Balanced Formulas, Minimal Formulas and Their Proofs

Sachio Hirokawa

November 7, 1991

Research Institute of Fundamental Information Science
Kyushu University 33
Fukuoka 812, Japan
E-mail: hirokawa@ec.kyushu-u.ac.jp    Phone: 092-771-4161
BALANCED FORMULAS, MINIMAL FORMULAS
AND THEIR PROOFS

SACHI HIROKAWA *
Department of Computer Science
College of General Education
Kyushu University

Abstract

According to the formulas-as-types notion, an implicational formula can be identified with a type of a $\lambda$-term which represents a proof of the formula in implicational fragment of intuitionistic logic. A formula is balanced iff no type variable occurs more than twice in it. It is known that balanced formulas have unique proofs. In this paper, it is shown that closed $\lambda$-terms in $\beta$-normal form having balanced types are BCK-$\lambda$-terms in which each variable occurs at most once. A formula is BCK-minimal iff it is BCK-provable and it is not a non-trivial substitution instance of other BCK-provable formula. It is also shown that the set BCK-minimal formulas is identical to the set of principal type-schemes of BCK-$\lambda$-terms in $\beta$-$\eta$-normal form.

1 Introduction

We study the structure of normal proof figures of implicational formulas provable in BCK-logic, in which each assumption can be used at most once.

*Supported by a Grant-in-Aid for Encouragement of Young Scientists No.02740115 and No.03750298 of the Ministry of Education.
According to the formulas-as-types notion [11], the set of BCK-formulas is identical to the set of types of closed BCK-\(\lambda\)-terms. So we use the words 'types' and 'formulas' with the same meaning.

In [13], Komori raised some conjectures among which is the following.

"If a formula is BCK-minimal then its normal form proof is unique."

Here a BCK-minimal formula is a BCK-provable formula which is not a non-trivial substitution instance of other BCK-provable formula. This conjecture was proved independently by Wronski [17], by Hirokawa [7] and by Tatsuta [16]. Wronski proved the conjecture as a corollary of the coherence theorem in cartesian closed category. Babaev and Soloviev [1] and Mints [15] formulated the theorem as follows and gave simple proofs for it.

"If a formula is balanced then its normal form proof is unique."

Here a formula is balanced iff no type variable occurs more than twice. Balanced formulas are called as formulas with one-two-property in [6, 12]. We denote the set of balanced formulas as \(F_{1,2}\). The author learned a result by Jaskowski [12] from P. Idzjak which states that

\[ BCK \cap F_{1,2} = LJ \cap F_{1,2}. \]

By analysis of the type assignment figures, we prove that if a \(\lambda\)-term in \(\beta\)-normal form has balanced type, then it is a BCK-\(\lambda\)-term. This gives a direct proof for Jaskowski’s result.

The problem by Komori, the proof by Wronski and Tatsuta and the coherence theorem concerns the uniqueness of proof figure not in \(\beta\)-normal form but in \(\beta-\eta\)-normal form. A precise statement of Wronski’s answer is as follows.

"If \(\alpha\) is a BCK-minimal then the closed BCK-\(\lambda\)-term in \(\beta-\eta\)-normal form which has \(\alpha\) as its type is unique."

On the other hand, the author’s solution [7] for Komori’s problem is as follows.

2
"If \( \alpha \) is a BCK-minimal then the closed BCK-\( \lambda \)-term in \( \beta \)-normal form which has \( \alpha \) as its type is unique."

We prove that a type-scheme \( \alpha \) of a closed BCK-\( \lambda \)-term is BCK-minimal iff \( \alpha \) is a principal type-scheme of some closed BCK-\( \lambda \)-term in \( \beta-\eta \)-normal form. This clarifies the difference between \( \beta \)-normal forms and \( \beta-\eta \)-normal form.

