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出版情報：Journal of physics. A, Mathematical and general. 37 (49), pp.11819-11839, 2004-12-10. Institute of Physics

MHF Preprint Series

Kyushu University
21st Century COE Program
Development of Dynamic Mathematics with
High Functionality

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MHF 2004-16

(Received June 2, 2004)

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Resonance and web structure in discrete soliton systems: the two-dimensional Toda lattice and its fully discrete and ultra-discrete versions

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Abstract. We present a class of solutions of the two-dimensional Toda lattice equation, its fully discrete analogue and its ultra-discrete limit. These solutions demonstrate the existence of soliton resonance and web-like structure in discrete integrable systems such as differential-difference equations, difference equations and cellular automata (ultra-discrete equations).

2 June 2004

PACS numbers: 02.30.Jr, 05.45.Yv

To be submitted to : *J. Phys. A: Math. Gen.*

1. Introduction

The discretization of integrable systems is an important issue in mathematical physics. A discretization process in which the dependent variables are discretized in addition to the independent variables is known as "ultra-discretization". One of the most important ultra-discrete soliton systems is the so-called "soliton cellular automaton" (SCA) [12, 16, 17]. A general method to obtain the SCA from discrete soliton equations was proposed in Refs. [6, 18] and involves using an appropriate limiting procedure.

Another issue which has received renewed interest in recent years is the phenomenon of soliton resonance, which was first discovered for the Kadomtsev-Petviashvili (KP) equation [8] (see also Refs. [7, 11]). More general resonant solutions possessing a web-like structure have recently been observed in a coupled KP (cKP) system [4, 5] and for the KP equation itself [1]. In particular, the Wronskian formalism was used in Ref. [1] to classify a class of resonant solutions of KP which also satisfy the Toda lattice hierarchy. It was also conjectured in Ref. [1] that resonance and web structure are not limited to KP and cKP, but rather they are a generic feature of integrable systems whose solutions can be expressed in terms of Wronskians.

The aim of this paper is to study soliton resonance and web structure in discrete soliton systems. In particular, by studying a class of soliton solutions of the two-dimensional Toda lattice (2DTL) equation, of its fully discrete version, and of their ultra-discrete analogue which

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was recently introduced by Nagai et al [9, 10], we show that an analogue to the class of solutions studied in Ref. [1] can be defined for all three of these systems, and that a similar type of resonant solutions with web-like structure is produced as a result in all three of these systems. To our knowledge, this is the first time that resonant behavior and web structure are observed in discrete soliton systems. These results also confirm that soliton resonance and web-like structure are general features of two-dimensional integrable systems whose solutions can be expressed via the determinant formalism.

2. Resonance and web structure in the two-dimensional Toda lattice equation

We start by considering the two-dimensional Toda lattice (2DTL) equation,

$$\frac{\partial^2}{\partial x \partial t} Q_n = V_{n+1} - 2V_n + V_{n-1} \,, \tag{2.1}$$

with $Q_n(x,t) = \log[1 + V_n(x,t)]$. Equation (2.1) can be written in bilinear form

$$\frac{\partial^2 \tau_n}{\partial x \partial t}\, \tau_n - \frac{\partial \tau_n}{\partial t}\, \frac{\partial \tau_n}{\partial x} = \tau_{n+1}\, \tau_{n-1} - \tau_n^2 \tag{2.2}$$

through the transformation

$$V_n(x,t) = \frac{\partial^2}{\partial x \partial t} \log \tau_n(x,t) \,. \tag{2.3}$$

It is well-known that some solutions of the 2DTL equation can be written via the Casorati determinant form $\tau_n = \tau_n^{(M)}$ [3], with

$$\tau_n^{(M)} = \begin{vmatrix} f_n^{(1)} & \cdots & f_{n+M-1}^{(1)} \\ \vdots & \ddots & \vdots \\ f_n^{(M)} & \cdots & f_{n+M-1}^{(M)} \end{vmatrix} \,, \tag{2.4}$$

where $\{f_n^{(1)}(x,t), \ldots, f_n^{(M)}(x,t)\}$ is a set of M linearly independent solutions of the linear equations

$$\frac{\partial f_n}{\partial x} = f_{n+1} \,, \qquad \qquad \frac{\partial f_n}{\partial t} = -f_{n-1} \,, \tag{2.5}$$

for $1 \le i \le M$. For example, a two-soliton solution of the 2DTL is obtained by the set $\{f^{(1)}, f^{(2)}\}$, with

$$f_n^{(i)} = \mathrm{e}^{\theta_n^{(2i-1)}} + \mathrm{e}^{\theta_n^{(2i)}} \,, \qquad i = 1, 2 \,, \tag{2.6}$$

where the phases $\theta^{(j)}$ are given by linear functions of (n, x, t),

$$\theta_n^{(j)}(x,t) = n \log p_j + p_j x - \frac{1}{p_j} t + \theta_0^{(j)} \,, \qquad j = 1, \ldots, 4 \,, \tag{2.7}$$

with $p_1 < p_2 < p_3 < p_4$. Equation (2.6) can be extended to the M-soliton solution with $\{f^{(1)}, \cdots, f^{(M)}\}$.

On the other hand, solutions of the 2DTL equation can also be obtained by the set of τ functions $\{\tau_n^{(M)} \mid M = 1, \ldots, N\}$ with the choice of f-functions,

