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https://doi.org/10.15017/1448928

出版情報：九州大学教養部数学雑誌. 5 (1), pp.1-20, 1967-03. General Education Department, Kyushu University
バージョン：
権利関係：

Math. Rep.
V-1, 1967.

ON CERTAIN TYPES OF RECURRENT EVENTS

By

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[1967. 11. 22]

§ 1. Introduction and preliminaries.

W. Feller [1], [2] treated generally, with many interesting examples, recurrent events in a sequence of repeated trials each of which can result only in denumerably many events. The present paper is also concerned with the same problems of a recurrent event $\mathcal{E}$ as Feller's, but this paper exclusively deals with the case where probability u_n that $\mathcal{E}$ occurs at the n-th trial satisfies certain difference equations.

1. 1. *Formulation of problems.* We take up a sequence of infinite trials each of which can result only in at most denumerably many events. The state space may be assumed to be $S=\{1, 2,\cdots\}$, without loss of generalities, in such case as above. Two probabilities that a certain event $\mathcal{E}$[1] occurs in this sequence of infinite trials are defined as follows:

(1. 1. 1) $\quad u_n=\mathrm{P}(\mathcal{E}$ occurs at the n-th trial)

(1. 1. 2) $\quad f_n=\mathrm{P}(\mathcal{E}$ occurs for the first time at the n-th trial)

$n=1, 2, \cdots$, and for convenience we put $u_0=1$, and $f_0=0$.

The following definitions are supposed to be known conceptions to readers, as they are explained in the following parts of Feller [1], Chapter XIII.

(1) $\mathcal{E}$ is a ***recurrent event***: p. 282, Definition 1,

(2) Recurrent event $\mathcal{E}$ is ***persistent*** or ***transient***: p. 283, Definition 2,

(3) Recurrent event $\mathcal{E}$ is ***periodic*** or ***non-periodic***: p. 284, Definition 3.

Readers may also refer to interesting examples of recurrent events of various kinds given in the foregoing chapter of Feller [1].

Now we take an interest in the following two cases, where $\{u_n\}$, corresponding to the recurrent event $\mathcal{E}$, satisfies each of the difference equations

1) For example: $\mathcal{E}=\{1, 2, 3, 4\}$ (set), $\mathcal{E}=(1, 2, 3, 4)$ (sequence), $\mathcal{E}=(E_1, E_1, E_2,)=(\{1, 2\}, \{1, 2\}, \{2, 3\})$ (sequence of sets), etc..

given below.

CASE [I]: where $\{u_n\}$ satisfies a difference equation

$$\text{(I)} \qquad \sum_{j=0}^{k} a_j u_{n-j}=0, \qquad n\geq k+1,$$

CASE [II]: where $\{u_n\}$ satisfies a difference equation

$$\text{(II)} \qquad \sum_{j=0}^{k} a_j u_{n-j}=p, \qquad n\geq k+1.$$

We assume in either case that $\{a_n\}_{n=0,1,2,\cdots,k}$ and p ($\neq 0$) are known constants, and that $a_0=1$.

The *problems* which we should take up here are

(1) investigating the *properties of a recurrent event* $\mathcal{E}$ corresponding to the equation (I) or (II) (for instance, whether $\mathcal{E}$ is transient or persistent; if persistent, the calculations of mean recurrence time $\mu=\sum_{n=1}^{\infty} n f_n$ and its variance $\sigma^2=\sum_{n=1}^{\infty}(n-\mu)^2 f_n$; etc.)

(2) seeking the *conditions for the existence of persistent recurrent event* $\mathcal{E}$ corresponding to the equations (I) or (II),

(3) showing interesting *examples* relating Cases [I] and [II]; investigating the *behavior of* $\{u_n\}$ *as* $n\to\infty$ and other probabilistic problems.

1.2. *Known results.* Given below are known results to be used to study the above problems. The *generating function* of sequences $\{u_n\}$ and $\{f_n\}$ defined in (1.1.1) and (1.1.2) are represented as follows:

$$(1.2.1) \qquad U(s)=\sum_{n=0}^{\infty} s^n u_n$$

$$(1.2.2) \qquad F(s)=\sum_{n=0}^{\infty} s^n f_n.$$

1°) If $\sum_{n=1}^{\infty} f_n=1$, then

$$(1.2.3) \qquad \mu=\sum_{n=1}^{\infty} n f_n=\lim_{s\uparrow 1} F'(s)$$

$$(1.2.4) \qquad \sigma^2=\sum_{n=1}^{\infty}(n-\mu)^2 f_n=\lim_{s\uparrow 1} F''(s)+\mu-\mu^2.$$

2°) If $\mathcal{E}$ is a recurrent event, the following relations hold,

$$(1.2.5) \qquad u_n=f_0\, u_n+f_1\, u_{n-1}+f_2\, u_{n-2}+\dots+f_n\, u_0,$$

$$(1.2.6) \qquad U(s)=1/(1-F(s)), \quad |s|<1.$$

3°) For $\mathcal{E}$ to be transient, it is necessary and sufficient that $u=\sum_{n=0}^{\infty} u_n$ is finite (i.e. $u=U(1)<+\infty$). Furthermore, the following relation holds,

$$f=\sum_{n=0}^{\infty} f_n=(u-1)/u<1.$$

4°) Let $\mathcal{E}$ be persistent and non-periodic, then u_n tends to $1/\mu$, as $n\to\infty$, where μ is given by (1.2.3), and if $\mu=+\infty$, $u_n\to 0$.

5°) If $\mathcal{E}$ is persistent and has period $\lambda>1$, then $u_{\lambda n'}$ tends to λ/μ as $n'\to\infty$, and $u_n=0$ for every n not divisible by λ.

6°) If s_ν (multiplicity r_ν) ($\nu=1, 2, \cdots, m$) denotes the root of the characteristic equation

$$\tilde{A}(s)\equiv s^k A(s^{-1})=\sum_{j=0}^{k} a_j\ s^{k-j}=0 \tag{1.2.7}$$

of the equation (I), then the general solutions of the equation (I) and (II) are respectively given by

$$u_n=\sum_{\nu=1}^{m}(c_{\nu 1}+c_{\nu 2}\ n+c_{\nu 3}\ n^2+\cdots+c_{\nu\, r_\nu}\ n^{r_\nu-1})s_\nu^n, \tag{1.2.8}$$

$$u_n=\sum_{\nu=1}^{m}(c_{\nu 1}+c_{\nu 2}\ n+c_{\nu 3}\ n^2+\cdots+c_{\nu\, r_\nu}\ n^{r_\nu-1})s_\nu^n+p/(\sum_{j=0}^{k}a_j). \tag{1.2.9}$$

The function $A(s)$ in the equation (1.2.7) is given in Lemma 2.1.1 which will appear later.

1.3. *Summary.* In next section we shall first take up the Case [I] and get necessary conditions for existence a recurrent event corresponding to the equation (I), formulas to calculate the mean recurrence time μ and its variance σ^2, behavior of $\{u_n\}$ as $n\to\infty$ and a sufficient condition of existence of non-periodic persistent recurrent event corresponding to the equation (I), etc..

In § 3, we shall investigate the same matters as § 2 in the Case [II], and get the results corresponding to that of § 2. Furthermore interesting examples will be shown.

In § 4, we shall remark that using the results of § 2 and § 3 problems which are treated as related problems for recurrent events in Feller [1] and [2], can be investigated and almost every Feller's results may be obtained in our case.

§ 2. Case [I].

2.1. The following are to be discussed in this section under the assumption that $\{u_n\}$, corresponding to a certain recurrent event $\mathcal{E}$, which was defined in (1.1.1) satisfies the equation

$$\sum_{j=0}^{k} a_j\ u_{n-j}=0,\quad n\geq k+1. \tag{I}$$

(i) The condition for $\mathcal{E}$ to be persistent or transient.