2 Balanced types and BCK-\( \lambda \)-terms

We use the terminology in Hindley [4] for type assignment figures to \( \lambda \)-terms. We say TA-figures instead of type assignment figures. Types are constructed from type variables and ‘\( \to \)’. We use the letters \( a, b, c, \cdots \) for type variables, \( \alpha, \beta, \gamma, \cdots \) for types, \( x, y, z, \cdots \) for term variables and \( L, M, N, \cdots \) for \( \lambda \)-terms. The set of type variables in a type \( \alpha \) is denoted by \( \text{var}(\alpha) \). The set of free variables in a \( \lambda \)-term \( M \) is denoted by \( \text{FV}(M) \).

We write \( B \vdash M : \alpha \) for type assignment \( \alpha \) to \( \lambda \)-term \( M \) from an assumption set \( B = \{ x_1 : \alpha_1, \cdots, x_n : \alpha_n \} \). Note that the set \( \{ x_1, \cdots, x_n \} \) of the subjects of \( B \) is identical to \( \text{FV}(M) \). When \( (B, \alpha) \) is not a non-trivial substitution of any \( (B', \alpha') \) such that \( B' \vdash M : \alpha' \), it is said to be a principal type assignment and the pair \( (B, \alpha) \) is called a principal-pair. In such a case, we write \( B \models M : \alpha \). When \( M \) is a closed-\( \lambda \)-term, \( \alpha \) is called a principal-type-scheme of \( M \). Given a set of closed \( \lambda \)-terms \( T \), \( \text{ts}(T) \) and \( \text{pts}(T) \) denotes the set of type-schemes of terms in \( T \) and the set of principal type-schemes of terms in \( T \).

A BCK-\( \lambda \)-term is a \( \lambda \)-term in which each variable occurs at most once. We consider only BCK-\( \lambda \)-terms in \( \beta \)-normal form.

**Definition 1 (BCK-\( \lambda \)-term)** The set of BCK-\( \lambda \)-terms is defined inductively as follows.

1. If \( x \) is a variable then \( x \) is a BCK-\( \lambda \)-term.

2. If \( M \) and \( N \) are BCK-\( \lambda \)-term and \( \text{FV}(M) \cap \text{FV}(N) = \emptyset \) then \((M \cdot N)\) is a BCK-\( \lambda \)-term.

3. If \( M \) is a BCK-\( \lambda \)-term and \( x \) is a variable then \((\lambda x . M)\) is a BCK-\( \lambda \)-term.
Definition 2 The core of a type \( \alpha \), denoted by \( \text{core}(\alpha) \), is the rightmost type variable in \( \alpha \).

We refer [6] to the history of the one-two-property and the two-property.

Definition 3 (balanced types) A type is balanced iff each type variable in the type occurs at most twice. The set of balanced types is denoted by \( F_{1,2} \).

Balanced formulas are identical to formulas with one-two-property in [6, 12]. We denote by \( BCK \) and \( BCK-\beta \) the set of closed \( BCK-\lambda \)-terms and the set of closed \( BCK-\lambda \)-terms in \( \beta \)-normal form respectively. We write \( ts(T) \) for the set of type-schemes of terms in \( T \), and write \( pts(T) \) for the set of principal type-schemes of terms in \( T \). Since \( BCK-\lambda \)-terms have types [6], we have \( pts(BCK) = pts(BCK-\beta) \). The fact that

\[
pts(BCK-\beta) \subseteq F_{1,2}
\]

was known to Belnap [2]. It is also known that

\[
pts(BCI) = ts(BCI) \cap F_2.
\]

(See [14, 6, 9].) Here \( BCI \) is the set closed \( BCI-\lambda \)-terms in which each variable occurs exactly once, and \( F_2 \) is the set of types with two-property, i.e., the set of types in which each type variable occurs twice. So it would be tempting to hope \( pts(BCK) = ts(BCK) \cap F_{1,2} \). But this is not the case. Characterizations of \( pts(BCK) \) and \( pts(BCI) \) are shown in [9, 10].

