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A DELIVERY-LAG INVENTORY CONTROL PROCESS WITH EMERGENCY AND NON-STATIONARY STOCHASTIC DEMANDS

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§1. Introduction.

In [1] and [2] authors investigated one-commodity inventory model which, in addition to regular ordering of arbitary amount of stock to be delivered one period later, there is also provided emergency ordering with instantaneous delivery of stock which is naturally more expensive than regular ordering. The model presented here is the generalization of the one discussed in [1] and [2]. We consider an n-period, one-commodity dynamic inventory model with non-stationary stochastic demands, and with a k (constant) period-lag delivery of regular orders, and with k "emergency" orders, characterized by delivery lags $\mu_1=0$, $\mu_2=1$, $\cdots$, $\mu_k=k-1$. We shall assume the emergency quantity $m^i_{r,n-j+1}$ of the time lag r for the period j, at the begining of which the demand density is given by φ_i, can be any positive amount not exceeding an upper bound m_{u_r}, $0\leqq m^i_{r,n-i+1}\leqq m_{u_r}$ $(i=1, 2, \cdots, m, j=1, 2, \cdots, n, r=0, 1, \cdots, k-1$ for $n\geqq k$, $r=0, 1, \cdots, n-1$ for $n<k)$, and the regular order is $m^i_{k,n-j+1}\geqq 0$ $(i=1, 2, \cdots m, j=1, 2, \cdots, n)$. This upper bound will be the same for each period. The cumulative demand in each period is a non-negative random variable whose distribution may change from period to period according to a Markov transition law with matrix $P=\|p_{ij}\|$ $(i, j=1, 2, \cdots, m)$ where $p_{ij}\geqq 0$ and $\sum_{j=1}^{m} p =1$ for each i. It is assume that the demand density remains unchanged during one period. In other word, when the demand in a given period is φ_i in the following period the demand density change to φ_j with probability p_{ij} $(i, j=1, 2, \cdots, m)$. The technique utilized in this paper was partly suggested by [3], [4], [5], and [6].

The inventory periods I_1, I_2, $\cdots$, I_n are numbered from left to right. At begining of the j-th period $(j=1, 2, \cdots, n)$ two action have to be taken (i) placing k emergency order, (ii) issuing a regular order to be delivered at the end of the $j+k-1$ period. The delivery lag $\lambda=k$ is constant throughout the rest of the paper. We impose the following conditions on the model.

(1.1) There is backlogging of excess demand.

(1.2) The known distribution function of demand is absolutely continuous with respect to the Lebesgue measure. The density will be denote by $\varphi_i(\xi)$ $(i=1, 2, \cdots, m)$.

(1.3) The holding cost function $h(\eta)$ and the penalty cost function $p(\eta)$ are twice differentiable, positive convex function for positive arguments.

(1.4) There is credit function $v(\eta)$ defined by

$$v(\eta)=\begin{cases} v\eta & \eta\geqq 0 \\ 0 & \eta<0 \end{cases}$$

The reduced penalty cost, the net penalty cost, is defined in the following way. If at the begining of some period the order of size z to be delivered at the end of the period, has been known and a demand $D=\xi$ occures, then the net penalty cost for this period is

$$p(\xi-y)-v[\min(z,\ \xi-y)]\ ,$$

where y is the starting stock level of that period.

(1.5) There is a concave, twice differentiable salvage gain function $w(\eta)$ that is increasing for $\eta>0$, and is zero for $\eta\leqq 0$.

(1.6) The ordering cost function $c_k(\eta_k)$ for regular orders to be delivered k period later is given by

$$c_k(\eta_k)=\begin{cases} c_k\eta_k & \eta_k\geqq 0\ , \\ 0 & \eta_k<0\ . \end{cases}$$

The ordering cost function $c_j(\eta_j)$ for emergency orders to be delivered j period later is given by

$$c(\eta_j)=\begin{cases} c_j\eta_j & 0<\eta_j\leqq m_{u_j}\ , \\ 0 & \eta_j\leqq 0 \quad ; \quad (j=0,\ 1,\ \cdots,\ k-1)\ , \end{cases}$$

with $c_0>c_1>\cdots>c_k>0$.

(1.7) (a) $\alpha^j \lim_{\eta\to\infty} w'(\eta)=\alpha^j \lim_{\eta\to\infty}\int_0^\eta w'(\eta-\xi)\varphi_i(\xi)d\xi<c_j<\alpha^{j-1}v\ ,$

(b) $w'(0)<v\ ,$ (c) $0<\alpha\leqq 1\ ,$ $(j=1,\ 2,\ \cdots,\ k;\ i=1,\ 2,\ \cdots, m)$,

(1.8) $$L(\eta,\ \varphi_i)-v\int_0^\eta (\eta-\xi)\varphi_i(\xi)d\xi \qquad \text{is convex,}$$

where $L(\eta,\ \varphi_i)$, the expected one-period loss arising from penalty and holding costs, is given by

(1.9) $$L(\eta,\ \varphi_i)=\begin{cases} \int_0^\eta h(\eta-\xi)\varphi_i(\xi)d\xi+\int_\eta^\infty p(\xi-\eta)\varphi_i(\xi)d\xi & \eta>0\ , \\ \int_0^\infty p(\xi-\eta)\varphi_i(\xi)d\xi & \eta\leqq 0\ . \end{cases}$$

We shall assume that all integrals occuring in this paper exist, and are finite, and that all integration and differentation where needed can be interchange. This impose certain restrictions on the class of demands densities.

§2. The Dynamic Model with a Linear Ordering Cost—general case.

Let $f_n(x, x_1, \cdots, x_{k-1}; \varphi_i)$ denote the total discounted expected loss for an n-period inventory model, where the demand density in first period is φ_i, x is the inventory on hand after the receipt of prior orders at the begining of the first period, x_i $(i=1, 2, \cdots, k-1)$ is the amount of the stock on order that is to be received at the end of period i, and an optimal ordering policy is followed in period 1, 2, $\cdots$, n. We obtain from the principle of optimality

(2.1) $$f_n(x, x_1, \cdots, x_{k-1}; \varphi_i) = \min_{m_{u_0}\geqq m_0\geqq 0, \cdots, m_{u_{k-1}}\geqq m_{k-1}\geqq 0, m_k\geqq 0} \Big\{ \sum_{j=0}^{k} c_j m_j + L_i(x+m_0, \varphi_i) - V(x+m_0, x_1+m_1, \varphi_i) + \alpha \sum_{i_1=1}^{m} p_{ii_1} \int_0^\infty f_{n-1}(x+x_1+m_0+m_1-t, x_2+m_2, \cdots, x_{k-1}+m_{k-1}, m_k; \varphi_{i_1}) \varphi_i(t)dt \Big\} \qquad n\geqq k,\ k\geqq 2 .$$

(2.2) $$f_n(x, x_1, \cdots, x_n; \varphi_i) = \min_{m_{u_0}\geqq m_0\geqq 0, \cdots, m_{u_{n-1}}\geqq m_{n-1}\geqq 0, m_{u_n}\geqq m_n\geqq 0} \Big\{ \sum_{j=0}^{n} c_j m_j + L(x+m_0, \varphi_i) - V(x+m_0, x_1+m_1, \varphi_i) + \alpha \sum_{i_1=1}^{u} p_{ii_1} \int_0^\infty f_{n-1}(x+x_1+m_0+m_1-t, x_2+m_2, \cdots, x_n+m_n; \varphi_{i_1}) \varphi_i(t)dt \Big\} \qquad n<k,\ k\geqq 2 .$$

$$\int_0^\infty f_0(u-t, \varphi_{i_1})\varphi_i(t)\,dt = -\int_0^u w(u-t)\varphi_i(t)dt ,$$

where

(2.3) $$V(x, z, \varphi_i) = \begin{cases} v\int_0^{x+z} (t-x)\varphi_i(t)\,dt + vz\int_{x+z}^\infty \varphi_i(t)dt & x+z>0,\quad x\leqq 0 , \\ v\int_x^{x+z} (t-x)\varphi_i(t)\,dt + vz\int_{x+z}^\infty \varphi_i(t)dt & x+z>0,\quad x>0 , \\ vz & x+z\leqq 0 . \end{cases}$$

From the above the relation it is evident that $V(x, z, \varphi_i)$ represents the expected one-period credit cost when the starting stock level is given by x and the size of order to delivered at the end of the period by z. It will be convenient to introduce the function

(2.4) $$V(u, \varphi_i) = \begin{cases} -vu + v\int_0^u (u-\xi)\varphi_i(\xi)d\xi & u\geqq 0 , \\ -vu & u<0 . \end{cases}$$

Then $V(x, z, \varphi_i)$ can be written

$$V(x, z, \varphi_i) = -V(x+z, \varphi_i) + V(x, \varphi_i) \ . \tag{2.5}$$

§3. The Dynamic Model with a Linear Ordering Cost—k-period case.

We assume throughout the rest of the paper that the interval in ordering of stock is k-period. Then we have for $n \geq k$

$$\begin{aligned}
(3.1)\quad f_n(x\,;\,\varphi_i) = & \min_{m_{u_0}\geqq m_0\geqq 0,\cdots,m_{u_{k-1}}\geqq m_{k-1}\geqq 0, m_k\geqq 0} \Big\{\sum_{j=0}^{k} c_j m_j + L(x+m_0, \varphi_i) \\
& + \alpha\sum_{i_1=1}^{m} p_{ii_1}\int_0^\infty L(x+m_0+m_1-t, \varphi_{i_1})\varphi_i(t)dt + \cdots + \alpha^{k-1}\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots \\
& \sum_{i_{k-1}=1}^{m} p_{ii_1}p_{i_1i_2}\cdots p_{i_{k-2}i_{k-1}}\int_0^\infty\cdots\int_0^\infty L(x+m_0+m_1+\cdots+m_{k-1}-t-t_1\cdots \\
& -t_{k-2},\varphi_{i_{k-1}})\varphi_i(t)\varphi_{i_1}(t_1)\cdots\varphi_{i_{k-2}}(t_{k-2})dtdt_1\cdots dt_{k-2} + [Vx(+m_0+m_1, \varphi_i) \\
& -V(x+m_0, \varphi_i)] + \alpha\sum_{i_1=1}^{m} p_{ii_1}\int_0^\infty [V(x+m_0+m_1+m_2-t, \varphi_{i_1}) - V(x+m_0 \\
& +m_1-t, \varphi_{i_1})]\varphi_i(t)dt + \cdots + \alpha^{k-1}\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_{k-1}=1}^{m} p_{ii_1}p_{i_1i_2}\cdots p_{i_{k-2}i_{k-1}}\int_0^\infty\cdots \\
& \int_0^\infty [V(x+m_0+m_1+\cdots+m_k-t-t_1\cdots-t_{k-2}, \varphi_{i_{k-1}}) - V(x+m_0+m_1+\cdots \\
& +m_{k-1}-t-t_1\cdots-t_{k-2}, \varphi_{i_{k-1}})]\varphi_i(t)\varphi_{i_1}(t_1)\cdots\varphi_{i_{k-2}}(t_{k-2})dtdt_1\cdots dt_{k-2} \\
& +\alpha^k\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_k=1}^{m} p_{ii_1}p_{i_1i_2}\cdots p_{i_{k-1}i_k}\int_0^\infty\cdots\int_0^\infty f_{n-k}(x+m_0+m_1+\cdots+m_k-t \\
& -t_1\cdots-t_{k-1},\varphi_{i_k})\varphi_i(t)\varphi_{i_1}(t_1)\cdots\varphi_{i_{k-1}}(t_{k-1})dtdt_1\cdots dt_{k-1}\Big\} \\
= & \min_{x\leqq u_0\leqq x+m_{u_0}}\Big\{[(c_0-c_1)u_0 + L(u_0, \varphi_i) - V(u_0, \varphi_i) + \min_{u_0\leqq u_1\leqq u_0+m_{u_1}}\Big\{[(c_1-c_2)u_1 \\
& + L_1(u_1, \varphi_i) + V(u_1, \varphi_i) - V_1(u_1, \varphi_i)] + \min_{u_1\leqq u_2\leqq u_1+m_{u_2}}\Big\{\cdots + \min_{u_{k-2}\leqq u_{k-1}\leqq u_{k-1}+m_{u_{k-1}}} \\
& \Big\{[(c_{k-1}-c_k)u_{k-1} + L_{k-1}(u_{k-1}, \varphi_i) + V_{k-2}(u_{k-1}, \varphi_i) - V_{k-1}(u_{k-1}, \varphi_i)] \\
& + \min_{u_{k-1}\leqq u_k}[c_k u_k + V_{k-1}(u_k, \varphi_i) + f_{n-k,k}(u_k, \varphi_i)]\}\cdots\Big\} - c_0 x \\
= & \min_{x\leqq u_0\leqq x+m_u.} G_0(u_0, \varphi_i) - c_0 x\ , \qquad i=1, 2, \cdots, m,
\end{aligned}$$

