A NOTE ON THE EFFICIENCY OF TAMURA'S $ Q $

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A NOTE ON THE EFFICIENCY OF TAMURA'S Q

By

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1. Introduction

Let $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots Y_n$ be random samples from the symmetric and continuous c. d. f. $F(x)$ and $G(x)=F(x/\theta)$ respectively.

For testing the statistical hypothesis $H: \theta=1$ against the alternative $AH: \theta>1$ Tamura [3] has proposed the following test statistics.

$$Q_s^{(1)}=\frac{1}{\binom{m}{s}\binom{n}{2}}\Sigma^*\phi(x_{\alpha_1}, \cdots, x_{\alpha_s}; y_{\beta_1}, y_{\beta_2}) \tag{1}$$

where

$$\phi(x_1, x_2, \cdots, x_s; y_1, y_2)=\begin{cases} 1 & \text{for } y_1<x_1, \cdots, x_s<y_2 \text{ or } y_2<x_1, x_2, \cdots, x_s<y_1, \\ 0 & \text{otherwise}. \end{cases}$$

and the summation Σ^* extends over all subscripts α, β such that $1\leqq\alpha_1<\cdots<\alpha_s\leqq m$, $1\leqq\beta_1<\beta_2\leqq n$.

Among the statistics $Q_s^{(1)}$, $s=1, 2, \cdots$, the interesting one would be $Q_1^{(1)}$ and $Q_2^{(1)}$. It has been proved that these two statistics have the same Pitman efficiencies. Thus $Q_1^{(1)}$ would be more practical than $Q_2^{(1)}$, since it is very easy to compute.

To make our investigation more precise we shall also consider the following statistics which are the same types of $Q_s^{(1)}$.

$$Q_s^{(2)}=\frac{1}{\binom{n}{s}\binom{m}{2}}\Sigma^{**}\phi(y_{\alpha_1}, \cdots, y_{\alpha_s}; x_{\beta_1}, x_{\beta_2}), \qquad s=1, 2, \tag{2}$$

where Σ^{**} extends over all subscripts α, β such that $1\leqq\alpha_1<\cdots<\alpha_s\leqq n$, $1\leqq\beta_1<\beta_2\leqq m$.

The purpose of this paper is to make further comparisons to these test statistics, which will make us to recomend $Q_1^{(2)}$ instead of $Q_1^{(1)}$ or $Q_2^{(1)}$ in practical situations.

In section 2 we shall consider the comparison of $Q_s^{(i)}$, $i, s=1, 2$ from the view point of the Bahadur asymptotic efficiency [1]. As pointed by Bahadur [2], Bahadur asymptotic efficiency has some pitfalls since it is an approximate measure of efficiency. Thus our results in section 2 might not be enough reliable. Therefore we shall in section 3 compute the small sample power of these test statistics and give comparisons of these powers for a specific distribution.

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2. Comparison of $Q_s^{(i)}$, $i, s=1, 2$ by the Bahadur asymptotic efficiency

Let denote the mean of $Q_s^{(i)}$, $i, s=1, 2$ by $\mu_{i,s}(\theta)$. Then we get

(4) $$\mu_{1,s}(\theta)=2\iint_{x<y}[F(y)-F(x)]^s dG(x)d(y),$$

(5) $$\mu_{2,s}(\theta)=2\iint_{x<y}[G(y)-G(x)]^s dF(x)dF(y).$$

Especially in the null case we get

(6) $$\mu_{1,s}(1)=\mu_{2,s}(1)=\frac{2}{(s+1)(s+2)}.$$

Let $m=\rho N$, $n=(1-\rho)N$ and denoting the asymptotic variance of $Q_s^{(i)}$ under null hypothesis by $\sigma_{i,s}^2$, then following the manner of Tamura [3] we get

(7) $$\sigma_{i,s}^2=\frac{1}{(s+1)^2}\frac{8}{\rho(1-\rho)}\Big[\frac{1}{2s+3}-\frac{2}{(s+2)^2}+\frac{[(s+1)!]^2}{(2s+3)!}\Big], \qquad s=1, 2.$$

Let normalize the statistics $Q_s^{(i)}$ as follows.