Jaskowski [12] proved that \( BCK \)-logic is strong enough to prove the balanced formulas. He proved

\[
ts(BCK) \cap F_{1,2} = IL \cap F_{1,2},
\]

where \( IL \) is the set of types of closed-\( \lambda \)-terms or equivalently the set of provable implicational formulas in intuitionistic logic. This result means that if one-two-formula is provable in \( IL \) then it is provable in \( BCK \)-logic. In the rest of this section, we prove the following theorem which states that if a balanced is provable in \( IL \) then its proof is in \( BCK \)-logic.

Theorem 1 For any type \( \alpha \in F_{1,2} \) and a closed \( \lambda \)-term \( M \) in \( \beta \)-normal form, if \( \vdash M : \alpha \) then \( M \) is a \( BCK-\lambda \)-term.
Proof. We prove the following claim which includes the theorem as a special case where the assumption set is empty.

Claim: Let $M$ be a $\lambda$-term in $\beta$-normal form and $B \vdash M : \alpha$. If $(B, \alpha)$ has the one-two-property then $M$ is a BCK-$\lambda$-term.

Here, a pair $(B, \alpha)$ of an assumption set $B = \{x_1 : \alpha_1, \cdots, x_n : \alpha_n\}$ and $\alpha$ is balanced iff $\alpha_1 \rightarrow \cdots \rightarrow \alpha_n \rightarrow \alpha$ is balanced. We prove this claim by induction on $M$.

1. $M = x$. Then $x$ is a BCK-$\lambda$-term.

2. $M = \lambda x. N$. Then we have $\alpha = \beta \rightarrow \gamma$. A TA-figure for $B \vdash M : \alpha$ has the following form.

$$
\begin{array}{c}
[x : \gamma] \\
\vdots \\
N : \beta \\
\overline{\lambda x. N : \gamma \rightarrow \beta}
\end{array}
$$

2.1 $x \in FV(N)$. Then we have $B \cup \{x : \gamma\} \vdash N : \beta$. Since $(B, \gamma \rightarrow \beta)$ has the one-two-property, so does $(B \cup \{x : \gamma\}, \beta)$. By induction hypothesis, $N$ is a BCK-$\lambda$-term. Therefore $\lambda x. N$ is a BCK-$\lambda$-term.

2.2 $x \notin FV(N)$. Then we have $B \vdash N : \beta$. Since $(B, \gamma \rightarrow \beta)$ has the one-two-property, so does $(B, \beta)$. Therefore $N$ is a BCK-$\lambda$-term. So is $\lambda x.N$.

3. $M = xM_1 \cdots M_n (n \geq 1)$.

First we claim that $x \notin FV(M_i)$ for $i = 1, \cdots, n$. Assume that $x \in FV(M_i)$. Then a TA-figure for $B \vdash xM_1 \cdots M_n : \alpha$ has the following form.

$$
\begin{array}{c}
x : \xi \quad \cdots \\
\vdots \\
M_i : \alpha_i \\
\overline{xM_1 \cdots M_i \cdots M_n : \alpha}
\end{array}
$$

Here $\xi = \alpha_1 \rightarrow \cdots \rightarrow \alpha_i \rightarrow \cdots \rightarrow \alpha_n \rightarrow \alpha$. Let $b = \text{core}(\alpha)$. We derive a contradiction by case analysis depending on whether $b \in \text{var}(\alpha_i)$ or not.

Case (i) $b \in \text{var}(\alpha_i)$. Then $b$ occurs twice in $x : \alpha_1 \rightarrow \cdots \rightarrow \alpha_i \rightarrow \cdots \rightarrow \alpha_n \rightarrow \alpha$ and occurs once in $xM_1 \cdots M_n : \alpha$. Thus $b$ occurs three times in $(B, \alpha)$. This contradicts that $(B, \alpha)$ has the one-two-property.

