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A SMALL SAMPLE ROBUST COMPETITOR OF HODGES-LEHMANN ESTIMATE

By

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§ 1. Introduction and Summary.

The purpose of this paper is to present a new robust estimate of location in small sample. The pencils considered are normal and double exponential. With respect to this family of pencils, the proposed estimate is, in the case of sample sizes $N=4\sim 20$, shown to be the most robust in the class of estimates which include sample mean, best linear unbiased estimate for double exponential distribution and $D_{N,i}$, $i=1, 2, 3$ which are closely related to the Hodges-Lehmann estimate and defined in section 3.

Let $X_1, X_2, \cdots, X_N$ be a random sample from a population with symmetric distribution $F(x-\theta)$, where θ is a location parameter and let $X_{N,1} \leqq X_{N,2} \leqq \cdots \leqq X_{N,N}$ be its order statistics. Denote by $\mathfrak{A}$ the set of all pairs (i, j) such that $1 \leqq i < j \leqq N$ and by $\mathfrak{B}$ the subset of $\mathfrak{A}$ such that $1 \leqq i < j \leqq N$, $i+j=N+1$. For each $(i, j) \in \mathfrak{A}, \mathfrak{B}$, form the mean $M_{ij}=(X_{N,i}+X_{N,j})/2$.

Among many robust estimates of location, an interesting one would be the Hodges-Lehmann estimate $T_N = \text{med}\,\{M_{ij}\,;\,(i, j) \in \mathfrak{A}\}$. Though the asymptotic robustness of the estimate has been investigated by many authors such as Hodges-Lehmann [4], Høyland [6], Bickel [1] and others, the investigation of its small sample robustness is still insufficient.

Generally, in large sample it would not be so difficult for us to know the population pencil from which the random sample has obtained. Therefore in this case the problem of robust estimation would not be considered so significant.

To investigate the small sample robustness of T_N, Hodges [4] proposed a closely related estimate D_N;

$$D_N = \text{med}\,\{M_{ij}\,;\,(i, j) \in \mathfrak{B}\}\,. \tag{1.1}$$

Our new estimate $L_{N,p}$ is defined as follows:

$$L_{N,p} = \sum_{i_1<i_2<\cdots<i_p} \text{med}\,(X_{i_1}, X_{i_2}, \cdots, X_{i_p}) \Big/ \binom{N}{p}\,. \tag{1.2}$$

The proposed estimate $L_{N,p}$ is also closely related with T_N (or D_N), but unlike the T_N (or D_N), $L_{N,p}$ is the mean of the medians, $\binom{N}{p}$ in number, of p-tuples $(X_{i_1}, X_{i_2}, \cdots, X_{i_p})$ obtained from the original random sample. Though the estimate $L_{N,p}$ given by

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(1.2) is a general form, in this paper we shall discuss $L_{N,p}$ only when $p=3$. Some properties of D_N and $L_{N,p}$ are discussed in section 2. The robustness of these estimates is investigated in section 3 in the case of sample sizes $N=3\sim 20$, and we obtain our main result in Theorem 3.1.

Intuitively we present a conjecture that the robustness of the estimates T_N, D_N and $L_{N,p}$ is due to our combining different criteria of estimation such as mean and median. Therefore in the problems of robust estimation it seems preferable to combine more than two criteria of estimation.

The problems of these and others will be attacked in a forthcoming paper.

§ 2. Some properties of the estimates.

The estimate defined in section 1 generates a class of linear order statistics, for example, when $N=7$ we have four estimates: $L_{7,1}=\bar{x}$, $L_{7,2}=L_{7,3}=(5X_{7,2}+8X_{7,3}+9X_{7,4}+8X_{7,5}+5X_{7,6})/35$, $L_{7,4}=L_{7,5}=(2X_{7,3}+3X_{7,4}+2X_{7,5})/7$ and $L_{7,6}=L_{7,7}=\text{med } X_i$. We shall, in this paper, discuss $L_{n,p}$ only when $p=3$.

When $p=3$, the simple calculation leads $L_{N,p}$ as follows,

$$L_N(x)=\binom{N}{3}^{-1}\sum_{i=2}^{N-1}(i-1)(N-i)X_{N,i}\,. \tag{2.1}$$

LEMMA. (i) *The estimate $L_N(x)$ is invariant:*

$$L_N(x+a)=L_N(x)+a\,, \tag{2.2}$$

$$L_N(ax)=aL_N(x)\,, \qquad \textit{for any real } a\,. \tag{2.3}$$

(ii) *The distribution of the estimate $L_N(X)$ is symmetric about θ, therefore $L_N(X)$ is an unbiased estimate of θ.*

PROOF. As $\sum_{i=2}^{N-1}(i-1)(N-i)=\binom{N}{3}$, (i) is obvious. (ii) By (i) we may assume without loss of generality that $\theta=0$. Further F is assumed to be symmetric about zero. Since the random variable X and $-X$ then have the same distribution, $L_N(X)$ and $L_N(-X)$ have the same distribution. But by (i), $L_N(-X)=-L_N(X)$. We therefore obtain the required result.

The theorem below states a close relation between the estimate D_N and Hodges-Lehmann estimate T_N.

THEOREM 2.1. (i) *We shall put $t_N^*=\text{med}\,\{m_{i,j};\,(i,j)\in\mathfrak{A}\}$ and $d_N^*=\text{med}\,\{m_{i,j};\,(i,j)\in\mathfrak{B}\}$, where $m_{ij}=(EX_{N,i}+EX_{N,j})/2$ and E means the expectation. Then $t_N^*=d_N^*=\theta$.*

(ii) *In the case $N=4l+2$, $4l+3$, $l=1, 2, \cdots$, the estimate T_N and D_N turn to the same type:*

$$T_N,\ D_N=\frac{X_{N,2l+1-r}+X_{N,2l+2+r}}{2}\,, \qquad \textit{when} \quad N=4l+2 \tag{2.4}$$

$$=\frac{X_{N,2l+2-r}+X_{N,2l+2+r}}{2}\,, \qquad \textit{when} \quad N=4l+3\,,$$

$$r=0, 1, 2, \cdots.$$

PROOF. As the estimates T_N and D_N are translation invariant, we may assume without loss of generality that $\theta=0$. (i) Since F is assumed to be symmetric about zero, $EX_{N,i}=-EX_{N,N+1-i}$. Thus the fact $m_{ij}<0$ leads $m_{N+1-i,N+1-j}>0$ shows $t_N^*=d_N^*$ immediately. (ii) In the case $N=4l+2$, $4l+3$, $\#\mathfrak{A}$ is $(2l+1)(4l+1)$, $(2l+1)(4l+3)$, respectively, and $\#\mathfrak{B}$ is $(2l+1)$. Since both $\#\mathfrak{A}$ and $\#\mathfrak{B}$ are odd, the estimate T_N and D_N must have type $(X_{N,i_0}+X_{N,j_0})/2$, especially D_N is given by the r.h.s. of (2.4). Moreover necessary and sufficient condition for $ET_N=0$ in this case is $i_0+j_0=N+1$. Thus we obtain the desired result.