where

$$\begin{aligned}
& G_{kn}(u_k, \varphi_i) = c_k u_k + V_{k-1}(u_k, \varphi_i) + f_{n-k,k}(u_k, \varphi_i), \qquad n\geq k \\
& G_{jn}(u_j, \varphi_i) = (c_j-c_{j+1})u_j + L_j(u_j, \varphi_i) + V_{j-1}(u_j, \varphi_i) - V_j(u_j, \varphi_i) + H_{jn}(u_j, \varphi_i), \\
& \qquad\qquad j=0, 1, \cdots, k-1\,;\ n\geq k\ , \\
& H_{k-1,n}(u_{k-1}, \varphi_i) = \min_{u_{k-1}\leqq u_k} G_{kn}(u_k, \varphi_i),\ H_{jn}(u_j, \varphi_i) = \min_{u_j\leqq u_{j+1}\leqq u_j+m_{u_{j+1}}} G_{j+1,n}(u_{j+1},\varphi_i),
\end{aligned}$$

$$j=0,\ 1,\ \cdots,\ k-2\ ,$$

$$i_0=i,\ x+m_0=u_0,\ x+m_0+m_1=u_1,\ \cdots,\ x+m_0+m_1+\cdots+m_k=u_k\ ,$$

(3.2)
$$L_l(x,\ \varphi_i)=\alpha\sum_{j=1}^{m} p_{ij}\int_0^\infty L_{l-1}(x-t,\ \varphi_j)\varphi_i(t)dt\ ,$$

$$V_l(x,\ \varphi_i)=\alpha\sum_{j=1}^{m} p_{ij}\int_0^\infty V_{l-1}(x-t,\ \varphi_j)\varphi_i(t)dt\ ,$$

$$W_l(x,\ \varphi_i)=\alpha\sum_{j=1}^{m} p_{ij}\int_0^x W_{l-1}(x-t,\ \varphi_j)\varphi_i(t)dt\ ,$$

$$f_{n-k,l}(x,\varphi_i)=\alpha\sum_{j=1}^{m} p_{ij}\int_0^\infty f_{n-k,l-1}(x-t,\ \varphi_j)\varphi_i(t)dt\ ,$$

$$f_{n-k,0}(x,\ \varphi_i)=f_{n-k}(x,\ \varphi_i)\ ,$$

$$L_{-1}(x,\ \varphi_i)=0,\ V_0(x,\ \varphi_i)=V(x,\ \varphi_i),\ L_0(x,\ \varphi_i)=L(x,\ \varphi_i),$$

$$f_0(x,\ \varphi_i)\ =-W_0(x,\ \varphi_i)=-w(x)\ .$$

It is noticed that $f_{0k}(x,\ \varphi_i)=-W_k(x,\ \varphi_i)$ for $x>0$. From the method as in the $n\geqq k$

(3.3)
$$\begin{aligned} f_n(x\ ;\ \varphi_i)=&\min_{x\leqq u_0\leqq x+m_{u_0}}\Big\{[(c_0-c_1)u_0+L(u_0,\ \varphi_i)-V(u_0,\ \varphi_i)]\\ &+\min_{u_0\leqq u_1\leqq u_0+m_{u_1}}\Big\{[(c_1-c_2)u_1+L_1(u_1,\ \varphi_i)+V(u_1,\ \varphi_i)-V_1(u_1,\ \varphi_i)]\\ &+\min_{u_1\leqq u_2\leqq u_1+m_{u_2}}\Big\{\cdots+\min_{u_{n-2}\leqq u_{n-1}\leqq u_{n-2}+m_{u_{n-1}}}\Big\{[(c_{n-1}-c_n)u_{n-1}\\ &+L_{n-1}(u_{n-1},\ \varphi_i)+V_{n-2}(u_{n-1},\ \varphi_i)-V_{n-1}(u_{n-1},\ \varphi_i)]\\ &+\min_{u_{n-1}\leqq u_n\leqq u_{n-1}+m_{u_n}}\Big\{[c_nu_n+V_{n-1}(u_n,\ \varphi_i)-W_n(u_n,\ \varphi_i)]\Big\}\cdots\Big\}-c_0x \end{aligned}$$

$$n<k,\ k\geqq 2\ ,$$

where $G_{jn}(u_j.\ \varphi_i)$ and $H_{jn}(u_j,\ \varphi_i)$ in (3.2) are given by

(3.4)
$$\begin{aligned} &G_{nn}(u_n,\ \varphi_i)=c_nu_n+V_{n-1}(u_n,\ \varphi_i)-W_n(u_n,\ \varphi_i)\\ &G_{jn}(u_j,\ \varphi_i)=(c_j-c_{j+1})u_j+L_j(u_j,\ \varphi_i)+V_{j-1}(u_j,\ \varphi_i)-V_j(u_j,\ \varphi_i)\\ &\qquad +H_{jn}(u_j,\ \varphi_i) \qquad\qquad j=0,\ 1,\ \cdots,\ n-1,\ n<k\ ,\\ &H_{jn}(u_j,\ \varphi_i)=\min_{u_j\leqq u_{j+1}\leqq u_j+m_{u_{j+1}}} G_{j+1,n}(u_{j+1},\ \varphi_i)\ \ j=0,\ 1,\ \cdots,\ n-1\ ;\ n<k. \end{aligned}$$

We cite known results in [2] that will be needed in the analysis that follows.

LEMMA A. (i) *If $g\ (x)$ is convex function on the real line, then for* $0\leqq m<\infty$

$$h(x)=\min_{x\leqq y\leqq x+m} g(y)$$

is convex. If $g(x)$ is bounded from below, so is $h(x)$.

(ii) (a) *If $g(x)$ is convex and bounded from below, then*

$$k(x)=\inf_{x\leqq y} g(y)$$

is convex and bounded from below. (b) *If $g(x)$ is convex, unbounded from below, but non-decreasing, then*

$$k(x)=\inf_{x\leqq y} g(y)=\min_{x\leqq y} g(y)=g(x)\ .$$

THEOREM 3.1. *Under assumption* (1.1)~(1.8), *the optimal expected loss function $f_n(x, \varphi_i)$ $(i=1, 2, \cdots, m)$ is convex.*

The proof is given by induction on the number n.

Case (a) $n\leq k$. Then from (3.4)

$$(3.5)\quad \begin{cases} G'_{11}(0, \varphi_i)=c_1+V'(0, \varphi_i)-W'_1(0, \varphi_i)=c_1-v<0\ , \\ \lim\limits_{u_1\to\infty} G'_{11}(u_1, \varphi_i)=c_1+\lim\limits_{u_1\to\infty} V'(u_1, \varphi_i)-\lim\limits_{u_1\to\infty} W'_1(u_1, \varphi_i)=c_1-\alpha\lim\limits_{u_1\to\infty} w'(u_1)>0 \end{cases}$$

$$(3.6)\quad G_{11}(u_1, \varphi_i)=\begin{cases} (c_1-v)u_1+v\int_0^u (u_1-t)\varphi_i(t)dt-\alpha\int_0^u w(u-t)\varphi_i(t)dt & u_1>0\ , \\ (c_1-v)u_1 & u_1<0\ . \end{cases}$$

The function $G_{11}(u_1, \varphi_i)$ is convex by (1.5) and (1.7b), attains a minimum at some positive $\bar{x}_{11}(\varphi_i)$, and increases for u_1 large enough. Hence $H_{01}(u_0, \varphi_i)$ is convex, bounded from below and increasing for u_0 large enough by Lemma A. Condition (1.8) establishes convexity of $L(u_0, \varphi_i)-V(u_0, \varphi_i)$. Hence $G_{01}(u_0, \varphi_i)$ is convex, since it is the sum of convex functions. By Lemma A $f_1(x, \varphi_i)$ is convex. Although $G_{01}(u_0, \varphi_i)$ is increasing, at least for u_0 large enough, $f_1(x, \varphi_i)$ can be convex decreasing, if

$$(3.7)\quad \lim_{u_0\to\infty} G'_{01}(u_0, \varphi_i)-c_0=\lim_{u_0\to\infty} L'(u_0, \varphi_i)-\alpha\lim_{u_0\to\infty}\int_0^{u_0} w'(u_0-t)\varphi_i(t)dt\leqq 0$$

Assume that the theorem is true for $n-1$, and that $V_{n-2}(u, \varphi_i)-W_{n-1}(u, \varphi_i)$, $L_{n-2}(u, \varphi_i)-V_{n-2}(u, \varphi_i)$ and $V_{n-2}(u, \varphi_i)$ is convex. Then from (3.2) and (3.4)

$$(3.8)\quad \begin{cases} G'_{nn}(0, \varphi_i)=c_n+V'_{n-1}(0, \varphi_i)-W'_n(0, \varphi_i)=c_n-\alpha^{n-1}v<0, \\ \lim\limits_{u_n\to\infty} G'_{nn}(u_n, \varphi_i)=c_n+\lim\limits_{u_n\to\infty} V'_{n-1}(u_n, \varphi_i)-\lim\limits_{u_n\to\infty} W'_n(u_n, \varphi_i) \\ \qquad =c_n-\alpha^n\lim\limits_{u_n\to\infty} w'(u_n)>0\ , \\ W_j(0, \varphi_i)=W'_j(0, \varphi_i)=0\ , \qquad j=1, 2, \cdots, n\ . \end{cases}$$

$$(3.9)\quad \begin{aligned} G_{nn}(u_n, \varphi_i)&=c_nu_n+V_{n-1}(u_n, \varphi_i)-W_n(u_n, \varphi_i) \\ &=c_nu_n+\alpha\sum_{i_1=1}^m p_{ii_1}\int_0^\infty [V_{n-2}(u_n-t, \varphi_{i_1})-W_{n-1}(u_n-t, \varphi_{i_1})]\varphi_i(t)dt\ . \end{aligned}$$

The function $G_{nn}(u_n, \varphi_i)$ is convex, since it is the sum of convex function, attains a minimum at some positive $\bar{x}_{nn}(\varphi_i)$, and increase for u_n large enough.

