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Note on Stein neighborhoods of $\mathbf{C}^k \times \mathbf{R}^l$

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Introduction

In §1 we investigate properties of Stein neighborhoods of $\boldsymbol{C}^k \times \boldsymbol{R}^l := \{(z, w) \in \boldsymbol{C}^k \times \boldsymbol{C}^l; \mathrm{Im} w_i = 0, 1 \leqq i \leqq l\}$ in $\boldsymbol{C}^k \times \boldsymbol{C}^l$ (Theorem 1). As a consequence we show that $\boldsymbol{C}^k \times \boldsymbol{R}^l$ has no Stein neighborhood bases in $\boldsymbol{C}^k \times \boldsymbol{C}^l$ (Corollary 1). We take an open neighborhood $U := \{(z, w) \in \boldsymbol{C}^2; |\mathrm{Im} w| < (1+|z|)^{-1}\}$ of $\boldsymbol{C} \times \boldsymbol{R}$. In §2 we find a $\bar{\partial}$-closed (0, 1)-form which is not $\bar{\partial}$-exact in any open neighborhood of $\boldsymbol{C} \times \boldsymbol{R}$ in U (Lemma 5). Let $U = \{\Delta_i\}$ be an open covering of $\boldsymbol{C}$ and $f_{ij}(z, t)$ real analytic functions in $(\Delta_i \cap \Delta_j) \times \boldsymbol{R}$ so that f_{ij} is holomorphic in $z \in (\Delta_i \cap \Delta_j)$ and $f_{ij} + f_{jk} = f_{ik}$. Such f_{ij} are called Cousin data depending on the parameter $t \in \boldsymbol{R}$. We say that $\{f_i\}$ is a solution for Cousin data f_{ij}, if $\{f_i\}$ satisfies the following properties. (i) $f_i : \Delta_i \times \boldsymbol{R} \to \boldsymbol{C}$ is real analytic and holomorphic in $z \in \Delta_i$. (ii) $f_{ij} = f_j - f_i$ in $(\Delta_i \cap \Delta_j) \times \boldsymbol{R}$. We ask whether there exists a solution $\{f_i\}$ for given Cousin data f_{ij}. By the result of [1] and [2] we have $\{g_i\}$ satisfies the following statements (i) g_i: $\Delta_i \times \boldsymbol{R} \to \boldsymbol{C}$ is of class C^∞ and holomorphic in $z \in \Delta_i$. (ii) $f_{ij} = g_j - g_i$ in $(\Delta_i \cap \Delta_i) \times \boldsymbol{R}$. In §2, using the $\bar{\partial}$-closed (0, 1)-form obtained in Lemma 5, we make Cousin data depending real analytically on the parameter $t \in \boldsymbol{R}$ which have no solutions (Proposition 2 and Theorem 2). In §3 we treat the Cauchy-Riemann equation $\dfrac{\partial}{\partial \bar{z}} f(z, t) = g(z, t)$ when $g(z, t)$ is real analytic in $\boldsymbol{C} \times \boldsymbol{R}$. Piccinini [9] showed that for some $g(z, t)$ the above equation has no global solution. As an application of Lemma 5 and proposition 2 we shall give another proof of the Piccinini's result (Theorem 3).

1. Stein neighborhoods of $\boldsymbol{C}^k \times \boldsymbol{R}^l$ in $\boldsymbol{C}^k \times \boldsymbol{C}^l$

We use the following notations throughout this paper. We put $\|a\| := \max\{|a_i|; 1 \leqq i \leqq m\}$ for an m-tuple $a = (a_1, \cdots, a_m)$. And the notation {equalities and inequalities involving functions $h_1, \cdots, h_m$} denotes the set of all points in the intersection of the domains of definition of $h_1, \cdots, h_m$ satisfying the given equalities and inequalities. Let $z=(z_1, \cdots, z_k)$ be the coordinate of C^k and $w=(w_1, \cdots, w_l)$ the coordinate of C^l.

We recall the following lemma by [7] which is also implicitly due to [6, Lemma 9].