All the estimates in this paper could be written as linear order statistics as follows.

$$\hat{\theta}=\sum_{i=1}^N l_i X_{N,i}\,,\qquad l_{N+1-i}=l_i\,,\qquad \sum_{i=1}^N l_i=1\,.$$

The variance of $\hat{\theta}$ is given, for even and odd sample sizes respectively, by:

$$N=2m:\quad V(\hat{\theta})=2\sum_{i=1}^m l_i^2(C_{i,i}+C_{i,N+1-i})+4\sum_{i=1}^{m-1}\sum_{j=i+1}^m l_i l_j(C_{ij}+C_{i,N+1-j}) \tag{2.5}$$

$$N=2m+1:\quad V(\hat{\theta})=2\sum_{i=1}^m l_i^2(C_{i,i}+C_{i,N+1-i})+4\sum_{i=1}^{m-1}\sum_{j=i+1}^m l_i l_j(C_{i,j}+C_{i,N+1-j})$$

$$+l_{m+1}^2 C_{m+1,m+1}+4l_{m+1}\sum_{i=1}^m l_i C_{i,m+1}$$

where

$$C_{i,i}=V(X_{N,i})=\int x^2\frac{N!}{(i-1)!\,(N-i)!}F(x)^{i-1}(1-F(x))^{N-i}dF(x) \tag{2.6}$$

$$C_{i,j}=\mathrm{Cov}\,(X_{N,i},\,X_{N,j})=\iint_{x<y} xy\frac{N!}{(i-1)!\,(j-i-1)!\,(N-j)!}$$

$$F(x)^{i-1}(F(y)-F(x))^{j-i-1}(1-F(y))^{N-j}dF(x)dF(y)\,.$$

Especially the variance of the estimate $L_N(x)$ is given by the theorem below.

THEOREM 2.2. *When $N\geqq 3$, the variance of the estimate $L_N(x)$ is given by:*

$$V(L_N(X))=\frac{36A_1}{N(N-1)}+\frac{144(N-3)B_1}{N(N-1)}+\frac{36(N-3)(N-4)}{N(N-1)(N-2)}(A_2+6B_2) \tag{2.7}$$

where

$$A_i=\int x^2F^i(x)(1-F(x))^i dF(x)\,,\qquad i=1,2 \tag{2.8}$$

$$B_i=\iint_{x<y} xyF(x)(F(y)-F(x))^{i-1}(1-F(y))dF(x)dF(y)\,,\qquad i=1,2\,.$$

PROOF. From lemma 1, we may assume without loss of generality that $\theta=0$.

$$V(L_N(X))=EL_N^2(X)=Q_1/\binom{N}{3}^2+2Q_2/\binom{N}{3}^2\,, \tag{2.9}$$

where

$$Q_1=\sum_{i=2}^{N-1}(i-1)^2(N-i)^2EX_{N,i}^2\,,$$

$$Q_2=\sum_{j=3}^{N-1}\sum_{i=2}^{j-1}(i-1)(j-1)(N-i)(N-j)EX_{N,i}X_{N,j}\,.$$

(i) $$Q_1=\sum_{i=2}^{N-1}(i-1)^2(N-i)^2\int x^2 \frac{N!}{(i-1)!(N-i)!} F(x)^{i-1}(1-F(x))^{N-i}dF(x).$$

Since

$$\sum_{i=2}^{N-1}\frac{(i-1)^2(N-i)^2}{(i-1)!(N-i)!} F(X)^{i-1}(1-F(x))^{N-i}=\frac{F(x)^2(1-F(x))^2}{(N-5)!}+\frac{(N-2)F(x)(1-F(x))}{(N-3)!},$$

we get $Q_1=N!(N-2)A_1/(N-3)!+N!A_2/(N-5)!$, where $A_i, i=1,2$ are defined by (2.8).

(ii) $$Q_2=\sum_{j=3}^{N-1}\sum_{i=2}^{j-1}\iint_{x<y} xy\frac{N!(i-1)(j-1)(N-i)(N-j)}{(i-1)!(j-i-1)!(N-j)!} F(x)^{i-1}(F(y)-F(x))^{j-i-1}$$

$$\cdot(1-F(x))^{N-j}dF(x)dF(y).$$

Because we get the relation:

$$\sum_{i=2}^{j-1}\frac{(i-1)(N-i)}{(i-1)!(j-i-1)!} F(x)^{i-1}(F(y)-F(x))^{j-i-1}$$

$$=F(x)F(y)^{j-3}\cdot N-2)/(j-3)!-F(x)^2F(y)^{j-4}/(j-4)!,$$

then

$$\sum_{j=3}^{N-1}\sum_{i=2}^{j-1}\frac{(i-1)(j-1)(N-i)(N-j)}{(i-1)!(j-i-1)!(N-j)!} F(x)^{i-1}(F(y)-F(x))^{j-i-1}(1-F(x))^{N-j}$$

$$=\sum_{j=0}^{N-4}\frac{(j+2)(N-2)}{(N-4-j)!j!} F(x)F(y)^j(1-F(y))^{N-3-j}$$

$$-\sum_{j=0}^{N-5}\frac{(j+3)}{j!(N-5-j)!} F(x)^2F(y)^j(1-F(y))^{N-4-j}$$

$$=\frac{N-2}{(N-5)!} F(x)F(y)(1-F(y))+\frac{2(N-2)}{(N-4)!} F(x)(1-F(y))$$

$$-\frac{1}{(N-6)!} F(x)^2(1-F(y))F(y)-\frac{3}{(N-5)!} F(x)^2(1-F(y)).$$

Then using the relation

$$\iint_{x<y} xyF(x)F(y)(1-F(y))dF(x)dF(y)=\iint_{x<y} xyF(x)^2F(y)(1-F(y))dF(x)dF(y)$$

we obtain $Q_2=2N!(N-2)B_1/(N-4)!+3N!B_2/(N-5)!$, where $B_i, i=1,2$ are defined by (2.8). Thus from (i), (ii) and (2.9) we get our desired result.

COROLLARY.

(2.10) $$V(L_N(X))=6V(X_{3,2})/N(N-1)$$
$$+6(N-3)[\mathrm{Cov}(X_{4,2}, X_{4,3})-(EX_{4,2})^2]/N(N-1)$$
$$+3(N-3)(N-4)[2V(X_{5,3})+\mathrm{Cov}(X_{5,2}, X_{5,4})-3(EX_{5,2})^2]/5N(N-1).$$

PROOF. Since $F(x)$ is assumed to be symmetric, $EX_{k,i}=-EX_{k,k+1-i}$ and $\mathrm{Cov}(X_{k,i}, X_{k,j})=\mathrm{Cov}(X_{k,k+1-i}, X_{k,k+1-j})$ for any sample size K. Thus from (2.6) and (2.7) we obtain (2.10).