Case (ii) $b \notin \text{var}(\alpha_i)$. Consider the sub-TA-figure $P_i$ with the end-formula $M_i : \alpha_i$. Note that $b$ occurs at the assumption $x : \alpha_1 \rightarrow \cdots \rightarrow \alpha_n \rightarrow \alpha$ and
that $b$ does occur at the end-formula $M_i : \alpha_i$. Traverse the sequence of TA-formulas from the assumption $x : \alpha_1 \rightarrow \cdots \rightarrow \alpha_n \rightarrow \alpha$ to $M_i : \alpha_i$. Then we would find a point where $b$ disappear from the predicate of a TA-formula. The lowest TA-formulas, among that sequence, which contains $b$ and below which $b$ does not appear, is the minor premiss of an $(\rightarrow E)$. Let the TA-formula be $Q : \gamma$. Then $P_i$ has the following form.

\[
\frac{y : \gamma_1 \rightarrow \cdots \gamma_k \rightarrow \gamma \rightarrow \delta \quad R_1 : \gamma_1 \rightarrow \cdots \rightarrow R_k : \gamma_k \quad Q : \gamma (\rightarrow E) \quad x : \alpha_1 \rightarrow \cdots \rightarrow \alpha_n \rightarrow \alpha}{yR_1 \cdots R_k Q : \delta}
\]

\[
M_i : \alpha_i
\]

Since $b$ does not occur from $yR_1 \cdots R_k Q : \delta$ to $M_i : \alpha_i$, $y : \gamma_1 \rightarrow \cdots \gamma_k \rightarrow \gamma \rightarrow \delta$ is not discharged. Therefore $y \in FV(M_i)$. Thus $y \in FV(xM_1 \cdots M_n)$. Therefore $b$ occurs in the core of $x : \alpha_1 \rightarrow \cdots \rightarrow \alpha_n \rightarrow \alpha$, in $\gamma$ of $y : \gamma_1 \rightarrow \cdots \gamma_k \rightarrow \gamma \rightarrow \delta$ and in $xM_1 \cdots M_n : \alpha$. A contradiction. This completes the first claim.

Secondly we claim that $M_i$ is a BCK-$\lambda$-term. It suffices to show that $(B_i, \alpha_i)$ has the one-two-property, where $B_i$ is the assumption set for the sub-TA-figure whose end-formula is $M_i : \alpha_i$. If this is proved, we can apply induction hypothesis for $(B_i, \alpha_i)$ obtaining that $M_i$ is a BCK-$\lambda$-term. Let $b$ be a type variable in $(B_i, \alpha_i)$. Then we have the following inequalities where $\#b : \cdots$ means the number of occurrences of $b$ in assumption sets.

\[
\#b : (B_i, \alpha_i) \leq \sum_{j=1}^{n} \#b : (B_j, \alpha_j)
\]

\[
\leq \#b : ( ((\cup_{j=1}^{n} B_j) \cup \{x : \alpha_1 \rightarrow \cdots \alpha_n \rightarrow \alpha \} )
\]

\[
\leq \#b : B
\]

\[
\leq \#b : (B, \alpha)
\]

\[
\leq 2
\]

Thus $(B_i, \alpha_i)$ has the one-two-property. This completes the second claim.

Thirdly we claim that $FV(M_i) \cap FV(M_j) = \emptyset$ for $i \neq j$. Assume that $z \in FV(M_i) \cap FV(M_j)$ for $i \neq j$. Let $z$ be the leftmost free variable in $M_i$ which occurs also in $FV(M_j)$. Let $\gamma$ be the subject to $z$ and $b = \text{core}(\gamma)$. 