(ii) The calculations of μ and σ^2 when $\mathcal{E}$ is persistent.

(iii) The necessary condition for the existence of a recurrent event $\mathcal{E}$ corresponding to the equation (I).

(iv) Some remarks when $\mathcal{E}$ is a periodic recurrent event.

There may be other simpler methods for the proofs of lemmas and the development of discussion, but consideration has been taken here to make the necessary conditions in (iii) as strong as possible.

First we derive the following lemma.

LEMMA 2.1.1. *For generating function $U(s)$ of $\{u_n\}$ which satisfies the equation* (I), *the following relation holds*:

$$U(s)\,A(s)=H(s),\quad |s|<1 \tag{2.1.1}$$

where

$$A(s)=\sum_{n=0}^{k} s^n a_n, \tag{2.1.2}$$

$$H(s)=\sum_{n=0}^{k} s^n \sum_{n=0}^{n} u_{n-j}\ a_j=\sum_{n=0}^{k} s^n h_n \tag{2.1.3}$$

and

$$h_n=\sum_{j=0}^{n} a_j\ u_{n-j}=\sum_{j=0}^{n} u_j\ a_{n-j},\ n=0,\ 1,\ 2, \cdots, k. \tag{2.1.4}$$

PROOF. If we add the equations $\sum_{j=0}^{n} a_j\ n_{n-j}=\sum_{j=0}^{n} a_j\ u_{n-j},\ n=0,\ 1,\ 2, \cdots,\ k$, before the equation (I), and multiply the both sides of n-th equation by s^n and sum up by all n, then we get the relation (2.1.1), where $A(s)$, $H(s)$ and $\{h_n\}$ are given by (2.1.2), (2.1.3) and (2.1.4). Q.E.D.

As $A(s)$ and $H(s)$ in the relation (2.1.1) are the polynomials of degree at most k in s, they can be expressed as follows;

$$A(s)=\sum_{n=0}^{k}(s-1)^n A^{(n)}(1)/n!=\sum_{n=0}^{k}(s-1)^n A_n \tag{2.1.5}$$

$$H(s)=\sum_{n=0}^{k}(s-1)^n H^{(n)}(1)/n!=\sum_{n=0}^{k}(s-1)^n H_n \tag{2.1.6}$$

where

$$A_n=A^{(n)}(1)/n!=\sum_{\nu=n}^{k}\binom{\nu}{n}a_\nu \text{ and } H_n=H^{(n)}(1)/n!=\sum_{\nu=n}^{k}\binom{\nu}{n}h_\nu,\quad n=0, 1, 2, \cdots, k. \tag{2.1.7}$$

Now let (A, l) and (H, m) (l, $m=0$, 1, 2, $\cdots$, k) be the following conditions:

(A, l) $\quad A_0=A_1=A_2=\cdots=A_{l-1}=0,\ A_l\neq 0,$

(H, m) $\quad H_0=H_1=H_2=\cdots=H_{m-1}=0,\ H_m\neq 0.$

LEMMA 2.1.2. *Under conditions* (A, l) *and* (H, m), *we have*

(2.1.8) $$U(s)=\sum_{n=m}^{k}(s-1)^n H_n/\sum_{n=l}^{k}(s-1)^n A_n.$$

Hence

1°) *if* $l<m$, *then* $\lim_{s\uparrow 1} U(s)=0$,

2°) *if* $l=m$, *then*

(i) $0<\lim_{s\uparrow 1} U(s)<+\infty$, *when* $A_l H_l>0$,

(ii) $-\infty<\lim_{s\uparrow 1} U(s)<0$, *when* $A_l H_l<0$,

3°) *if* $l>m$, *then*

(i) $\lim_{s\uparrow 1} U(s)=\infty$, *when* $(-1)^{l-m}A_l H_m>0$,

(ii) $\lim_{s\uparrow 1} U(s)=-\infty$, *when* $(-1)^{l-m}A_l H_m<0$.

PROOF. We can derive $U(s)=H(s)/A(s)$, $|s|<1$, from (2.1.1). Substitute (2.1.5) and (2.1.6) into it, and we easily get 1°), 2°) and 3°) from conditions (A, l) and (H, m). Q.E.D.

Now, we get $U(s)>1$ $(0<s<1)$ from the definition of $\{u_n\}$, so that 1°); 2°), (ii) and 3°), (ii) are contradictions. Furthermore, we must assume that $H_l/A_l\geq 1$ in the case of 2°), (i). Therefore, for the existence of a recurrent event $\mathcal{E}$ corresponding to the equation (I), one of the following two conditions must be satisfied:

$(\mathrm{T}')_{\mathrm{I}}$ (A, l) *and* (H, l) *hold and* $H/_l A_l\geq 1$,

$(\mathrm{P}')_{\mathrm{I}}$ (A, l) *and* (H, m) $(l>m)$ *hold and* $(-1)^{l-m}H_m/A_l>0$.

LEMMA 2.1.3. 1°) *If condition* $(\mathrm{T}')_{\mathrm{I}}$ *holds, the following expression is made possible,*

(2.1.9) $$F(s)=1-A(s)/H(s)=1-\sum_{\nu=l}^{k}(s-1)^{\nu-l}A_\nu/\sum_{\nu=l}^{k}(s-1)^{\nu-l}H_\nu$$

(2.1.10) $$\lim_{s\uparrow 1}F(s)=1-A_l/H_l.$$

2°) *If condition* $(\mathrm{P}')_{\mathrm{I}}$ *holds, the following expression is made possible,*

(2.1.11) $$F(s)=1-A(s)/H(s)=1-\sum_{\nu=l}^{k}(s-1)^{\nu-l}A_\nu/\sum_{\nu=m}^{k}(s-1)^{\nu-m}H_\nu$$

(2.1.12) $$\lim_{s\uparrow 1}F(s)=\sum_{n=1}^{\infty}f_n=1,$$

(2.1.13) $$\mu=\sum_{n=1}^{\infty}n f_n=\begin{cases}-A_l/H_{l-1}, & if \quad m=l-1,\\ 0, & if \quad m<l-1,\end{cases}$$

(2.1.14) $$\sigma^2=\lim_{s\uparrow 1}F''(s)+\mu-\mu^2=\{2(A_l H_l-A_{l+1}H_{l-1})-A_l H_{l-1}-A_l^2\}/H_{l-1}^2,\ if\ m=l-1.$$

PROOF. As the existence of recurrent event $\mathcal{E}$ corresponding to equation (I) is assumed in this section, generating function $F(s)$ of $\{f_n\}$ corresponding to $\mathcal{E}$ is, from (1.2.4), represented as follows;

$$F(s)=1-1/U(s)=1-A(s)/H(s).$$

Substitute (2.1.5) and (2.1.6) into the above formula and take conditions $(\mathrm{T}')_{\mathrm{I}}$ and $(\mathrm{P}')_{\mathrm{I}}$ into consideration, and we get (2.1.9) and (2.1.11) in accordance with the respective conditions. Furthermore, if $s \uparrow 1$, we get (2.1.10) and (2.1.12). Differentiating (2.1.11) under condition $(\mathrm{P}')_{\mathrm{I}}$, we get

$$F(s)=\frac{-A_l H_m(l-m)(s-1)^{l-m-1}+\cdots+\mathrm{const.}(s-1)^{2(k-m-1)}}{H_m^2+2H_m H_{m+1}(s-1)+\cdots+\mathrm{const.}(s-1)^{2(k-m)}}. \tag{2.1.15}$$

Hence we have (2.1.13) from § 1.2, 1°).