LEMMA 1. *Let $\pi: S \to C^k \times C^l$ be a (unramified Riemann) domain of holomorphy over $C^k \times C^l (k, l \geqq 1)$, $\Delta_r := \{(w_1, \cdots, w_l) \in C^l; |w_j - a_j| < r_j, 1 \leqq j \leqq l\}$ where $r=(r_1, \cdots, r_l), r_j > 0$ and $(a_1, \cdots, a_l) \in C^l$ and let $\varepsilon = (\varepsilon_1, \cdots \varepsilon_l)$ for $\varepsilon_j \geqq 0$ $(1 \leqq j \leqq l)$. Further assume there exist an open subset V_1 of S and $\delta > 0$ such that $\pi | V_1$ is biholomorphic into $C^k \times C^l$ and $\pi(V_1) \supset (C^k \times \Delta_r) \cup \{\|z\| < \delta\} \times \Delta_{r+\varepsilon}$, where $\Delta_{r+\varepsilon} := \{|w_j - a_j| < r_j + \varepsilon_j, 1 \leqq j \leqq l\}$. Then there exists an open subset V_2 of S with $V_1 \subset V_2$ such that $\pi | V_2$ is biholomorphic into $C^k \times C^l$ and $\pi(V_2) \supset C^k \times \Delta_{r+\varepsilon}$.*

PROOF. We may assume $a_1 = \cdots = a_l = 0$. Let $f \in H^0(S, \mathcal{O}_s)$. Then f can be expanded in the power series: $f|(\pi|V_1)^{-1}(C^k \times \Delta_r)(x) = \sum_{\nu, \mu} a_{\nu\mu}(z \cdot \pi(x))^{\nu}(w \cdot \pi(x))^{\mu}$, *where* $(z \cdot \pi(x))^{\nu} = (z_1 \cdot \pi(x))^{\nu_1} \cdots (z_k \cdot \pi(x))^{\nu_k}$ and $(w \cdot \pi(x))^{\mu} = (w_1 \cdot \pi(x))^{\mu_1} \cdots (w_l \cdot \pi(x))^{\mu_l}$. Then the power series $F(z, w) := \sum_{\nu, \mu} a_{\nu\mu} z^{\nu} w^{\mu}$ converges in $(C^k \times \Delta_r) \cup \{\|z\| < \delta\} \times \Delta_{r+\varepsilon}$. We put $D_d := (\{\|z\| < d\} \times \{|w_j| < r_j, 1 \leqq j \leqq l\}) \cup (\{\|z\| < \delta\} \times \{|w_j| r_j + \varepsilon_j, 1 \leqq j \leqq l\})$ for $d > \delta$, The envelope of holomorphy of D_d is the smallest logarithmically convex complete Reinhardt domain $\hat{D}_d := \{\|z\| < d, |w_j| < r_j + \varepsilon_j, \log |w_j| - \log r_j < \frac{\log d - \log \|z\|}{\log d - \log \delta}(\log r_j + \varepsilon_j) - \log r_j)\}$ which contains D_d (for instance see [10]). This implies that $\cup_{d > \delta} \hat{D}_d = C^k \times \Delta_{r+\varepsilon}$. Since F can be continued holomorphically to $\hat{D}_d$ for any $d > \delta$, hence F converges in $C^k \times \Delta_{\varepsilon}$. We can find an open subset V_2 of S with the required properties, for $\pi: S \to C^k \times C^l$ is a domain of holomorphy (for instance see [6, Theorem 18, p. 55]).

PROPOSITION 1. *Let D be an open and connected Stein subset of $C^k \times C^l$. If there exists a non-empty open subset V of C^l such that $C^k \times V \subset D$, then one can find a Stein open subset V^* of* C^l so *that $D = C^k \times V^*$.*

PROOF. Let $(z^1, w^1) \in D - C^k \times V$. We take a fixed point $(z^0, w^0) \in C^k \times V$ and a continuous curve $(z(t), w(t)): [0, 1] \to D$ such that $(z(0), w(0)) = (z^0, w^0)$ *and* $(z(1), w(1)) = (z^1, w^1)$. We put $\delta := \inf \{d((z(t), w(t)), \partial D);$

$0 \leqq t \leqq 1\}$, where $d((z(t), w(t)), \partial D)$ is the distance from $(z(t), w(t))$ to the boundary of D. Then $\bigcup_{0 \leqq t \leqq 1}\{\|z-z(t)\| < \delta/\sqrt{k+l},\ \|w-w(t)\| < \delta/\sqrt{k+l}\} \subset D$. We choose $0=t_0<t_1<\cdots<t_s=1$ so that $(z(t_{i+1}), w(t_{i+1})) \in \{\|z-z(t_i)\| < \delta/\sqrt{k+l},\ \|w-w(t_i)\| < \delta/\sqrt{k+l}\}$ for $0 \leqq i \leqq s-1$. Since $(z(0), w(0)) \in \boldsymbol{C}^k \times V$ and $\{\|z-z(0)\| < \delta/\sqrt{k+l},\ \|w-w(0)\| < \delta/\sqrt{k+l}\} \subset D$, there exists $\varepsilon > 0$ such that $\boldsymbol{C}^k \times \{\|w-w(0)\| < \varepsilon\} \cup \{\|z-z(0)\| < \delta/\sqrt{k+l},\ \|w-w(0)\| < \delta/\sqrt{k+l}\} \subset D$. Then by Lemma 1 we have $\boldsymbol{C}^k \times \{\|w-w(0)\| < \delta/\sqrt{k+l}\} \subset D$. We put $\delta_1 := \min\{\|w-w(t_1)\|; \|w-w(0)\| = \delta/\sqrt{k+l}\} > 0$. Then $\boldsymbol{C}^k \times \{\|w-w(t_1)\| < \delta_1\} \subset D$. By Lemma 1 $\boldsymbol{C}^k \times \{\|w-w(t_1)\| < \delta/\sqrt{k+l}\} \subset D$. Repeating this argument on t_i $(2 \leqq i \leqq s)$, finally we have $\boldsymbol{C}^k \times \{\|w-w^1\| < \delta/\sqrt{k+l}\} \subset D$.

