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Number of Proofs for Implicational Formulas(MATHEMATICAL LOGIC AND ITS APPLICATIONS)

Hirokawa Sachio Department of Computer Science, College of General Education, Kyushu University

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Number of Proofs for Implicational Formulas

広川 佐千男 (Sachio Hirokawa)*

An algorithm is shown which determines the number $0, 1, \dots, \infty$ of normal form proofs for implicational formulas. The number of proofs had not been studied well. Concerning to BCK-logic, it is proved by Komori and Hirokawa [3] that the number is identical to the number of BCK-minimal formulas of α . For general implicational formulas in intuitionistic logic, Ben-Yelles [1] showed an algorithm which enumerates all the normal form proofs for α when α has finitely many proofs. But we cannot use the algorithm to decide whether α has infinitely many proofs or not. We show a limit of proof search to decide whether α has infinitely many proofs.

Given an implicational formula α , we denote by $|\alpha|$ the number of occurrences of propositional variables and the implicational symbol ' \rightarrow '. We consider proof figures in the intuitionistic logic in Natural Deduction System (NJ) [4]. We denote by $proof(\alpha)$ the set of normal form proofs of α . The cardinality of $proof(\alpha)$ is denoted by $\#proof(\alpha)$. The depth of a thread in a proof π is the number of minimum formula occurrences in the thread. The depth of π , denoted by $depth(\pi)$, is the maximal depth among all the threads in π . According to the formulae-as-types correspondence [2], a normal form proof π can be represented by a closed λ -term M in β -normal form . Then the $depth(\pi)$ is identical to the depth of Böhm-tree of M.

Theorem 1 Given an implicational formula α ,

 $\sharp proof(\alpha) = \infty$

^{*}Department of Computer Science, College of General Education, Kyushu University, Fukuoka 810 JAPAN. (e-mail: hirokawa@ec.kyushu-u.ac.jp) Supported by a Grant-in-Aid for Encouragement of Young Scientists No.02740115 of the Ministry of Education.

- (1) $depth(\pi) \leq |\alpha| \ 2^{|\alpha|+1}$ and
- (2) π contains a thread in which a formula ξ occurs twice as minimum formula occurrence.

$$\pi \left\{ \begin{array}{ccc} & \vdots \\ & \xi \\ & \vdots \\ & \xi \\ & \vdots \\ & \alpha \end{array} \right\} \pi_2$$

Outline of proof. If-part is trivial. In fact, we can replace π_1 by π_2 . We can apply this rewriting successively. Thus we have $\# proof(\alpha) = \infty$. To prove only-if-part, assume that $\# proof(\alpha) = \infty$. Then there is a proof $\pi \in proof(\alpha)$ which contains a thread with depth $\geq 2 d$, where $d = |\alpha| 2^{|\alpha|}$. Then the thread contains more than 2 d minimum formula occurrences. Let ξ be an arbitrary minimum formula occurrence in the thread and $\{\delta_1, \dots, \delta_n\}$ the assumption set for the sub-proof for ξ . By the sub-formula property, all of $\xi, \delta_1, \dots, \delta_n$ are sub-formulas of α . So we have at most d such pairs $(\xi, \{\delta_1, \dots, \delta_n\})$. Since the depth of the thread is longer than 2 d, it contains three occurrences of the same minimum formula occurrence ξ with the same assumption set $\{\delta_1, \dots, \delta_n\}$. Let π_1, π_2 , and π_3 be sub-proof for such occurrences of ξ which π_i appears above $\pi_{i+1}(i = 1, 2)$. Then we can replace π_2 by π_1 obtaining a smaller proof of α . We can apply this transformation until we obtain a proof of α with depth $\leq 2 d$.

Theorem 2 There is an algorithm which determines $\sharp proof(\alpha)$ for implicational formula α .

Proof. Consider the set of normal form proofs of α with depth $\leq |\alpha| 2^{|\alpha|+1}$. Note that the set is finite. If this set contains some π which satisfies (2) of Theorem 1, then $\# proof(\alpha) = \infty$. Otherwise $\# proof(\alpha)$ is finite.

Theorem 1 without (1) is proved in Ben-Yelles [1]. Proof of Theorem 1 would remind some readers the similarity to the proof of *uvwxy-theorem* and infinity test for context free languages. Further work shall be necessary on this similarity.

References

- Ben-Yelles, C.-B., Type-assignment in the lambda-calculus, PhD thesis, University College, Swansea, 1979.
- Howard, W.A., The formulae-as-types notion of construction, in: Hindley and Seldin Ed., To H.B. Curry, Essays on Combinatory Logic, Lambda Calculus and Formalism (Academic Press, 1980) 479-490.
- [3] Komori, Y., Hirokawa, S., Number of proofs for BCK-formulas, preprint.
- [4] Prawitz, D., Natural Deduction (Almqvist and Wiksell, Stockholm, 1965)