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https://doi.org/10.15017/1448939

出版情報：九州大学教養部数学雑誌. 7 (2), pp.23-35, 1970-09. College of General Education, Kyushu University

Math. Rep.
VII-2, 1970.

CERTAIN OPERATORS OF RIEMANN-LIOUVILLE TYPE AND STATIONARY PROCESSES

By

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[1969. 8. 15.]

1. Introduction.

T. Hida [1] has proposed the problem that "what properties has the stationary process $x(t)\equiv x(t,\omega)$, which satisfies the formal equation

(1.1) $$D^{\alpha}x(t)=\dot{\xi}(t),$$

where $D\equiv d/dt$, $0<\alpha<1$ and $\dot{\xi}(t)$ is a white noise". K. Yosida [2] had interpreted and studied the operators such as D^{α}, which proposed by A. V. Balakrishnan [3], as the fractional powers of infinitesimal generators of the semi-groups of bounded linear operators g_t $(t>0)$ on a Banach space $\boldsymbol{X}$ into $\boldsymbol{X}$.

In this note, we shall interprete and study the operators such as D^{α}, $(D+\beta I)^{\alpha}$, $(D+\beta I)^{-\alpha}$, as the special cases of shift-commutative linear operators which are defined by T. Seguchi [4] and [5]. And then we shall refer to the properties of stationary processes which satisfy the stochastic functional equations concerning to the operators such as above S-C.L.Op.s of Riemann-Liouville type.

2. Preliminaries.

DEFINITION 2.1. Let $\boldsymbol{F}$ be the all complex- (or real-) valued functions defined on $\boldsymbol{R}$[1], and let $(\Lambda, \boldsymbol{D}, \{\tau_t\}_{t\in R})$ be such that:

(1) $\{\tau_t\}_{t\in R}$ is the shift operator defined on $\boldsymbol{F}$; i.e. for any $f\in\boldsymbol{F}$ and for any $t\in\boldsymbol{R}$

$$\tau_t f(\cdot)\equiv f(\cdot+t).$$

1) $R\equiv(-\infty, \infty)$: "the set of all real numbers".

(2) $\boldsymbol{D}$ is a non-trivial complex (or real) linear sub-space of $\boldsymbol{F}$, i.e. $\boldsymbol{D}\neq\{0\}$, and satisfies the condition

(L 1) $\tau_t\boldsymbol{D}\subset\boldsymbol{D}$, for any $t\in\boldsymbol{R}$.

(3) Λ is a linear operator defined on $\boldsymbol{D}$ and satisfies the condition

(L 2) $\Lambda\boldsymbol{D}\subset\boldsymbol{F}\equiv\mathscr{D}(\tau_t)$[2].

(4) Between the linear operator Λ and the shift τ_t, the condition

(L 3) for any $f\in\boldsymbol{D}$ and for any $t\in\boldsymbol{R}$

$$\tau_t(\Lambda f)=\Lambda(\tau_t f)$$

holds.

Then we shall call the linear operator Λ is the *shift-commutative linear operator* (S-C.L.Op.) defined on $\boldsymbol{D}$ w.r.t. τ_t. The notation $\widetilde{\boldsymbol{\Lambda}}$ means the class of all S-C.L.Op.s defined above and $\widetilde{\mathscr{D}}(\Lambda)$ means the domain $\boldsymbol{D}$ of $\Lambda\in\widetilde{\boldsymbol{\Lambda}}$.

Let $\Lambda\in\widetilde{\boldsymbol{\Lambda}}$, and $_\Lambda f(t)\equiv(_\Lambda f)(t)$ be the value of $_\Lambda f$ at t, then the following relations hold; there exist two functions such that for any $\lambda\in\mathscr{A}(\Lambda)\equiv\{\lambda;\ e^{i\lambda\cdot}\in\widetilde{\mathscr{D}}(\Lambda),\ \lambda\in\boldsymbol{R}\}$ and $z\in\mathscr{B}(\Lambda)\equiv\{z;\ e^{z\cdot}\in\widetilde{\mathscr{D}}(\Lambda),\ z\in C\}$[3],

(2.1) $\Lambda e^{i\lambda t}\equiv e^{i\lambda t}G_\Lambda(\lambda),\quad \Lambda e^{zt}\equiv e^{zt}C_\Lambda(z),$ for every $t\in\boldsymbol{R}$.

DEFINITION 2.2. The above two functions $G_\Lambda(\lambda)$ and $C_\Lambda(z)$ are called the *generating function* (g. f.) and the *characterizing function* (c. f.) of S-C. L. Op. $\Lambda\in\widetilde{\boldsymbol{\Lambda}}$, respectively.

Let us use the following notations:

$\mathscr{S}\equiv$" the space of all *rapidly decreasing functions* in the sense of L. Schwartz [6]".