$$f_n^{(i)} = f_{n+i-1} \,, \qquad 1 < i \le M \le N \,, \tag{2.8}$$

with

$$f_n = \sum_{j=1}^{N} \mathrm{e}^{\theta_n^{(j)}} \,. \tag{2.9}$$

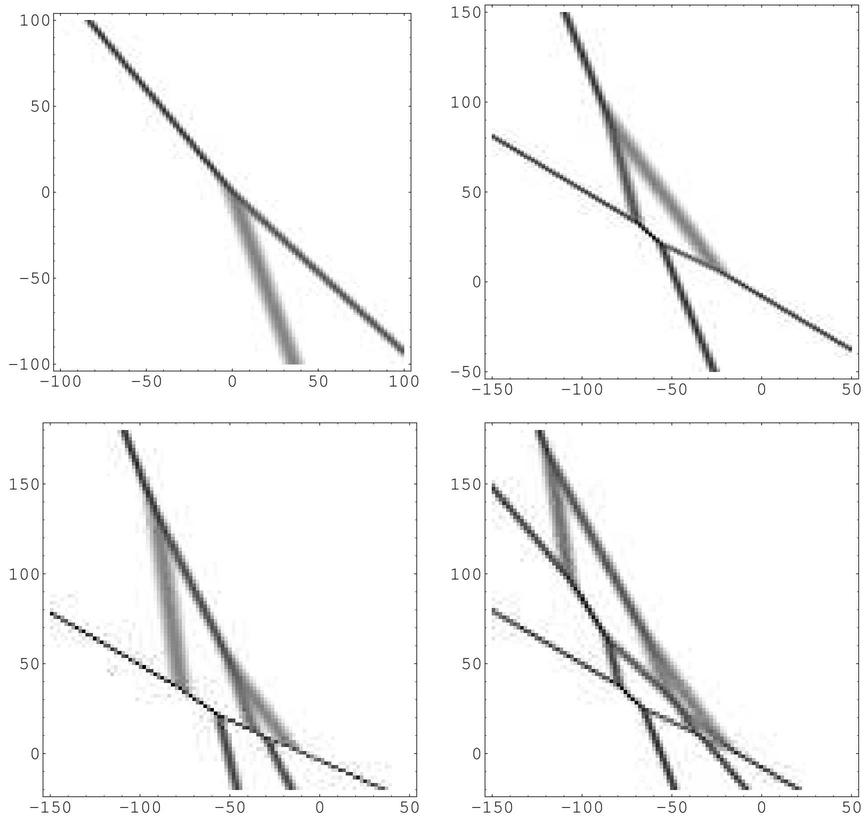

Figure 1. Resonant solutions of the two-dimensional Toda lattice: (a) (2,1)-soliton solution (i.e., a Y-junction) at $t = 0$, with $M = 1$, $N = 3$, $p_1 = 1/4$, $p_2 = 1/2$, $p_3 = 2$; (b) (2,2)-soliton solution at $t = 14$, with $M = 2$, $N = 4$, $p_1 = 1/8$, $p_2 = 1/2$, $p_3 = 1$ and $p_4 = 4$; (c) (3,2)-soliton solution at $t = 10$, with $M = 2$, $N = 5$, $p_1 = 1/10$, $p_2 = 1/5$, $p_3 = 1/2$, $p_4 = 1$ and $p_5 = 6$; (d) (3,3)-soliton solution at $t = 14$, with $M = 3$, $N = 6$, $p_1 = 1/10$, $p_2 = 1/4$, $p_3 = 1/2$ and $p_4 = 1$, $p_5 = 2$, $p_6 = 4$. In all cases the horizontal axis is n and the vertical axis is x, and each figure is a plot of $q(n, x, t)$ in logarithmic grayscale. Note that the values of n in the horizontal axis are discrete.

and with the phases $\theta_n^{(j)}$, $1 \le i \le N$ still given by Eq. (2.7). If the f-functions are chosen according to eq. (2.8), the τ function $\tau_n^{(M)}$ is then given by the Hankel determinant

$$\tau_n^{(M)} = \begin{vmatrix} f_n & \cdots & f_{n+M-1} \\ \vdots & \ddots & \vdots \\ f_{n+M-1} & \cdots & f_{n+2M-2} \end{vmatrix}, \qquad (2.10)$$

for $1 \le M \le N$. It should be noted that, even when the set of functions $\{f_n^{(i)}\}_{i=1}^M$ is chosen according to Eq. (2.8), no derivatives appear in the τ function, and therefore Eq. (2.10) cannot be considered a Wronskian in the same sanse as for the KP equation (cf. Eq. (1.9) in Ref. [1]). Nonetheless, this choice produces a similar outcome as in the KP equation. Indeed, similarly to Ref. [1], we have the following:

Lemma 2.1 *Let f_n be given by Eq. (2.9), with $\theta_n^{(j)}$ ($j = 1, \ldots, N$) given by Eq. (2.7). Then, for $1 \le M \le N-1$ the τ function defined by the Hankel determinant (2.10) has the form*

$$\tau_n^{(M)} = \sum_{1 \le i_1 < \cdots < i_M \le N} \Delta(i_1, \ldots, i_M) \ \exp\left(\sum_{j=1}^{M} \theta_n^{(i_j)} \right), \tag{2.11}$$

where $\Delta(i_1, \ldots, i_M)$ is the square of the van der Monde determinant,

$$\Delta(i_1, \ldots, i_M) = \prod_{1 \le j < l \le M} (p_{i_j} - p_{i_l})^2 \,.$$

Proof. Apply the Binet-Cauchy theorem to Eq. (2.10), as in Ref. [1]. □

An immediate consequence of Lemma 2.1 is that the τ function $\tau_n^{(M)}$ is positive definite, and therefore all the solutions generated by it are non-singular. Like its analogue in the KP equation [1], the above τ function produces soliton solutions of resonant type with web structure. More precisely, we conjecture that, like its analogue in the KP equation, the above τ function produces an (N_-, N_+)-soliton solution, that is, a solution with $N_- = N - M$ asymptotic line solitons as $x \to -\infty$ and $N_+ = M$ asymptotic line solitons as $x \to \infty$. As an example, in Fig. 1 we show a (2,1)-soliton solution (also called a Y-shape solution, or a Y-junction), a (2,2)-soliton solution, a (2,3)-soliton solution and a (3,3)-soliton solution.

Note that even when $N_+ = N_- = M$, the interaction pattern of resonant soliton solutions differs from that of ordinary M-soliton solutions. As seen from Fig. 1, the resonant solutions of the 2DTL obtained from Eq. (2.10) are very similar to the solitons of the KP and coupled KP equation [1, 4, 5], where such solutions were called "spider-web" solitons. (In contrast, an ordinary M-soliton solution produces a simple pattern of M intersecting lines.) The web structure manifests itself in the number of bounded regions, the number of vertices and the number of intermediate solitons, which are respectively $(N_- - 1)(N_+ - 1)$, $2N_-N_+ - N$ and $3N_-N_+ - 2N$ for an (N_-, N_+)-soliton solution [1]. (In contrast, an ordinary M-soliton solution has $(M-1)(M-2)/2$ bounded regions and $M(M-1)/2$ interaction vertices.) Finally, it should be noted that, as in the KP equation, only the Y-shape solution is a traveling wave solution. All other resonant solutions (as well as ordinary M-soliton solutions with $M \ge 3$) have a time-dependent shape, as shown in Ref. [1].