Furthermore, when $m=l-1$, differentiating (2.1.15) once more, we have

(2.1.16)

$$F''(s)=\frac{2A_l H_{l-1}^2 H_l-2A_{l+1}H_{l-1}^3+(s-1)\{\text{polynomial of degree at most } 4(k-l)\}}{H_{l-1}^4+(s-1)\{\text{polynomial of degree at most } 4(k-l)+3\}}.$$

Hence we have (2.1.14) from § 1.2, 1°). Q.E.D.

When condition $(\mathrm{T}')_{\mathrm{I}}$ holds, and if $H_l/A_l=1$, it is derived from (2.1.10) of Lemma 2.1.3 that $\lim_{s\uparrow 1} F(s)=\sum_{n=0}^{\infty} f_n=1-A_l/H_l=0$. Since $f_n \geq 0$ $(n=1, 2, \cdots)$ it follows that $f_n=0$ $(n=1, 2, \cdots)$. It means that recurrent event $\mathcal{E}$ never occurs in a finite trial. Accordingly the case where $H_l/A_l=1$ is meaningless, so that this case is excluded. Hence, in order that $\mathcal{E}$ may be a transient recurrent event, condition

$(\mathrm{T})_{\mathrm{I}}$ (A, l) *and* (H, l) *hold and* $H_l/A_l>1$

must be satisfied, instead of $(\mathrm{T}')_{\mathrm{I}}$.

When condition $(\mathrm{P}')_{\mathrm{I}}$ is satisfied, we know from (2.1.12) that $\mathcal{E}$ is a persistent recurrent event. But as (2.1.3) is obtained as a result of the calculation of $\mu=\sum n f_n$, it is required from § 1.2, 1°) that $\mu \geq 1$ is satisfied. Accordingly in order that $\mathcal{E}$ may be a persistent recurrent event, condition

$(\mathrm{P})_{\mathrm{I}}$ (A, l) *and* (H, $l-1$) *hold and* $1 \geq -H_{l-1}/A_l>0$

must be satisfied, instead of $(\mathrm{P}')_{\mathrm{I}}$.

From the preceding, we get the following theorem.

THEOREM 2.1.1. *When probabilities* $\{u_n\}$ *corresponding to recurrent event* $\mathcal{E}$ *satisfy the equation*

$$\sum_{j=0}^{k} a_j\, u_{n-j}=0, \quad a_0=1, \quad n \geq k+1, \tag{I}$$

the conditions $(\mathrm{T})_{\mathrm{I}}$ *or* $(\mathrm{P})_{\mathrm{I}}$ *should be satisfied.*

Furthermore

1°) *if condition* $(\mathrm{T})_{\mathrm{I}}$ *is satisfied, then* $\mathcal{E}$ *is a transient recurrent event, and*

$$f=\sum_{n=1}^{\infty}f_n=1-A_l/H_l, \tag{2.1.17}$$

2°) *if condition* $(\mathrm{P})_{\mathrm{I}}$ *is satisfied, then* $\mathcal{E}$ *is a persistent recurrent event, and mean recurrence time* μ *and its variance* σ^2 *are always existent in the following forms:*

$$\mu=-A_l/H_{l-1} \tag{2.1.18}$$

and

$$\sigma^2=\{2(A_lH_l-A_{l+1}H_{l-1})-A_lH_{l-1}-A_l^2\}/H_{l-1}^2. \tag{2.1.19}$$

Some remarks on periodic case. Let us suppose that $\{u_n\}$, corresponding to periodic recurrent event $\mathcal{E}$ with period λ, satisfies equation (I). Then $\{u_n\}$ is such that

$$\begin{cases} 0\leq u_{\lambda n'}\leq 1, & n'=0,\ 1,\ 2,\cdots, \\ u_{\lambda n'+\nu}=0, & \nu=1,\ 2,\cdots,\ \lambda-1;\ n'=0,\ 1,\ 2,\cdots. \end{cases} \tag{2.1.20}$$

Therefore it is required to satisfy the following equations instead of equation (I),

$$\begin{cases} u_{\lambda n'}=-\sum_{j=1}^{k'}a_{\lambda j}\ u_{\lambda(n'-j)}=0, & n'\geq k'+1, \\ a_\nu\ u_{\lambda n'+\nu}=-\sum_{j=0}^{k'-1}a_{\lambda j+\nu}\ u_{\lambda(n'-j)}=0, & \nu=1,\ 2,\cdots,\ \lambda-1;\ n'\geq k', \end{cases} \tag{2.1.21}$$

where we suppose $k=\lambda k'$.

As stated in § 2. 1, $\{u_n\}$ and its behavior are known by equation (I), and $A(s)$ and $H(s)$. In a periodic case, however, it is enough for only $\{u_{\lambda n'}\}$ and its behavior to be known, so that we may well consider the upper expressions alone of (2. 1. 20) and (2. 1. 21). Consequently if we assume λ to be one unit time and put $u_n^{(\lambda)}\equiv u_{\lambda n}$, then discussions can be reduced to a non-periodic case:

$$\sum_{j=0}^{k'}a_{\lambda j}\ u_{n-j}^{(\lambda)}=0,\quad n\geq k'+1, \tag{I$^\lambda$}$$

$$A_\lambda(s)=\sum_{n=0}^{k'}s^n a_{\lambda n}, \tag{2.1.22}$$

$$H_\lambda(s)=\sum_{n=0}^{k'}s^n\sum_{j=0}^{n}a_{\lambda j}\ u_{n-j}^{(\lambda)}\equiv\sum_{n=0}^{k'}s^n h_n^{(\lambda)} \tag{2.1.23}$$

where

$$h_n^{(\lambda)}=\sum_{j=0}^{n}a_{\lambda j}\ u_{n-j}^{(\lambda)},\quad n=0,\ 1,\ 2,\cdots,k', \tag{2.1.24}$$

and we may have the same results as were showed or shall be showed later, by taking § 1. 2, 5°) etc. into considerations. Accordingly, we shall only consider the case of non-periodic in the following § 2. 2.

Furthermore the following is to be noted, when coefficient $\{a_n\}$ of the equation (I) is periodic (i.e. $a_{\lambda n'}$ may be not 0, $n'=0, 1, 2, \cdots, k'=k/\lambda$; $a_n=0$, $k>n\neq\lambda n'$), the lower expression of (2.1.21) never fails to hold. Therefore, when we discuss a periodic recurrent event, we may discuss the problem as a non-periodic case as stated above. However, attention is to be paid (especially after § 2.2) to the fact that there are cases where non-periodic recurrent event exists even when $\{a_n\}$ is periodic.

2.2. We shall discuss, without the existence of recurrent event $\mathcal{E}$ as a premise, the following matters when the equation

$$\text{(I)} \qquad \sum_{j=0}^{k} a_j\, u_{n-j}=0, \quad n\geq k+1$$

satisfies condition $(\mathrm{P})_{\mathrm{I}}$.

(i) The behavior of $\{u_n\}$, especially as $n\to\infty$,

(ii) A sufficient condition for the existence of a recurrent event $\mathcal{E}$ corresponding to the equation (I).

It is to be noted that since $(\mathrm{P})_{\mathrm{I}}$ holds, it follows that $A(1)=\sum_{n=0}^{k} a_n=0$. Furthermore, we introduce the following condition regarding $A(s)$:

$(\mathrm{R})_{\mathrm{I}}$ *The absolute values of the roots of equation* $A(s)=0$ *excepting* 1 *are larger than* 1.[2)]

First we derive the following lemma.