$\boldsymbol{M}\equiv\left\{\mu;\int_{-\infty}^{\infty}|\varphi(\lambda)|^2\mu(d\lambda)<\infty,\text{ for every }\varphi\in\mathscr{S},\text{ non-negative measures}\right\}.$

$\boldsymbol{M}_k\equiv\left\{\mu;\int_{-\infty}^{\infty}\mu(d\lambda)/(1+\lambda^2)^k<\infty,\ \mu\in\boldsymbol{M}\right\}.$

$\mathrm{L}^2(\mu;\mathscr{S})\equiv\left\{f;\int_{-\infty}^{\infty}|f(\lambda)\varphi(\lambda)|^2\mu(d\lambda)>\infty,\text{ for every }\varphi\in\mathscr{S}\right\}.$

$\mathrm{L}^2(\mu;k)\equiv\left\{f;\int_{-\infty}^{\infty}|f(\lambda)|^2/(1+\lambda^2)^k\mu(d\lambda)<\infty\right\}.$

$\Omega(\mathfrak{B},\boldsymbol{P})\equiv$" the basic *probability space*".

$\boldsymbol{L}^2(\Omega)\equiv\{x;\ E[|x(\omega)|^2]<\infty\}$.

$\mathfrak{S}\equiv$" the totality of (weakly) *stationary distributions*".

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2) $\mathscr{D}(*)$ means the *domain* of a operator "$*$".

3) C means "the set of all complex numbers".

$\mathfrak{S}^{\circ}\equiv$" the totality of (weakly) *stationary processes*".

$\mathfrak{S}_k\equiv\{x\in\mathfrak{S};\ \mu_x\in \boldsymbol{M}_k\}$.

$X_R\equiv\{x(t,\ \omega),\ t\in R;\ \omega\in\Omega\}$.

$X_{\mathscr{S}}\equiv\{x(\varphi,\ \omega),\ \varphi\in\mathscr{S};\ \omega\in\Omega\}$.

$X\equiv X_R$ or $X_{\mathscr{S}}$, if we need not to distinguish X_R *and* $X_{\mathscr{S}}$.

$L^2(X)\equiv L^2\{X\}\subset L^2(\Omega)$.[4)]

$L^2(X;\ a,\ b)\equiv L^2(X_R;\ a,\ b)$ or $L^2(X_{\mathscr{S}};\ a,\ b)$, $\quad L^2(X;\ t)\equiv L^2(X;\ -\infty,\ t)$,

$L^2(X_R;\ a,\ b)\equiv L^2\{x(t,\ \omega),\ a\leq t\leq b;\ \omega\in\Omega\}$,

$L^2(X_{\mathscr{S}};\ a,\ b)\equiv L^2\{x(\varphi,\ \omega),\ \varphi\in\mathscr{S}_{ab};\ \omega\in\Omega\}$,

$\mathscr{S}_{ab}\equiv\{\varphi;\ \text{car.}(\varphi)\in[a,\ b],\ \varphi\in\mathscr{S}\}$.

$\varLambda_-\equiv$"S-C.L.Op.s which are dependent only on the past".

$N_\varLambda\equiv\{\lambda;\ G_\varLambda(\lambda)=0\}$.

$dZ_x(\lambda)\equiv dZ_x(\lambda,\ \omega)\equiv$"the corresponding *orthogonal random measure* to x".

$\mu_x\equiv\mu_x(d\lambda)\equiv$" the corresponding *spectral measure* to x".

$F_\alpha(t)\equiv$" the *continuous primitive* of $e^{-\alpha t}f(t)$ as the function of t".

$$[\beta/(D-\alpha I)]f(t)\equiv-\beta\int^0 e^{\alpha u}\tau_{-u}f(t)du, \tag{2.2}$$

where $\alpha,\ \beta\in C,\ \neq 0$ and

$$\int^0 e^{\alpha u}\tau_{-u}f(t)du\equiv-\int^t e^{\alpha(t-s)}f(s)ds\equiv-e^{\alpha t}F_\alpha(t)$$

$$=\begin{cases}\displaystyle\int_{-\infty}^0 e^{\alpha u}\tau_{-u}f(t)du=\int_t^\infty e^{\alpha(t-s)}f(s)ds, & \text{if } \Re\ \alpha>0,\\ \displaystyle-\int_0^\infty e^{\alpha u}\tau_{-u}f(t)du=\int_{-\infty}^t e^{\alpha(t-s)}f(s)ds, & \text{if } \Re\ \alpha<0,\\ -e^{\alpha t}F_\alpha(t), & \text{if } \Re\ \alpha=0.\end{cases}$$

$$(1/[(D-\alpha I)(D-\alpha I)])f(t)\equiv-(1/b)\int^0 e^{au}\sin bu\ \tau_{-u}f(t)du, \tag{2.3}$$

where $\alpha\equiv a+ib$ $(a,\ b\in R,\ \neq 0)$.

DEFINITION 2.3. Let $T_t(t\in R)$ be the *shift operator* on $\mathfrak{S}$, such that for any $x\in\mathfrak{S}$, for every $\varphi\in\mathscr{S}$ and for every $t\in R$ the relation

$$T_tx(\varphi)\equiv x(\tau_{-t}\varphi)$$

holds, where τ_t is the shift defined on $\boldsymbol{F}$.