By Lemma A, $H_{n-1,n}(u_{n-1}, \varphi_i)$ is convex and increasing for u_{n-1} large enough. By (3.2), (3.4), and the induction hypothesis, $L_{n-1}(u, \varphi_i)-V_{n-1}(u, \varphi_i)$ and $V_{n-2}(u, \varphi_i)$ are convex. Hence $G_{n-1,n}(u_{n-1}, \varphi_i)$ is convex and increasing for u_{n-1} large enough. By Lemma A, (3.4), it is proved inductively that $H_{jn}(u_j, \varphi_i)$ and $G_{jn}(u_j, \varphi_i)$ $(j=0, 1, \cdots, n-2, n\leqq k, i=1, 2, \cdots, m)$ are convex, and increasing for u_j large enough. Hence $f_n(x, \varphi_i)$ is convex by Lemma A. Although $G_{0n}(u_0, \varphi_i)$ is increasing, at least for u_0 large enough, $f_n(x, \varphi_i)$ can be convex decreasing, if

$$\begin{aligned}(3.10)\qquad \lim_{u_0\to\infty} G'_{0n}(u_0, \varphi_i)-c_0 &= -c_1+\lim_{u_0\to\infty} L'(u_0, \varphi_i)+\lim_{u_0\to\infty} G'_{1n}(u_0, \varphi_i)\\ &= -c_n+\lim_{u_0\to\infty} L'(u_0, \varphi_i)+\lim_{u_0\to\infty} L'_1(u_0, \varphi_i)+\cdots\\ &\quad +\lim_{u_0\to\infty} L'_{n-1}(u_0, \varphi_i)+\lim_{u\to\infty} G'_{nn}(u_0, \varphi_i)\\ &= \sum_{j=0}^{n-1}\lim_{u_0\to\infty} L'_j(u, \varphi_i)-\alpha^n\lim_{u_0\to\infty} w'(u_0)\leqq 0\,.\end{aligned}$$

Case (b) $n\geqq k+1$. $n=k+1$. Then from (3.2)

$$\begin{aligned}(3.11)\qquad G_{k,k+1}(u_k, \varphi_i) &= c_k u_k+V_{k-1}(u_k, \varphi_i)+f_{1k}(u_k, \varphi_i)\\ &= c_k u_k+V_{k-1}(u_k, \varphi_i)+\alpha^k\sum_{i_1=1}^{m}\cdots\sum_{i_k=1}^{m} p_{ii_1}\cdots p_{i_{k-1}i_k}\int_0^\infty\cdots\\ &\quad \int_0^\infty f_1(u_k-t\cdots-t_{k-1}, \varphi_{i_k})\varphi_i(t)\cdots\varphi_{i_{k-1}}(t_{k-1})\,dt\cdots dt_{k-1}\end{aligned}$$

is convex, since it is the sum of convex function by case (a). By Lemma A and case (a), $H_{j,k+1}(u_j, \varphi_i)$ and $G_{j,k+1}(u_j, \varphi_i)$ $(j=0, 1, \cdots, k-1)$ are convex. Hence $f_{k+1}(x, \varphi_i)$ is convex.

Assume that the theorem is true for $n-1$. Then from (3.2)

$$\begin{aligned}(3.12)\qquad G_{kn}(u_k, \varphi_i) &= c_k u_k+V_{k-1}(u_k, \varphi_i)+f_{n-k,k}(u_k, \varphi_i)\\ &= c_k u_k+V_{k-1}(u_k, \varphi_i)+\alpha^k\sum_{i_1=1}^{m}\cdots\sum_{i_k=1}^{m} p_{ii_1}\cdots p_{i_{k-1}i_k}\int_0^\infty\cdots\\ &\quad \int_0^\infty f_{n-k}(u_k-t\cdots-t_{k-1}, \varphi_{i_k})\varphi_i(t)\cdots\varphi_{i_{k-1}}(t_{k-1})\,dt\cdots dt_{k-1}\end{aligned}$$

is convex, since it is the sum of three convex functions, By Lemma A, $H_{k-1,n}(u_{k-1}, \varphi_i)$ is convex and so is $G_{k-1,n}(u_{k-1}, \varphi_i)$. By Lemma A and case (a), $H_{jn}(u_j, \varphi_i)$ and $G_{jn}(u_j, \varphi_i)$ $(j=0, 1, \cdots, n-1)$ are convex. Hence $f_n(x, \varphi_i)$ is convex.

Remark. There exists the positive $\bar{x}_{nn}(\varphi_i)$ determined by

$$(3.13)\qquad G_{nn}(\bar{x}_{nn}(\varphi_i), \varphi_i)=0 \qquad n=1, 2, \cdots, k;\ i=1, 2, \cdots, m.$$

If $\varphi_i(t)>0$ for $t>0$ $(i=1, 2, \cdots, m)$ then $\bar{x}_{nn}(\varphi_i)$ is uniquely determined for $i=1, 2, \cdots, m$. In this case $G_{nn}(u_n, \varphi_i)$ is strictly convex for $u_n>0$.

THEOREM 3.2. *Let assnmption* (1.1)~(1.8) *hold. If*

$$(3.14)\quad \begin{array}{ll}\text{(a) } \lim_{t\to-\infty} L'(t,\ \varphi_i)+c_0-c_1+v<0, & \text{(b) } \lim_{t\to\infty} L'(t,\ \varphi_i)-\alpha\lim_{t\to\infty} w'(t)>0,\\ & i=1,\ 2,\ \cdots,\ m,\\ \text{(c) } v<p'(0),\ p(0)=h(0), & \text{(d) } \varphi_i(t)>0 \text{ for } t>0,\\ & i=1,\ 2,\ \cdots,\ m.\end{array}$$

are satisfied, then (i) *there exists a unique positive* $\bar{x}_{jn}(\varphi_i)$ *determined by*

$$(3.15)\quad \begin{array}{l} G'_{jn}(\bar{x}_{jn}(\varphi_i),\varphi_i)=0 \quad j=1,\ 2,\ \cdots,\ n;\ n=1,\ 2,\cdots,\ k-1;\ i=1,\ 2,\ \cdots,\ m,\\ G_{jn}(\bar{x}_{jn}(\varphi_i),\ \varphi_i)=0 \quad j=1,\ 2,\ \cdots,\ k;\ n\geqq k;\ i=1,\ 2,\ \cdots,\ m.\end{array}$$

(ii) *there exists a unique finite* $\bar{x}_{0n}(\varphi_i)$ *determined by*

$$(3.16)\quad G'_{0n}(\bar{x}_{0n}(\varphi_i),\ \varphi_i)=0 \qquad i=1,\ 2,\ \cdots,\ m.$$

(iii) $f_n(x,\ \varphi_i)$ *is convex, decreasing for x small enough, increasing for x large enough.*

Proof (by induction). $n\leqq k$. The existence of $\bar{x}_{nn}(\varphi_i)>0$ has been shown before under less stringent conditions.
Since

$$\begin{aligned}\lim_{u_0\to\infty} G'_{01}(u_0,\ \varphi_i)&=(c_0-c_1)+\lim_{u_0\to\infty} L'(u_0,\ \varphi_i)+\lim_{n_0\to\infty} H'_{01}(u_0,\ \varphi_i)\\ &=(c_0-c_1)+\lim_{u_0\to\infty} L'(u_0,\ \varphi_i)+\lim_{u_0\to\infty} G'_{11}(u_0,\ \varphi_i)>0\end{aligned}$$

and

$$\lim_{u_0\to-\infty} G'_{01}(u_0,\ \varphi_i)=(c_0-c_1)+v+\lim_{u_0\to-\infty} L'(u_0,\ \varphi_i)<0$$

it follows that $\bar{x}_{01}(\varphi_i)$ is finite. The uniqueness assertions are consequences of (3.14c, d).
(it is noticed that $G'_{01}(u,\ \varphi_i)<0$ is not assured). If $p(\eta)$ is linear, $\bar{x}_{01}(\varphi_i)$ is even pesitive. Thus we have

$$(3.17)\quad f'_1(x,\ \varphi_i)=\begin{cases} G'_{01}(x+m_{u_0},\ \varphi_i)-c_0 & x+m_{u_0}<\bar{x}_{01}(\varphi_i),\\ -c_0 & x<\bar{x}_{01}(\varphi_i)<x+m_{u_0},\\ G'_{01}(x,\ \varphi_i)-c_0 & \bar{x}_{01}(\varphi_i)<x,\end{cases}$$

hence, $\lim_{x\to-\infty} f'_1(x,\ \varphi_i)<0$, moreover by (3.14 b), $\lim_{x\to\infty} f'_1(x,\ \varphi_i)>0$, and also by Theorem 3.1 $f_1(x,\ \varphi_i)$ is convex.
That $\bar{x}_{11}(\varphi_i)>0$ implies

$$(3.13)\quad \begin{aligned} G'_{01}(0,\ \varphi_i)-c_0&=-c_1+L'(0,\ \varphi_i)-V'(0,\ \varphi_i)+H'_{01}(0,\ \varphi_i)\\ &=v-c_1-\int_0^\infty p'(t)\varphi_i(t)dt<v-c_1-p'(0)<0.\end{aligned}$$

Hence $f'_1(0,\ \varphi_i)<0$.