LEMMA 2.2.1. *If* $\{a_n\}$ *and* $\{u_n\}$ $(n=0, 1, 2, \cdots, k)$ *are so given as to satisfy conditions* $(\mathrm{P})_{\mathrm{I}}$ *and* $(\mathrm{R})_{\mathrm{I}}$, *then* $\{u_n\}$ *which satisfies the equation* (I) *is convergent as* $n\to\infty$ *and*

$$\text{(2.2.1)} \qquad u_n\to -H_{l-1}/A_l.$$

PROOF. Condition $(\mathrm{R})_{\mathrm{I}}$ makes it possible by § 1.2, 6°) that $U(s)$ for general solution $\{u_n\}$ of the equation (I) is convergent at $|s|<1$. Condition $(\mathrm{P})_{\mathrm{I}}$ makes (A, l) and (H, $l-1$) hold, so that, from Lemma 2.1.2, the following reduction is effected:

$$\text{(2.2.2)} \qquad U(s)=H(s)/A(s)=\sum_{j=l-1}^{k}(s-1)^{j-l+1}H_j\Big/\sum_{j=l}^{k}(s-1)^{j-l+1}A_j$$
$$=P(s)+\hat{H}(s)/\hat{A}(s),$$

where $\hat{H}(s)/\hat{A}(s)$ is irreducible rational function, and {degree of $\hat{A}(s)$} = {degree of $H(s)$} +1, and $P(s)$ is a polynomial in s with degree at most $k_0=$ {degree of $H(s)$} − {degree of $A(s)$}.

It is known that, from the fact that (A, l) and (H, $l-1$) hold and from

2) In a persistent periodic case, the condition $(\mathrm{R})_{\mathrm{I}}$ is not satisfied.

(2.2.2), the roots of $\hat{A}(s)=0$ are parts of roots of $A(s)=0$ and $s=1$ is a simple root. Now let the root of $\hat{A}(s)=0$ excepting $s=1$ be s_ν (multiplicity r_ν), $\nu=1, 2, \cdots, m$, then $\hat{H}(s)/\hat{A}(s)$ can be decomposed into partial fractions as follows;

(2.2.3) $$\hat{H}(s)/\hat{A}(s)=\hat{H}(s)/\{(s-1)(s-s_1)^{r_1}(s-s_2)^{r_2}\cdots(s-s_m)^{r_m}\}$$
$$=\frac{\rho_{01}}{1-s}+\sum_{\nu=1}^{m}\left\{\frac{\rho_{\nu1}}{s_\nu}\sum_{n=0}^{\infty}\binom{-1}{n}\left(-\frac{s}{s_\nu}\right)^n+\cdots+\frac{\rho_{\nu r_\nu}}{s_\nu^{r_\nu}}\sum_{n=0}^{\infty}\binom{-r_\nu}{n}\left(-\frac{s}{s_\nu}\right)^n\right\}$$

where $\rho_{01}, \rho_{\nu1}, \rho_{\nu2}, \cdots, \rho_{\nu r_\nu}$ ($\nu=1, 2, \cdots, m$) are constants.

Therefore, (2.2.3) can be reduced, with $\binom{-m}{n}=(-1)^n\binom{n+m-1}{m-1}$, in the following form:

$$\hat{H}(s)/\hat{A}(s)=\sum_{n=0}^{\infty}\rho_{01}s^n+\sum_{\nu=1}^{m}\sum_{n=0}^{\infty}\left\{\binom{n}{0}\frac{\rho_{\nu1}}{s_\nu^{n+1}}+\binom{n+1}{1}\frac{\rho_{\nu2}}{s_\nu^{n+2}}+\cdots+\binom{n+r_\nu-1}{r_\nu-1}\frac{\rho_{\nu r_\nu}}{s_\nu^{n+r_\nu}}\right\}s^n.$$

From the above expression and (2.2.2) we have, for $n\geq k_0$,

(2.2.4) $$u_n=\rho_{01}+\sum_{\nu=1}^{m}\left\{\binom{n}{0}\frac{\rho_{\nu1}}{s_\nu^{n+1}}+\binom{n+1}{1}\frac{\rho_{\nu2}}{s_\nu^{n+2}}+\cdots+\binom{n+r_\nu-1}{r_\nu-1}\frac{\rho_{\nu r_\nu}}{s_\nu^{n+r_\nu}}\right\}.$$

Since $|s_\nu|>1$, u_n tends to ρ_{01}, as $n\to\infty$, where ρ_{01} is given by

$$\rho_{01}=\lim_{s\uparrow1}(1-s)\hat{H}(s)/\hat{A}(s)=-H_{l-1}/A_l. \qquad \text{Q.E.D.}$$

In Lemma 2.2.1 we introduced condition $(\mathrm{R})_\mathrm{I}$ with regard to the equation $A(s)=0$. Now let us consider the sufficient conditions to satisfy $(\mathrm{R})_\mathrm{I}$. When $A(s)$ satisfies condition (A, l), $l\geq1$, $a_k\neq0$ and $s=1$ is a root of equation $A(s)=0$ and its multiplicity is l. If we put

(2.2.5) $$A(s)=(s-1)^lA^*(s),$$

then, since $A^*(s)$ is a polynomial in s with degree $(k-l)$, we can put

(2.2.6) $$A^*(s)=\sum_{n=0}^{k-l}a_n^*s^n.$$

Especially when $l=1$,

(2.2.7) $$A^*(s)=\sum_{n=0}^{k-1}\Big(\sum_{j=n+1}^{k}a_j\Big)s^n,$$

i.e. $a_n^*=\sum_{j=n+1}^{k}a_j$ ($n=0, 1, 2, \cdots, k-1$).

Because $A^*(s)=0$ has not $s=1$ as its root, the *sufficient conditions to satisfy condition* $(\mathrm{R})_\mathrm{I}$, i.e. the sufficient conditions for the absolute value of arbitrary root s_ν of the equation $A^*(s)=0$ to be larger than 1 are given by the following:[3)]

1°) If (A, l) holds, then $|a_0^*|<\sum_{j=1}^{k-l}|a_j^*|$. Especially if (A, l) holds, then

3) See, for example, Fujihara [3], p.p. 484-493.

$|\sum_{j=1}^{k} a_j| > \sum_{n=1}^{k-1} |\sum_{j=n+1}^{k} a_j|,$

2°) If (A, l) holds, then $a_0^* > a_1^* > \cdots > a_{k-l}^* > 0$ or $a_0^* < a_1^* < \cdots < a_{k-l}^* < 0$. Especially if (A, l) holds, then $0 < a_k < a_k + a_{k-1} < \cdots < \sum_{j=1}^{k} a_j$ or $0 > a_k > a_k + a_{k-1} > \ldots > \sum_{j=1}^{k} a_j$.

3°) If (A, l) holds, then a_n^* $(n=0, 1, 2, \cdots, k-l)$ are the same sign and the least value of a_{k-l+1}^*/a_{k-l}, $a_{k-l+2}^*/a_{k-l+3}^*, \cdots, a_0^*/a_1^*$ is larger than 1. Especially if (A, l) holds, then $\sum_{j=n}^{k} a_j (n=1, 2, \cdots, k)$ are the same sign and the least value of $(a_k + a_{k-1})/a_k$, $(a_k + a_{k-1} + a_{k-2})/(a_k + a_{k-1}), \cdots, (\sum_{j=1}^{k} a_j)/(\sum_{j=2}^{k} a_j)$ is larger than 1.

Next, from equation (I) we can introduce (k, k)-matrix $\boldsymbol{B}_l$, such that

$$\boldsymbol{B}_l = \begin{pmatrix} b_1 & b_2 & \ldots & b_{k-1} & b_k \\ 1 & & O & 0 & 0 \\ & 1 & & \vdots & \vdots \\ & O & \ddots & 1 & 0 \end{pmatrix} \tag{2.2.8}$$

where $b_n = -a_n$, $(n=1, 2, \ldots, k)$. Then matrix $\boldsymbol{B}_l$ can be regarded as the linear transformation from k-dimensional vector space into k-dimensional vector space. And that is equivalent to the equation (I), if we put

$\boldsymbol{u}_n \equiv \begin{pmatrix} u_n \\ u_{n-1} \\ \vdots \\ u_{n-k+1} \end{pmatrix}$ $(n=k, k+1, k+2, \cdots)$ and represent

$$\boldsymbol{u}_n = \boldsymbol{B}_l \boldsymbol{u}_{n-1}, \quad n \geq k+1, \tag{2.2.9}$$

where it is noted that $\{b_n\}$ satisfies $\sum_{n=1}^{k} b_n = 1$.