Let $K(t)$ be a right continuous step-function defined on R, and has finite jumping points, i. e.

4) $L^2\{*\}$ means the L^2-space generated by "$*$".

$$K(t)\equiv\sum_{t_n\leq t} k_n,\quad \sum_n |k_n|<\infty,\quad k_n\in R \text{ or } C,\quad n=1,2,\cdots.$$

And for any $x\in\mathfrak{S}$ and for every $\varphi\in\mathscr{S}$, let us put

$$Ax(\varphi)\equiv\sum_n T_{t_n}x(\varphi)k_n=\sum_n x(\tau_{-t_n}\varphi)k_n=\int_{-\infty}^{\infty}x(\tau_{-t}\varphi)dK(t). \tag{2.4}$$

The all these operators A such as above will be called the *shift-commutative linear operators* on $\mathfrak{S}$, and are denoted class A_S.

PROPOSITION 2.1 (Proposition 3.1.1).[5] *If the above operator A on $\mathfrak{S}$ defined by* (2.4) *is corresponding to the S-C.L.Op. $A\in\widetilde{A}$ such that, for any $f\in F$*

$$Af(\cdot)\equiv\sum_n \tau_{tn}f(\cdot)k_n=\sum_n f(\cdot+t_n)k_n=\int_{-\infty}^{\infty}f(\cdot+s)dK(s)=\int_{-\infty}^{\infty}\tau^s f(\cdot)dK(s). \tag{2.5}$$

Then we have the following results:

(1°) *G_A and C_A of $A\in A$ defined above are given by*

$$G_A(\lambda)=\sum_n e^{i\lambda t_n}k_n=\int_{-\infty}^{\infty}e^{i\lambda t}dK(t),$$

$$C_A(z)=\sum_n e^{zt_n}k_n=\int_{-\infty}^{8}e^{zt}dK(t).$$

(2°) *Especially, if $x\in\mathfrak{S}_0$, then for the corresponding usual stationary process $x\in\mathfrak{S}^\circ$, $Ax(t)$ is given by*

$$Ax(t)\equiv\sum_n \tau_{t_n}x(t)k_n=\sum_n x(t+t_n)k_n=\int_{-\infty}^{\infty}x(t+s)dK(s)=\int_{-\infty}^{\infty}\tau_s x(t)dK(s).$$

DEFINITION 2.4. If $x\in\mathfrak{S}$ has the orthogonal random measure and the spectral measure dZ and μ respectively, and if S-C.L.Op. $A\in\widetilde{A}$ has the g.f. $G_A\in L^2(\mu;\mathscr{S})$, then the corresponding S-C.L.Op. A in $\mathfrak{S}$ to $A\in\widetilde{A}$ is defined by, for every $\varphi\in\mathscr{S}$

$$Ax(\varphi)=\int_{-\infty}^{\infty}(\mathscr{F}\varphi)(\lambda)G_A(\lambda)dZ(\lambda). \tag{2.6}$$

The domain of A in $\mathfrak{S}$ is denoted by $\mathscr{D}(A)\equiv\{x;\ G_A\in L^2(\mu_x;\mathscr{S})\}$, The class of all operators A in $\mathfrak{S}$ defined above and whose domain in $\mathfrak{S}$ is not empty, is called the *shift-commutative linear operators* (S-C.L.Op.s) in $\mathfrak{S}$ and denoted by $\boldsymbol{A}$.

REMARK. Here we are treating the weakly stationary processes, i.e. stationary process x is regarded as a point of the closed linear sub-manifold of the Hilbert space $L^2(\Omega)$. Hence, in its representation (2.6), it

5) The inside of "( )" shows the the number of result in T. Seguchi [4].

will be in question that the *difference of measure zero.* Here we wish to point out that the above definition of Λx give the answer to this question. That is, as the g. f. G_Λ of S-C. L. Op. Λ is uniquely determined by the operator Λ, hence the representation (2. 6) is unique, and even if $G_\Lambda \simeq G_\Gamma$ (μ_x) (in this case $\Lambda x(\varphi)=\int_{-\infty}^{\infty}(\mathscr{F}\varphi)(\lambda)G_\Lambda(\lambda)\,dZ_x(\lambda)=\int_{-\infty}^{\infty}(\mathscr{F}\varphi)(\lambda)G_\Gamma(\lambda)\,dZ_x(\lambda)$ $=\Gamma x(\varphi)$ is considered as the same point in $L^2(\Omega)$), we must consider that $\Lambda x \neq \Gamma x$.