3. Resonance and web structure in the fully discrete 2D Toda lattice equation

We now consider a fully discrete analogue of the 2DTL equation (2.1), namely

$$\begin{aligned} &\Delta_l^+ \Delta_m^- Q_{l,m,n} = V_{l,m-1,n+1} - V_{l+1,m-1,n} - V_{l,m,n} + V_{l+1,m,n-1} \,, \\ &V_{l,m,n} = (\delta\kappa)^{-1} \log[1 + \delta\kappa\,(\exp Q_{l,m,n} - 1)] \,, \end{aligned} \tag{3.1}$$

with $l, m, n \in \mathbb{Z}, l$ and m being the discrete analogues of the time t and space x coordinates, respectively, and where Δ_l^+ and Δ_m^- are forward and backward difference operators defined by

$$\Delta_l^+ f_{l,m,n} = \frac{f_{l+1,m,n} - f_{l,m,n}}{\delta}, \tag{3.2}$$

$$\Delta_m^- f_{l,m,n} = \frac{f_{l,m,n} - f_{l,m-1,n}}{\kappa}. \tag{3.3}$$

Equation (3.1), which is the discrete analogue of Eq. (2.1), can be written in bilinear form [2] in a manner similar to Eq. (2.2):

$$\begin{aligned} &(\Delta_l^+ \Delta_m^- \tau_{l,m,n})\, \tau_{l,m,n} - (\Delta_l^+ \tau_{l,m,n})\, \Delta_m^- \tau_{l,m,n} \\ &\qquad = \tau_{l,m-1,n+1} \tau_{l+1,m,n-1} - \tau_{l+1,m-1,n} \tau_{l,m,n}, \end{aligned} \tag{3.4}$$

with $Q_{l,m,n}$ related to $\tau_{l,m,n}$ by the transformation $V_{l,m,n} = \Delta_l^+ \Delta_m^- \log \tau_{l,m,n}$, i.e.,

$$Q_{l,m,n} = \log \frac{\tau_{l+1,m+1,n-1}\tau_{l,m,n+1}}{\tau_{l+1,m,n}\tau_{l,m+1,n}} \,. \tag{3.5}$$

Note that $Q_{l,m,n} = \log[1 + (\mathrm{e}^{\delta\kappa\, V_{l,m,n}} - 1)/\delta\kappa]$. Special solutions of Eq. (3.4) (which is the discrete analogue of Eq. (2.2)) are obtained when the τ function $\tau_{l,m,n}$ is expressed in terms of a Casorati determinant $\tau_{l,m,n} = \tau_{l,m,n}^{(M)}$ as [2]

$$\tau_{l,m,n}^{(M)} = \begin{vmatrix} f_{l,m,n}^{(1)} & f_{l,m,n+1}^{(1)} & \cdots & f_{l,m,n+M-1}^{(1)} \\ f_{l,m,n}^{(2)} & f_{l,m,n+1}^{(2)} & \cdots & f_{l,m,n+M-1}^{(2)} \\ \vdots & \vdots & & \vdots \\ f_{l,m,n}^{(M)} & f_{l,m,n+1}^{(M)} & \cdots & f_{l,m,n+M-1}^{(M)} \end{vmatrix} , \tag{3.6}$$

where each of the functions $\{f_{l,m,n}^{(i)}, i = 1, 2, \cdots, M\}$ satisfies the following discrete dispersion relations:

$$\Delta_l^+ f_{l,m,n} = f_{l,m,n+1} , \tag{3.7}$$

$$\Delta_m^- f_{l,m,n} = -f_{l,m,n-1} . \tag{3.8}$$

If we take as a solution for Eqs. (3.7) and (3.8) the functions

$$f_{l,m,n}^{(i)} = \phi(p_i) + \phi(q_i) \,, \tag{3.9}$$

with

$$\phi(p) = p^n (1 + \delta p)^l (1 + \kappa p^{-1})^{-m} \,, \tag{3.10}$$

the τ function (3.6) yields a M-soliton solution for the discrete 2DTL Eq. (3.4).

As in the continuous 2DTL, however, solutions of Eq. (3.4) can also be obtained when we consider the τ function defined by the Hankel determinant

$$\tau_{l,m,n}^{(M)} = \begin{vmatrix} f_{l,m,n} & f_{l,m,n+1} & \cdots & f_{l,m,n+M-1} \\ f_{l,m,n+1} & f_{l,m,n+2} & \cdots & f_{l,m,n+M} \\ \vdots & \vdots & & \vdots \\ f_{l,m,n+M-1} & f_{l,m,n+M} & \cdots & f_{l,m,n+2M-2} \end{vmatrix} , \tag{3.11}$$

where

$$f_{l,m,n} = \sum_{i=1}^{N} \alpha_i \, \phi(p_i) \,, \tag{3.12}$$

which corresponds to choosing

$$f_{l,m,n}^{(i)} = f_{l,m,n+i-1} \,, \tag{3.13}$$

for $i = 1, \ldots, M$. Without loss of generality, we can label the parameters p_i so that $0 < p_1 < p_2 < \cdots < p_{N-1} < p_N$. Then, as in the continuous 2DTL, we have the following:

Lemma 3.1 *Let $f_{l,m,n}$ be given by Eq. (3.13). Then, for $1 \le M \le N-1$, the τ function defined by the Hankel determinant (3.11) has the form*

$$\tau_n^{(M)} = \sum_{1 \le i_1 < \cdots < i_M \le N} \Delta(i_1, \ldots, i_M) \prod_{j=1}^{M} \alpha_{i_j} \, \phi(p_{i_j}) \tag{3.14}$$

where $\Delta(i_1, \ldots, i_M)$ is the square of the van der Monde determinant,

$$\Delta(i_1, \ldots, i_M) = \prod_{1 \le j < l \le M} (p_{i_j} - p_{i_l})^2 , . \tag{3.15}$$