We assume that the theorem is true for $n-1$ and that is true for n and $j=n,\ n-1,\ \cdots,\ n-r(r\leqq n-2)$. That $\bar{x}_{n-r,n}(\varphi_i)>0$ implies

$$(3.19)\quad H'_{n-(r+1),n}(0,\ \varphi_i)=\begin{cases} G'_{n-r,n}(m_{u_{n-r}},\ \varphi_i)<0 & 0<\bar{x}_{n-r,n}(\varphi_i)-m_{u_{n-r}}\\ 0 & \bar{x}_{n-r,n}(\varphi_i)-m_{u_{n-r}}<0<\bar{x}_{n-r,n}(\varphi_i)\end{cases}$$

hence

$$(3.20)\qquad G'_{n-(r+1),n}(0,\ \varphi_i)=(c_{n-(r+1)}-c_{n-r})+L'_{n-(r+1)}(0,\ \varphi_i)+V'_{n-(r+2)}(0,\ \varphi_i)$$
$$-V'_{n-(r+1)}(0,\ \varphi_i)+H'_{n-(r+1),\,n}(0,\ \varphi_i)$$
$$\leqq(c_{n-(r+1)}-\alpha^{n-(r+2)}v)-c_{n-r}-\alpha^{n-(r+1)}(p'(0)-v)<0\,.$$

On the other side

$$(3.21)\qquad \lim_{u_{n-(r+1)}\to\infty} G'_{n-(r+1),n}(u_{n-(r+1)},\ \varphi_i)=(c_{n-(r+1)}-c_{n-r})$$
$$+\lim_{u_{n-(r+1)}\to\infty} L'_{n-(r+1)}(u_{(n-r+1)},\ \varphi_i)+\lim_{u_{n-(r+1)}\to\infty} G'_{n-r,n}(u_{n-(r+1)},\ \varphi_i)>0\,.$$

There exists $\bar{x}_{n-(r+1),n}(\varphi_i)>0$ such that $G'_{n-(r+1),n}\ (\bar{x}_{n-(r+1),n},\ \varphi_i)=0$. Uniqueness follows from (3.14 c, d).

Since

$$\lim_{u_0\to-\infty} G'_{0n}(u_0,\ \varphi_i)=(c_0-c_1)+\lim_{u_0\to-\infty} L'(u_0,\ \varphi_i)-\lim_{u_0\to-\infty} V'(u_0,\ \varphi_i)+\lim_{u_0\to-\infty} H'_{0n}(u_0,\ \varphi_i)$$
$$<(c_0-c_1)+\lim_{u_0\to-\infty} L'(u_0,\ \varphi_i)+v<0$$

and

$$\lim_{u_0\to\infty} G'_{0n}(u_0,\ \varphi_i)=(c_0-c_1)+\lim_{u_0\to\infty} L'(u_0,\ \varphi_i)-\lim_{u_0\to\infty} V'(u_0,\ \varphi_i)+\lim_{u_0\to\infty} H'_{0n}(u_0,\ \varphi_i)$$
$$=(c_0-c_1)+\lim_{u_0\to\infty} L'(u_0,\ \varphi_i)+\lim_{u_0\to\infty} G'_{1n}(u_0,\ \varphi_i)>0\,.$$

it follows that there is finite $\bar{x}_{0n}(\varphi_i)$ such that $G'_{0n}(\bar{x},\ \varphi_i)=0$. Uniqueness is shown as before. Since

$$(3.22)\qquad f'_n(x,\ \varphi_i)=\begin{cases} G'_{0n}(x+m_{u_0},\ \varphi_i)-c_0 & x+m_{u_0}<\bar{x}_{0n}(\varphi_i), \\ -c_0 & x<\bar{x}_{0n}(\varphi_i)<x+m_{u_0}, \\ G_{0n}(x,\ \varphi_i)-c_0 & \bar{x}_{0n}(\varphi_i)<x. \end{cases}$$

it follows that

$$(3.23)\qquad \lim_{x\to\infty} f'_n(x,\varphi_i)=\lim_{x\to\infty} G'_{0n}(x,\varphi_i)-c_0=-c_1+\lim_{x\to\infty} L'(x,\ \varphi_i)+\lim_{x\to\infty} G'_{1n}(x,\ \varphi_i)$$
$$=\sum_{j=0}^{n-1}\lim_{x\to\infty} L'_j(x,\ \varphi_i)-\alpha^n\lim_{x\to\infty} w'(x)$$
$$>\lim_{x\to\infty} L'(x,\ \varphi_i)-\alpha\lim_{x\to\infty} w'(x)>0.$$

by (3.14 b), and $\lim_{x\to-\infty} f'_n(x,\ \varphi_i)<-c_0<0$.

By Theorem 3.1 $f_n(x,\ \varphi_i)$ is convex. Hence the last assertion is true. That $\bar{x}_{1n}(\varphi_i)>0$ implies

$$(3.24)\qquad G'_{0n}(0,\ \varphi_i)-c_0=-c_1+L'(0,\ \varphi)-V'(0,\ \varphi)+H'_{on}(0,\ \varphi_i)$$
$$=v-c_1-\int_0^\infty p'(t)\varphi_i(t_i)dt_i<v-c_1-p'(0)<0.$$

Hence $f'_n(0,\ \varphi_i)<0$.

Case (b). $n\geqq k+1$.

Since

(3.25) $$\lim_{u_k\to\infty} G'_{k,k+1}(u_k, \varphi_i) = c_k + \lim_{u_k\to\infty} V'_{k-1}(u_k, \varphi_i) + \lim_{u_k\to\infty} f'_{1k}(u_k, \varphi_i)$$
$$= c_k + \alpha^k \lim_{u_k\to\infty} \sum_{i_1=1}^{m}\cdots\sum_{i_k=1}^{m} p_{ii_1}\cdots p_{i_{k-1}i_k}\int_0^\infty\cdots$$
$$\int_0^\infty f'_1(u_k - t - t_1 \cdots - t_{k-1}, \varphi_{i_k})\varphi_i(t)\cdots\varphi_{i_{k-1}}(t_{k-1})dt\cdots dt_{k-1} > 0,$$

and

(3.26) $$G'_{k,k+1}(0, \varphi_i) = c_k + V'_{k-1}(0, \varphi_i) + f'_{1k}(0, \varphi_i)$$
$$= c_k - \alpha^{k-1}v + \alpha^k \sum_{i_1=1}^{m}\cdots\sum_{i_k=1}^{m} p_{ii_1}\cdots p_{i_{k-1}i_k}\int_0^\infty\cdots$$
$$\int_0^\infty f'_1(-t - t_1 - \cdots - t_{k-1}, \varphi_{i_k})\varphi_i(t)\cdots\varphi_{k-1}(t_{k-1})dt\cdots dt_{k-1}$$
$$< c_k - \alpha^{k-1}v + \alpha^k f'_1(0, \varphi_{i_k}) < 0.$$

By (3.14 c, d), $G_{k,k+1}(u_k, \varphi_i)$ is strictly convex for $u_k > 0$. By lemma A, case (a), it is proved inductively that $\lim_{uj\to\infty} G'_{jk+1}(u_j, \varphi_i) > 0$ for $j=0, 1, \cdots, k-1$, $G'_{jk+1}(0, \varphi_i) < 0$ for $j=1, 2, \cdots, k-1$ and $\lim_{u_0\to-\infty} G'_{0k+1}(u_0, \varphi_i) < 0$. There exists $\bar{x}_{j,k+1}(\varphi_i) > 0$ such that $G'_{j,k+1}(\bar{x}_{j,k+1}(\varphi_i), \varphi_i) = 0$. Uniqueness follows from (3.14 c, d) (iii) is shown as in the case (a). Hence the theorem holds for $k+1$. It is easily seen, as in the case (a), that $f'_{k+1}(0, \varphi_i) < 0$. Assume that the theorem is true for $n-1$. Then

(3.27) $$f'_{n-k}(x, \varphi_i) = \begin{cases} G'_{0,n-k}(x + m_{u_0}, \varphi_i) - c_0 & x + m_{u_0} < \bar{x}_{0,n-k}(\varphi_i), \\ -c_0 & x < \bar{x}_{0,n-k}(\varphi_i) < x + m_{u_0}, \\ G'_{0,n-k}(x, \varphi_i) - c_0 & \bar{x}_{0,n-k}(\varphi_i) < x \end{cases}$$

implies $f'_{n-k}(x, \varphi_i) < 0$ for $x \leqq 0$, hence

$$G'_{kn}(0, \varphi_i) = c_k + V'_{k-1}(0, \varphi_i) + f'_{n-k,k}(0, \varphi_i) < c_k - \alpha^{k-1}v < 0$$

(3.28) $$\lim_{u_k\to\infty} G'_{kn}(u_k, \varphi_i) = c_k + \lim_{u_k\to\infty} V'_{k-1}(u_k, \varphi_i) + \lim_{u_k\to\infty} f'_{n-k,k}(u_k, \varphi_i) > 0.$$

There exists $\bar{x}_{kn}(\varphi_i) > 0$ such that $G'_{kn}(u_k, \varphi_i) = 0$. Uniqueness follows from (3.14 c, d). The assertion of (i) and (ii) for $j = k-1$ $k-2, \cdots, 0$. is shown as before. The last assertion is shown as in the case (a). Moreover we obtain from the discussion of Theorem 3.2 the following theorem.

THEOREM 3.3. *If conditions of Theorem 3.2. are satisfied, then the optimal ordering policy is of the following form.*

(i) $n < k$

$$m^i_{0n}(x) = \begin{cases} \bar{x}_{0n}(\varphi_i) - x & \bar{x}_{0n}(\varphi_i) - m_{u_0} < x < \bar{x}_{0n}(\varphi_i), \\ 0 & \bar{x}_{0n}(\varphi_i) < x, \\ m_{u_0} & \bar{x}_{0n}(\varphi_i) - m_{u_0} > x. \end{cases} \qquad n = 1, 2, \cdots, k-1;\ i = 1, 2, \cdots, m,$$

(3.29)

$$m_{jn}^{i}(u_{j-1})=\begin{cases}\bar{x}_{jn}(\varphi_i)-u_{j-1} & \bar{x}_{jn}(\varphi_i)-m_{u_j}<u_{j-1}<\bar{x}_{jn}(\varphi_i),\\ 0 & \bar{x}_{jn}(\varphi_i)<u_{j-1},\\ m_{u_j} & \bar{x}_{jn}(\varphi_i)-m_{u_j}>u_{j-1}.\end{cases}\quad n=1,2,\cdots,k-1;\ i=1,2,\cdots,m,\ j=1,\cdots,n,$$

(ii) $n\geqq k$

$$m_{0n}^{i}(x)=\begin{cases}\bar{x}_{0n}(\varphi_i)-x & \bar{x}_{0n}(\varphi_i)-m_{u_0}<x<\bar{x}_{0n}(\varphi_i),\\ 0 & \bar{x}_{0n}(\varphi_i)<x,\\ m_{u_0} & \bar{x}_{0n}(\varphi_i)-m_{u_0}>x.\end{cases}\quad i=1,\ 2,\ \cdots,\ m,$$

$$(3.30)\quad m_{jn}^{i}(u_{j-1})=\begin{cases}\bar{x}_{jn}(\varphi_i)-u_{j-1} & \bar{x}_{jn}(\varphi_i)-m_{u_j}<u_{j-1}<\bar{x}_{jn}(\varphi_i),\\ 0 & \bar{x}_{jn}(\varphi_i)<u_{j-1},\\ m_{u_j} & \bar{x}_{jn}(\varphi_i)-m_{u_j}>u_{j-1}.\end{cases}\quad i=1,2,\cdots,m;\ j=1,2,\cdots,k-1,$$

$$m_{kn}^{i}(u_{k-1})=\begin{cases}\bar{x}_{kn}(\varphi_i)-u_{k-1} & u_{k-1}<\bar{x}_{kn}(\varphi_i),\\ 0 & u_{k-1}>\bar{x}_{kn}(\varphi_i).\end{cases}\quad i=1,\ 2,\ \cdots,\ m,$$

where $\bar{x}_{jn}(\varphi_i)$, $\bar{x}_{0n}(\varphi_i)$, *are determined satisfying* (3.15) *and* (3.16) *respectively, and* u_j *is given by* (3.2).