We regard $\boldsymbol{R}^l$ as the real analytic submanifold $\boldsymbol{C}^l \cap \{\mathrm{Im} w_j = 0,\ 1 \leqq j \leqq l\}$ of $\boldsymbol{C}^l$.

LEMMA 2. *Let G be a non-empty open subset $\boldsymbol{R}^l$ and D a Stein open neighborhood of $\boldsymbol{C}^k \times G$ in $\boldsymbol{C}^k \times \boldsymbol{C}^l$. Then, for any $u^0 = (u_1^0, \cdots, u_l^0) \in G$, there exists an open neighborhood $V(u^0)$ of u^0 in $\boldsymbol{C}^l$ such that $\boldsymbol{C}^k \times V(u^0) \subset D$.*

PROOF. First we prove the lemma in the case $l=1$. In this case we may suppose $G = \{u \in \boldsymbol{R}; a < u < b\}$, where $-\infty \leqq a < b \leqq \infty$ and D is a Stein open neighborhood of $\boldsymbol{C}^k \times G$ in $\boldsymbol{C}^k \times \boldsymbol{C}$. We denote by D_1 the connected component of $\{(z, w) \in D;\ \mathrm{Re}\ w \in G\}$ which intersects $\boldsymbol{C}^k \times G$. Then D_1 is also a Stein open neighborhood of $\boldsymbol{C}^k \times G$ in $\boldsymbol{C}^k \times \boldsymbol{C}$. We put $D_1(+1) := D_1 \cup \boldsymbol{C}^k \times \{w;\ \mathrm{Im} w > 0,\ \mathrm{Re}\ w \in G\}$, $D_1(-1) := D_1 \cup \boldsymbol{C}^k \times \{w;\ \mathrm{Im} w < 0,\ \mathrm{Re}\ w \in G\}$. Let $D_1(+1, t) := \{(z, w); (z, w) \in D_1(+1),\ a+t < \mathrm{Re} w < b-t\}$ for $0 < t < (b-a)/2$. It is easy to check that $D_1(+1, t)$ is a locally Stein open subset of $\boldsymbol{C}^2$. Then $D_1(+1, t)$ is a Stein open set. By the result of [3] $D_1(+1) = \bigcup_{0 < t < (b-a)/2} D_1(+1, t)$ is also a Stein open set. Similary we can show that $D_1(-1)$ is also a Stein open set. Let $u^0 \in G$. We put $\varepsilon := (1/2) \min\{u^0 - a, b - u^0\}$. Then $\boldsymbol{C}^k \times \{w \in \boldsymbol{C}; |w + (u^0 + \sqrt{-1}\varepsilon)| < \varepsilon\} \subset D_1(+1)$ and there exists δ such that $0 < \delta < \varepsilon$ and $\{\|z\| < \delta\} \times \{|w - u^0| < \delta\} \subset D_1 \subset D_1(+1)$. This means that $\boldsymbol{C}^k \times \{w \in \boldsymbol{C}; |w - (u^0 + \sqrt{-1}\varepsilon)| < \varepsilon\} \cup \{\|z\| < \delta\} \times \{|w - (u^0 + \sqrt{-1}\varepsilon)| < \sqrt{\varepsilon^2 + \delta^2}\} \subset D_1 \subset D_1(+1)$. By Lemma 1 we have $\boldsymbol{C}^k \times \{|w - (u^0 + \sqrt{-1}\varepsilon)| < \sqrt{\varepsilon^2 + \delta^2}\} \subset D_1(+1)$ and then

$$(1.1) \qquad \boldsymbol{C}^k \times \{\mathrm{Im} w \leqq 0,\ |w - u^0| < \sqrt{\varepsilon^2 + \delta^2} - \varepsilon\} \subset D_1.$$

