A NOTE ON THE EFFICIENCY OF TAMURA'S \$ Q \$

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A NOTE ON THE EFFICIENCY OF TAMURA'S Q

By

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1. Introduction

Let X_1, X_2, \dots, X_m and Y_1, Y_2, \dots, Y_n be random samples from the symmetric and continuous c. d. f. F(x) and $G(x) = F(x/\theta)$ respectively.

For testing the statistical hypothesis $H: \theta = 1$ against the alternative $AH: \theta > 1$ Tamura [3] has proposed the following test statistics.

(1)
$$Q_s^{(1)} = \frac{1}{\binom{m}{s}\binom{n}{2}} \Sigma^* \phi(x_{\alpha_1}, \cdots, x_{\alpha_s}; y_{\beta_1}, y_{\beta_2})$$

where

$$\phi(x_1, x_2, \dots, x_s; y_1, y_2) = \begin{cases} 1 & \text{for } y_1 < x_1, \dots, x_s < y_2 \text{ or } y_2 < x_1, x_2, \dots, x_s < y_1, \\ 0 & \text{otherwise.} \end{cases}$$

and the summation Σ^* extends over all subscripts α , β such that $1 \leq \alpha_1 < \cdots < \alpha_s \leq m$, $1 \leq \beta_1 < \beta_2 \leq n$.

Among the statistics $Q_s^{(1)}$, $s = 1, 2, \cdots$, the interesting one would be $Q_1^{(1)}$ and $Q_2^{(1)}$. It has been proved that these two statistics have the same Pitman efficiencies. Thus $Q_1^{(1)}$ would be more practical than $Q_2^{(1)}$, since it is very easy to compute.

To make our investigation more precise we shall also consider the following statistics which are the same types of $Q_s^{(1)}$.

(2)
$$Q_s^{(2)} = \frac{1}{\binom{n}{s}\binom{m}{2}} \Sigma^{**}\phi(y_{\alpha_1}, \cdots, y_{\alpha_s}; x_{\beta_1}, x_{\beta_2}), \quad s = 1, 2,$$

where Σ^{**} extends over all subscripts α , β such that $1 \leq \alpha_1 < \cdots < \alpha_s \leq n$, $1 \leq \beta_1 < \beta_2 \leq m$.

The purpose of this paper is to make further comparisons to these test statistics, which will make us to recommend $Q_1^{(2)}$ instead of $Q_1^{(1)}$ or $Q_2^{(1)}$ in practical situations.

In section 2 we shall consider the comparison of $Q_s^{(i)}$, i, s = 1, 2 from the view point of the Bahadur asymptotic efficiency [1]. As pointed by Bahadur [2], Bahadur asymptotic efficiency has some pitfalls since it is an approximate measure of efficiency. Thus our results in section 2 might not be enough reliable. Therefore we shall in section 3 compute the small sample power of these test statistics and give comparisons of these powers for a specific distribution.

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2. Comparison of $Q_s^{(i)}$, *i*, s = 1, 2 by the Bahadur asymptotic efficiency

Let denote the mean of $Q_s^{(i)}$, *i*, s=1, 2 by $\mu_{i,s}(\theta)$. Then we get

(4)
$$\mu_{1,s}(\theta) = 2 \iint_{x < y} [F(y) - F(x)]^s dG(x) d(y)$$

(5)
$$\mu_{2,s}(\theta) = 2 \iint_{x < y} [G(y) - G(x)]^s dF(x) dF(y) .$$

Especially in the null case we get

(6)
$$\mu_{1,s}(1) = \mu_{2,s}(1) = \frac{2}{(s+1)(s+2)}$$

Let $m = \rho N$, $n = (1-\rho)N$ and denoting the asymptotic variance of ${}^{I}_{\Delta}Q_{s}^{(i)}$ under null hypothesis by $\sigma_{i,s}^{2}$, then following the manner of Tamura [3] we get

(7)
$$\sigma_{i,s}^2 = \frac{1}{(s+1)^2} \frac{8}{\rho(1-\rho)} \left[\frac{1}{2s+3} - \frac{2}{(s+2)^2} + \frac{[(s+1)!]^2}{(2s+3)!} \right], \quad s = 1, 2.$$

Let normalize the statistics $Q_s^{(i)}$ as follows.

$$Q_{\scriptscriptstyle N,s}^{\scriptscriptstyle (i)} = \; rac{\sqrt{N} \left(Q_s^{\scriptscriptstyle (i)} - \mu_s(1)
ight)}{\sigma_{i,s}} \;$$
 , $i = 1, 2$.