From the tables of Sarhan and Greenberg [7] and Govindarajuru [3], (2.10) is given, respectively normal and double exponential distribution by:

(i) normal distribution: $f(x)=1/\sqrt{2\pi}\exp(-x^2/2)$

(2.11) $V(L_N(x))=[2.69203+0.88637(N-3)+0.17302(N-3)(N-4)/(N-2)]/N(N-1)$.

(ii) double exponential distribution: $f(x)=1/2\exp(-|x|)$

(2.12) $V(L_N(x))=[3.83334+1.21074(N-3)+0.21910(N-3)(N-4)/(N-2)]/N(N-1)$.

(iii) logistic distribution: $f(x)=\exp(-x)/(1+\exp(-x))^2$

(2.13) $V(L_N(x))=[7.73921+2.52168(N-3)+0.47838(N-3)(N-4)/(N-2)]/N(N-1)$.

The Table 1 below gives the numerical values of (2.11), (2.12) and (2.13) for sample sizes $N=3\sim20$.

TABLE 1. Variances of the estimate $L_N(x)$ for the normal, double exponential and the logistic distribution.

N	normal	d. exp.	logistic	N	normal	d. exp.	logistic
3	.44867	.63889	1.28987	12	.09026	.12354	.25666
4	.29820	.42035	.85507	13	.08315	.11367	.23635
5	.22901	.32004	.65507	14	.07707	.10531	.21903
6	.18702	.25981	.53406	15	.07184	.09803	.20408
7	.15841	.21910	.47035	16	.06726	.09172	.19105
8	.13751	.18959	.39183	17	.06324	.08618	.17959
9	.12155	.16718	.34610	18	.05967	.08128	.16942
10	.10894	.14954	.31003	19	.05648	.07689	.16035
11	.09872	.13529	.28081	20	.05361	.07296	.15220

§ 3. Robustness of the estimate.

The situation similar as given by Crow and Siddiqui [2] is assumed in this section to show the robustness of the estimate $L_N(x)$.

Let $X_1, X_2, \cdots, X_N$ be a random sample drawn from the population with distribution $F(x-\theta)$, where F is known to belong to the calss of pencils $\mathfrak{F}$ which consist of normal and double exponential pencils of distribution but it is not known which particular pencil it belongs. When a pencil is specified, one can usually obtain an efficient (at least, asymptotically efficient) estimate of θ. For example, in the case of the normal pencil the mean is the efficient and in the case of the double exponential pencil the median is asymptotically efficient (since we are considering in this paper the small sample case, the BLUE (best linear unbiased estimate) for double exponential distribution given by Govindarajuru [3] is used as the efficient estimate for the double exponential pencil).

But the troubles arise in the above situation because non of the efficient estimates for individual pencils for $\mathfrak{F}$ may be satisfactory for other pencils (see FIG. 1).

In this section we shall show that the estimate $L_N(x)$ is, in some sence, a satisfactory estimate of θ for both members of $\mathfrak{F}$.

Let $\hat{\Theta}$ be a class of estimates and $\sigma^2[\hat{\theta}\,;F]$ the variance of $\hat{\theta}\in\hat{\Theta}$ for specific F.

Denote by B_N the BLUE for double exponential distribution. Then the relative efficiency of $\hat{\theta}$ is defined, respectively normal and double exponential pencil, by:

normal pencil:

$$\text{reff}\,[\hat{\theta}:F]=\sigma^2[\bar{x}:F]/\sigma^2[\hat{\theta}:F] \tag{3.1}$$

double exponential pencil:

$$\text{reff}\,[\hat{\theta}:F]=\sigma^2[B_N:F]/\sigma^2[\hat{\theta}:F]\,. \tag{3.2}$$

DEFINITION. We shall call the estimate $\hat{\theta}^*$ the most robust estimate of location with respect to $(\hat{\Theta}, \mathfrak{F})$ when $\hat{\theta}=\hat{\theta}^*$ attains the robust criterion below.

$$\underset{\hat{\theta}\in\hat{\Theta}}{\text{Max}}\ \underset{F\in\mathfrak{F}}{\text{Min}}\ \text{reff}\,[\hat{\theta}:F]\,. \tag{3.3}$$

Now the estimate D_N has the types M_{ij}, $(i, j)\in\mathfrak{B}$ when $N=4l+2$, $4l+3$ and $M^*_{ijkh}=(M_{ij}+M_{kh})/2$, (i, j), $(k, h)\in\mathfrak{B}$ when $N=4l$, $4l+1$. TABLE 4 gives the relative efficiencies of all kinds of estimates M_{ij}, $(i, j)\in\mathfrak{B}$ and M^*_{ijkl}, (i, j), $(k, l)\in\mathfrak{B}$. Since the direct investigation of the estimate D_N is difficult in small sample case. The estimate D_{Ni}, $i=1, 2, 3$, instead of D_N are considered, which has the same type as D_N and selected from the TABLE 4 in order to minimize the variances for the normal distribution (D_{N1}), for the double exponential distribution (D_{N2}) and to satisfy the robust criterion (3.3) (D_{N3}), where $\hat{\Theta}$ consists of all types of estimates M_{ij}, M^*_{ijkl}, respectively.

The estimates D_{Ni}, $i=1, 2, 3$ are given by TABLE 2.

TABLE 2. Estimates D_{Ni}, $i = 1, 2, 3$.

N	D_{N1}	D_{N2}	D_{N3}	N	D_{N1}	D_{N2}	D_{N3}
3	(1, 3)	(1, 3)	(1, 3)	12	(2, 5, 8, 11)	(5, 6, 7, 8)	(3, 6, 7, 10)
4	(1, 2, 3, 4)	(1, 2, 3, 4)	(1, 2, 3, 4)	13	(2, 5, 9, 12)	(5, 6, 8, 9)	(4, 6, 8, 10)
5	(1, 2, 4, 5)	(1, 2, 4, 5)	(1, 2, 4, 5)	14	(4, 11)	(7, 8)	(5, 10)
6	(2, 5)	(3, 4)	(2. 5)	15	(4, 12)	(7, 9)	(6, 10)
7	(2, 6)	(3, 5)	(2, 6)	16	(3, 7, 10, 14)	(7, 8, 9, 10)	(4, 8, 9, 13)
8	(1, 3, 6, 8)	(3, 4, 5, 6)	(2, 4, 5, 7)	17	(3, 7, 11, 15)	(7, 8, 10, 11)	(5, 7, 11, 13)
9	(2, 4, 6, 8)	(3, 4, 6, 7)	(3, 4, 6, 7)	18	(5, 14)	(8, 11)	(7, 12)
10	(3, 8)	(5, 6)	(4, 7)	19	(6, 14)	(9, 11)	(7, 13)
11	(3, 9)	(5, 7)	(4, 8)	20	(3, 8, 13, 18)	(8, 10, 11, 13)	(5, 10, 11, 16)

(i, j) means the estimate $(X_{N,i}+X_{N,j})/2$.
(i, j, k, l) means the estimate $(X_{N,i}+X_{N,j}+X_{N,k}+X_{N,l})/4$.

