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The best constant of Sobolev inequality corresponding to biharmonic operator on a disk

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Abstract

We generalize our recent results on Sobolev inequalities to multi-variable case. In particular, investigating Green function of biharmonic operator on a disk, we find the best constant of the corresponding Sobolev inequality. This is a joint work with Yoshinori Kametaka of Osaka University and Alexander P. Veselov of Loughbourough University.

1 Introduction

The Sobolev inequality

$$||u||_{L^q(\Omega)} \leq C||\nabla u||_{L^p(\Omega)}, \quad u \in W^{M,p}(\Omega), \qquad \Omega \subset \mathbb{R}^N \tag{1}$$

played crucial roles in the development of modern theory of differential equations. In our recent papers [3, 5], we find a systematic approach of finding the least constant C_0 of Sobolev inequality (1) in a special case

$$q = \infty, p = 2, N = 1, \Omega = (a, b) \subset \mathbb{R},$$

or equivalently,

$$\left(\sup_{a<x<b} |u(x)|\right)^2 \leq C \int_a^b |u^{(M)}(x)|^2 dx. \tag{2}$$

Details are given in Figure 1. By investigating Green function for a given boundary value problems of ODE, the best constant C_0 and the best function $u_0(x)$ which attains "=" in the Sobolev inequality is given as follows:

$$C_0 = \sup_y G(y, y) = G(y_0, y_0), \quad u_0(x) = G(x, y_0).$$

We also generalize our results to discrete case [8], L^p case [6], estimate of $|u^j(x)|$ [7]. In this paper, we consider the multi-variable version of the Sobolev inequality starting from boundary value problem of biharmonic operator $\Delta^2 = (\partial_x^2 + \partial_y^2)^2$ on a disk.

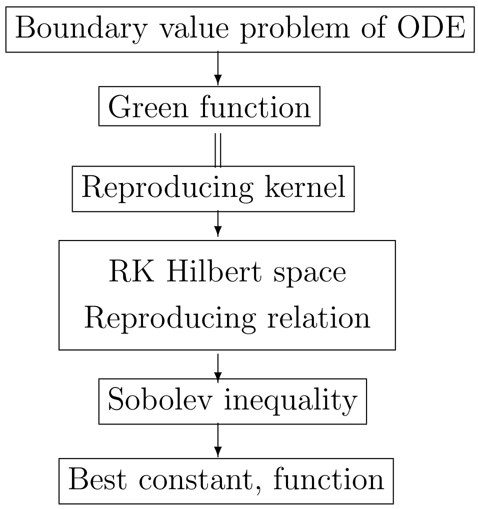

Figure 1: Procedures to find the best constant

2 Green function for biharmonic operator on a disk

We consider the following boundary value problem for biharmonic operator on a disk:

$$\begin{cases} \Delta^2 u = f(x) \quad (x = (x_1, x_2),\ |x| < R) \\ u(R\xi) = 0 \quad (|\xi| = 1) \\ Du(x)\Big|_{x=R\xi} = 0 \quad (|\xi| = 1) \end{cases} \tag{1}$$

where

$$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 = r^{-2}(D^2 + \partial_\theta^2), \quad D = x_1\partial_{x_1} + x_2\partial_{x_2} = r\partial_r$$
$$x_1 = r\cos\theta,\ x_2 = r\sin\theta.$$

The solution is given by

$$u(x, y) = u(x_1, x_2, y_1, y_2) = \int_{|y|<R} G(x, y) f(y) dy$$
$$\Leftrightarrow u(r, \theta) = \int_0^R \int_0^{2\pi} G(r, s, \theta, \varphi) f(s, \varphi) d\varphi s ds \quad (y_1 = s\cos\varphi,\ y_2 = s\sin\varphi),$$

where $G = G(x, y) = G(r, s, \theta, \varphi)$ is Green function given by [2]

$$G(r, s, \theta, \varphi) = G(r, s, \theta - \varphi) = \frac{rs}{8} \int_{\rho_0}^{\rho_1} \{\rho + \rho^{-1} - (\rho_1 + \rho_1^{-1})\} Q(\rho, \xi \cdot \eta) \rho^{-1} d\rho$$

$$\xi = x/|x|, \quad \eta = y/|y|, \quad \xi \cdot \eta = \cos(\theta - \varphi)$$

$$\rho_0 = R^{-2} rs, \ \rho_1 = \frac{r \wedge s}{r \vee s}$$

$$Q(r, \lambda) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2\lambda r + r^2}$$

It should be noted that $Q(r, \lambda)$ is a generating function of Chebysheff polynomials.