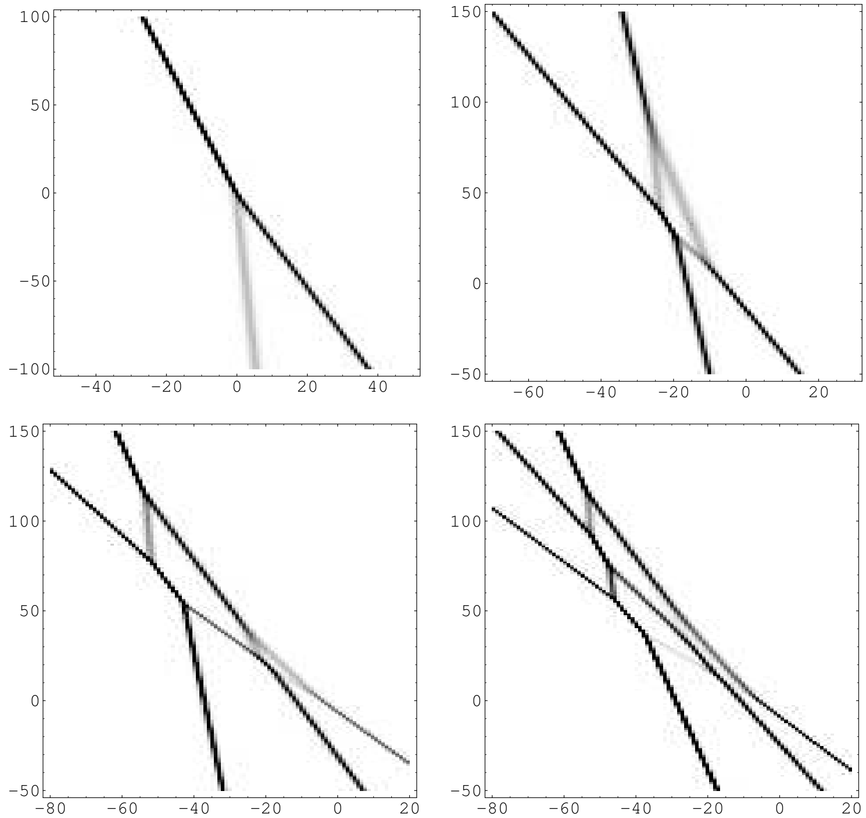

Figure 2. Resonant solutions of the fully discrete two-dimensional Toda lattice: (a) (2,1)-soliton solution (i.e., a Y-junction) at $l = 0$, with $p_1 = 1/10$, $p_2 = 1/2$, $p_3 = 10$, (b) (2,2)-soliton solution at $t = 40$, with $p_1 = 1/10$, $p_2 = 1/2$, $p_3 = 2$, $p_4 = 15$; (c) (3,2)-soliton solution at $t = 80$, with $p_1 = 1/20$, $p_2 = 1/2$, $p_3 = 2$, $p_4 = 10$, $p_5 = 60$; (d) (3,3)-soliton solution at $t = 80$, with $p_1 = 1/20$, $p_2 = 1/2$, $p_3 = 2$, $p_4 = 10$, $p_5 = 20$, $p_6 = 120$. In all cases $\delta = \kappa = 1/4$; the horizontal axis is n and the vertical axis is n, and each figure is a plot of $Q_{l,m,n}$ in logarithmic grayscale. Note that the values of both m and n in the horizontal and vertical axes are discrete.

Proof. Again, the result follows by applying the Binet-Cauchy theorem to the Hankel determinant (3.11). □

Note that, unlike its counterpart in the continuous 2DTL, the τ function in Eq. (3.11) cannot be written in terms of a Wronskian, since no derivatives appear. However, as in the continuous 2DTL, the τ function thus defined is positive definite, and therefore all the solutions generated by it are non-singular. Like its analogue in the continuous 2DTL, the above τ function produces soliton solutions of resonant type with web structure, and we conjecture that, like in the continous case, an (N_-, N_+)-soliton with $N_- = N - M$ and $N_+ = M$ is created. As an example, in Fig. 2 we show a (2,1)-soliton solution, a (2,2)-soliton solution, a (2,3)-soliton solution and a (3,3)-soliton solution. As in the continuous case, however, a full characterization of the solution remains a problem for further study. The resonant solutions of the fully discrete 2DTL provide the basis for the construction of the resonant solution of the ultra-discrete 2DTL, as is shown in the next two sections.

4. The ultra-discrete two-dimensional Toda lattice

We now turn our attention to an ultra-discrete analogue of the 2DTL equation. Using Eqs. (3.2) and (3.3), we first write the 2DTL Eq. (3.4) in bilinear form as

$$(1-\delta\kappa)\,\tau_{l+1,m,n}\tau_{l,m+1,n} - \tau_{l+1,m+1,n}\tau_{l,m,n} + \delta\kappa\ \tau_{l,m,n+1}\tau_{l+1,m+1,n-1} = 0\,, \tag{4.1}$$

We define the difference operator Δ' as

$$\Delta' = \mathrm{e}^{-\partial_n}(\Delta_n^+ - \Delta_t^+)(\Delta_n^+ - \Delta_m^+)\,, \tag{4.2}$$

where from here on the symbols Δ_l^+, Δ_m^+ and Δ_n^+ will be used to denote the difference operators

$$\Delta_l^+ = \mathrm{e}^{\partial_l} - 1\,, \qquad \Delta_m^+ = \mathrm{e}^{\partial_m} - 1\,, \qquad \Delta_n^+ = \mathrm{e}^{\partial_n} - 1\,, \tag{4.3}$$

and where the shift operators e^{∂_l}, e^{∂_m} and e^{∂_n} are defined by $\mathrm{e}^{\partial_n} f_{l,m,n} = f_{l,m,n+1}$ etc. That is,

$$\Delta' f_{l,m,n} = f_{l+1,m+1,n-1} + f_{l,m,n+1} - f_{l+1,m,n} - f_{l,m+1,n}\,. \tag{4.4}$$

Using Eqs. (4.3) and (4.2), we can rewrite Eq. (4.1) as

$$(1-\delta\kappa) + \delta\kappa \exp\left[\Delta' \log \tau_{l,m,n}\right] = \exp[\Delta_t^+\Delta_m^+ \log \tau_{l,m,n}], \tag{4.5}$$

which becomes, taking a logarithm and applying Δ' (assuming $\delta\kappa \neq 1$),

$$\Delta' \log\left[1 + \frac{\delta\kappa}{1-\delta\kappa}\exp(\Delta' \log \tau_{l,m,n})\right] = \Delta_t^+\Delta_m^+\Delta' \log \tau_{l,m,n}\,. \tag{4.6}$$

We now take an ultra-discrete limit of Eq. (4.6) following Refs. (2.1) [9, 10]. This is accomplished by choosing the lattice intervals as

$$\delta_\varepsilon = \mathrm{e}^{-r/\varepsilon}, \qquad \kappa_\varepsilon = \mathrm{e}^{-s/\varepsilon}, \tag{4.7}$$

where $r, s \in \mathbb{Z}_{\geq 0}$ are some predetermined integer constants, and by defining

$$v_{l,m,n}^\varepsilon = \Delta' \varepsilon \log \tau_{l,m,n}^\varepsilon\,, \tag{4.8}$$

Taking the limit $\varepsilon \to 0^+$ in Eq. (4.6) and noting that $\lim_{\varepsilon\to 0^+} \varepsilon \log(1 + \mathrm{e}^{X/\varepsilon}) = \max(0, X)$, we then obtain

$$\Delta_t^+\Delta_m^+ v_{l,m,n} = \Delta' \max(0, v_{l,m,n} - r - s), \tag{4.9}$$

where $v_{l,m,n} = \lim_{\varepsilon\to 0^+} v_{l,m,n}^\varepsilon$. That is, using Eq. (4.8),

$$v_{l,m,n} = \Delta' \lim_{\varepsilon\to 0^+} \varepsilon\, \tau_{l,m,n}^\varepsilon\,. \tag{4.10}$$

Equation (4.9) is the ultra-discrete analogue of the 2DTL equation, and can be considered a cellular automaton in the sense that $v_{l,m,n}$ takes on integer values.