We next show that the analogue of a theorem, which is given in [3], holds true for comparing the critical numbers asociated with two sequences of demand densities under our model. We say that φ is stochastically smaller than ψ and we write $\varphi\subset\psi$ if $\int_0^x\varphi(t)dt\geqq\int_0^x\psi(t)dt$ for all x.

THEOREM 3.4. *Let conditions of Theorem* 3.2 *hold. If two sequences of demand densities* $\varphi_1, \varphi_2, \cdots, \varphi_m$, *and* $\psi_1, \psi_2, \cdots, \psi_m$, *are such that* $\varphi_i\subset\psi_i$ *for* $i=1, 2, \cdots, m$, *then*

(a) $\bar{x}_{jn}(\varphi_i)\leqq\bar{x}_{jn}(\psi_i)$ $\quad j=0, 1, \cdots, n;\ n=1, 2, \cdots, k-1;\ i=1, 2, \cdots, m,$

(3.31) $\quad\bar{x}_{jn}(\varphi_i)\leqq\bar{x}_{jn}(\psi_i)$ $\quad j=0, 1, \cdots, k;\ n\geqq k;\ i=1, 2, \cdots, m,$

and

(b) $f_n'(x, \varphi_i)\geqq f_n'(x, \psi_i)$ for all x, $\quad i=1, 2, \cdots, m.$

Proof. We establish the proof by induction on n. There are two posibilities requiring separate treatment, $n\leqq k$ and $n\geqq k+1$.

Case (a) $n\leqq k$. When $n=1$, we consider demand densities φ_i and ψ_i, where $\varphi_i\subset\psi_i$. Integrating by parts, we obtain from (3.4)

$$G_{11}'(u,\varphi_i)=\begin{cases}(c_1-v)+(v-\alpha w'(0))\Phi_i(u)-\alpha\int_0^u w''(u-t)\Phi_i(t)dt & u<0,\\ (c_1-v) & u<0,\end{cases}$$

(3.32)
$$G'_{01}(u,\ \varphi_i)-H'_{01}(u,\ \varphi_i)=\begin{cases}(c_0-c_1+v)+(h'(0)+p'(0)-v)\Phi_i(u) \\ \quad -p'(0)+\int_0^u h''(u-t)\Phi_i(t)dt \\ \quad -\int_u^\infty p''(t-u)(1-\Phi(t_i))dt & u>0 \\ (c_0-c_1+v)-p'(u)-\int_0^\infty p''(t-u)(1-\Phi_i(t))dt \\ & u<0,\end{cases}$$

where

$$\Phi_i(t)=\int_0^t \varphi_i(\eta)d\eta \qquad i=1,\ 2,\ \cdots,\ m.$$

A similar expression is obtain for $G'_{11}(u,\ \psi_i)$ and $G'_{01}(u,\ \psi_i)$, we simply replace $\Phi_i(t)$ in (3.32) by $\Psi_i(t)$, where

$$\Psi_i(t)=\int_0^t \psi_i(\eta)d\eta \qquad i=1,\ 2,\ \cdots,\ m.$$

Since $\varphi_i\subset\psi_i$, it follows by definition that $\Phi_i(t)\geqq\Psi_i(t)$ for all $t\geqq0$; hence, we get from (1.7 b), (1.3), (1.5), and (3.14)

(3.33) $G'_{11}(u_1,\ \varphi_i)\geqq G'_{11}(u_1,\ \psi_i)$ for all u_1 $\qquad i=1,2,\cdots,m,$

and

$$G'_{01}(u_0,\ \varphi_i)-H'_{01}(u_0,\ \varphi_i)\geqq G'_{01}(u_0,\ \psi_i)-H'_{01}(u_0,\ \psi_i) \text{ for all } u_0 \qquad i=1,2,\cdots,m.$$

By (3.33) and (3.4), $H_{01}(u_0,\ \varphi_i)\geqq H_{01}(u_0,\ \psi_i)$ for all u_0. Hence

(3.34) $G'_{01}(u_0,\ \varphi_i)\geqq G'_{01}(u_0,\ \psi_i)$ for all u_0 $\qquad i=1,2,\cdots,m.$

Hence, it follows immeditely from (3.33), (3.34) and Theorem 3.2, that

(3.35)
$$\begin{aligned}&\bar{x}_{11}(\varphi_i)\leqq\bar{x}_{11}(\psi_i)\\&\bar{x}_{01}(\varphi_i)\leqq\bar{x}_{01}(\psi_i)\end{aligned} \qquad i=1,2,\cdots,m.$$

If we compare (3.33), (3.34), and (3.35) with (3.17) and the correspondnig equation for ψ_i

(3.36)
$$f'_1(x,\ \psi_i)=\begin{cases}G'_{01}(x+m_{u_0},\ \psi_i)-c_0 & x+m_{u_0}<\bar{x}_{01}(\psi_i),\\ -c_0 & x<\bar{x}_{01}(\psi_i)<x+m_{u_0},\\ G'_{01}(x,\ \psi_i)-c_0 & \bar{x}_{01}(\psi_i)<x,\end{cases}$$

it is clear that

(3.37) $f'_1(x,\ \varphi_i)\geqq f'_1(x,\ \psi_i)$ for all x.

Thus we have proved the theorem for the one-period model.

Assume that we proved

$$\bar{x}_{jn}(\varphi_i)\leqq\bar{x}_{jn}(\psi_i) \qquad j=0,\ 1,\ \cdots,\ n-1;\ i=1,\ 2,\ \cdots,\ m,$$

$$f'_{n-1}(x,\ \varphi_i)\geqq f'_{n-1}(x,\ \psi_i) \text{ for all } x \qquad i=1,\ 2,\ \cdots,\ m,$$

$$G'_{j,n-1}(u_j, \varphi_i)-H'_{j,n-1}(u_j, \varphi_i)\geqq G'_{j,n-1}(u_j, \psi_i)-H'_{j,n-1}(u_j, \psi_i) \text{ for all } u_j, \quad j=0, 1, \cdots, n-2;\ i=1, 2, \cdots, m,$$

$$G'_{j,n-1}(u_j, \varphi_i)\geqq G'_{j,n-1}(u_j, \psi_i) \text{ for all } u_j, \quad j=0, 1, \cdots, n-1;\ i=1, 2, \cdots, m,$$

for any m pair of demand densities φ_i and ψ_i satisfying $\varphi_i \subset \psi_i$. Recalling (3.9) and differentiating (3.4) with respect to u_j yields

$$(3.38)\quad \begin{aligned} G'_{nn}(u_n, \varphi_i) &= c_n + V'_{n-1}(u_n, \varphi_i) - W'_n(u_n, \varphi_i) \\ &= c_n - \alpha c_{n-1} + \alpha\sum_{i_1=1}^{m} p_{ii_1}\int_0^{\infty}[c_{n-1}+V'_{n-2}(u_n-t, \varphi_{i_1}) \\ &\quad - W'_{n-1}(u_n-t, \varphi_{i_1})]\varphi_i(t)dt \\ &= c_n - \alpha c_{n-1} + \alpha\sum_{i_1=1}^{m} p_{ii_1}\int_0^{\infty} G'_{n-1,n-1}(u_n-t, \varphi_{i_1})\varphi_i(t)dt. \end{aligned}$$

Using our inductive assumption, integrating by parts, we get

$$(3.39)\quad \begin{aligned} G'_{nn}(u_n, \varphi_i) &\geqq c_n - \alpha c_{n-1} + \alpha\sum_{i_1=1}^{m} p_{ii_1}\int_0^{\infty} G'_{n-1,n-1}(u_n-t, \psi_{i_1})\varphi_i(t)dt \\ &= c_n - \alpha c_{n-1} + \alpha\sum_{i_1=1}^{m} p_{ii_1}G'_{n-1,n-1}(u_n, \psi_{i_1}) \\ &\quad - \alpha\sum_{i_1=1}^{m}\int_0^{\infty} G''_{n-1,n-1}(u_n-t, \psi_{i_1})(1-\Phi_i(t))dt \\ &\geqq c_n - \alpha c_{n-1} + \alpha\sum_{i_1=1}^{m} p_{ii_1}G'_{n-1,n-1}(u_n, \psi_{i_1}) \\ &\quad - \alpha\sum_{i_1=1}^{m} p_{ii_1}\int_0^{\infty} G''_{n-1,n-1}(u_n-t, \psi_{i_1})(1-\psi_i(t))dt \\ &= G'_{nn}(u_n, \psi_i) \text{ for all } u_n \qquad i=1, 2, \cdots, m. \end{aligned}$$

Hence

$$\bar{x}_{nn}(\varphi_i)\leqq \bar{x}_{nn}(\psi_i) \qquad i=1, 2, \cdots, m.$$

Assuming that $G'_{jn}(u_j, \varphi_i)\geqq G_{jn}(u_j, \psi_i)$ for all u_j and $j=n, n-1, \cdots, n-r+1$ $(1\leqq r\leqq n)$. Using the inductive assumption and (3.4), we obtain from the simple calculation

$$(3.40)\quad H'_{jn}(u_j, \varphi_i)\geqq H'_{jn}(u_j, \psi_i) \text{ for all } u_j \quad j=n-1, n-2, \cdots, n-r\,(1\leqq r\leqq n).$$