Next we derive corresponding Sobolev inequality together with its best estimate. We start with the sesquilinear form, which is proved to be an inner product of certain Hilbert space H later.

$$(u, v)_H := \int_{|x|<R} (\Delta u)(\Delta v) dx.$$

Then we have

$$\begin{aligned}
&(u(\cdot), G(\cdot, y))_H = \int_{|x|<R} (\Delta u(x))(\Delta G(x, y)) dx \\
&= \int_{|x|<R} u(x)(\Delta^2 G(x, y)) dx \\
&+ \int_{|x|=R} \{(\frac{\partial}{\partial n} u(x)) \Delta G(x, y) - u(x)(\frac{\partial}{\partial n} \Delta G(x, y))\} dx
\end{aligned}$$

where we have used Gauss-Green theorem. Considering that $\Delta^2 G(x, y) = \delta(x - y)$, we obtain

$$(u(\cdot), G(\cdot, y))_H = u(y) + \int_{|x|=R} \{(\frac{\partial}{\partial n} u(x)) \Delta G(x, y) - u(x)(\frac{\partial}{\partial n} \Delta G(x, y))\} dx.$$

In the above expressions, $\partial/\partial n$ denotes

$$\frac{\partial \psi}{\partial n} = \nabla \psi \cdot n = \cos\theta \frac{\partial \psi}{\partial x_1} + \sin\theta \frac{\partial \psi}{\partial x_2} = \frac{\partial \psi}{\partial r}.$$

From the above discussions, we have the following theorem.

Theorem 1 *Let H be Hilbert space defined by*

$$\begin{aligned}
H = \Big\{ u(x) = u(r, \theta) \ \Big| \ &\int_{|x|<R} |u(x)|^2 dx, \ \int_{|x|<R} |\Delta u(x)|^2 dx < \infty, \\
&u(R, \theta) = 0, \ \partial_r u(r, \theta)|_{r=R} = 0 \Big\}
\end{aligned}$$

which is equipped with an inner product

$$(u, v)_H := \int_{|x|<R} (\Delta u)(\Delta v) dx$$

Then $G(x, y)$ is a reproducing kernel of H, in other words, the following two properties hold.

$$(i)\ G(x, y) \in H \quad (|y| < R : \textit{fixed}) \Leftrightarrow G(x, y)\Big|_{|x|=R} = \partial_r G(x, y)\Big|_{|x|=R} = 0$$

$$(ii)\ (u(\cdot), G(\cdot, y))_H = \int_{|x|<R} (\Delta u(x))(\Delta G(x, y)) dx = u(y)$$

Concerning the best constant of Sobolev inequality, we have the following theorem:

Theorem 2 *There exists a positive constant C such that the following Sobolev inequality holds for any function $u(x) \in H$:*

$$\sup_{|y|<R} |u(y)|^2 \leq C \int_{|y|<R} |\Delta u(y)|^2 dy. \tag{2}$$

Among such C, the best constant C_0 is given by

$$C_0 = \sup_{0 \leq s < R} G(s, s, \varphi - \varphi) = G(0, 0, 0) = \frac{R^2}{16\pi}.$$

If one replaces C by C_0 in (2)*, the equality holds for*

$$u_0(y) = \kappa \left(-(y_1^2 + y_2^2) \log \frac{R^2}{y_1^2 + y_2^2} + R^2 - y_1^2 - y_2^2 \right)$$

Proof of Theorem 2 : Applying Cauchy-Schwarz inequality to the reproducing relation (ii), we have

$$|u(y)| = |(u(\cdot), G(\cdot, y))_H| \leq \|u\|_H \cdot \|G(\cdot, y)\|_H$$
$$|u(y)|^2 \leq G(y, y)(u, u)_H$$
$$\sup_{|y|<R} |u(y)|^2 \leq \sup_{|y|<R} G(y, y) \int_{|y|<R} |\Delta u(y)|^2 dy$$

which proves (2). Best constant C_0 is given by

$$C_0 = \sup_{|y|<R} G(y, y) = \sup_{0 \leq s < R} G(s, s, \varphi - \varphi)$$

From Kametaka formula, we have

$$G(s,s,0) = \frac{s^2}{16\pi}\int_{R^{-2}s^2}^{1}(\rho+\rho^{-1}-2)\frac{1-\rho^2}{1-2\rho+\rho^2}\rho^{-1}d\rho$$

$$= \frac{s^2}{16\pi}\left[-\rho^{-1}-\rho\right]_{R^{-2}s^2}^{1} = \frac{1}{16\pi}\left\{-2s^2+R^2+\frac{s^4}{R^2}\right\} = \frac{1}{16\pi R^2}(R^2-s^2)^2$$

and therefore

$$C_0 = \sup_{0\le s<R} G(s,s,\varphi-\varphi) = G(0,0,0) = \frac{R^2}{16\pi}$$

We finally find the best function which attains "=" in (2).

$$u_0(y) = u_0(s) = C\lim_{r\to+0} G(r,s,\theta-\varphi)$$

$$= C\lim_{r\to+0} rs\int_{R^{-2}rs}^{rs^{-1}}(\rho^2+1-sr^{-1}\rho-rs^{-1}\rho)\frac{1-\rho^2}{1-2\rho\cos(\theta-\varphi)+\rho^2}\rho^{-2}d\rho$$