Let us briefly discuss ordinary soliton solutions of the ultra-discrete 2DTL Eq. (4.9). As shown in Refs. [9, 10], soliton solutions for the ultra-discrete 2DTL Eq. (4.9) are obtained by an ultra-discretization of the soliton solution of the discrete 2DTL Eq. (3.4). For example, a one-soliton solution for Eq. (3.4) is given by

$$\tau_{l,m,n} = 1 + \eta_1, \tag{4.11}$$

with

$$\eta_i = \alpha_i \frac{\phi(p_i)}{\phi(q_i)}\,, \tag{4.12}$$

and where

$$\phi(p) = p^n(1+\delta p)^l(1+\kappa p^{-1})^{-m} \tag{4.13}$$

as before. We introduce a new dependent variable

$$\rho^{\varepsilon}_{l,m,n} = \varepsilon \log \tau_{l,m,n}, \tag{4.14}$$

and new parameters $P_1, Q_1, A_1 \in \mathbb{Z}$ as

$$\mathrm{e}^{P_1/\varepsilon} = p_1, \qquad \mathrm{e}^{Q_1/\varepsilon} = q_1, \qquad \mathrm{e}^{A_1/\varepsilon} = \alpha_1. \tag{4.15}$$

Taking the limit $\varepsilon \to 0^+$, we then obtain

$$\rho_{l,m,n} = \max(0, \Theta_1)\,, \tag{4.16}$$

where

$$\begin{aligned}\Theta_i = A_i + n\,(P_i - Q_i) + l\{\max(0, P_i - r) - \max(0, Q_i - r)\}\\ +m\{\max(0, -Q_i - s) - \max(0, -P_i - s)\}\,,\end{aligned} \tag{4.17}$$

with $\mathrm{e}^{-r/\varepsilon} = \delta$ and $\mathrm{e}^{-s/\varepsilon} = \kappa$ as before, and where $\rho_{l,m,n} = \lim_{\varepsilon\to 0^+} \rho^{\varepsilon}_{l,m,n}$. According to Eqs. (4.10) and (4.14), the one-soliton solution for Eq. (4.9) is then given by

$$v_{l,m,n} = \rho_{l+1,m+1,n-1} + \rho_{l,m,n+1} - \rho_{l+1,m,n} - \rho_{l,m+1,n}\,. \tag{4.18}$$

Using a similar procedure we can construct a two-soliton solution. Equation (3.4) admits a two-soliton solution given by

$$\tau_{l,m,n} = 1 + \eta_1 + \eta_2 + \theta_{12}\eta_1\eta_2, \tag{4.19}$$

with

$$\theta_{12} = \frac{(p_2 - p_1)(q_1 - q_2)}{(q_1 - p_2)(q_2 - p_1)}, \tag{4.20}$$

and where $\eta_i = \alpha_i \phi(p_i)/\phi(q_i)$ as before ($i = 1, 2$). In order to take the ultra-discrete limit of the above solution, we suppose without loss of generality that the soliton parameters satisfy the inequality

$$0 < p_1 < p_2 < q_2 < q_1\,. \tag{4.21}$$

Introducing again the dependent variable $\rho^{\varepsilon}_{l,m,n} = \varepsilon \log \tau_{l,m,n}$, as well as integer parameters P_i, Q_i and A_i as

$$\mathrm{e}^{P_i/\varepsilon} = p_i, \qquad \mathrm{e}^{Q_i/\varepsilon} = q_i, \qquad \mathrm{e}^{A_i/\varepsilon} = \alpha_i \tag{4.22}$$

($i = 1, 2$), and taking the limit of small ε, we obtain

$$\rho_{l,m,n} = \max(0, \Theta_1, \Theta_2, \Theta_1 + \Theta_2 + P_2 - Q_2), \tag{4.23}$$

where Θ_i ($i = 1, 2$) was defined in Eq. (4.17), with $\rho_{l,m,n} = \lim_{\varepsilon\to 0^+} \rho^{\varepsilon}_{l,m,n}$ again, and where $v_{l,m,n}$ is obtained from $\rho_{l,m,n}$ using Eq. (4.18). Note that $P_1 < P_2 < Q_2 < Q_1$.

More in general, starting from Eqs. (3.6) and (3.9) (with $0 < p_1 < p_2 < \cdots < p_N < q_N < q_{N-1} < \cdots < q_1$) and repeating the same construction, one obtains the N-soliton solution of the ultra-discrete 2DTL Eq. (4.9) as [10]

$$\rho_{l,m,n} = \max_{\mu=0,1}\left[\sum_{1\le i\le N} \mu_i \Theta_i + \sum_{1\le i<i'\le N} \mu_i \mu_{i'} (P_{i'} - Q_{i'})\right] \tag{4.24}$$

where $\max_{\mu=0,1}$ indicates maximization over all possible combinations of the integers $\mu_i = 0, 1$, with $i = 1, \ldots, N$. Again, $v_{l,m,n}$ is obtained from $\rho_{l,m,n}$ via Eq. (4.18).

Ordinary soliton solutions corresponding to the above choices were presented in Ref. [9, 10]. In the next section we show how this basic construction can be generalized to obtain resonant soliton solutions.

```
 m ..2.....2.....     m ..2....11.....     m ..2....2......     m ..2...2.......
   ...2....2.....       ...2...11.....       ...2...2......       ...2..2.......
   ....2...2.....       ....2..11.....       ....2..2......       ....2.2.......
 5 ....2...2.....     5 ....2..11.....     5 ....2..2......     5 ....2.2.......
   .....2..2.....       .....2.11.....       .....2.2......       .....22.......
   ......2.2.....       ......211.....       ......22......       ......4.......
   ......2.2.....       ......211.....       ......22......       ......4.......
   .......22.....       .......31.....       .......5......       .......4......
 0 ........4.....     0 .......13.....     0 .......31.....     0 .......5......
   ........4.....       ........5.....       ........5.....       .......13.....
   .........4....       ........22....       ........4.....       ........5.....
   .........5....       .........5....       .........4....       ........22....
   .........13...       .........31...       .........5....       .........5....
-5 ..........5...    -5 ..........5...    -5 .........13...    -5 .........31...
   ..........22..       ..........4...       ..........5...       ..........5...
   ...........5..       ...........4..       ..........22..       ..........4...
           0     n              0     n              0     n              0     n
```

Figure 3. A $(2,1)$-resonant soliton solution (i.e., a Y-junction) for the ultradiscrete 2DTL Eq. (4.9), with $P_1 = -5$, $P_2 = 1$, $P_3 = 4$, $r = 3$, $s = 1$, $A_1 - A_3 = 1$, $A_2 - A_3 = 1$: (a) $l = 0$, (b) $l = 2$, (c) $l = 4$, (d) $l = 6$. The dots indicate zero values of $v_{l,m,n}$.