6
We derive a contradiction by case analysis depending on whether \( b \) occurs in \( \alpha_i \) and in \( \alpha_j \) or not.

\[
x : \alpha_1 \to \cdots \to \alpha_n \to \alpha \quad \cdots P_i \quad \{ \begin{array}{c}
z : \gamma \\
\vdots \\
M_i : \alpha_i
\end{array} \quad \cdots \quad P_j \quad \{ \begin{array}{c}
z : \gamma \\
\vdots \\
M_j : \alpha_j
\end{array} \\

\]

\[x M_1 \cdots M_n : \alpha\]

Case (i) \( b \in \text{var}(\alpha_i) \) and \( b \in \text{var}(\alpha_j) \). Then \( b \) occurs twice in \( x : \alpha_1 \to \cdots \to \alpha_i \to \cdots \to \alpha_j \to \cdots \to \alpha_n \to \alpha \) and once in \( z : \gamma \). By the first claim \( x \neq z \). Therefore \( b \) occurs three times in \((B, \alpha)\). This is a contradiction.

Case (ii) \( b \) occurs only one of \( \alpha_i \) and \( \alpha_j \). We can assume that \( b \in \text{var}(\alpha_i) \) and \( b \notin \text{var}(\alpha_j) \). Consider the sequence of TA-formulas from the assumption \( z : \gamma \) to \( M_j : \alpha_j \). Let \( R : \beta \) be the lowest TA-formula along the sequence such that \( b \in \text{var}(\beta) \) and \( b \) does not occur below \( R : \beta \). Since \( M_j \) is in \( \beta \)-normal form, \( R : \beta \) is the minor premiss of an \((\to E)\).

\[
y : \beta_1 \to \cdots \to \beta_k \to \beta \to \delta \quad Q_1 : \beta_1 \quad \cdots \quad Q_k : \beta_k \\
\begin{array}{c}
\vdots \\
yQ_1 \cdots Q_k : \gamma \to \delta \\
\vdots \\
R : \beta (\to E)
\end{array}
\]

\[M_j : \alpha_j\]

Since \( b \) does not occur from \( yQ_1 \cdots Q_k R : \delta \) to \( M_j : \alpha_j \), \( y : \beta_1 \to \cdots \to \beta_k \to \beta \to \delta \) is not discharged. Therefore \( y \in \text{FV}(M_j) \). By the first claim \( x \neq y \) and \( x \neq z \). By the second claim \( y \neq z \). Therefore we have three occurrences of \( b \) in \( y : \beta_1 \to \cdots \to \beta_k \to \beta \to \delta \), in \( x : \alpha_1 \to \cdots \to \alpha_i \to \cdots \to \alpha_n \to \alpha \) and in \( z : \gamma \). A contradiction.

Case (iii) \( b \notin \text{var}(\alpha_i) \) and \( b \notin \text{var}(\alpha_j) \). Let \( R : \beta \) be the same TA-formula in case (ii). (See above figure.) Let \( S : \nu \) be the lowest TA-formula, in the sequence of TA-formulas from \( z : \gamma \) to \( M_i : \alpha_i \), such that \( b \in \text{var}(\nu) \) and \( b \) does not occur below \( S : \nu \).

\[
u : \xi_1 \to \cdots \to \xi_i \to \nu \to \mu \quad S_1 : \xi_1 \quad \cdots \quad S_i : \xi_i \\
\begin{array}{c}
\vdots \\
uS_1 \cdots S_i \to \mu \\
\vdots \\
S : \nu (\to E)
\end{array}
\]

\[M_i : \alpha_i\]
Remember that $z$ is the leftmost free variable in $M_i$ which occurs also in $FV(M_j)$. Thus $u \neq y$. By the second claim, $M_i$ and $M_j$ are BCK-$\lambda$-terms. Therefore $z \neq u$ and $z \neq y$. Thus $b$ occurs three times in $u : \xi_1 \rightarrow \cdots \rightarrow \xi_i \rightarrow \nu \rightarrow \mu, y : \beta_1 \rightarrow \cdots \rightarrow \beta_k \rightarrow \beta \rightarrow \delta$ and $z : \gamma$. A contradiction. This completes the third claim.