Similarly, since $C^k\times\{|w-(u^0-\sqrt{-1}\varepsilon)|<\varepsilon\}\subset D_1(-1)$ and $\{\|z\|<\delta\}\times\{|w-(u^0-\sqrt{-1}\varepsilon)|<\sqrt{\varepsilon^2+\delta^2}\}\subset D_1(-1)$, we have, by Lemma 1, $C^k\times\{|w-u^0|<\sqrt{\varepsilon^2-\delta^2}-\varepsilon\}\subset D_1(-1)$ and then

$$(1.2)\qquad C^k\times\{\operatorname{Im} w\geqq 0,\ |w-u^0|<\sqrt{\varepsilon^2+\delta^2}-\varepsilon\}\subset D_1.$$

From (1.1) and (1.2) we complete the proof of the lemma in the case $l=1$. Next we assume $l=2$. Let $u^0=(u_1^0, u_2^0)\in G\subset R^2$. We choose $\gamma>0$ so that $G_1:=\{u=(u_1, u_2); (u_1, u_2)\in R^2, |u_i-u_i^0|<\gamma\ i=1, 2\}\subset G$ and $\{\|z\|<\gamma\}\times\{(w_1, w_2); (w_1, w_2)\in C^2, |w_i-u_0^i|<\gamma\ i=1, 2\}\subset D$. For $u_2\in R$ satisfying $|u_2-u_2^0|<\gamma$, we put $D(u_2):=\{(z, w_1)\in C^k\times C; (z, w_1, u_2)\in D, |\operatorname{Re}w_1-u_1^0|<\gamma\}$ and for $\theta=\pm 1$ $D(u_2, \theta):=D(u_2)\cup C^k\times\{w_1\in C; \theta\operatorname{Im}w_1\geqq 0, |\operatorname{Re}w_1-u_1^0|<\gamma\}$. Then $D(u_2, \theta)$ is a domain of holomorphy for $\theta=\pm 1$ and $u_2\in R$ satisfying $|u_2-u_2^0|<\gamma$. And we have $C^k\times\{w_1\in C;\ |w_1-(u_1+\sqrt{-1}\gamma/3)|<\gamma/3\}\subset D(u_2, +1)$ and $\{\|z\|<\gamma\}\times\{w_1\in C; |w_1-(u_1+\sqrt{-1}\gamma/3)|<2\gamma/3\}\subset D(u_2, +1)$ for $(u_1, u_2)\in\{(u_1, u_2)\in R^2; |u_1-u_1^0|<\gamma/3, |u_2-u_2^0|<\gamma\}$. By Lemma 1 we obtain $C^k\times\{w_1\in C; |w_1-(u_1+\sqrt{-1}\gamma/3)|<2\gamma/3\}\subset D(u_2, +1)$ for $(u_1, u_2)\in\{(u_1, u_2)\in R^2; |u_1-u_1^0|<\gamma/3, |u_2-u_2^0|<\gamma\}$. Similarly for $\theta=-1$ we have $C^k\times\{w_1\in C; |w_1-(u_1-\sqrt{-1}\gamma/3)|<2\gamma/3\}\subset D(u_2, -1)$ for $(u_1, u_2)\in\{(u_1, u_2)\in R^2; |u_1-u_2^0|<\gamma/3, |u_2-u_2^0|<\gamma\}$. Put $U_1:=\{w_1\in C; |\operatorname{Re}w_1-u_1^0|<\gamma/3, |\operatorname{Im}w_1|<\gamma/3\}$. Then $C^k\times U_1\times\{u_2\in R; |u_2-u_2^0|<\gamma\}\subset D$. Consider the domains $D^*:=\{(z, w_1, w_2)\in D; w_1\in U_1, |\operatorname{Re}w_2-u_2^0|<\gamma\}$ and for $\theta=\pm 1$ $D^*(\theta):=D^*\cup C^k\times U_1\times\{w_2\in C;\ \theta\operatorname{Im}w_2\geqq 0, |\operatorname{Re}w_2-u_2^0|<\gamma\}$. Then $D^*(\theta)$ is a domain of holomorphy for $\theta=\pm 1$. Since $C^k\times U_1\times\{|w_2-(u_2^0+\sqrt{-1}\theta\gamma/2)|<\gamma/2\}\subset D^*(\theta)$ and $\{\|z\|<\gamma\}\times U_1\times\{|w_2-(u_2^0+\sqrt{-1}\theta\gamma/2)|<\gamma\}\subset D^*(\theta)$, by Lemma 1 we have $C^k\times U_1\times\{|w_2-u_2^0|<\gamma/2\}\subset D^*$. This completes the proof of the lemma in the case $l=2$. For $l\geqq 3$ we can prove the lemma similarly to the case $l=2$.

The following theorem is a consequence of Proposition 1 and Lemma 2.