$$Q_{N,s}^{(i)}=\frac{\sqrt{N}(Q_s^{(i)}-\mu_s(1))}{\sigma_{i,s}}, \qquad i=1, 2.$$

Then we get under non null hypothesis

(8) $$E\Big[\frac{Q_{N,s}^{(i)}}{\sqrt{N}}\Big]=\frac{1}{\sigma_{i,s}}(\mu_{i,s}(\theta)-\mu_{i,s}(1)).$$

Thus by using Chebychev's inequality and the definition of the asymptotic slope, the asymptotic slope $C(Q_s^{(i)};\theta)$ of statistic $Q_s^{(i)}$ is obtained, after some calculations, as follows.

$$C(Q_s^{(i)}:\theta)=\frac{1}{\sigma_{i,s}^2}(\mu_{i,s}(\theta)-\mu_{i,s}(1))^2.$$

Then from (4), (5) and (7) we get

(9) $$C(Q_1^{(1)}:\theta)=720\rho(1-\rho)\Big(\int FGdG-\frac{1}{3}\Big)^2,$$

(10) $$C(Q_2^{(1)}:\theta)=180\rho(1-\rho)\Big(\int F^2dG-\frac{1}{3}\Big)^2,$$

(11) $$C(Q_1^{(2)}:\theta)=720\rho(1-\rho)\Big(\int FGdF-\frac{1}{3}\Big)^2,$$

(12) $$C(Q_2^{(2)}:\theta)=180\rho(1-\rho)\Big(\int G^2dF-\frac{1}{3}\Big)^2.$$

By integration by part it can be easily shown

$$C(Q_1^{(1)}:\theta)=C(Q_2^{(2)}:\theta), \qquad C(Q_2^{(1)}:\theta)=C(Q_1^{(2)}:\theta).$$

Further

$$C(Q_2^{(1)}:\theta)-C(Q_1^{(1)}:\theta)=180\rho(1-\rho)\Big[\int F^2dG-2\int FGdG+\frac{1}{3}\Big]\Big[\int F^2dG+2\int FGdG-1\Big]$$

$$=180\rho(1-\rho)\Big[\int (F-G)^2dG\Big]\Big[\Big(\int F^2Gd-\frac{1}{3}\Big)+2\Big(\int GFdG-\frac{1}{3}\Big)\Big].$$

But it is seen that $\int F^2dG$ is an increasing function of $\theta>0$ for $G(x)=F(x/\theta)$. Thus we get

$$\int F^2dG>\frac{1}{3}\qquad \text{for } \theta>1.$$

Similarly

$$\int FGdG>\frac{1}{3}\qquad \text{for } \theta>1.$$

Thus we get

$$C(Q_2^{(1)}:\theta)>C(Q_1^{(1)}:\theta)\qquad \text{for } \theta>1.$$

Namely it has been proved that for testing the hypothesis $H:\theta=1$ *against the alternative* $AH:\theta>1$,

(i) *Bahadur asymptotic efficiency of* $Q_2^{(1)}$ *and* $Q_1^{(1)}$ *are respectively equivalent to that of* $Q_1^{(2)}$ *and* $Q_2^{(2)}$,

(ii) $Q_2^{(1)}$ *(or* $Q_1^{(2)}$*) is more efficient than* $Q_1^{(1)}$ *(or* $Q_2^{(2)}$*)*

or equivalently

(ii)′ $Q_1^{(2)}$ *(or* $Q_2^{(1)}$*) is more efficient than* $Q_1^{(1)}$ *(or* $Q_2^{(2)}$*).*

Thus against Tamura's proposal, $Q_1^{(2)}$ instead of $Q_1^{(1)}$ would be recomended in the practical situations.

3. Small sample comparisons of $Q_s^{(i)}$, $i, s=1, 2$

Since the results given in the section 2 are asymtotic and approximate, behaviours of statistics $Q_{N,s}^{(i)}$, $i, s=1, 2$ must be discussed in small sample. We cannot unfortunately deal with them in the general form, therefore we only check in the simple and special cases. When $m=n=4$, the orderings of X's and Y's which have larger values of $Q_1^{(i)}$, $i=1, 2$ are respectively given in the following table.