5. Resonance and web structure in the ultra-discrete 2D Toda lattice equation

Following Ref. [1], we now construct more general solutions of the ultra-discrete 2DTL equation (4.9) which display soliton resonance and web structure.

We first consider the case of a $(2,1)$-soliton for Eq. (3.4), which is given by

$$\tau_{l,m,n} = \xi_1 + \xi_2 + \xi_3 \,, \tag{5.1}$$

where

$$\xi_i = \alpha_i \phi(p_i) \tag{5.2}$$

$(i = 1, 2, 3)$, with

$$\phi(p) = p^n (1 + \delta p)^l (1 + \kappa p^{-1})^{-m} \tag{5.3}$$

as before, and where again we take $0 < p_1 < p_2 < p_3$. As in the previous section, we introduce the new dependent variable

$$\rho^{\varepsilon}_{l,m,n} = \varepsilon \log \tau_{l,m,n} \,, \tag{5.4}$$

and new parameters as

$$\mathrm{e}^{P_i/\varepsilon} = p_i, \qquad \mathrm{e}^{A_i/\varepsilon} = \alpha_i \tag{5.5}$$

$(i = 1, 2, 3)$, with $\mathrm{e}^{-r/\varepsilon} = \delta$ and $\mathrm{e}^{-s/\varepsilon} = \kappa$ as before. Taking the limit $\varepsilon \to 0^+$, we then obtain

$$\rho_{l,m,n} = \max(R_1, R_2, R_3) \tag{5.6}$$

where $\rho_{l,m,n} = \lim_{\varepsilon \to 0^+} \rho^{\varepsilon}_{l,m,n}$ as before, but where now

$$R_i = A_i + nP_i + l \max(0, P_i - r) - m \max(0, -P_i - s) \tag{5.7}$$

$(i = 1, 2, 3)$. Figure 3 shows that this solution, which again can be called a $(2,1)$-soliton, is a Y-shape solution. Note however that the $(2,1)$-soliton in Fig. 3 looks like a $(1,2)$-soliton, in the sense that there are 2 solitons for large positive m and only one for large negative m. In general, a (N_-, N_+)-soliton of the discrete 2DTL equation (3.1) leads to a (N_+, N_-)-soliton of Eq. (4.9) when taking the ultra-discrete limit. We also note that, interestingly, an L-shape solution can be is obtained instead of Y-shape solution for different solution parameters. An

```
 m ........4.....    m .......13.....    m .......31.....    m .......4......
   ........4.....      .......13.....      .......31.....      .......4......
   ........4.....      .......13.....      .......31.....      .......4......
 5 ........4.....    5 .......13.....    5 .......31.....    5 .......4......
   ........4.....      .......13.....      .......31.....      .......4......
   ........4.....      .......13.....      .......31.....      .......4......
   ........4.....      .......13.....      .......31.....      .......4......
   ........4.....      .......11.....      .......31.....      .......4......
 0 ........4.....    0 .......13.....    0 .......31.....    0 .......5......
   ........4.....      ........5.....      ........5.....      .......13.....
   .........4....      ........22....      ........4.....      ........5.....
   .........5....      .........5....      .........4....      ........22....
   .........13...      .........31...      .........5....      .........5....
-5 ..........5...   -5 ..........5...   -5 .........13...   -5 .........31...
   ..........22..      ..........4...      ..........5...      ..........5...
   ...........5..      ...........4..      ..........22..      ..........4...
           0    n              0    n              0    n              0    n
```

Figure 4. An L-shape $(2, 1)$-resonant soliton solution for Eq. (4.9), with $P_1 = -5$, $P_2 = -1$, $P_3 = 4$, $r = 3$, $s = 1$, $A_1 - A_3 = 1$, $A_2 - A_3 = 1$: (a) $l = 0$, (b) $l = 2$, (c) $l = 4$, (d) $l = 6$.

example of such a L-shape soliton is shown in Fig. 4. No analogue of this solution exists in the 2DTL and its fully discrete version.

Next, we consider the case of a $(2, 2)$-soliton for Eq. (3.4) following Ref. [1]. Let us consider the following τ function

$$\tau_{l,m,n} = \begin{vmatrix} f_{l,m,n} & f_{l,m,n+1} \\ f_{l,m,n+1} & f_{l,m,n+2} \end{vmatrix} , \tag{5.8}$$

where

$$f_{l,m,n} = \xi_1 + \xi_2 + \xi_3 + \xi_4 , \tag{5.9}$$

where ξ_i $(i = 1, \ldots, 4)$ is again defined as in Eq. (5.3), and where $0 < p_1 < p_2 < p_3 < p_4$ holds. We introduce again the new parameters $\mathrm{e}^{P_k/\varepsilon} = p_k$ and $\mathrm{e}^{A_k/\varepsilon} = \alpha_k$ $(k = 1, \ldots, 4)$ and the new dependent variable $\rho^{\varepsilon}_{l,m,n} = \varepsilon \log \tau_{l,m,n}$. Taking the limit $\varepsilon \to 0^+$, we then obtain

$$\rho_{l,m,n} = \max_{1 \le i < j \le 4} (K_{ij} + 2P_j) , \tag{5.10}$$

where $\rho_{l,m,n} = \lim_{\varepsilon \to 0^+} \rho^{\varepsilon}_{l,m,n}$, as before, and

$$K_{ij} = R_i + R_j , \tag{5.11}$$

and with R_j given by Eq. (5.7) as before. Figure 5 shows the temporal evolution of a $(2, 2)$-soliton solution. Note the appearance of a hole in Fig. 5.