THEOREM 1. *Let G be a non-empty open and connected subset of R^l. Then D is a Stein open and connected neighborhood of $C^k\times G$ in $C^k\times C^l$ if and only if there exists a Stein open and connected neighborhood V of G in C^l such that $D=C^k\times V$.*

Putting $G:=R^l$ in Theorem 1, we have a corollary of Theorem 1.

COROLLARY 1. *If $k, l\geqq 1$, then $C^k\times R^l$ has no Stein neighborhood bases in $C^k\times C^l$.*

PROOF. Take the open neighborhood $U := \{(z, w); \sum_{j=1}^{l} |\mathrm{Im} w_j| \leqq (1 + \sum_{i=1}^{k} |z_i|)^{-1}\}$ of $C^k \times R^l$ in $C^k \times C^l$. Then there is no open and non-empty subset V of C^l such that $C^k \times V \subset U$. This means by Theorem 1 that we cannot find a Stein open neighborhood D of $C^k \times R^l$ so that $C^k \times R^l \subset D \subset U$.

2. The first Cousin problems depending real analytically on a parameter

Let $\{U_i\}$ be a Stein open covering of Stein manifold X and $g_{ij}(z, t)$: $U_i \cap U_j \times R \to C$ be of class C^∞ and holomorphic in $z \in U_i \cap U_j$ so that $g_{ij} + g_{jk} = g_{ik}$. Then g_{ij} are called Cousin data depending differenciably on the parameter $t \in R$. By the result of [1] and [2] we have $H^p(X, \mathcal{O}^F) = 0 (p \geqq 1)$, where F denotes the Frechet space of all C^∞ functions on R and $\mathcal{O}^F$ is the sheaf of germs of F-valued holomorphic functions on X. Then we get $\{g_i$: $U_i \times R \to C$; g_i are of class C^∞ in $U_i \times R$ and holomorphic in $z \in U_i\}$ so that $g_{ij} = g_j - g_i$; in other words, the first Cousin problem on X depending differenciably on the parameter t has a solution.

In this section we consider the first Cousin problem depending real analytically on a parameter.

In the rest of this paper we denote by (z, w) the coordinate of C^2.

LEMMA 3. *Let V be a connected and simply connected open subset of C^2 with $C \times \{\mathrm{Im} w \leqq 0\} \subset V$ and $V(z) := \{w; (z, w) \in V\}$ for $z \in C$. Then $C \times \bigcup_{z \in C} V(z)$ is the envelope of holomorphy of V.*

PROOF. Let $\pi: \hat{V} \to C^2$ be the envelope of holomorphy of V with the injection $i: V \to \hat{V}$. We take a point $w^* \in \bigcup_{z \in C} V(z)$. Then we have $z^* \in C$ such that $(z^*, w^*) \in V$. Let $T := \{z(t), w(t)) \in V; t \in [0, 1]\}$ be a continuous curve from $(z(0), w(0)) = (0, -\sqrt{-1}) \in C \times \{\mathrm{Im}\ w \leqq 0\} \subset V$ to $(z(1), w(1)) = (z^*, w^*)$. Using the result of Lemma 1 and applying the technique of the proof of proposition 1 to the curve T, we have an open subset W^* of $\hat{V}$ with $i(V) \subset W^*$ and an open neighoborhood V^* of $\{\mathrm{Im}\ w \leqq 0\} \cup \{w(t); t \in [0, 1]\}$ in C so that W^* is mapped homeomorphically onto $C \times V^*$ by π. Since V is simply connected, the subset $(\pi|_{w^*})^{-1}(T)$ doesn't depend on the choice of curves in V from $(0, -\sqrt{-1})$ to (z^*, w^*) and then we have an open subset W of $\hat{V}$ with $i(V) \subset W$ which is mapped homeomorphically onto $C \times \bigcup_{z \in C} V(z)$. For $C \times \bigcup_{z \in C} V(z)$ is a domain of holomorphy, we have the assertion of the lemma.

LEMMA 4. *Let D be a connected and simply connected open neighborhood of $\boldsymbol{C}\times\boldsymbol{R}$ in $\boldsymbol{C}^2$ and $D(z):=\{w;(z,w)\in D\}$ for $z\in\boldsymbol{C}$. Then $\boldsymbol{C}\times\bigcup_{z\in C}D(z)$ is the envelope of holomorphy of D.*