Hence from inductive assumption and (3.4), we have

$$\begin{aligned} & G'_{n-r,n}(u_{n-r}, \varphi_i) - H_{n-r,n}(u_{n-r}, \varphi_i) \\ &= (c_{n-r}-c_{n-r+1}) + L'_{n-r}(u_{n-r}, \varphi_i) + V'_{n-r-1}(u_{n-r}, \varphi_i) - V'_{n-r}(u_{n-r}, \varphi_i) \\ &= (c_{n-r}-c_{n-r+1}) - \alpha(c_{n-r-1}-c_{n-r}) \\ &\quad + \alpha\sum_{i_1=1}^{m} p_{ii_1}\int_0^{\infty}[G'_{n-r-1,n-1}(u_{n-r}-t, \varphi_{i_1}) - H'_{n-r,n-1}(u_{n-r}-t, \varphi_{i_1})]\varphi_i(t)dt \\ &\geqq (c_{n-r}-c_{n-r+1}) - \alpha(c_{n-r-1}-c_{n-r}) \\ &\quad + \alpha\sum_{i_1=1}^{m} p_{ii_1}[G'_{n-r-1,n-1}(u_{n-r}, \psi_{i_1}) - H'_{n-r-1,n-1}(u_{n-r}, \psi_{i_1})] \end{aligned}$$

$$-\alpha \sum_{i_1=1}^{m} p_{ii_1} \int_0^{\infty} [G''_{n-r-1,n-1}(u_{n-r}-t,\ \psi_{i_1})$$
$$-H''_{n-r-1,n-1}(u_{n-r}-t,\ \psi_{i_1})](1-\Phi_i(t))dt$$
$$\geqq (c_{n-r}-c_{n-r+1})-\alpha(c_{n-r-1}-c_{n-r})$$
$$+\alpha \sum_{i_1=1}^{m} p_{ii_1}[G'_{n-r-1,n-1}(u_{n-r},\ \psi_{i_1})-H'_{n-r-1,n-1}(u_{n-r},\ \psi_{i_1})]$$
$$-\alpha \sum_{i_1=1}^{m} p_{ii_1} \int_0^{\infty} [G''_{n-r-1,n-1}(u_{n-r}-t,\ \psi_{i_1})$$
$$-H''_{n-r-1,n-1}(u_{n-r}-t,\ \psi_{i_1})](1-\Psi_i(t))dt$$
$$=G'_{n-r,n}(u_{n-r},\ \psi_i)-H'_{n-r,n}(u_{n-r},\ \psi_i) \text{ for all } u_{n-r} \qquad i=1, 2, \cdots, m.$$

$$G'_{1n}(u_1,\ \varphi_i)-H'_{1n}(u_1,\ \varphi_i)$$
$$=(c_1-c_2)-\alpha(c_0-c_1)$$
$$+\alpha \sum_{i_1=1}^{m} p_{ii_1} \int_0^{\infty} [G'_{0,n-1}(u_1-t,\ \varphi_{i_1})-H'_{0,n-1}(u_1-t,\ \varphi_{i_1})]\varphi_i(t)dt+V'(u_1,\ \varphi_i)$$
$$\geqq G'_{1n}(u_1,\ \psi_i)-H'_{1n}(u_1,\ \psi_i) \text{ for all } u_1 \qquad i=1, 2, \cdots, m.$$

$$G'_{0n}(u_0,\ \varphi_i)=(c_0-c_1)+L'(u_0,\ \varphi_i)-V'(u_0,\ \varphi_i)+H'_{0n}(u_0,\ \varphi_i)$$
$$\geqq (c_0-c_1)+L'(u_0,\ \psi_i)-V'(u_0,\ \psi_i)+H'_{0n}(u_0,\ \psi_i)$$
$$=G'_{0n}(u_0,\ \psi_i) \text{ for all } u_0. \qquad i=1, 2, \cdots, m.$$

From (3.40) and (3.41), we get

$$G'_{n-r,n}(u_{n-r},\ \varphi_i)\geqq G'_{n-r,n}(u_{n-r},\ \psi_i) \text{ for all } u_{n-r} \qquad i=1, 2, \cdots, m.$$

The same argument used for the case $n=1$, now readily yields

$$\bar{x}_{jn}(\varphi_i)\leqq \bar{x}_{jn}(\psi_i) \qquad j=0, 1, \cdots, n;\ i=1, 2, \cdots, m,$$

and

$$f'_n(x,\ \varphi_i)\geqq f'_n(x,\ \psi_i) \text{ for all } x \qquad i=1, 2, \cdots, m.$$

Case (b) $n\geqq k+1$. We begin the proof by showing that

(3.42) $V'_{l-1}(u,\ \varphi_i)\geqq V'_{l-1}(u,\ \psi_i)$ for all u $\quad l=1, 2, \cdots, k;\ i=1, 2, \cdots, m.$

(3.43) $f_{jl}(u,\ \varphi_i)\geqq f'_{jl}(u,\ \psi_i)$ for all u $\quad l=1, 2, \cdots, k;\ j=1, 2, \cdots, n-k;$
$i=1, 2, \cdots, m.$

We first establish (3.42) for the case $l=1$. From (2.4), we have

$$V'(u,\varphi_i)=v(\Phi_i(u)-1)$$
$$\geqq v(\Psi_i(u)-1)$$
$$=V'(u,\ \psi_i) \text{ for all } u\geqq 0, \qquad i=1, 2, \cdots, m,$$

and

$$V'(u,\ \psi_i)=-v=V'(u,\ \psi_i) \text{ for all } u<0 \qquad i=1, 2, \cdots, m.$$

Assuming that (3.42) holds for $l-1$, $(1\leqq l-1<k)$, we advance the induction to l. Invoking the inductive assumption and (3.2), integrating by parts, we have

$$(3.44)\quad V'_{k-1}(u,\ \varphi_i)=\alpha\sum_{i_1=1}^{m}p_{ii_1}\int_0^\infty V'_{k-2}(u-t,\ \varphi_{i_1})\varphi_i(t)dt$$
$$\geqq\alpha\sum_{i_1=1}^{m}p_{ii_1}\int_0^\infty V'_{k-2}(u-t,\ \psi_{i_1})\varphi_i(t)dt$$
$$=\alpha\sum_{i_1=1}^{m}p_{ii_1}\int_0^\infty V'_{k-2}(u,\ \psi_{i_1})-\alpha\sum_{i_1=1}^{m}p_{ii_1}\int_0^\infty V''_{k-2}(u-t,\ \psi_{i_1})(1-\Phi_i(t))dt$$
$$\geqq\alpha\sum_{i_1=1}^{m}p_{ii_1}V'_{k-2}(u,\ \psi_{i_1})-\alpha\sum_{i_1=1}^{m}p_{ii_1}\int_0^\infty V'_{k-2}(u-t,\ \psi_{i_1})(1-\Psi_i(t))dt$$
$$=V'_{k-1}(u,\ \psi_i)\ \text{ for all } u. \qquad i=1,2,\cdots,m,$$

which proves (3.42).

We establish (3.43) by induction on l and j. Suppose first $l=1$, $j=1$, then from (3.2) and case (a),

$$f'_{11}(u,\ \varphi_i)=\alpha\sum_{i_1=1}^{m}p_{ii_1}\int_0^\infty f'_1(u-t,\ \varphi_{i_1})\varphi_i(t)dt$$
$$\geqq\alpha\sum_{i_1=1}^{m}p_{ii_1}\int_0^\infty f'_1(u,\ \psi_{i_1})-\alpha\sum_{i_1=1}^{m}p_{ii_1}\int_0^\infty f''_1(u-t,\ \psi_{i_1})(1-\Psi_i(t))dt$$
$$=f'_{11}(u,\ \psi_i)\ \text{ for all } u \qquad i=1,2,\cdots,m.$$

We verify (3.43) for the integer $l=1$ and $j-1(1\leqq j-1<n-k)$ in a manner analogous to that for $l=1$ and $j=1$. Now supposing that (3.43) is true for the integer $l-1(1\leqq l-1<k)$ and $j=1, 2, \cdots, n-k$, we extend the induction to integer l. It follows from the inductive assumption, (3.2) and case (a) that for all u, we have

$$(3.45)\quad f'_{jl}(u,\ \varphi_i)=\alpha\sum_{i_1=1}^{m}p_{ii_1}\int_0^\infty f'_{j,l-1}(u-t,\ \varphi_{i_1})\varphi_i(t)dt$$
$$\geqq\alpha\sum_{i_1=1}^{m}p_{ii_1}f'_{j,l-1}(u,\ \psi_{i_1})-\alpha\sum_{i_1=1}^{m}p_{ii_1}\int_0^\infty f''_{j,l-1}(u-t,\ \psi_{i_1})(1-\Psi_i(t))dt)$$
$$=f'_{jl}(u,\ \psi_i)\ \text{for all } u \qquad l=1,2,\cdots,k;\ j=1,2,\cdots,n-k;\ i=1,2,\cdots,m,$$

which proves (3.43).

Since

$$G'_{jn}(u_j,\ \varphi_i)-H'_{jn}(u_j,\ \varphi_i)=G'_{j,n-1}(u_j,\ \varphi_i)-H'_{j,n-1}(u_j,\ \varphi_i)\ \text{ for all } u_j$$
$$j=0,1,\cdots,n-1;\ n\geqq k+1;\ i=1,2,\cdots,m.$$

Using relations (3.42) and (3.43), we can be prove the theorem for the $n\geqq k+1$ in the way as in the case (a).

COROLLARY. *In Theorem* 3.4, *if* $\psi_i(x)=\varphi_i(x-a_i)$ *for positive* $a_i>0$, *then*

$$\bar{x}_{nn}(\varphi_i)+\sum_{l=0}^{n-1}a_{i_l}=\bar{x}_{nn}(\psi_i) \qquad i_l=1,2,\cdots,m;\ l=0,1,\cdots,n-1;\ n\leqq k,$$

$$\text{(a)}\quad \begin{aligned}
&\bar{x}_{jn}(\varphi_i)+\sum_{l=0}^{j-1}a_{i_l}<\bar{x}_{jn}(\psi_i)<\bar{x}_{jn}(\varphi_i)+\sum_{l=0}^{n-1}a_{i_l} && j=1,2,\cdots,n-1;\ i_l=1,2,\cdots,m;\ l=0,1,\cdots,n-1;\ k\geqq n\geqq 2,\\
&\bar{x}_{0n}(\varphi_i)+a_i\leqq\bar{x}_{0n}(\psi_i)<\bar{x}_{0n}(\varphi_i)+\sum_{l=0}^{n-1}a_{i_l} && i_l=1,2,\cdots,m;\ l=0,1,\cdots,n-1;\ n\leqq k,\\
&\bar{x}_{01}(\varphi_i)+a_i=\bar{x}_{01}(\psi_i) && i=1,2,\cdots,m;\ n\leqq k.
\end{aligned}$$

$$\text{(b)}\quad \begin{aligned}
&\bar{x}_{kn}(\varphi_i)+\sum_{l=0}^{k-1}a_{i_l}\leqq\bar{x}_{kn}(\psi_i)\leqq\bar{x}_{kn}(\varphi_i)+\sum_{l=0}^{n-1}a_{i_l} && i_l=1,2,\cdots,m;\ l=0,1,\cdots,n-1;\ n\geqq k+1,\\
&\bar{x}_{jn}(\varphi_i)+\sum_{l=0}^{j-1}a_{i_l}<\bar{x}_{jn}(\psi_i)<\bar{x}_{jn}(\varphi_i)+\sum_{l=0}^{n-1}a_{i_l} && j=1,2,\cdots,k-1;\ i_l=1,2,\cdots,m;\ l=0,1,\cdots,n-1;\ n\geqq k+1,\\
&\bar{x}_{0n}(\varphi_i)+a_i\leqq\bar{x}_{0n}(\psi_i)<\bar{x}_{0n}(\varphi_i)+\sum_{l=0}^{n-1}a_{i_l} && i_l=1,2,\cdots;\ l=0,1,\cdots,n-1;\ n\geqq k+1.
\end{aligned}$$

$$\text{(c)}\begin{cases}
f_1'(x-a_i,\ \varphi_i)=f_1'(x,\ \psi_i)\quad\text{for all } x, & i=1,2,\cdots,m,\\
f_n'(x-a_i,\ \varphi_i)\geqq f_n'(x,\ \psi_i)\geqq f_n'(x-a_i-a_{i_1}\cdots-a_{i_{n-1}},\ \varphi_i)\quad\text{for all } x, & i_l=1,2,\cdots,m;\ l=0,1,\cdots,n-1.
\end{cases}$$