PROPOSITION 2. 2. (1°) *If $x \in \mathfrak{S}$ has an orthogonal random measure dZ_x and a spectral measure μ_x, then for any $\Lambda \in \boldsymbol{\Lambda}$ satisfying $G_\Lambda \in \mathrm{L}^2(\mu_x\,;\mathscr{S})$, expressions*

$$\Lambda x(\varphi)=\int_{-\infty}^{\infty}(\mathscr{F}\varphi)(\lambda)G_\Lambda(\lambda)dZ_x(\lambda),$$

$$r_{\Lambda x}(\varphi)=\int_{-\infty}^{\infty}(\mathscr{F}\varphi)(\lambda)\,|\,G_\Lambda(\lambda)\,|^2\mu_x(d\lambda), \tag{2.7}$$

hold, for every $\varphi \in \mathscr{S}$, where $r_{\Lambda x}(\varphi)$ is the covariance distribution of Λx.

(2°) *Especially, if $x \in \mathfrak{S}_0$ and $G_\Lambda \in \mathrm{L}^2(\mu_x\,;0)$, then for the corresponding usual stationary process $x \in \mathfrak{S}^\circ$ to $x \in \mathfrak{S}_0$, the following expressions hold*:

$$\Lambda x(t)=\int_{-\infty}^{\infty}e^{i\lambda t}G_\Lambda(\lambda)dZ_x(\lambda), \tag{2.8}$$

$$r_{\Lambda x}(t)=\int_{-\infty}^{\infty}e^{i\lambda t}\,|\,G_\Lambda(\lambda)\,|^2\mu_x(d\lambda), \tag{2.9}$$

where dZ_x and μ_x are those of $x \in \mathfrak{S}_0$.

PROPOSITION 2. 3 (Theorem 3. 4. 4). *Let Λ, Γ, $\Delta \in \boldsymbol{\Lambda}$ (or $\widetilde{\boldsymbol{\Lambda}}$), then we have*

(1°) *if $\Delta = a\Lambda + b\Gamma$ then $G_\Delta = aG_\Lambda + bG_\Gamma$ and $C_\Delta = aC_\Lambda + bC_\Gamma$,*

(2°) *if $\Delta = \Lambda\Gamma$ then $G_\Delta = G_\Lambda G_\Gamma = G_\Gamma G_\Lambda$ and $C_\Delta = C_\Lambda C_\Gamma = C_\Gamma C_\Lambda$,*

(3°) *if there exists the inverse operator $\Lambda^{-1} \in \boldsymbol{\Lambda}$ (or $\widetilde{\boldsymbol{\Lambda}}$), then $G_{\Lambda^{-1}} = 1/G_\Lambda$ and $C_{\Lambda^{-1}} = 1/C_\Lambda$, where Λ^{-1} is an operator such that $\Lambda\Lambda^{-1} = I$ (identity operator).*

PROPOSITION 2. 4. (1°) *If $\Lambda \in \widetilde{\boldsymbol{\Lambda}}$ is given by*

$$\Lambda f(\cdot) \equiv \int_a^b \tau_{-s} f(\cdot)\,dK(s),$$

then $\Lambda \in \boldsymbol{\Lambda}$ corresponding to the above $\Lambda \in \widetilde{\boldsymbol{\Lambda}}$ can be expressed by

$$\Lambda x(\varphi) \equiv \int_a^b T_{-s}x(\varphi)\,dK(s) = \int_a^b x(\tau_s\varphi)\,dK(s),$$

where $K \in \mathrm{BV}[a, b]$[6] $(-\infty \leq a < b \leq +\infty)$, $x \in \mathscr{D}(\Lambda)$.

(2°) *Especially, when* $x \in \mathfrak{S}^{\circ}$, *then*

$$\Lambda x(t) \equiv \int_a^b \tau_{-s} x(t) dK(s) = \int_a^b x(t-s) dK(s).$$

PROPOSITION 2.5 (Theorem 4.2.3). (1°) *Let* x *and* $y \in \mathfrak{S}$ *satisfy the equation*

$$\Lambda y = x,$$

where $\Lambda \in \boldsymbol{\Lambda}_{-}$. *Then for every* $t \in R$

$$\boldsymbol{L}^2(Y;\ t) \supset \boldsymbol{L}^2(X;\ t)$$

holds.

(2°) *Furthermore, if there exists the inverse operator* $\Lambda^{-1} \in \boldsymbol{\Lambda}_{-}$, *then*

$$\boldsymbol{L}^2(Y;\ t) = \boldsymbol{L}^2(X;\ t)$$

holds, for every $t \in R$.

PROPOSITION 2.6 (Theorem 5.2.1). *Let* $x \in \mathfrak{S}_0$, $\dot{\xi}$ *be a differentiation in the sense of random distribution of a homogeneous orthogonal random measure* $d\xi$ *and* $\Lambda \in \boldsymbol{\Lambda}$. *Then*

(1°) *the relation*

$$\Lambda x = \dot{\xi} \tag{2.10}$$

holds if and only if the conditions (G 1) $N_\Lambda = \phi$ *and* (G 4°) $1/G_\Lambda \in \mathrm{L}^2(m;\ 0)$[7] *are satisfied.*

(2°) *any* $x \in \mathfrak{S}_0$ *satisfying* (2.10) *is given by uniquely*

$$x(\varphi) = \Lambda^{-1}\dot{\xi}(\varphi) = \int_{-\infty}^{\infty} (\mathscr{F}\varphi)(\lambda) d\tilde{\xi}(\lambda)/G_\Lambda(\lambda),$$

where $d\tilde{\xi}$ *is the homogeneous orthogonal random measure of* $\dot{\xi} \in \mathfrak{S}_1$.