From the above considerations we have

LEMMA 2.2.2.[4] *If conditions* (P)$_{\text{I}}$ *and* (R)$_{\text{I}}$ *are satisfied, then* $\lim_{n\to\infty} \boldsymbol{B}_l^n = \tilde{\boldsymbol{B}}_l$ *is existent and represented as*

$$\tilde{\boldsymbol{B}}_l = \begin{pmatrix} \beta_1 & \beta_2 \cdots \beta_k \\ \beta_1 & \beta_2 \cdots \beta_k \\ \vdots & \vdots \quad \vdots \\ \beta_1 & \beta_2 \cdots \beta_k \end{pmatrix} \tag{2.2.10}$$

where

$$\beta_n = \sum_{j=n}^{k} b_j / \sum_{j=1}^{k} j\, b_j \qquad (n=1, 2, \cdots, k) \tag{2.2.11}$$

and $\{\beta_n\}$ *satisfies the relation*

$$\sum_{n=1}^{k} \beta_n = 1. \tag{2.2.12}$$

PROOF. It follows from $u_n \to -H_{l-1}/A_l \equiv \tilde{u}$ $(n\to\infty)$ that $\boldsymbol{u}_n$ tends to

4) The authors would like to express our appreciation to Professor T. Tsuda for various advices in this lemma.

$\tilde{\boldsymbol{u}}=\begin{pmatrix}\tilde{u}\\ \tilde{u}\\ \vdots\\ \tilde{u}\end{pmatrix}=\lim_{n\to\infty} B_I^n \boldsymbol{u}_k$, as $n\to\infty$. Hence it is clear that $\tilde{B}_I=\lim_{n\to\infty} B_I^n$ is existent, and that, for arbitrary n, $m>0$,

(2.2.13) $$B_I^n\tilde{B}_I=\tilde{B}_IB_I^m=\tilde{B}_I \text{ and } \tilde{B}_I^2=\tilde{B}_I$$

should hold.

Now if we put

$$\tilde{B}_I=\begin{pmatrix}\beta_{11} & \beta_{12}\cdots\beta_{1k}\\ \beta_{21} & \beta_{22}\cdots\beta_{2k}\\ \vdots & \vdots \quad\quad \vdots\\ \beta_{k1} & \beta_{k2}\cdots\beta_{kk}\end{pmatrix}$$

then it follows from $B_I\tilde{B}_I=\tilde{B}_I$ that $\beta_{mj}=\beta_{nj}\equiv\beta_j$ $(1\leq j,\ m,\ n\leq k)$. Hence (2.2.10) holds.

Again it follows from $\tilde{B}_I^2=\tilde{B}_I$ that $\sum_{j=1}^{k}\beta_j=1$.

Next, from $\tilde{B}_I=\tilde{B}_IB_I$ we get

$$\beta_1=\beta_1b_1+\beta_2,\ \beta_2=\beta_1b_2+\beta_3,\ \cdots,\ \beta_{k-1}=\beta_1b_{k-1}+\beta_k \text{ and } \beta_k=\beta_1b_k.$$

If we reduce the above equations, using $\sum_{j=1}^{k}\beta_j=1$ and $\sum_{j=1}^{k}b_j=1$, then we have (2.2.11). Q.E.D.

The following lemma will be evident if we pay attention to $\boldsymbol{u}_{m+n}=B_I^m\boldsymbol{u}_n \to\tilde{\boldsymbol{u}}=\tilde{B}_I\boldsymbol{u}_n$ as $m\to\infty$.

LEMMA 2.2.3. *The following relation is effected between limiting value $\tilde{u}$ of u_n $(n=1, 2, \cdots)$, which satisfies the equation* (I) *under conditions* $(\mathrm{P})_\mathrm{I}$ *and* $(\mathrm{R})_\mathrm{I}$, *and* u_n, $u_{n-1}, \cdots, u_{n-k+1}$ $(n\geq k+1)$;

(2.2.14) $$\tilde{u}=\beta_1\ u_n+\beta_2\ u_{n-1}+\cdots+\beta_k\ u_{n-k+1}.$$

This result means that, when we consider in the k-dimensional space, all points $\tilde{\boldsymbol{u}}_n(n\geq k+1)$ are moving on the hyper-plane which is passing the point $\tilde{\boldsymbol{u}}$ and its normal vector is $\boldsymbol{\beta}=\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_k\end{pmatrix}$.

From Lemmas 2.2.1∼3, we come to the conclusion that

THEOREM 2.2.1. *If $\{a_n\}$ and $\{u_n\}$ $(n=0, 1, 2, \cdots, k)$ are so given as to satisfy conditions* $(\mathrm{P})_\mathrm{I}$ *and* $(\mathrm{R})_\mathrm{I}$, *then $\{u_n\}$ which satisfies the equation* (I) *tends to $\tilde{u}$ as $n\to\infty$. Besides, if we consider the linear transformation with (k, k) matrix B_I defined by* (2.2.8) *then the points $\{\boldsymbol{u}_n;\ n\geq k+1\}$ move on the hyper-plane, which passes the point $\tilde{\boldsymbol{u}}$ and whose normal vector is $\boldsymbol{\beta}$, and converge to the point $\tilde{\boldsymbol{u}}$.*

From Theorem 2.2.1 the following theorem is derived.

THEOREM 2.2.2. *When* $(P)_I$ *and* $(R)_I$ *hold, there exists a non-periodic persistent recurrent event* $\mathcal{E}$ *corresponding to* $\{u_n\}$ *which satisfies the equation* (I).

PROOF. If we put $u_1=u_2=\cdots=u_k=\tilde{u}$ where $\tilde{u}$ is a constant value such that $0<\tilde{u}<1$, it is clear from $\sum_{n=1}^{k} b_n=1$ that $u_n=\tilde{u}$ $(n=1, 2,\cdots)$.

Furthermore, $F(s)$ is expressed as follows by $F(s)=1-1/U(s)$,

$$F(s)=1-1/(1+\sum_{n=1}^{\infty}\tilde{u}s^n)=1-(1-s)/\{1-(1-\tilde{u})s\}$$
$$=\sum_{n=1}^{\infty}\tilde{u}(1-\tilde{u})^{n-1}s^n, \qquad |s|<1.$$

Hence we have $f_n=\tilde{u}\ (1-\tilde{u})^{n-1}$, $0<f_n<1$ and $\sum_{n=1}^{\infty}f_n=1$. Therefore, there exists a persistent non-periodic recurrent event $\mathcal{E}$ which makes the above $\{u_n\}$ and $\{f_n\}$ be corresponding probabilities of (1.1.1) and (1.1.2). Q.E.D..

§ 3. Case [II].

3.1. In this section the same matters as were treated in (i), (ii), (iii) and (iv) of § 2.1 will be discussed under the assumption that $\{u_n\}$, corresponding to a certain recurrent event $\mathcal{E}$, satisfies the equation

$$\text{(II)} \qquad \sum_{j=0}^{k}a_j\ u_{n-j}=p, \quad n\geq k+1.$$

Consideration has been taken also here so as to make the necessary condition of (iii) in particular as strong as possible in spite of the possibility of simpler proof of results and development of discussion.

