Note on cohomology groups of the product space C^n x P_m

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Note on cohomology groups of the product space $C^n \times P_m$

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1. Introduction

Grauert and Remmert [2] proved a vanishing theorem for relatively compact pseudoconvex domains of the product space $C^n \times P_m$. Fujiki and Hironaka (Fujiki [1]) obtained a vanishing theorem for relatively compact pseudoconvex domains in an analytic space with positive line bundles.

The purpose of this paper is to prove that the condition "relatively compact" in the vanishing theorems by Fujiki-Hironaka and Grauert-Remmert is necessary.

2. Infinite dimensional cohomology groups of the product space $C^n \times P_m$

Let $(X, \mathcal{O})$ be a reduced analytic space and B a holomorphic line bundle over X defined by a holomorphic cocycle $(\{U_i\}, \{e_{ij}(x)\})_{i,j \in I}$, where $\{U_i\}_{i \in I}$ denotes an open covering of X and $e_{ij}(x)$ is a nowhere vanishing holomorphic function on $U_i \cap U_j$ with $e_{ij}e_{jk}=e_{ik}$ on $U_i \cap U_j \cap U_k$. We say that a collection $h=\{h_i\}$ of C^∞ functions h_i: $U_i \to \{x \in \boldsymbol{R} \mid x>0\}$ is a hermitian metric along the fibres of B if

$$h_j/h_i = |e_{ij}|^2 \text{ on } U_i \cap U_j \quad i,\ j \in I.$$

A holomorphic line bundle B is called positive if there exists a hermitian metric $h=\{h_i\}$ such that $-\log h_i$ is strictly plurisubharmonic on U_i $i \in I$.

An analytic space $(X, \mathcal{O})$ is said to be weakly (resp. strongly) 1-complete (with respect to ψ), if there exists a weakly (resp. strictly) plurisubharmonic function ψ of calss C^∞ on X such that

$$X_c := \{x \in X;\ \psi(x) < c\} \Subset X$$

for any $c \in \boldsymbol{R}$. The following theorem was obtained by Fujiki and Hironaka.

THEOREM (Fujiki [1]). *Let* $(X, \mathcal{O})$ *be a weakly* 1-*complete analytic* *space, B a positive line bundle over X and $\mathscr{F}$ a coherent analytic sheaf on X. Then for any $c\in\boldsymbol{R}$, there exists a positive integer k_0 such that*

$$H^i(X_c, \mathscr{F}\otimes B^k)=0 \text{ for } i\geqq 1 \text{ and } k\geqq k_0.$$

We have a question for the above theorem. Can we take the whole space X instead of X_c in the conclusion of the above theorem? A similar problem is raised with respect to the vanishing theorem for relatively compact pseudoconvex domains of the product space $\boldsymbol{C}^n\times \boldsymbol{P}_m$ by Grauert-Remmert [2]. Is the condition "relatively compact" necessary? The following is to answer negatively this questions.

We take a plurisubharmonic function

$$\psi(z_1, \cdots, z_n, x)=\sum_{i=1}^{n}|z_i|^2 \qquad (z_1, \cdots, z_n, x)\in \boldsymbol{C}^n\times \boldsymbol{P}_m.$$

Then $X:=\boldsymbol{C}^n\times\boldsymbol{P}_m$ is weakly 1-complete with respect to ψ. Let H be a hyperplane bundle over $\boldsymbol{P}_m$ and π_1 (resp. π_2) the projection of X onto $\boldsymbol{C}^n$ (resp. $\boldsymbol{P}_m$). Since H is a positive line bundle over $\boldsymbol{P}_m$, there exists a hermitian metric $h=\{h_i\}$ such that $-\log h_i$ is strictly plurisubharmonic. The pull-back $B:=\pi_2{}^*H$ is a positive line bundle over X with respect to the hermitian metric $e^{-\psi}h=\{e^{-\psi}h_i\pi_2\}$.

We have the following theorem.

THEOREM. *There exists a coherent analytic sheaf $\mathscr{F}$ on $X:=\boldsymbol{C}^n\times\boldsymbol{P}_m$ such that*

$$\dim_C H^m(X, \mathscr{F}\otimes\underline{B}^k)=\infty$$

for any positive integer k.

PROOF. Let $\{x_\alpha\}_{\alpha=1}^{\infty}$ be a discrete sequence in $\boldsymbol{C}^n$. We put

$$A_\alpha=\pi_1^{-1}(x_\alpha)\subset X.$$

A_α is biholomorphic onto $\boldsymbol{P}_m$. We take a sequence $\{s_\alpha\colon \alpha=1,2,\cdots\}$ of positive integers satisfying

$$\sup\{\alpha; s_\alpha=p\}=\infty$$

for any positive integer p. Let $\mathscr{I}_\alpha$ be the ideal sheaf of $A_\alpha\subset X$. We have a coherent analytic sheaf

$$\mathscr{F}=\bigcup_{\alpha=1}^{\infty}(\mathscr{O}/\mathscr{I}_\alpha\otimes\underline{B}^{-m-1-s_\alpha})$$

on X. Then, for any positive integer k

$$H^m(X, \mathscr{F}\otimes\underline{B}^k)=\sum_{\alpha=1}^{\infty}H^m(A_\alpha, \mathscr{O}/\mathscr{I}_\alpha\otimes\underline{B}^{-m-1+k-s_\alpha})$$

$$\geqq\sum_{\alpha\in\{\beta:k=s_\beta\}}H^m(A_\alpha, \mathscr{O}/\mathscr{I}_\alpha\otimes\underline{B}^{-m-1})$$

Since $\sup\{\beta; k=s_\beta\}=\infty$ and

$$\dim_{\boldsymbol{C}} H^m(A_\alpha, \mathcal{O}/\mathcal{I}_\alpha \otimes \underline{B}^{-m-1}) = \dim_{\boldsymbol{C}} H^m(P^m, \underline{H}^{-m-1})$$
$$=1$$

we have $\dim_{\boldsymbol{C}} H^m(X, \mathcal{F} \otimes \underline{B}^k)=\infty$.

The above coherent analytic sheaf $\mathcal{F}$ on X is not locally free. We have the following guestion with respect to the above theorem.

QUESTION. Can we fined a holomorphic vector bundle E over $X=\boldsymbol{C}^n \times \boldsymbol{P}_m$ such that for any positive integer k

$$\sum_{i=1}^{n+m} \dim_{\boldsymbol{C}} H^i(X, \underline{E} \otimes \underline{B}^k) \neq 0 \ ?$$

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