(3°) $x \in \mathfrak{S}^{\circ}$, *corresponding to* $x \in \mathfrak{S}_0$, *is represented by*

$$x(t) = \int_{-\infty}^{\infty} e^{i\lambda t} d\tilde{\xi}(\lambda)/G_\Lambda(\lambda),$$

$$r_x(t) = \int_{-\infty}^{\infty} e^{i\lambda t}/|G_\Lambda(\lambda)|^2 d\lambda.$$

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6) $\mathrm{BV}[a, b]$ means "the all real- or complex-valued functions of bounded variations in $[a, b]$."

7) m means "the *usual Lebesgue measure* on R", i.e. $m(d\lambda) \equiv d\lambda$.

The above all notations, notions, definitions and propositions are in T. Seguchi [4], or can be easily proved.

3. S-C.L.Op.s of Riemann-Liouville type belonging to $\widetilde{\Lambda}$.

[1°] *S-C.L.Op.* $(D+\beta I)^{-\alpha}$.

We shall first consider the case that the operator Λ defined by the following form: For $\alpha>0$, $\beta>0$ (α, $\beta \in R$)

$$\Lambda f(t) \equiv 1/\Gamma(\alpha)\int_{-\infty}^{t}(t-s)^{\alpha-1}e^{-\beta(t-s)}f(s)ds \tag{3.1}$$

$$=1/\Gamma(\alpha)\int_{0}^{\infty}u^{\alpha-1}e^{-\beta u}\tau_{-u}f(t)du.$$

The following formula holds, for any $\alpha>0$ ($\alpha \in R$), and $z \in C$ such as $\mathfrak{Re}\ z>0$

$$\int_{0}^{\infty}e^{-zt}t^{\alpha-1}dt=\Gamma(\alpha)/z^{\alpha}. \tag{3.2}$$

If we put $f(t)\equiv e^{i\lambda t}(\lambda \in R)$, we have from (3.1)

$$\Lambda e^{i\lambda t}=1/\Gamma(\alpha)\int_{-\infty}^{t}(t-s)^{\alpha-1}e^{-\beta(t-s)}e^{i\lambda s}ds$$

$$=e^{i\lambda t}/\Gamma(\alpha)\int_{0}^{\infty}u^{\alpha-1}e^{-(\beta+i\lambda)u}\ du$$

$$=e^{i\lambda t}/(\beta+i\lambda).$$

Hence, from the above formula, we can see that the operator Λ belongs to the class $\widetilde{\Lambda}$ of S-C.L.Op.s and the g.f. of Λ is $G_{\Lambda}(\lambda)=(i\lambda+\beta)^{-\alpha}$.

If $\alpha=1$ then we can easily assure the operator Λ defined by (3.1) is the operator $(D+\beta I)^{-1}$, which is given by (2.2). Hence, we shall denote the operator Λ defined by (3.1) $\Lambda\equiv(D+\beta I)^{-\alpha}$.

[2°] *S-C.L.Op.* $(D+\beta I)^{\alpha}$.

Next, we shall consider the following case; $0<\alpha<1$, $\beta>0$ (α, $\beta \in R$). Using the formula (3.2) and

$$\Gamma(1-\alpha)=-\alpha\Gamma(-\alpha), \qquad 0<\alpha<1, \tag{3.3}$$

we have

(3. 4) $$\int_0^\infty u^{-\alpha-1}(1-e^{-(\beta+i\lambda)u})du$$

$$=-1/\alpha\,[u^{-\alpha}(1-e^{-(\beta+i\lambda)u})]\Big|_0^\infty-[(\beta+i\lambda)/\alpha]\int_0^\infty u^{-\alpha}e^{-(\beta+i\lambda)u}du$$

$$=-[(\beta+i\lambda)/\alpha]\int_0^\infty u^{(1-\alpha)-1}e^{-(\beta+i\lambda)u}du$$

$$=-[(\beta+i\lambda)/\alpha]\,[\Gamma(1-\alpha)]/(\beta+i\lambda)^{1-\alpha}$$

$$=-[\Gamma(1-\alpha)/\alpha]\,(\beta+i\lambda)^{\alpha}$$

$$=\Gamma(-\alpha)\,(\beta+i\lambda)^{\alpha}.$$

On the other hand, if we put $u=t-s$, from (3. 4) the following formula holds:

(3. 5) $$\int_0^\infty u^{-\alpha-1}(1-e^{-(\beta+i\lambda)u})du$$

$$=\int_{-\infty}^t (t-s)^{-\alpha-1}(1-e^{-(\beta+i\lambda)(t-s)})ds$$

$$=e^{-i\lambda t}\int_{-\infty}^t (t-s)^{-\alpha-1}(e^{i\lambda t}-e^{-\beta(t-s)}e^{i\lambda s})ds.$$

Hence, from (3. 4) and (3. 5), if we consider the operator Λ such that

(3. 6) $$\Lambda f(t)\equiv 1/\Gamma(-\alpha)\int_{-\infty}^t (t-s)^{-\alpha-1}\,[f(t)-e^{-\beta(t-s)}f(s)]\,ds$$

$$=1/\Gamma(-\alpha)\int_0^\infty u^{-\alpha-1}(I-e^{-\beta u}\tau_{-u})f(t)du,$$

then, putting $f(t)\equiv e^{i\lambda t}$, we get

(3. 7) $$\Lambda e^{i\lambda t}=1/\Gamma(-\alpha)\int_{-\infty}^t (t-s)^{-\alpha-1}(e^{i\lambda t}-e^{-\beta(t-s)}e^{i\lambda s})ds$$

$$=e^{i\lambda t}/\Gamma(-\alpha)\int_{-\infty}^t (t-s)^{-\alpha-1}(1-e^{-(\beta+i\lambda)(t-s)})ds$$

$$=e^{i\lambda t}(\beta+i\lambda)^{\alpha}.$$

The formula (3. 7) shows that the operator Λ defined by (3. 6) is belonging to the class $\widetilde{\Lambda}$ and its g. f. is $(i\lambda+\beta)^{\alpha}$. In accordance with [1°], we shall denote the above operator $\Lambda\equiv(D+\beta I)^{\alpha}$.

[3°] *S-C.L.Op.* D^{α}.

If $f(t)$ be the function such that the right hand side of formula (3. 6) is meaningful, and let $0<\alpha<1$, then we have

$$(3.8)\qquad \lim_{\beta\downarrow 0} 1/\Gamma(-\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha-1}[f(t)-e^{-\beta(t-s)}f(s)]ds$$
$$=1/\Gamma(-\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha-1}[f(t)-f(s)]ds$$
$$=1/\Gamma(-\alpha)\int_{0}^{\infty}u^{-\alpha-1}(I-\tau_{-u})f(t)du.$$

Let the operator, which is defined by the right hand side of (3.8), be D^α, then from (3.7) we can easily see that

$$D^\alpha e^{i\lambda t}=e^{i\lambda t}(i\lambda)^\alpha.$$

Hence we see that $D^\alpha \in \widetilde{\Lambda}$, $G_{D^\alpha}(\lambda)=(i\lambda)^\alpha$, $C_{D^\alpha}(z)=z^\alpha$. We shall call the operator D^α the ***differential operator of order*** α.

[4°] For general $\alpha\in R$, there exists an integer n such that $\alpha\equiv n+\alpha'$ $(0<\alpha'<1)$. Then above three operators are given by the followings:

(1) $\qquad (D+\beta I)^{-\alpha}f(t)\equiv(D+\beta I)^{-n}(D+\beta I)^{-\alpha'}f(t)$,

(2) $\qquad (D+\beta I)^{\alpha}f(t)\equiv(D+\beta I)^{n}(D+\beta I)^{\alpha'}f(t)$,

(3) $\qquad D^\alpha f(t)\equiv D^n D^{\alpha'}f(t)$,

where if $n>0$, $(D+\beta I)^{-n}$, $(D+\beta I)^{n}$ and D^n are given by the definition of differential operator D and the integral operators (2.2) and (2.3) and Proposition 2.3, and if $n>0$, the operator D^{-n} is defined as the following; let $F(t)$ be a continuous primitive of $f(t)$ at t, satisfying $F(0)=0$ and let $D^{-1}f(t)\equiv F(t)$, then from Proposition 2.3 we can see that $D^{-n}\equiv\underbrace{D^{-1}D^{-1}\cdots D^{-1}}_{n}$.

[5°] From Proposition 2.3 and [1°]—[4°] in this section, we can easily see that for any $\alpha\in R$

(1) $\qquad [(D+\beta I)^{\alpha}]^{-1}=(D+\beta I)^{-\alpha}$,

(2) $\qquad [(D+\beta I)^{-\alpha}]^{-1}=(D+\beta I)^{\alpha}$,

(3) $\qquad (D^\alpha)^{-1}=D^{-\alpha}$.