$\sigma^2[\bar{x}:F]$ for the normal distribution is $1/N$ and $\sigma^2[B_N:F]$ for the double exponential distribution has been given by Govindarajuru [3]. The numerical value of $\sigma^2[\bar{x}:F]$ for the double exponential distribution and $\sigma^2[B_N:F]$ for the normal distribution is obtained from (2.5), using the table of Sarhan and Greenberg [7] and Govindarajuru [3].

The variances of the estimates $(\bar{x}, B_N, L_N(x))$, $(D_{N1}, D_{N2}, L_N(x))$ and $(\bar{x}, D_{N3}, L_N(x))$ for the normal and the double exponential distribution, in the case $N=4\sim20$, are

TABLE 3. Relative efficiencies of the estimates $\bar{X}$, B_N, $L_N(x)$ and D_{Ni}, $i=1,2,3$.

estimates	pencil \ N	3	4	5	6	7	8	9	10	11
$\bar{X}$	d. exp.	.8842	.8310	.7923	.7645	.7427	.7256	.7115	.6995	.6892
$B_N(x)$	normal	.9035	.8875	.8647	.8398	.8391	.8289	.8163	.8097	.8051
$L_N(x)$	normal	.7429	.8384	.8733	.8912	.9018	.9090	.9141	.9179	.9209
	d. exp.	.9227	.9885	.9902	.9807	.9685	.9568	.9457	.9355	.9262
D_{N1}	normal	.9203	1.0000	.9737	.8651	.8486	.9252	.9376	.8403	.8324
	d. exp.	.6765	.8310	.6786	.7561	.6458	.5035	.7961	.6978	.6297
D_{N2}	normal	.9203	1.0000	.9737	.7764	.8180	.8350	.8734	.7231	.7613
	d. exp.	.6765	.8310	.6786	.9766	.9398	.9864	.9306	.9635	.9668
D_{N3}	normal	.9203	1.0000	.9737	.8651	.8486	.9006	.8734	.8104	.8241
	d. exp.	.6765	.8310	.6786	.7561	.6458	.8646	.9306	.9102	.8461

estimates	pencil \ N	12	13	14	15	16	17	18	19	20
$\bar{X}$	d. exp.	.6770	.6725	.6662	.6602	.6544	.6491	.6454	.6410	.6370
$B_N(x)$	normal	.7994	.7961	.7925	.7884	.8138	.7798	.7787	.7735	.7716
$L_N(x)$	normal	.9233	.9251	.9268	.9280	.9292	.9302	.9311	.9315	.9327
	d. exp.	.9179	.9105	.9040	.8977	.8918	.8865	.8821	.8779	.8731
D_{N1}	normal	.9381	.9333	.8302	.8255	.9259	.9274	.8262	.8219	.9273
	d. exp.	.6719	.6216	.6667	.6180	.7363	.6977	.6460	.7393	.6321
D_{N2}	normal	.7807	.8095	.6986	.7314	.7467	.7727	.7453	.7118	.7683
	d. exp.	.9939	.9737	.9587	.9724	.9927	.9858	.9663	.9726	.9904
D_{N3}	normal	.8976	.8680	.8169	.7903	.8828	.8685	.7909	.8018	.8741
	d. exp.	.8670	.9241	.8358	.9148	.8647	.8682	.8974	.8577	.8621

reff $[\bar{X}:F]=1$, F: normal   reff $[\bar{X}:F]=1$, F: double exponential

plotted in FIG. 1, FIG. 2, FIG. 3, respectively. These figures show the robust features of each estimates.

The relative efficiencies of the estimates $\bar{x}$, B_N, $L_N(x)$ and D_{Ni}, $i=1, 2, 3$, in the case $N=3\sim20$, for the normal and the double exponential distribution is shown in TABLE 3. We can read off from the table the estimate which satisfy the robust criterion (3.3) and have the result that the estimate $L_N(x)$ is the most robust in the case $N=4\sim20$.

Thus we obtain the concluding theorem.

THEOREM 3.1. *Let* $\hat{\Theta}$, $\mathfrak{F}$ *be the class of esitmates and pencils such that* $\hat{\Theta}=\{\bar{x}, B_N, L_N(x), D_{Ni}, i=1, 2, 3\}$, $\mathfrak{F}=\{$*pencil of normal distribution, pencil of double exponential distribution*$\}$. *Then* $L_N(x)$ *is, in the case* $N=4\sim20$, *the most robust estimate of location with respect to* $(\hat{\Theta}, \mathfrak{F})$.

Since the robustness of the estimate D_N is explained satisfactorily by the estimates D_{Ni}, $i=1, 2, 3$, the theorem above shows that our estimate $L_N(x)$ is more robust than the estimate D_N. Moreover the theorem 2.1 and the definition of D_{Ni}, $i=1, 2, 3$, show $L_N(x)$ is more robust than Hodges-Lehmann estimate when $N=4l+2$, $4l+3$, $l=1, 2, \cdots$. Remark that there is the close relations between the Hodges-Lehmann estimate and D_N, which are investigated by Hodges [4]. Thus our estimate $L_N(x)$ is concluded to be

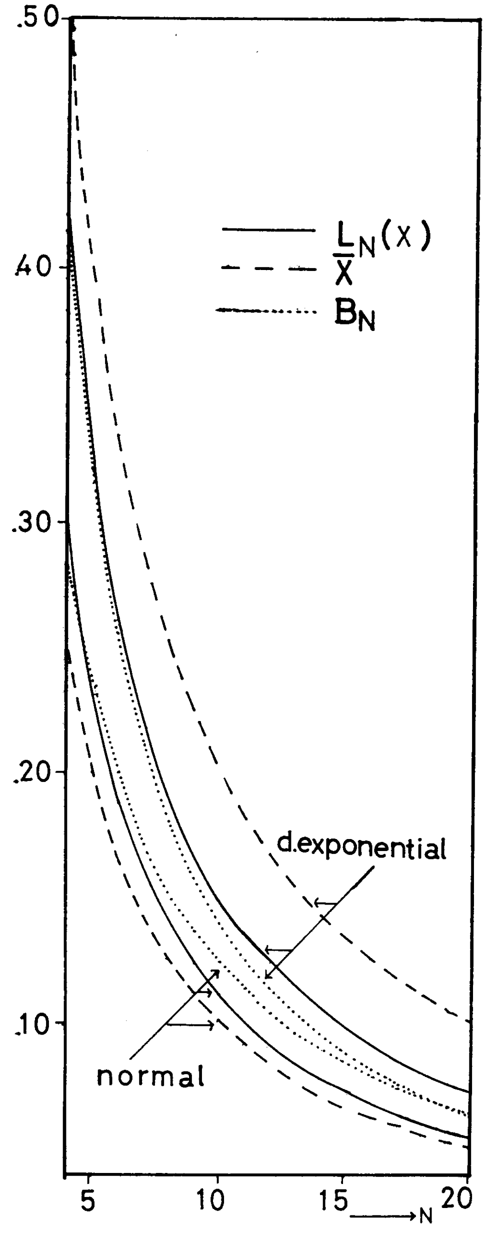

FIG. 1.