Then we get under non null hypothesis

(8)
$$E\left[\frac{Q_{N,s}^{(i)}}{\sqrt{N}}\right] = \frac{1}{\sigma_{i,s}} \left(\mu_{i,s}(\theta) - \mu_{i,s}(1)\right).$$

Thus by using Chebychev's inequality and the definition of the asymptotic slope, the asymptotic slope $C(Q_s^{(i)}; \theta)$ of statistic $Q_s^{(i)}$ is obtained, after some calculations, as follows.

$$C(Q_s^{(i)}:\theta) = \frac{1}{\sigma_{i,s}^2} (\mu_{i,s}(\theta) - \mu_{i,s}(1))^2.$$

Then from (4), (5) and (7) we get

(9)
$$C(Q_1^{(1)}:\theta) = 720\rho(1-\rho)\left(\int FGdG - \frac{1}{3}\right)^2,$$

(10)
$$C(Q_2^{(1)}:\theta) = 180\rho(1-\rho)\left(\int F^2 dG - \frac{1}{3}\right)^2,$$

(11)
$$C(Q_1^{(2)}:\theta) = 720\rho(1-\rho)\left(\int FGdF - \frac{1}{3}\right)^2,$$

(12)
$$C(Q_2^{(2)}:\theta) = 180\rho(1-\rho)\left(\int G^2 dF - \frac{1}{3}\right)^2.$$

By integration by part it can be easily shown

$$C(Q_1^{(1)}:\theta) = C(Q_2^{(2)}:\theta), \qquad C(Q_2^{(1)}:\theta) = C(Q_1^{(2)}:\theta).$$

Further

$$C(Q_{2}^{(1)}:\theta) - C(Q_{1}^{(1)}:\theta) = 180\rho(1-\rho) \Big[\int F^{2}dG - 2 \int FGdG + \frac{1}{3} \Big] \Big[\int F^{2}dG + 2 \int FGdG - 1 \Big]$$

= 180\rho(1-\rho) \Big[\int (F-G)^{2}dG \Big] \Big[\Big(\int F^{2}Gd - \frac{1}{3} \Big) + 2 \Big(\int GFdG - \frac{1}{3} \Big) \Big].

But it is seen that $\int F^2 dG$ is an increasing function of $\theta > 0$ for $G(x) = F(x/\theta)$. Thus we get

$$\int F^2 dG > \frac{1}{3} \quad \text{for } \theta > 1.$$

$$\int FG dG > \frac{1}{2} \quad \text{for } \theta > 1.$$

Similarly

$$\int FGdG > \frac{1}{3}$$
 for $\theta > 1$

Thus we get

 $C(Q_2^{\scriptscriptstyle (1)}:\theta) > C(Q_1^{\scriptscriptstyle (1)}:\theta) \qquad ext{for } \theta > 1$.

Namely it has been proved that for testing the hypothesis $H: \theta = 1$ against the alternative $AH: \theta > 1$,

(i) Bahadur asymptotic efficiency of $Q_2^{(1)}$ and $Q_1^{(1)}$ are respectively equivalent to that of $Q_1^{(2)}$ and $Q_2^{(2)}$,

(ii) $Q_2^{(1)}$ (or $Q_1^{(2)}$) is more efficient than $Q_1^{(1)}$ (or $Q_2^{(2)}$) or equivalently

(ii)' $Q_1^{(2)}$ (or $Q_2^{(1)}$) is more efficient than $Q_1^{(1)}$ (or $Q_2^{(2)}$).

Thus against Tamura's proposal, $Q_1^{(2)}$ instead of $Q_1^{(1)}$ would be recommended in the practical situations.

3. Small sample comparisons of $Q_s^{(i)}$, *i*, s = 1, 2

Since the results given in the section 2 are asymptotic and approximate, behaviours of statistics $Q_{N,s}^{(i)}$, i, s = 1, 2 must be discussed in small sample. We cannot unfortunately deal with them in the general form, therefore we only check in the simple and special cases. When m = n = 4, the orderings of X's and Y's which have larger values of $Q_1^{(i)}$, i = 1, 2 are respectively given in the following table.