PROOF. Let π: $\hat{D}\rightarrow\boldsymbol{C}^2$ be the envelope of holomorphy of D with the injection $j:D\rightarrow\hat{D}$. Since $j(D)\cup(\hat{D}-\pi^{-1}(\boldsymbol{C}\times\boldsymbol{R}))$ is pseudoconvex and $j(D)\cup(\hat{D}-\pi^{-1}(\boldsymbol{C}\times\boldsymbol{R}))\subset\hat{D}$, we have $\pi^{-1}(\boldsymbol{C}\times\boldsymbol{R})\subset j(D)$. In the following argument θ denotes $+1$ or -1. We put $\hat{D}(\theta):=\pi^{-1}(\boldsymbol{C}\times\{\theta\mathrm{Im}w<0\})\cup j(D)$. We make a Riemann domain $D^*(\theta)$ out of the disjoint union $\hat{D}(\theta)\cup\boldsymbol{C}\times\{\theta\mathrm{Im}w>0\}$. To do so we identify $p\in\hat{D}(\theta)$ and $(z,w)\in\boldsymbol{C}\times\{\theta\mathrm{Im}w>0\}$ if $p\in j(D)$ and $\pi(p)=(z,w)$. The map π extends to $D^*(\theta)$ if we define it as the identity on $\boldsymbol{C}\times\{\theta\mathrm{Im}w>0\}$. Easily we can check that $D^*(\theta)$ is p_7-convex in the sense of [3]. Since any holomorphic function in $D\cup\boldsymbol{C}\times\{\theta\mathrm{Im}w>0\}$ can be continued holomorphically to $D^*(\theta)$, $D^*(\theta)$ is the envelope of holomorphy of $D\cup\boldsymbol{C}\times\{\theta\mathrm{Im}w>0\}$. Note that $D\cup\boldsymbol{C}\times\{\theta\mathrm{Im}w>0\}$ is simply connected. Applying Lemma 3 to $D^*(\theta)$ for each $\theta=\pm1$, we complete the proof of the lemma.

We have the following fundamental lemma in the rest of this paper.

LEMMA 5. *Let $U:=\{(z,w)\in\boldsymbol{C}^2;|\mathrm{Im}w|<(1+|z|)^{-1}\}$. Then there exists a $\bar{\partial}$-closed C^∞ form Ψ of type $(0,1)$ on U such that for any open neighborhood V of $\boldsymbol{C}\times\boldsymbol{R}$ in $\boldsymbol{C}^2$ with $V\subset U$ $\Psi|_V$ is not $\bar{\partial}$-exact in V.*

PROOF. We put $U(n):=\{w;|\mathrm{Im}w|<(1+n)^{-1}\}$ and take a holomorphic function $f_n(w)$ in $U(n)$ which has no holomorphic continuation through any boundary point of $U(n)$ for $n=1,2,\cdots$. Let Π be the projection $\boldsymbol{C}^2\in(z,w)\rightarrow w\in\boldsymbol{C}$. Since $T_n:=\{(z,w)\in U;\Pi(z,w)\notin U(n)\}\cup\{(z,w)\in U;|z-n|\geqq 1/2\}$ and $\{n\}\times U(n)$ are disjoint and closed in U, we can find a C^∞ function ϕ_n: $U\rightarrow[0,1]$ so that $\phi_n=0$ in a neighborhood of T_n in U and $\phi_n=1$ in a neighborhood of $\{n\}\times U(n)$ in U. We take $\Psi(z,w):=\sum_{n=1}^{\infty}(z-n)^{-1}f_n(w)\bar{\partial}\phi_n$ which is a well-defined (0.1)-form of class C^∞ in U. We obtain $\bar{\partial}\Psi=0$. We assume that there exist an open neighborhood V of $\boldsymbol{C}\times\boldsymbol{R}$ in $\boldsymbol{C}^2$ with $V\subset U$ and a C^∞ function ψ on V such that $\Psi|_V=\bar{\partial}\psi$. Putting $G_n(z):=(z-n)^{-1}(1-\exp 2\pi\sqrt{-1}z)$, we get a C^∞ function $F(z,w):=\sum_{n=1}^{\infty}G_n(z)f_n(w)\phi_n(z,w)-(1-\exp 2\pi\sqrt{-1}z)\psi(z,w)$. We have $\bar{\partial}F=(1-\exp 2\pi\sqrt{-1}z)(\sum_{n=1}^{\infty}(z-n)^{-1}f_n(w)\bar{\partial}\phi_n-\bar{\partial}\psi)=0$. Then F is holomorphic in V and $F(n,w)=-2\pi$

$\sqrt{-1} f_n(w)$. We take a connected and simply connected open neighborhood V^* of $\boldsymbol{C}\times\boldsymbol{R}$ in $\boldsymbol{C}^2$ with $V^*\subset V$ and put $V^*(z):=\{w;(z,w)\in V^*\}$. By Lemma 4 there exists a holomorphic fuction F^* in $\boldsymbol{C}\times\bigcup_{z\in C} V^*(z)$ such that $F^*|_V=F$. If n is sufficiently large, $\bigcup_{z\in C} V^*(z)-U(n)\neq\phi$. Since $F^*(n,w)$ is holomorphic in $w\in\bigcup_{z\in C}V^*(z)$, this contradicts that f_n cannot be continued holomorphically beyond $U(n)$.