LEMMA 3.1.1. *Generating function* $U(s)$ *of* $\{u_n\}$, *which satisfies the equation* (II), *satisfies the following relation,*

$$\text{(3.1.1)} \qquad U(s)\ A(s)=K(s), \quad |s|<1$$

where $A(s)$ *and* $H(s)$ *are the same as* (2.1.2) *and* (2.1.3), *and*

$$\text{(3.1.2)} \qquad K(s)=H(s)+ps^{k+1}/(1-s).$$

PROOF. We can prove this lemma in the same way as that of Lemma 2.1.1.

As in § 2.1, $A(s)$ and $H(s)$ can be represented in (2.1.5) and (2.1.6). The same symbols and assumptions or conditions as in § 2 are also employed here.

In the same way as we get Lemma 2.1.2 from Lemma 2.1.1, we have, from Lemma 3.1.1, the following lemma.

LEMMA 3.1.2. *If conditions* (A, l) *and* (H, m) *are satisfied, then*

$$\text{(3.1.3)} \qquad U(s)=K(s)/A(s)=(ps^{k+1}-\sum_{\nu=m}^{k}(s-1)^{\nu+1}H_\nu)/(-\sum_{\nu=l}^{k}(s-1)^{\nu+1}A_\nu).$$

Hence we have

1°) *If* $(-1)^l p A_l > 0$, *then* $\lim_{s \uparrow 1} U(s) = \infty$,

2°) *If* $(-1)^l p A_l < 0$, *then* $\lim_{s \uparrow 1} U(s) = -\infty$.

The case of 2°) is contradictory for the same reasons as were stated immediately following Lemma 2.1.2 in § 2.1.

As nothing but either 1°) or 2°) is possible in Case [II], we get the following corollary.

COROLLARY. *If a recurrent event which corresponds to the equation* (II) *exists, then it is persistent. The necessary condition for it is that*

$(P')_{II}$ (A, l) *and* (H, m) *hold, and* $(-1)^l p A_l > 0$.

LEMMA 3.1.3. *If condition* $(P')_{II}$ *holds, then*

$$F(s) = 1 - (1-s) A(s) / \{ps^{k+1} + (1-s) H(s)\} \to 1, \quad (s \uparrow 1) \tag{3.1.4}$$

and

$$\mu = \begin{cases} \sum_{n=0}^{k} a_n / p, & \text{if (A, 0) holds,} \\ 0, & \text{if (A, } l) \ (l > 1) \text{ holds.} \end{cases} \tag{3.1.5}$$

If (A, 0) *holds, then*

$$\sigma^2 = 2A'(1)/p - 2A(1)\{p(k+1)) - H(1)\}/p^2 + A(1)/p - A^2(1)/p^2. \tag{3.1.6}$$

PROOF. We get (3.1.4) from 1°) *in* § 1.2 and (3.1.1), and it follows from $p \neq 0$ that $\lim_{s \uparrow 1} F(s) = 1$. Differentiating (3.1.4) we have

(3.1.7)

$$F'(s) = \frac{\{A(s) - (1-s)A'(s)\}\{ps^{k+1} + (1-s)H(s)\} + (1-s)A(s)\{ps^{k+1} + (1-s)H(s)\}'}{\{ps^{k+1} + (1-s)\, H(s)\}^2}$$

From (3.1.7) and § 1.2, 1°) we have (3.1.5) which is a formula to obtain mean recurrence time μ. Furthermore, if (A, 0) holds and we differentiate (3.1.7) again and employ (1.2.4) in § 1.2, 1°), then we get (3.1.6) which is a formula to obtain σ^2. Q.E.D.

If $\mathcal{E}$ is a non-periodic persistent recurrent event, then it may be supposed that $\lim_{s \uparrow 1} F'(s) = \lim_{n \to \infty} 1/u_n \geq 1$, so that the necessary condition for the existence of a recurrent event corresponding to the equation (II) is

$(P)_{II}$ (A, 0) *and* $A(1)/p = \sum_{n=0}^{k} a_n / p \geq 1$ *hold,*

where the case when an equality holds is a trivial one, i.e. $\{f_n\} = \{0, 1, 0, 0, \dots\}$.

From the preceding, we have the following theorem.

THEOREM 3.1.1. *When probabilities* $\{u_n\}$ *corresponding to a recurrent event* $\mathcal{E}$

satisfy the equation

(II) $$\sum_{j=0}^{k} a_j\ u_{n-j}=p, \quad n\geq k+1,$$

the condition $(\mathrm{P})_{\mathrm{II}}$ *should be satisfied.*

Hence, in this case the recurrent event $\mathcal{E}$ *is only persistent, and mean recurrence time* μ *and its variance* σ^2 *are always existent in the following forms:*

(3.1.8) $$\mu=\sum_{n=0}^{k} a_n/p$$

and

(3.1.9) $$\sigma^2=2A'(1)/p-2A(1)\{p(k+1)-H(1)\}/p^2+A(1)/p-A^2(1)/p^2.$$

respectively.

Remarks on periodic case. We don't know whether a periodic recurrent event $\mathcal{E}$, which corresponds to the equation (II), exists or not. However, when there exists a periodic recurrent event $\mathcal{E}$ with period λ, then discussions can be reduced to a non-periodic case by considering λ to be one unit time and putting $u_n^{(\lambda)}\equiv u_{\lambda n}$. Accordingly, we shall consider only the case of non-periodic hereafter.

3.2. In the same way as in §2.2, we shall discuss, without the existence of recurrent event $\mathcal{E}$ as a premise, the following two matters when the equation

(II) $$\sum_{j=0}^{k} a_j\ u_{n-j}=p, \quad n\geqq k+1,$$

satisfies the condition $(\mathrm{P})_{\mathrm{II}}$.

(i) The behavior of $\{u_n\}$, especially as $n\to\infty$.

(ii) A sufficient condition for the existence of a recurrent event $\mathcal{E}$ which corresponds to equation (II).

Here, too, we introduce the following condition regarding the roots of equation $A(s)=0$.

$(\mathrm{R})_{\mathrm{II}}$ *The absolute values of the roots of* $A(s)=0$ *are larger than* 1.[5)]

First we derive the following lemma.

LEMMA 3.2.1. *If* a_n $(n=0,1,2,\cdots,k)$ *and* p *satisfy conditions* $(\mathrm{P})_{\mathrm{II}}$ *and* $(\mathrm{R})_{\mathrm{II}}$, *then* $\{u_n\}$ *which satisfies the equation* (II) *is such that*

(3.2.1) $$\lim_{n\to\infty} u_n=p/(\sum_{j=0}^{k} a_j).$$

PROOF. Lemma 3.1.1 holds even now as it is a result independent of recurrent event $\mathcal{E}$. Condition $(\mathrm{R})_{\mathrm{II}}$ makes it proved by general solution of

5) In a persistent periodic case, the condition $(\mathrm{R})_{\mathrm{II}}$ is not satisfied.

the equation (II), which was given by (1.2.9) in § 1.2, 6°), that $U(s)$ for $\{u_n\}$, an arbitrary solution of the equation (II), is converged at $|s|<1$. Hence we have

(3.2.2) $$U(s)=K(s)/A(s)=C(s)/\{(1-s)A(s)\},$$

where

(3.2.3) $$C(s)=(1-s)\,H(s)+ps^{k+1}=\sum_{\nu=0}^{k+1}(s-1)^{\nu}c_{\nu},\ c_{\nu}=C^{(\nu)}(1)/n!.$$

From $(P)_{\text{II}}$, $A(1)\neq 0$ and from $p\neq 0$, $C(1)\neq 0$, so that we have

$$U(s)=\sum_{\nu=0}^{k+1}(s-1)^{\nu}c_{\nu}/-\sum_{\nu=0}^{k}(s-1)^{\nu+1}A_{\nu}$$
$$=-c_{k+1}/A_k-\{c_0+\sum_{\nu=1}^{k}(s-1)^{\nu}(c_{\nu}A_k-c_{k+1}A_{\nu-1})/A_k\}/\sum_{\nu=1}^{k}(s-1)^{\nu+1}A_{\nu}.$$

The denominator of the reduced second term of the above expression has $s=1$ as its simple root, and it is evident from $(R)_{\text{II}}$ that the absolute values of the other roots are larger than 1.

