THE NUMBER OF PROOFS FOR A BCK-FORMULA

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In this note, we give a necessary and sufficient condition for a BCK-formula to have the unique normal form proof.

We call implicational propositional formulas formulas for short. BCK-formulas are the formulas which are derivable from axioms \( B = (a \rightarrow b) \rightarrow (c \rightarrow a) \rightarrow c \rightarrow b \), \( C = (a \rightarrow b \rightarrow c) \rightarrow b \rightarrow a \rightarrow c \), and \( K = a \rightarrow b \rightarrow a \) by substitution and modus ponens. It is known that the property of being a BCK-formula is decidable (Jaskowski [11, Theorem 6.5], Ben-Yelles [3, Chapter 3, Theorem 3.22], Komori [12, Corollary 6]). The set of BCK-formulas is identical to the set of provable formulas in the natural deduction system with the following two inference rules.

\[
\begin{align*}
\frac{\gamma \vdash \delta}{\gamma \rightarrow \delta} \quad & (\rightarrow I) \\
\frac{\gamma \rightarrow \delta \quad \gamma}{\delta} \quad & (\rightarrow E).
\end{align*}
\]

Here \( \gamma \) occurs at most once in \( (\rightarrow I) \). By the formulae-as-types correspondence [10], this set is identical to the set of type-schemes of closed BCK-\( \lambda \)-terms. (See [5].) A BCK-\( \lambda \)-term is a \( \lambda \)-term in which no variable occurs twice. Basic notions concerning the type assignment system can be found [4]. Uniqueness of normal form proofs has been known for balanced formulas. (See [2, 14].) It is related to the coherence theorem in cartesian closed categories. A formula is balanced when no variable occurs more than twice in it. It was shown in [8] that the proofs of balanced formulas are BCK-proofs. Relevantly balanced formulas were defined in [9], and it was proved that such formulas have unique normal form proofs. Balanced formulas are included in the set of relevantly balanced formulas. We show a necessary and sufficient condition for a BCK-formula to have a unique

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normal form proof using the following notion of minimality. The notion of BCK-minimality was introduced by Komori [13]. A formula \( \alpha \) is called a trivial substitution instance of \( \beta \) iff \( \alpha \) is a substitution instance of \( \beta \) and \( \beta \) is a substitution instance of \( \alpha \).

**Definition 1.** A formula is **BCK-minimal** iff it is a BCK-formula and it is not a nontrivial substitution instance of another BCK-formula. A BCK-formula \( \beta \) is a **minimal formula** of \( \alpha \) iff \( \beta \) is BCK-minimal and \( \alpha \) is a substitution instance of \( \beta \).

It is clear that a BCK-minimal formula is a principal type-scheme of a closed BCK-\( \lambda \)-term.

We identify two \( \lambda \)-terms when they are \( \alpha \)-convertible. Similarly, two types are identified when one is a trivial substitution instance of the other.

**Lemma 1 ([7]).** If two closed BCK-\( \lambda \)-terms in \( \beta \eta \)-normal form have the same principal type, then they are identical.

**Lemma 2 ([8]).** A BCK-formula is BCK-minimal iff it is a principal type-scheme of a closed BCK-\( \lambda \)-term in \( \beta \eta \)-normal form.

**Theorem 1.** Given a BCK-formula \( \alpha \), the number of closed BCK-\( \lambda \)-terms in \( \beta \eta \)-normal form which has type \( \alpha \) is identical to the number of minimal formulas of \( \alpha \).

**Proof.** Let \( \alpha \) be a BCK-formula. We denote by \( \text{proof}(\alpha) \) the set of closed BCK-\( \lambda \)-terms in \( \beta \eta \)-normal form which have type \( \alpha \) and we denote by \( \text{min}(\alpha) \) the set of minimal formulas of \( \alpha \). We define a function from \( \text{proof}(\alpha) \) to \( \text{min}(\alpha) \) and show that it is surjective and injective. Let \( M \in \text{proof}(\alpha) \). Then \( M \) has type \( \alpha \). By the principal type-scheme theorem (Theorem 15.26 of [4]), \( M \) has a principal type-scheme. We denote it by \( \text{pts}(M) \). Since \( M \) is in \( \beta \eta \)-normal form, \( \text{pts}(M) \) is minimal by Lemma 2. So we have \( \text{pts}(M) \in \text{min}(\alpha) \). Thus \( \text{pts} \) is a function from \( \text{proof}(\alpha) \) to \( \text{min}(\alpha) \). Injectivity of \( \text{pts} \) is immediate from Lemma 1. To prove the surjectivity, let \( \beta \in \text{min}(\alpha) \) and apply Lemma 2 to \( \beta \). Then there is a closed BCK-\( \lambda \)-term \( N \) in \( \beta \eta \)-normal form whose principal type-scheme is \( \beta \). Therefore \( \text{pts} \) is surjective. □

One consequence of the theorem is that a BCK-formula \( \alpha \) has only a finite number of normal form proofs. In fact, we can enumerate all the minimal formulas instead of \( \lambda \)-terms. Given a formula \( \gamma \), we denote by \( s_0(\gamma) \) the set of formulas \( \beta \) such that \( \gamma \) is a substitution instance of \( \beta \). Since we identify trivial substitution instances, the set \( s_0(\gamma) \) is finite. Next we denote by \( s(\gamma) \) the set of BCK-formulas in \( s_0(\gamma) \). Since BCK-provability is decidable, we can enumerate the elements of \( s(\gamma) \) from \( s_0(\gamma) \). Finally note that \( \beta \) is BCK-minimal iff \( s(\beta) = \{ \beta \} \). Therefore we have

\[
\text{min}(\alpha) = \{ \beta \in s_0(\alpha) \mid s(\beta) = \{ \beta \} \}.
\]

Thus we can enumerate all the elements of \( \text{min}(\alpha) \).

Akama [1] showed that the number of cut-free proof (in sequent calculus) for a BCK-formula is finite.

**Corollary 1.** A BCK-formula has a unique proof in \( \beta \eta \)-normal form iff it has a unique minimal formula.

**References**


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