Like in the two-dimensional Toda lattice (2.1) and its fully discrete version (3.1), we now consider more general resonant solutions for the ultra-discrete 2DTL (4.9). We start from the general τ function defined in Eq. (3.11), and introduce again the parameters $\mathrm{e}^{P_k/\varepsilon} = p_k$ and $e^{A_k/\varepsilon} = \alpha_k$ $(k = 1, 2, \ldots, N)$ and the variable $\rho^{\varepsilon}_{l,m,n} = \varepsilon \log \tau_{l,m,n}$, together with $\mathrm{e}^{-r/\varepsilon} = \delta$ and $\mathrm{e}^{-s/\varepsilon} = \kappa$. Taking the limit $\varepsilon \to 0^+$, we then obtain the following solution of the ultra-discrete 2DTL (4.9):

$$\rho_{l,m,n} = \max_{1 \le i_1 < \cdots < i_M \le N} \left[\sum_{j=1}^{M} R_{i_j} + 2 \sum_{j=2}^{M} (j-1) P_{i_j} \right] , \tag{5.12}$$

where again $\lim_{\varepsilon \to 0^+} \rho^{\varepsilon}_{l,m,n} = \rho_{l,m,n}$, with the maximum being taken among all possible combinations of the indices i_j $(j = 1, \ldots, M)$, and where once more we have

$$R_i = A_i + nP_i + l \max(0, P_i - r) - m \max(0, -P_i - s) . \tag{5.13}$$

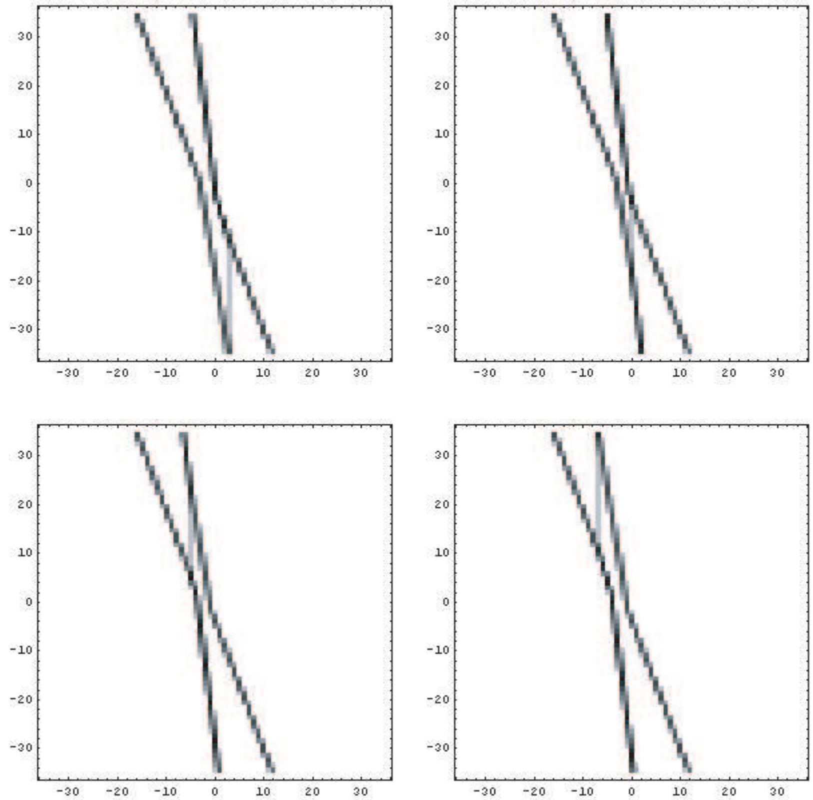

Figure 5. Snapshots illustrating the temporal evolution of a $(2, 2)$-resonant soliton solution for Eq. (4.9), with $P_1 = -7$, $P_2 = -5$, $P_3 = 1$, $P_4 = 3$, $r = 4$, $s = 2$, $A_1 = -10$, $A_2 = -6$, $A_3 = 0$, $A_4 = 2$: (a) $l = -12$; (b) $l = -7$; (c) $l = 4$; (d) $l = 7$. As in Figs. 3 and 4, the horizontal axis is n and the vertical axis is m. Since the interaction extends over a wider range of values of m and n, the solution is now plotted in grayscale, in a similar way as in Figs. 1 and 2; the values of $v_{l,m,n}$ however are still discrete, as in Figs. 3 and 4.

Equation (5.12) produces complicated soliton resonance solutions. As an example, in Fig. 6 and Fig. 7 we show the snapshots of the time evolution of a $(3, 3)$-resonant soliton solution and a $(4, 4)$-resonant soliton solution. Indeed, as in the 2DTL and in its fully discrete analogue, we conjecture that Eq. (5.12) yields the (N_-, N_+)-soliton solution of the ultra-discrete 2DTL Eq. (4.9), with $N_+ = M$ and $N_- = N - M$. It should be noted however that the interaction patterns in the ultra-discrete system differ somewhat from their analogues in the differential-difference and fully discrete cases. In particular, low-amplitude interaction arms may disappear when considering the ultra-discrete limit. Note also that the specific interactions which are produced in the ultra-discrete limit depend on the value of the parameters r and s, and different kinds of solutions may appear for different values of r and s. In particular, large values of r and s tend to result in the production of several vertical solitons, as shown in Figs. 6 and 7. In order to preserve the soliton count in these cases, all the outgoing vertical solitons should be counted as one, as should the incoming ones. In this sense, a set of outgoing or incoming vertical lines can be considered as a bound state composed of several solitons. A full characterization of these phenomena and their parameter dependence is however outside the scope of this work, and is a subject for future research.