Let the latter roots be denoted as s_{ν} (multiplicity r_{ν}), $(\nu=1, 2, \cdots, m)$, and if we reduce the preceding expression into partial fractions expansion in the same way as (2.2.3), then we have

$$U(s)=-c_{k+1}/A_k+\rho_{01}/(1-s)+\sum_{\nu=1}^{m}\{\rho_{\nu1}/(s_{\nu}-s)+\ldots+\rho_{\nu r_{\nu}}/(s_{\nu}-s)^{r_{\nu}}\}$$
$$=-c_{k+1}/A_k+\rho_{01}\sum_{n=0}^{\infty}s^n+\sum_{\nu=1}^{m}(\rho_{\nu1}/s_{\nu}\sum_{n=0}^{\infty}\binom{-1}{n}(-s/s_{\nu})^n+\ldots+(\rho_{\nu r_{\nu}}/s_{\nu}^{r_{\nu}})$$
$$\times\sum_{n=0}^{\infty}\binom{-r_{\nu}}{n}(-s/s_{\nu})^n),\quad |s|<1<\min_{\nu}.|s_{\nu}|.$$

Furthermore, in the same way as calculated § 2.2, for $n\geq 1$,

$$u_n=\rho_{01}+\sum_{\nu=1}^{m}\left(\rho_{\nu1}/s_{\nu}^{n+1}\binom{n}{0}+\rho_{\nu2}/s_{\nu}^{n+2}\binom{n+1}{1}+\cdots+\rho_{\nu r_{\nu}}/s_{\nu}^{n+r_{\nu}}\binom{n+r_{\nu}-1}{r_{\nu}-1}\right)$$

Hence if $n\to\infty$, as $|s_{\nu}|>1$, we have

$$u_n\to\rho_{01}=\lim_{s\uparrow 1}(1-s)\,U(s)=c_0/A_0=C(1)/A(1)=p/(\sum_{n=0}^{k}a_n).\qquad \text{Q.E.D.}$$

Next, as in § 2.2, from the equation (II) we can introduce $(k+1, k+1)$ -matrix B_{II}, such that

(3.2.3) $$B_{\text{II}}=\begin{pmatrix}1 & 0 & 0 & \cdots & 0\\ 1 & b_1 & b_2 & \cdots & q_k\\ & 1 & & & \\ O & & \ddots & O & \\ & & & 1 & 0\end{pmatrix}$$

where $b_n=-a_n$, $n=1, 2, \cdots, k$. Then B_{II} can be regarded as the linear transformation from $(k+1)$-dimensional vector space into $(k+1)$-dimensional vector space, which is equivalent to equation (II), if we put

$$\boldsymbol{u}_n=\begin{pmatrix} p \\ u_n \\ u_{n-1} \\ \vdots \\ u_{n-k+1} \end{pmatrix}$$ $(n=k,\ k+1,\ k+2,\cdots)$ and represent

$$\boldsymbol{u}_{n+1}=B_{II}\boldsymbol{u}_n,\ n\geq k+1. \tag{2.2.4}$$

From Lemma 3.2.1, if $(\mathrm{P})_{\mathrm{II}}$ and $(\mathrm{R})_{\mathrm{II}}$ hold, then it follows that as $n\to\infty$, $\boldsymbol{u}_n$ converges to $\tilde{\boldsymbol{u}}$ $(0<\tilde{u}<1)$. This makes the following lemma hold in the same manner as in the case of Lemma 2.2.2.

LEMMA 3.2.2. *Udern the conditions* $(\mathrm{P})_{\mathrm{II}}$ *and* $(\mathrm{R})_{\mathrm{II}}$,

1°) $\lim_{n\to\infty} B^n_{\mathrm{II}}=\tilde{B}_{\mathrm{II}}$ *exists, and for any integers n, m*>0

$$B^m_{\mathrm{II}}\tilde{B}_{\mathrm{II}}=\tilde{B}_{\mathrm{II}}B^n_{\mathrm{II}}=\tilde{B}_{\mathrm{II}},\ \textit{and}\ \tilde{B}^2_{\mathrm{II}}=\tilde{B}_{\mathrm{II}}. \tag{3.2.5}$$

2°) *Matrix* $\tilde{B}_{\mathrm{II}}$ *is given by*

$$\tilde{B}_{\mathrm{II}}=\begin{pmatrix} 1 & \\ 1/\sum_{n=0}^{k} a_n & \\ \vdots & \Large 0 \\ 1/\sum_{n=0}^{k} a_n & \end{pmatrix}. \tag{3.2.6}$$

Lemma 3.2.2, as well as Theorem 3.1.1 and Lemma 3.2.1, shows that $\{u_n\}$, which satisfies the equation (II), tends to $p/\sum_{n=0}^{k} a_n$ independently of the existence of a recurrent event corresponding to it, and that its limiting value is independent of $(u_1, u_2, \cdots, u_k)$. Furthermore, it is known as a result of Lemma 3.2.2, that $\{\boldsymbol{u}_n\}$ tends to $\tilde{\boldsymbol{u}}$ as $n\to\infty$.

From Lemmas 3.2.1 and 3.2.2, we came to the conclusion that

THEOREM 3.2.1. *If* $\{a_n\}_{n=0,1,2,\cdots,k}$ *and p of the equation* (II) *are so given as to satisfy the conditions* $(\mathrm{P})_{\mathrm{II}}$ *and* $(\mathrm{R})_{\mathrm{II}}$, *then* $\{u_n\}$ *which satisfies the equation* (II) *tends to* $p/\sum_{j=0}^{k} a_j$ *as* $n\to\infty$, *and this limiting value is independent of initial values* $(u_1, u_2, \ldots, u_k)$.

Furthermore, we have

THEOREM 3.2.2. *If* $\{a_n\}_{n=0,1,2,\cdots,k}$ *and p are so given as to satisfy the conditions* $(\mathrm{P})_{\mathrm{II}}$ *and* $(\mathrm{R})_{\mathrm{II}}$, *then there exists a non-periodic persistent recurrent event* $\mathcal{E}$ *corresponding to the equation* (II).

PROOF. From the condition $(\mathrm{P})_{\mathrm{II}}$, excepting the trivial case $u_n=\tilde{u}=p/\sum_{j=0}^{k} a_j=1$, we have $0<p/\sum_{j=0}^{k} a_j<1$. Hence if we put $u_n=\tilde{u}=p/\sum_{j=0}^{k} a_j$, $n=1, 2, \cdots, k$, then we easily see from the equation (II) $u_n=\tilde{u}$, $n=1, 2, \cdots$, and $U(s)=1+\sum_{n=1}^{\infty}\tilde{u}s^n$ (where $u_0=1$).

We define the function $F(s)$ formally by

$$F(s)=1-1/U(s)=1-1/(1+\sum_{n=1}^{\infty}\tilde{u}s^n)=1-(1-s)/(1-(1-\tilde{u})s)$$
$$=\sum_{n=1}^{\infty}\tilde{u}(1-\tilde{u})^{n-1}s^n, \qquad |s|<1,$$

and if we consider $F(s)$ is the generating function of $\{f_n\}$, then we get $f_n=\tilde{u}\ (1-\tilde{u})^{n-1}$, $n=1, 2, \cdots$. Hence

$$0<f_n<1,\ n=1,\ 2, \cdots, \text{ and } \sum_{n=1}^{\infty}f_n=1.$$

Therefore, there exists a persistent non-periodic recurrent event $\mathcal{E}$ which makes the above $\{u_n\}$ and $\{f_n\}$ be corresponding probabilities of (1.1.1) and (1.1.2). Q.E.D.