Variances of the estimates $L_N(x)$, $\bar{x}$ and B_N for the normal and the double exponential distribution.

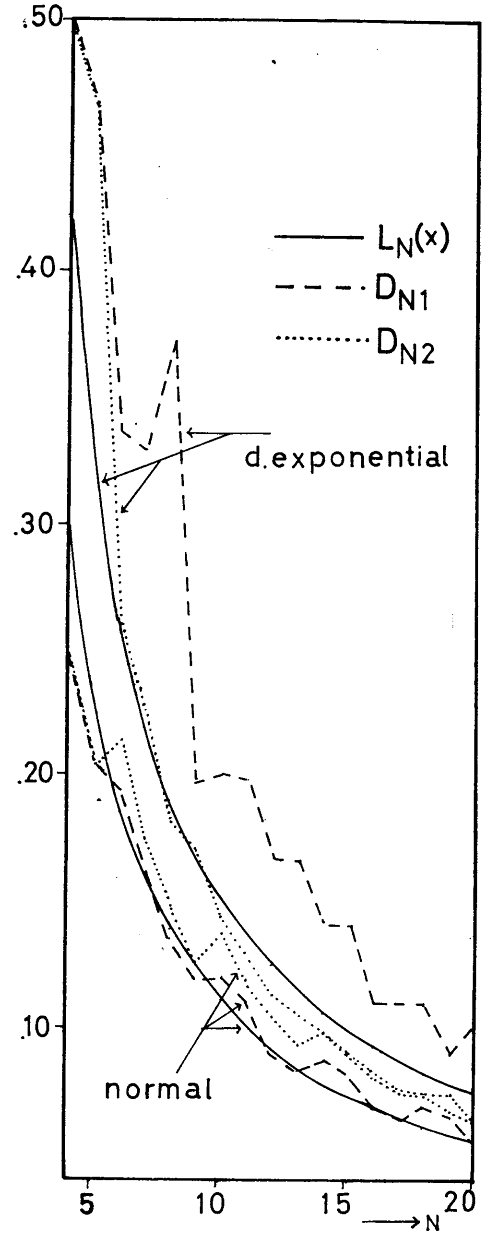

FIG. 2.

Variances of the estimates $L_N(x)$, D_{N1} and D_{N2} for the normal and the double exponential distribution. The estimates D_{Ni}, $i=1, 2$ are given by TABLE 2.

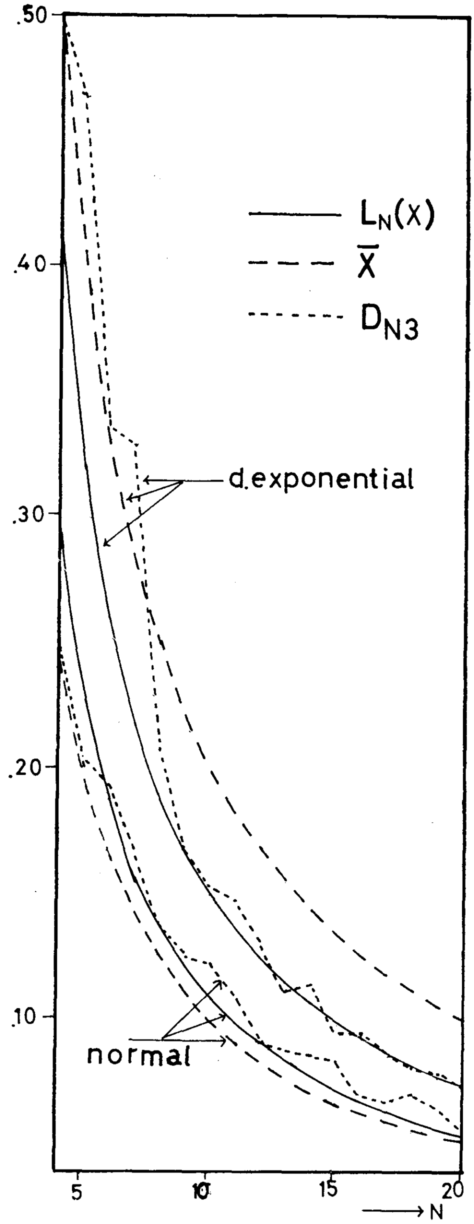

FIG. 3.

Variances of the estimates $L_N(x)$, $\bar{x}$ and D_{N3} for the normal and the double exponential distribution. The estimate D_{N3} is given by TABLE 2.

a strong competitor of the Hodges-Lehmann estimate in small sample.

§ 4. Acknowledgement.

The authour should like to express his hearty thanks to Prof. T. Kitagawa, Kyushu University and Prof. M. Okamoto, Iowa State University and Osaka University for their generous help and encouragement during the course of the entire work.

TABLE 4. Rerative efficiencies of the estimates

$M_{ij}=(X_{N,i}+X_{N,j})/2\,,\quad i+j=N+1\,,$

$M^*_{ijkl}=(X_{N,i}+X_{N,j}+X_{N,k}+X_{N,l})/4\,,\quad i+j=N+1\,,\quad k+l=N+1\,.$

In the table, (i, j) means the estimate $(X_{N,i}+X_{N,j})/2$ and (i, j, k, l) means the estimate $(X_{N,i}+X_{N,j}+X_{N,k}+X_{N,l})/4$.