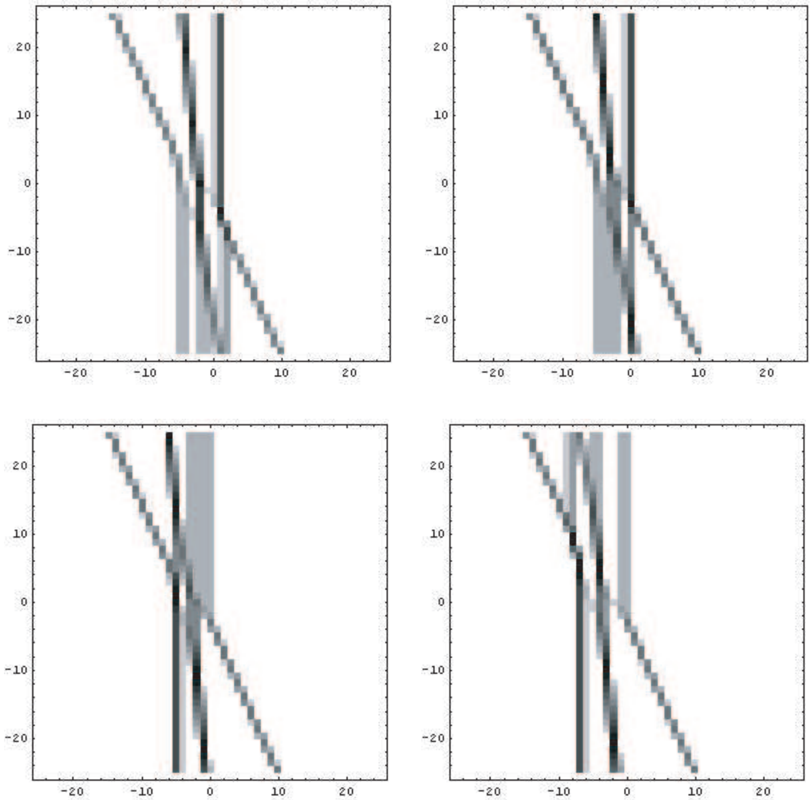

Figure 6. Snapshots illustrating the temporal evolution of a $(3,3)$-resonant soliton solution for Eq. (4.9), with $P_1 = -10$, $P_2 = -7$, $P_3 = -5$, $P_4 = -1$, $P_5 = 4$, $P_6 = 5$, $r = 7$, $s = 4$, $A_1 = -8$, $A_2 = -6$, $A_3 = 0$, $A_4 = 2$, $A_5 = 4$, $A_6 = 7$: (a) $l = -15$, (b) $l = -10$, (c) $l = 0$, (c) $l = 10$.

6. Conclusions

We have demonstrated the existence of soliton resonance and web structure in discrete soliton systems by presenting a class of solutions of the two-dimensional Toda lattice (2DTL) equation, its fully discrete version and their ultra-discrete analogue. Soliton resonance and web structure had been previously found for nonlinear partial differential equations such as the KP and cKP systems. Since the 2DTL is a differential-difference equation, its fully discrete version a difference equation and their ultra-discrete version a cellular automaton, however our findings show that these phenomena are general features of two-dimensional integrable systems whose solutions are expressed in determinant form.

A full characterization of these solutions, including the study of asymptotic amplitudes and velocities, the resonance condition and the analysis of the intermediate patterns of interactions in all three cases is outside the scope of this work, and remains as a problem for further research. Of particular interest is the ultra-discrete 2DTL, where new types of solutions such as the L-shape soliton shown in Fig. 4 appear. We should also note that the solutions of the ultra-discrete 2DTL arise as a result of the properties of the maximum function, and therefore their study might require the use of techniques from tropical algebraic geometry, which is a subject of current research [13, 14, 15].

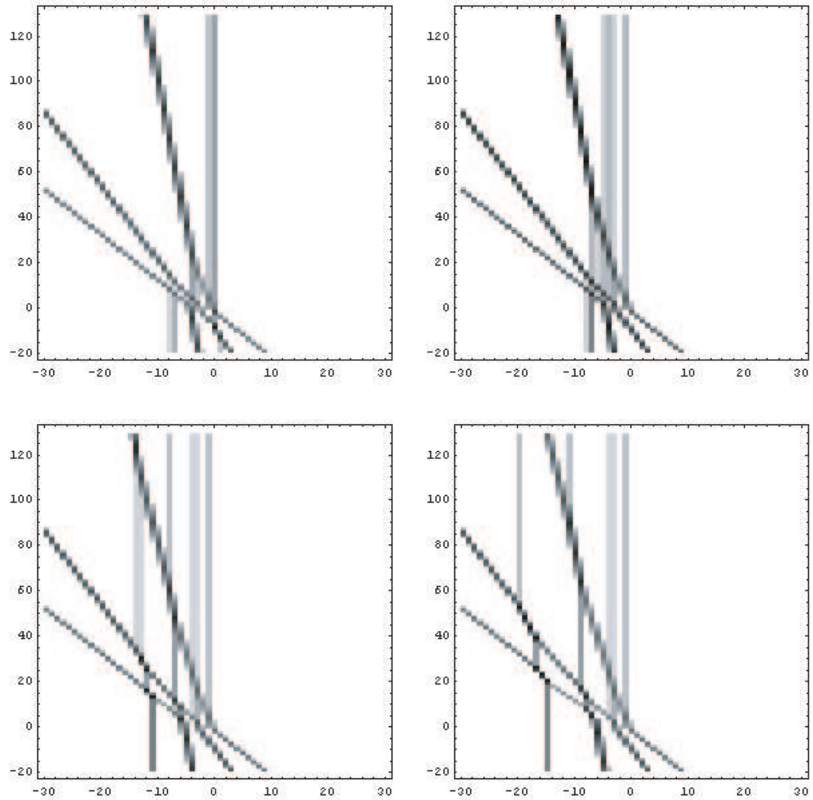

Figure 7. Snapshots illustrating the temporal evolution of a $(4, 4)$-resonant soliton solution for Eq. (4.9), with $P_1 = -15$, $P_2 = -12$, $P_3 = -9$, $P_4 = -3$, $P_5 = 1$, $P_6 = 1$, $P_7 = 4$, $P_8 = 7$, $r = 7$, $s = 4$, $A_1 = -8$, $A_2 = -6$, $A_3 = 0$, $A_4 = 2$, $A_5 = 4$, $A_6 = 7$, $A_7 = 8$, $A_8 = 10$: (a) $l = -10$, (b) $l = 0$, (c) $l = 10$, (d) $l = 20$.

Finally, we note that the class of solutions presented in this work is just one of the possible choices that yield resonance and web structure. Just like with the KP and cKP equations, the class of soliton solutions of each of the systems we have considered (namely, the 2DTL and its fully discrete and its ultra-discrete analogues) is much wider, and includes also partially resonant solutions. The solutions described in this work represent the extreme case in which all of the interactions among the various solitons are resonant, whereas ordinary soliton solutions represent the opposite case where none of the interactions among the solitons are resonant. Inbetween these two situations, a number of intermediate cases exist in which only some of the interactions are resonant. As in the case of the KP equation, the study of these partially resonant solutions remains an open problem.

Acknowledgements

We thank M. J. Ablowitz, Y. Kodama, J. Matsukidaira and A. Nagai for many helpful discussions. K.M. acknowledges support from the Rotary foundation and the 21st Century COE program "Development of Dynamic Mathematics with High Functionality" at Faculty of Mathematics, Kyushu University. G.B. was partially supported by the National Science Foundation, under grant number DMS-0101476.

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