From the first, second and the third claim, $x M_1 \cdots M_n$ is a BCK-$\lambda$-term.

\section{BCK-minimal types and BCK-$\lambda$-terms in $\beta$-$\eta$-normal form}

A BCK-minimal type is a type of a closed BCK-$\lambda$-term which is not a non-trivial substitution instance of any type of closed BCK-$\lambda$-term. The notion was defined by Komori in [13], where he conjectured that any BCK-minimal formula has the unique normal form proof. We can state the conjecture in terms of type assignment to BCK-$\lambda$-terms as follows.

Let $M$ and $N$ be closed BCK-$\lambda$-terms in $\beta$-normal form and let $\alpha$ be a BCK-minimal type. If $\vdash M : \alpha$ and $\vdash N : \alpha$ then $M =_\eta N$.

This conjecture was proved independently by Wronski [17], by Hirokawa [7] and by Tatsuta [16]. Wronski derived it from the following theorem by Mints [15].

\textbf{Theorem 2} ([15]) Let $M$ and $N$ be closed $\lambda$-terms in $\beta$-normal form and let $\alpha$ be a balanced type. If $\vdash M : \alpha$ and $\vdash N : \alpha$ then $M =_\eta N$.

We [7] proved the conjecture as a corollary of the following theorem.

\textbf{Theorem 3} ([7]) Let $M$ and $N$ be closed $\lambda$-term in $\beta$-normal form and $\alpha$ be a type. If $\vdash M : \alpha$ and $\vdash N : \alpha$ then $M = N$.

Note that if $M$ has a BCK-minimal type $\alpha$ as its type-scheme, then $\alpha$ is a principal type-scheme of $M$. Thus Komori's conjecture is a corollary of the above theorem. Moreover $\eta$-convertibility in the Komori's conjecture is replaced by equality. In [7], we left the following conjecture concerning to the BCK-minimality and $\eta$-normal form.
A type $\alpha$ is BCK-minimal iff $\alpha$ is a principal type-scheme of some closed BCK-$\lambda$-term in $\beta$-$\eta$-normal form.

Note that $pts(BCK-\beta-\eta) \subseteq ts(BCK-\beta)$. In fact, $\alpha = (a \rightarrow b) \rightarrow a \rightarrow b$ is a principal type-scheme of $BCK-\lambda$-term $\lambda x y . x y$. By Theorem 3, $\lambda x y . x y$ is a unique $BCK-\lambda$-term in $\beta$-normal form. Since $\lambda x y . x y$ is not in $\eta$-normal form, $\alpha \notin pts(BCK-\beta-\eta)$.

**Theorem 4 (Characterization of BCK-minimal types)**

$$pts(BCK-\beta-\eta) = BCK-\text{minimal}.$$  

**Remark 1** We proved, in [9], $pts(BCI-\beta-\eta) = BCI$-minimal. The proof of $pts(BCK-\beta-\eta) \supseteq BCK$-minimal is essentially the same to that of $pts(BCI-\beta-\eta) \supseteq BCI$-minimal. But the proof for the opposite inclusion is different to that for $BCI-\lambda$-terms in [9].

**Proof of $pts(BCK-\beta-\eta) \supseteq BCK$-minimal.** Let $\alpha$ be a $BCK$-minimal type. Then there is a closed BCK-$\lambda$-term $M$ in $\beta$-normal form such that $\vdash M : \alpha$. Let $P$ be a TA-figure for $\vdash M : \alpha$. Assume that $M$ is not in $\eta$-normal form. Then $M$ contains some subterm $\lambda x . N x$ such that $x \notin FV(N)$. Since $M$ is in $\beta$-normal form, so is $\lambda x . N x$. Therefore $N$ has the form $N = y N_1 \cdots N_n$ and $P$ has the following form.