4. S-C.L.Op.s of Riemann-Liouville type belonging to Λ.

From the preceding Section 3, Definition 2.4 and Propositions 2.2 and 2.4, we can immediately obtain the followings:

[1°] For any $x\in\mathfrak{S}$ such as $(i\lambda+\beta)^{-\alpha}\in L^2(\mu_x;\mathscr{S})$, $\alpha>0$, $\beta>0$ $(\alpha, \beta\in R)$

$$(4.1)\qquad (D+\beta I)^{-\alpha}x(\varphi)\equiv\int_{-\infty}^{\infty}(\mathscr{F}\varphi)(\lambda)(i\lambda+\beta)^{-\alpha}dZ_x(\lambda)$$
$$=1/\Gamma(\alpha)\int_0^{\infty}u^{\alpha-1}e^{-\beta u}T_{-u}x(\varphi)du$$
$$=1/\Gamma(\alpha)\int_0^{\infty}u^{\alpha-1}e^{-\beta u}x(\tau_u\varphi)\ du.$$

[2°] For any $x\in\mathfrak{S}$ such as $(i\lambda+\beta)^{\alpha}\in L^2(\mu_x;\mathscr{S})$, $0<\alpha<1$, $\beta>0$ (α, $\beta\in R$)

$$(4.2)\qquad (D+\beta I)^{\alpha}x(\varphi)\equiv\int_{-\infty}^{\infty}(\mathscr{F}\varphi)(\lambda)(i\lambda+\beta)^{\alpha}dZ_x(\lambda)$$
$$=1/\Gamma(-\alpha)\int_0^{\infty}u^{-\alpha-1}(I-e^{-\beta u}T_{-u})x(\varphi)du$$
$$=1/\Gamma(-\alpha)\int_0^{\infty}u^{-\alpha-1}[x(\varphi)-e^{-\beta u}x(\tau_u\varphi)]du.$$

[3°] For any $x\in\mathfrak{S}$ such as $(i\lambda)^{\alpha}\in L^2(\mu_x;\mathscr{S})$, $0<\alpha<1$ ($\alpha\in R$)

$$(4.3)\qquad D^{\alpha}x(\varphi)\equiv\int_{-\infty}^{\infty}(\mathscr{F}\varphi)(\lambda)(i\lambda)^{\alpha}dZ_x(\lambda)$$
$$=1/\Gamma(-\alpha)\int_0^{\infty}u^{-\alpha-1}(I-T_{-u})x(\varphi)du$$
$$=1/\Gamma(-\alpha)\int_0^{\infty}u^{-\alpha-1}[x(\varphi)-x(\tau_u\varphi)]du.$$

[4°] The results of preceding Section 3, [4°] and [5°] are all obtained for the S-C.L.Op.s in $\mathfrak{S}$ except the case $D^{-\alpha}$ ($\alpha>0$). And for $x\in\mathfrak{S}$, from Proposition 2.2 the results are trivial, furthermore, for the covariance functions or covariance distributions the results are trivial from Proposition 2.2 and Definition 2.4, hence we shall omit to show the results. As for the operator D^{α} ($\alpha>0$), from Proposition 2.3, the following relation is not satisfied;

$$(D^{\alpha})^{-1}=D^{-\alpha},$$

(This relation does not hold in the case of $D^{\alpha}\in\boldsymbol{A}$, but in the Section 3, we defined formally that $D^{-1}f(t)\equiv F(t)$. In this case, from Proposition 2.3 the definition of $(D^{\alpha})^{-1}$ can not possible, as $G_{D^{\alpha}}(\lambda)$ has a zero on pure imaginary axis, i.e. $G_{D^{\alpha}}(0)=0$ hence the relation $(D^{\alpha})^{-1}=D^{-\alpha}$ does not hold from Proposition 2.3.)

5. Applications to the functional equations.

Let us consider the following stochastic functional equations:

$$(5.1)\qquad (D+\beta I)^{-\alpha}y=x, \quad \alpha>0,\ \beta>0\ (\alpha,\ \beta\in R),$$

$$(5.2)\qquad (D+\beta I)^{\alpha}y=x, \quad 0<\alpha<1,\ \beta>0\ (\alpha,\ \beta\in R),$$

$$(5.3)\qquad D^{\alpha}y=x, \quad 0<\alpha<1,\ (\alpha\in R),$$

where x and $y\in\mathfrak{S}$.

The above equations are meanigful if and only if the following relations

$$(5.1)'\qquad (i\lambda+\beta)^{-\alpha}dZ_y(\lambda)=dZ_x(\lambda),$$

$$(5.2)'\qquad (i\lambda+\beta)^{\alpha}dZ_y(\lambda)=dZ_x(\lambda),$$

$$(5.3)'\qquad (i\lambda)^{\alpha}dZ_y=dZ_x(\lambda)$$

hold for any $\lambda\in R$. Then from the preceding Section 4, [4°], we can see for the equations (5.1) and (5.2) the following relations hold:

$$(5.1)''\qquad y=((D+\beta I)^{-\alpha})^{-1}x=(D+\beta I)^{\alpha}x,$$

$$(5.2)''\qquad y=((D+\beta I)^{\alpha})^{-1}x=(D+\beta I)^{-\alpha}x.$$

Furthermore, from the definitions of above operators $(D+\beta I)^{\alpha}$ and $(D+\beta I)^{-\alpha}$ (Section 4, [1°] and [2°]) and Proposition 2.5, we can obtain the following **result**:

$$(5.4)\qquad (D+\beta I)^{-\alpha}=((D+\beta I)^{\alpha})^{-1}\in\Lambda_-,$$

$$(5.5)\qquad (D+\beta I)^{\alpha}=((D+\beta I)^{-\alpha})^{-1}\in\Lambda_-,$$

$$(5.6)\qquad L^2(Y;\ t)=L^2(X;\ t),\ \textit{for every } t\in R.$$

The other results are obtained from Section 4 in T. Seguchi [4].