Let the size α of test be 1/70, then the critical regions of $Q_s^{(1)}$, s = 1, 2 contain only an ordering YYXXXYY and that of $Q_1^{(2)}$ and $Q_2^{(2)}$ are constructed by above five and seventeen orderings in the table respectively in the randomized form. In the case $\alpha = 5/70$, the critical regions of both $Q_2^{(1)}$ and $Q_1^{(2)}$ are constructed by above five orderings and that of $Q_1^{(1)}$ and $Q_2^{(2)}$ are constructed respectively by above six and seventeen ordering in the randomized form. When F(x) is symmetrical, symmetric orderings have the same probability, for example

$$P_r(YYXXXYXY) = P_r(YXYXXXYY)$$
.

Now we assume that F(x) be the uniform distribution in (-1/2, 1/2), then after some computation we get

$$\begin{split} P_r(YYXXXYY) &= 72 \iint G^2(x) [F(y) - F(x)]^2 [1 - G(y)]^2 dF(x) dF(y) \\ &= -\frac{1}{70\theta^4} \Big[-\frac{105}{4} (\theta - 1)^4 + 42(\theta - 1)^3 + 28(\theta - 1)^2 + 8(\theta - 1) + 1 \Big] \quad \text{for } \theta > 1 \,. \end{split}$$

$Q_1^{(1)}$		$Q_2^{(1)}$		$Q_1^{(2)}$		$Q_2^{(2)}$	
	value of		value of		value of		value of
ordering	$4\binom{4}{2}Q_1^{(1)}$	ordering	$\binom{4}{2}\binom{4}{2}Q_2^{(1)}$	ordering	$4\binom{4}{2}Q_{1}^{(2)}$	ordering	$\binom{4}{2}\binom{4}{2}Q_2^{(2)}$
YYXXXXYY	1.6	YYXXXXYY	24	YYXXXXYY	0	YYXXXXYY	0
YYXXXYXY	15	YYXXXYXY	18	YYYXXXXY	0	YYXXXYXY	0
YXYXXXYY	15	YYYXXXXY	18	YXXXXYYY	0	YYXXYXXY	0
YYXXYXXY	14	YXYXXXYY	18	XXXXYYYY	0	YYXYXXY	0
YXXYXXYY	14	YXXXXYYY	18	YYYYXXXX	0	YYYXXXXY	0
YXYXXYXY	1.4	YYXXYXXY	15	YYXXXYXY	3	YXYXXXYY	0
YYXYXXXY	13	YYXYXXXY	15	YYXYXXXY	3	YXXYXXYY	0
YXXXYXYY	13	YXXYXXYY	15	YXYXXXYY	3	YXXXYXYY	0
YXXYXYXY	1.3	YXXXYXYY	15	YXXXYXYY	3	YXXXXYYY	0
YXYXYXXY	13			XYXXXYYY	3	XYXXXYYY	0
YYYXXXXY	12			YYYXXXYX	3	YYYXXXYX	0
YXYYXXXY	12			XXXYXYYY	3	XXYXXYYY	0
YXXYYXXY	12			YYYXYXXX	3	YYYXXYXX	0
YXXXYYXY	12		-	YYXXYXXY	4	XXXYXYYY	0
YXXXXYYY	12			YXXYXXYY	4	YYYXYXXX	0
XYYXXXYY	12			XXYXXYYY	4	XXXXYYYY	0
YYXXXYYX	12			YYYXXYXX	4	YYYYXXXX	0
:	1 :			:		÷	

.

Table. Ordering of $Q_1^{(j)}$, i, s = 1, 2, m = n = 4.

From the similar computations we get for $\theta > 1$

$$P_{r}(YYYXXXY) = \frac{1}{70\theta^{4}} \left(\frac{35}{2} a^{4} + 28a^{3} + 21a^{2} + 8a + 1 \right),$$

$$P_{r}(YYXXYXY)$$

$$P_{r}(YYXXYXY)$$

$$P_{r}(YYXYXXY)$$

$$P_{r}(YYXYXXY) = \frac{1}{70\theta^{4}} (21a^{3} + 21a^{2} + 8a + 1),$$

$$P_{r}(YXYXXYXY) = \frac{1}{70\theta^{4}} (14a^{2} + 8a + 1),$$

$$P_{r}(YYYYXXXX) = \frac{1}{70\theta^{4}} \left(\frac{35}{8} a^{4} + 7a^{3} + 7a^{2} + 4a + 1 \right),$$

$$P_{r}(YYYXXXYX)$$

$$P_{r}(YYYXXYX)$$

$$P_{r}(YYYXXYX)$$

$$P_{r}(YYYXXYX)$$

$$P_{r}(YYYXXYX)$$

$$P_{r}(YYYXXYX)$$

where $a = \theta - 1$.