Proof (by induction). Case (a). $n\leqq k$. When $n=1$, we obtain from (3.4) upon making a simple change variable that

$$\text{(3.46)}\quad \begin{aligned}
G_{11}'(u_1,\ \psi_i)&=c_1+V'(u_1,\ \psi_i)-W_1'(u_1,\ \psi_i)\\
&=c_1+V'(u_1-a_i,\ \varphi_i)-W_1'(u_1-a_i,\ \varphi_i)\\
&=G_{11}'(u_1-a_i,\ \varphi_i)\quad\text{for all } u_1,\qquad i=1,2,\cdots,m.
\end{aligned}$$

$$\text{(3.47)}\quad H_{01}'(u_0,\ \psi_i)=\begin{cases}
G_{11}'(u_0+m_{u1},\ \psi_i) & u_0+m_{u1}<\bar{x}_{11}(\psi_i),\\
0 & u_0<\bar{x}_{11}(\psi_i)<u_0+m_{u1},\\
G_{11}'(u_0,\ \psi_i) & u_0>\bar{x}_{11}(\psi_i),
\end{cases}$$

$$=\begin{cases}
G_{11}'(u_0-a_i+m_{u1},\ \varphi_i) & u_0-a_i+m_{u1}<\bar{x}_{11}(\varphi_i)\\
0 & u_0-a_i<\bar{x}_{11}(\varphi_i)<u_0-a_i+m_{u1},\\
G_{11}'(u_0-a_i,\ \varphi_i) & u_0-a_i>\bar{x}_{11}(\varphi_i),
\end{cases}$$

$$=H_{01}'(u_0-a_i,\ \varphi_i)\quad\text{for all } u_0,\qquad i=1,2,\cdots,m.$$

$$\text{(3.48)}\quad \begin{aligned}
G_{01}'(u_0,\ \psi_i)&=(c_0-c_1)+L'(u_0,\ \psi_i)-V'(u_0,\ \psi_i)+H_{01}'(u_0,\ \psi_i)\\
&=(c_0-c_1)+L'(u_0-a_i,\ \varphi_i)-V'(u_0-a_i,\ \varphi_i)+H_{01}'(u_0-a_i,\ \varphi_i)\\
&=G_{01}'(u_0-a_i,\ \varphi_i)\quad\text{for all } u_0,\qquad i=1,2,\cdots,m.
\end{aligned}$$

Hence it follows immediately from (3.46), (3.47), (3.48), (3.17), (3.36) and Theorem 3.2. that

$$\begin{aligned}
&\bar{x}_{11}(\varphi_i)+a_i=\bar{x}_{11}(\psi_i) && i=1,2,\cdots,m,\\
&\bar{x}_{01}(\varphi_i)+a_i=\bar{x}_{01}(\psi_i) && i=1,2,\cdots,m,\\
&f_1'(x-a_i,\ \varphi_i)=f_1'(x,\ \psi_i)\quad\text{for all } x, && i=1,2,\cdots,m.
\end{aligned}$$

When $n=2$, we get from (3.4), (3.46), (3.47), (3.48), and strict convexity of $L(u, \psi_i)-V(u, \psi_i)$ and $V(u, \psi_i)$

$$\begin{aligned}
G'_{22}(u_2, \psi_i) &= c_2+\alpha\sum_{i_1=1}^{m} p_{ii_1}\int_0^\infty V'(u_2-t, \psi_{i_1})\psi_i(t)dt \\
&\quad -\alpha\sum_{i_1=1}^{m} p_{ii_1}\int_0^\infty W'_1(u_2-t, \psi_{i_1})\psi_i(t)dt \\
&= c_2+\alpha\sum_{i_1=1}^{m} p_{ii_1}\int_0^\infty V'(u_2-a_{i_1}-t, \varphi_{i_1})\varphi_i(t-a_i)dt \\
&\quad -\alpha\sum_{i_1=1}^{m} p_{ii_1}\int_0^\infty W'_1(u_2-a_{i_1}-t, \varphi_{i_1})\varphi_i(t-a_i)dt \\
&= c_2+\alpha\sum_{i_1=1}^{m} p_{ii_1}\int_0^\infty V'(u_2-a_i-a_{i_1}-t, \varphi_{i_1})\varphi_i(t)dt \\
&\quad -\alpha\sum_{i_1=1}^{m} p_{ii_1}\int_0^\infty W'_1(u_2-a_i-a_{i_1}-t, \varphi_{i_1})\varphi_i(t)dt \\
&= G'_{22}(u_2-a_i-a_{i_1}, \varphi_i) \text{ for all } u_2, \qquad i=1, 2, \cdots, m.
\end{aligned}$$

(3.49)
$$\begin{aligned}
H'_{12}(u_1, \psi_i) &= H'_{12}(u_1-a_i-a_{i_1}, \varphi_i) \text{ for all } u_1, \qquad i=1, 2, \cdots, m. \\
G'_{12}(u_1, \psi_i) &= (c_1-c_2)+L'_1(u_1, \psi_i)+V'(u_1, \psi_i)-V'_1(u_1, \psi_i)+H'_{12}(u_1, \psi_i) \\
&= (c_1-c_2)+L'_1(u_1-a_i-a_{i1}, \varphi_i)+V'(u_1-a_i, \varphi_i) \\
&\quad -V'_1(u_1-a_i-a_{i1}, \varphi_i)+H'_{12}(u_1-a_i-a_{i_1}, \varphi_i) \\
&\geqq G'_{12}(u_1-a_i-a_{i_1}, \varphi_i) \text{ for all } u_1, \\
G'_{12}(u_1, \psi_i) &< G'_{12}(u_1-a_i, \varphi_i) \text{ for all } u_1, \qquad i=1, 2, \cdots, m. \\
H'_{02}(u_0-a_i, \varphi_i) &\geqq H'_{02}(u_0, \psi_i)\geqq H'_{02}(u_0-a_i-a_{i_1}, \varphi_i) \text{ for all } u_0, \\
&\qquad i=1, 2, \cdots, m. \\
G'_{02}(u_0, \psi_i) &= (c_0-c_1)+L'(u_0, \psi_i)-V'(u_0, \psi_i)+H'_{02}(u_0, \psi_i) \\
&= (c_0-c_1)+L'(u_0-a_i, \varphi_i)-V'(u_0-a_i, \varphi_i)+H'_{02}(u_0, \psi_i) \\
&\leqq G'_{02}(u_0-a_i, \varphi_i) \text{ for all } u_0, \\
G'_{02}(u_0, \psi_i) &\geqq (c_0-c_1)+L'(u_0-a_i, \varphi_i)-V'(u_0-a_i, \varphi_i)+H'_{02}(u_0-a_i, \varphi_i) \\
&> (c_0-c_1)+L'(u_0-a_i-a_{i_1}, \varphi_i)-V'(u_0-a_i-a_{i_1}, \varphi_i) \\
&\quad +H'_{02}(u_0-a_i-a_{i_1}, \varphi_i) \\
&= G'_{02}(u_0-a_i-a_{i_1}, \varphi_i) \text{ for all } u_0, \qquad i=1, 2, \cdots, m.
\end{aligned}$$

Recalling $V'(u, \varphi_0)=-v$ for all $u\leqq 0$, $x_{12}(\varphi_i)>0$ implies that

$$\begin{aligned}
&G'_{12}(a_i, \psi_i)=G'_{12}(-a_{i_1}, \varphi_i)<0, \\
&G'_{12}(u_1, \psi_i)=G'_{12}(u_1-a_i-a_{i_1}, \varphi_i) \text{ for all } u_1\leqq a_i, \\
&G'_{12}(u_1, \psi_i)>G'_{12}(u_1-a_i-a_{i_1}, \varphi_i) \text{ for all } u_1>a_i.
\end{aligned}$$

Hence $\bar{x}_{12}(\psi_i)<\bar{x}_{12}(\varphi_i)+a_i+a_{i_1}$. We get the theorem for $n=2$ from above relations. The equality and inequality in part (c) depend on a relationship between $\bar{x}_{02}(\psi_i)$, $\bar{x}_{02}(\varphi_i)+a_i$, and $\bar{x}_{02}(\varphi_i)+a_i+a_{i_1}-m_{u_0}$. For example, if $\bar{x}_{02}(\varphi_i)+a_i>x>\bar{x}_{02}(\varphi_i)+a_i+a_{i_1}-m_{u_0}$, then

$$f_2'(x-a_i,\ \varphi_i)=f_2'(x,\ \psi_i)=f_2'(x-a_i-a_{i_1},\ \varphi_i).$$

Assuming that Corollary is true for $n-1$, and that

$$G'_{n-1,n-1}(u_{n-1},\ \psi_i)=G'_{n-1,n-1}(u_{n-1}-a_i-a_{i_1}-\cdots-a_{i_{n-2}},\ \varphi_i)\ \text{for all}\ u_{n-1},\quad i_l=1,2,\cdots,m;\ l=0,1,\cdots,n-2,$$

$$L'_{n-1}(u,\ \psi_i)=L'_{n-1}(u-a_i-a_{i_1}-\cdots-a_{i_{n-1}},\ \varphi_i)\ \text{for all}\ u,\quad i_l=1,2,\cdots,m;\ l=0,1,\cdots,n-1, \tag{3.50}$$

$$V'_{n-1}(u,\psi_i)=V'_{n-1}(u-a_i-a_{i_1}-\cdots-a_{i_{n-1}},\ \varphi_i)\ \text{for all}\ u,\quad i_l=1,2,\cdots,m;\ l=0,1,\cdots,n-1,$$

, then we have from the (3.4)

$$\begin{aligned} G'_{nn}(u_n,\ \psi_i)&=(c_n-\alpha c_{n-1})+\alpha\sum_{i_1=1}^{m}p_{ii_1}\int_0^\infty[c_{n-1}+V'_{n-2}(u_n-t,\ \psi_{i_1})\\ &\quad-W'_{n-1}(u_n-t,\ \psi_{i_1})]\psi_i(t)dt\\ &=(c_n-\alpha c_{n-1})+\alpha\sum_{i_1=1}^{m}p_{ii_1}\int_0^\infty G'_{n-1,n-1}(u_n-t,\ \psi_{i_1})\psi_i(t)dt\\ &=(c_n-\alpha c_{n-1})+\alpha\sum_{i_1=1}^{m}p_{ii_1}\int_0^\infty G'_{n-1,n-1}(u_n-a_{i_1}-a_{i_2}-\cdots\\ &\quad-a_{i_{n-1}}-t,\ \varphi_{i_1})\varphi_i(t-a_i)dt\\ &=(c_n-\alpha c_{n-1})+\alpha\sum_{i_1=1}^{m}p_{ii_1}\int_0^\infty G'_{n-1,n-1}(u_n-a_i-a_{i_1}-\cdots\\ &\quad-a_{i_{n-1}}-t,\ \varphi_{i_1})\varphi_i(t)dt\\ &=G'_{nn}(u_n-a_i-a_{i_1}-\cdots-a_{i_{n-1}},\ \varphi_i)\ \text{for all}\ u_n,\quad i=1,2,\cdots,m. \end{aligned}\tag{3.51}$$

