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On the ideal generated by complex cobordism classes of projective space bundles over CP^2

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Dedicated to Professor Seiya Sasao on his sixtieth birthday

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1. Let $F_{MU_*}(x,y) = \sum a_{ij}^{MU} x^i y^j (\in MU_*[[x,y]])$ be the universal formal group law over the complex cobordism ring MU_*. For the formal group law $F(x,y) = (x+y-2axy)/(1-(a^2+b^2)xy)$ with real numbers a, b, we have a ring homomorphism of MU_* into the real numbers field R

$$\varphi_{a,b} : MU_* \longrightarrow R$$

which is called the multiplicative genus associated with the formal group law $F(x,y)$ (cf. [1], [2], [3], [6]). In [2] we showed that the ideal I generated by elements α in $MU_* \otimes Q$ which satisfy $\varphi_{\frac{\delta}{2},\frac{\sqrt{3}\delta}{2}}(\alpha) = 0$ for any rational number δ is contained in the ideal J in $MU_* \otimes Q$ generated by cobordism classes of complex projective space bundles over the complex projective 2-space CP^2. The aim of this paper is to prove that the ideal I coincides with the ideal J.

2. Let $q(x)$ be a formal power series in $R[[x]]$. The symmetric power series $q(t_1)q(t_2)\cdots q(t_n)$ is described by elementary symmetric polynomials $\sigma_1, \sigma_2, \cdots, \sigma_n$ of variables t_i. Let

$$P_q(\sigma_1, \sigma_2, \cdots, \sigma_n) = q(t_1)q(t_2)\cdots q(t_n).$$

For an n-dimensional complex vector bundle ξ over X, define the cohomology class $P_q(\xi) \in H^*(X; R)$ by

$$P_q(c_1(\xi), c_2(\xi), \cdots, c_n(\xi))$$

where $c_i(\xi) (\in H^{2i}(X))$ is the i-th Chern class of ξ. Let M be a weakly almost complex closed manifold. The Whitney sum $\tau'(M) = \tau(M) \oplus \varepsilon^k$ of the tangent bundle $\tau(M)$ and a suitable k-dimensional real trivial vector bundle ε^k is equipped with the complex structure. We then evaluate $P_q(\tau'(M))$ on the fundamental class $[M]$ to get a multiplicative genus

$$\Phi_q : MU_* \longrightarrow R$$
$$\Phi_q([M]) = < P_q(\tau'(M)) , [M] > .$$

For a formal group law $f(x,y)$, let $l(x)$ be the formal power series in $R[[x]]$ with the leading term x which satisfies $l(f(x,y)) = l(x) + l(y)$. This is called the logarithm of $f(x,y)$. The formal power series

$$q(x) = \frac{x}{l^{-1}(x)}$$

induces the multiplicative genus

$$\Phi_q : MU_* \longrightarrow R$$

which coincides with the multiplicative genus $\varphi : MU_* \longrightarrow R$ associated with the formal group law $f(x,y)$. The logarithm for the formal group law $(x+y-2axy)/(1-(a^2+b^2)xy)$ is given by

$$l(x) = \int_0^x \frac{1}{1-2ax+(a^2+b^2)x^2}dx$$

and the formal power series $q(x)$ provided to the multiplicative genus $\varphi_{a,b}$ is given by

$$q(x) = x(a + b\cot bx).$$

The values of the multiplicative genus $\varphi_{a,b}$ for the complex cobordism class of complex projective n-spaces CP^n are easily computed as follows:

PROPOSITION 1. *If $\alpha = a + b\sqrt{-1}$ and $\beta = a - b\sqrt{-1}$, then*

$$\varphi_{a,b}([CP^n]) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}.$$

and

$$\varphi_{a,0}([CP^n]) = (n+1)a^n.$$

Proof. We see the logarithm $l(x)$ of $\varphi_{a,b}$ to be

$$\begin{aligned} l(x) &= \int_0^x \frac{1}{(\alpha-\beta)x}\left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right)dx \\ &= \int_0^x \sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{\alpha - \beta} x^{n-1} dx \\ &= \sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{n(\alpha-\beta)} x^n . \end{aligned}$$