N	estimate	normal	d. exp.	N	estimate	normal	d. exp.
4	(2. 3)	.8384	.9891	10	(3. 8)	.8403	.6978
	(1. 4)	.8384	.4965		(2. 9)	.7752	.4331
	(1. 2. 3. 4)	1.0000	.8310		(1. 10)	.5391	.1701
5	(2. 4)	.8668	.8965		(4. 5. 6. 7)	.8078	.9936
	(1. 5)	.7666	.3830		(3. 5. 6. 8)	.8803	.9308
	(1. 2. 4. 5)	.9937	.6786		(2. 5. 6. 9)	.9272	.7759
6	(3. 4)	.7764	.9766		(1. 5. 6. 10)	.8826	.4585
	(2. 5)	.8651	.7561		(3. 4. 7. 8)	.8865	.8583
	(1. 6)	.7058	.3091		(2. 4. 7. 9)	.9381	.7249
	(2. 3. 4. 5)	.9055	.9590		(1. 4. 7. 10)	.8969	.4402
	(1. 3. 4. 6)	.9625	.6627		(2. 3. 8. 9)	.8532	.6069
	(1. 2. 5. 6)	.9055	.5591		(1. 3. 8. 10)	.8658	.3939
7	(3. 5)	.8180	.9398		(1. 2. 9. 10)	.7698	.3121
	(2. 6)	.8486	.6458	11	(5. 7)	.7613	.9668
	(1. 7)	.6543	.2578		(4. 8)	.8241	.8461
	(2. 3. 5. 6)	.9225	.9018		(3. 9)	.8324	.6297
	(1. 3. 5. 7)	.9501	.5773		(2. 10)	.7500	.3885
	(1. 2. 6. 7)	.9482	.4707		(1. 11)	.5098	.1523
8	(4. 5)	.7432	.9685		(4. 5. 7. 8)	.8370	.9609
	(3. 6)	.8372	.8609		(3. 5. 7. 9)	.9009	.8812
	(2. 7)	.8262	.5580		(2. 5. 7. 10)	.9371	.7247
	(1. 8)	.6101	.2205		(1. 5. 7. 11)	.8766	.4209
	(3. 4. 5. 6)	.8350	.9864		(3. 4. 8. 9)	.8903	.7856
	(2. 4. 5. 7)	.9006	.8646		(2. 4. 8. 10)	.9314	.6588
	(1. 4. 5. 8)	.9218	.5436		(1. 4. 8. 11)	.8757	.3977
	(2. 3. 6. 7)	.9218	.7673		(2. 3. 9. 10)	.8766	.5457
	(1. 3. 6. 8)	.9252	.5035		(1. 3. 9. 11)	.8355	.3536
	(1. 2. 7. 8)	.8463	.4041		(1. 2. 10. 11)	.6032	.2799
9	(4. 6)	.7604	.9599	12	(6. 7)	.7089	.9602
	(3. 7)	.8429	.7758		(5. 8)	.7873	.9356
	(2. 8)	.8004	.4884		(4. 9)	.8313	.7815
	(1. 9)	.5721	.1922		(3. 10)	.8207	.5722
	(3. 4. 6. 7)	.8734	.9306		(2. 11)	.7256	.3515
	(1. 2. 4. 6)	.9376	.7961		(1. 12)	.4840	.1379
	(1. 4. 6. 9)	.9129	.4893		(5. 6. 7. 8)	.7807	.9939
	(2. 3. 7. 8)	.9107	.6903		(4. 6. 7. 9)	.8457	.9602
	(1. 3. 7. 9)	.8960	.4430		(3. 6. 7. 10)	.8976	.8670
	(1. 2. 8. 9)	.6433	.3528		(2. 6. 7. 11)	.9215	.7009
10	(5. 6)	.7231	.9635		(1. 6. 7. 12)	.8474	.7013
	(4. 7)	.8104	.9102		(4. 5. 8. 9)	.8552	.9288

TABLE 4. Continued.

N	estimate	normal	d. exp.	N	estimate	normal	d. exp.
12	(3. 5. 8. 10)	.9104	.8229	14	(4. 6. 9. 11)	.8826	.8774
	(2. 5. 8. 11)	.9381	.6718		(3. 6. 9. 12)	.9213	.7803
	(1. 5. 8. 12)	.8632	.3861		(2. 6. 9. 13)	.9309	.6214
	(3. 4. 9. 10)	.8033	.7191		(1. 6. 9. 14)	.8307	.4172
	(2. 4. 9. 11)	.8289	.6010		(4. 5. 10. 11)	.8709	.7900
	(1. 4. 9. 12)	.7742	.3123		(3. 5. 10. 12)	.9123	.7105
	(2. 3. 10. 11)	.7478	.5024		(2. 5. 10. 13)	.9249	.5763
	(1. 3. 10. 12)	.7095	.3202		(1. 5. 10. 14)	.8293	.3281
	(1. 2. 11. 12)	.5976	.2534		(3. 4. 11. 12)	.8729	.6091
13	(6. 8)	.7437	.9700		(2. 4. 11. 13)	.8892	.5077
	(5. 9)	.8061	.8884		(1. 4. 11. 14)	.8059	.3046
	(4. 10)	.8323	.7208		(2. 3. 12. 13)	.8151	.3258
	(3. 11)	.8078	.5233		(1. 3. 12. 14)	.7532	.1756
	(2. 12)	.7029	.3209		(1. 2. 13. 14)	.5348	.1497
	(1. 13)	.4610	.1258	15	(7. 9)	.7314	.9724
	(5. 6. 8. 9)	.8095	.9737		(6. 10)	.7903	.9148
	(4. 6. 8. 10)	.8680	.9241		(5. 11)	.8245	.7822
	(3. 6. 8. 11)	.9133	.8267		(4. 12)	.8255	.6180
	(2. 6. 8. 12)	.9299	.6618		(3. 13)	.7792	.4453
	(1. 6. 8. 13)	.8414	.3679		(2. 14)	.6611	.2729
	(4. 5. 9. 10)	.8659	.8473		(1. 15)	.4219	.2729
	(3. 5. 9. 11)	.9144	.7644		(6. 7. 9. 10)	.7894	.9821
	(2. 5. 9. 12)	.9333	.6216		(5. 7. 9. 11)	.8422	.9483
	(1. 5. 9. 13)	.8469	.3551		(4. 7. 9. 12)	.8870	.8730
	(3. 4. 10. 11)	.8819	.6601		(3. 7. 9. 13)	.9187	.7747
	(2. 4. 10. 12)	.9047	.5508		(2. 7. 9. 14)	.9187	.6073
	(1. 4. 10. 13)	.8287	.3306		(1. 7. 9. 15)	.8095	.3262
	(2. 3. 11. 12)	.8359	.4522		(5. 6. 10. 11)	.8443	.8871
	(1. 3. 11. 13)	.7791	.2925		(4. 6. 10. 12)	.8917	.8271
	(1. 2. 12. 13)	.6763	.2313		(3. 6. 10. 13)	.9251	.7327
14	(7. 8)	.6986	.9587		(2. 6. 10. 14)	.9277	.5816
	(6. 9)	.7702	.9520		(1. 6. 10. 15)	.8184	.3185
	(5. 10)	.8169	.8358		(4. 5. 11. 12)	.8730	.7358
	(4. 11)	.8302	.6667		(3. 5. 11. 13)	.9087	.6602
	(3. 12)	.7933	.4815		(2. 5. 11. 14)	.9137	.6452
	(2. 13)	.6807	.2952		(1. 5. 11. 15)	.8105	.3040
	(1. 14)	.4402	.1157		(3. 4. 12. 13)	.8742	.5637
	(6. 7. 8. 9)	.7612	.9937		(2. 4. 12. 14)	.8730	.4698
	(5. 7. 8. 10)	.8188	.9754		(1. 4. 12. 15)	.7838	.2818
	(4. 7. 8. 11)	.8686	.9136		(2. 3. 13. 14)	.7960	.4593
	(3. 7. 8. 12)	.9051	.8082		(1. 3. 13. 15)	.7290	.2487
	(2. 7. 8. 13)	.9119	.6389		(1. 2. 14. 15)	.6269	.1958
	(1. 7. 8. 14)	.8151	.3475	16	(8. 9)	.6915	.9567
	(5. 6. 9. 10)	.8302	.9352		(7. 10)	.7562	.9061