$$\begin{align*}
P_1 \left\{ &y : \alpha_1 \rightarrow \cdots \rightarrow \alpha_n \rightarrow \beta \rightarrow \gamma, \quad N_1 : \alpha_1 \quad \cdots \quad N_n : \alpha_n \quad x : \beta \\
&\frac{y N_1 \cdots N_n x : \gamma \quad \lambda x . y N_1 \cdots N_n x : \beta \rightarrow \gamma}{\vdash x : \beta}
\right\} P_2
\end{align*}$$

Let $P_1$ be the sub-TA-figure with the end-formula $y N_1 \cdots N_n : \beta \rightarrow \gamma$ and let $P_2$ be the sub-TA-figure with the end-formula $\lambda x . y N_1 \cdots N_n x : \beta \rightarrow \gamma$. In $P_1$ replace $\beta \rightarrow \gamma$ by a new type variable $c$. Let $P_1^*$ be the result of $P_1$. Note that in the rest of $P_2$ the predicate $\beta \rightarrow \gamma$ of $\lambda x . y N_1 \cdots N_n x : \beta \rightarrow \gamma$ does not connect to the predicate of any major premiss of $(\rightarrow I)$. This is proved by induction on $\vdash M : \alpha$. Below the type-assignment formula $\lambda x . y N_1 \cdots N_n x : \beta \rightarrow \gamma$.
β → γ in $P$, replace the connection of $β → γ$ by $c$ and replace the occurrence of the subterm $λx.yN_1 \cdots N_n x$ by $yN_1 \cdots N_n$. After that rewriting, replace $P_2$ by $P^*_1$. Then we obtain a TA-figure for $t_0 = M[λx.yN_1 \cdots N_n x/yN_1 \cdots N_n] : δ$ where $α = δ[c := β → γ]$. This contradicts the minimality of $α$. Therefore $M$ is in $η$-normal form.

To prove $pts(BCK-β-η) ⊆ BCK-minimal$ we need the following lemma which is a special case of the subject expansion theorem [4].

**Lemma 1** Let $M$ and $N$ be closed BCK-$λ$-terms in $β$-normal form and $||- M : α$. If $N$ is η-reducible to $M$ then $||- N : αθ$ for some substitution $θ$. 

**Proof.** We perform a transformation to the TA-figure $P$ for $||- M : α$. It is the reverse transformation we applied in the proof of $pts(BCK - β - η) ⊇ BCK - minimal$. The proof is by induction on the number of η-reductions. It suffices to prove the case where $N$ is η-reducible to $M$ by one-step. Since $N$ is η-reducible to $M$, $N$ contains a η-redex $λx.Qx$ such that $x \notin FV(Q)$. Since $N$ is in $β$-normal form, $Q$ has the form $Q = yQ_1 \cdots Q_n$. The contractum of $λx.yQ_1 \cdots Q_n x$ is $yQ_1 \cdots Q_n$. Note that the occurrence $yQ_1 \cdots Q_n$ is not in a function part, i.e., $yQ_1 \cdots Q_n$ does not occur as $yQ_1 \cdots Q_n R$ in $M$. If the contractum occurred in that form, then η-redex in $N$ has the form $(λy.Q_1 \cdots Q_n x)R$. This contradicts the $β$-normality of $N$. Therefore $P$ has the following form.

\[
\begin{align*}
  yQ_1 \cdots Q_n x & : b \\
  \vdots & \\
  M & : α
\end{align*}
\]