Since

$$l(x) = \sum_{n=0}^{\infty} \frac{\varphi_{a,b}([CP^n])}{n+1} x^{n+1} \qquad [4],$$

the proposition follows. □

Let ξ be an n-dimensional smooth complex vector bundle over an oriented closed manifold M. The tangent bundle $\tau(CP(\xi))$ over the complex projective space bundle $CP(\xi)$ associated with ξ is described as follows:

$$\tau(CP(\xi)) \oplus 1_C \cong \pi^* \tau(M) \oplus \bar{\eta}_\xi \otimes \pi^* \xi,$$

where $\bar{\eta}_\xi$ is the conjugate bundle of the canonical line bundle η_ξ over $CP(\xi)$ and $\pi : CP(\xi) \longrightarrow M$ is the projection. By virtue of the splitting principle of complex vector bundle we describe the total Chern class $c(\pi^*\xi)$ as

$$c(\pi^*\xi) = (1 + x_1) \cdots (1 + x_n).$$

Let $z = c_1(\bar{\eta}_\xi)$. In [5] Ochanine investigates

$$\begin{aligned} P_q(\bar{\eta}_\xi \otimes \pi^* \xi) &= q(z + x_1) \cdots q(z + x_n) \\ &= \frac{z + x_1}{l^{-1}(z + x_1)} \cdots \frac{z + x_n}{l^{-1}(z + x_n)} \end{aligned}$$

to get that the coefficient of z^{n-1} in this polynomial is

$$\sum_{i=1}^{n} \prod_{k \neq i} \frac{1}{l^{-1}(x_k - x_i)}.$$

Proposition 2. *For the logarithm of* $(x + y - 2axy)/(1 - (a^2 + b^2)xy)$

$$l(x) = \int_0^x \frac{1}{1 - 2ax + (a^2 + b^2)x^2} dx.$$

we have

$$\sum_{i=1}^{n} \prod_{k \neq i} \frac{1}{l^{-1}(x_k - x_i)} = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = a + b\sqrt{-1}$ *and* $\beta = a - b\sqrt{-1}$.

Proof. We get

$$\frac{1}{l^{-1}(u)} = a + b \cot bu.$$

Write

$$G_k(y_1, y_2, \cdots, y_k) = \sum_{i=1}^{k} \prod_{j \neq i} \cot(y_j - y_i).$$

Then

$$G_2(y_1, y_2) = 0 \qquad \text{and} \qquad G_3(y_1, y_2, y_3) = -1.$$

Applying $\cot\alpha \cot\beta = \cot(\alpha - \beta)(\cot\beta - \cot\alpha) - 1$, we have

$$\begin{aligned} & G_k(y_1, \cdots, y_k) \\ &= \cot(y_{k-1} - y_k)\{G_{k-1}(y_1, \cdots, y_{k-2}, y_k) - G_{k-1}(y_1, \cdots y_{k-2}, y_{k-1})\} \\ &\quad -G_{k-2}(y_1, \cdots, y_{k-2}). \end{aligned}$$

By inductive reasoning it follows that $G_k(y_1, \cdots, y_k)$ is constant and in the sequel we obtain

$$G_k(y_1, y_2, \cdots, y_k) = -G_{k-2}(y_1, y_2, \cdots, y_{k-2})$$

and

$$G_{2t}(y_1, y_2, \cdots, y_{2t}) = 0$$

$$G_{2t+1}(y_1, y_2, \cdots, y_{2t+1}) = (-1)^t.$$

We utilize these results to calculate the following:

$$\begin{aligned} & \sum_{i=1}^{n} \prod_{k \neq i} (a + b \cot b(x_k - x_i)) \\ &= na^{n-1} + \{\sum_{i_1 < i_2} G_2(bx_{i_1}, bx_{i_2})\} a^{n-2} b + \cdots \\ &\quad +\{\sum_{i_1 < \cdots < i_s} G_s(bx_{i_1}, \cdots, bx_{i_s})\} a^{n-s} b^{s-1} + \cdots \\ &= na^{n-1} - \binom{n}{3} a^{n-3} b^2 + \cdots + (-1)^t \binom{n}{2t+1} a^{n-2t-1} b^{2t} + \cdots \\ &= \frac{\alpha^n - \beta^n}{\alpha - \beta} \end{aligned}$$

