\in N}\frac{1-\lambda^2}{\det(I_4-\lambda g)} & (1)\\
&= \frac{1-\lambda^2}{120}\{\frac{1}{(1-\lambda)^4}+\frac{1}{(1+\lambda)^4}+\frac{30}{(1+\lambda^2)^2} & (2)\\
&\quad +\frac{20}{(1-\lambda+\lambda^2)^2}+\frac{20}{(1+\lambda+\lambda^2)^2}\\
&\quad +\frac{12}{(\lambda-\exp(\pi i/5))^2(\lambda-\exp(-\pi i/5))^2}\\
&\quad +\frac{12}{(\lambda-\exp(2\pi i/5))^2(\lambda-\exp(-2\pi i/5))^2}\\
&\quad +\frac{12}{(\lambda-\exp(3\pi i/5))^2(\lambda-\exp(-3\pi i/5))^2}\\
&\quad +\frac{12}{(\lambda-\exp(4\pi i/5))^2(\lambda-\exp(-4\pi i/5))^2}\}\\
&= 1+13\lambda^{12}+21\lambda^{20}+25\lambda^{24}+31\lambda^{30}+\cdots
\end{aligned}$$

Therefore every orbit $x^N := \{x^w \mid w \in N\}$ is spherical 11-design in $\mathbb{S}^3$ for any $x \in \mathbb{S}^3$.

By Fischer-Type inequality, if X is spherical 11-design in $\mathbb{S}^3$, then the cardinality of X is at least 112. Thus the stabilizer subgroup N_x of any single point $x \in \mathbb{S}^3$ is trivial. Since 120 points spherical 11-design in $\mathbb{S}^3$ is unique, every N-orbit is the 600-cell. The orbit $x^{W(H_4)}$ is disjoint union of N-orbits. Therefore this proposition is proved. □

Thus the orbits of the group H_4 are non-rigid spherical 11-designs except the 600-cell.

In the case of the group E_8, if the E_8 root system is removed from the orbit of the corner vectors, then the remaining set is also spherical 7-design in $\mathbb{S}^7$. The reflection group of E_8 does not have the subgroup like N which appeared in proof of the Proposition 3.1.

Problem 3.1. *Let v_i be corner vectors for $i = 2, 3, \ldots, 8$ and $W(E_8)$ denote reflection group E_8. Is the orbit $X := \{v_i^w \mid w \in W(E_8)\}$ disjoint union of orthogonal transformations of the E_8 root system?*

By using computer, we checked that the orbits of v_i for $i = 2, 7, 8$ are disjoint union of orthogonal transformations of the E_8 root system.

Remark:

(i) In the case of group D_4, one of the orbit of corner vectors is a cross polytope which is a tight 3-design in $\mathbb{S}^3$. The orbits of corner vectors are disjoint union of orthogonal transformations of the cross polytope.

(ii) In the case of groups $A_n (n \geq 3)$, one of the orbits of corner vectors is a

regular simplex which is a tight 2-design in $\mathbb{S}^{n-1}$. Some orbits of corner vectors are **not** disjoint union of orthogonal transformations of the regular simplex.
(iii) In the case of groups $B_n(n \geq 3)$, $C_n(n \geq 3)$ and $D_n(n \geq 5)$, one of the orbits of corner vectors is a cross polytope which is a tight 3-design in $\mathbb{S}^{n-1}$. Some orbits of corner vectors are **not** disjoint union of orthogonal transformations of the cross polytope.
(vi) In the case of group H_3, one of the orbits of corner vectors is the icosahedron which is a tight 5-design in $\mathbb{S}^2$. Some orbits of corner vectors are **not** disjoint union of orthogonal transformations of the icosahedron.
(v) In the case of group E_6, one of the orbits of corner vectors is a tight 4-design in $\mathbb{S}^5$. Some orbits of corner vectors are **not** disjoint union of orthogonal transformations of the tight 4-design.
(iv) In the case of group E_7, one of the orbits of corner vectors is a tight 5-design in $\mathbb{S}^6$. Some orbits of corner vectors are **not** disjoint union of orthogonal transformations of the tight 5-design.

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