TABLE 4. Continued.

N	estimate	normal	d. exp.	N	estimate	normal	d. exp.
16	(6. 11)	.8036	.8711	17	(5. 8. 10. 13)	.8634	.9128
	(5. 12)	.8271	.7323		(4. 8. 10. 14)	.8977	.8368
	(4. 13)	.8181	.5753		(3. 8. 10. 15)	.9188	.7269
	(3. 14)	.7636	.4142		(2. 8. 10. 16)	.9060	.5605
	(2. 15)	.6411	.2536		(1. 8. 10. 17)	.7798	.2924
	(1. 16)	.4048	.0995		(6. 7. 11. 12)	.8247	.9128
	(7. 8. 9. 10)	.7467	.9927		(5. 7. 11. 13)	.8685	.8682
	(6. 8. 9. 11)	.7982	.9832		(4. 7. 11. 14)	.9060	.7992
	(5. 8. 9. 12)	.8446	.9392		(3. 7. 11. 15)	.9274	.6977
	(4. 8. 9. 13)	.8828	.8647		(2. 7. 11. 16)	.9159	.5430
	(3. 8. 9. 14)	.9084	.7546		(1. 7. 11. 17)	.7882	.2539
	(2. 8. 9. 15)	.9006	.5860		(5. 6. 12. 13)	.8584	.7884
	(1. 8. 9. 16)	.7852	.3088		(4. 6. 12. 14)	.8963	.7311
	(6. 7. 10. 11)	.8096	.9523		(3. 6. 12. 15)	.9202	.6458
	(5. 7. 10. 12)	.8585	.9109		(2. 6. 12. 16)	.9102	.3062
	(4. 7. 10. 13)	.8993	.8410		(1. 6. 12. 17)	.7872	.2783
	(3. 7. 10. 14)	.9259	.7363		(4. 5. 13. 14)	.8647	.6431
	(2. 7. 10. 15)	.9191	.5748		(3. 5. 13. 15)	.8909	.5757
	(1. 7. 10. 16)	.8003	.3060		(2. 5. 13. 16)	.8869	.4661
	(5. 6. 11. 12)	.8527	.9381		(1. 5. 13. 17)	.7717	.2645
	(4. 6. 11. 13)	.8954	.7776		(3. 4. 14. 15)	.8376	.4901
	(3. 6. 11. 14)	.9246	.6874		(2. 4. 14. 16)	.8384	.4083
	(2. 6. 11. 15)	.9205	.5446		(1. 4. 14. 17)	.7406	.2448
	(1. 6. 11. 16)	.8023	.2972		(2. 3. 15. 16)	.7568	.3341
	(4. 5. 12. 13)	.8705	.6868		(1. 3. 15. 17)	.6837	.1936
	(3. 5. 12. 14)	.9006	.6155		(1. 2. 16. 17)	.5833	.1707
	(2. 5. 12. 15)	.9006	.4976	18	(9. 10)	.6864	.9573
	(1. 5. 12. 16)	.7913	.2830		(8. 11)	.7453	.9663
	(3. 4. 13. 14)	.8502	.5244		(7. 12)	.7909	.8974
	(2. 4. 13. 15)	.8551	.4370		(6. 13)	.8201	.7811
	(1. 4. 13. 16)	.7617	.2620		(5. 14)	.8262	.6460
	(2. 3. 14. 15)	.7759	.3577		(4. 15)	.8012	.5049
	(1. 3. 14. 16)	.7235	.2313		(3. 16)	.7345	.3630
	(1. 2. 15. 16)	.6045	.1828		(2. 17)	.6171	.2223
17	(8. 10)	.7206	.9720		(1. 18)	.3752	.0872
	(7. 11)	.7757	.9328		(8. 9. 10. 11)	.7355	.9917
	(6. 12)	.8122	.8260		(7. 9. 10. 12)	.7820	.9876
	(5. 13)	.8270	.6877		(6. 9. 10. 13)	.8249	.9560
	(4. 14)	.8088	.5380		(5. 9. 10. 14)	.8620	.9008
	(3. 15)	.7481	.3868		(4. 9. 10. 15)	.8925	.8213
	(2. 16)	.6222	.2369		(3. 9. 10. 16)	.9070	.7085
	(1. 17)	.3892	.0929		(2. 9. 10. 17)	.8953	.5415
	(7. 8. 10. 11)	.7727	.9858		(1. 9. 10. 18)	.7585	.2786
	(6. 8. 10. 12)	.8201	.9634		(7. 8. 11. 12)	.7932	.9637

TABLE 4. Continued.