Let the size α of test be 1/70, then the critical regions of $Q_s^{(1)}$, $s=1, 2$ contain only an ordering $YYXXXXYY$ and that of $Q_1^{(2)}$ and $Q_2^{(2)}$ are constructed by above five and seventeen orderings in the table respectively in the randomized form. In the case $\alpha=5/70$, the critical regions of both $Q_2^{(1)}$ and $Q_1^{(2)}$ are constructed by above five orderings and that of $Q_1^{(1)}$ and $Q_2^{(2)}$ are constructed respectively by above six and seventeen ordering in the randomized form. When $F(x)$ is symmetrical, symmetric orderings have the same probability, for example

$$P_r(YYXXXYXY)=P_r(YXYXXXYY).$$

Now we assume that $F(x)$ be the uniform distribution in $(-1/2, 1/2)$, then after some computation we get

$$P_r(YYXXXXYY)=72\iint G^2(x)[F(y)-F(x)]^2[1-G(y)]^2dF(x)dF(y)$$

$$=\frac{1}{70\theta^4}\Big[\frac{105}{4}(\theta-1)^4+42(\theta-1)^3+28(\theta-1)^2+8(\theta-1)+1\Big]\quad \text{for } \theta>1.$$

Table. Ordering of $Q_1^{(i)}$, $i, s = 1, 2$, $m = n = 4$.

$Q_1^{(1)}$ ordering	value of $4\binom{4}{2}Q_1^{(1)}$	$Q_2^{(1)}$ ordering	value of $\binom{4}{2}\binom{4}{2}Q_2^{(1)}$	$Q_1^{(2)}$ ordering	value of $4\binom{4}{2}Q_1^{(2)}$	$Q_2^{(2)}$ ordering	value of $\binom{4}{2}\binom{4}{2}Q_2^{(2)}$
$YYXXXXYY$	16	$YYXXXXYY$	24	$YYXXXXYY$	0	$YYXXXXYY$	0
$YYXXXYXY$	15	$YYXXXYXY$	18	$YYYXXXXY$	0	$YYXXXYXY$	0
$YXYXXXYY$	15	$YYYXXXXY$	18	$YXXXXYYY$	0	$YYXXYXXY$	0
$YYXXYXXY$	14	$YXYXXXYY$	18	$XXXXYYYY$	0	$YYXYXXXY$	0
$YXXYXXYY$	14	$YXXXXYYY$	18	$YYYYXXXX$	0	$YYYXXXXY$	0
$YXYXXYXY$	14	$YYXXYXXY$	15	$YYXXXYXY$	3	$YXYXXXYY$	0
$YYXYXXXY$	13	$YYXYXXXY$	15	$YYXYXXXY$	3	$YXXYXXYY$	0
$YXXXYXYY$	13	$YXXYXXYY$	15	$YXYXXXYY$	3	$YXXXYXYY$	0
$YXXYXYXY$	13	$YXXXYXYY$	15	$YXXXYXYY$	3	$YXXXXYYY$	0
$YXYXYXXY$	13	⋮		$XYXXXYYY$	3	$XYXXXYYY$	0
$YYYXXXXY$	12			$YYYXXXYX$	3	$YYYXXXYX$	0
$YXYYXXXY$	12			$XXXYXYYY$	3	$XXYXXYYY$	0
$YXXYYXXY$	12			$YYYXYXXX$	3	$YYYXXYXX$	0
$YXXXYYXY$	12			$YYXXYXXY$	4	$XXXYXYYY$	0
$YXXXXYYY$	12			$YXXYXXYY$	4	$YYYXYXXX$	0
$XYYXXXYY$	12			$XXYXXYYY$	4	$XXXXYYYY$	0
$YYXXXYYX$	12			$YYYXXYXX$	4	$YYYYXXXX$	0
⋮				⋮		⋮	