6. Applications to the representations of certain stationary processes.

Now let us consider the following stochastic functional equations

$$(6.1)\qquad (D+\beta I)^{-\alpha}x=\dot{\xi}, \quad \alpha>0,\ \beta>0\ (\alpha,\ \beta\in R),$$

$$(6.2)\qquad (D+\beta I)^{\alpha}x=\dot{\xi}, \quad 0<\alpha<1,\ \beta>0\ (\alpha,\ \beta\in R),$$

$$(6.3)\qquad D^{\alpha}x=\dot{\xi}, \quad 0<\alpha<1\ (\alpha\in R),$$

where $x\in\mathfrak{S}_0$, $\dot{\xi}\in\mathfrak{S}_1$ which is the differentiation in the sense of random distribution of the homogeneous orthogonal random measure $d\xi$.

From Proposition 2.6 the equations (6. 1), (6. 2) and (6. 3) are meaningful if and only if the conditions

$$1/(i\lambda+\beta)^{\alpha'} \in L^2(m;\, 0), \quad N_A=\phi, \qquad (6.4)$$
$$(\beta\geq 0,\ \alpha' \in R,\ \lambda \in R)$$

are satisfied, where A is $(D+\beta I)^{-\alpha}$ in (6. 1), $(D+\beta I)^{\alpha}$ in (6. 2), D^{α} in (6. 3) and $\alpha'=\alpha$ or $-\alpha$, $\alpha>0$.

But the last condition of (6. 4) is not satisfied in the case of D^{α}, hence the equation (6. 3) is not meaningful.

For (6. 2), if $1/2<\alpha<1$ the conditions of (6. 4) are all satisfied and otherwise the first condition of (6. 4) is not satisfied. Hence the equation (6. 2) is meaningful if and only if the case that $1/2<\alpha<1$.

For (6. 1), the first condition of (6. 4) is not satisfied, hence the equation (6. 1) is not meaningful.

From the above discussions, finally, among the above equations, *only the equation* (6. 2) *is meaningful in the case* $1/2<\alpha<1$, $\beta>0$ $(\alpha,\ \beta \in R)$. In this case, we can obtain, from Section 3, [5°] and Propositions 2.2, 2.5, 2.6, the following **results**: (1) *There exists unique solution of equation* (6. 2) *and is represented by*

$$\begin{aligned} x(\varphi) &= (D+\beta I)^{-\alpha}\dot{\xi}(\varphi) \\ &= 1/\Gamma(-\alpha)\int_0^{\infty} u^{-\alpha-1}(I-e^{-\beta u}T_{-u})\dot{\xi}(\varphi)du \\ &= \int_{-\infty}^{\infty} (\mathscr{F}\varphi)(\lambda)(i\lambda+\beta)^{-\alpha}d\tilde{\xi}(\lambda), \end{aligned} \qquad (6.5)$$

where $d\tilde{\xi}(\lambda)$ *is the homogeneous orthogonal random measure corresponding to the white noise* $\dot{\xi}$, *and the covariance distribution of* x *is given by*

$$r_x(\varphi) = \int_{-\infty}^{\infty} (\mathscr{F}\varphi)(\lambda)\,|i\lambda+\beta|^{-2\alpha}d\lambda. \qquad (6.6)$$

(2) *Let* $x \in \mathfrak{S}^c$ *be the stationary process corresponding to the solution* $x \in \mathfrak{S}_0$ *of the equation* (6. 2), *then we have, from Proposition* 2.2 *and Definition* 2.4

$$\begin{aligned} x(t) &= 1/\Gamma(-\alpha)\int_0^{\infty} u^{-\alpha-1}(I-e^{-\beta u}\tau_{-u})d\xi(u) \\ &= \int_{-\infty}^{\infty} e^{i\lambda t}(i\lambda+\beta)^{-\alpha}d\tilde{\xi}(\lambda), \end{aligned} \qquad (6.7)$$

$$r_x(t)=\int_{-\infty}^{\infty} e^{i\lambda t}|i\lambda+\beta|^{-2\alpha}d\lambda, \tag{6.8}$$

where $d\tilde{\xi}$ *is that of relation* (6.5). (3) *Furthermore, in this case, the Paley-Wiener's condition is satisfied in addition to the conditions* (6.4), *by the same way as Proposition* 2.6, *we can see that the canonical representation of* $x \in \mathfrak{S}^{\circ}$ *satisfying the equation* (6.2) *is the first term of the expression* (6.7).

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