Let $\mathscr{U}=\{\varDelta_i;i\in I\}$ be a locally finite open coverring of $\boldsymbol{C}$ and $f_{ij}(z,t)$ (complex valued) real analytic functions in $(\varDelta_i\cap\varDelta_j)\times\boldsymbol{R}$ which are holomorphic in $z\subset\varDelta_i\cap\varDelta_j$. If $f_{ij}+f_{jk}=f_{ik}$ in $(\varDelta_i\cap\varDelta_j\cap\varDelta_k)\times\boldsymbol{R}$, then f_{ij} are called Cousin data depending real analytically on the parameter t. If there exist real analytic functions f_i in $\varDelta_i\times\boldsymbol{R}$ such that f_i are holomorphic in $z\in\varDelta_i$ and $f_{ij}=f_j-f_i$, we say that $\{f_i\}$ is a solution for the Cousin data f_{ij}.

PROPOSITION 2. *Let $\mathscr{U}=\{\varDelta_i;\varDelta_i\Subset\boldsymbol{C},\ i\in I\}$ be a locally finite open covering of $\boldsymbol{C}$. Then there exist Cousin data f_{ij} for $\mathscr{U}$ depending real analytically on the parameter $t\in\boldsymbol{R}$ without solutions.*

PROOF. Let U and Ψ be as in Lemma 5. We put $d_i:=\sup\{|z|;z\in\varDelta_i\}$ and $U_i:=\{(z,w);z\in\varDelta_i, |\mathrm{Im}w|<(1+d_i)^{-1}\}$. Since U_i is a Stein open subset of U, we have a C^∞ function Φ_i in U_i so that $\bar\partial\Phi_i=\Psi$. Let $\Phi_{ij}:=\Phi_j-\Phi_i$ in $U_i\cap U_j$ and $f_{ij}:=\Phi_{ij}|(\varDelta_i\cap\varDelta_j)\times\{\mathrm{Im}w=0\}$. Then f_{ij} are Cousin data depending real analytically on the parameter $t\in\boldsymbol{R}$, where $t:=\mathrm{Re}\ w$. Suppose $\{f_i\}$ is a solution for f_{ij}. Since $f_i(z,t)$ is real analytic in $\varDelta_i\times\boldsymbol{R}$ and holomorphic in $z\in\varDelta_i$, there exist an open neighborhood V_i of $\varDelta_i\times\boldsymbol{R}$ in C^2 and a holomorphic function F_i in V_i such that $F_i|_{\varDelta_i\times R}=f_i$. Hence we have an open subset V of U so that $\boldsymbol{C}\times\boldsymbol{R}\subset V\subset(\bigcup_{i\in I}U_i)\cap(\bigcup_{i\in I}V_i)$ and $\Phi_{ij}=F_j-F_i$ in $V_i\cap V_j\cap V$. Putting $\Phi=\Phi_i-F_i$, we have a C^∞ function Φ on V so that $\Psi|_V=\bar\partial\Phi$. This contradicts the statement of Lemma 5.

Let $\mathscr{U}^*=\{\varDelta^*_\alpha;\alpha\in A\}$ be a refinement of $\mathscr{U}$. Then we get a mapping $\rho:A\to I$ so that $\varDelta^*_\alpha\subset\varDelta_{\rho(\alpha)}$ for $\alpha\in A$. Let Ψ, $\{\Phi_i\}$, $\{f_{ij}\}$ be as in the proof of Proposition 2. Assume that the Cousin data $f^*_{\alpha\beta}:=f_{\rho(\alpha)\rho(\beta)}|\varDelta^*_\alpha\cap\varDelta^*_\beta$ for U^* has a solution $\{f^*_\alpha\}$. Then there exist an open neighborhood W^*_α of $\varDelta^*_\alpha\times\boldsymbol{R}$ in $\boldsymbol{C}^2$ and a holomorphic function H_α in W^*_α such that $H_\alpha|\varDelta^*_\alpha{}_{\times R}=f^*_\alpha$. Then we have an open neighborhood V of $\boldsymbol{C}\times\boldsymbol{R}$ in C^2 so that $\Phi_{\rho(\alpha)\rho(\beta)}=\Phi_{\rho(\beta)}-\Phi_{\rho(\alpha)}=H_\beta-H_\alpha$ in $W^*_\alpha\cap W^*_\beta\cap V$. Putting $\Lambda:=\Phi_{\rho(\alpha)}-H_\alpha$, we have a C^∞ function Λ on V so that $\bar\partial\Lambda=\Psi$. This contradicts the conclusion of Lemma 5.