Putting $\rho = r\sigma,\ d\rho = rd\sigma$,

$$u_0(s) = C\lim_{r\to+0} s\int_{R^{-2}s}^{s^{-1}}(r^2\sigma^2+1-s\sigma-r^2s^{-1}\sigma)\frac{1-r^2\sigma^2}{1-2r\sigma\cos(\theta-\varphi)+r^2\sigma^2}\sigma^{-2}d\sigma$$

$$= C\,s\int_{R^{-2}s}^{s^{-1}}\left\{\lim_{r\to+0}(r^2\sigma^2+1-s\sigma-r^2s^{-1}\sigma)\frac{1-r^2\sigma^2}{1-2r\sigma\cos(\theta-\varphi)+r^2\sigma^2}\sigma^{-2}\right\}d\sigma$$

$$= C\int_{R^{-2}s}^{s^{-1}}(-s^2\sigma^{-1}+s\sigma^{-2})d\sigma$$

$$= C\left(-s^2\log\frac{R^2}{s^2}+R^2-S^2\right) = C\left(-(y_1^2+y_2^2)\log\frac{R^2}{y_1^2+y_2^2}+R^2-y_1^2-y_2^2\right)$$

3 Green function for biharmonic operator on a 3D sphere

We next consider the following boundary value problems for biharmonic operator on a 3D sphere [4]:

$$\begin{cases} \Delta^2 u = f(x) \quad (x=(x_1,x_2,x_3),\ |x|<R) \\ u(R\xi) = 0 \quad (|\xi|=1) \\ Du(x)\Big|_{x=R\xi} = 0 \quad (|\xi|=1) \end{cases} \tag{1}$$

where

$$\Delta = \partial_{x_1}^2+\partial_{x_2}^2+\partial_{x_3}^2, \quad D = x_1\partial_{x_1}+x_2\partial_{x_2}+x_3\partial_{x_3} = r\partial_r$$

$$x_1 = r\sin\theta\cos\varphi,\ x_2 = r\sin\theta\sin\varphi,\ x_3 = r\cos\theta$$

Solution is given by

$$u(x,y) = u(x_1,x_2,x_3,y_1,y_2,y_3) = \int_{|y|<R} G(x,y)f(y)dy$$
$$\Leftrightarrow u(r,\theta) = \int_0^R \int_0^\pi \int_0^{2\pi} G(r,s,\theta,\tilde{\theta},\varphi,\tilde{\varphi})f(s,\tilde{\theta},\tilde{\varphi})r\sin^2\tilde{\theta}d\tilde{\varphi}d\tilde{\theta}ds$$

where Green function G is given by

$$G(r,s,\theta,\tilde{\theta},\varphi,\tilde{\varphi}) = \frac{(rs)^{\frac{1}{2}}}{8}\int_{\rho_0}^{\rho_1}\{\rho+\rho^{-1}-(\rho_1+\rho_1^{-1})\}Q(\rho,\lambda)\rho^{-\frac{1}{2}}d\rho$$
$$\xi = x/|x|, \quad \eta = y/|y|,$$
$$\lambda = \xi\cdot\eta = \sin\theta\sin\tilde{\theta}\cos(\varphi-\tilde{\varphi})+\cos\theta\cos\tilde{\theta}$$
$$\rho_0 = R^{-2}rs, \ \rho_1 = \frac{r\wedge s}{r\vee s}$$
$$Q(r,\lambda) = \frac{1}{4\pi}\frac{1-r^2}{(1-2\lambda r+r^2)^{\frac{3}{2}}}$$

The following theorem shows that the Green function is a reproducing kernel of a certain Hilbert space.

Theorem 1 *Let H be Hilbert space defined by*

$$H = \Big\{u(x) = u(x_1,x_2,x_3) = u(r,\theta,\varphi) \ \Big| \ \int_{|x|<R}|u(x)|^2dx, \ \int_{|x|<R}|\Delta u(x)|^2dx < \infty, \ u(R,\theta,\varphi) = 0, \ \partial_r u(r,\theta,\varphi)|_{r=R} = 0\Big\}$$

which is equipped with an inner product

$$(u,v)_H := \int_{|x|<R}(\Delta u)(\Delta v)dx$$

Then $G(x,y)$ is a reproducing kernel of H, in other words, the following two properties hold.

$$(i) \ G(x,y)\in H \quad (|y|<R : \textit{fixed}) \Leftrightarrow G(x,y)\Big|_{|x|=R} = \partial_r G(x,y)\Big|_{|x|=R} = 0$$
$$(ii) \ (u(\cdot),G(\cdot,y))_H = \int_{|x|<R}(\Delta u(x))(\Delta G(x,y))dx = u(y)$$

Taking the similar procedures as 2D case, we have the following theorem:

Theorem 2 *There exists a positive constant C such that the following Sobolev inequality holds for any function $u(x) \in H$:*

$$\sup_{|y|<R} |u(y)|^2 \leq C \int_{|y|<R} |\Delta u(y)|^2 dy. \tag{2}$$

Among such C, the best constant C_0 is given by

$$C_0 = \sup_{0 \leq s < R} G(s,s) = G(0,0) = \frac{R}{16\pi}.$$

If one replaces C by C_0 in (2), the equality holds for

$$u(y) = C\left(R - \sqrt{y_1^2 + y_2^2 + y_3^2}\right)^2$$

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