Thus the power of $Q_s^{(i)}$, denoted by $\gamma_s^{(i)}$, is given for $\alpha = 1/70$

(13)
$$\gamma_1^{(1)} = \gamma_2^{(1)} = \left(\frac{105}{4}a^4 + 42a^3 + 28a^2 + 8a + 1\right)/70\theta^4 \quad \text{for } \theta > 1,$$

(14)
$$\gamma_1^{\scriptscriptstyle(2)} = (70a^4 + 112a^3 + 84a^2 + 32a + 5)/350\theta^4$$
 for $\theta > 1$,

(15)
$$\gamma_2^{(2)} = (70a^4 + 280a^3 + 252a^2 + 104a + 17)/1190\theta^4$$
 for $\theta > 1$

Comparing (13), (14) and (15) we get

(16)
$$\gamma_1^{(1)} = \gamma_2^{(1)} > \gamma_1^{(2)} > \gamma_2^{(2)}$$
 for $\theta > 1$.

In the case $\alpha = 5/70$ we get

(17)
$$\gamma_1^{(1)} = \left(\begin{array}{c} 315\\4 \end{array} a^4 + 336a^3 + 322a^2 + 120a + 15 \right)/210\theta^4 \text{ for } \theta > 1$$

(18)
$$\gamma_1^{(2)} = (70a^4 + 112a^3 + 84a^2 + 32a + 5)/70\theta^4$$
 for $\theta > 1$,

(19)
$$\gamma_2^{(1)} = \left(\frac{245}{4}a^4 + 140a^3 + 112a^2 + 40a + 5\right)/70\theta^4 \quad \text{for } \theta > 1,$$

(20)
$$\gamma_2^{(2)} = (70a^4 + 280a^3 + 252a^2 + 104a + 17)/238\theta^4$$
 for $\theta > 1$.

Comparing (17), (18), (19) and (20) we get

(21)
$$\begin{pmatrix} \gamma_1^{(2)} > \gamma_2^{(1)} > \gamma_1^{(1)} > \gamma_2^{(2)} & \text{for } \theta > 5.047 ,\\ \gamma_2^{(1)} > \gamma_1^{(2)} > \gamma_1^{(1)} > \gamma_2^{(2)} & \text{for } 1.863 < \theta \le 5.047 ,\\ \gamma_2^{(1)} > \gamma_1^{(1)} > \gamma_1^{(2)} > \gamma_2^{(2)} & \text{for } 1 < \theta \le 1.863 , \end{cases}$$

(16) and (21) support the results in section 2. Namely, let denote by $B(T^{(1)}:T^{(2)})$ the Bahadur asymptotic efficiency of $T^{(1)}$ relative to $T^{(2)}$, then we have following correspondence between the results of section 2 and section 3.

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Results in section 2

Results in section 3

$B(Q_1^{(1)}:Q_2^{(1)}) < 1$	$\gamma_1^{_{(1)}}{<}\gamma_2^{_{(1)}}$
$B(Q_1^{(2)}:Q_2^{(2)})>1$	$\gamma_{1}^{(2)} > \gamma_{2}^{(2)}$
$B(Q_1^{(1)}:Q_1^{(2)}) < 1$	$\gamma_1^{(1)} < \gamma_1^{(2)}$
$B(Q_2^{(1)}:Q_2^{(2)})>1$	$\gamma_{2}^{(1)} > \gamma_{2}^{(2)}$
$B(Q_1^{(1)}:Q_2^{(2)})=1$	$\gamma_1^{(1)} > \gamma_2^{(2)}$
$B(Q_2^{(1)}:Q_1^{(2)})=1$	$\gamma_2^{(1)} \gtrsim \gamma_1^{(2)}$.

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