□

PROPOSITION 3. *Let ξ be an n-dimensional smooth complex vector bundle over a weakly almost complex closed manifold M. Then*

$$\varphi_{a,b}([CP(\xi)]) = \varphi_{a,b}([M])\varphi_{a,b}([CP^{n-1}]).$$

Proof. Let $q(u) = u(a + b\cot bu)$. Then

$$\begin{aligned}\varphi_{a,b}([CP(\xi)]) &= <\pi^* P_q(\tau(M)) P_q(\bar{\eta}_\xi \otimes \pi^* \xi)\, ,\, [CP(\xi)]> \\ &= <\pi^* P_q(\tau(M)) \frac{\alpha^n - \beta^n}{\alpha - \beta} z^{n-1}\, ,\, [CP(\xi)]> \\ &= \frac{\alpha^n - \beta^n}{\alpha - \beta} < z^{n-1} \pi^* P_q(\tau(M))\, ,\, [CP(\xi)]>\end{aligned}$$

The fundamental homology class of $CP(\xi)$ is the dual of $\pi^*([M]^*)z^{n-1}$, where $[M]^*$ denotes the fundamental cohomology class of M. Thereby we have

$$\varphi_{a,b}([CP(\xi)]) = \frac{\alpha^n - \beta^n}{\alpha - \beta} < P_q(\tau(M))\, ,\, [M]> .$$

This and Proposition 1 complete the proof of Propositon 3. □

For a rational number δ, we have the multiplicative genus $\varphi_{\frac{1}{2}\delta, \frac{\sqrt{3}}{2}\delta} : MU_* \longrightarrow Q$, where Q is the field of rational numbers, and so we obtain the main theorem.

THEOREM 4. *The ideal J in $MU_* \otimes Q$ generated by cobordism classes of complex projective space bundles over CP^2 conicides with the ideal I in $MU_* \otimes Q$ consisting of cobordism classes α which satisfy*

$$\varphi_{\frac{1}{2}\delta, \frac{\sqrt{3}}{2}\delta}(\alpha) = 0$$

for any rational number δ.

Proof. Let ξ be a smooth complex vector bundle over CP^2 with $\dim \xi = n$. It follows from Proposition 1 and Proposition 3 that

$$\varphi_{\frac{1}{2}\delta, \frac{\sqrt{3}}{2}\delta}([CP(\xi)]) = \varphi_{\frac{1}{2}\delta, \frac{\sqrt{3}}{2}\delta}([CP^2]) \varphi_{\frac{1}{2}\delta, \frac{\sqrt{3}}{2}\delta}([CP^{n-1}]) = 0$$

and $J \subset I$. On the other hand in [2] we proved $I \subset J$. Hence $I = J$. □

References

[1] F. HIRZEBRUCH, T. BERGER AND R. JUNG: Manifolds and Modular forms, Max-Plank-Institute für Mathematik, Bonn, 1992

[2] M. KAMATA: *The multiplicative genus associated with the formal group law* $(x+y-2axy)/(1-(a^2+b^2)xy)$, to appear in Osaka J. Math.

[3] P. S. LANDWEBER: Elliptic Curves and Modular Forms in Algebraic Topology, Lecture Notes in Math. vol.**1326**, Berlin Heiderberg New York : Springer,1988.

[4] S. P. NOVIKOV: *The method of algebraic topology from the viewpoint of cobordism theories* (Russian), Izvestija Akademi Nauk SSSR Serija Matematičeskaja **31** (1967), 855-951.

[5] S. OCHANINE: *Sur les generes multiplicatifs definis par des integrales elliptiques,* Topology **26** (1987), 143-151.

[6] D. QUILLEN: *On the formal group laws of unoriented and complex cobordism theory,* Bull. Amer. Math. Soc. **75** (1969), 1293-1298.

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