Then we have the following theorem.

THEOREM 2. *Let A be the vector space of all real analytic functions on $\boldsymbol{R}$ endowed with the natural locally convex topology and $\mathcal{O}^A$ the sheaf of germs of A-valued holomorphic functions on $\boldsymbol{C}$. Then $H^1(\boldsymbol{C}, \mathcal{O}^A)$: $=\text{ind}\lim H^1(\mathfrak{U}, \mathcal{O}^A)\neq 0$, where $\mathfrak{U}$ runs through the set of all locally finite open coverings of $\boldsymbol{C}$.*

PEMARK. Let F be a Frechet space and $\mathcal{O}^F$ the sheaf of germs of F-valued holomorphic functions on a Stein space. Then $H^i(X, \mathcal{O}^F)=0$, $i\geqq 1$ ([1], [2]).

3. Piccinini's result for Cauchy-Riemann equations depending real analytically on a parameter

As an application of Proposition 2 we can give another proof of Piccinini's theorem ([9]).

THEOREM 3. *There exists a real analytic function $g(z,t)$ in $\boldsymbol{C}\times\boldsymbol{R}$ such that one cannot find a real analytic function $f(z,t)$ in $\boldsymbol{C}\times\boldsymbol{R}$ satisfying $\frac{\partial}{\partial\bar{z}}f(z,t)=g(z,t)$ in $\boldsymbol{C}\times\boldsymbol{R}$.*

PROOF. Let f_{ij} be the Cousin data as in the proof of Proposition 2. We denote by $\mathcal{A}$ the sheaf of germs of real analytic functions on $\boldsymbol{C}\times\boldsymbol{R}$. Then, by the result of [4] and [8] we have $H^1(\boldsymbol{C}\times\boldsymbol{R}, \mathcal{A})=H^1(\{\Delta_i\times\boldsymbol{R}\}, \mathcal{A})=0$. There exists $\{g_i\in H^0(\Delta_i\times\boldsymbol{R}, \mathcal{A})\}$ such that $f_{ij}(z,t)=g_j(z,t)-g_i(z,t)$ in $(\Delta_i\cap\Delta_j)\times\boldsymbol{R}$. Since $\frac{\partial f_{ij}}{\partial\bar{z}}=0$, we have a real analytic function $g(z,t)=\frac{\partial g_i(z,t)}{\partial\bar{z}}$ in $\boldsymbol{C}\times\boldsymbol{R}$. If we find a real analytic function $f(z,t)$ in $\boldsymbol{C}\times\boldsymbol{R}$ so that $\frac{\partial f(z,t)}{\partial\bar{z}}=g(z,t)$ in $\boldsymbol{C}\times\boldsymbol{R}$, then we have a solution $\{f_i:=g_i-f\}$ for Cousin data f_{ij} depending real analytically on t. This contradicts the statement of Proposition 2.

REMARK. In Theorem 3 we can replace $\frac{\partial f(z,t)}{\partial\bar{z}}=g(z,t)$ by $\frac{\partial f(z,t)}{\partial z}=g(z,t)$ and $(\partial^2/\partial x^2+\partial^2/\partial y^2)f(z,t)=g(z,t)$, where $z=x+\sqrt{-1}y$.

RREMARK. Let $g(z, t)$ be a real analytic function in $C\times R$ and D_n: $=\{|z|<n\}$ for $n=1, 2, \cdots$. We put $f_n(z, t) := (2\pi\sqrt{-1})^{-1}\iint_{D_n} g(\zeta,t)/(\zeta-z)\, d\zeta\wedge d\bar{\zeta}$ for $(z, t)\in D_n\times R$. Then $\partial f_n/\partial\bar{z}=f$ in $D_n\times R$. We can show that f_n is real analytic in $D_n\times R$ and $f_m-f_n (m>n)$ is holomorphic in $z\in D_n$. By the result of [2, Theorem C] the restriction map $H^0(C, \mathcal{O}^F)\rightarrow H^0(D_n, \mathcal{O}^F)$ has a dense image, where F denotes the Frechet space of all C^∞ functions in R and $\mathcal{O}^F$ is the sheaf of germs of F-valued holomorphic functions on C. Applying this approximation theorem to the local solution $\{f_n\}$, we have a C^∞ function $f^*(z, t)$ in $C\times R$ so that $f^*(z, t)$ is real analytic in z and $\frac{\partial f^*(z, t)}{\partial\bar{z}}=g(z, t)$ in $C\times R$. This implies that we cannot have Oka-Weil approximation property for holomorphic functions with values in the locally convex space of all real analytic functions in R.

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