From the definition of $H'_{n-1,n}(u_{n-1},\ \varphi_i)$, and $H'_{n-1,n}(u_{n-1},\ \psi_i)$, $H'_{n-1,n}(u_{n-1},\ \psi_i)=H'_{n-1,n}(u_{n-1}-a_i-a_{i_1}-\cdots-a_{i_{n-1}},\ \varphi_i)$ for all u_{n-1}, Hence get from (3.4)

$$\begin{aligned} G'_{n-1,n}(u_{n-1},\ \psi_i)&=(c_{n-1}-c_n)+L'_{n-1}(u_{n-1},\ \psi_i)\\ &\quad+V'_{n-2}(u_{n-1},\ \psi_i)-V'_{n-1}(u_{n-1},\ \psi_i)+H'_{n-1,n}(u_{n-1},\ \psi_i)\\ &=(c_{n-1}-c_n)+L'_{n-1}(u_{n-1}-a_i-a_{i_1}-\cdots-a_{i_{n-1}},\ \varphi_i)\\ &\quad+V'_{n-2}(u_{n-1}-a_i-a_{i_1}-\cdots-a_{i_{n-2}},\ \varphi_i)\\ &\quad-V'_{n-1}(u_{n-1}-a_i-a_{i_1}-\cdots-a_{i_{n-1}},\ \varphi_i)+H'_{n-1,n}(u_{n-1},\psi_i)\\ &\geqq G'_{n-1,n}(u_{n-1}-a_i-a_{i_1}-\cdots-a_{i_{n-1}},\ \varphi_i)\ \text{for all}\ u_{n-1}, \end{aligned}\tag{3.52}$$

$$G'_{n-1,n}(u_{n-1},\ \psi_i)<G'_{u-1,n}(u_{n-1}-a_i-a_{i_1}-\cdots-a_{i_{n-2}},\ \varphi_i)\quad\text{for all}\ u_{n-1},\quad i_l=1,2,\cdots,m;\ l=0,1,\cdots,n-2.$$

Hence we have, from the discussion as before,

$$\bar{x}_{nn}(\psi_i)=\bar{x}_{nn}(\varphi_i)+\sum_{l=0}^{n-1}a_{i_l}\qquad i_l=1,2,\cdots,m;\ l=0,1,\cdots,n-1,$$

$$\bar{x}_{n-1,n}(\varphi_i)+\sum_{l=0}^{n-2}a_{i_l}<\bar{x}_{nn}(\psi_i)<\bar{x}_{nn}(\varphi_i)+\sum_{l=0}^{n-1}a_{i_l},\qquad i_l=1,2,\cdots,m;\ l=0,1,\cdots,n-1.$$

We assume that Corollary holds for n and $j=n, n-1, \cdots, n-r+1$, and that

$$G'_{jn}(u_j-a_i-a_{i_1}-\cdots-a_{i_{j-1}}, \varphi_i)>G_{jn}(u_j, \psi_i)\geqq G'_{jn}(u_j-a_i-a_{i_1}-\cdots-a_{i_j}, \varphi_i)$$
$$\text{for all } u_j, \quad j=n-2, \cdots, n-r+1 \; i_l=1, 2, \cdots, n-1; \; l=0, 1, \cdots, n-1.$$

Then we obtain from (3.4)

$$H'_{jn}(u_j-a_i-a_{i_1}-\cdots-a_{i_{j-1}}, \varphi_i)\geqq H'_{jn}(u_j, \psi_i)\geqq H'_{jn}(u_j-a_i-a_{i_1}-\cdots-a_{i_j}, \varphi_i) \tag{3.53}$$
$$\text{for all } u_j,\; j=n-2, \cdots, n-r$$
$$i_l=1, 2, \cdots, m; \; l=0, 1, \cdots, n-1.$$

Hence, from the inductive assumption and (3.53) we have

$$\begin{aligned}
G'_{n-r,n}(u_{n-r}, \psi_i) &= (c_{n-r}-c_{n-r+1})+L'_{n-r}(u_{n-r}, \psi_i)+V'_{n-r-1}(u_{n-r}, \psi_i)\\
&\quad -V'_{n-r}(u_{n-r}, \psi_i)+H'_{n-r,n}(u_{n-r}, \psi_i)\\
&=(c_{n-r}-c_{n-r+1})+L'_{n-r}(u_{n-r}-a_i-a_{i1}-\cdots-a_{i_{n-r}}, \varphi_i)\\
&\quad +V'_{n-r-1}(u_{n-r}-a_i-a_{i_1}-\cdots-a_{i_{n-r-1}}, \varphi_i)\\
&\quad -V'_{n-r}(u_{n-r}-a_i-a_{i_1}-\cdots-a_{i_{n-r}}, \varphi_i)+H'_{n-r,n}(u_{n-r}, \psi_i)\\
&\geqq(c_{n-r}-c_{n-r+1})+L'_{n-r}(u_{n-r}-a_i-a_{i_1}-\cdots-a_{i_{n-r}}, \varphi_i)\\
&\quad +V'_{n-r-1}(u_{n-r}-a_i-a_{i_1}-\cdots-a_{i_{n-r-1}}, \varphi_i)\\
&\quad -V'_{n-r}(u_{n-r}-a_i-a_{i_1}-\cdots-a_{i_{n-r}}, \varphi_i)\\
&\quad +H'_{n-r,n}(u_{n-r}-a_i-a_{i_1}-\cdots-a_{i_{n-r}}, \varphi_i)\\
&\geqq G'_{n-r,n}(u_{n-r}-a_i-a_{i_1}-\cdots-a_{i_{n-r}}, \varphi_i) \qquad \text{for all } u_{n-r}
\end{aligned}$$
$$G'_{n-r,n}(u_{n-r}, \psi_i)<G'_{n-r,n}(u_{n-r}-a_i-a_{i_1}-\cdots-a_{i_{n-r-1}}, \varphi_i) \quad \text{for all } u_{n-r},$$
$$i_l=1, 2,\cdots, m; \; l=0, 1,\cdots, n-1.$$

From the same argument used for the case $n=1$, $n=2$, we obtain Corollary for $n\leqq k$.

Case (b) $n\geqq k+1$. We have

$$f'_{1l}(u-a_i-a_{i_1}-\cdots-a_{i_l}, \varphi_i)=f_{1l}(u, \psi_i) \qquad i_r=1, 2, \cdots, m; \; r=0, 1,$$
$$\text{for all } u, \qquad \cdots, l; \; l=1, 2, \cdots, k.$$

$$\begin{aligned}
f'_{jl}(u-a_i-a_{il}-\cdots-a_{il}, \varphi_i)&\geqq f'_{jl}(u, \psi_i) \quad \text{for all } u,\\
&\geqq f'_{jl}(u-a_i-a_{i_1}-\cdots-a_{i_{j+l-1}}, \varphi_i) \quad j=2, \cdots, n-k, \; i_r=1,\\
&\qquad 2, \cdots, m; \; r=0, 1, \cdots, l; \; l=1, 2, \cdots, k,
\end{aligned} \tag{3.54}$$

which can be proved inductively from the results of case (a) and (3.2). We can prove Corollary for the case under consideration by (3.54) in the same method as for the case (a). The detail of the proof are omitted. We established the proof of Corollary.

Specially $\psi_i(x)=\varphi_i(x-a)$, then we get

$$\text{(a}')\;\begin{cases}
\bar{x}_{nn}(\varphi_i)+na=\bar{x}_{nn}(\psi_i) & i=1, 2, \cdots, m; \; n\leqq k,\\
\bar{x}_{jn}(\varphi_i)+ja<\bar{x}_{jn}(\psi_i)<\bar{x}_{jn}(\varphi_i)+na & j=1, 2, \cdots, n-1; \; k\geqq n\geqq 2; \; i=1, 2, \cdots, m,
\end{cases}$$

$$\begin{cases} \bar{x}_{0n}(\varphi_i)+a \leqq \bar{x}_{0n}(\psi_i) < \bar{x}_{jn}(\varphi_i)+na & i=1,\ 2,\ \cdots,\ m\,;\ n \leqq k, \\ \bar{x}_{01}(\varphi_i)+a=\bar{x}_{01}(\psi_i) & i=1,\ 2,\ \cdots,\ m\,;\ n \leqq k. \end{cases}$$

$$(\mathrm{b}')\ \begin{cases} \bar{x}_{kn}(\varphi_i)+ka \leqq \bar{x}_{kn}(\psi_i) \leqq \bar{x}_{kn}(\varphi_i)+na, & i=1,\ 2,\ \cdots,\ m\,;\ n \geqq k+1, \\ \bar{x}_{jn}(\varphi_i)+ja < \bar{x}_{jn}(\psi_i) < \bar{x}_{jn}(\varphi_i)+na, & j=1,\ 2,\ \cdots,\ k-1\,;\ i=1,\ 2,\ \cdots,\ m\,;\ n \geqq k+1, \\ \bar{x}_{0n}(\varphi_i)+a \leqq \bar{x}_{0n}(\psi_i) < \bar{x}_{0n}(\varphi_i)+na, & i=1,\ 2,\ \cdots,\ m\,;\ n \geqq k+1. \end{cases}$$

$$(\mathrm{c}')\ \begin{cases} f_1'(x-a,\ \varphi_i)=f_1'(x,\ \psi_i) & \text{for all } x,\quad i=1,\ 2,\ \cdots,\ m. \\ f_n'(x-a,\ \varphi_i) \geqq f_n'(x_1,\ \psi_i) \geqq f_n'(x-na,\ \varphi_i) & \text{for all } x,\quad i=1,\ 2,\ \cdots,\ m. \end{cases}$$

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Reference

[1] BARANKIN, E. W. A Delivery-Lag Inventory Model with Emergency Provision (The Single-Period Case). Naval Res. Logist. Quart., 1961, **8** (3), 285-311.

[2] DANIEL, K. H., A Delivery-Lag Inventory Model with Emergency. Chapter 2 in SCARF, H. E., D. M. GILFORD, and M. W. SHELLY (eds.), Multistage Inventory Models and Techniques, Stanford, Calif: Stanford Univ Press, 1963.

[3] KARLIN, S. Dynamic Inventory Policy with Varying Stochastic Demands. Management Sci., 1960, **6** (3), 231-258.

[4] FUKUDA, Y. Optimal Policy for the Inventory Problem with Negotiable Leadtime. Management Sci., 1964, **10** (4), 690-708.

[5] ARROW, K. J., S. KARLIN, and H. SCARF. Studies in the Mathematical Theory of Inventory and Production. Stanford, Calif: Stanford Univ. Press, 1958.

[6] IGLEHART, D. and S. KARLIN. Optimal Policy for Dynamic Inventory Process with Non-Stationary Stochastic Demands. Studies in Applid Probability and Management Science. Stanford, Calif: Stanford Univ. Press, 1962.

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