Here $b$ is a type variable. Let $P_1$ be the sub-TA-figure whose end-formula is $yQ_1 \cdots Q_n x : b$. Let $c$ and $d$ be new type variables. First rewrite all the occurrences of $b$ by $c → d$. Let $P^*_1$ be the result of $P_1$ by this rewriting and $P^*$ be the result of $P$. Next replace this $P^*_1$ by the following $P_2$. Finally in $P^*$, rewrite the occurrence of $yQ_1 \cdots Q_n$ below $y : yQ_1 \cdots Q_n : c → d$ by
\[ \lambda x.yQ_1 \cdots Q_nx. \] This becomes a TA-figure for \[ \mathord{\vdash}M : \alpha[b : c \to d] . \]

\[ P_1^* \left\{ \begin{array}{l}
\vdash yQ_1 \cdots Q_n : c \to d \\
\vdash x : c \\
\vdash yQ_1 \cdots Q_nx : d \\
\vdash \lambda x.yQ_1 \cdots Q_nx : c \to d
\end{array} \right\} \]

\[ P_2 \]

\[ N[yQ_1 \cdots Q_n/\lambda x.yQ_1 \cdots Q_nx] : \alpha[b := c \to d] \]

**Proof of \( \text{pts}(\text{BCK}-\beta-\eta) \subseteq \text{BCK-minimal} \)** Let \( \alpha \in \text{pts}(\text{BCK}-\beta-\eta) \) and \( \mathord{\vdash}M : \alpha \) for a closed BCK-\( \lambda \)-term \( M \) in \( \beta-\eta \)-normal form. To prove the minimality of \( \alpha \), assume \( \alpha = \beta\theta_1 \) for some substitution \( \theta_1 \) and \( \beta \) where \( \beta \) is a type-scheme of a closed BCK-\( \lambda \)-term \( N \) in \( \beta \)-normal form, i.e., \( \vdash N : \beta \). By the principal type-scheme theorem \([4, 8]\), there is a type \( \gamma \) and a substitution \( \theta_2 \) such that \( \mathord{\vdash}N : \gamma \) and \( \beta = \gamma\theta_2 \). Therefore \( \gamma\theta_2\theta_1 \). Since \( \vdash N : \beta \), it follows that \( \vdash N : \alpha \). Since \( \alpha \) is a p.t.s. of closed BCK-\( \lambda \)-term in \( \beta \)-normal form, \( \alpha \) is balanced. Thus we have \( \vdash M : \alpha \) and \( \vdash N : \alpha \) for balanced type \( \alpha \). By Theorem 2 we have \( M =_\eta N \). Since \( M \) is in \( \eta \)-normal form, \( N \) is \( \eta \)-reducible to \( M \). By Lemma 1 we have \( \gamma = \alpha\theta \) for some substitution \( \theta \). Thus we have \( \alpha = \gamma\theta_1\theta_2 \) and \( \gamma = \alpha\theta \). Therefore \( \theta_1\theta_2 \) and \( \theta \) is trivial. Thus \( \theta_1 \) is trivial. Thus \( \alpha \) is BCK-minimal. \]

**References**


6. Hindley, J.R., BCK and BCI logics, condensed detachment and the 2-
property, a summary, Report, University of Wolongon, Aug 1990.

7. Hirokawa, S., Principal types of BCK-lambda terms, submitted to The-

8. Hirokawa, S., Principal type assignment to lambda terms, submitted to


10. Hirokawa, S., Relevance graph of BCK-formula Manuscript, Nov 1990,

11. Howard, W.A., The formulae-as-types notion of construction, in: Hind-
ley and Seldin Ed., To H.B. Curry, Essays on Combinatory Logic,

12. Jaskowski, S., Über Tautologien, in welchen keine Variable mehr lls

on Semigroups, Sakado 1986, (Josai University, Sakado 1987) 5-11.

14. Meyer, R.K., Bunder, M.W., Condensed detachment and combinators,
Report TR-ARP-8/88, (Research School of the Social Sciences, Aus-
tralian National University, P.O.Box 4, Camberra, A.C.T., Australia,
1988).

15. Mints, G.E., A simple proof of the coherence theorem for cartesian closed
categories, Manuscript 1982.

16. Tatsuta, M., Uniqueness of normal proofs in implicational logic,