3.3. *Examples.* Here we take up a case of independent trials which is the special case treated in this paper.

Now let us assume that $P(\{i\})=p_i>0$ for each $i\in S$, and that the recurrent event $\mathcal{E}^*$ be denoted by one finite pattern $(E_1, E_2, \cdots, E_M)$ (where $E_j\subset S$, $j=1, 2, \cdots, M$) or finite sum of such finite patterns. It is known by Nakayama [5] that all the above types of recurrent events are persistent and such a recurrent event $\mathcal{E}^*$ can be decomposed into

$$\mathcal{E}^*=\mathcal{E}_1\dot{+}\mathcal{E}_2\dot{+}\cdots\dot{+}\mathcal{E}_N$$

i.e. the sum of recurrent events of the mutually disjoint recurrent events such as $\mathcal{E}_j=(i_{j1}, i_{j2}, \cdots, i_{jM_j})$, $j=1, 2, \cdots, N$, and moreover by Kitagawa and Seguchi [4], §3 that if the mean recurrence times of $\mathcal{E}^*$ and $\mathcal{E}_j$ is denoted by μ and μ_j $(j=1, 2, \cdots, N)$ respectively the following relation holds:

$$1/\mu=\sum_{j=1}^{N}1/\mu_j.$$

Therefore, if we calculate the mean recurrence time μ and its variance σ^2 of the following type of a recurrence event $\mathcal{E}=(i_1, i_2, \cdots, i_M)$ then for any recurrent event $\mathcal{E}^*$ as above, at least the mean recurrence time of $\mathcal{E}^*$ can be calculated. Hence we take up the following examples.

EXAMPLE 1 (General case). Let the recurrent event $\mathcal{E}$ be denoted by the pattern $\mathcal{E}=(i_1, i_2, \cdots, i_M)$ then the equation which is satisfied by $\{u_n\}$ corresponding to the recurrent event $\mathcal{E}$ is given by

$$\sum_{j=0}^{M-1}a_j\ u_{n-j}=p, \quad n\geq M$$

where

$$p=p_{i_1}p_{i_2}\cdots p_{i_M}$$
$$a_0=1$$
$$a_j=\begin{cases} p_{i_{j+1}}p_{i_{j+2}}\cdots p_{i_M}, & \text{if } (i_1, i_2, \cdots, i_j)=(i_{M-j+1}, i_{M-j-2}, \cdots, i_M) \\ 0 & \text{if } (i_1, i_2, \cdots, i_j)\neq(i_{M-j+1}, i_{M-j+2}, \cdots, i_M) \end{cases}$$
$$j=1, 2, \cdots, M-1.$$

As $u_0=1$, $u_1=0$, $u_2=0, \cdots, u_{M-1}=0$, $u_M=p$, we get

$$A(1)=H(1)=\sum_{n=0}^{M-1}a_n.$$

Hence from Theorem 3.2.1, we get

$$\mu=\sum_{n=0}^{N-1}a_n/p$$
$$\sigma^2=2\sum_{n=1}^{M-1}na_n/p-2\sum_{n=1}^{M-1}a_n(pM-\sum_{n=1}^{M-1}a_n)/p^2+\sum_{n=0}^{M-1}a_n/p-(\sum_{n=0}^{M-1}a_n)^2/p^2$$
$$=2\sum_{n=1}^{M-1}na_n/p+(1-2M)\sum_{n=0}^{M-1}a_n/p+(\sum_{n=0}^{M-1}a_n)^2/p^2.$$

Though following examples are included by Example 1, but more easily can be calculated the μ and σ^2. Hence we take up as our examples.

EXAMPLE 2. Case where $i_1=i_2=\cdots=i_M=i$.

This case has already been solved in Feller [1], Chapter VIII, because it is regarded as the case of the success run of length M in the sequence of Bernoulli trials.

However, we will calculate μ and σ^2 in our own way. The equation which $\{u_n\}$ satisfies is such that

$$\sum_{j=0}^{M-1}p^j\, u_{n-j}=p^M, \quad n\geq M,$$

where $p=p_i$, and thus the condition of Theorem 3.1.1 holds.

As $A(s)=H(s)=(1-p^Ms^M)/(1-ps)$, we get $A(1)=H(1)=(1-p^M)/q$ and $A'(1)=(-Mp^Mq+p(1-p^M))/q^2$, where $q=1-p$.

Therefore,

$$\mu=A(1)/p^M=(1-p^M)/(p^Mq)$$
$$\sigma^2=2A'(1)/p^M-2A(1)(p^MM-H(1))/p^{2M}+A(1)/p^M-A^2(1)/p^{2M}$$
$$=1/(qp^M)^2-(2M+1)/(qp^M)-p/q^2.$$

EXAMPLE 3. Case where $\max\{j;\ i_1=i_j\}\leq[M/2]$.[6] In this case, for any $1\leq k\leq M-1$ $(i_1, i_2, \cdots, i_k)\neq(i_{M-k+1}, i_{M-k+2}, \cdots, i_M)$. Hence the equation which $\{u_n\}$ satisfies is such that

6) [ ] is the notation of Gauss.

$$u_n=\begin{cases} p_{i_1}p_{i_2}\cdots p_{i_M}\equiv p, & n\geq M \\ 0, & n<M. \end{cases}$$

Therefore we get $A(s)=H(s)=1$, $A'(s)=0$ and

$$\mu=A(1)/p=1/p$$
$$\sigma^2=2A'(1)/p-2A(1)(pM-A(1))/p^2+A(1)/p-A^2(1)/p^2$$
$$=-2M/p+1/p+1/p^2.$$

§ 4. Appendix.

We have discussed the cases where $\{u_n\}$ corresponding to recurrent event $\mathcal{E}$ satisfies the equations (I) or (II), and obtained (1) the necessary conditions for the existence of recurrent event $\mathcal{E}$ correspending to the equation (I) or (II) are $(T)_{I}$ or $(P)_{I}$ in Case [I], and $(P)_{II}$ in Case [II], (2) if $(T)_{I}$ is satisfied, then (if recurrent event exists) $\mathcal{E}$ is a transient recurrent event, (3) if $(P)_{I}$ or $(P)_{II}$ is satisfied, then (if recurrent event exists) $\mathcal{E}$ is a persistent recurrent event and always means recurrence time μ and its variance σ^2 exists, (4) the formula which calculate μ and σ^2, (5) another attendant results.

Feller [1] and [2] proceeds to discuss probabilistic behaviors relating the number N_n of occurrences of $\mathcal{E}$ in n trials and number $T^{(r)}$ of trials up to and including the r-th occurrence of $\mathcal{E}$, their probability distributions and asymptotic behaviors as $n\to\infty$, and especially various limit theorems, etc.. Where the existence of mean recurrence time μ and its variance σ^2 is an essential condition, and the values of μ and σ^2 are playing an important role.

As were stated in the first part of this section, it has been proved that μ and σ^2 exist when the recurrent event $\mathcal{E}$ corresponding to equation (I) or (II) is persistent, and the formula to calculate the values of μ and σ^2 have also been obtained. Accordingly most of the results in Feller [1] and [2] which hold with regard to $T^{(r)}$ and N_n, hold in our cases, too. Consequently it is possible by means of the substitution of the values of μ and σ^2 to concrete reductions of Feller's various calculative results. No further mentions shall be made in this regard, which is only too evident as long as we avail ourselves of the results of Feller [1] and [2].

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