From the similar computations we get for $\theta>1$

$$P_r(YYYXXXXY)=\frac{1}{70\theta^4}\left(\frac{35}{2}a^4+28a^3+21a^2+8a+1\right),$$

$$\left.\begin{array}{l}P_r(YYXXXYXY)\\ P_r(YYXXYXXY)\\ P_r(YYXYXXXY)\end{array}\right)=\frac{1}{70\theta^4}(21a^3+21a^2+8a+1),$$

$$P_r(YXYXXYXY)=\frac{1}{70\theta^4}(14a^2+8a+1),$$

$$P_r(YYYYXXXX)=\frac{1}{70\theta^4}\left(\frac{35}{8}a^4+7a^3+7a^2+4a+1\right),$$

$$\left.\begin{array}{l}P_r(YYYXXXYX)\\ P_r(YYYXXYXX)\\ P_r(YYYXYXXX)\end{array}\right)=\frac{1}{70\theta^4}(7a^3+7a^2+4a+1),$$

where $a=\theta-1$.

Thus the power of $Q_s^{(i)}$, denoted by $\gamma_s^{(i)}$, is given for $\alpha=1/70$

(13) $$\gamma_1^{(1)}=\gamma_2^{(1)}=\left(\frac{105}{4}a^4+42a^3+28a^2+8a+1\right)/70\theta^4 \quad \text{for } \theta>1,$$

(14) $$\gamma_1^{(2)}=(70a^4+112a^3+84a^2+32a+5)/350\theta^4 \quad \text{for } \theta>1,$$

(15) $$\gamma_2^{(2)}=(70a^4+280a^3+252a^2+104a+17)/1190\theta^4 \quad \text{for } \theta>1.$$

Comparing (13), (14) and (15) we get

(16) $$\gamma_1^{(1)}=\gamma_2^{(1)}>\gamma_1^{(2)}>\gamma_2^{(2)} \quad \text{for } \theta>1.$$

In the case $\alpha=5/70$ we get

(17) $$\gamma_1^{(1)}=\left(\frac{315}{4}a^4+336a^3+322a^2+120a+15\right)/210\theta^4 \quad \text{for } \theta>1,$$

(18) $$\gamma_1^{(2)}=(70a^4+112a^3+84a^2+32a+5)/70\theta^4 \quad \text{for } \theta>1,$$

(19) $$\gamma_2^{(1)}=\left(\frac{245}{4}a^4+140a^3+112a^2+40a+5\right)/70\theta^4 \quad \text{for } \theta>1,$$

(20) $$\gamma_2^{(2)}=(70a^4+280a^3+252a^2+104a+17)/238\theta^4 \quad \text{for } \theta>1.$$

Comparing (17), (18), (19) and (20) we get

(21) $$\left(\begin{array}{ll}\gamma_1^{(2)}>\gamma_2^{(1)}>\gamma_1^{(1)}>\gamma_2^{(2)} & \text{for } \theta>5.047,\\ \gamma_2^{(1)}>\gamma_1^{(2)}>\gamma_1^{(1)}>\gamma_2^{(2)} & \text{for } 1.863<\theta\leqq 5.047,\\ \gamma_2^{(1)}>\gamma_1^{(1)}>\gamma_1^{(2)}>\gamma_2^{(2)} & \text{for } 1<\theta\leqq 1.863,\end{array}\right.$$

(16) and (21) support the results in section 2. Namely, let denote by $B(T^{(1)}:T^{(2)})$ the Bahadur asymptotic efficiency of $T^{(1)}$ relative to $T^{(2)}$, then we have following correspondence between the results of section 2 and section 3.

Results in section 2	Results in section 3
$B(Q_1^{(1)}:Q_2^{(1)})<1$	$\gamma_1^{(1)}<\gamma_2^{(1)}$
$B(Q_1^{(2)}:Q_2^{(2)})>1$	$\gamma_1^{(2)}>\gamma_2^{(2)}$
$B(Q_1^{(1)}:Q_1^{(2)})<1$	$\gamma_1^{(1)}<\gamma_1^{(2)}$
$B(Q_2^{(1)}:Q_2^{(2)})>1$	$\gamma_2^{(1)}>\gamma_2^{(2)}$
$B(Q_1^{(1)}:Q_2^{(2)})=1$	$\gamma_1^{(1)}>\gamma_2^{(2)}$
$B(Q_2^{(1)}:Q_1^{(2)})=1$	$\gamma_2^{(1)}\gtrless\gamma_1^{(2)}$.

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