N	estimate	normal	d. exp.
18	(6. 8. 11. 13)	.8386	.9336
	(5. 8. 11. 14)	.8770	.8798
	(4. 8. 11. 15)	.9085	.8047
	(3. 8. 11. 16)	.9251	.6961
	(2. 8. 11. 17)	.9130	.5343
	(1. 8. 11. 18)	.7722	.2766
	(6. 7. 12. 13)	.8374	.8712
	(5. 7. 12. 14)	.8770	.8251
	(4. 7. 12. 15)	.9236	.7579
	(3. 7. 12. 16)	.9282	.6608
	(2. 7. 12. 17)	.9175	.5985
	(1. 7. 12. 18)	.7765	.2709
	(5. 6. 13. 14)	.8607	.7438
	(4. 6. 13. 15)	.8953	.6894
	(3. 6. 13. 16)	.9160	.6081
	(2. 6. 13. 17)	.9070	.4806
	(1. 6. 13. 18)	.7712	.2615
	(4. 5. 14. 15)	.8607	.6040
	(3. 5. 14. 16)	.8825	.5411
	(2. 5. 14. 17)	.8798	.4377
	(1. 5. 14. 18)	.7544	.2483
	(3. 4. 15. 16)	.8249	.4602
	(2. 4. 15. 17)	.8274	.3832
	(1. 4. 15. 18)	.7211	.2297
	(2. 3. 16. 17)	.7443	.3135
	(1. 3. 16. 18)	.6643	.2027
	(1. 2. 17. 18)	.5679	.1601
19	(9. 11)	.7118	.9726
	(8. 12)	.7634	.9454
	(7. 13)	.8018	.8577
	(6. 14)	.8219	.7393
	(5. 15)	.8206	.6092
	(4. 16)	.7898	.4754
	(3. 17)	.7186	.3418
	(2. 18)	.4618	.2093
	(1. 19)	.3618	.0821
	(8. 9. 11. 12)	.7590	.9883
	(7. 9. 11. 13)	.8031	.9740
	(6. 9. 11. 14)	.8430	.9349
	(5. 9. 11. 15)	.8767	.8778
	(4. 9. 11. 16)	.9038	.7969
	(3. 9. 11. 17)	.9148	.8172
	(2. 9. 11. 18)	.8080	.5200
	(1. 9. 11. 19)	.7525	.2649
19	(7. 8. 12. 13)	.8080	.9323
	(6. 8. 12. 14)	.8498	.8976
	(5. 8. 12. 15)	.8855	.8435
	(4. 8. 12. 16)	.9148	.7688
	(3. 8. 12. 17)	.9277	.6644
	(2. 8. 12. 18)	.8180	.5079
	(1. 8. 12. 19)	.7612	.2617
	(6. 7. 13. 14)	.8430	.8252
	(5. 7. 13. 15)	.8811	.7822
	(4. 7. 13. 16)	.9100	.7181
	(3. 7. 13. 17)	.9244	.6256
	(2. 7. 13. 18)	.8168	.4853
	(1. 7. 13. 19)	.7623	.2556
	(5. 6. 14. 15)	.8595	.6944
	(4. 6. 14. 16)	.8900	.6503
	(3. 6. 14. 17)	.9069	.5735
	(2. 6. 14. 18)	.8055	.4533
	(1. 6. 14. 19)	.7547	.2487
	(4. 5. 15. 16)	.8525	.7426
	(3. 5. 15. 17)	.8709	.5094
	(2. 5. 15. 18)	.7793	.4123
	(1. 5. 15. 19)	.7357	.2338
	(3. 4. 16. 17)	.8105	.4333
	(2. 4. 16. 18)	.7357	.3608
	(1. 4. 16. 19)	.7013	.2163
	(2. 3. 17. 18)	.6650	.2952
	(1. 3. 17. 19)	.6438	.1908
	(1. 2. 18. 19)	.5142	.1508
20	(10. 11)	.6808	.9562
	(9. 12)	.7351	.9700
	(8. 13)	.7792	.9154
	(7. 14)	.8100	.8176
	(6. 15)	.8240	.6998
	(5. 16)	.8163	.5753
	(4. 17)	.7802	.4488
	(3. 18)	.7052	.3226
	(2. 19)	.5738	.1975
	(1. 20)	.3498	.0775
	(9. 10. 11. 12)	.7259	.9893
	(8. 10. 11. 13)	.7683	.9904
	(7. 10. 11. 14)	.8078	.9663
	(6. 10. 11. 15)	.8435	.9229
	(5. 10. 11. 16)	.8741	.8621
	(4. 10. 11. 17)	.8967	.7800

TABLE 4. Concluded.

N	estimate	normal	d. exp.	N	estimate	normal	d. exp.
20	(3. 10. 11. 18)	.9038	.6667	20	(4. 7. 14. 17)	.9096	.6803
	(2. 10. 11. 19)	.8758	.5074		(3. 7. 14. 18)	.9205	.5925
	(1. 10. 11. 20)	.7328	.2530		(2. 7. 14. 19)	.8948	.5340
	(8. 9. 12. 13)	.7791	.9713		(1. 7. 14. 20)	.7491	.2415
	(7. 9. 12. 14)	.8201	.9483		(5. 6. 15. 16)	.8585	.6643
	(6. 9. 12. 15)	.8575	.9066		(4. 6. 15. 17)	.8859	.6145
	(5. 9. 12. 16)	.8897	.8478		(3. 6. 15. 18)	.8988	.5419
	(4. 9. 12. 17)	.9136	.7682		(2. 6. 15. 19)	.8770	.4281
	(3. 9. 12. 18)	.9215	.6581		(1. 6. 15. 20)	.7391	.2327
	(2. 9. 12. 19)	.8929	.4975		(4. 5. 16. 17)	.8443	.5374
	(1. 9. 12. 20)	.7452	.2682		(3. 5. 16. 18)	.8596	.4811
	(7. 8. 13. 14)	.8212	.8952		(2. 5. 16. 19)	.8435	.3892
	(6. 8. 13. 15)	.8597	.8579		(1. 5. 16. 20)	.7186	.2207
	(5. 8. 13. 16)	.8930	.8052		(3. 4. 17. 18)	.7980	.4088
	(4. 8. 13. 17)	.9181	.7331		(2. 4. 17. 19)	.7893	.3405
	(3. 8. 13. 18)	.9273	.6321		(1. 4. 17. 20)	.6837	.2041
	(2. 8. 13. 19)	.8994	.4825		(2. 3. 18. 19)	.7053	.3226
	(1. 8. 13. 20)	.7502	.2478		(1. 3. 18. 20)	.6268	.1801
	(6. 7. 14. 15)	.8490	.7866		(12. 19. 20)	.5307	.1421
	(5. 7. 14. 16)	.8832	.7420				

Refferences

[1] BICKEL, P. J. (1965). *On some robust estimates of location.* Ann. Math. Statist., **36**, 847–858.

[2] CROW, E. L. and SIDDIQUI, M. M. (1967). *Robust estimation in location.* Journal of the American Statistical Association, **62**, 353–389.

[3] GOVINDARAJURU, Z. (1966). *Best linear estimates under systematic cencoring of the parameters of a double exponential population.* Journal of the American Statistical Association, **61**, 248–258.

[4] HODGES, J. L. JR. (1965). *Efficiency in normal samples and tolerans of extreme values for some estimates of location.* Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability. Berkeley and Los Angeles, University of California Press, 163–186.

[5] HODGES, J. L. JR. and LEHMANN, E. L. (1963). *Estimates of location based on rank tests.* Ann. Math. Statist., **34**, 598–611.

[6] HφYLAND, A. (1964). *Robustness of Hodges-Lehmann estimates for shift.* Ann. Math. Statist., **36**, 174–197.

[7] SARHAN, A. E. and GREENBERG, B. G. (ed.) (1962). *Contribution to order statistics.* John Wiley and Sons, Inc., New York.